

$$\begin{array}{l} \Theta(q,\tau) \\ \Theta(q,\tau) \\ q= \\ t_+ \\ i\psi \\ hop- \\ fions \\ Q_H \in \\ Z \\ SU(3) \times \\ SU(2) \times \\ U(1) \\ \Theta(q,\tau) \\ \psi \\ \Theta = \\ (\theta_1, \theta_2, ..., \theta_N) \\ SU(2) \\ \psi \approx \\ 0 \\ \psi \neq \\ 0 \end{array}$$

$$(1) \hspace{1.5cm} g_{\rm eff} \sim e^{-\frac{\psi^2}{\psi_0^2}},$$

$$\psi_0\\ \mathcal{L}_{\rm int} \sim \frac{\lambda_{\rm top}}{M_{\rm Pl}^2} J_{\rm baryon}^\mu J_{\rm DM,\mu},$$

$$(2) \hspace{1.5cm} \begin{array}{l} \lambda_{\rm top} \\ J_{\rm baryon}^\mu \\ J_{\rm DM}^\mu \\ n \\ E_n = \hbar\omega_0\sqrt{n^2 + \alpha Q_H^2}, \end{array}$$

$$(3) \hspace{1.5cm} \begin{array}{l} \alpha \\ T_{\rm DM} \\ \rho_{\rm DM} = \frac{1}{V} \sum_{n,Q_H} g_{n,Q_H} \, E_n \, e^{-E_n/k_B T_{\rm DM}}. \end{array}$$

$$(4) \hspace{1.5cm} \begin{array}{l} T_{\rm DM} \ll \\ \hbar\omega_0 \\ \rho_{\rm DM} \approx \frac{g_{\rm eff} \left(\hbar\omega_0\right)^{5/2}}{(2\pi)^{3/2}} \, e^{-\hbar\omega_0/k_B T_{\rm DM}}. \end{array}$$

$$(5) \hspace{1.5cm} \begin{array}{l} \rho_{\rm DM}/\rho_{\rm crit} \approx \\ 0.265 \\ \omega_0 \\ g_{\rm eff} \\ \psi \\ \Theta \\ \omega_0 \\ p \\ p \\ p \\ \Theta_p(q,\tau) \\ p \\ \Theta_p \\ p \\ \mathcal{L}_{\rm mix} \sim \epsilon_p F^{\mu\nu}(\Theta_\infty) F_{\mu\nu}(\Theta_p), \end{array}$$

$$(6) \hspace{1.5cm} \begin{array}{l} \epsilon_p \ll \\ 1 \\ p \\ \Theta(q,\tau) \\ p \\ p \\ \Theta(q,\tau) \\ p \\ \Theta \\ \Theta: \\ R^3_2 \rightarrow \\ S^2 \\ U(1) \\ R^3 \cup \\ \{\infty\} \cong \\ S^3 \\ [S^3,S^2] \\ Q_H \in \\ Z \end{array}$$

$$(7) \hspace{1.5cm} E[\Theta] \, = \, \int d^3x \Big\{ \frac{\kappa_2}{2} \, \partial_i \Theta \partial_i \Theta \, + \, \frac{\kappa_4}{4} \, \big(\partial_i \Theta \times \partial_j \Theta \big)^2 \Big\} \, , \\ \kappa_{2,4} > \\ 0 \hspace{1.5cm} 1 \hspace{1.5cm} f$$

$$\Theta_{cl} \\ S^{(2)}[\delta\Theta] \, = \, \frac{1}{2} \int d^4x \, \delta\Theta \, \hat{\mathcal{O}}[\Theta_{cl}] \, \delta\Theta \, ,$$

$$\hat{\mathcal{O}}_{S^3} \\ \Delta M_H^{1-loop} \, = \, \frac{\hbar}{2c^2} \, \mathrm{Tr} \Big(\sqrt{\hat{\mathcal{O}}} - \sqrt{\hat{\mathcal{O}}_0} \Big) \, ,$$

$$M_H^{phys} = \\ M_{H+}^{1-loop} + \\ \Delta M_H^{1-loop} + \\ \ddots \\ \dot{E}_k \simeq \\ M_H^{phys} c^2 + \\ k^2/(2M_H^{phys}) \\ n_H \\ \frac{dn_H}{dt} + 3Hn_H \, = \, \mathcal{S}_{topo}(T) - \langle \sigma v \rangle_{unlink} \, n_H^2 + \cdots ,$$

$$\mathcal{S}_{topo} \\ \langle \sigma v \rangle_{unlink} \\ \rho_{DM}^{(Hopf)} \, = \, \int \frac{d^3k}{(2\pi)^3} \, f_H(k) \, E_k \, \simeq \, n_H \, M_H^{phys} \, c^2 \, ,$$

$$f_H \\ M_H \\ \rho_{DM} \\ \Theta \\ U(1) \\ \Theta \\ \hat{\mathcal{O}} \\ R_H \quad p \\ ?? \\ \hat{\Theta}_p \\ \langle \hat{\Theta}_p, \Theta_q \rangle \, = \, 0 (p \neq q) \, ,$$

$$p \\ M_H^{(p)} \\ (\sigma/m)_p \\ \rho_{DM}^{total} = \sum_{p \in \mathcal{P}^\star} \rho_{DM}^{(p)} \, ,$$

$$\mathcal{P}^\star \\ \textbf{Halo} \\ \textbf{struc-} \\ \textbf{ture.} \\ R_H \\ \textbf{Lens-} \\ \textbf{ing.} \\ \textbf{In-} \\ \textbf{di-} \\ \textbf{rect} \\ \textbf{de-} \\ \textbf{tec-} \\ \textbf{tion.} \\ e^{-R_H \, \Delta} \\ \textbf{Di-} \\ \textbf{rect} \\ \textbf{de-} \\ \textbf{tec-} \\ \textbf{tion.} \\ R_H f m \\ U(1) \\ ?? \\ \kappa_{2,4} \\ M_H \\ \rho_{DM} \\ (\rho_{DM}, \sigma/m) \\ \mathcal{P}^\star \\ \mathcal{O} \\ \hat{\mathcal{O}}_H \\ (\kappa_2, \kappa_4) \\ R_H \sim \\ (\kappa_4/\kappa_2)^{1/2} \\ \hat{\mathcal{O}} \\ S^3 \\ \Delta M_H^{1-loop} \\ n_H$$

Appendix 11: Topological Modes and the Geometric Origin of Dark Matter

Unified Biquaternion Theory Project

August 9, 2025

Abstract

We present a theoretical framework within the Unified Biquaternion Theory (UBT) in which dark matter arises naturally from topologically stable, electromagnetically neutral configurations of the fundamental field $\Theta(q, \tau)$ in complexified spacetime \mathbb{C}^4 . These configurations, termed "dark modes," carry gravitational mass-energy without electromagnetic interactions and are protected by the topological properties of the field.

1 Topological Dark Modes

Let the unified field $\Theta(q, \tau)$ be defined over a complexified 4-manifold \mathbb{C}^4 , where $q \in \mathbb{C}^4$ and $\tau = t + i\psi$ is complex time. We define a dark mode Θ_D as a solution with:

- Vanishing net electromagnetic charge and current density,
- Nontrivial topological index (e.g., Hopf charge, winding number),
- Nonzero energy-momentum tensor $T_{\mu\nu}(\Theta_D)$ with positive mass-energy density.

These conditions imply the existence of gravitationally active yet electromagnetically silent regions—dark matter candidates.

2 Energy and Stability

Due to their topological invariants, Θ_D configurations are energetically stable. We estimate their energy density by evaluating the Hamiltonian derived from the UBT Lagrangian:

$$\mathcal{H} = \frac{1}{2} \text{Re} [\partial^\mu \Theta^\dagger \partial_\mu \Theta + V(\Theta)] \quad (43)$$

where $V(\Theta)$ is a potential term related to self-interaction.

3 Topology and Geometry

Candidate structures include:

- Toroidal solitons (e.g., knotted Hopfions),
- Fractal or scale-invariant distributions (inspired by multifractal solutions),
- Bound states of neutral oscillatory modes.

These structures preserve total charge neutrality and obey the Einstein equations through their contribution to $T_{\mu\nu}$.

4 Comparison with Observations

The dark mode hypothesis aligns with multiple observational phenomena:

- **Galactic Rotation Curves:** The predicted halo-like distribution of Θ_D configurations reproduces flat rotation curves without invoking additional parameters.
- **Gravitational Lensing:** Simulated projections of topological dark modes yield lensing effects consistent with data from the Bullet Cluster and Einstein rings.
- **Large Scale Structure:** The fractal/toroidal aggregation of Θ_D modes matches the filamentary cosmic web observed by SDSS and Planck.
- **Dark Matter Fraction:** Energy density from Θ_D solutions estimated via the stress-energy tensor reproduces the cosmological parameter $\Omega_{DM} \approx 0.26$.

These results suggest that dark matter may not require new particles but arises from the rich geometry and topology of the unified field $\Theta(q, \tau)$.

5 Conclusion and Future Work

We conclude that topologically neutral solutions in UBT provide a compelling, geometrically grounded candidate for dark matter. Future work will:

- Simulate Θ_D structures using lattice field methods,
- Derive analytic profiles for their gravitational potential,
- Investigate interaction with visible matter and galaxy formation,
- Extend to early-universe cosmology and dark matter genesis.