Hexylvania

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Contents

Abstract

Analyzing the relation between adjacency of tessellated hexagons and the minimum quantity of numbers required to fill 91 of said hexagons we will attempt to determine and prove the minimum quantity of numbers required to satisfy rules around adjacency and repetition of numbers. We will explore mainly immediate adjacency, adjacency separated by one shared hexagon, changing how large the difference in adjacent hexagons must be, and combining aforementioned rules.

1 Introduction

The fictional country of Hexylvania that is comprised of 91 adjacent evenly-spaced states shaped as hexagons is attempting to construct a series of radio towers. The problem arises when different radio towers interfere with another depending on their proximity. In this paper we will explore how to assign values to hexagons such that they follow certain adjacency rules that will differ from section to section. For the sake of ease, in presented diagrams we will mainly use different colours to represent different numbers, only adding numbers to display where the individual relation between hexagons is necessary or when diagrams become to complex for colours alone. When describing diagrams, we will refer to the “north” of the diagram as that, that is closest to the top of the page and determine other directions accordingly. Further when discussing how close two hexagons are in proximity of each other, we will use the term “jacency” where 1-jacency refers to hexagons that while do not share a border with one another, are both immediately adjacent with a common hexagon. Whatever level of jacency will determine how many hexagons will be permitted between two hexagons of the same number. When looking at Hexylvania, we see that because the “radius” of the country is five states, such that every hexagon will be 10-jacent to all other states. We will define a “cluster” as some group of hexagons where each hexagon is some level of adjacency to all the others. We will explore the relation between jacency and the minimum size of the spectrum — which we will define as the quantity of required numbers to satisfy the jacency rules. We will provide solutions that will not only work for the 91 states of Hexylvania, but as a general solution for an arbitrarily large number of hexagons.

2 Adjacency and Variations

2.1 Immediate Adjacency

Analyzing the case of Hexylvania where no adjacent state can have the same frequency or colour, we will look for the minimum size for the spectrum and how to edit our result such that each hexagon is different by some value k.

2.1.1 Minimizing the Spectrum

When looking to find an upper bound for the minimum possible size of the spectrum, we can borrow from the four-colour theorem. The theorem states that you can colour any map, where shapes/objects are simply connected, with a maximum of four colours [1]. Applying

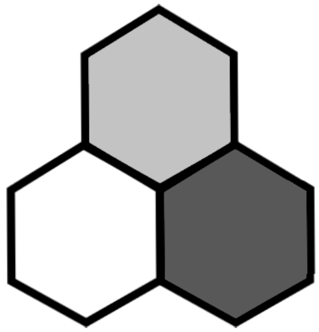

Figure 1: Two Hexagon of Varying Colour

this too Hexylvania we get an upper bound of four numbers or a range of three. Dismissing the case that we could complete this task with only one number, we move on to attempting to fill Hexylvania with two distinct numbers. In Figure 1 we see that if you try to colour a cluster of three hexagons that are all adjacent, that after placing the two colours, the third will be adjacent to both other hexagons and will result in having to break either the 1-jacency rule or the 2-number rule. Choosing to break the 2-number rule, we more on to our analysis of 3-numbers.

Evidently, we can fill any cluster of three (as seen in Figure 1) with three numbers and have them follow the 1-jacency rule. From there, if we "coat" that cluster in another layer of hexagons as we did in Figure 2. We see that no dark-grey hexagon touch all three of the original grey. Seeing that none of the dark-grey can never be adjacent to more than two of the grey, we can number the dark-grey hexagons by what ever grey hexagon they are not adjacent too. At this moment we are only concerned with the instance of where the hexagons are immediately adjacent to each other, so the middle grey blocks will not have any direct effect on those outside of its coat/those directly surrounding it. Because of this we can continue this process, reselecting what hexagons will be considered our "middle grey" cluster, filling Hexylvania. We see here that we were able to successfully fill the 91 hexagons without any two being adjacent and sharing the same colour, which culminates in Figure 3. This leads us to the conclusion that we need a minimum spectrum of three for Hexylvanias broadcasting system to function.

2.1.2 Differing by K

Furthering the problem, if we set the problem as such that two adjacent states must differ by some number k we can determine that our spectrum must be at least size $2k$. In Figure 4 we see some hexagon has been assigned some number n, from there we can assign one of the other hexagons $n+k$. Next, we know that the remaining blank block must differ again by k, so it must be either $n-k$ or $n+k+k$. In the case of $n-k$, we get that the spectrum is of size $(n+k)-(n-k)=2k$ and for $n+2k$ we get $(n+2k)-(n)=2k$. Concluding we need numbers in the range $(n+2k, 2k)$.

Figure 2: Group of Hexagons Showing that No Hexagon is Adjacent to More Than Two Inner Three

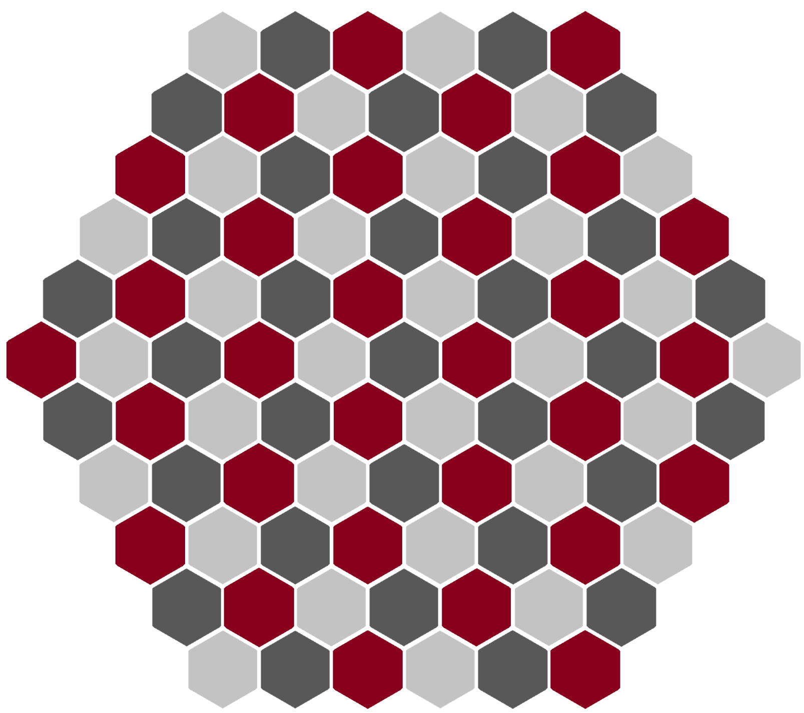

Figure 3: Hexylvania Coloured Following the Rules of 1-Jacency

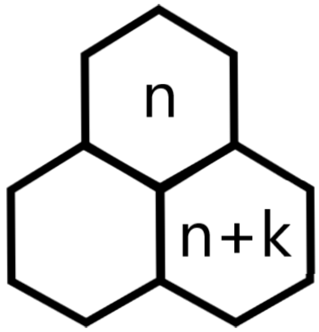

Figure 4: Group of Hexagons Showing Adjacency

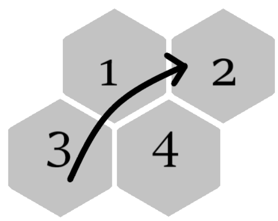

Figure 5: Group of Hexagons Showing 2-jacency

2.2 2-Jacency

Moving on to the case of where states can be the same as those immediately adjacent to it, but not those that are two-jacent to it. Meaning the minimum number of steps or hops you can take to a cell is one. For example, Figure 5 would be illegal because while hexagons 1 and 4 are adjacent to both 2 and 3, hexagons 2 and 3 are 2-jacent to each other. We could choose to state that because you can move from say hex 3 to 4 to hex 1, making hexagon 1 and 3 2-jacent from each other while also being immediately adjacent. For the sake of this paper we will consider lower levels of adjacency to over-ride rules made for higher orders of adjacency.

We can show that the largest collection of hexagons we can create where each is immediately adjacent to each other is 3 by showing that if we add another hexagon to a cluster of two that we will either get a string of three where the two on opposing ends are 2-jacent from each other (making it illegal) or we recreate the group of three in the shape discussed above. If we attempt to attach a fourth hexagon to the cluster, we get again the shape shown in Figure 5 — no matter where you attach the hexagon you can rotate the cluster such that you get found in Figure 5 or a straight string of three hexagons with a branch of one. This time we cannot rely on the four-colour theorem because that only applies to immediate adjacency. To show that we can not use two or three numbers we will rely on a more visual proof. In Figure 6 we see a part of Hexylvania, for this example we will consider the brown hexagons those that have yet to be assigned numbers (Hexagons are numbered

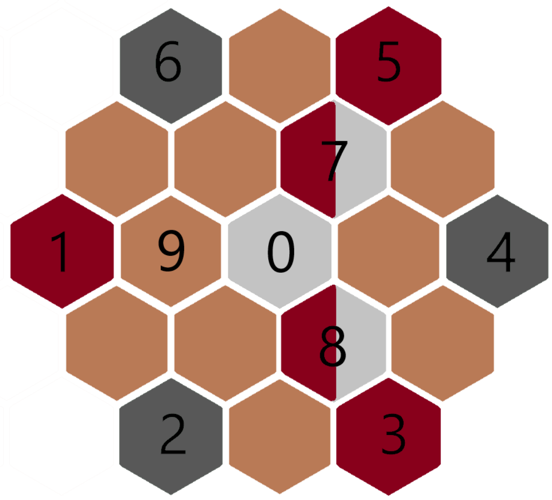

Figure 6: Group of Hexagons Showing Need for Four Numbers

for sake of reference). We can arbitrarily assign the centre hexagon some number (shown in grey), from there we know the hex in space labeled 1 must be some number different than that in 0 (this eliminates the case that we can use only one number). Moving on, in hex 2 we see that it is 2-jacent to both hex 1 and hex 0, so it must be some different number again, this time displayed in dark-grey. This disproves the idea that we can only use two numbers. We can continue to determine hexagons 3 through 6 by what numbers the hex is and is not 2- jacent too. We now see in blocks 7 and 8 that they are 2-jacent to hexagons 2,4 and 6 which are all numbered dark-grey, in still attempting to use three numbers then they must either be grey or burgundy. From there we see that hexagon 9, is 2-jacent to 7 and 8, so it cannot be grey or burgundy, as well as 2 and 6, so it cannot be dark-grey either. Therefore, we get a situation that no matter what, hexagon 9 must be a fourth number — disproving the idea that we can only use three numbers.

Moving forward, we can show that the smallest the spectrum possible is four elements or have a range of three. We can create a horizontal "wall" relying on the triangular shape we created earlier and proved that was the largest shape that we can make where all blocks are 1-jacent, but none are 2-jacent to each other. Rotating every other cluster 180° we can ensure that it will be two hexagons high and arbitrarily long with no holes or gaps — but will be 6 hexagons wide before repetition. For ease of reference, in Figure 7 we will number every block some different number (represented blue, burgundy, grey, and dark-grey) repeating after every four clusters. Knowing that the wall is 6 wide, any given number in the wall will be at least 4 hexagons away from the closest reoccurrence. We can then copy this wall and translate it in the horizontal direction such that each number is directly above the number that is two to the right of it in the row below. Doing this produces what we see in Figure 7.

After translating the wall, we see that it will be 3-jacent to the closest cluster of the same number. If the triangular piece if facing downwards (such that the portion with two to the horizontal to each other is to the north of the Figure), we see that it will be separated from its other occurrences by two hexagons (or be 3-jacent). This holds true as well when the

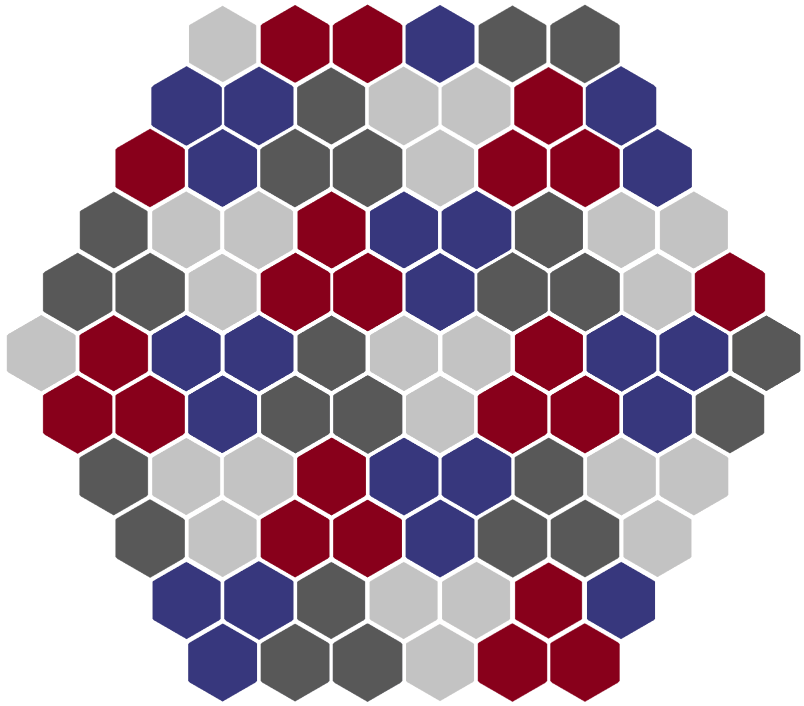

Figure 7: Hexylvania Coloured to Show 2-jacency

base is facing south, for instance, burgundy's base is separated from the row below its tip by two.

By numbering each of the clusters zero through three we see that it reflects a modular system of four. For example, clusters that are four to the right some chosen cluster will be the same number. $0 \equiv 4(\text{mod}4)$. From this we know that by translating the triangular shape two to the right that every other row will be the same as the original, $0 + 2(2k) \equiv 4(\text{mod}4)$. We therefore get that every repetition of two joining walls will be the exact same.

2.2.1 Differing by K

Again, moving on to determine how small we can make the spectrum we can again rely on the portion of Hexylvania as displayed in Figure 6. We know that no matter how we distribute the number assignments we will get that we must use at least four numbers. Following the logic as discussed in the previous time in section 2.1.2 we get that each number must differ by exactly k, or whatever value we say each 2-jacent hexagon must differ by. Arranging this, we see that our greatest value will be $n + 3k$ where we define n as the smallest accepted value. We then get that the spectrum must range from n to $n + 3k$ giving a range size of $3k$.

3 Other Values in the Spectrum

Imagining a scenario where the people of Hexylvania want to use some radio frequency that is not a multiple of k but is still within the bounds of the given rules, be it one or two jacency

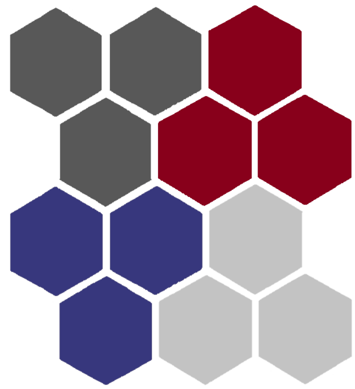

Figure 8: Group of 12 Hexagons following 2-jacency

here we will analyze if this is possible and if not, how to enable it.

3.1 One Jacency

Remembering that in the case of 1-jacency we may have a minimum spectrum size of 3 different values. Assuming that they each vary from each other by a factor of k, we will assume the three used values are $0, k$ and $2k$. From here, looking at the shape shown in Figure 10, if the middle point is the centre value, k, because it is adjacent to all other hexagons it is not possible for the other hexes to be $k+p, p \neq 0$ and still remain in the given spectrum.

If the value in hex 1 is the smallest value, we get that the second placed value (say in hex 2) can be any value in the range $(k, 2k)$. After placing some value $k+p, 0<p<k$ we get that both hexagons immediately adjacent to both hex 1 and 2 must be a minimum of $2k+p$ — which is greater than $2k$ for all instances where $p \neq 0$. The case where the greatest value is placed in hex 1 is logically equivalent to the case where the minimum value is placed there and shows us the same result.

3.2 Two Jacency

When we consider 2-jacency, we can show that we need increments of k for proper adjacency and to follow the rules proposed in section 2.2. If we assume the smallest value is placed in the blue group of three hexagons in Figure 8 we get that the remaining three clusters must be a minimum of k greater than it because all of the clusters are 2-jacent to at least one blue hexagon. Again, following from the same methodology from section 3.1, we will place $k+p, 0<p<k$ in one of the three remaining clusters and because that cluster is also 2-jacent to both remaining clusters, they must be assigned at minimum of $2k+p$ and $3k+p$ which puts it out of our range. The cases where $k, 2k$, or $3k$ are placed first are logically equivalent and result in the same conclusion.

If we attempt to circumvent this and place a new hexagon somewhere around Figure 8, we get that it will either be 2-jacent to either two or three of the clusters. In the case that it is 2-jacent to all but one of the clusters, we see that we are again left with no choice but to assign it the same value as the one it is not 2-jacent from. To show this, we can list the combinations of what values it is 2-jacent too (and therefor must be k different). We get that there are four possible combinations, $\{\{0, k, 2k\}, \{0, k, 3k\}, \{0, 2k, 3k\}, \{k, 2k, 3k\}\}$. Each time we have no choice but to place the missing value.

In the case where the new cluster is placed and is only 2-jacent to two of the previous clusters, there is some freedom in what we assign the new hexagon. Say if 0 and 3k are placed in the blue and grey clusters in Figure 8 and a new hexagon is placed just south of those blocks, it can be labeled any number that is able to be written $k + p, 0 \leq p \leq k$. Of course, if we attempt to continue and place more hexagons we quickly come to find that we no longer have the ability to find a number that is between $k+p$ and $3k$ that has a difference of k with both of them unless $p = 0$.

3.3 Greater Variance

Considering it is not possible to place a new value that is not a multiple of k in Hexylvania and remain using the minimum possible spectrum we will allow for greater possible variance in the spectrum. We see we no longer have to follow the ridged pattern where all values had to be dividable by k. We get that we can assign the hexagons in Figure 8 values of $\{a, k + a + b, 2k + a + b + c, 3k + a + b + c + d\}$ where $a, b, c, d \geq 0$ and $a + b + c + d \leq k$. Not only that, but because we know that the second smallest value will be at minimum $k + a$ we have complete control to reassign any cluster assigned a to any value in range $(0, a + b)$. Similarly, we can reassign any value primarily given $3k + a + b + c + d$ to some new number from range $(2k+a+b+c, 4k)$. We can do each of these without having to reassign any of the other clusters of hexagons, to any or all of the specified values. This is reflected in Figure 9 where we increased the value directly in the middle of Hexylvania to utilize our new possible range of $(0, 4)$ and it does not affect the other clusters which remain k in the difference from the altered cluster. We can also reassign the middle two values depending on the values of $a, b, c,$ and d — but it is more reliant on surrounding values for what will be accepted.

4 More on Interference

When we consider a two-stage interference system where adjacent states must differ by two and 2-jacent states must differ by one, we get a more complex example than those previously discussed. We can show that we need a minimum spectrum of 9 by again using a visual proof. In discussion, we will assume that the smallest value is 0, and we will denote the largest as $K - 1$. When looking to prove this we will be first analyzing the instance where neither the smallest nor largest value are in the centre hex (numbered in Figure 10 as hex 1).

We know that because all shown hexagons are 2-jacent to each other they must differ by at least one, so we must have a minimum spectrum greater than or equal to seven. Further we know that because the value in hexagon 1 is not 0 or $K - 1$ then it must be some value in-between. Hex 1 is immediately adjacent to each of the other shown hexagons meaning that all hexes must differ in value from hex 1 by at least 2 — implying that there are two unused values, $m - 1$ and $m + 1$. In this case we see that there must be seven used values

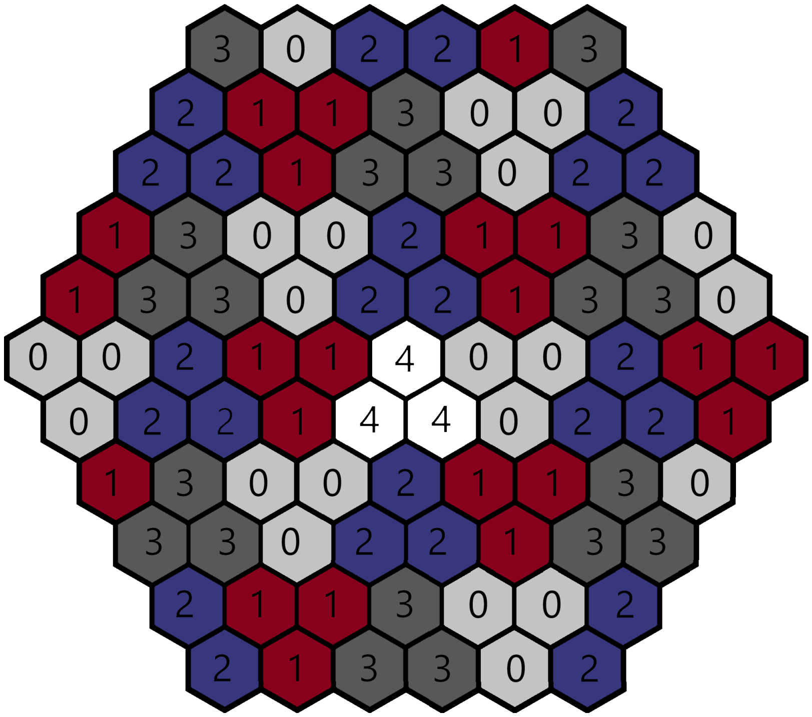

Figure 9: Hexagons following 2-jacency with Spectrum of 4

and the two unused values, totaling to nine, such that $k = 9$.

We can say without loss of generality that the example creates a modular system. It does not matter what values are used but rather their relation to those closest to them. If there are two hexagons immediately adjacent to each other and are given values of 7 and 0 no matter what we add to each, they will always be 2 in the difference from being congruent modular nine. Applying this idea, we can arbitrarily add values to the on the condition that we add them to all hexagons assigned numbers. Of note of course, is that the values 0 and $k-1$ will no longer be considered two in the difference, this is something we will just accept. With this information we can move on to the instance of where either 0 or $k-1$ are in hex

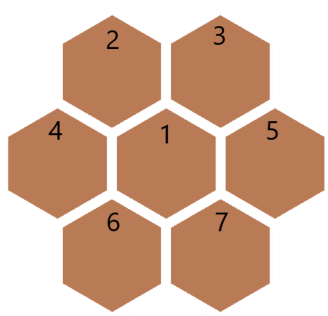

Figure 10: Group of Seven Hexagons

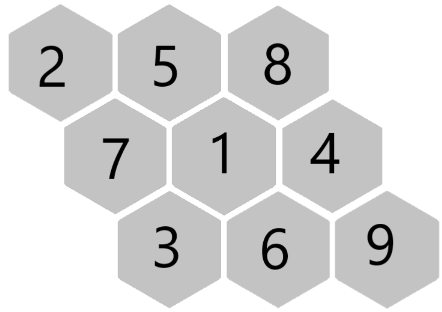

Figure 11: Group of Nine Hexagons Where Adjacent Blocks Differ by Two and 2-jacent Differ by One

1. By simply adding to every value we get that it the inside value is no longer the smallest or greatest value listed, and we get back to the case we discussed first, where some middle value is assigned to the inside.

We can use this to fill Hexylvania with numbers ranging from 1-9. In order to create a pattern that repeats we will create a three column and three row block seen in Figure 11 where each hex in each row is incremented by three, and values that are to the North-East to each hexagon are decreased by 2 each time. This will create Figure 11. In this case, because it is a modular system and our rate of increase divides k, so every three horizontal hex's values will repeat and will always be 3 in the difference. Meanwhile those in North-East direction repeat every nine rows. This way, there will also always be a difference of two in those immediately adjacent, and a difference of one for every 2-jacent hex. This culminates to create Figure 12.

5 Combing Jacency Rules

In the final section, we will explore the relation between figures such that adjacency rules add up, as opposed to replacing one another.

5.1 One and Two Jacency

For instance, in our first example, we will explore one and two jacency. To explain this concept, we will refer back to Figure 5. We see that hex 3 is 1-jacent to hexagons 1 and 4 but also it is 2-jacent to hexagon 2 — so hex 3 must be a different colour than all others shown. First, we will create a tessalatable shape that we do not have to rotate that can follow one and two jacency. Again, referring to Figure 10 we see that if we place some value in any of the hexagons that it will either be one or two jacent to any of the other blocks. We therefore get that we need a minimum of seven unique numbers. Further analyzing this, we see that the hexagons in the top row will be 3-jacent from any placed directly south of the figure. This is true for any of the edge hexagons — we can reuse the number for a newly placed hexagon that is on the opposite side of the cluster. We can essentially use our group

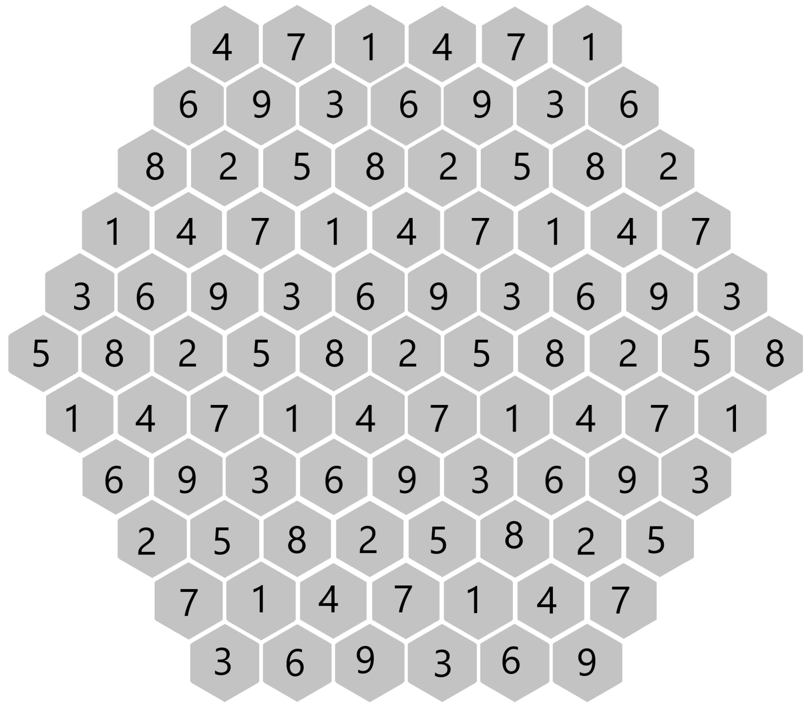

Figure 12: Hexylvania's States Numbered to show Adjacent Blocks That Differ by Two and 2-jacent Differ by One

of hexagons as a barrier so that we can replace values that have been used, remaining in our spectrum of seven numbers.

Further we also see any value placed in the centre (in hex 1) will not be able to be adjacent to any of the other shown hexagons. In order to reuse the centre value, we will need some other value to place between Figure 10 and the newly placed value.

We see that in Figure 13 the colours represent the repetition of the seven hexagons and are solely added to aid the reader in their interpretation of the Figure. If we follow any of the numbered hexagons, we see that each is 3-jacent to the closest hexagon with the same number. Thus proving we need a minimum of seven numbers. We know that this solution is not unique, but rather again depends on the proximity of each hexagon to the same. We can switch any number and as long as consistency is applied, Hexylvanias radio towers will still be able to function.

5.2 One, Two, and Three Jacency

Continuing with this, we can add the rule that numbers may not be repeated for hexagons that are 3-jacent as well. Similar to section 5.1, we will start with creating a legal tessalatable shape that we do not have to rotate.

Seeing in Figure 14 that every hexagon is one, two, or three jacent to every other hexagon in the figure. The hexagons in the southern most, still have influence on those in the north. From this we get that we need a minimum spectrum of 12 hexagons in order to satisfy one,

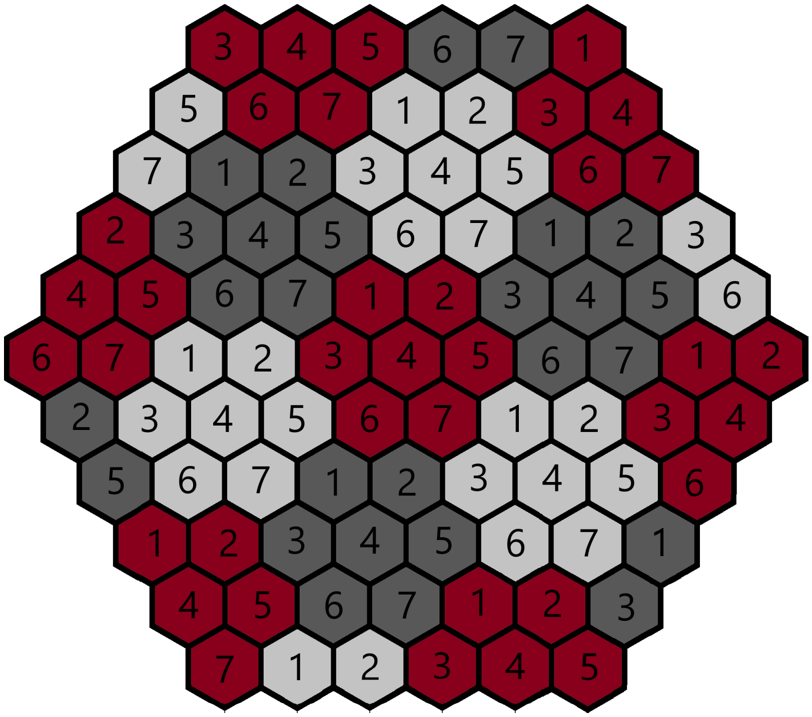

Figure 13: Hexylvania With One and Two Jacent Rules Applied

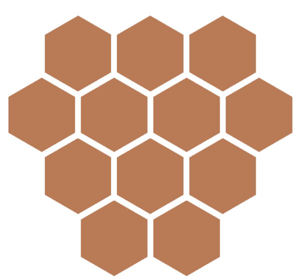

Figure 14: Group of Twelve Hexagons

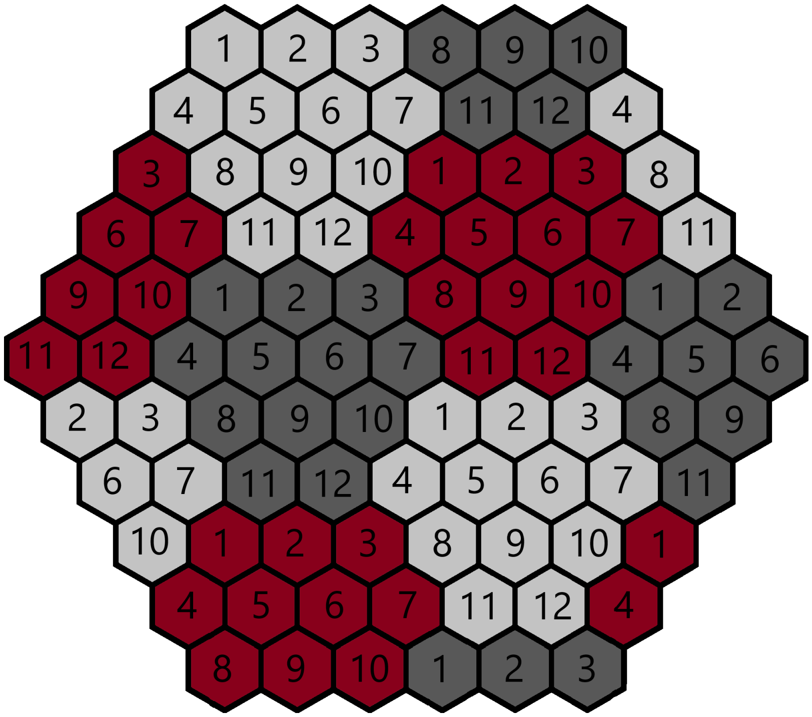

Figure 15: Hexylvania With One and Two and Three Jacent Rules Applied

two, and three jacency. By tessellating this shape, we get the answer seen in Figure 15.

Following the logic and reasoning in section 5.1, we see that the shape acts as a barrier for newly added hexagons. We can add the southern most hexagons to the north side of Figure 14 legally, and the western most to the east wall and vice-versa. Following any one number we see that it is 4-jacent to the closest hexagon assigned the same number/is in the same relative position in the tessellated shape.

Similar to previous discussed examples, the answer is not unique. Numbers assigned are of no relevance but rather the repetition of numbers. From this we can switch which hexagons are assigned what numbers such that it stays legal and within the spectrum.

5.3 Greater Values of Jacency

Continuing on this we will search for some equation that predicts the minimum number of hexagons needed for up to n-jacency.

5.3.1 Even Values

We will start with examples of where n is some even natural number. Searching for some base case, we see that if we permit 0-jacency, or where any two adjacent states may share the same value, we get that we still need a minimum spectrum with one element. Analyzing the case discussed in section 5.1, we know the minimum quantity of numbers needed will be seven. The smallest possible shape we created in section 5.1 resembles that of 0-jacency

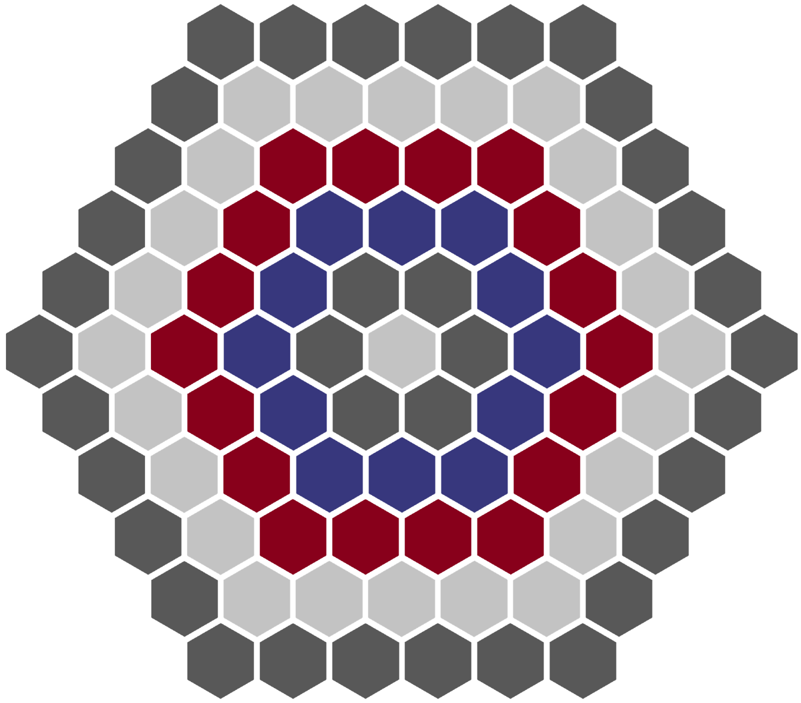

Figure 16: Solutions to Odd-Jacency Problems

(a single hexagon) except it has a "coat" wrapped around it. Following this idea, if we surround the cluster shown in Figure 10 in another layer of hexagons, such that the original hexagons shown are now adjacent to six hexagons. Exploring this new shape, we see that each hexagon is now either one, two, three, or four jacent to all other hexagons and is the solution to the 1, 2, 3, and 4 jacency problem. Continuing this and adding a new layer/coat to the previous answer gives us the sequence of solutions of $\{1, 7, 19, 37, 61, 91...\}$. Noting that the list of solutions can be expressed as $\{1, 1+6, 1+6+12, 1+6+12+18...\}$ which can be expressed by the following equation.

$$k = 1 + \sum_{i=1}^{n} 6i$$

Which we can write as

$$1 + \sum_{i=1}^{n} 6i = 1 + \frac{6n(n+1)}{2}$$

. From this we can derive the equation $k = 3n^2 + 3n + 1$ Where n is the proximity of the border of the cluster to the centre hexagon. We can note of course that this is the crystal ball sequence for hexagonal lattice which our solution follows the same pattern of construction. [2]

We can also show this visually. In Figure 16 we see each solution to the current jacency problem. The hexagon in the middle representing 0-jacency, blue represents 4-jacency, and the outside represents 10-jacency where we need 91 values — or a unique value for each

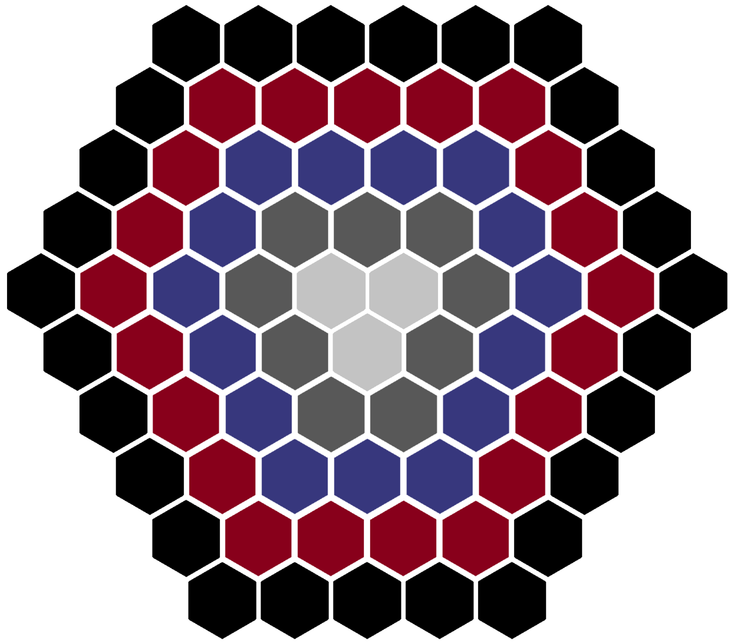

Figure 17: Solutions to Odd-Jacency Problems

state of Hexylvania. Analyzing this figure, we see that for each solution each hexagon will be n-jacent to all hexagons in the proposed solution where $n = 2i$ where i represents the number jacency to the middle hexagon from the border of the solution.

5.3.2 Odd Values

Similarly, we can continue our search to find some equation to determine adjacency up to some odd number. Referring back to the beginning of the paper, in section 2.1 that our spectrum required three numbers, while in section 5.2 we determined for up to 3-jacency, that we have a minimum of twelve. Again, if we replicate the idea shown in section 5.3.1 by coating the cluster shown in Figure 14, we can create the smallest sized cluster that follows the rules of one, two, and three jacency with 27 hexagons. Observing this pattern we see that it follows that created by $3i^2$. Following this, we see that the equation produces $(3, 12, 27, 48, 75...)$.

We can show this visually. In Figure 17 we see each new row is the solution to the each iteration of the jacency problem for odd numbers. The middle light grey represents the solution for 1-jacency, dark grey represents 1, 2, and 3 jacenecy and so on. We see that each time, it if we each value is n-jacent to every other hexagon in its row and those inside it where $n = 2i + 1$ where i is the jacency to the middle three hexagons.

6 Conclusion

Looking at aforementioned examples, we see that 1-jacency needs a minimum spectrum of range $2k$ comprised of exactly three elements, 2-jacency needs a spectrum of range $3k$ with exactly four elements — however we did not explore further adjacency. When setting the rule that immediate adjacent must be two difference and 2-jacent must be one in the different, we see it creates a modular nine system. Further, we found the equations $3i^2$ and $3n^2 + 3n + 1$ describing the minimum number of elements for odd and even n-jacency where jacency less than n is also not allowed. We however did not explore where multiple values are assigned to singular hexagons, nor did we construct a three-stage interference scheme.

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