$\Theta(q,\tau)$
$\Theta(q,\tau)$
q
$=$
$t+$
$i\psi$
hop-
fions
$Q_H \in$
Z
$SU(3)\times$
$SU(2)\times$
$U(1)$
$\Theta(q,\tau)$
ψ
$\Theta =$
$(\theta_1, \theta_2, ..., \theta_N)$
$SU(2)$
$\psi \approx$
0
$\psi \neq$
0

$$g_{\text{eff}} \sim e^{-\frac{\psi^2}{\psi_0^2}}, \tag{1}$$

ψ_0

$$\mathcal{L}_{\text{int}} \sim \frac{\lambda_{\text{top}}}{M_{\text{Pl}}^2} J^{\mu}_{\text{baryon}} J_{\text{DM},\mu}, \tag{2}$$

λ_{top}
J^{μ}_{baryon}
J^{μ}_{DM}
n

$$E_n = \hbar\omega_0 \sqrt{n^2 + \alpha Q_H^2}, \tag{3}$$

α
T_{DM}

$$\rho_{\text{DM}} = \frac{1}{V} \sum_{n,Q_H} g_{n,Q_H}\, E_n\, e^{-E_n/k_B T_{\text{DM}}}. \tag{4}$$

$T_{\text{DM}} \ll$
$\hbar\omega_0$

$$\rho_{\text{DM}} \approx \frac{g_{\text{eff}}\,(\hbar\omega_0)^{5/2}}{(2\pi)^{3/2}}\, e^{-\hbar\omega_0/k_B T_{\text{DM}}}. \tag{5}$$

$\rho_{\text{DM}}/\rho_{\text{crit}} \approx$
0.265
ω_0
g_{eff}
ψ
Θ
ω_0
p
p
p
$\Theta_p(q,\tau)$
p
Θ_p
$p =$
∞

$$\mathcal{L}_{\text{mix}} \sim \epsilon_p F^{\mu\nu}(\Theta_\infty) F_{\mu\nu}(\Theta_p), \tag{6}$$

$\epsilon_p \ll$
1
p
$\Theta(q,\tau)$
p
p
$\Theta(q,\tau)$
p
Θ
$\Theta :$
$R^3 \to$
S^2
$U(1)$
$R^3 \cup$
$\{\infty\} \cong$
S^3
$[S^3, S^2]$
$Q_H \in$
Z

$$E[\Theta] \;=\; \int d^3x \Big\{ \frac{\kappa_2}{2}\, \partial_i\Theta\partial_i\Theta + \frac{\kappa_4}{4}\, \big(\partial_i\Theta\times\partial_j\Theta\big)^2 \Big\}, \tag{7}$$

$\kappa_{2,4} >$
0

Θ_{cl}

$$S^{(2)}[\delta\Theta] \;=\; \frac{1}{2}\int d^4x\,\delta\Theta\,\hat{\mathcal{O}}[\Theta_{cl}]\,\delta\Theta\,, \tag{11}$$

$\hat{\mathcal{O}}$
S^3

$$\Delta M_H^{1-loop} \;=\; \frac{\hbar}{2c^2}\,\mathrm{Tr}\Big(\sqrt{\hat{\mathcal{O}}}-\sqrt{\hat{\mathcal{O}}_0}\Big)\,, \tag{12}$$

$M_H^{phys} = M_H + \Delta M_H^{1-loop} + \cdots$
α
$E_k \simeq M_H^{phys} c^2 + k^2/(2M_H^{phys})$
n_H

$$\frac{dn_H}{dt}+3Hn_H \;=\; \mathcal{S}_{topo}(T) - \langle\sigma v\rangle_{unlink}\,n_H^2 + \cdots, \tag{13}$$

$\mathcal{S}_{topo}$
$\langle\sigma v\rangle_{unlink}$

$$\rho_{DM}^{(Hopf)} \;=\; \int \frac{d^3k}{(2\pi)^3}\,f_H(k)\,E_k \;\simeq\; n_H\,M_H^{phys}c^2\,, \tag{14}$$

f_H
M_H
ρ_{DM}
Θ
$U(1)$
Θ
$\hat{\mathcal{O}}$
R_H p
??
Θ_p

$$\langle\Theta_p,\Theta_q\rangle \;=\; 0 (p\neq q)\,, \tag{15}$$

p
$M_H^{(p)}$
$(\sigma/m)_p$

$$\rho_{DM}^{total} \;=\; \sum_{p\in\mathcal{P}^\star}\rho_{DM}^{(p)}\,, \tag{16}$$

$\mathcal{P}^\star$
Halo structure.
R_H
Lensing.
Indirect detection.
$\sim e^{-R_H\,\Delta}$
Direct detection.
$R_H fm$
$U(1)$
??
$\kappa_{2,4}$
M_H
ρ_{DM}
$(\rho_{DM},\sigma/m)$
$\mathcal{P}^\star$
α
??
Q_H
(κ_2,κ_4)
$R_H \sim (\kappa_4/\kappa_2)^{1/2}$
$\hat{\mathcal{O}}$
S^3
ΔM_H^{1-loop}
n_H

Appendix 11: Topological Modes and the Geometric Origin of Dark Matter

Unified Biquaternion Theory Project

August 9, 2025

Abstract

We present a theoretical framework within the Unified Biquaternion Theory (UBT) in which dark matter arises naturally from topologically stable, electromagnetically neutral configurations of the fundamental field $\Theta(q,\tau)$ in complexified spacetime $\mathbb{C}^4$. These configurations, termed "dark modes," carry gravitational mass-energy without electromagnetic interactions and are protected by the topological properties of the field.

1 Topological Dark Modes

Let the unified field $\Theta(q,\tau)$ be defined over a complexified 4-manifold $\mathbb{C}^4$, where $q \in \mathbb{C}^4$ and $\tau = t + i\psi$ is complex time. We define a dark mode Θ_D as a solution with:

- Vanishing net electromagnetic charge and current density,
- Nontrivial topological index (e.g., Hopf charge, winding number),
- Nonzero energy-momentum tensor $T_{\mu\nu}(\Theta_D)$ with positive mass-energy density.

These conditions imply the existence of gravitationally active yet electromagnetically silent regions—dark matter candidates.

2 Energy and Stability

Due to their topological invariants, Θ_D configurations are energetically stable. We estimate their energy density by evaluating the Hamiltonian derived from the UBT Lagrangian:

$$\mathcal{H} = \frac{1}{2}\mathrm{Re}\left[\partial^\mu \Theta^\dagger \partial_\mu \Theta + V(\Theta)\right] \tag{43}$$

where $V(\Theta)$ is a potential term related to self-interaction.

3 Topology and Geometry

Candidate structures include:

- Toroidal solitons (e.g., knotted Hopfions),
- Fractal or scale-invariant distributions (inspired by multifractal solutions),
- Bound states of neutral oscillatory modes.

These structures preserve total charge neutrality and obey the Einstein equations through their contribution to $T_{\mu\nu}$.

4 Comparison with Observations

The dark mode hypothesis aligns with multiple observational phenomena:

- **Galactic Rotation Curves:** The predicted halo-like distribution of Θ_D configurations reproduces flat rotation curves without invoking additional parameters.
- **Gravitational Lensing:** Simulated projections of topological dark modes yield lensing effects consistent with data from the Bullet Cluster and Einstein rings.
- **Large Scale Structure:** The fractal/toroidal aggregation of Θ_D modes matches the filamentary cosmic web observed by SDSS and Planck.
- **Dark Matter Fraction:** Energy density from Θ_D solutions estimated via the stress-energy tensor reproduces the cosmological parameter $\Omega_{DM} \approx 0.26$.

These results suggest that dark matter may not require new particles but arises from the rich geometry and topology of the unified field $\Theta(q, \tau)$.

5 Conclusion and Future Work

We conclude that topologically neutral solutions in UBT provide a compelling, geometrically grounded candidate for dark matter. Future work will:

- Simulate Θ_D structures using lattice field methods,
- Derive analytic profiles for their gravitational potential,
- Investigate interaction with visible matter and galaxy formation,
- Extend to early-universe cosmology and dark matter genesis.