

Noether  $\rightarrow$   $\alpha$  v0.4: Ab-initio fixing of  $L_\psi$ ,  $Z$ ,

$g_5$

with derivation of the resummed  $V_{\text{eff}}$  and massless limit

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## 1 Setup: 5D Action, Geometry, BC

We consider  $M^4 \times S_\psi^1$  with coordinates  $x^\mu$  and  $\psi \sim \psi + L_\psi$ . Let  $\Theta(x, \psi)$  denote the unified (biquaternionic) field and  $A_M(x, \psi)$  a  $U(1)$  gauge field. The 5D action (natural units  $c = \hbar = 1$ ) is

$$S = \int d^4x \int_0^{L_\psi} d\psi \sqrt{|g_5|} \left[ g^{MN} (D_M \Theta)^\dagger (D_N \Theta) - V(\Theta) - \frac{1}{4} g^{MR} g^{NS} F_{MN} F_{RS} \right], \quad (1)$$

with  $D_M = \partial_M + ig_5 A_M$ ,  $F_{MN} = \partial_M A_N - \partial_N A_M$ . We allow a warped background

$$ds^2 = e^{2A(\psi)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(\psi)} d\psi^2, \quad \sqrt{|g_5|} = e^{4A(\psi) + B(\psi)}. \quad (2)$$

Boundary conditions (BC) along  $S_\psi^1$  may be periodic or twisted (phase  $\delta$ ).

**Holonomy (Wilson line).** The gauge-invariant holonomy (Hosotani parameter) is

$$\theta_H \equiv g_5 \oint_{S_\psi^1} A_\psi d\psi = g_5 \int_0^{L_\psi} d\psi A_\psi(x, \psi). \quad (3)$$

Large gauge transformations shift  $\theta_H \rightarrow \theta_H + 2\pi n$ ,  $n \in \mathbb{Z}$ .

## 2 Reduction and Canonical Normalization

Assume the photon zero-mode  $A_\mu^{(0)}(x)$  is independent of  $\psi$  and normalized canonically in 4D. Reducing the gauge kinetic term in (1) gives

$$S_{\text{gauge}} \supset -\frac{1}{4} \int d^4x Z F_{\mu\nu}^{(0)} F_{(0)}^{\mu\nu}, \quad Z \equiv \int_0^{L_\psi} d\psi e^{B(\psi)-2A(\psi)}, \quad (4)$$

so the canonically normalized 4D photon is  $A_\mu^{(0)} \rightarrow A_\mu^{(0)}/\sqrt{Z}$ . The covariant derivative contributes the interaction

$$\int d^4x \int_0^{L_\psi} d\psi \sqrt{|g_5|} J^\mu A_\mu \longrightarrow \int d^4x g_4 J_{(0)}^\mu A_\mu^{(0)}, \quad g_4 = \frac{g_5}{\sqrt{Z}}, \quad (5)$$

where  $J^\mu$  is the Noether current density and  $J_{(0)}^\mu$  its overlap with the photon zero-mode. Therefore the fine-structure constant at a reference scale  $\mu_0$  is

$$\boxed{\alpha(\mu_0) = \frac{g_4^2}{4\pi} = \frac{g_5^2}{4\pi Z}}. \quad (6)$$

## 3 Noether Charge and Current Matching

Global  $U(1)$ :  $\Theta \rightarrow e^{i\lambda}\Theta$  yields the 5D Noether current

$$J^M = i [\Theta^\dagger (D^M \Theta) - (D^M \Theta)^\dagger \Theta] \Big|_{A=0}. \quad (7)$$

We fix the normalization of  $\Theta$  such that the fundamental charged excitation has unit Noether charge

$$Q \equiv \int d^3x \int_0^{L_\psi} d\psi \sqrt{|g_5|} J^0 = \pm 1. \quad (8)$$

This fixes the overall scale entering the coupling (5); no additional free normalization survives.

## 4 Wilson Line and Quantization

On the compact  $\psi$ -cycle the holonomy (3) is physical. For a field with  $U(1)$  charge  $q$  and BC phase  $\delta$  the KK momenta are shifted by

$$p_\psi^{(n)} = \frac{2\pi}{L_\psi} (n + a), \quad a \equiv \frac{q\theta_H}{2\pi} + \delta, \quad n \in \mathbb{Z}. \quad (9)$$

Stationary vacua (Hosotani mechanism) are determined dynamically and may select nontrivial  $\theta_H^*$ . Large gauge invariance enforces periodicity  $\theta_H \sim \theta_H + 2\pi$ ; stable vacua satisfy  $\partial V_{\text{eff}}/\partial\theta_H = 0$ .

## 5 One-Loop Effective Potential $V_{\text{eff}}(L_\psi, \theta_H)$

For each field  $j$  with spin-statistics sign  $\sigma_j = \pm 1$  (boson  $+1$ , fermion  $-1$ ), degeneracy  $d_j$ , mass  $m_j$ , charge  $q_j$ , and twist  $\delta_j$ , the one-loop contribution is

$$V_j(L_\psi, \theta_H) = \frac{\sigma_j d_j}{2} \int \frac{d^4 p}{(2\pi)^4} \sum_{n \in \mathbb{Z}} \ln \left[ p^2 + m_j^2 + \left( \frac{2\pi}{L_\psi} \right)^2 (n + a_j)^2 \right], \quad (10)$$

$$a_j \equiv \frac{q_j \theta_H}{2\pi} + \delta_j. \quad (11)$$

Using standard contour/Matsubara techniques (equivalently Poisson resummation) one obtains an exact resummed form

$$\boxed{V_j(L_\psi, \theta_H) = \sigma_j d_j \int \frac{d^4 p}{(2\pi)^4} \frac{1}{L_\psi} \ln \left( 1 - 2e^{-L_\psi \omega_j(p)} \cos 2\pi a_j + e^{-2L_\psi \omega_j(p)} \right)}, \quad (12)$$

where  $\omega_j(p) = \sqrt{p^2 + m_j^2}$ . The total potential is

$$V_{\text{eff}}(L_\psi, \theta_H) = \sum_j V_j(L_\psi, \theta_H). \quad (13)$$

**Massless scaling (correction).** In the massless limit  $m_j \rightarrow 0$ ,  $V_j$  scales as  $1/L_\psi^5$  (not  $1/L_\psi^4$ ). This is consistent with dimensional analysis in 5D.

## 6 Determining $Z$ (Warped Case)

If the vacuum back-reacts and generates a nontrivial warp, the 4D gauge kinetic factor is

$$Z = \int_0^{L_\psi} d\psi \, e^{B(\psi) - 2A(\psi)} |\xi_0(\psi)|^2, \quad (14)$$

where  $\xi_0(\psi)$  is the photon zero-mode profile (constant in flat space). The warp factors  $A(\psi)$ ,  $B(\psi)$  and  $\xi_0(\psi)$  follow from the coupled background equations of motion derived from (1) (e.g. via a first-order BPS system if available). In flat space,  $\xi_0(\psi) \equiv 1$  and  $Z = L_\psi$ . In general we write

$$Z = L_\psi f(\tau, \text{BC}), \quad (15)$$

where  $f$  encodes the modular parameter  $\tau$  (complex-time sector) and boundary data.

## 7 Putting It Together: $\alpha$ from UBT

Combining (6), the stationarity conditions and (14),

$$\boxed{\alpha(\mu_0) = \frac{g_5^2}{4\pi Z^\star} = \frac{g_5^2}{4\pi L_\psi^\star f^\star(\tau, \text{BC})}, \quad (L_\psi^\star, \theta_H^\star) \text{ solve } \partial V_{\text{eff}} = 0.} \quad (16)$$

Here  $g_5$  is fixed by Noether normalization (8) (unit charge for the fundamental excitation) and current matching (5). Thus, given the field content, BC, and UBT potential  $V(\Theta)$ , the pair  $(L_\psi^\star, \theta_H^\star)$  and the warp  $A, B$  are determined and *no free fit survives* in  $\alpha$ . Standard QED running then connects  $\alpha(\mu_0)$  to experimental scales.

## 8 Electron Mass Consistency (Sketch)

The same background must produce the lightest charged KK eigenvalue

$$m_e^2 = \lambda_{\min} \left[ -e^{-B} \partial_\psi (e^{B-2A} \partial_\psi \cdot) + m_\Theta^2 e^{-2A} + \dots \right], \quad (17)$$

for the appropriate (spinor) sector and BC, including the holonomy shift (9). This provides a nontrivial cross-check: the *same*  $(L_\psi^\star, \theta_H^\star, A, B)$  that gives  $\alpha$  must also yield  $m_e$ .

## Appendix A: Derivation of the Resummed Form (12)

Starting from (10), define  $\omega = \sqrt{p^2 + m^2}$  and write

$$\sum_{n \in \mathbb{Z}} \ln \left[ (n+a)^2 + \left( \frac{L_\psi \omega}{2\pi} \right)^2 \right] = \sum_{n \in \mathbb{Z}} \left\{ \ln \left( n+a + i \frac{L_\psi \omega}{2\pi} \right) + \ln \left( n+a - i \frac{L_\psi \omega}{2\pi} \right) \right\}. \quad (18)$$

Using the product representation for  $\sin(\pi z)$  and standard finite-temperature/Matsubara algebra, one obtains (up to  $p$ - and  $m$ -independent constants that drop out)

$$\sum_{n \in \mathbb{Z}} \ln \left[ (n+a)^2 + b^2 \right] = \ln \left( 1 - 2e^{-2\pi b} \cos 2\pi a + e^{-4\pi b} \right) + \text{const}, \quad (19)$$

where  $b = L_\psi \omega / (2\pi)$ . Restoring factors and integrating over  $p$ , we get (12):

$$V(L_\psi, \theta_H) = \sigma d \int \frac{d^4 p}{(2\pi)^4} \frac{1}{L_\psi} \ln \left( 1 - 2e^{-L_\psi \omega(p)} \cos 2\pi a + e^{-2L_\psi \omega(p)} \right). \quad (20)$$

An equivalent route is to differentiate with respect to  $m^2$ , perform the geometric series or Poisson resummation, and integrate back.

## Appendix B: Massless Limit and Polylogarithms

For  $m \rightarrow 0$ , set  $\omega(p) = |p|$  and use the factorization

$$\ln(1 - 2ct + t^2) = \ln(1 - e^{i\theta}t) + \ln(1 - e^{-i\theta}t) = - \sum_{k=1}^{\infty} \frac{2 \cos(k\theta)}{k} t^k, \quad c = \cos \theta, \quad t = e^{-L_\psi |p|}. \quad (21)$$

Then

$$V_{\text{massless}}(L_\psi, \theta_H) = -\frac{2\sigma d}{L_\psi} \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k} \int \frac{d^4 p}{(2\pi)^4} e^{-kL_\psi |p|} \quad (22)$$

$$= -\frac{2\sigma d}{L_\psi} \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k} \frac{2\pi^2}{(2\pi)^4} \int_0^\infty dp p^3 e^{-kL_\psi p} \quad (23)$$

$$= -\frac{2\sigma d}{L_\psi} \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k} \frac{2\pi^2}{(2\pi)^4} \cdot \frac{3!}{(kL_\psi)^4} \quad (24)$$

$$= -\frac{3\sigma d}{2\pi^2 L_\psi^5} \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^5} = -\frac{3\sigma d}{2\pi^2 L_\psi^5} \text{Re Li}_5(e^{i\theta}), \quad \theta = 2\pi a. \quad (25)$$

Thus, in 5D the massless contribution scales as  $1/L_\psi^5$  with a universal (non-fitted) coefficient.

**Massive case (asymptotics).** For  $m > 0$ , using  $\ln(1 - 2ct + t^2) = -2 \sum_{k \geq 1} \cos(k\theta) t^k/k$  with  $t = e^{-L_\psi \omega}$  and the known integral

$$\int \frac{d^4 p}{(2\pi)^4} e^{-kL_\psi \sqrt{p^2 + m^2}} = \frac{m^3}{(2\pi)^2 (kL_\psi)^3} K_3(kmL_\psi), \quad (26)$$

one finds the exact series

$$V(L_\psi, \theta_H; m) = -\frac{2\sigma d}{L_\psi} \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k} \frac{m^3}{(2\pi)^2 (kL_\psi)^3} K_3(kmL_\psi), \quad (27)$$

which reduces to the polylogarithmic result above as  $m \rightarrow 0$  (using  $K_\nu(z) \sim 2^{\nu-1} \Gamma(\nu) z^{-\nu}$ ).