

Noether $\Rightarrow \alpha$ (clean route, UBT-only)

One-page derivation

September 1, 2025

Setup

Let Θ carry a global $U(1)$ phase symmetry generated by Q . The 5D background is $M^4 \times S_\psi^1$ with metric $ds^2 = e^{2A(\psi)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(\psi)} d\psi^2$. Noether's theorem gives a conserved current J^M ; gauging the symmetry amounts to

$$\partial_M \rightarrow D_M \equiv \partial_M + i g_5 Q A_M, \quad (1)$$

and (either fundamental or emergent) gauge dynamics produces $-\frac{1}{4g_5^2} \int \sqrt{|g_5|} F_{MN} F^{MN}$.

Dimensional reduction

Expand the photon in 4D zero-mode $A_\mu^{(0)}(x)$ with profile $\xi_0(\psi)$:

$$A_\mu(x, \psi) = \xi_0(\psi) A_\mu^{(0)}(x) + \dots, \quad Z^\star \equiv \int_0^{L_\psi^\star} e^{B-2A} |\xi_0(\psi)|^2 d\psi. \quad (2)$$

Canonical normalization gives

$$\frac{1}{g_4^2} = \frac{Z^\star}{g_5^2}, \quad \Rightarrow \quad \alpha \equiv \frac{g_4^2}{4\pi} = \frac{g_5^2}{4\pi Z^\star}. \quad (3)$$

Holonomy fixes g_5

The vacuum background $A_\psi(\psi)$ is characterized by the gauge-invariant holonomy

$$\theta_H = g_5 \oint_{S_\psi^1} A_\psi d\psi = g_5 \mathcal{I}_\psi, \quad \mathcal{I}_\psi \equiv \int_0^{L_\psi^\star} A_\psi(\psi) d\psi. \quad (4)$$

Large gauge invariance implies $\theta_H \sim \theta_H + 2\pi n$. Minimizing $V_{\text{eff}}(L_\psi, \theta_H)$ in UBT picks a stationary value θ_H^\star and a vacuum length L_ψ^\star together with $A_\psi(\psi)$. Hence

$$\boxed{g_5 = \frac{\theta_H^\star}{\mathcal{I}_\psi^\star}, \quad \mathcal{I}_\psi^\star = \int_0^{L_\psi^\star} A_\psi(\psi) d\psi.} \quad (5)$$

Final relation

Combining the pieces,

Noether $\Rightarrow \alpha$ (UBT-only)

$$\alpha(\mu_0) = \frac{1}{4\pi} \frac{\theta_H^{\star 2}}{(\mathcal{I}_\psi^\star)^2 Z^\star}.$$

All quantities are determined by the *same* UBT vacuum (Θ, A_ψ, A, B) —no external field list or tunable input.

Notes and special cases

- **Flat zero-mode:** $A = B = 0, \xi_0 = \text{const} \Rightarrow Z^\star = L_\psi^\star$.
- **Mild warp:** $A(\psi) = \varepsilon \cos(2\pi\psi/L)$ gives $Z^\star = L I_0(2\varepsilon)$ (modified Bessel), a purely geometric factor.
- **Constant A_ψ :** $A_\psi = A_0 \Rightarrow \mathcal{I}_\psi^\star = A_0 L_\psi^\star$.
- **Discreteness:** $\theta_H^\star \in 2\pi\mathbb{Z}$ (large gauge). Nontrivial vacua often select $\theta_H^\star = \pi$.

How UBT fixes the inputs (no tuning)

1. Solve vacuum EOM $\Rightarrow (L_\psi^\star, \theta_H^\star, A(\psi), B(\psi), A_\psi(\psi))$.
2. Compute $Z^\star = \int e^{B-2A} |\xi_0|^2 d\psi$ from the Maxwell zero-mode equation on that background.
3. Compute $\mathcal{I}_\psi^\star = \int_0^{L_\psi^\star} A_\psi d\psi$ from the same vacuum $A_\psi(\psi)$.
4. Insert into the boxed formula to get α .

Comment. Pokud má UBT fraktální strukturu ve směru ψ , promítne se *jen* do Z^\star a/nebo \mathcal{I}_ψ^\star . Rovnice pro α zůstává stejná; fraktálnost nahradí konstantní profil vhodnou efektivní mírou v integrálech výše.