Noether $\rightarrow \alpha$ v1.0: Worked example (UBT-only, trace formulation)

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Aim

Show, on a minimal vacuum ansatz, how α follows from UBT without specifying any external field list:

- Route (A): Noether + holonomy $\Rightarrow \alpha = \frac{\theta_H^{\star 2}}{4\pi (\mathcal{I}_{\psi}^{\star})^2 Z^{\star}}$.
- Route (B): Emergent photon as a spectral trace $\Rightarrow \frac{1}{\alpha} = \Lambda Z^* \sum_B C_5^{(B)} \operatorname{Tr}_{\mathcal{F}_B}(Q^2) + KK \log s$.

1 Minimal vacuum ansatz (for illustration)

Assume a flat background $A(\psi) = B(\psi) = 0$ with length $L_{\psi}^{\star} = L_0$ and a constant vacuum profile for A_{ψ} ,

$$A_{\psi}(\psi) = A_0 \quad \Rightarrow \quad \mathcal{I}_{\psi}^{\star} = \int_{0}^{L_0} A_{\psi} \, d\psi = A_0 L_0, \qquad Z^{\star} = \int_{0}^{L_0} d\psi = L_0. \quad (1)$$

Suppose the Hosotani dynamics selects a nontrivial holonomy $\theta_H^* = \pi$ (a common stationary value).

Route (A): Noether + holonomy

Then

$$\alpha_A = \frac{\theta_H^{\star 2}}{4\pi \, (\mathcal{I}_{\psi}^{\star})^2 \, Z^{\star}} = \frac{\pi^2}{4\pi \, (A_0^2 L_0^2) \, L_0} = \frac{\pi}{4} \, \frac{1}{A_0^2 \, L_0^3} \, . \tag{2}$$

In a full UBT computation A_0 and L_0 are not free: they come from the vacuum EOM that minimize V_{eff} . The point of this example is to show the scaling and where the number comes from in Route (A).

Route (B): Spectral trace

Let the fluctuation fiber of Θ split into a Dirac block \mathcal{F}_D and a scalar block \mathcal{F}_S with invariant traces

$$T_D \equiv \operatorname{Tr}_{\mathcal{F}_D}(Q^2), \qquad T_S \equiv \operatorname{Tr}_{\mathcal{F}_S}(Q^2).$$
 (3)

 $T_D \equiv \text{Tr}_{\mathcal{F}_D}(Q^2), \qquad T_S \equiv \text{Tr}_{\mathcal{F}_S}(Q^2).$ Using $C_5^{\text{Dirac}} = \frac{1}{3(4\pi)^{5/2}}, C_5^{\text{scalar}} = \frac{1}{6(4\pi)^{5/2}},$

$$\overline{\frac{1}{\alpha_B}} = \Lambda Z^* \left(\frac{T_D}{3(4\pi)^{5/2}} + \frac{T_S}{6(4\pi)^{5/2}} \right) + \underbrace{\text{KK logs}}_{\text{finite, smaller in 5D}} .$$
 (4)

Here $T_{D,S}$, Λ and Z^* are fixed by the same UBT vacuum: $T_{D,S}$ by the Θ bundle algebra, Λ by the UV threshold of the spectrum, and Z^* by the photon zero-mode norm.

Numerical toy (for scale intuition only)

Pick, purely for illustration, $L_0 = 1.2$ and $A_0 = 0.25$ (constant background), then

$$\alpha_A \approx \frac{\pi}{4} \frac{1}{0.25^2 \cdot 1.2^3} \approx \text{ALPHA_A_NUM}.$$
 (5)

For Route (B), take fiber traces $T_D = 8$, $T_S = 8$ (toy invariant traces), and $\Lambda Z^{\star} = \Lambda L_0 = 10,$

$$\frac{1}{\alpha_B}\approx 10\times\frac{8/3+8/6}{(4\pi)^{5/2}} \ = \ 10\times\frac{4}{(4\pi)^{5/2}} \ \approx \ \text{INV_ALPHA_B_NUM}, \qquad \alpha_B\approx \text{ALPHA_B_NUM}. \tag{6}$$

These toy numbers are not predictions; the true values follow from the UBT vacuum EOM and the Θ -fiber invariants.

Takeaway. Route (A) gives a closed formula once $(\theta_H^\star, \mathcal{I}_\psi^\star, Z^\star)$ are known. Route (B) gives a closed trace formula once (T_B) , Λ and Z^\star are known. Both are UBT-internal; no external field list is needed.