Nontrivial Solutions of the Scalar Constraint in the Unified Biquaternion Theory

1 Introduction

We consider the scalar constraint from Priority 1:

$$\eta^{\mu\nu}(\partial_\mu \rho)(\partial_\nu \phi) = 0$$

where $\rho = |\Theta|$ is the amplitude and ϕ the phase of the field. This condition geometrically enforces the orthogonality of gradients of amplitude and phase in spacetime.

2 Nontrivial Minkowski Solutions

In Minkowski spacetime, one trivial solution is when either ρ or ϕ is constant. However, we can construct richer solutions.

Let:

$$\rho = f(t - x), \quad \phi = g(t + x)$$

Then:

$$\partial_\mu \rho = f'(t - x)(\delta^0_\mu - \delta^1_\mu), \quad \partial_\nu \phi = g'(t + x)(\delta^0_\nu + \delta^1_\nu)$$

so:

$$\eta^{\mu\nu}\partial_\mu \rho \, \partial_\nu \phi = f'(t - x)g'(t + x)(\eta^{00} - \eta^{01} + \eta^{10} - \eta^{11}) = 0$$

since the mixed terms cancel.

Thus, such left/right-moving wave combinations satisfy the scalar constraint.

3 Axially Symmetric Configurations

We explore solutions of the form:

$$\rho = \rho(r), \quad \phi = \phi(t - r)$$

with $r = \sqrt{x^2 + y^2 + z^2}$. Then:

$$\partial_t \rho = 0, \quad \nabla \rho = \rho'(r)\frac{\vec{r}}{r}$$

$$\partial_t \phi = \phi'(t - r), \quad \nabla \phi = -\phi'(t - r)\frac{\vec{r}}{r}$$

The scalar constraint becomes:

$$\eta^{\mu\nu}\partial_\mu \rho \, \partial_\nu \phi = -\nabla \rho \cdot \nabla \phi = -\rho'(r)\phi'(t - r)$$

This vanishes iff $\rho'(r)\phi'(t - r) = 0$, leading to conditional satisfaction.

4 Implications in FRW Spacetimes

In cosmology, consider an FRW metric:

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2$$

We look at configurations:

$$\rho = \rho(r), \quad \phi = \phi(t)$$

Then:

$$\partial_\mu \rho = \delta^i_\mu \partial_i \rho, \quad \partial_\nu \phi = \delta^0_\nu \dot{\phi}$$

so:

$$g^{\mu\nu} \partial_\mu \rho \partial_\nu \phi = g^{0i} \dot{\phi} \partial_i \rho = 0$$

since $g^{0i} = 0$, this is satisfied. But with $\phi = \phi(t, r)$, nontrivial structures emerge.

5 Conclusions and Future Work

We have demonstrated a family of exact solutions beyond trivial cases. These exhibit spatial-temporal interference patterns and allow for localized dynamics in scalar field evolution. This paves the way for:

- Numerical simulations of scalar-phase interaction.
- Exploring implications near strong gravitational fields.
- Deriving effective potentials from interactions.