

# Noether $\rightarrow \alpha$ v1.0: Worked example (UBT-only, trace formulation)

Draft for UBT Project

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## Aim

Show, on a minimal vacuum ansatz, how  $\alpha$  follows from UBT *without* specifying any external field list:

- Route (A): Noether + holonomy  $\Rightarrow \alpha = \frac{\theta_H^{\star 2}}{4\pi(\mathcal{I}_\psi^\star)^2 Z^\star}$ .
- Route (B): Emergent photon as a spectral trace  $\Rightarrow \frac{1}{\alpha} = \Lambda Z^\star \sum_B C_5^{(B)} \text{Tr}_{\mathcal{F}_B}(Q^2) + \text{KK logs}$ .

## 1 Minimal vacuum ansatz (for illustration)

Assume a flat background  $A(\psi) = B(\psi) = 0$  with length  $L_\psi^\star = L_0$  and a constant vacuum profile for  $A_\psi$ ,

$$A_\psi(\psi) = A_0 \quad \Rightarrow \quad \mathcal{I}_\psi^\star = \int_0^{L_0} A_\psi d\psi = A_0 L_0, \quad Z^\star = \int_0^{L_0} d\psi = L_0. \quad (1)$$

Suppose the Hosotani dynamics selects a nontrivial holonomy  $\theta_H^\star = \pi$  (a common stationary value).

## Route (A): Noether + holonomy

Then

$$\alpha_A = \frac{\theta_H^{\star 2}}{4\pi (\mathcal{I}_\psi^\star)^2 Z^\star} = \frac{\pi^2}{4\pi (A_0^2 L_0^2) L_0} = \frac{\pi}{4} \frac{1}{A_0^2 L_0^3}. \quad (2)$$

In a full UBT computation  $A_0$  and  $L_0$  are not free: they come from the vacuum EOM that minimize  $V_{\text{eff}}$ . The point of this example is to show the scaling and where the number comes from in Route (A).

## Route (B): Spectral trace

Let the fluctuation fiber of  $\Theta$  split into a Dirac block  $\mathcal{F}_D$  and a scalar block  $\mathcal{F}_S$  with invariant traces

$$T_D \equiv \text{Tr}_{\mathcal{F}_D}(Q^2), \quad T_S \equiv \text{Tr}_{\mathcal{F}_S}(Q^2). \quad (3)$$

Using  $C_5^{\text{Dirac}} = \frac{1}{3(4\pi)^{5/2}}$ ,  $C_5^{\text{scalar}} = \frac{1}{6(4\pi)^{5/2}}$ ,

$$\frac{1}{\alpha_B} = \Lambda Z^\star \left( \frac{T_D}{3(4\pi)^{5/2}} + \frac{T_S}{6(4\pi)^{5/2}} \right) + \underbrace{\text{KK logs}}_{\text{finite, smaller in 5D}}. \quad (4)$$

Here  $T_{D,S}$ ,  $\Lambda$  and  $Z^\star$  are *fixed* by the same UBT vacuum:  $T_{D,S}$  by the  $\Theta$ -bundle algebra,  $\Lambda$  by the UV threshold of the spectrum, and  $Z^\star$  by the photon zero-mode norm.

## Numerical toy (for scale intuition only)

Pick, purely for illustration,  $L_0 = 1.2$  and  $A_0 = 0.25$  (constant background), then

$$\alpha_A \approx \frac{\pi}{4} \frac{1}{0.25^2 \cdot 1.2^3} \approx \text{ALPHA\_A\_NUM}. \quad (5)$$

For Route (B), take fiber traces  $T_D = 8$ ,  $T_S = 8$  (toy invariant traces), and  $\Lambda Z^\star = \Lambda L_0 = 10$ ,

$$\frac{1}{\alpha_B} \approx 10 \times \frac{8/3 + 8/6}{(4\pi)^{5/2}} = 10 \times \frac{4}{(4\pi)^{5/2}} \approx \text{INV\_ALPHA\_B\_NUM}, \quad \alpha_B \approx \text{ALPHA\_B\_NUM}. \quad (6)$$

These toy numbers *are not predictions*; the true values follow from the UBT vacuum EOM and the  $\Theta$ -fiber invariants.

**Takeaway.** Route (A) gives a closed formula once  $(\theta_H^*, \mathcal{I}_\psi^*, Z^*)$  are known. Route (B) gives a closed trace formula once  $(T_B)$ ,  $\Lambda$  and  $Z^*$  are known. Both are UBT-internal; no external field list is needed.