# Nontrivial Solutions of the Scalar Constraint in the Unified Biquaternion Theory

#### 1 Introduction

We consider the scalar constraint from Priority 1:

$$\eta^{\mu\nu}(\partial_{\mu}\rho)(\partial_{\nu}\phi) = 0$$

where  $\rho = |\Theta|$  is the amplitude and  $\phi$  the phase of the field. This condition geometrically enforces the orthogonality of gradients of amplitude and phase in spacetime.

### 2 Nontrivial Minkowski Solutions

In Minkowski spacetime, one trivial solution is when either  $\rho$  or  $\phi$  is constant. However, we can construct richer solutions.

Let:

$$\rho = f(t-x), \quad \phi = g(t+x)$$

Then:

$$\partial_{\mu}\rho = f'(t-x)(\delta_{\mu}^0 - \delta_{\mu}^1), \quad \partial_{\nu}\phi = g'(t+x)(\delta_{\nu}^0 + \delta_{\nu}^1)$$

so:

$$\eta^{\mu\nu}\partial_{\mu}\rho\,\partial_{\nu}\phi = f'(t-x)g'(t+x)(\eta^{00} - \eta^{01} + \eta^{10} - \eta^{11}) = 0$$

since the mixed terms cancel.

Thus, such left/right-moving wave combinations satisfy the scalar constraint.

## 3 Axially Symmetric Configurations

We explore solutions of the form:

$$\rho = \rho(r), \quad \phi = \phi(t - r)$$

with  $r = \sqrt{x^2 + y^2 + z^2}$ . Then:

$$\partial_t \rho = 0, \quad \nabla \rho = \rho'(r) \frac{\vec{r}}{r}$$

$$\partial_t \phi = \phi'(t-r), \quad \nabla \phi = -\phi'(t-r)\frac{\vec{r}}{r}$$

The scalar constraint becomes:

$$\eta^{\mu\nu}\partial_{\mu}\rho\,\partial_{\nu}\phi = -\nabla\rho\cdot\nabla\phi = -\rho'(r)\phi'(t-r)$$

This vanishes iff  $\rho'(r)\phi'(t-r)=0$ , leading to conditional satisfaction.

## 4 Implications in FRW Spacetimes

In cosmology, consider an FRW metric:

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2$$

We look at configurations:

$$\rho = \rho(r), \quad \phi = \phi(t)$$

Then:

$$\partial_{\mu}\rho = \delta^{i}_{\mu}\partial_{i}\rho, \quad \partial_{\nu}\phi = \delta^{0}_{\nu}\dot{\phi}$$

so:

$$g^{\mu\nu}\partial_{\mu}\rho\partial_{\nu}\phi = g^{0i}\dot{\phi}\partial_{i}\rho = 0$$

since  $g^{0i} = 0$ , this is satisfied. But with  $\phi = \phi(t, r)$ , nontrivial structures emerge.

#### 5 Conclusions and Future Work

We have demonstrated a family of exact solutions beyond trivial cases. These exhibit spatial-temporal interference patterns and allow for localized dynamics in scalar field evolution. This paves the way for:

- Numerical simulations of scalar-phase interaction.
- Exploring implications near strong gravitational fields.
- Deriving effective potentials from interactions.