Noether $\to \alpha$ v0.3: Fixing L_ψ, Z and g_5 ab-initio

Draft for UBT Project

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1 Setup: 5D Action, Geometry, BC

We consider $M^4 \times S^1_\psi$ with coordinates x^μ and $\psi \sim \psi + L_\psi$. Let $\Theta(x,\psi)$ denote the unified (biquaternionic) field and $A_M(x,\psi)$ a $U(1)$ gauge field. The 5D action (natural units $c = \hbar = 1$) is

$$S = \int d^4x \int_0^{L_\psi} d\psi \, \sqrt{|g_5|} \left[g^{MN}(D_M\Theta)^\dagger(D_N\Theta) - V(\Theta) - \frac{1}{4} g^{MR} g^{NS} F_{MN} F_{RS} \right], \tag{1}$$

with $D_M = \partial_M + i g_5 A_M$, $F_{MN} = \partial_M A_N - \partial_N A_M$. We allow a warped background

$$ds^2 = e^{2A(\psi)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(\psi)} d\psi^2, \qquad \sqrt{|g_5|} = e^{4A(\psi)+B(\psi)}. \tag{2}$$

Boundary conditions (BC) along S^1_ψ may be periodic or twisted (phase δ).

Holonomy (Wilson line). The gauge-invariant holonomy (Hosotani parameter) is

$$\theta_H \equiv g_5 \oint_{S^1_\psi} A_\psi \, d\psi = g_5 \int_0^{L_\psi} d\psi \, A_\psi(x,\psi). \tag{3}$$

Large gauge transformations shift $\theta_H \to \theta_H + 2\pi n$, $n \in \mathbb{Z}$.

2 Reduction and Canonical Normalization

Assume the photon zero-mode $A_\mu^{(0)}(x)$ is independent of ψ and normalized canonically in 4D. Reducing the gauge kinetic term in (1) gives

$$S_{\text{gauge}} \supset -\frac{1}{4}\int d^4x\, Z\, F^{(0)}_{\mu\nu} F^{\mu\nu}_{(0)}, \qquad Z \equiv \int_0^{L_\psi} d\psi\, e^{B(\psi)-2A(\psi)}, \tag{4}$$

so the canonically normalized 4D photon is $A_\mu^{(0)} \to A_\mu^{(0)}/\sqrt{Z}$. The covariant derivative contributes the interaction

$$\int d^4x \int_0^{L_\psi} d\psi\, \sqrt{|g_5|}\; J^\mu A_\mu \quad \longrightarrow \quad \int d^4x\; g_4\, J^\mu_{(0)} A^{(0)}_\mu, \qquad g_4 = \frac{g_5}{\sqrt{Z}}, \tag{5}$$

where J^μ is the Noether current density and $J^\mu_{(0)}$ its overlap with the photon zero-mode. Therefore the fine-structure constant at a reference scale μ_0 is

$$\boxed{\alpha(\mu_0) \;=\; \frac{g_4^2}{4\pi} \;=\; \frac{g_5^2}{4\pi\, Z}}. \tag{6}$$

3 Noether Charge and Current Matching

Global $U(1)$: $\Theta \to e^{i\lambda}\Theta$ yields the 5D Noether current

$$J^M = i\left[\Theta^\dagger (D^M\Theta) - (D^M\Theta)^\dagger \Theta\right]\Big|_{A=0}. \tag{7}$$

We fix the normalization of Θ such that the fundamental charged excitation has unit Noether charge

$$Q \equiv \int d^3x \int_0^{L_\psi} d\psi\, \sqrt{|g_5|}\, J^0 = \pm 1. \tag{8}$$

This fixes the overall scale entering the coupling (5); no additional free normalization survives.

4 Wilson Line and Quantization

On the compact ψ-cycle the holonomy (3) is physical. For a field with $U(1)$ charge q and BC phase δ the KK momenta are shifted by

$$p_\psi^{(n)} = \frac{2\pi}{L_\psi}\Big(n + a\Big), \qquad a \equiv \frac{q\,\theta_H}{2\pi} + \delta, \qquad n \in \mathbb{Z}. \tag{9}$$

Stationary vacua (Hosotani mechanism) are determined dynamically and may select nontrivial $\theta_H^\star$. Large gauge invariance enforces periodicity $\theta_H \sim \theta_H + 2\pi$; stable vacua satisfy $\partial V_{\text{eff}}/\partial\theta_H = 0$.

5 One-Loop Effective Potential $V_{\text{eff}}(L_\psi, \theta_H)$

For each field j with spin-statistics sign $\sigma_j = \pm 1$ (boson $+1$, fermion -1), degeneracy d_j, mass m_j, charge q_j, and twist δ_j, the one-loop contribution is

$$V_j(L_\psi, \theta_H) = \frac{\sigma_j d_j}{2} \int \frac{d^4p}{(2\pi)^4} \sum_{n\in\mathbb{Z}} \ln\Big[p^2 + m_j^2 + \big(\tfrac{2\pi}{L_\psi}\big)^2 \big(n + a_j\big)^2\Big], \tag{10}$$

$$a_j \equiv \frac{q_j\,\theta_H}{2\pi} + \delta_j. \tag{11}$$

Using standard contour/Matsubara techniques (equivalently Poisson resummation) one obtains an exact resummed form

$$V_j(L_\psi, \theta_H) \;=\; \sigma_j d_j \int \frac{d^4p}{(2\pi)^4}\,\frac{1}{L_\psi}\; \ln\Big(1 - 2e^{-L_\psi \omega_j(p)} \cos 2\pi a_j + e^{-2L_\psi \omega_j(p)}\Big), \tag{12}$$

where $\omega_j(p) = \sqrt{p^2 + m_j^2}$. The total potential is

$$V_{\text{eff}}(L_\psi, \theta_H) \;=\; \sum_j V_j(L_\psi, \theta_H). \tag{13}$$

The vacuum $(L_\psi^\star, \theta_H^\star)$ solves

$$\frac{\partial V_{\text{eff}}}{\partial\theta_H} = 0, \qquad \frac{\partial V_{\text{eff}}}{\partial L_\psi} = 0. \tag{14}$$

These conditions fix L_ψ and θ_H *dynamically* in terms of the field content and BC, with no fits.

Massless limit and modular dependence. For $m_j L_\psi \ll 1$ one can expand (12) to obtain polylogarithms $\mathrm{Li}_5(e^{\pm i2\pi a_j})$, reproducing the familiar $1/L_\psi^4$ scaling and periodic dependence on a_j. Massive fields yield exponentially suppressed Bessel-function tails for large $m_j L_\psi$.

6 Determining Z (Warped Case)

If the vacuum back-reacts and generates a nontrivial warp, the 4D gauge kinetic factor is

$$Z \;=\; \int_0^{L_\psi} d\psi\; e^{B(\psi)-2A(\psi)}\,|\xi_0(\psi)|^2, \tag{15}$$

where $\xi_0(\psi)$ is the photon zero-mode profile (constant in flat space). The warp factors $A(\psi), B(\psi)$ and $\xi_0(\psi)$ follow from the coupled background equations of motion derived from (1) (e.g. via a first-order BPS system if available). In flat space, $\xi_0(\psi) \equiv 1$ and $Z = L_\psi$. In general we write

$$Z \;=\; L_\psi\, f(\tau, \mathrm{BC}), \tag{16}$$

where f encodes the modular parameter τ (complex-time sector) and boundary data.

7 Putting It Together: α from UBT

Combining (6), (14) and (15),

$$\boxed{\alpha(\mu_0) \;=\; \frac{g_5^2}{4\pi\, Z^\star} \;=\; \frac{g_5^2}{4\pi\, L_\psi^\star f^\star(\tau, \mathrm{BC})}, \qquad (L_\psi^\star, \theta_H^\star)\ \text{solve}\ \partial V_{\mathrm{eff}} = 0.} \tag{17}$$

Here g_5 is fixed by Noether normalization (8) (unit charge for the fundamental excitation) and current matching (5). Thus, given the field content, BC, and UBT potential $V(\Theta)$, the pair $(L_\psi^\star, \theta_H^\star)$ and the warp A, B are determined and *no free fit survives* in α. Standard QED running then connects $\alpha(\mu_0)$ to experimental scales.

8 Electron Mass Consistency (Sketch)

The same background must produce the lightest charged KK eigenvalue

$$m_e^2 \;=\; \lambda_{\min}\Big[- e^{-B}\partial_\psi\big(e^{B-2A}\partial_\psi\,\cdot\,\big) + m_\Theta^2 e^{-2A} + \cdots \Big], \tag{18}$$

for the appropriate (spinor) sector and BC, including the holonomy shift (9). This provides a nontrivial cross-check: the *same* $(L_\psi^\star, \theta_H^\star, A, B)$ that gives α must also yield m_e.

9 Checklist to Avoid Numerology

- Use (12) (or its Poisson/Bessel form) *exactly*; minimize (14) without inserting data.
- Fix Noether charge to $Q = \pm 1$ via (8); this removes arbitrary rescalings of Θ.
- Compute Z from (4) or (15); do not set $Z = L_\psi$ unless flat space is justified by the background EOM.
- Only after obtaining $\alpha(\mu_0)$ ab-initio apply perturbative running to compare to $\alpha(M_Z)$.