The Slapshot Curve

AM 2130	Paper Two
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Abstract

We create and model a “Slapshot curve” by finding what position relative to two horizontally placed points will create the greatest angle between them. We will compare numerical and analytical methods of finding the desired position and graph the answers found. For the analytical method we will approach the task as an optimization problem.

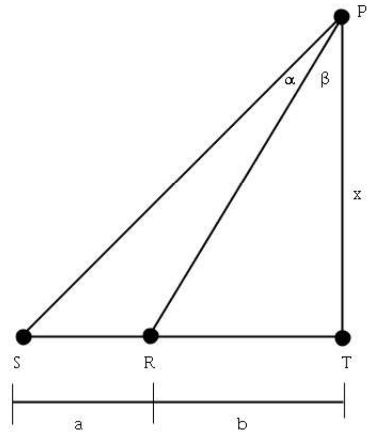
1 Introduction

In the problem presented, character Tina is searching for the point in which the angle between the furthest goal post and the closest goal post is the greatest - given she can only skate in the direction of play.

We can represent the problem as seen in figure 1, x is the distance from the goal line, a is the distance between the goal posts, and b is the horizontal distance between Tina and the closer goal post given $x = 0$. From this we get two triangles, $\triangle PQV$ and $\triangle PRV$ from these, angles $\angle\alpha$ and $\angle\beta$ are created. To find where the angle on the net is largest, or where $\angle\theta$ is the greatest given some constant b, we have multiple methods to approach said task. As discussed in the mathematical details section we approach the task analytically. Creating an equation to find θ we can differentiate the equation treating it as an optimization problem. Further, in the programming details section, we can approach the task numerically, incrementing x, and checking to see if θ is bigger than the previous iterations of x.

For purposes of this paper, we will use dimensions used by NCAA rinks[**?**] with a goal crease of 8 feet in diameter.

Figure 1: Diagram displaying general case of problem



2 Mathematical Details

2.1 Left and Right of the Net

We can treat finding the maximum of $\alpha - \beta$ as an optimization problem. So we can find the maximums from where the slope reaches critical points or where the slope is equal to zero. We will represent α as:

$$\arctan \frac{(a + (b - \frac{a}{2}))}{x}$$

and β as:

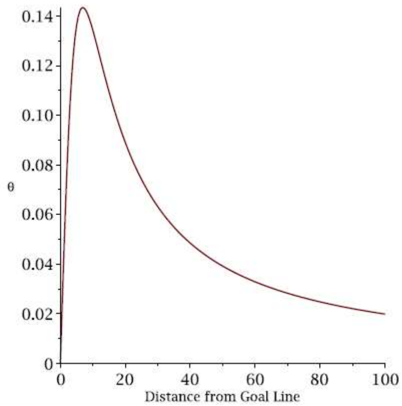
$$\arctan \frac{(b - \frac{a}{2})}{x}$$

By subtracting $\frac{a}{2}$ from b we measure from the centre of the net as opposed to the closest goal post. This will centre any graph derived from the equation. The equations for both α and β will be defined on the set of positive real numbers for any points where $b > \frac{a}{2}$ and $x > 0$. From this equation we can represent some $\theta = \alpha - \beta$ as:

$$\theta = \arctan \frac{(a + (b - \frac{a}{2}))}{x} - \arctan \frac{(b - \frac{a}{2})}{x}$$

The extreme value theorem tells us that some point in a closed range will be a maximum.[3] Knowing that $arctan f(x)$ will be defined for all values where $f(x)$ is defined, we see that θ will be defined for all $x \neq 0$. We know the derivative of this equation will give us the slope at some point. From this, when the derivative is equal to zero, x will be some point where the graph is horizontal or where the slope changes from negative to positive - either a local minimum or local maximum. As shown in figure 2 the line for θ only reaches a maximum once (this is proved later).

Figure 2: θ at some constant a and b



As shown in Maple:

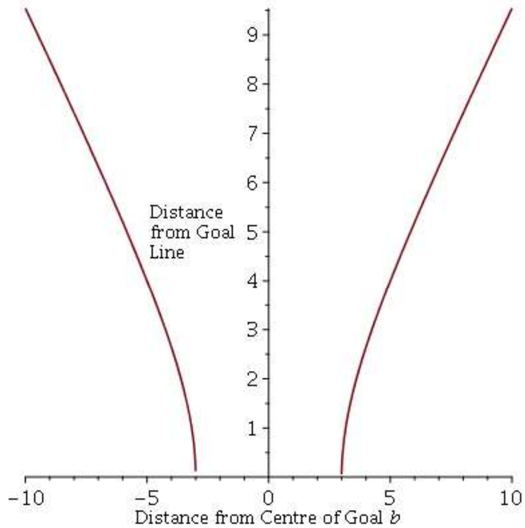
$$
\begin{aligned}
\theta = &\quad \arctan \frac{(a+(b-\frac{a}{2}))}{x} - \arctan \frac{(b-\frac{a}{2})}{x}, a, x > 0, b > 3 \\
\frac{\mathrm{d}\theta}{\mathrm{d}x} = &\quad \frac{b-\frac{a}{2}}{(b-\frac{a}{2})^2+x^2} - \frac{b-\frac{a}{2}+a}{(b-\frac{a}{2}+a)^2+x^2} \\
0 = &\quad \frac{b-\frac{a}{2}}{(b-\frac{a}{2})^2+x^2} - \frac{b+\frac{a}{2}}{(b+\frac{a}{2})^2+x^2} \\
0 = &\quad (b-\tfrac{a}{2})((b+\tfrac{a}{2})^2+x^2) - (b+\tfrac{a}{2})((b-\tfrac{a}{2})^2+x^2) \\
0 = &\quad -a(b-\tfrac{a}{2})^2 - a^2(b-\tfrac{a}{2}) + ax^2 \\
0 = &\quad -(b-\tfrac{a}{2})^2 - a(b-\tfrac{a}{2}) + x^2 \\
0 = &\quad -(b-\tfrac{a}{2})^2 - a(b-\tfrac{a}{2}) + x^2 \\
x^2 = &\quad (b-\tfrac{a}{2})^2 + a(b-\tfrac{a}{2}) \\
x^2 = &\quad b^2 - \frac{a^2}{4} \\
x = &\quad \pm\sqrt{b^2 - \frac{a^2}{4}} \\
x = &\quad \sqrt{b^2 - \frac{a^2}{4}}
\end{aligned}
$$

Given that any game will have the same sized net, we will use a 6 foot wide net as standard for the North American Rinks:

$$x = \sqrt{b^2 - 9}$$

Tina is not be able to shoot on the net when she is past the goal line, so we understand that x must always be greater than zero. Further, b will be constant for any instance of this problem, so we get only one point where the slope of the line is equal to zero - when $x = \sqrt{b^2 - 9}$. This shows that the graph for θ will only reach a critical point once in the range we are concerned of - and we see it is a maximum from figure 2.

Figure 3: Plot of where $\theta' = 0$



2.2 In Front of the Crease

In hockey, players cannot enter the goal crease or the semicircle around the net, because of this, Tina will not be able to shoot at any point in the crease. We show that if the ideal point for Tina is in the crease, that the best place for her to shoot is on the edge of the crease. We can show this by replacing x^2 with $b^2 - 9 - c$ in $x^2 - b^2 + 9$ where c is the distance from the ideal point, we get that the slope of the graph will always be in the direction of the polarity of $-c$. When approaching the ideal point from an arbitrarily large distance, we see from the slope that θ will increase until we get to the ideal point. Considering we are stopped from reaching that point, we get that Tina should shoot from as close to it as possible - at the edge of the crease.

This point can be found again with simple trigonometry. Assuming she is on the edge of the crease, she must be the radius of the circle in distance from the centre of the net. This is illustrated by figure 4 By knowing the horizontal distance from Tina and the centre of the net, we create a triangle from which we can find the distance from Tina to the goal line. Using Pythagorean theorem, we can find the distance from Tina to the goal, d, is equal to $\sqrt{r^2 - c^2}$. Analyzing these two cases, we can plot them together and create the graph seen in figure 5.

3 Program Details

The code written in Maple brute forces the problem at hand. It breaks down the problem into two cases, when Tina is to the left and right of the crease, and when she is in front of the crease. For the first case, the program starts testing one foot from the goal line ($x = 1$)

Figure 4: Diagram of instances where Tina is in Front of the Crease

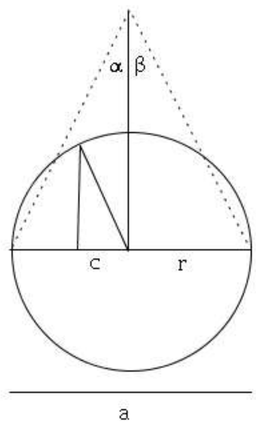


Figure 5: Solution found from Analytical Solution

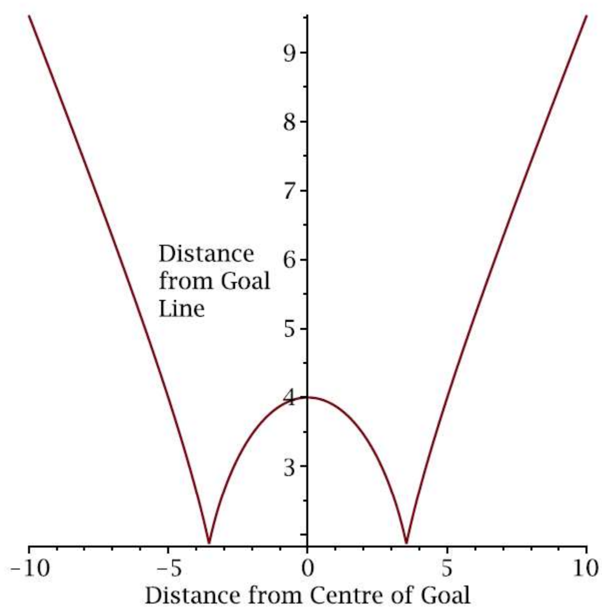
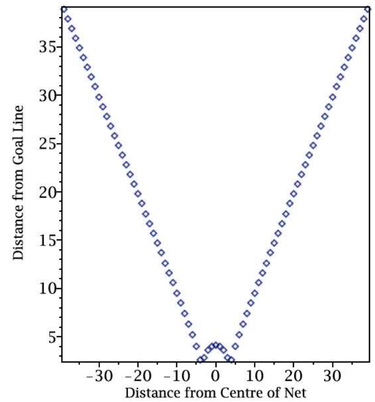


Figure 6: Solutions Produced by Numerical Solution



and tests every 10cm (incrementing x by 0.1). The program compares if θ is larger or smaller than previous iterations, saving x as w when it finds that x produces a larger θ. The program continues until $x = 178$ or until Tina would be shooting from the opposite goal line in a North American rink. It repeats this for $-42 < b < -4$ and $4 < b < 42$, $b \in \mathbb{Z}$. These dimensions were chosen to reflect an official rink which is 85 feet wide or 42.5 feet from the centre of the rink.

Similarly, the code approaches case two - when Tina is in front of the crease, with the same method. Starting at $\sqrt{r^2 - c^2}$ (the formula calculated in section 2.2), we again test if θ is greater than previous iterations and save x as w when it is. Keeping with North American standards, $r = 4$ such that, $\sqrt{r^2 - c^2} = \sqrt{16 - c^2}$

This culminates with data points we can use to graph figure 6.

4 Results and Analysis

Comparing the two graphs produced by the analytical approach and the numerical approach (shown in figures 5 and 6), we see that they create the same shape - mapping the same regions on the plane. The distance from the goal line Tina must be to shoot is determined by how close she is to the closest goal post and the width of the net. We can compare the two results, the found result from the Maple code in section 7.1 and the results calculated from the equations found in section 2. Looking at table 1 we see that the values are within acceptable range of 20cm to each other. This would have been caused by the program only checking every 10cm and being unable to compare values on the edge of the crease. When we overlay the two graphs, we see that they overlap as demonstrated in figure 7. We see they reaffirm each other.

Further, following the same explanation given in section 2.2, if Tina starts at some location past the ideal point, she should shoot where she is as opposed to approaching the goal line

Figure 7: Numerical and Analytical Solutions Plotted Together

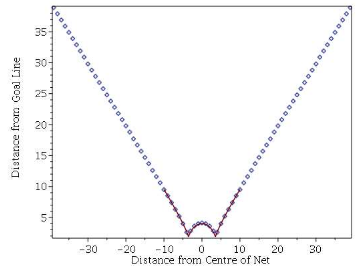
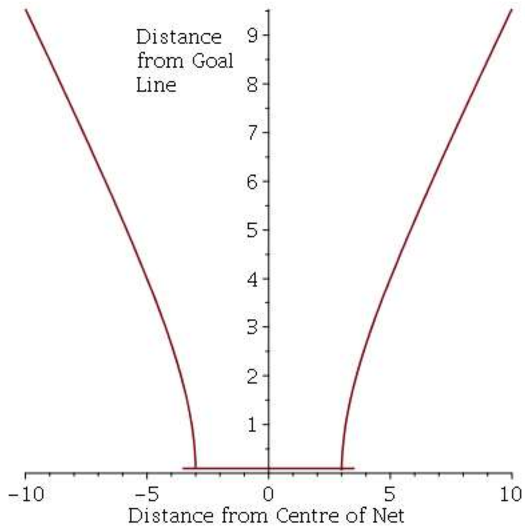


Figure 8: Solution as applied to a Soccer Pitch



any further. From the slope of θ' that θ will only continue to decrease as she approaches the goal line. Regarding Bob's soccer team, when given a player with a similar difficulty handling the ball, the results are very close to Tina's case. The analytically and numerically obtained solution still applies for when the player is to the left and the right of the net - what is different is that there is no goal crease the players are not allowed to enter.[2] Rather than force the player to shoot from some specific point, the soccer player can directly approach the net. With the same reason discussed in section 2.2 – the player should get as close to the net as possible in order to have the greatest angle to score on the net. This forms the graph in figure 8.

5 Conclusion

Throughout the paper, two methods of finding the optimal position for Tina to take a shot on net were discussed. Numerical and Analytical methods both returned similar responses - which are reflected in the graph in figure 7. The program written in 7.1 incrementally tests

Table 1: Comparison of Values found from Numerical and Analytical Methods

b	x Found	x Calculated	$\|\Delta x\|$	Acceptable Range?
-39	38.9	38.88	0.02	Yes
-24	23.8	23.81	0.01	Yes
-5	4.0	4.00	0.00	Yes
-1	4.0	3.87	0.13	Yes
3	2.8	2.65	0.15	Yes
17	16.7	16.73	0.03	Yes
35	34.9	34.87	0.03	Yes

and compares values - it stores some found value x for some instance b and continues for each foot of the width of the hockey rink. The analytical method finds some equation for the angle we are searching for, and then derives the equation and finds the critical point as to return a simple equation we can use to find x on an instance by instance basis. All in all, these methods are applicable to similar problems - save for the area of the goal crease.

References

[1] Piotrowske, S. (2012). NCAA Men's and Women's Ice Hockey Rules and Interprestations. Indianapolis, Indiana, pp.11-18.

[2] Biggs, S. (2019). When Can a Soccer Player Enter the Goal Box? Livestrong.com. [online] LIVESTRONG.COM. Available at: https://www.livestrong.com/article/424519-when-can-a-soccer-player-enter-the-goal-box/ [Accessed 21 Feb. 2019].

[3] Renze, John and Weisstein, Eric W. "Extreme Value Theorem." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/ExtremeValueTheorem.html

6 Acknowledgments

Thank you to Mr. Luke Battcock - who helped with my understanding of hockey and soccer.

7 Appendix

7.1 Maple Code

```
xmax := 178;
a := 6;
mylist1 := array(1 .. 84);
mylist2 := array(1 .. 84);
GeneralCase := proc (a, b, xmax, mylist1, mylist2)
local x, c1, c2, w, temp;
w := 1; for x by .1 to xmax do
c1 := convert(arctan((a+b)/w)-arctan(b/w), float);
c2 := convert(arctan((a+b)/x)-arctan(b/x), float);
if c1 < c2 then
w := x
end if
end do;
temp := b+43;
mylist1[temp] := b+3;
mylist2[temp] := w
end proc;
InFrontOfCrease := proc (a, b, xmax, mylist1, mylist2)
local x, c1, c2, w, temp;
w := sqrt(17-(b+3)^2);
for x from w by .1 to xmax do
c1 := convert(arctan((a+b)/w)-arctan(b/w), float);
c2 := convert(arctan((a+b)/x)-arctan(b/x), float);
if c1 < c2 then w := x end if end do;
temp := b+43;
mylist1[temp] := b+3;
mylist2[temp] := w
end proc;
for b from -42 to -5 do
GeneralCase(a, b, xmax, mylist1, mylist2)
end do;
for b from -4 to 4 do
InFrontOfCrease(a, b, xmax, mylist1, mylist2)
end do;
for b from 5 to 42 do
GeneralCase(a, b, xmax, mylist1, mylist2)
end do
```