Noether $\Rightarrow \alpha$ (clean route, UBT-only)

One-page derivation

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Setup

Let Θ carry a global $U(1)$ phase symmetry generated by Q. The 5D background is $M^4 \times S^1_\psi$ with metric $ds^2 = e^{2A(\psi)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(\psi)}d\psi^2$. Noether's theorem gives a conserved current J^M; gauging the symmetry amounts to

$$\partial_M \;\to\; D_M \equiv \partial_M + i\, g_5\, Q\, A_M, \tag{1}$$

and (either fundamental or emergent) gauge dynamics produces $-\frac{1}{4g_5^2}\int\sqrt{|g_5|}F_{MN}F^{MN}$.

Dimensional reduction

Expand the photon in 4D zero-mode $A^{(0)}_\mu(x)$ with profile $\xi_0(\psi)$:

$$A_\mu(x,\psi) = \xi_0(\psi)\, A^{(0)}_\mu(x) + \cdots\,, \qquad Z^\star \equiv \int_0^{L^\star_\psi} e^{B-2A}\, |\xi_0(\psi)|^2\, d\psi. \tag{2}$$

Canonical normalization gives

$$\frac{1}{g_4^2} \;=\; \frac{Z^\star}{g_5^2}\,, \qquad \Rightarrow \quad \alpha \equiv \frac{g_4^2}{4\pi} \;=\; \frac{g_5^2}{4\pi\, Z^\star}. \tag{3}$$

Holonomy fixes g_5

The vacuum background $A_\psi(\psi)$ is characterized by the gauge-invariant holonomy

$$\theta_H \;=\; g_5 \oint_{S^1_\psi} A_\psi\, d\psi \;=\; g_5\, \mathcal{I}_\psi, \qquad \mathcal{I}_\psi \equiv \int_0^{L_\psi} A_\psi(\psi)\, d\psi. \tag{4}$$

Large gauge invariance implies $\theta_H \sim \theta_H + 2\pi n$. Minimizing $V_{\text{eff}}(L_\psi, \theta_H)$ in UBT picks a stationary value $\theta^\star_H$ and a vacuum length $L^\star_\psi$ together with $A_\psi(\psi)$. Hence

$$\boxed{g_5 \;=\; \frac{\theta^\star_H}{\mathcal{I}^\star_\psi}\,, \qquad \mathcal{I}^\star_\psi = \int_0^{L^\star_\psi} A_\psi(\psi)\, d\psi\ .} \tag{5}$$

Final relation

Combining the pieces,

Noether $\Rightarrow \alpha$ (UBT-only)

$$\boxed{\alpha(\mu_0) \;=\; \frac{1}{4\pi}\,\frac{\theta_H^{\star 2}}{(\mathcal{I}_\psi^\star)^2\, Z^\star}\,.}$$

All quantities are determined by the *same* UBT vacuum (Θ, A_ψ, A, B)—no external field list or tunable input.

Notes and special cases

- **Flat zero-mode:** $A = B = 0$, $\xi_0 = \text{const} \Rightarrow Z^\star = L_\psi^\star$.
- **Mild warp:** $A(\psi) = \varepsilon \cos(2\pi\psi/L)$ gives $Z^\star = L\, I_0(2\varepsilon)$ (modified Bessel), a purely geometric factor.
- **Constant A_ψ:** $A_\psi = A_0 \Rightarrow \mathcal{I}_\psi^\star = A_0\, L_\psi^\star$.
- **Discreteness:** $\theta_H^\star \in 2\pi\mathbb{Z}$ (large gauge). Nontrivial vacua often select $\theta_H^\star = \pi$.

How UBT fixes the inputs (no tuning)

1. Solve vacuum EOM $\Rightarrow (L_\psi^\star, \theta_H^\star, A(\psi), B(\psi), A_\psi(\psi))$.
2. Compute $Z^\star = \int e^{B-2A}|\xi_0|^2 d\psi$ from the Maxwell zero-mode equation on that background.
3. Compute $\mathcal{I}_\psi^\star = \int_0^{L_\psi^\star} A_\psi\, d\psi$ from the same vacuum $A_\psi(\psi)$.
4. Insert into the boxed formula to get α.

Comment. Pokud má UBT fraktální strukturu ve směru ψ, promítne se *jen* do $Z^\star$ a/nebo $\mathcal{I}_\psi^\star$. Rovnice pro α zůstává stejná; fraktálnost nahradí konstantní profil vhodnou efektivní mírou v integrálech výše.