Noether $\to \alpha$ v1.0: Worked example (UBT-only, trace formulation)

Draft for UBT Project

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Aim

Show, on a minimal vacuum ansatz, how α follows from UBT *without* specifying any external field list:

- Route (A): Noether + holonomy $\Rightarrow$ $\alpha = \dfrac{\theta_H^{\star 2}}{4\pi(\mathcal{I}_\psi^\star)^2 Z^\star}$.
- Route (B): Emergent photon as a spectral trace $\Rightarrow$ $\dfrac{1}{\alpha} = \Lambda Z^\star \sum_B C_5^{(B)} \, \mathrm{Tr}_{\mathcal{F}_B}(Q^2) +$ KK logs.

1 Minimal vacuum ansatz (for illustration)

Assume a flat background $A(\psi) = B(\psi) = 0$ with length $L_\psi^\star = L_0$ and a constant vacuum profile for A_ψ,

$$A_\psi(\psi) = A_0 \quad \Rightarrow \quad \mathcal{I}_\psi^\star = \int_0^{L_0} A_\psi \, d\psi = A_0 L_0, \qquad Z^\star = \int_0^{L_0} d\psi = L_0. \quad (1)$$

Suppose the Hosotani dynamics selects a nontrivial holonomy $\theta_H^\star = \pi$ (a common stationary value).

Route (A): Noether + holonomy

Then

$$\boxed{\alpha_A \;=\; \frac{\theta_H^{\star 2}}{4\pi\,(\mathcal{I}_\psi^\star)^2\,Z^\star} \;=\; \frac{\pi^2}{4\pi\,(A_0^2 L_0^2)\,L_0} \;=\; \frac{\pi}{4}\,\frac{1}{A_0^2\,L_0^3}}. \tag{2}$$

In a full UBT computation A_0 and L_0 are not free: they come from the vacuum EOM that minimize V_{eff}. The point of this example is to show the scaling and where the number comes from in Route (A).

Route (B): Spectral trace

Let the fluctuation fiber of Θ split into a Dirac block $\mathcal{F}_D$ and a scalar block $\mathcal{F}_S$ with invariant traces

$$T_D \equiv \mathrm{Tr}_{\mathcal{F}_D}(Q^2), \qquad T_S \equiv \mathrm{Tr}_{\mathcal{F}_S}(Q^2). \tag{3}$$

Using $C_5^{\mathrm{Dirac}} = \dfrac{1}{3(4\pi)^{5/2}}$, $C_5^{\mathrm{scalar}} = \dfrac{1}{6(4\pi)^{5/2}}$,

$$\boxed{\frac{1}{\alpha_B} \;=\; \Lambda Z^\star \left(\frac{T_D}{3(4\pi)^{5/2}} + \frac{T_S}{6(4\pi)^{5/2}}\right) + \underbrace{\text{KK logs}}_{\text{finite, smaller in 5D}}}. \tag{4}$$

Here $T_{D,S}$, Λ and $Z^\star$ are *fixed* by the same UBT vacuum: $T_{D,S}$ by the Θ-bundle algebra, Λ by the UV threshold of the spectrum, and $Z^\star$ by the photon zero-mode norm.

Numerical toy (for scale intuition only)

Pick, purely for illustration, $L_0 = 1.2$ and $A_0 = 0.25$ (constant background), then

$$\alpha_A \approx \frac{\pi}{4}\,\frac{1}{0.25^2 \cdot 1.2^3} \;\approx\; \texttt{ALPHA\_A\_NUM}. \tag{5}$$

For Route (B), take fiber traces $T_D = 8$, $T_S = 8$ (toy invariant traces), and $\Lambda Z^\star = \Lambda L_0 = 10$,

$$\frac{1}{\alpha_B} \approx 10\times\frac{8/3 + 8/6}{(4\pi)^{5/2}} \;=\; 10\times\frac{4}{(4\pi)^{5/2}} \;\approx\; \texttt{INV\_ALPHA\_B\_NUM}, \qquad \alpha_B \approx \texttt{ALPHA\_B\_NUM}. \tag{6}$$

These toy numbers *are not predictions*; the true values follow from the UBT vacuum EOM and the Θ-fiber invariants.

Takeaway. Route (A) gives a closed formula once $(\theta_H^\star, \mathcal{I}_\psi^\star, Z^\star)$ are known. Route (B) gives a closed trace formula once (T_B), Λ and $Z^\star$ are known. Both are UBT-internal; no external field list is needed.