Noether $\to \alpha$ v0.4: Ab-initio fixing of L_ψ, Z, g_5

with derivation of the resummed V_{eff} and massless limit

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1 Setup: 5D Action, Geometry, BC

We consider $M^4 \times S^1_\psi$ with coordinates x^μ and $\psi \sim \psi + L_\psi$. Let $\Theta(x,\psi)$ denote the unified (biquaternionic) field and $A_M(x,\psi)$ a $U(1)$ gauge field. The 5D action (natural units $c = \hbar = 1$) is

$$S = \int d^4x \int_0^{L_\psi} d\psi \sqrt{|g_5|} \left[g^{MN}(D_M\Theta)^\dagger(D_N\Theta) - V(\Theta) - \frac{1}{4} g^{MR} g^{NS} F_{MN} F_{RS} \right], \tag{1}$$

with $D_M = \partial_M + i g_5 A_M$, $F_{MN} = \partial_M A_N - \partial_N A_M$. We allow a warped background

$$ds^2 = e^{2A(\psi)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(\psi)} d\psi^2, \qquad \sqrt{|g_5|} = e^{4A(\psi)+B(\psi)}. \tag{2}$$

Boundary conditions (BC) along S^1_ψ may be periodic or twisted (phase δ).

Holonomy (Wilson line). The gauge-invariant holonomy (Hosotani parameter) is

$$\theta_H \equiv g_5 \oint_{S^1_\psi} A_\psi \, d\psi = g_5 \int_0^{L_\psi} d\psi \, A_\psi(x,\psi). \tag{3}$$

Large gauge transformations shift $\theta_H \to \theta_H + 2\pi n$, $n \in \mathbb{Z}$.

2 Reduction and Canonical Normalization

Assume the photon zero-mode $A_\mu^{(0)}(x)$ is independent of ψ and normalized canonically in 4D. Reducing the gauge kinetic term in (1) gives

$$S_{\text{gauge}} \supset -\frac{1}{4}\int d^4x\, Z\, F_{\mu\nu}^{(0)} F_{(0)}^{\mu\nu}, \qquad Z \equiv \int_0^{L_\psi} d\psi\, e^{B(\psi)-2A(\psi)}, \tag{4}$$

so the canonically normalized 4D photon is $A_\mu^{(0)} \to A_\mu^{(0)}/\sqrt{Z}$. The covariant derivative contributes the interaction

$$\int d^4x \int_0^{L_\psi} d\psi\, \sqrt{|g_5|}\; J^\mu A_\mu \quad\longrightarrow\quad \int d^4x\; g_4\, J_{(0)}^\mu A_\mu^{(0)}, \qquad g_4 = \frac{g_5}{\sqrt{Z}}, \tag{5}$$

where J^μ is the Noether current density and $J_{(0)}^\mu$ its overlap with the photon zero-mode. Therefore the fine-structure constant at a reference scale μ_0 is

$$\boxed{\alpha(\mu_0) \;=\; \frac{g_4^2}{4\pi} \;=\; \frac{g_5^2}{4\pi\, Z}}. \tag{6}$$

3 Noether Charge and Current Matching

Global $U(1)$: $\Theta \to e^{i\lambda}\Theta$ yields the 5D Noether current

$$J^M = i\left[\Theta^\dagger(D^M\Theta) - (D^M\Theta)^\dagger\Theta\right]\Big|_{A=0}. \tag{7}$$

We fix the normalization of Θ such that the fundamental charged excitation has unit Noether charge

$$Q \equiv \int d^3x \int_0^{L_\psi} d\psi\, \sqrt{|g_5|}\, J^0 = \pm 1. \tag{8}$$

This fixes the overall scale entering the coupling (5); no additional free normalization survives.

4 Wilson Line and Quantization

On the compact ψ-cycle the holonomy (3) is physical. For a field with $U(1)$ charge q and BC phase δ the KK momenta are shifted by

$$p_\psi^{(n)} = \frac{2\pi}{L_\psi}\Big(n + a\Big), \qquad a \equiv \frac{q\,\theta_H}{2\pi} + \delta, \qquad n \in \mathbb{Z}. \tag{9}$$

Stationary vacua (Hosotani mechanism) are determined dynamically and may select nontrivial $\theta_H^\star$. Large gauge invariance enforces periodicity $\theta_H \sim \theta_H + 2\pi$; stable vacua satisfy $\partial V_{\text{eff}}/\partial\theta_H = 0$.

5 One-Loop Effective Potential $V_{\text{eff}}(L_\psi, \theta_H)$

For each field j with spin-statistics sign $\sigma_j = \pm 1$ (boson $+1$, fermion -1), degeneracy d_j, mass m_j, charge q_j, and twist δ_j, the one-loop contribution is

$$V_j(L_\psi, \theta_H) = \frac{\sigma_j d_j}{2} \int \frac{d^4p}{(2\pi)^4} \sum_{n\in\mathbb{Z}} \ln\Big[p^2 + m_j^2 + \big(\tfrac{2\pi}{L_\psi}\big)^2 \big(n + a_j\big)^2\Big], \tag{10}$$

$$a_j \equiv \frac{q_j\,\theta_H}{2\pi} + \delta_j. \tag{11}$$

Using standard contour/Matsubara techniques (equivalently Poisson resummation) one obtains an exact resummed form

$$\boxed{V_j(L_\psi, \theta_H) \;=\; \sigma_j d_j \int \frac{d^4p}{(2\pi)^4}\, \frac{1}{L_\psi}\, \ln\Big(1 - 2e^{-L_\psi \omega_j(p)} \cos 2\pi a_j + e^{-2L_\psi\omega_j(p)}\Big),} \tag{12}$$

where $\omega_j(p) = \sqrt{p^2 + m_j^2}$. The total potential is

$$V_{\text{eff}}(L_\psi, \theta_H) \;=\; \sum_j V_j(L_\psi, \theta_H). \tag{13}$$

Massless scaling (correction). In the massless limit $m_j \to 0$, V_j scales as $1/L_\psi^5$ (not $1/L_\psi^4$). This is consistent with dimensional analysis in 5D.

6 Determining Z (Warped Case)

If the vacuum back-reacts and generates a nontrivial warp, the 4D gauge kinetic factor is

$$Z \;=\; \int_0^{L_\psi} d\psi\; e^{B(\psi) - 2A(\psi)}\, |\xi_0(\psi)|^2, \tag{14}$$

where $\xi_0(\psi)$ is the photon zero-mode profile (constant in flat space). The warp factors $A(\psi), B(\psi)$ and $\xi_0(\psi)$ follow from the coupled background equations of motion derived from (1) (e.g. via a first-order BPS system if available). In flat space, $\xi_0(\psi) \equiv 1$ and $Z = L_\psi$. In general we write

$$Z \;=\; L_\psi\, f(\tau, \mathrm{BC}), \tag{15}$$

where f encodes the modular parameter τ (complex-time sector) and boundary data.

7 Putting It Together: α from UBT

Combining (6), the stationarity conditions and (14),

$$\boxed{\alpha(\mu_0) \;=\; \frac{g_5^2}{4\pi\, Z^\star} \;=\; \frac{g_5^2}{4\pi\, L_\psi^\star f^\star(\tau, \mathrm{BC})}, \qquad (L_\psi^\star, \theta_H^\star) \text{ solve } \partial V_{\mathrm{eff}} = 0.} \tag{16}$$

Here g_5 is fixed by Noether normalization (8) (unit charge for the fundamental excitation) and current matching (5). Thus, given the field content, BC, and UBT potential $V(\Theta)$, the pair $(L_\psi^\star, \theta_H^\star)$ and the warp A, B are determined and *no free fit survives* in α. Standard QED running then connects $\alpha(\mu_0)$ to experimental scales.

8 Electron Mass Consistency (Sketch)

The same background must produce the lightest charged KK eigenvalue

$$m_e^2 \;=\; \lambda_{\min}\Big[- e^{-B}\partial_\psi\big(e^{B-2A}\partial_\psi\,\cdot\,\big) + m_\Theta^2 e^{-2A} + \cdots \Big], \tag{17}$$

for the appropriate (spinor) sector and BC, including the holonomy shift (9). This provides a nontrivial cross-check: the *same* $(L_\psi^\star, \theta_H^\star, A, B)$ that gives α must also yield m_e.

Appendix A: Derivation of the Resummed Form (12)

Starting from (10), define $\omega = \sqrt{p^2 + m^2}$ and write

$$\sum_{n\in\mathbb{Z}} \ln\Big[\big(n+a\big)^2 + \big(\tfrac{L_\psi \omega}{2\pi}\big)^2\Big] = \sum_{n\in\mathbb{Z}} \Big\{ \ln\big(n+a+i\tfrac{L_\psi\omega}{2\pi}\big) + \ln\big(n+a-i\tfrac{L_\psi\omega}{2\pi}\big)\Big\}. \quad (18)$$

Using the product representation for $\sin(\pi z)$ and standard finite-temperature/Matsubara algebra, one obtains (up to p- and m-independent constants that drop out)

$$\sum_{n\in\mathbb{Z}} \ln\Big[\big(n+a\big)^2 + b^2\Big] = \ln\Big(1 - 2e^{-2\pi b}\cos 2\pi a + e^{-4\pi b}\Big) + \text{const}, \quad (19)$$

where $b = L_\psi\omega/(2\pi)$. Restoring factors and integrating over p, we get (12):

$$V(L_\psi, \theta_H) \;=\; \sigma d \int \frac{d^4p}{(2\pi)^4}\, \frac{1}{L_\psi}\, \ln\Big(1 - 2e^{-L_\psi\omega(p)}\cos 2\pi a + e^{-2L_\psi\omega(p)}\Big). \quad (20)$$

An equivalent route is to differentiate with respect to m^2, perform the geometric series or Poisson resummation, and integrate back.

Appendix B: Massless Limit and Polylogarithms

For $m \to 0$, set $\omega(p) = |p|$ and use the factorization

$$\ln\big(1-2c\,t+t^2\big) = \ln\big(1 - e^{i\theta}t\big) + \ln\big(1 - e^{-i\theta}t\big) = -\sum_{k=1}^{\infty} \frac{2\cos(k\theta)}{k}\, t^k, \qquad c = \cos\theta,\ t = e^{-L_\psi|p|}. \quad (21)$$

Then

$$V_{\text{massless}}(L_\psi, \theta_H) = -\frac{2\sigma d}{L_\psi} \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k} \int \frac{d^4p}{(2\pi)^4}\, e^{-kL_\psi |p|} \tag{22}$$

$$= -\frac{2\sigma d}{L_\psi} \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k} \frac{2\pi^2}{(2\pi)^4} \int_0^\infty dp\, p^3\, e^{-kL_\psi p} \tag{23}$$

$$= -\frac{2\sigma d}{L_\psi} \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k} \frac{2\pi^2}{(2\pi)^4} \cdot \frac{3!}{(kL_\psi)^4} \tag{24}$$

$$= -\frac{3\,\sigma d}{2\pi^2 L_\psi^5} \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^5} \;=\; -\frac{3\,\sigma d}{2\pi^2 L_\psi^5}\ \mathrm{Re}\ \mathrm{Li}_5\big(e^{i\theta}\big), \qquad \theta = 2\pi a. \tag{25}$$

Thus, in 5D the massless contribution scales as $1/L_\psi^5$ with a universal (non-fitted) coefficient.

Massive case (asymptotics). For $m > 0$, using $\ln(1 - 2ct + t^2) = -2\sum_{k\geq 1} \cos(k\theta)\, t^k/k$ with $t = e^{-L_\psi \omega}$ and the known integral

$$\int \frac{d^4p}{(2\pi)^4}\, e^{-kL_\psi \sqrt{p^2+m^2}} = \frac{m^3}{(2\pi)^2 (kL_\psi)^3} K_3(kmL_\psi), \tag{26}$$

one finds the exact series

$$V(L_\psi, \theta_H; m) = -\frac{2\sigma d}{L_\psi} \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k} \frac{m^3}{(2\pi)^2 (kL_\psi)^3} K_3(kmL_\psi), \tag{27}$$

which reduces to the polylogarithmic result above as $m \to 0$ (using $K_\nu(z) \sim 2^{\nu-1}\Gamma(\nu) z^{-\nu}$).