Appendix I: Variational Derivation and Einstein Compatibility in Biquaternionic Gravity

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Abstract

We complete the variational derivation of field equations from the biquaternionic gravitational action proposed in the Unified Biquaternion Theory (UBT). We then explicitly demonstrate the compatibility of the resulting equations with the Einstein vacuum field equations in the real-valued limit. This appendix establishes that General Relativity emerges as a special case of the broader algebraic structure.

1 Recap: The Biquaternionic Action

We begin with the action functional:

$$S[e,\omega] = \int \det(e)\,\mathrm{Re}\left[\mathrm{ScalarPart}(e^\mu_a e^\nu_b R^{ab}_{\mu\nu})\right]\, d^4x$$

where:

- e^μ_a: biquaternionic tetrad
- ω^{ab}_μ: biquaternionic spin connection
- $R^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + [\omega^{ac}_\mu, \omega^{cb}_\nu]$: curvature tensor

2 Variation with Respect to Tetrad

Let:

$$\mathcal{R} := \mathrm{ScalarPart}(e^\mu_a e^\nu_b R^{ab}_{\mu\nu})$$

Then:

$$\delta S = \int \left(\delta \det(e) \cdot \mathrm{Re}[\mathcal{R}] + \det(e) \cdot \mathrm{Re}[\delta\mathcal{R}]\right) d^4x$$

Using:

$$\delta \det(e) = \det(e) \cdot e^\mu_a \delta e^a_\mu$$

$$\delta\mathcal{R} = \mathrm{ScalarPart}\left((\delta e^\mu_a) e^\nu_b R^{ab}_{\mu\nu} + e^\mu_a (\delta e^\nu_b) R^{ab}_{\mu\nu}\right)$$

We obtain:

$$\delta S = \int \delta e^\mu_a \cdot \det(e) \cdot \left[\mathrm{Re}(\mathcal{R}) \cdot e^\mu_a + \mathrm{Re}\left(\mathrm{ScalarPart}(e^\nu_b R^{ab}_{\mu\nu}) + \mathrm{ScalarPart}(e^\nu_b R^{ba}_{\nu\mu})\right)\right] d^4x$$

3 Field Equations

Demanding $\delta_e S = 0$ for arbitrary variations gives:

$$\boxed{\mathrm{Re}\left(\mathrm{ScalarPart}(e_b^\nu R_{\mu\nu}^{ab}) + \mathrm{ScalarPart}(e_b^\nu R_{\nu\mu}^{ba})\right) + \mathrm{Re}(\mathcal{R}) \cdot e_a^\mu = 0}$$

4 Compatibility with Einstein Gravity

Assume the real-valued limit:

- $e_a^\mu \in \mathbb{R}$, $\omega_\mu^{ab} \in \mathbb{R}$
- Define $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$

  Then:

  $$\mathrm{ScalarPart}(e_b^\nu R_{\mu\nu}^{ab}) = e_b^\nu R_{\mu\nu}^{ab} \quad \text{and} \quad \mathcal{R} = e_a^\mu e_b^\nu R_{\mu\nu}^{ab} = R$$

  Therefore, the equation becomes:

  $$e_b^\nu R_{\mu\nu}^{ab} + e_b^\nu R_{\nu\mu}^{ba} + R e_a^\mu = 0$$

  Using symmetrization and projection:

  $$R_{\mu a} := e_b^\nu R_{\mu\nu}^{ab} \Rightarrow E_a^\mu := R_{\mu a} - \tfrac{1}{2} e_a^\mu R = 0 \Rightarrow G_{\mu\nu} = 0$$

5 Variation of the Determinant $\det(e)$

In the biquaternionic formalism, we define the determinant of the tetrad field via an antisymmetric volume form:

$$\det(e) := \frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} \,\mathrm{Re}\left[e_\mu^a e_\nu^b e_\rho^c e_\sigma^d\right]$$

To compute its variation, we use the linearity of the real part and antisymmetric contraction to write:

$$\begin{aligned}\delta \det(e) &= \frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} \,\mathrm{Re}\left[(\delta e_\mu^a) e_\nu^b e_\rho^c e_\sigma^d + e_\mu^a (\delta e_\nu^b) e_\rho^c e_\sigma^d + \ldots\right] \\ &= \mathrm{Re}\left[\mathrm{Tr}_{\mathrm{antisym}}\left(\delta e \cdot (\text{triple product of } e)\right)\right]\end{aligned}$$

This motivates a compact expression:

$$\boxed{\delta \det(e) = \det(e) \cdot \mathrm{Re}\left[(e^{-1})_\mu^a \delta e_\mu^a\right]}$$

where $(e^{-1})_\mu^a$ denotes the inverse tetrad defined via antisymmetric projection, compatible with the real scalar volume measure. This identity is fundamental for the variational principle in the presence of non-commutative and non-associative field variables like biquaternions.

6 Compatibility with Einstein Gravity in the Real Limit

To demonstrate that our biquaternionic field equation reduces to General Relativity in the real-valued limit, we examine the field equation obtained from variation:

$$e_b^\nu R_{\mu\nu}^{ab} + e_b^\nu R_{\nu\mu}^{ba} + \mathcal{R} \cdot e_a^\mu = 0$$

Due to the antisymmetry properties of the Riemann curvature tensor:

$$R_{\mu\nu}^{ab} = -R_{\nu\mu}^{ab}, \quad R_{\mu\nu}^{ab} = -R_{\mu\nu}^{ba}$$

the first two terms combine symmetrically upon contraction. Defining the Ricci tensor in frame-index form:

$$R_\mu^a := e_b^\nu R_{\mu\nu}^{ab}$$

we obtain:

$$e_b^\nu R_{\mu\nu}^{ab} + e_b^\nu R_{\nu\mu}^{ba} = 2R_\mu^a$$

Substituting into the field equation gives:

$$2R_\mu^a + \mathcal{R} \cdot e_a^\mu = 0$$

Multiplying both sides by the inverse tetrad e_ν^a, we project into coordinate indices:

$$2R_{\mu\nu} + \mathcal{R} \cdot g_{\mu\nu} = 0$$

Here we used the identities:

$$R_{\mu\nu} := R_\mu^a e_\nu^a, \qquad g_{\mu\nu} := e_\mu^a e_\nu^a$$

and note that $\mathcal{R} := e_a^\mu e_b^\nu R_{\mu\nu}^{ab}$ corresponds to the scalar curvature R up to sign conventions. Thus:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

which is the Einstein field equation in vacuum. This demonstrates that the real part of the biquaternionic variational principle is fully compatible with classical General Relativity.

7 Conclusion

The field equations of the UBT reduce to Einstein's equations in the real-valued limit. This confirms that General Relativity is a special case embedded in the more general biquaternionic formulation, and the remaining components of the master equation encode extended physics beyond GR.