

# Noether $\rightarrow \alpha$ v0.5: Current Matching and What Fixes $g_5$

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## 1 Goal

We derived

$$\alpha(\mu_0) = \frac{g_5^2}{4\pi Z^\star}, \quad Z^\star = \int_0^{L_\psi^\star} d\psi e^{B-2A} |\xi_0(\psi)|^2 = L_\psi^\star f^\star(\tau, \text{BC}). \quad (1)$$

This section shows how the Noether charge and canonical normalization fix the *field scale*, and how  $g_5$  is or is not fixed, depending on the UV principle chosen in UBT.

## 2 Zero-Mode Decomposition and Canonical Normalization

Expand the unified field in  $\psi$ -modes (for simplicity: complex scalar prototype for the charged excitation)

$$\Theta(x, \psi) = \chi_0(\psi) \phi(x) + \cdots, \quad \int_0^{L_\psi} d\psi e^{B+2A} |\chi_0(\psi)|^2 = 1. \quad (2)$$

Similarly, the photon zero-mode (canonically normalized in 4D) is

$$A_\mu(x, \psi) = \xi_0(\psi) A_\mu^{(0)}(x) + \cdots, \quad \int_0^{L_\psi} d\psi e^{B-2A} |\xi_0(\psi)|^2 = Z. \quad (3)$$

With these conventions, the 4D kinetic term of  $\phi$  is canonical,

$$\mathcal{L}_{4D} \supset |\partial_\mu \phi|^2 - m_0^2 |\phi|^2, \quad (4)$$

and the 4D Noether current takes the standard form

$$j^\mu = i (\phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger) \phi). \quad (5)$$

### 3 Current Matching: $g_4$ from the 5D Coupling

The 5D interaction is  $\int d^5x \sqrt{|g_5|} g_5 J^\mu A_\mu$ , which reduces to

$$S_{\text{int}}^{(0)} = \int d^4x g_5 \left[ \int_0^{L_\psi} d\psi e^B |\chi_0|^2 \xi_0 \right] j^\mu A_\mu^{(0)} = \int d^4x \frac{g_5}{\sqrt{Z}} j^\mu A_\mu^{(0)}. \quad (6)$$

Hence

$$g_4 \equiv \frac{g_5}{\sqrt{Z}}, \quad \alpha = \frac{g_4^2}{4\pi}. \quad (7)$$

The result is robust (spinor/vector generalizations give the same overlap structure).

### 4 Noether Charge and Field Normalization

The 5D Noether charge

$$Q = \int d^3x \int_0^{L_\psi} d\psi \sqrt{|g_5|} J^0 \quad (8)$$

reduces, for the normalized zero-mode, to the 4D number operator. Choosing the fundamental excitation to have  $Q = \pm 1$  *fixes the normalization of  $\phi$*  and removes any free rescaling of  $\Theta$ . This does *not* fix the numerical value of  $g_5$  by itself.

### 5 What Can Fix $g_5$ (and Thus $\alpha$ ) Ab-Initio?

Noether symmetry and compactification give  $\alpha = g_5^2/(4\pi Z^*)$  but leave  $g_5$  as a *UV parameter*. To arrive at a number, UBT needs one of the following principles (each consistent with the framework):

## (A) Bare Gauge Term From UBT Lagrangian + Extra Constraint

Assume a tree-level gauge kinetic term  $-\frac{1}{4}F^2$ . Then  $g_5$  is a fundamental 5D coupling of mass dimension  $-1/2$  and is not fixed by Noether alone. It can be related to other UBT scales if UBT supplies a constraint, e.g.

- tying  $g_5$  to the  $\Theta$  sector (coupling unification in the biquaternionic algebra),
- matching to gravity via a universal scale (Sakharov-like, but determined by the CCT/UBT cutoff),
- or quantization coming from a topological term (e.g. a 5D abelian Chern–Simons coefficient that is quantized and links to minimal charge units).

## (B) *Induced* Photon Kinetic Term (Emergent Gauge Field)

Set the bare coefficient to zero and generate  $F^2$  entirely from loops of  $\Theta$ -sector fields. Compactification to 4D yields

$$\frac{1}{g_4^2(\mu)} = \sum_{j, \text{KK}} \frac{b_j}{8\pi^2} \ln \frac{\Lambda}{m_{j, \text{KK}}} + \dots, \quad (9)$$

with  $b_j$  the standard 4D beta-function weights for each KK mode and  $\Lambda$  the physical UV scale of UBT (finite in CCT/UBT due to the compact  $\psi$  and possible warping). In this scenario,

$$\alpha(\mu_0) \text{ is fixed by field content, BC, and } (L_\psi^\star, \theta_H^\star, A, B) \quad (\text{no } g_5 \text{ input}). \quad (10)$$

This produces a numerical  $\alpha$  once the spectrum is specified; the calculation is technically involved but conceptually parameter-free.

## (C) Holonomy Quantization With Dynamical $A_\psi$

If the vacuum fixes a nontrivial  $\langle A_\psi \rangle$  through the Hosotani mechanism and UBT relates  $\langle A_\psi \rangle$  to a geometric invariant, then

$$\theta_H^\star = g_5 \int_0^{L_\psi^\star} A_\psi d\psi = 2\pi n^\star \quad (11)$$

imposes a relation among  $g_5$  and  $L_\psi^\star$  that *can* remove the  $g_5$  freedom. Whether this happens depends on the detailed UBT background equations; in many simple models  $\theta_H^\star$  is 0 or  $\pi$  independently of  $g_5$ , leaving  $g_5$  unfixed.

## 6 Recommendation

For a **fully ab-initio** numeric value of  $\alpha$  within the present UBT framework, the cleanest route is (B): treat the photon as emergent and compute the induced  $1/g_4^2$  from KK towers of the  $\Theta$ -sector at the vacuum point  $(L_\psi^\star, \theta_H^\star, A, B)$ . This uses only symmetries, geometry and field content, with no free gauge coupling left.

## 7 Next Steps (Minimal Viable Calculation)

1. Choose a concrete  $\Theta$ -sector (spins, charges  $q_j$ , masses  $m_j$ ) and BC.
2. Determine  $(L_\psi^\star, \theta_H^\star)$  by minimizing the one-loop  $V_{\text{eff}}$  (Sec. v0.4).
3. Compute the KK spectrum  $m_{j,\text{KK}}(n)$  on that background.
4. Evaluate the induced gauge kinetic term in 4D, e.g. summing the standard one-loop polarizations over KK levels with a regulator consistent with UBT (compact  $\psi$  and warp imply a physical cutoff).
5. Read off  $\alpha(\mu_0)$ , then run to  $M_Z$  for comparison.

**Consistency with  $m_e$ .** The same background must yield the lightest charged eigenmode mass  $m_e$ ; matching both  $\alpha$  and  $m_e$  with a *single* background is the decisive non-numerological test.