Reasoners Resemble Spiders

AM 2130	Paper Three
Submitted by:	David Martin
	201724804
Submitted to:	D. Dyer
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Contents

Abstract

We analyze the function, $f(x) = \lambda \sin(\pi x)$ as a mathematical feedback loop. That is that, whatever output is generated by the function, is put back into the function replacing the previous input. This iterative process is continued until a favorable result has been achieved or until realization that one will not be found. This paper will look at analytical and numerical methods of determining what conditions the function converges and to what values or becomes chaotic and attempts to explain why it does so. Further displaying results in cobweb diagrams and bifurcation diagrams.

1 Introduction

In this paper we explore the mathematical feedback loop, $x_{n+1} = \lambda \sin(\pi x_n)$ and how it responds as n becomes arbitrarily large. A feedback loop in this case is a functions ability to have the generated output as a valid input. For the purposes of this paper, both arbitrarily chosen variables λ and x_0 will be bound between 0 and 1 (inclusive) and both belong to the set of real numbers. As discussed below, as λ approaches 1, the outputted x_i becomes erratic and seemingly chaotic, following no pattern. A chaotic function is one that outputs a wildly different output depending on a mild change the inputted numbers. For this paper, x_0 will be the first term inputted into the function, x_i will represent some arbitrary output from the function, and x_n will represent the value the function converges to — should it exist. Further, $f(x)$ will represent the general function while $f(x_i)$ will represent some instance of it, and $f(x_n)$ will represent some instance that we know or assume converges to some point. Our main objectives are to understand why $f(x)$ converges, why it converges to where it does, isolate any special cases of the function, and most importantly, discuss chaos and how it applies to the function at hand. Finally analyzing $f(x)$ using cobweb and bifurcation diagrams.

2 Analytical Analysis

Depending on the chosen λ, $f(x)$ will either converge on some point, chaotically output an x between $(0, 1)$, or oscillate between points. If we assume n is arbitrarily large, and has reached a point of convergence, we can denote $f(x_n)$ as $f(x_n) = x_n = \lambda \text{sin}(\pi x_n)$. We can therefore solve for λ, as $\lambda = \frac{x_n}{\text{sin}(\pi x_n)}$, we will name this $\alpha(x)$. Knowing λ is bounded between 0 and 1, if we solve for $\lambda = 1$, we get $x_i = \sin(\pi x_i)$, where $x_i = 0$ and $x_i \approx 0.64$. From this we get can divide the bounded region into two sections, $0 \leq \lambda < 0.64$ and $0.64 \leq \lambda \leq 1$.

2.1 Convergence

To define convergence, it is the property held by some series and functions that as they approach an infinite limit, the output or next term of the series becomes arbitrarily close to some number. In terms of $f(x)$, we are searching for where x_i approaches x_n, should x_n exist. [1]

This section will mainly deal with the first region, where $0 \leq \lambda < 0.64$. Starting our search for points of convergence, we know if $x_0 = 0.5$, $\sin(\pi x_0) = 1$. Because the sine term of the function is equal to one, whatever value we set λ equal to will determine x_1. By setting $\lambda = x_0 = 0.5$, we get a infinite loop of where $x_n = 0.5$, where the point of convergence x_n is 0.5. This case is one of the simplest points of convergence of $f(x)$.

2.1.1 Converges to Zero

We can further explore the function $\alpha(x)$. By taking the derivative of the function we get

$$\begin{aligned} \alpha^{'}(x) &= \frac{\sin \pi x - \pi x \cos \pi x}{\sin^2 \pi x} \\ \alpha^{'}(x) > 0, &= (\forall x)|0 < x < 1 \\ \lim_{x \to 0} \alpha(x) &= \beta \end{aligned}$$

Where β is equal to a local minimum. Because $\alpha(x)$ is undefined at $x = 0$, we will substitute $x = 10^{-16}$, or approximately zero. We get $\lambda(10^{-16}) = 0.3182098862$, we will denote this A. In order for any $f(x)$ to converge at some point, x_n must equal $\lambda \sin(\pi x_{n-1})$ because x_{n-1} is so close to x_n we can say $x_n = \lambda \sin(\pi x_n)$ or $\lambda = \frac{x_n}{\sin(\pi x_n)}$ — or $\lambda(x_n)$. Where $\alpha(x)$ is undefined where $\lambda \leq A$, we therefore know that $f(x)$ will forever descend for any $\lambda \leq A$ — such that it converges at the lower bound of 0. This is displayed later in the numerical section as well.

Building on this we know that given $x_i = 1 \implies i = 0$ or $\lambda = 1$. We can break it into two cases, $\lambda \neq 1$, and $\lambda = 1$, we will look at the first case first. For the product of two real numbers multiplied together to equal 1, we know that one must be the inverse of the other. For $\lambda \sin(\pi x) = 1$, λ must equal $\frac{1}{\sin(\pi x)}$. Because $\sin(\pi x)$ is bound between 0 and 1, we get for $\lambda > 1$ which is not allowed by definition of the problem — but we can arbitrarily choose x_0, leaving it open it being set as 1. Moving on to the second case, where $\lambda = 1$. We therefore get that $x_i = \sin(\pi x_{i-1}) = 1$, inferring $x_{i-1} = 0.5$. This implies that any value that can be found recursively using the function $x_i = \frac{\arcsin(x_{i+1})}{\pi}$ where $x_0 = 0.5$. This is important because we know that it is only these instances that $f(x_n)$ will ever equal 0. For every other case, we get $f(x)$ approaches 0.

2.1.2 General Convergence

We can start by looking at the equation $g(x) = \sin(\pi x)$ bounded between the points $x = 0$ and $x = 1$, the equation crests or hits it's amplitude at $x = 0.5, g(x) = 1$. By multiplying g(x) by some constant (this will become λ) we get that the crest becomes that constant. By applying this to $f(x)$ we see that crest is $f(0.5) = \lambda$ — so if $f(x_n)$ exists, it is less than or equal to λ.

Knowing this we can move on to showing where the function converges. Using the same logic as above, if $f(x_n)$ exists, then know $\alpha(x)$ must also exist between the predetermined bounds. λ must be equal to or less than 1, so if we solve for where $\alpha(x) = 1$, we get $x \approx 0.72$. This infers that the greatest value λ can equal and $f(x)$ still converge is ≈ 0.64 — we will refer to this value as B.

For any λ value in the region bound between A and B we can represent as $\frac{x}{\sin(\pi x)}$. We are interested in showing that $\frac{\sin(\pi x_i)}{\sin(\pi x_n)} = 1$ so that when we multiply the sines by x_n, we get $x_{i+1} = x_n$. As defined in the problem, the range for $\sin(\pi x)$ is very limited (due to $\sin(\pi x) \in \{0,1\} \forall x \in \{0,1\}$). We know that when we divide these two values that it will output some value less than 1 or such that when we multiply it by x_n it will be less than 1. We know that x_i will approach x_n because of its repetitive occurrence in λ which we multiply the sine term for each iteration of the function. We know that $\sin(\pi x_i)$ will output some value greater than x_i (for $x_i < B$) so for each iteration of the function when we divide $\sin(\pi x_i)$ by $\sin(\pi x_n)$ it will be closer to 1 each time, which when we multiply it by x_n, we show that x_i approaches x_n. For cases where $x_i \geq B$, we can show $\sin(\pi x_i) = \sin(\pi 1 - x_i)$.

$$\begin{aligned}
\sin(\pi - x_i) &= \sin(A - B) \\
\sin(A - B) &= \sin(A\pi)\cos(B\pi) - \cos(A\pi)\sin(B\pi) \\
\sin(\pi - B) &= \sin(\pi)\cos(B) - \cos(\pi)\sin(B) \\
\sin(\pi - B) &= -(-\sin(B)) \\
\sin(\pi - x_i) &= \sin(x_i).
\end{aligned}$$

From this we show that x_0 does not matter unless it falls into some special case.

2.1.3 Other Cases

There are examples that converge that can be considered base cases. For example, given $x_0 = 0$ or $x_0 = 1$, we get $x_1 = \lambda \sin(\pi x_0) = 0\lambda = 0$. This will create a loop where $x_i = 0 \forall i$ in the sequence will always be equal to 0, regardless of λ. When exploring the function numerically, we will have to ensure that $0 \neq x_0 \neq 1$ else the function will converge to 0, regardless of λ. Further, given $\lambda = 0$ we get another case where x_i will be equal to 0 for any $i > 0$.

Through $\alpha(x)$ we can now arbitrarily choose at what point we want $f(x)$ to converge. For example, $\lambda = \frac{0.5}{\sin(\pi 0.5)} = 0.5$ (as discussed above). We can show that this will converge with to our chosen x_n with mathematical induction. For this proof, we will choose $x_0 = x_c$ and $\lambda = \frac{x_c}{\sin(\pi x_c)}$.

$$\begin{aligned}
x_1 &= \lambda(\sin(\pi x_0)) \\
x_1 &= (\frac{x_c}{\sin(\pi x_c)})(\sin(\pi x_c)) \\
x_1 &= x_c
\end{aligned}$$

We will assume the inductive step, of $x_n = x_c$ and show it implies $x_{n+1} = x_c$

$$\begin{aligned}
x_n &= x_c \\
x_{n+1} &= \lambda(\sin(\pi x_n)) \\
x_{n+1} &= (\frac{x_c}{\sin(\pi x_c)})(\sin(\pi x_c)) \\
x_{n+1} &= x_c
\end{aligned}$$

However from this we can find an infinite amount of points where $f(x)$ will converge. This does not fully encompass every value from 0 to 1. For example, we know that $\lambda(0.74) =$

Table 1: Analysis of $f(x) = 0.92... \sin(\pi x)$ for varying x_0

x_0	$i = 1$	$i = 1001$	$i = 10001$
0.5	0.95	0.9085	0.2008
0.55	0.9383	0.2978	0.4282
0.45	0.9383	0.9489	0.9026
0.475	0.9471	0.5607	0.6147
0.5125	0.9493	0.6841	0.9442

1.015... Because λ is bounded between 0 and 1, this cannot be. So any value where $x_c \geq 0.74$, $\lambda(0)$ or $\lambda(1)$, $\alpha(x)$ is undefined. (Counter to this we know that $f(x)$ where $\lambda = 0$ or will converge to $x_n = 0$.)

As well, we can find points of convergence for λ that would otherwise not converge. As discussed above, $\lambda \geq B$ do not converge on any certain point — but if we know some x_i, then we can force $f(x)$ to converge. For instance, given $\lambda(0.72) \approx 0.93444$ (this is well above B) given $x_0 = 0.72$ will converge to 0.72. Using the formula $x_i = \frac{\arcsin(x_{i+1})}{\pi}$, we can again find an arbitrarily large number of instances that will converge to any chosen point.

2.2 Chaos

When λ grows greater than B, we see that because it no longer falls into the discussed ranges, that the function no longer converges; the function instead becomes chaotic. We define a chaotic function as a function that produces an erratically different results based on initial conditions but yet is still contained between bounds. We see that $f(x)$ fits these definitions

Again, relying on $\alpha(x)$ to determine if this function will converge, we show that it cannot because $x_n \frac{\sin(\pi x_i)}{\sin(\pi x_n)}$ can never equal x_n such that $\frac{\sin(\pi x_i)}{\sin(\pi x_n)}$ can never equal one. By solving for where $\alpha(x) = B$ we find the approximate point of 0.653, so from $\lambda(0.653)$ to $\lambda(B)$ will not converge. When we increase λ past B we see that the generated sequence oscillates between points. Taking $\lambda = 0.75$ as an example, we see it switches from 0.5401765790106284 to 0.7440337660428007 every iteration of the function. Why is this possible? Denoting $p \approx 0.540$ and $q \approx 0.744$ we can say that $p = \lambda \sin(\pi q)$ and $q = \lambda \sin(\pi p)$. By solving for λ and setting the two equations equal to each other, we can derive the equation $\frac{q}{\sin(\pi p)} = \lambda = \frac{p}{\sin(\pi q)}$. This logic holds true for a sequence with any number of terms. Testing the equation with values stated above we get $0.535879485879 = \lambda = 0.535879485879$. We see that no value less than B can hold true such that $\lambda \sin(\pi p) = q$ and $\lambda \sin(\pi q) = p$ — the lowest occurring example of this is B, because all values less than B converge.

3 Numerical Analysis

To begin with a numerical approach, we can analyze Figure 1. By using a variation of the code from section 6.1, what is displayed is x_{10000} for 100 different evenly spaced occurrences of λ. By testing if $x_{9999} = x_{10000}$ within acceptable range we can determine if the given lambda converges or becomes chaotic. We use $i = 10000$ as an arbitrarily large number where $f(x)$ converges.

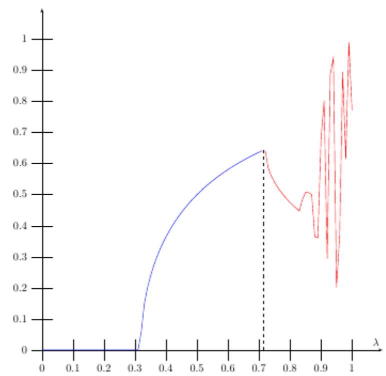

Figure 1: Examination of x_{10000} for increasing λ

If $x_{9999} = x_{10000}$ — or when we assume $f(x)$ to converge, the line is graphed in blue, else where the line is shown it represents where $f(x)$ is chaotic. From $\lambda = 0$ till $\lambda = 0.32$ or A we see that the points all converge to 0. This reaffirms what was discussed in section 2.1.1, that any point below A converges to 0. Further, in the region of the graph represented in red we see it take on erratic movements up and down the y-axis. This shows the functions chaotic behavior, which was predicted in section 2.2 where we discussed how $f(x)$ would discretely bounce from value to value, showing no preference for any x.

As discussed in section 2.1.2 we see that from $\lambda = A$ until $\lambda = B$ all converge but at varying points from 0 to B (not inclusive).

3.1 More About Convergence

Looking for notable points where x_n converges, we are most interested in λ. We can show this with a variety of ways. The simplest we see in Figure 2 where we map ten different $f(x)$ where λ is constant but x_0 increments by 0.1. Here λ is defined with the formula discussed in analytical analysis, $\alpha(x)$ where $x = 1/3$ — meaning that $f(x)$ will converge to $1/3$.

In Figure 2, we see that $x_0 = 0.1$ and $x_0 = 0.9$ overlap, (the same for $x_0 = 0.2$ and 0.8, $x_0 = 0.3$ and 0.7, $x_0 = 0.4$ and 0.6). The proof for why $\sin(\pi x) = \sin(\pi(1-x))$ is discussed in section 2.1.2. We see how the maximum value achieved was for when $x_0 = 0.5$ where x_1 (the first value graphed) becomes equal to $\alpha(x)$ or about 0.38. We see here that regardless of x_0, $f(x)$ all converges to the same point $1/3$.

On a different note, we can plot $f^n(x)$ where $f^n(x)$ is the number of iteration that have transpired on $f(x)$; for example, $f^2(x) = \lambda \sin(\pi\lambda \sin(\pi x_i))$. This will give an idea of if and where the function converges. By using a variation of the code seen in section 6.1, we can plot the graph seen in Figure 3.

In this Figure, and any similar to follow, red represents $f(x)$, blue represents $f^2(x)$, brown is $f^3(x)$, orange is $f^4(x)$ and finally green is $f^5(x)$. We see that after each iteration of $f(x)$, the range of the maximum and minimum outputted x shrinks, and the number of times the function reaches a maximum increases by 1. Isolating $f^5(x)$ produces the graph seen in Figure 4.

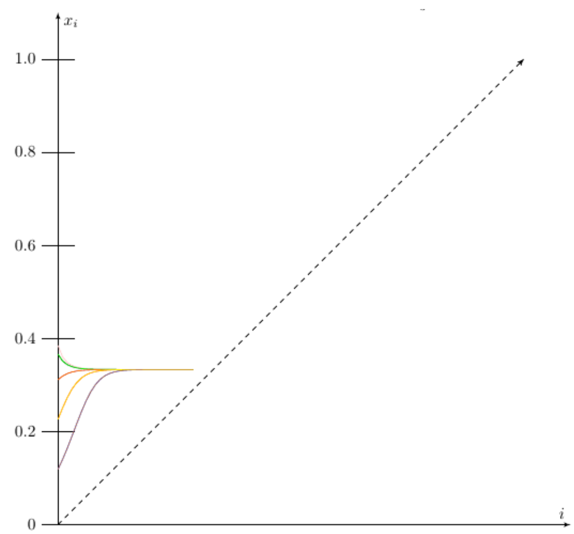

Figure 2: $f(x) = \lambda(\sin(\pi x)), \lambda = \frac{1}{3\sin\left(\frac{\pi}{3}\right)}$

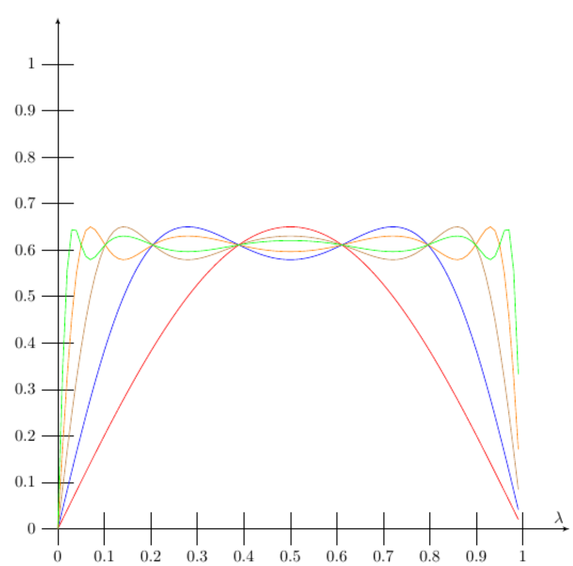

Figure 3: Graph of $f(x), f^2(x), f^3(x), f^4(x)$, and $f^5(x), \lambda = 0.65$

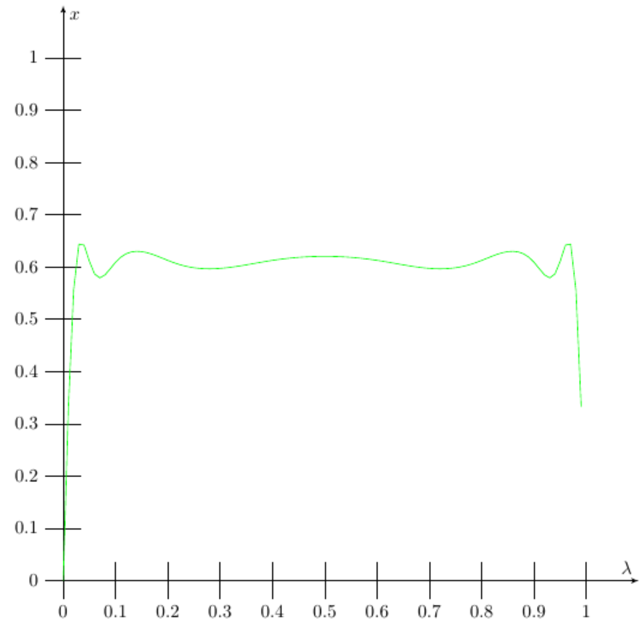

Figure 4: Graph of $f^5(x), \lambda = 0.65$

This shows more specifically how the sinusoidal shape becomes more linear. By solving for $\alpha(x) = 0.65$ we see that $x \approx 0.611$ which aligns with where $f^5(x)$ becomes more linear. We can compare this graph with a more chaotic one produced by the same methods, but where $\lambda = 1$.

The colours represent the same iteration of $f(x)$ as in Figure 3. Here we see drastic changes from the previous graphs. λ does not restrict the range in that where outputted x may have a maximum or minimum — for this reason the sinusoidal shape stays and becomes more condensed. This becomes more evident in Figure 6.

4 Results and Analysis

We can analyze the results found above with a cobweb diagram. Cobweb diagrams are used to display iterative functions, by plotting $(x_i, x_{i+1}), (x_{i+1}, x_{i+1})$ for every instance i. We only ever change either the independent or dependent variable at any given step – so because of this, we get either only vertical or horizontal moves, creating a box like shape. If the function converges to any point, the boxes created via graphing will get smaller and smaller around the point of convergence. Otherwise, if the function becomes chaotic, the diagram does as well. [**?**]First, we will look at the function where $\lambda = 0.7$ and converges to $x_n \approx 0.6366$.

We see here in Figure 7 that as x_i approaches x_n that the box like shape of the graph becomes smaller and smaller — identifying the point in question. For ease of reference, a 45 degree angle is plotted with a dashed line. Having every other point on the graph be at point (x_{i+1}, x_{i+1}) means every other point in the graph will be plotted along this line. Further every other point is connected with the line plotted representing $0.7 \sin(\pi x)$. We see from

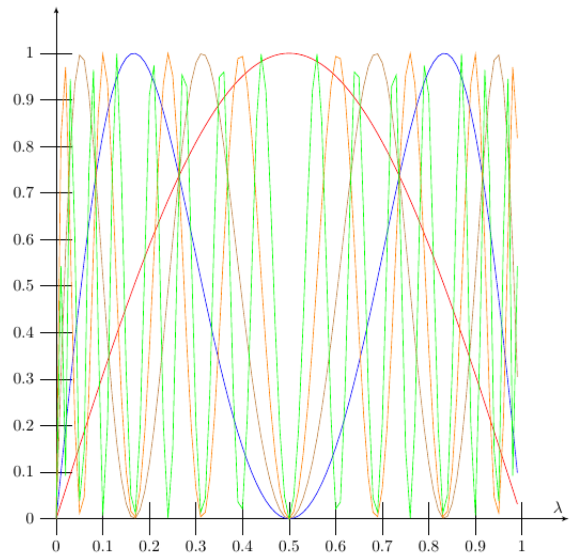

Figure 5: Graph of $f(x), f^2(x), f^3(x), f^4(x)$, and $f^5(x), \lambda = 1$

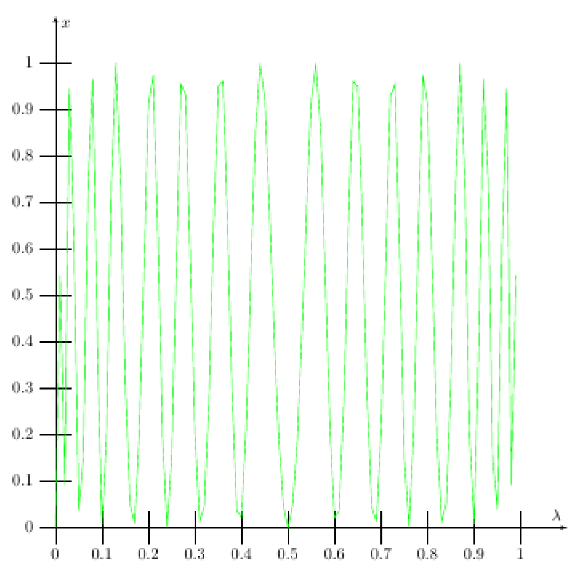

Figure 6: Graph of $f^5(x), \lambda = 1$

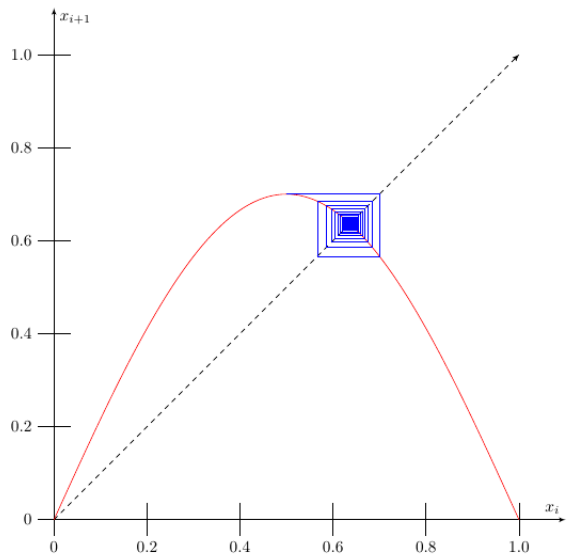

Figure 7: Cobweb Diagram for $\lambda = 0.7$

the diagram that the boxes converge to the point where $y = 0.7\sin(\pi x)$ and $y = x$ meet. Building on this, we can analyze the diagram that is created for when $f(x) = 0.8\sin(\pi x)$ shown in Figure 8.

As for this diagram we see that as x_i approaches its potential x_n but it never reaches it. The function $0.8\sin(\pi x)$ is not convergent but rather oscillates between multiple values — as discussed possible in Section 2.2. By solving numerically, we know that these two points are $x_n = 0.7975659894354807$ and 0.4751634347793161, where the top left corner is equal to $(0.4752, f(0.4752))$ and the bottom right is $(0.7976, f(0.7976))$. Furthermore, if we build on this again, we get to the graph from Figure 9.

In this diagram we see the function as it has gone chaotic. It neither circles nor converges upon any point, but rather it fills the space it can — as shown in Figure 9. It bounces from the sine function and the function $y = x$.

Continuing on this premise, we can show in what regions $f(x)$ is convergent, oscillating, or chaotic. By plotting the first 100 points after 10000 iterations, we will get that any repeated points will overlap on the graph, but any new points will display individually. We show this in Figure 10. We see again that, as described in section 2, points where $\lambda \leq B$ all converge to one point. After this is where we see where any $f(x)$ with a λ greater than B will oscillate between multiple points or contain more than 100 terms. We observe that as λ increases, the number of terms increases drastically. More importantly, we see that from 0.64 to 0.81 that is oscillates between two values. From after $\lambda = 0.81$ we see that the output becomes more chaotic and more spread out. From this we get three regions, $0 \leq \lambda \leq A$, $A < \lambda < B$, and $B \leq \lambda \leq 1.00$ where the function converges, oscillates, and is chaotic respectfully.

Analyzing the function $x_{i+1} = \lambda\sin(\pi x)$ we see how it can both converge or create a chaotic sequence — but this chaos is far from random. After analyzing the instance where

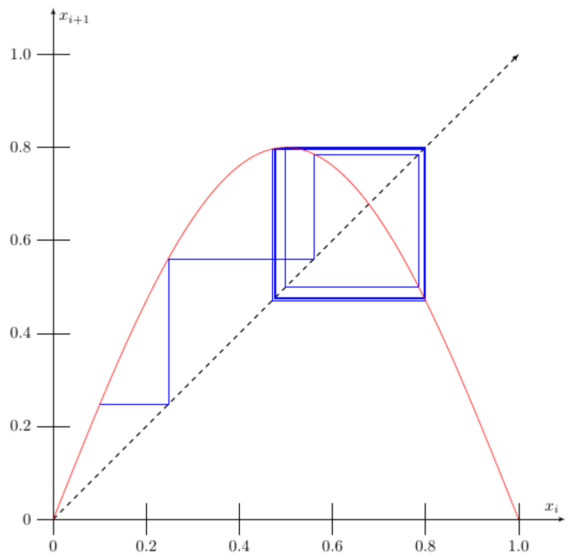

Figure 8: Cobweb Diagram for $\lambda = 0.8$

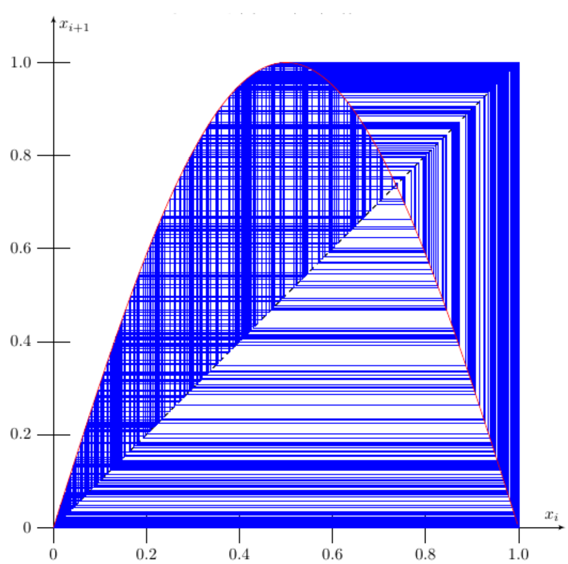

Figure 9: Cobweb Diagram for $\lambda = 1.0$

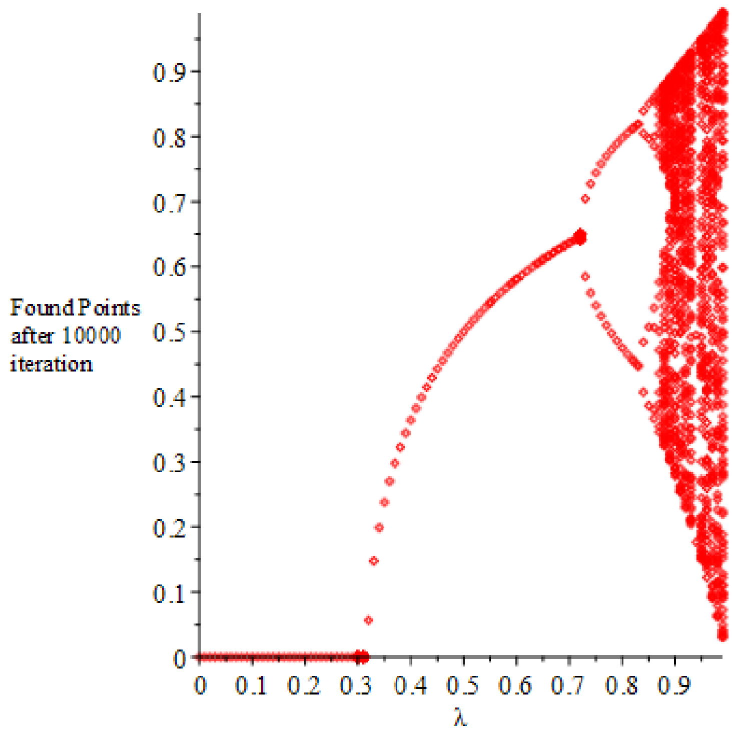

Figure 10: Bifurcation Diagram

$\lambda = 0.80$, we see it iterates between 0.475163434779316444 and 0.79756598943548806. It repeats with accuracy of $\pm 1 \times 10^{-15}$ every two iterations (This of course could be a rounding error on the part of the computer). The more we increase λ, we see that the frequency that the generated sequence repeats with this accuracy decreases and the number of iterations between each repetition increases. We will define k as the number of iterations before the pattern repeats. In the table in Figure 2 we see a variety of λ's and the corresponding k for 5 digits after the decimal place.

For $\lambda = 0.74$ (the closest displayed value to B) we see that sequence only contains two values that the function oscillates back and forth. Moving onward to $\lambda = 0.85$ we see that $k = 4$, so it repeats every fourth iteration. What is not displayed is that if we increased

Table 2: Exploration of Repetition in Chaotic Pattern

λ	k
0.74	2
0.85	4
0.89	174
0.90	1143
0.91	4561
0.92	112231
1.00	436083

Table 3: Exploration of Repetition when $\lambda = 0.991$

n	$f(x_n)$	$f(x_n) - f(x_{n+k})$
n	0.112723	0
$n+1$	0.343655	0
$n+2$	0.873844	0
$n+3$	0.382562	0
$n+4$	0.924315	0
$n+5$	0.233417	0
$n+164030$	0.112723	0
$n+164031$	0.343654	0.000001
$n+164032$	0.873843	0.000001
$n+164033$	0.382565	-0.000003
$n+164034$	0.924318	-0.000003
$n+164035$	0.233409	0.000008
$n+477166$	0.112723	0
$n+477167$	0.343653	0.000002
$n+477168$	0.873842	0.000002
$n+477169$	0.382568	-0.000006
$n+477170$	0.924321	-0.000006
$n+477171$	0.233398	0.000019

our scrutiny, so that we are looking for where it is accurate to 16 decimal places, k would increase to 8. Every second iteration is different by $\sim 10^{-15}$ before returning to the original value. This discrepancy is further exacerbated as λ approaches 1.

As we see in the table in Figure 3, the number of terms between each repetition is not constant. This table was generated by iterating through $x_{i+1} = 0.991\sin(\pi x_i)$ where $x_0 = 0.8$ with a variation of the code seen in 6.1. It tests each x_i after $i = 1001$ against x_{1001} to see if it is equal within acceptable bounds ($\pm 10^{-7}$).

The program finds that $x_{1001} = 0.112723$ (rounding to six decimal places). It then finds 0.112723 again after 164030 more iterations, and then at $n = 478173$, and at $n = 960898$. From this, we find $k_1 = 164035, k_2 = 313136, k_3 = 482714$. This discrepancy in the length of k seems counter to the idea that the function is simply iterating through a set sequence. Also not displayed is that the if we require another digit of accuracy, the first repetition is when $i = 960898$ or 5.857 times k_1. Following this leap, if we required 9 digits of accuracy we would find $k_1 \approx 5.6$ million.

5 Conclusion

Through this discussion the iterative function $f(x) = \lambda \sin(\pi x)$ and have analyzed where it converges, oscillates, and becomes chaotic. For any input λ between 0.00 and value assigned A (about 0.32) will converge to 0, while values of λ from A to B converge at some point between 0 and λ. The derivation of the equation $\alpha(x) = \frac{x}{\sin(\pi x)}$ allowed for further analytical methods of intrigue take place. The code seen in section 6.1 allowed for a brute force

Table 4: Exploration of Repetition when $\lambda = 1$

n	$f(x_n)$	$f(x_{n+436083})$	$f(x_n) - f(x_{n+k})$
n	0.951929	0.951928	0.000001
$n+1$	0.1504453	0.150447	-0.000001
$n+2$	0.455236	0.455241	-0.000005
$n+3$	0.990128	0.990130	-0.000002
$n+4$	0.031008	0.031001	0.000007
$n+5$	0.097262	0.097238	0.000024

method of determining where $f(x)$ converges and where it does not. We then analyze the found outputs through cobweb diagrams that we use to see the correlation between $f(x)$ and the plot $y = x$. Next we used bifurcation diagram to find when the function oscillates between two outputs — and then comparing it with previously found values as to support analytical and numerical solutions. Finally we explored the chaotic region of the function and the possibility of rather than it being a series of randomly obtained numbers, but a pattern we are able to follow. This idea however falls short when analyzing the example $\lambda = 0.991$.

References

[1] "Convergence in Mathematics". Encyclopedia Britannica, 2019, https://www.britannica.com/science/convergence-mathematics.

[2] Hofstadter, D. (1981). Metamagical Themas. Scientific American, 245, pp.22-43.

6 Appendix

6.1 Code

```
from math import sin
from math import pi

L = float(input(``Please Input a Lambdia: "))
x = float(input(``Please Input a starting x value: "))
while True:
    for i in range (1): ##Increase range to jump ahead
        x = L*sin(pi*x)
    a = input(x)

#any variations of code described in the paper
#were to change the output of the program.
```