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David Xu

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Groupes d'isométries discrets de l'espace hyperbolique de dimension infinie

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Composition du jury

Fanny Kassel (Directrice de recherche CNRS) Institut des Hautes Études Scientifiques	Présidente
Beatrice Pozzetti (Professeure) Universität Heidelberg	Rapportrice
Pierre Py (Chargé de recherche CNRS) Université Grenoble Alpes	Rapporteur
Bruno Duchesne (Professeur) Université Paris-Saclay	Directeur de thèse
Gye-Seon Lee (Maître de conférence) Seoul National University	Co-directeur de thèse
Julien Maubon (Maître de conférence) Université de Lorraine	Examinateur

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Contents

Introduction en français

La théorie géométrique des groupes est une branche des mathématiques qui étudie les propriétés de groupes à travers leurs actions sur certains objets géométriques. Pour mieux appréhender les propriétés algébriques d'un groupe G, on peut s'intéresser à la géométrie et la topologie d'un espace X sur lequel il opère. Et à l'inverse, on peut également apprendre des choses sur les propriétés de l'espace X en faisant agir des groupes sur celui-ci.

L'objet qui nous intéressera est l'espace hyperbolique de dimension infinie. En toute dimension (finie), il existe une unique variété riemannienne complète et simplement connexe de courbure constante égale à 1, 0 ou -1. Ce sont respectivement la sphère $\mathbf{S}^n$, l'espace euclidien $\mathbf{R}^n$ et l'espace hyperbolique $\mathbf{H}^n$.

Comme pour les espaces euclidiens et les sphères, il existe un analogue en dimension infinie aux espaces hyperboliques $\mathbf{H}^n$. L'étude de cet espace (et d'autres espaces symétriques de type non compact et de dimension infinie) a été suggérée par Gromov dans [Gro93, Section 6.A] :

> *The spaces like this look as cute and sexy as their finite dimensional brothers and sisters but they have been for years shamefully neglected by geometers and algebraists alike.*

Ce à quoi il ajouta :

> *The question that interest us most about such spaces concerns discrete isometry groups Γ acting on them where the word "discrete" requires an explanation.*

L'objectif principal de cette thèse est d'étudier et de construire des groupes « discrets » agissant par isométries sur l'espace hyperbolique de dimension infinie.

On peut définir l'espace hyperbolique réel séparable de dimension infinie de la manière suivante : soit $\mathcal{H}$ un espace de Hilbert réel séparable muni d'une base hilbertienne $(e_i)_{i\in\mathbf{N}}$ et soit Q la forme quadratique non-dégénérée définie par

$$Q(x) = -x_0^2 + \sum_{i\geqslant 1} x_i^2,$$

où les x_i sont les coordonnées de $x \in \mathcal{H}$ dans la base $(e_i)_{i \in \mathbf{N}}$. Alors

$$\mathbf{H}^\infty = \{x \in \mathcal{H} \mid Q(x) = -1, x_0 > 0\}.$$

Comme en dimension finie, le groupe des isométries de $\mathbf{H}^\infty$ est la projectivisation du groupe des automorphismes linéaires d'un espace de Hilbert préservant une forme quadratique non-dégénérée de signature $(\infty, 1)$,

$$\mathrm{Isom}(\mathbf{H}^\infty) = \mathrm{PO}(\infty, 1).$$

Une différence majeure avec la dimension finie est l'absence de compacité locale. En effet, $\mathbf{H}^\infty$ est une variété riemannienne de dimension infinie, modelée sur un espace de Banach de dimension infinie dans lequel les boules fermées ne sont pas compactes. Il en va de même pour son groupe d'isométries $\mathrm{Isom}(\mathbf{H}^\infty)$. Cependant, $\mathbf{H}^\infty$ partage tout de même de nombreuses propriétés avec les espaces de dimension finie. Comme mentionné ci-dessus, c'est un espace symétrique riemannien de courbure constante égale à -1 (nous renvoyons à [Duc13, Duc15] pour plus de détails sur les espaces symétriques de rang fini et de dimension infinie). En tant que tel, c'est un espace $\mathrm{CAT}(-1)$, ce qui implique en particulier qu'il est Gromov-hyperbolique.

Groupes agissant sur l'espace hyperbolique de dimension infinie. La dimension infinie apparaît déjà dans la théorie des représentations unitaires où l'on considère des représentations dans le groupe orthogonal ou unitaire d'un espace de Hilbert. Nous nous intéresserons ici aux représentations dans le groupe $\mathrm{Isom}(\mathbf{H}^\infty)$ qui constituent d'autres exemples de représentations en dimension infinie ayant « une saveur géométrique très prononcée » ([DLP23]).

Un exemple important de groupe agissant sur l'espace hyperbolique de dimension infinie provient de la géométrie algébrique. Cantat a prouvé dans [Can11] que le groupe de Cremona $\mathrm{Bir}(\mathrm{P}^2(\mathbf{C}))$, groupe des transformations birationnelles du plan projectif complexe, peut être plongé dans un espace hyperbolique de dimension infinie (non-séparable), l'*espace de Picard-Manin*. Quelques références sur ce sujet incluent [CLdC13, Lon16, Lon19b, Lon19a, LU21].

Dans [BIM05], Burger, Iozzi et Monod ont étudié les représentations du groupe d'automorphismes $\mathrm{Aut}(\mathcal{T})$ d'un arbre $\mathcal{T}$ dans $\mathrm{Isom}(\mathbf{H}^\infty)$. Ils ont exhibé une famille à un paramètre de représentations continues et irréductibles $\mathrm{Aut}(\mathcal{T}) \to \mathrm{Isom}(\mathbf{H}^\infty)$ qui est accompagnée d'une famille de plongements quasi-isométriques de $\mathcal{T}$ dans $\mathbf{H}^\infty$. Delzant, Monod et Py ont obtenu des résultats similaires pour le groupe $\mathrm{PSL}(2, \mathbf{R})$ dans [DP12] et plus généralement pour $\mathrm{Isom}(\mathbf{H}^n) = \mathrm{PO}(n, 1)$ dans [MP14], où les représentations continues et irréductibles $\mathrm{Isom}(\mathbf{H}^n) \to \mathrm{Isom}(\mathbf{H}^\infty)$ ont été classifiées. Cette classification a en outre été étendue à toutes les représentations de $\mathrm{Isom}(\mathbf{H}^\infty)$ dans lui-même, voir [MP19].

Groupes discrets La théorie des groupes discrets agissant sur un espace hyperbolique est devenue un domaine de recherche important depuis les travaux de Schottky, Fuchs, Klein et Poincaré portant sur les groupes agissant sur les espaces hyperboliques de dimension 2 ou 3. En dimension 2, par exemple, les sous-groupes discrets de $\mathrm{PSL}(2,\mathbf{R}) \simeq \mathrm{Isom}^+(\mathbf{H}^2)$ sont appelés *groupes fuchsiens*. Ils induisent des pavages du plan hyperbolique $\mathbf{H}^2$ avec des régions polygonales. Il est également bien connu qu'un tel groupe est, à un indice fini près, le groupe fondamental d'une surface hyperbolique. Plus généralement, l'étude des variétés hyperboliques en dimension n est étroitement liée à celle des sous-groupes discrets de $\mathrm{PO}(n,1) = \mathrm{Isom}(\mathbf{H}^n)$. Une classe particulière de groupes discrets est celle des *réseaux* pour lesquels il existe un domaine fondamental de volume fini pour la mesure de Haar (on dit que le réseau est de *covolume fini*). Cependant, en dimension infinie, les propriétés de discrétude et de volume fini ne se généralisent pas immédiatement.

Dans ce manuscrit, nous définirons cinq notions de discrétude qui s'impliquent mutuellement selon la chaîne d'implications suivante, voir la Section II.3 et la Proposition II.3.11 :

$$\mathrm{SD} \Rightarrow \mathrm{MD} \Rightarrow \mathrm{WD} \Rightarrow \mathrm{COTD} \Rightarrow \mathrm{UOTD}\,.$$

Ces définitions sont toutes équivalentes en dimension finie, mais aucune des implications manquantes dans la chaîne n'est vraie en dimension infinie.

Notre objectif est de construire des sous-groupes de $\mathrm{Isom}(\mathbf{H}^\infty)$ qui sont discrets en un certain sens. Les deux sections principales de la thèse seront consacrées respectivement à la déformation des représentations convexes cocompactes dans $\mathrm{Isom}(\mathbf{H}^\infty)$ et à la construction de groupes de Coxeter agissant par réflexions sur $\mathbf{H}^\infty$.

Représentations convexes cocompactes

Les représentations convexes cocompactes de groupes fondamentaux de surfaces hyperboliques dans des groupes de Lie de rang un sont une généralisation naturelle d'une classe importante de représentations fuchsiennes. Si Γ est le groupe fondamental d'une surface fermée S (compacte, connexe, orientable et sans composante de bord) de genre au moins 2, classifier les structures hyperboliques sur S revient à classifier les représentations discrètes et fidèles de Γ dans $\mathrm{PSL}(2,\mathbf{R})$. Ces représentations sont définies à conjugaison près, sous l'action de $\mathrm{PSL}(2,\mathbf{R})$. Notons $\mathrm{DF}(\Gamma,\mathrm{PSL}(2,\mathbf{R}))$ l'espace des représentations discrètes et fidèles de Γ dans $\mathrm{PSL}(2,\mathbf{R})$. L'espace quotient

$$\mathcal{T}(S) = {}^{\mathrm{DF}(\Gamma,\mathrm{PSL}(2,\mathbf{R}))}\!\big/_{\mathrm{PSL}(2,\mathbf{R})}$$

est appelé *espace de Teichmüller de S* et est composé des représentations de Γ ayant une image cocompacte dans $\mathrm{PSL}(2,\mathbf{R})$. L'espace $\mathrm{DF}(\Gamma,\mathrm{PSL}(2,\mathbf{R}))$ est muni d'une topologie en tant que sous-ensemble de $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbf{R}))$, et $\mathcal{T}(S)$ hérite alors de la topologie quotient. Si $g \geqslant 2$ est le genre de S, alors $\mathcal{T}(S)$ est homéomorphe à $\mathbf{R}^{6g-6}$. Une propriété remarquable de cet espace de Teichmüller est qu'il correspond exactement à une composante connexe de la variété des caractères $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbf{R})) / \mathrm{PSL}(2, \mathbf{R})$.

Les représentations quasi-fuchsiennes ont été introduites comme une généralisation naturelle de ces représentations fuchsiennes. Si Γ est le groupe fondamental d'une surface hyperbolique compacte S, une représentation *quasi-fuchsienne* de Γ est une représentation discrète et fidèle $\rho : \Gamma \to \mathrm{PSL}(2, \mathbf{C})$ telle que l'ensemble limite de $\rho(\Gamma)$ est homéomorphe à un cercle. Ces représentations ont été étudiées en tant que déformations des représentations fuchsiennes de Γ par Ahlfors et Bers en utilisant des méthodes analytiques [AB60, Ahl64, Ber70]. Plus tard, Thurston a introduit de nouveaux outils géométriques dans cette étude et a montré que l'espace des représentations quasi-fuchsiennes coïncide avec celui des représentations convexes cocompactes de Γ dans $\mathrm{PSL}(2, \mathbf{C})$, c'est-à-dire des représentations dont l'image préserve un sous-ensemble convexe de $\mathbf{H}^3$ et agit de manière cocompacte sur celui-ci (voir [Thu22, Chapitre 8.7]). De plus, cet espace est ouvert dans la variété des caractères $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbf{C})) / \mathrm{PSL}(2, \mathbf{C})$.

Plus généralement, on peut s'intéresser aux représentations discrètes du groupe Γ dans les groupes d'isométries d'espaces hyperboliques de dimension plus grande, les $\mathrm{Isom}(\mathbf{H}^n)$. L'ensemble des représentations convexes cocompactes de Γ dans $\mathrm{Isom}(\mathbf{H}^n)$ est encore ouvert. Cette propriété est connue sous le nom de *stabilité* des représentations convexes cocompactes et est due à Marden pour $n = 3$ et à Thurston dans le cas général.

Nous serons intéressés par l'espace des représentations convexes cocompactes d'un groupe de surface Γ dans $\mathrm{Isom}(\mathbf{H}^\infty)$. Bien que $\mathbf{H}^\infty$ et $\mathrm{Isom}(\mathbf{H}^\infty)$ ne soient pas propres (les boules fermées ne sont pas compactes), il existe tout de même des sous-groupes convexes cocompactes dans $\mathrm{Isom}(\mathbf{H}^\infty)$. Il suffit de considérer un plongement totalement géodésique de $\mathbf{H}^n$ dans $\mathbf{H}^\infty$ par exemple. Cela induit un plongement au niveau des groupes d'isométries et tout réseau cocompact Γ dans $\mathrm{Isom}(\mathbf{H}^n)$ préserve alors cette copie de $\mathbf{H}^n$ dans $\mathbf{H}^\infty$ tout en agissant de manière cocompacte dessus. Grâce à la famille à un paramètre de représentations décrite par Monod et Py dans [MP14], il existe en fait beaucoup plus de manières de plonger $\mathrm{Isom}(\mathbf{H}^n)$ dans $\mathrm{Isom}(\mathbf{H}^\infty)$. Puis, en restreignant ces représentations à un sous-groupe discret Γ dans $\mathrm{Isom}(\mathbf{H}^n)$, on obtient un sous-groupe « fortement discret » (*strongly discrete*) de $\mathrm{Isom}(\mathbf{H}^\infty)$ agissant de manière irréductible sur $\mathbf{H}^\infty$. La « forte discrétude » sera définie en même temps que les autres notions de discrétude dans le Chapitre II. Si Γ est cocompact, alors son image dans $\mathrm{Isom}(\mathbf{H}^\infty)$ par les représentations de Monod et Py est convexe cocompacte.

En dimension finie, comme mentionnée plus tôt, l'espace des représentations convexes cocompactes d'un groupe de type fini Γ dans $\mathrm{Isom}(\mathbf{H}^n)$ est ouvert. La démonstration présentée dans [Can21, Théorème 11.4] repose sur le fait que les représentations convexes cocompactes coïncident avec les représentations ρ dont les applications orbitales τ_ρ :

$\gamma \mapsto \rho(\gamma)(x_0)$ sont des plongements quasi-isométriques dans $\mathbf{H}^n$, où $x_0 \in \mathbf{H}^n$ est un point base quelconque. Cette correspondance est encore vraie pour les représentations convexes cocompactes dans $\mathrm{Isom}(\mathbf{H}^\infty)$.

Théorème A. Soit Γ un groupe de type fini et $\rho : \Gamma \to \mathrm{Isom}(\mathbf{H}^\infty)$ une représentation. Les deux propositions suivantes sont équivalentes.

1. Pour tout $x_0 \in \mathbf{H}^\infty$, l'application orbitale $\tau_\rho : \gamma \mapsto \rho(\gamma)(x_0)$ est un plongement quasi-isométrique Γ-équivariant de Γ dans $\mathbf{H}^\infty$, où Γ est muni de la distance des mots associée à un ensemble fini de générateurs.
2. L'image $\rho(\Gamma)$ est fortement discrète dans $\mathrm{Isom}(\mathbf{H}^\infty)$ et il existe un sous-ensemble fermé, convexe, localement compact et Γ-invariant $\mathcal{C} \subset \mathbf{H}^\infty$ sur lequel Γ agit de manière cocompacte.

Notons $\mathrm{QI}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$ l'ensemble des représentations de Γ dans $\mathrm{Isom}(\mathbf{H}^\infty)$ ayant des applications orbitales quasi-isométriques et $\mathrm{CC}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$ l'ensemble des représentations convexes cocompactes dans $\mathrm{Hom}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$. Ces deux ensembles sont définis respectivement par les deux points équivalents du Théorème A. On peut alors déduire que les représentations convexes cocompactes d'un groupe de type fini dans $\mathrm{Isom}(\mathbf{H}^\infty)$ est un ouvert dans l'espace de toutes les représentations.

Corollaire B. Si Γ est de type fini, alors $\mathrm{QI}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty)) = \mathrm{CC}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$ est ouvert dans $\mathrm{Hom}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$.

Il est déjà surprenant qu'il puisse exister des représentations convexes cocompactes irréductibles en dimension infinie puisque $\mathbf{H}^\infty$ n'est pas propre. Les constructions de [BIM05, DP12] produisent des familles de représentations convexes cobornées pour les groupes $\mathrm{Aut}(\mathcal{T})$ et $\mathrm{SL}(2, \mathbf{R})$. Cette étude a été étendue à tous les groupes $\mathrm{Isom}(\mathbf{H}^n)$ pour $n \geqslant 2$ dans [MP14], où il est montré également que ces familles sont naturellement associées à des sous-espaces localement compacts de $\mathbf{H}^\infty$ sur lesquels les groupes agissent de manière minimale. Ces représentations irréductibles de $\mathrm{Isom}(\mathbf{H}^n)$ sont appelées « exotiques » par Monod et Py. La stabilité des représentations convexes cocompactes permet alors l'usage de déformations pour construire de nouveaux sous-groupes convexes cocompacts de $\mathrm{Isom}(\mathbf{H}^\infty)$.

Le pliage est une technique développée par Johnson et Millson dans [JM87] pour déformer des représentations de certains réseaux de $\mathrm{PO}(n, 1)$ dans $\mathrm{PO}(m, 1)$ lorsque $m > n$. En utilisant ces pliages, nous pouvons montrer le résultat suivant.

Théorème C. Soit Γ le groupe fondamental d'une surface hyperbolique fermée. Si $\rho : \mathrm{Isom}(\mathbf{H}^2) \to \mathrm{Isom}(\mathbf{H}^\infty)$ est l'une des représentations décrites par Monod et Py, alors l'espace des déformations de $\rho|_\Gamma$, à conjugaison près dans $\mathrm{Isom}(\mathbf{H}^\infty)$, contient une famille à un paramètre de représentations qui sont deux à deux non-conjuguées et qui ne sont conjuguées à la restriction d'aucune représentation « exotiques » de $\mathrm{Isom}(\mathbf{H}^2)$.

Notre théorème montre donc que pour les réseaux cocompacts (sans torsion) dans $\mathrm{Isom}(\mathbf{H}^2)$, il y a beaucoup plus de représentations dans $\mathrm{Isom}(\mathbf{H}^\infty)$ que pour le groupe $\mathrm{Isom}(\mathbf{H}^2)$ tout entier, comme classifiées par Monod et Py, [MP14]. Cela peut être vu comme un résultat de non-rigidité ou de flexibilité pour les représentations des réseaux de $\mathrm{Isom}(\mathbf{H}^2)$ dans $\mathrm{Isom}(\mathbf{H}^\infty)$.

Pour montrer que les représentations obtenues par pliage ne sont pas conjuguées entre elles, nous avons besoin de décrire le centralisateur d'une isométrie loxodromique dans $\mathrm{Isom}(\mathbf{H}^\infty)$. À cette fin, nous nous intéressons d'abord au centralisateur d'un opérateur orthogonal d'un espace de Hilbert réel. Pour cela, nous passons par la théorie spectrale des opérateurs orthogonaux.

En outre, le fait que les représentations de groupes de surface $\Gamma < \mathrm{Isom}(\mathbf{H}^2)$ obtenues par déformations ne sont pas des restrictions des représentations « exotiques » $\mathrm{PSL}(2,\mathbf{R}) \to \mathrm{Isom}(\mathbf{H}^\infty)$ décrites par Monod et Py illustre le fait que les représentations de groupes de surface dans $\mathrm{Isom}(\mathbf{H}^\infty)$ semblent beaucoup plus riches que celles du groupe $\mathrm{PSL}(2,\mathbf{R})$ entier.

Groupes engendrés par des réflexions

Les groupes engendrés par des réflexions fournissent de nombreux exemples de groupes discrets agissant sur des espaces symétriques de coubure constante (sphères, espaces euclidiens et espaces hyperboliques). Ils jouent un rôle important dans la théorie de Lie grâce à leurs liens avec les systèmes de racines par exemple et ils sont étroitement liés à la théorie des groupes de Coxeter.

Un groupe de Coxeter est un groupe abstrait défini par une présentation de la forme

$$W = \langle S \mid \forall s,t \in S,\ (st)^{m_{s,t}} = 1 \rangle,$$

où $m_{s,s} = 1$ et $m_{s,t} = m_{t,s} \in \mathbf{N}_{\geqslant 2} \cup \{\infty\}$ pour $s \neq t$.

Les groupes de Coxeter admettent une représentation linéaire naturelle, la *représentation géométrique*, étudiée par Tits (voir [Bou02, Chapter 5]). À travers cette représentation, ils agissent sur un espace vectoriel par des réflexions linéaires. On peut noter que tout groupe de Coxeter admet également une action discrète et cocompacte sur un certain objet combinatoire appelé le *complexe de Davis*. De nombreuses propriétés géométriques d'un groupe de Coxeter W peuvent être déduites de son action sur le complexe de Davis associé Σ. Par exemple, lorsque Σ possède une métrique $\mathrm{CAT}(-1)$, alors W est Gromov-hyperbolique (voir [Mou88]). Les groupes de Coxeter Gromov-hyperboliques sont abondants, mais très peu peuvent agir de manière discrète et cocompacte sur un espace hyperbolique (voir par exemple [DH13]).

La théorie de Vinberg donne des conditions nécessaires et suffisantes pour qu'un groupe de Coxeter de type fini agisse par isométries sur un espace hyperbolique $\mathbf{H}^n$ en tant que sous-groupe discret de $\mathrm{Isom}(\mathbf{H}^n)$, via la représentation géométrique. Ces conditions, que nous notons ici (C), ne dépendent que des coefficients de la *matrice de Cartan* de W.

Un groupe de Coxeter ne peut pas agir de manière irréductible par réflexions sur $\mathbf{H}^\infty$ à travers la représentation géométrique s'il est de type fini. Pour un groupe de Coxeter W de type infini, les conditions pour la discrétude ne sont plus suffisantes. Cependant, il est encore possible de retrouver une action irréductible sur $\mathbf{H}^\infty$ en vérifiant ces conditions sur des sous-groupes de type de fini de W. Nous obtenons le critère suivant.

Proposition D. Soit $A = (a_{i,j})_{i,j\in\mathbf{N}}$ une matrice symétrique infinie telle que pour tout $i \in \mathbf{N}$, $a_{i,i} = 2$. Supposons qu'il existe $n_0 \in \mathbf{N}$ tel que pour tout $n \geqslant n_0$, la sous-matrice principale A_n satisfait les conditions (C) données par la théorie de Vinberg. Alors le groupe engendré par les réflexions associées à A agit sur l'espace hyperbolique de dimension infinie $\mathbf{H}^\infty$. De plus, cette action est irréducible.

Ce résultat permet de construire plusieurs familles de groupes de Coxeter de type infini agissant de manière irréductible sur $\mathbf{H}^\infty$. Nous exhibons des exemples en utilisant les trois familles de groupes de Coxeter sphériques irréductibles pouvant être définis pour un nombre arbitraire de générateurs : les groupes de Coxeter de type A_n, B_n et D_n. La construction débute avec un groupe de Coxeter de type fini agissant sur un certain espace hyperbolique $\mathbf{H}^n$ que l'on étend en ajoutant une infinité de générateurs de sorte que toutes les nouvelles relations soient du même type que celles des groupes de Coxeter sphériques. Par souci de concision, nous appelons ces exemples les groupes de Coxeter « sphériques à partir d'un certain rang ».

Concernant le caractère discret de ces groupes, nous pouvons montrer le résultat suivant.

Proposition E. Aucun des groupes de Coxeter « sphériques à partir d'un certain rang » $W < \mathrm{Isom}(\mathbf{H}^\infty)$ n'est discret pour la topologie compacte-ouverte sur $\mathrm{Isom}(\mathbf{H}^\infty)$.

Structure du manuscrit

Le manuscrit est organisé de la manière suivante.

- Le premier chapitre a pour but de donner quelques définitions de base sur les actions de groupes et les espaces à courbure négative. Nous y décrivons certaines propriétés des espaces Gromov-hyperboliques et des espaces $\mathrm{CAT}(\kappa)$.
- Le deuxième chapitre sert à introduire notre objet d'étude principal, l'espace hyperbolique de dimension infinie. Les différents modèles de $\mathbf{H}^\infty$ y sont décrits, ainsi

que certaines de leurs propriétés géométriques et topologiques. Nous introduisons le groupe d'isométries $\mathrm{Isom}(\mathbf{H}^\infty)$ et donnons une démonstration de la continuité de la longueur de translation pour la *topologie faible* qui sera définie en cours de route. De plus, nous présentons une autre description des isométries, appelée *matrices de Clifford*. Cette représentation matricielle généralise à toute dimension les célèbres isomorphismes $\mathrm{Isom}^+(\mathbf{H}^2) \simeq \mathrm{PSL}(2, \mathbf{R})$ et $\mathrm{Isom}^+(\mathbf{H}^3) \simeq \mathrm{PSL}(2, \mathbf{C})$.

- La troisième partie expose les résultats concernant les déformations de représentations convexes cocompactes d'un réseau cocompact $\Gamma < \mathrm{Isom}(\mathbf{H}^2)$ dans $\mathrm{Isom}(\mathbf{H}^\infty)$.
- La quatrième et dernière partie est consacrée à la construction de groupes de Coxeter agissant de manière irréductible sur $\mathbf{H}^\infty$ ainsi qu'aux exemples de groupes de Coxeter « sphériques à partir d'un certain rang ». Nous montrons également que les groupes de Coxeter affines irréductibles sont nécessairement de type fini.

Introduction

Geometric group theory is a branch of mathematics that studies the properties of groups through their actions on certain geometric objects. One can gain insight on algebraic properties of a group G by studying the geometric and topological properties of a space X it acts on. And conversely, one can also examine the properties of the space X by making groups act on it.

In this thesis, the object that we will be interested in is the infinite-dimensional hyperbolic space. In every finite dimension, there is a unique simply connected and complete Riemannian manifold of constant curvature 1, 0 or -1. These are respectively the sphere $\mathbf{S}^n$, the Euclidean space $\mathbf{R}^n$ and the hyperbolic space $\mathbf{H}^n$.

Similarly to Euclidean spaces and spheres, there is an infinite-dimensional analog of the hyperbolic n-spaces $\mathbf{H}^n$. The study of this space (and other infinite-dimensional symmetric spaces of non-compact type) was suggested by Gromov in [Gro93, Section 6.A]:

> *The spaces like this look as cute and sexy as their finite dimensional brothers and sisters but they have been for years shamefully neglected by geometers and algebraists alike.*

To which he added:

> *The question that interest us most about such spaces concerns discrete isometry groups Γ acting on them where the word "discrete" requires an explanation.*

The main objective of this thesis is to investigate and construct "discrete" groups acting by isometries on the infinite-dimensional hyperbolic space.

One can define the infinite-dimensional separable real hyperbolic space as follows: let $\mathcal{H}$ be a real separable Hilbert space with some Hilbert basis $(e_i)_{i\in\mathbf{N}}$ and let Q be the non-degenerate quadratic form defined by

$$Q(x) = -x_0^2 + \sum_{i\geqslant 1} x_i^2$$

where the x_i are the coordinates of $x \in \mathcal{H}$ in the basis $(e_i)_{i\in\mathbf{N}}$. Then

$$\mathbf{H}^\infty = \{x \in \mathcal{H} \mid Q(x) = -1, x_0 > 0\}.$$

As in finite dimension, the group of isometries of $\mathbf{H}^\infty$ is the projectivisation of the group of linear automorphisms of some Hilbert space preserving a non-degenerate quadratic form of signature $(\infty, 1)$,

$$\mathrm{Isom}(\mathbf{H}^\infty) = \mathrm{PO}(\infty, 1).$$

One major difference with finite dimension is the lack of local compactness. Indeed, $\mathbf{H}^\infty$ is a Riemannian manifold of infinite dimension, *i.e.* it is modelled on an infinite-dimensional Banach space, in which closed balls are not compact. The same is true for its group of isometries $\mathrm{Isom}(\mathbf{H}^\infty)$. Yet, $\mathbf{H}^\infty$ still possesses some common features with the finite-dimensional spaces. As mentioned above, it is a Riemannian symmetric space of constant curvature -1 (we refer to [Duc13, Duc15] for details on finite rank symmetric spaces of infinite dimension). As such, it is a $\mathrm{CAT}(-1)$ space, which in turn implies that it is Gromov-hyperbolic. All those properties have attracted a lot of interest over the last decades.

Groups acting on the infinite-dimensional hyperbolic space. Infinite dimension already appears in the theory of unitary representations where one considers representations into the orthogonal or unitary group of a Hilbert space. We will be interested here in representations into the group $\mathrm{Isom}(\mathbf{H}^\infty)$ which is another instance of infinite-dimensional representation having "a very strong geometric flavor" ([DLP23]).

An important example of group acting on the infinite-dimensional hyperbolic space comes from algebraic geometry. Cantat proved in [Can11] that the Cremona group $\mathrm{Bir}(\mathrm{P}^2(\mathbf{C}))$, the group of birational transformations of the complex projective plane, can be embedded into the isometry group of a (non-separable) infinite-dimensional hyperbolic space, the *Picard-Manin space*. A few other references on this subject include [CLdC13, Lon16, Lon19b, Lon19a, LU21].

In [BIM05], Burger, Iozzi and Monod studied representations of the automorphism group $\mathrm{Aut}(\mathcal{T})$ of a tree $\mathcal{T}$ into $\mathrm{Isom}(\mathbf{H}^\infty)$. They exhibited a one-parameter family of irreducible continuous representations $\mathrm{Aut}(\mathcal{T}) \to \mathrm{Isom}(\mathbf{H}^\infty)$ which comes along a family of quasi-isometric embeddings of $\mathcal{T}$ into $\mathbf{H}^\infty$. Delzant, Monod and Py obtained similar results for the group $\mathrm{PSL}(2, \mathbf{R})$ in [DP12] and more generally for $\mathrm{Isom}(\mathbf{H}^n) = \mathrm{PO}(n, 1)$ in [MP14], where the irreducible continuous representations $\mathrm{Isom}(\mathbf{H}^n) \to \mathrm{Isom}(\mathbf{H}^\infty)$ were classified. The classification was moreover extended to all the self-representations of $\mathrm{Isom}(\mathbf{H}^\infty)$, see [MP19].

Discrete groups. The theory of discrete groups acting on a hyperbolic space has become an important field of research since the works of Schottky, Fuchs, Klein and Poincaré about groups acting on the hyperbolic spaces of dimension 2 or 3. In dimension 2 for example, discrete subgroups of $\mathrm{PSL}(2,\mathbf{R}) \simeq \mathrm{Isom}^+(\mathbf{H}^2)$ are called *Fuchsian groups*. They induce tessellations of the hyperbolic plane $\mathbf{H}^2$ with polygonal regions. It is also well-known that such a group arise, up to finite index, as the fundamental group of a hyperbolic surface. More generally, the study of hyperbolic n-manifolds is closely related to that of discrete subgroups of $\mathrm{PO}(n,1) = \mathrm{Isom}(\mathbf{H}^n)$. A particular class of discrete groups is that of *lattices* for which there exists a fundamental domain of finite volume for the Haar measure (the lattice has *finite covolume*). Though, in infinite dimension, the properties of discreteness and of finite covolume become more involved.

Throughout the manuscript, we will define five notions of discreteness which imply each other according to the following chain of implications, see Section II.3 and Proposition II.3.11:

$$\mathrm{SD} \Rightarrow \mathrm{MD} \Rightarrow \mathrm{WD} \Rightarrow \mathrm{COTD} \Rightarrow \mathrm{UOTD}\,.$$

These definitions are all equivalent in finite dimension, but none of the implications which does not appear in the chain is true in infinite dimension.

Our purpose is to construct subgroups of $\mathrm{Isom}(\mathbf{H}^\infty)$ which are discrete for some definition. The two main sections will be devoted respectively to the deformation of convex-cocompact representations into $\mathrm{Isom}(\mathbf{H}^\infty)$ and to the construction of Coxeter groups acting by reflections on $\mathbf{H}^\infty$.

Convex-cocompact representations

Convex-cocompact representations of fundamental groups of hyperbolic surfaces into rank-one Lie groups have been largely studied as natural generalisations of a large family of Fuchsian representations. If Γ is the fundamental group of a closed surface S (compact, connected, orientable and without boundary components) of genus at least 2, classifying the hyperbolic structures on S is equivalent to classifying the discrete and faithful representations of Γ into $\mathrm{PSL}(2,\mathbf{R})$. These representations are defined up to the action by conjugation of $\mathrm{PSL}(2,\mathbf{R})$. Denote by $\mathrm{DF}(\Gamma,\mathrm{PSL}(2,\mathbf{R}))$ the space of discrete and faithful representations from Γ to $\mathrm{PSL}(2,\mathbf{R})$, the quotient space

$$\mathcal{T}(S) = {\mathrm{DF}(\Gamma,\mathrm{PSL}(2,\mathbf{R}))}\big/{\mathrm{PSL}(2,\mathbf{R})}$$

is called the *Teichmüller space of S* and consists of representations of Γ having cocompact image in $\mathrm{PSL}(2,\mathbf{R})$. The space $\mathrm{DF}(\Gamma,\mathrm{PSL}(2,\mathbf{R}))$ inherits a topology as a subset of $\mathrm{Hom}(\Gamma,\mathrm{PSL}(2,\mathbf{R}))$ and $\mathcal{T}(S)$ is then equipped with the quotient topology. If $g \geqslant 2$ is the genus of S, then $\mathcal{T}(S)$ is homeomorphic to $\mathbf{R}^{6g-6}$. One noticeable feature of the Teichmüller space is that it is a whole connected component in the character variety $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbf{R})) / \mathrm{PSL}(2,\mathbf{R})$.

Quasi-Fuchsian representations emerged as a natural generalisation of these Fuchsian representations. If Γ is the fundamental group of a compact hyperbolic surface S, a *quasi-Fuchsian* representation of Γ is a discrete and faithful representation $\rho : \Gamma \to \mathrm{PSL}(2,\mathbf{C})$ such that the limit set of $\rho(\Gamma)$ is homeomorphic to a circle. These representations were studied as deformations of Fuchsian representations of Γ by Ahlfors and Bers using analytic methods [AB60, Ahl64, Ber70]. Thurston later introduced some new geometric tools in this study and showed that the space of quasi-Fuchsian representations coincides with that of convex-cocompact representations of Γ into $\mathrm{PSL}(2,\mathbf{C})$, *i.e.* representations whose images preserve some convex subset of $\mathbf{H}^3$ and act cocompactly on it (see [Thu22, Chapter 8.7]). Moreover, this space is open in the character variety $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbf{C})) / \mathrm{PSL}(2,\mathbf{C})$.

More generally, one can consider discrete representations of the group Γ into the isometry groups of higher-dimensional hyperbolic spaces, $\mathrm{Isom}(\mathbf{H}^n)$. The set of convex-cocompact representations of Γ into $\mathrm{Isom}(\mathbf{H}^n)$ is again open. This property is called the *stability* of convex-cocompact representations, it is attributed to Marden for $n = 3$ and to Thurston for the general case.

We are interested in describing the space of convex-cocompact representations of a surface group Γ into $\mathrm{Isom}(\mathbf{H}^\infty)$. Although $\mathbf{H}^\infty$ and $\mathrm{Isom}(\mathbf{H}^\infty)$ are not proper (closed balls are not compact), convex-cocompact subgroups in $\mathrm{Isom}(\mathbf{H}^\infty)$ do exist. Take a totally geodesic embedding of $\mathbf{H}^n$ into $\mathbf{H}^\infty$, this comes with an embedding of their groups of isometries. Then any cocompact lattice Γ in $\mathrm{Isom}(\mathbf{H}^n)$ preserves this copy of $\mathbf{H}^n$ and acts cocompactly on it. Thanks to the one-parameter family of representations described by Monod and Py in [MP14], one can actually find more ways to embed $\mathrm{Isom}(\mathbf{H}^n)$ into $\mathrm{Isom}(\mathbf{H}^\infty)$. By restricting these representations to a discrete subgroup Γ of $\mathrm{Isom}(\mathbf{H}^n)$, we obtain a *strongly discrete* subgroup of $\mathrm{Isom}(\mathbf{H}^\infty)$ acting irreducibly on $\mathbf{H}^\infty$. Strong discreteness will be defined along with other notions of discreteness in Chapter II. If Γ is cocompact, its image in $\mathrm{Isom}(\mathbf{H}^\infty)$ via the representations of Monod and Py will be convex-cocompact.

In finite dimension, as mentioned above, the space of convex-cocompact representations of a finitely generated group Γ into $\mathrm{Isom}(\mathbf{H}^n)$ is open. The proof presented in [Can21, Theorem 11.4] relies on the fact that the convex-cocompact representations coincide with the representations ρ such that the orbit map $\tau_\rho : \gamma \mapsto \rho(\gamma)(x_0)$ is a quasi-isometric embedding in $\mathbf{H}^n$, where $x_0 \in \mathbf{H}^n$ is any base point. We prove that this correspondence also holds for convex-cocompact representations into $\mathrm{Isom}(\mathbf{H}^\infty)$.

Theorem A. Let Γ be a finitely generated group and $\rho : \Gamma \to \mathrm{Isom}(\mathbf{H}^\infty)$ be a representation. The two following assertions are equivalent.

1. For any $x_0 \in \mathbf{H}^\infty$, the orbit map $\tau_\rho : \gamma \mapsto \rho(\gamma)(x_0)$ is a Γ-equivariant quasi-isometric embedding from Γ into $\mathbf{H}^\infty$, where Γ is endowed with the word distance associated to a finite set of generators.
2. There is a closed, convex and locally compact Γ-invariant subset $\mathcal{C} \subset \mathbf{H}^\infty$ on which Γ acts cocompactly and $\rho(\Gamma)$ is strongly discrete in $\mathrm{Isom}(\mathbf{H}^\infty)$.

Denote by $\mathrm{QI}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$ the set of representations of Γ into $\mathrm{Isom}(\mathbf{H}^\infty)$ which have quasi-isometric orbit maps and let $\mathrm{CC}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$ be the set of convex-cocompact representations in $\mathrm{Hom}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$. These two sets are defined respectively by the two equivalent points of Theorem A. We can deduce that convex-cocompact representations of finitely generated groups into $\mathrm{Isom}(\mathbf{H}^\infty)$ form an open subset of the space of all the representations.

Corollary B. If Γ is finitely generated, then $\mathrm{QI}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty)) = \mathrm{CC}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$ is open in $\mathrm{Hom}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$.

It is already a little surprising that irreducible convex-cocompact representations could exist in infinite dimension since $\mathbf{H}^\infty$ is not proper. The constructions in [BIM05, DP12] displayed families of convex-cobounded representations for $\mathrm{Aut}(\mathcal{T})$ and $\mathrm{SL}(2, \mathbf{R})$. This study was extended to all the $\mathrm{Isom}(\mathbf{H}^n)$ for $n \geqslant 2$ in [MP14], where it was also proved that these families actually come along with some locally compact subset of $\mathbf{H}^\infty$ on which the groups act minimally. These irreducible representations of $\mathrm{Isom}(\mathbf{H}^n)$ are called "exotic" by Monod and Py. The stability of convex-cocompact representations then allows the use of deformations to construct new convex-cocompact subgroups of $\mathrm{Isom}(\mathbf{H}^\infty)$.

Using bendings, which are techniques developed by Johnson and Millson [JM87] to deform representations of some lattices of $\mathrm{PO}(n,1)$ into $\mathrm{PO}(m,1)$ with $m > n$, we are able to prove the following.

Theorem C. Let Γ be the fundamental group of a closed hyperbolic surface. If $\rho : \mathrm{Isom}(\mathbf{H}^2) \to \mathrm{Isom}(\mathbf{H}^\infty)$ is one of the representations described by Monod and Py, then the space of deformations of $\rho|_\Gamma$, up to conjugation in $\mathrm{Isom}(\mathbf{H}^\infty)$, contains a one-parameter family of representations which are not pairwise conjugate, nor conjugate to the restriction of any "exotic" representation of $\mathrm{Isom}(\mathbf{H}^2)$.

Our theorem actually shows that for (torsion-free) cocompact lattices in $\mathrm{Isom}(\mathbf{H}^2)$, there are much more representations than for the whole group $\mathrm{Isom}(\mathbf{H}^2)$ itself as classified by Monod and Py, [MP14]. This can be thought as some non-rigidity or flexibility result for representations of lattices of $\mathrm{Isom}(\mathbf{H}^2)$ into $\mathrm{Isom}(\mathbf{H}^\infty)$.

To prove that the representations obtained by bending are not conjugate to each other, we need to describe the centraliser of a loxodromic isometry in $\mathrm{Isom}(\mathbf{H}^\infty)$. For this purpose, we study the centraliser of an orthogonal operator of a real Hilbert space, making use of the spectral theory of orthogonal operators.

Besides, the fact that the representations of the surface groups $\Gamma < \mathrm{Isom}(\mathbf{H}^2)$ obtained by deformations do not arise as the restriction to Γ of any "exotic" representation $\mathrm{PSL}(2,\mathbf{R}) \to \mathrm{Isom}(\mathbf{H}^\infty)$ described by Monod and Py illustrates that the representations of surface groups in $\mathrm{Isom}(\mathbf{H}^\infty)$ are much richer than that of $\mathrm{PSL}(2,\mathbf{R})$.

Groups generated by reflections

Groups generated by reflections provide numerous instances of discrete groups acting on symmetric spaces of constant curvature (spheres, Euclidean spaces and hyperbolic spaces). They play an important role in Lie theory with their relations with root systems for example and are closely related to the theory of Coxeter groups.

A Coxeter group is an abstract group defined by a presentation of the type

$$W = \langle S \mid \forall s,t \in S,\ (st)^{m_{s,t}} = 1 \rangle,$$

where $m_{s,s} = 1$ and $m_{s,t} = m_{t,s} \in \mathbf{N}_{\geqslant 2} \cup \{\infty\}$ for $s \neq t$.

Coxeter groups admit a natural linear representation, the *geometric representation*, which was studied by Tits (see [Bou02, Chapter 5]). They act by linear reflections on some vector space via this representation. We mention also that every Coxeter group has a discrete and cocompact action on some combinatorial object called the *Davis complex*. Many geometric properties of a Coxeter group W can be deduced from its action on its associated Davis complex Σ. In particular, when Σ has a $\mathrm{CAT}(-1)$ metric, then W is Gromov-hyperbolic (see [Mou88]). Gromov-hyperbolic Coxeter groups are actually abundant, but very few of these can act discretely and cocompactly on a hyperbolic space (see for example [DH13]).

The theory of Vinberg gives necessary and sufficient conditions for a finitely generated Coxeter group to act by isometries on a hyperbolic space $\mathbf{H}^n$ as a discrete subgroup of $\mathrm{Isom}(\mathbf{H}^n)$ via the geometric representation. Denote these conditions by (C). They depend only on the entries of the *Cartan matrix* of W.

To make a Coxeter group act irreducibly by reflections on $\mathbf{H}^\infty$, one cannot remain in the world of finite type groups. For an infinitely generated Coxeter group W, the conditions for discreteness are no longer sufficient. However, it is still possible to recover an irreducible action on $\mathbf{H}^\infty$ by checking them on finitely generated subgroups of W. We obtained the following criterion.

Proposition D. Let $A = (a_{i,j})_{i,j\in\mathbf{N}}$ be an infinite symmetric matrix such that for all $i \in \mathbf{N}$, $a_{i,i} = 2$. Suppose that there exists $n_0 \in \mathbf{N}$ such that for all $n \geqslant n_0$, the principal submatrix A_n satisfies the conditions (C) given by Vinberg's theory. Then the group generated by reflections associated to A acts on the infinite-dimensional hyperbolic space $\mathbf{H}^\infty$. Moreover, this action is irreducible.

This result allow to construct several families of infinitely generated Coxeter groups acting irreducibly on $\mathbf{H}^\infty$. We provide examples using the three families of irreducible spherical Coxeter groups that are defined for any finite number of generators, namely the Coxeter groups of type A_n, B_n and D_n. The construction starts with a finitely generated Coxeter group acting on some hyperbolic space $\mathbf{H}^n$ that we extend by adding infinitely many generators such that all the new relations resemble those of spherical Coxeter groups. For brevity, we refer to these examples as "*eventually spherical*".

We were then concerned about their discreteness properties and could prove the following.

Proposition E. None of the "eventually spherical" Coxeter groups $W < \mathrm{Isom}(\mathbf{H}^\infty)$ is discrete for the compact-open topology on $\mathrm{Isom}(\mathbf{H}^\infty)$.

Structure of the thesis

The thesis is organised as follows.

- The first chapter is devoted to give some basic definitions about group actions and spaces of non-positive curvature. We describe some properties of Gromov-hyperbolic spaces and $\mathrm{CAT}(\kappa)$-spaces.
- The second chapter aims at introducing our main object of study, the infinite-dimensional hyperbolic space. The different models of $\mathbf{H}^\infty$ are defined as well as some basic geometric and topological properties. We also discuss its group of isometries $\mathrm{Isom}(\mathbf{H}^\infty)$ and provide a proof for the continuity of the translation length of isometries with respect to the *weak topology* that we define therein. Besides, we present another description of the isometries called the *Clifford matrices*. This matrix representation generalises to any dimension the well-known isomorphisms $\mathrm{Isom}^+(\mathbf{H}^2) \simeq \mathrm{PSL}(2, \mathbf{R})$ and $\mathrm{Isom}^+(\mathbf{H}^3) \simeq \mathrm{PSL}(2, \mathbf{C})$.
- The third chapter consists in the exposition of our results concerning the deformations of convex-cocompact representations from a cocompact lattice $\Gamma < \mathrm{Isom}(\mathbf{H}^2)$ into $\mathrm{Isom}(\mathbf{H}^\infty)$.
- The fourth and last chapter is devoted to the construction of Coxeter groups acting irreducibly on $\mathbf{H}^\infty$ as well as the examples of "eventually spherical" Coxeter groups. We also give a proof for the fact that irreducible affine Coxeter groups are necessarily finitely generated.

Chapter I

Preliminaries

This chapter gathers some basic material on which this thesis relies.

1 Group actions

An action of a group G on a set X will be denoted by $G \curvearrowright X$. If $x \in X$,

$$G \cdot x := \{g(x) \mid g \in G\}$$

denotes the *orbit* of x under the action of G.

If (X, d) is a metric space, the group G acts on X by isometries if for all $x, y \in X$ and $g \in G$, $d(g(x), g(y)) = d(x, y)$.

Definition 1.1. Let G be a topological group acting on a topological space X. Then the action $G \curvearrowright X$ is *continuous* if the map

$$\begin{array}{ccc} G \times X & \to & X \\ (g, x) & \mapsto & g(x) \end{array}$$

is continuous.

The continuity of the action $G \curvearrowright X$ implies in particular that for all $x_0 \in X$, the following orbit map is continuous:

$$\tau_{x_0} : \begin{array}{ccc} G & \to & X \\ g & \mapsto & g(x_0) \end{array}.$$

Definition 1.2. A *linear representation* of G on a vector space V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$.

A subspace $W \subset V$ is *invariant under the action of* G (or G-*invariant*) if for all $g \in G$, $\rho(g)(W) \subset W$.

When V is topological vector space, we say that the representation $\rho : G \to \mathrm{GL}(V)$ is *irreducible* if the only closed subspaces of V that are G-invariant are either $\{0\}$ or V. Otherwise, the representation is *reducible*.

A topological space X is *locally compact* if every point $x \in X$ admits a basis of compact neighbourhoods. If X is a metric space, it is *proper* if its closed and bounded subsets are compact.

Definition 1.3. An action of a group G on a locally compact space X is *cocompact* if there exists a compact subset $K \subset X$ such that the orbit of K covers X, $G \cdot K = X$.

Actually, when X is a topological space which is not locally compact, we still define a cocompact action of G in this way.

An action $G \curvearrowright X$ is *proper* if for all compact subsets $K \subset X$, the set $\{g \in G \mid g(K) \cap K \neq \emptyset\}$ is finite.

2 Metric spaces of negative curvature

In this section, we describe the notions of Gromov-hyperbolic spaces and $\mathrm{CAT}(\kappa)$ spaces. For metric spaces which are not Riemannian manifolds such as graphs or other combinatorial objects like complexes, these provide a substitute for the notions of negative curvature. Some classic references on this subject include [Gro87, CDP90, GdlH90, BH99]. We will also frequently refer to [DSU17] which emphasises the case of non-proper metric spaces.

2.1 Gromov-hyperbolic spaces

Definition 2.1. A metric space (X,d) is *geodesic* if every pair of points $x, y \in X$ can be joined by a continuous path of length $d(x,y)$, *i.e.* there is an isometric embedding $c : [0, d(x,y)] \subset \mathbf{R} \to X$ such that $c(0) = x$ and $c(d(x,y)) = y$.

The image of such a path is called a *geodesic segment*. It may not be unique, but we will denote it by $[x,y]$.

A *(geodesic) triangle* in X is the data of three points $x, y, z \in X$ and three geodesic segments $[x,y]$, $[y,z]$ and $[x,z]$ connecting these points. We will denote such a triangle by $\Delta(x,y,z)$.

Definition 2.2. Given a triangle $\Delta(x,y,z) = [x,y] \cup [y,z] \cup [z,x] \subset X$ and $\delta \geqslant 0$ in a geodesic metric space (X,d), a δ-*centre* of $\Delta(x,y,z)$ is a point $c \in X$ which is at distance at most δ from $[x,y]$, $[y,z]$ and $[z,x]$.

The space (X, d) is called *δ-hyperbolic* if every triangle in X has a δ-centre. It is called *Gromov-hyperbolic* if it is δ-hyperbolic for some $\delta \geqslant 0$.

Example 2.3.
- A bounded geodesic space is δ-hyperbolic where δ can be taken to be its diameter.
- A **R**-tree is a metric space $\mathcal{T}$ such that
  1. there is a unique geodesic segment joining each pair of points in $\mathcal{T}$;
  2. if $[x, y] \cap [y, z] = \{y\}$, then $[x, y] \cup [y, z] = [x, z]$.

  Any triangle in a **R**-tree is a tripod and the intersection of the segments is a 0-centre for the triangle. Therefore, every **R**-tree is 0-hyperbolic.
- The hyperbolic plane $\mathbf{H}^2$ is $\log(2)$-hyperbolic.
- The hyperbolic n-space $\mathbf{H}^n$ is also $\log(2)$-hyperbolic because every triangle in $\mathbf{H}^n$ spans a copy of $\mathbf{H}^2$ inside $\mathbf{H}^n$.
- The Euclidean space $\mathbf{R}^n$, for $n \geqslant 2$, is not δ-hyperbolic for any $\delta \geqslant 0$.

Definition 2.4. Let (X, d) be a metric space and let $x, y, z \in X$ be three points in X. The *Gromov product* is defined by

$$(y \cdot z)_x = \frac{1}{2}\left(d(x, y) + d(x, z) - d(y, z)\right).$$

The Gromov product $(y \cdot z)_x$ can be seen as an approximation of the distance between x and a geodesic joining y and z, $(y \cdot z)_x \simeq d(x, [y, z])$. More precisely, we have the following inequalities.

Proposition 2.5 ([GdlH90], Lemmas 2.17 and 2.20)**.** If (X, d) is a δ-hyperbolic metric space and $[y, z]$ is a geodesic in X, then

$$(y \cdot z)_x \leqslant d(x, [y, z]) \leqslant (y \cdot z)_x + 4\delta.$$

Gromov-hyperbolicity can also be defined using *slim triangles* and the so-called *Gromov inequality*, see Proposition 2.10.

Definition 2.6. A Gromov-hyperbolic space (X, d) is called *strongly hyperbolic* (see [DSU17, Definition 3.3.6]) if it satisfies the following condition: for all $x, y, z, o \in X$, we have

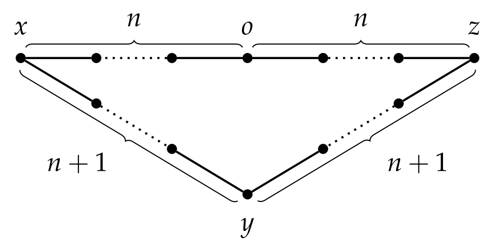
$$e^{-(x \cdot z)_o} \leqslant e^{-(x \cdot y)_o} + e^{-(y \cdot z)_o}.$$

Remark 2.7. In the remainder of the thesis, we will always consider that the Gromov-hyperbolic spaces are also strongly hyperbolic.

Strong hyperbolicity is in fact satisfied by every CAT(-1) space ([DSU17, Proposition 3.3.4]) and corresponds to the triangle inequality for the *metametric* that we define later for the particular case of $\mathbf{H}^\infty$.

Without strong hyperbolicity, most of the results we state in these preliminaries as regards the Gromov product and the Busemann functions remain true up to additive constants.

Example 2.8. Here is an example of a δ-hyperbolic space which does not satisfy the strong hyperbolicity condition. Consider the following graph where every edge has length 1 and such that $d(o,x) = d(o,z) = n$ and $d(y,x) = d(y,z) = n+1$ so that $o \in [x,z]$.



This graph is bounded, and hence it is δ-hyperbolic for some δ. Moreover, we have $(x \cdot z)_o = 0$, so $e^{-(x \cdot z)_o} = 1$ and $(x \cdot y)_o = (y \cdot z)_o = n$, so that

$$e^{-(x \cdot y)_o} + e^{-(y \cdot z)_o} = 2e^{-n} \underset{n \to \infty}{\longrightarrow} 0.$$

Notation. If (X,d) is a metric space, $S \subset X$ and $R \geqslant 0$, we denote the R-neighbourhood of S in X by

$$\mathcal{N}_R(S) = \{x \in X \mid d(x,S) \leqslant R\}.$$

Definition 2.9. If (X,d) is a metric space, $\Delta(x,y,z)$ a triangle in X and $\delta \geqslant 0$. Then $\Delta(x,y,z)$ is called *δ-slim* if each side of $\Delta(x,y,z)$ is contained in the union of the δ-neighbourhoods of the two other sides, *i.e.* the three following inclusions hold:

$$[x,y] \subset \mathcal{N}_\delta([y,z]) \cup \mathcal{N}_\delta([z,x]);$$
$$[y,z] \subset \mathcal{N}_\delta([x,y]) \cup \mathcal{N}_\delta([z,x]);$$
$$[z,x] \subset \mathcal{N}_\delta([x,y]) \cup \mathcal{N}_\delta([y,z]).$$

Proposition 2.10 ([GdlH90], Proposition 2.21)**.** Let (X,d) be a geodesic space. Then the following conditions are equivalent.

1. Every geodesic triangle in X has a δ-centre, for some $\delta \geqslant 0$.
2. Every geodesic triangle in X is δ-slim, for some $\delta \geqslant 0$ (this is also known as *Rips condition*).
3. There exists $\delta \geqslant 0$ such that for all $x,y,z,t \in X$,

$$(x \cdot z)_t \geqslant \min\{(x \cdot y)_t, (y \cdot z)_t\} - \delta.$$

This inequality is referred to as the *Gromov inequality* or the *four-point condition*.

Definition 2.11. Let (X, d) be a metric space and $K \geqslant 1$, $C \geqslant 0$ be some constants. A (K, C)-*quasi-geodesic* in X is a map $\rho : [a, b] \subset \mathbf{R} \to X$ such that for all $s, t \in [a, b]$,

$$\frac{1}{K}|s-t| - C \leqslant d(\rho(s), \rho(t)) \leqslant K|s-t| + C.$$

Proposition 2.12 ([GdlH90], Theorem 5.6)**.** Let $\delta \geqslant 0$, $K \geqslant 1$ and $C \geqslant 0$. There exists $R = R(\delta, K, C) \geqslant 0$, such that if ρ_1, ρ_2 are two (K, C)-quasi-geodesics in a δ-hyperbolic space (X, d) with same endpoints, then

$$\rho_1 \subset \mathcal{N}_R(\rho_2).$$

This property is called *Morse lemma* or *stability of quasi-geodesics*.

Remark 2.13. In particular, since a geodesic segment is a (K, C)-quasi-geodesic for any $K \geqslant 1$ and any $C \geqslant 0$, we can deduce that every quasi-geodesic in a hyperbolic metric space lies at bounded distance from some geodesic. This bound is uniform over all (K, C)-quasi-geodesics and depends only on K, C and δ.

Definition 2.14. Let (X, d_X) and (Y, d_Y) be two metric spaces and fix $K \geqslant 1$, $C \geqslant 0$. A map $f : X \to Y$ is a (K,C)-*quasi-isometric embedding* if for all $x_1, x_2 \in X$,

$$\frac{1}{K} d_X(x_1, x_2) - C \leqslant d_Y(f(x_1), f(x_2)) \leqslant K d_X(x_1, x_2) + C.$$

The map f is *coarsely surjective* if there is $D \geqslant 0$ such that for all $y \in Y$, there exists $x \in X$ such that $d_Y(f(x), y) \leqslant D$.

A (K, C)-quasi-isometric embedding which is coarsely surjective is a (K, C)-*quasi-isometry*.

Example 2.15.
- A $(1, 0)$-quasi-isometry is an isometry.
- A $(K, 0)$-quasi-isometry is a bilipschitz map.
- A quasi-geodesic is a quasi-isometric embedding of a segment $[a, b] \subset \mathbf{R}$.

Remark 2.16. Quasi-isometry is an equivalence relation on metric spaces.

Theorem 2.17 ([GdlH90], Theorem 5.12)**.** Let (X, d_X) and (Y, d_Y) be two quasi-isometric geodesic spaces. If X is Gromov-hyperbolic, then so is Y.

Hyperbolic groups

Let G be a group and S be a set of generators of G. We may assume that S is symmetric up to replacing S by $S \cup S^{-1}$ where $S^{-1} = \{s^{-1} \mid s \in S\}$. The *Cayley graph* of G with respect to S, denoted by $\mathrm{Cay}_S(G)$, is the graph (V, E) such that

- the set of vertices is $V = G$;
- the set of edges is $E = \{(g, h) \in G \mid g = sh \text{ for some } s \in S\}$.

The Cayley graph of a group is endowed with the *combinatorial metric* which assigns length one to each edge.

Remark 2.18. If the generating set S is finite, then $\mathrm{Cay}_S(G)$ is a locally finite graph.

Definition 2.19. A group G is *hyperbolic* (or *Gromov-hyperbolic*) if it is generated by a finite set S and its Cayley graph $\mathrm{Cay}_S(G)$ is a Gromov-hyperbolic metric space with respect to the combinatorial metric.

Proposition 2.20 ([BH99], Example I.8.17(3))**.** If S_1 and S_2 are two finite generating sets for G, then $\mathrm{Cay}_{S_1}(G)$ and $\mathrm{Cay}_{S_2}(G)$ are quasi-isometric. In particular, the hyperbolicity of $\mathrm{Cay}_S(G)$ is independent of the choice of the generating set S.

Example 2.21. Finitely generated free groups $\mathbf{F}_n$ are hyperbolic because the Cayley graphs associated to their usual system of generators are trees.

The following proposition is known as *Švarc-Milnor's lemma*.

Proposition 2.22 ([BH99], Proposition I.8.19)**.** Let G be a discrete group acting by isometries on a proper geodesic space (X, d). Suppose that G acts on X properly and cocompactly. Then G is generated by a finite set S and $\mathrm{Cay}_S(G)$ is quasi-isometric to X.

Corollary 2.23. Every discrete group acting on $\mathbf{H}^2$ properly and cocompactly is hyperbolic. In particular, if S_g is a hyperbolic closed surface of genus $g \geqslant 2$, then its fundamental group $\pi_1(S_g)$ is hyperbolic.

Boundaries of a hyperbolic space

In this part, we will define the *Gromov boundary* of a Gromov hyperbolic space. We will also define the *visual boundary* for a geodesic metric space. These two notions do not always coincide.

Definition 2.24. Let X be a Gromov-hyperbolic space. A sequence $(x_n)_{n\in\mathbf{N}}$ in X is called a *Gromov sequence* if for some base point $o \in X$,

$$(x_n \cdot x_m)_o \underset{n,m\to\infty}{\longrightarrow} \infty.$$

This definition does not depend on the base point o thanks to Proposition 2.5.

Definition 2.25. Let (X, d) be a Gromov-hyperbolic space and let $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ be two Gromov sequences. We say that $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ are equivalent if for all $o \in X$, $(x_n \cdot y_m)_o \underset{n,m\to\infty}{\longrightarrow} \infty$.

Definition 2.26. If (X, d) is a Gromov-hyperbolic metric space, its *Gromov boundary* $\partial_\infty X$ is the set of equivalence classes of Gromov sequences in X.

Notation. We will define other notions of boundary in what follows, but the notation $\overline{X}$ refers to $X \cup \partial_\infty X$ where $\partial_\infty X$ is the Gromov boundary of X. Those boundaries will coincide for the case of the infinite-dimensional hyperbolic space, therefore they will not be distinguished in the remainder of the thesis.

The Gromov product can be extended continuously to $\partial_\infty X$.

Definition 2.27. For $\xi, \eta \in \partial_\infty X$ and $y, o \in X$, let

$$(\xi \cdot \eta)_o := \inf \left\{ \liminf_{n,m\to\infty} (x_n \cdot y_m)_o \mid (x_n)_{n\in\mathbf{N}} \in \xi, (y_n)_{n\in\mathbf{N}} \in \eta \right\} \in [0, +\infty];$$
$$(\xi \cdot y)_o = (y \cdot \xi)_o := \inf \left\{ \liminf_{n\to\infty} (x_n \cdot y)_o \mid (x_n)_{n\in\mathbf{N}} \in \xi \right\} \in [0, +\infty).$$

Remark 2.28. We have $(\xi \cdot \eta)_o < \infty$ if and only if $\xi \neq \eta$.

Moreover, the Gromov product is continuous when seen as a function in its three variables.

Proposition 2.29 ([DSU17], Lemma 3.4.22)**.** Suppose that $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ are sequences in $\overline{X}$ that converge to points $x \in \overline{X}$ and $y \in \overline{X}$, respectively. Suppose also that $(z_n)_{n\in\mathbf{N}}$ is a sequence in X converging to $z \in X$. Then,

$$(x_n \cdot y_n)_{z_n} \underset{n\to\infty}{\longrightarrow} (x \cdot y)_z.$$

The space $\overline{X}$ is endowed with the topology $\mathcal{T}_\infty$ that can be defined by specifying the open sets.

Let $S \subset \overline{X}$. Then S is an open set for $\mathcal{T}_\infty$ if

- $S \cap X$ is open;
- and for every $\xi \in S \cap \partial_\infty X$, there exists $t \geqslant 0$ such that $N(\xi, t) \subset S$ where

$$N(\xi, t) = \{x \in \overline{X} \mid (x \cdot \xi)_o > t\}.$$

Observe that the restriction of $\mathcal{T}_\infty$ to X is compatible with the metric d on X and that for each $\xi \in \partial_\infty X$, the collection $\{N(\xi, t) \mid t \geqslant 0\}$ is a basis of neighbourhoods for $\mathcal{T}_\infty$ at ξ ([DSU17, Remark 3.4.15]).

Another boundary that can be defined for geodesic metric spaces is the following.

Definition 2.30. Let X be a Gromov-hyperbolic space. A *geodesic ray* in X is an isometric embedding of the interval $\mathbf{R}_+ = [0, \infty)$ into X.

Let $o \in X$. The *visual boundary* of X based at o, $\partial_{\mathrm{vis}}(X, o)$, is the set of equivalence classes of geodesic rays $\rho : \mathbf{R}_+ \to X$ with $\rho(0) = o$, where the equivalence relation is given by

$$\rho_1 \sim \rho_2 \Leftrightarrow \sup_{t \geqslant 0} d(\rho_1(t), \rho_2(t)) < \infty.$$

Proposition 2.31 ([GdlH90], Proposition 7.4)**.** Let X be a Gromov-hyperbolic space and $o \in X$. Then there is a natural map $\varphi_o : \partial_{\mathrm{vis}}(X, o) \to \partial_\infty X$ defined by $\varphi_o([\rho]) = [(\rho(n))_{n \in \mathbf{N}}]$. This map φ_o is injective.

If moreover X is complete and locally compact, then the map φ_o is a bijection.

Remark 2.32. In general, the map φ is not surjective when the space X is not complete or locally compact.

Example 2.33. Let $p \in \mathbf{H}^2$. The space $X = \mathbf{H}^2 \setminus \{p\}$ is not complete. We have $\partial_\infty X = \partial_\infty \mathbf{H}^2 \simeq \mathbf{S}^1$. However, let $\xi \in \partial_\infty X$ and $o \in X$ such that $p \in [o, \xi) \subset \mathbf{H}^2$, then ξ is not in the image of the map $\varphi_o : \partial_{\mathrm{vis}}(X, o) \to \partial_\infty X$.

Proposition 2.34 ([GdlH90], Proposition 7.9)**.** If X is a locally compact Gromov-hyperbolic space, then $\partial_\infty X$ is compact with the above topology.

Proposition 2.35 ([GdlH90], Proposition 7.14)**.** If $f : X \to Y$ is a quasi-isometry between two Gromov-hyperbolic spaces, then f induces a boundary map which is a homeomorphism between the Gromov boundaries $\partial_\infty f : \partial_\infty X \to \partial_\infty Y$.

Now if G is a finitely generated hyperbolic group and S is a finite generating set for G. Then $\mathrm{Cay}_S(G)$ is a locally finite graph. Its boundary $\partial_\infty \mathrm{Cay}_S(G)$ is compact by Proposition 2.34 and independent of S (up to homeomorphism) by Propositions 2.20 and 2.35. We will simply call it the *boundary* of G, and denote it by ∂G.

Horospheres and Busemann functions

Let (X, d) be a Gromov-hyperbolic space.

Definition 2.36. For every $x \in X$, let $\beta_z : X \times X \to \mathbf{R}$ be the Busemann function defined by

$$\beta_x(y, z) := d(x, y) - d(x, z).$$

Proposition 2.37 ([DSU17], Proposition 3.3.3)**.** For all $x, y, z \in X$, we have

1. $|\beta_x(y, z)| \leqslant d(y, z)$;
2. $\beta_x(y, z) = (z \cdot x)_y - (y \cdot x)_z$.

In view of Proposition 2.37 and Definition 2.27 which defines the Gromov product for points lying in the Gromov boundary, we may extend the definition of Busemann functions as well.

Definition 2.38. For $\xi \in \partial_\infty X$ and $y, z \in X$, we define

$$\beta_\xi(y,z) := (z \cdot \xi)_y - (y \cdot \xi)_z.$$

Lemma 2.39 ([DSU17], Lemma 3.4.10)**.** For $\xi \in \partial_\infty X$, $y, z \in X$ and all $(x_n)_{n\in\mathbf{N}} \in X^{\mathbf{N}}$ converging to ξ,

$$\beta_{x_n}(y,z) \underset{n\to\infty}{\longrightarrow} \beta_\xi(y,z).$$

Like the Gromov product, the Busemann functions are also continuous when viewed as functions in three variables.

Proposition 2.40 ([DSU17], Lemma 3.4.22)**.** Suppose that $(x_n)_{n\in\mathbf{N}}$ is a sequence in $\overline{X}$ that converges to a point $x \in \overline{X}$. Suppose also that $(y_n)_{n\in\mathbf{N}}$ and $(z_n)_{n\in\mathbf{N}}$ are sequences in X that converge to some points $y \in X$ and $z \in X$, respectively. Then,

$$\beta_{x_n}(y_n, z_n) \underset{n\to\infty}{\longrightarrow} \beta_x(y,z).$$

With these Busemann functions, we can define the horospheres.

Definition 2.41. Let X be a Gromov-hyperbolic space. A *horosphere* is a level set for a Busemann function, *i.e.* a set of the form

$$H_{\xi,t} = \{x \in X \mid \beta_\xi(o,x) = t\},$$

where $\xi \in \partial_\infty X$, $o \in X$ and $t \in \mathbf{R}$. A *horoball* is a set of the form $\{x \in X \mid \beta_\xi(o,x) > t\}$.

The point ξ is called the *center* of the horosphere or horoball. We have $\overline{H_{\xi,t}} \cap \partial_\infty X = \{\xi\}$.

The notion of Busemann functions is useful to describe "how close" we are from a given point ξ in the boundary. A point $x \in X$ is "closer to ξ" than the base point $o \in X$ if $\beta_\xi(o,x) > 0$. In that case, x belongs to some horoball $\{x \in X \mid \beta_\xi(o,x) > t\}$ with $t > 0$.

2.2 CAT(κ) spaces

Let (X, d_X) be a geodesic metric space. Let $x, y, z \in X$ be three points and consider a geodesic triangle $\Delta = \Delta(x,y,z) = [x,y] \cup [y,z] \cup [x,z]$ where $[x,y]$, $[y,z]$ and $[x,z]$ are three geodesic segments connecting these points. Note that these geodesic segments are not necessarily unique.

Since d_X is a metric, we have the following inequalities

$$d_X(x,y) \leqslant d_X(x,z) + d_X(z,y);$$
$$d_X(y,z) \leqslant d_X(y,x) + d_X(x,z);$$
$$d_X(x,z) \leqslant d_X(x,y) + d_X(y,z).$$

Therefore, if $Y = \mathbf{R}^2$ or $Y = \mathbf{H}^2$, there exists a geodesic triangle $\overline{\Delta} = \Delta(\overline{x}, \overline{y}, \overline{z})$ in Y such that $d_Y(\overline{x}, \overline{y}) = d_X(x,y)$, $d_Y(\overline{y}, \overline{z}) = d_X(y,z)$ and $d_Y(\overline{x}, \overline{z}) = d_X(x,z)$. Such a triangle $\overline{\Delta}$ is called a *comparison triangle* for Δ and is unique up to isometry in Y.

If $\overline{p} \in \overline{\Delta}$ belongs to the geodesic segment $[\overline{x}, \overline{y}]$, then $\overline{p}$ is a *comparison point* for $p \in [x,y]$ if $d_X(x,p) = d_Y(\overline{x}, \overline{p})$. We can define comparison points in $[\overline{y}, \overline{z}]$ and $[\overline{x}, \overline{z}]$ in a similar fashion.

For $\kappa = 0$ (respectively $\kappa = -1$), the triangle Δ is said to satisfy the CAT(κ) *inequality* if for all $a, b \in \Delta$ and all comparison points $\overline{a}, \overline{b} \in \overline{\Delta} \subset \mathbf{R}^2$ (resp. $\overline{\Delta} \subset \mathbf{H}^2$), we have

$$d_X(a,b) \leqslant d_Y(\overline{a}, \overline{b}).$$

Definition 2.42. For $\kappa = 0$ or $\kappa = -1$, a metric space (X,d) is CAT(κ) if it is a geodesic metric space such that all its triangles satisfy the CAT(κ) inequality.

Intuitively, a space X is CAT(0) or CAT(-1) means that its triangles are "thinner" than the corresponding triangles in the Euclidean plane $\mathbf{R}^2$ or the hyperbolic plane $\mathbf{H}^2$, respectively.

Example 2.43.
- The Euclidean spaces $\mathbf{R}^n$, for $n \geqslant 2$, are CAT(0) but not CAT(-1).
- Every $\mathbf{R}$-tree is a CAT(-1) space.
- Riemannian manifolds with sectional curvature bounded above by -1 are CAT(-1) space ([BH99, Theorem II.1.A.6]).
- The hyperbolic spaces $\mathbf{H}^n$ are CAT(-1).

Remark 2.44.
- One can actually define CAT(κ) where κ is any real number (we refer to [BH99, Part II]), but we will only be interested in properties of CAT(0) and CAT(-1) spaces.
- Note that Definition 2.42 does not require X to be complete. Complete CAT(0) spaces are sometimes called *Hadamard spaces*.
- This terminology "CAT" was coined by Gromov in ([Gro87, page 119]). It refers to the initials of Cartan, Alexandrov and Toponogov.
  Some French people also call it CHAT instead of CAT in honour of Hadamard (the word "chat" meaning "cat" in French, which is a nice coincidence).

Proposition 2.45 ([BH99], Proposition II.1.4(1))**.** If (X,d) is a CAT(κ) space ($\kappa = 0$ or -1), then it is *uniquely geodesic*, *i.e.* every pair of points are connected by a unique geodesic segment.

Theorem 2.46 ([BH99], Theorem II.1.12)**.** Let (X,d) be a metric space. If X is CAT(-1), then it is CAT(0).

See also [DSU17, Proposition 3.2.2].

Proposition 2.47 ([DSU17], Proposition 3.3.4)**.** If (X,d) is a CAT(-1) space, then it is Gromov-hyperbolic.

Remark 2.48. Neither of the two following implications is true:

$$\mathrm{CAT}(0) \not\Rightarrow \text{ Gromov-hyperbolic;}$$
$$\text{Gromov-hyperbolic } \not\Rightarrow \mathrm{CAT}(0).$$

Example 2.49.
- The Euclidean space $\mathbf{R}^n$, $n \geqslant 2$, is CAT(0) but not Gromov-hyperbolic.
- A bounded graph which is not a tree is Gromov-hyperbolic but not CAT(0) since every CAT(0) space is contractible (see [BH99, Proposition II.1.4(4)]).

Convex and bounded subsets of CAT(0) spaces enjoy some very useful geometric properties.

Proposition 2.50 ([BH99], Proposition II.2.4)**.** Let (X,d) be a CAT(0) space and let $C \subset X$ be a convex subset which is complete for the induced metric. Then
1. for every $x \in X$, there is a unique point $\pi_C(x) \in X$ such that $d(x,\pi_C(x)) = d(x,C) = \inf\{d(x,y) \mid y \in C\}$;
2. if $y \in [x,\pi_C(x)]$, then $\pi_C(y) = \pi_C(x)$;
3. the map $x \mapsto \pi_C(x)$ is a retraction from X to C which does not increase distance, *i.e.* for all $x,y \in X$, $d(\pi_C(x),\pi_C(y)) \leqslant d(x,y)$;
4. the map $x \in X \mapsto d(x,C)$ is convex.

The following statement is known as the *Cartan fixed point theorem*. This property is specific to non-positive curvature. It is false for example for groups acting by isometries on a sphere, where the orbits are always bounded.

Proposition 2.51 ([BH99], Corollary II.2.8)**.** If X is a complete CAT(0) space and Γ is a group acting by isometries of X with a bounded orbit, then the fixed points set of Γ, Fix(Γ), is a non-empty convex subspace of X.

Chapter II

The infinite-dimensional hyperbolic space and its isometry group

This chapter focuses on the main object of this thesis, namely the infinite-dimensional hyperbolic space, and its group of isometries.

This space is the infinite-dimensional analog of the classically studied $\mathbf{H}^n$. We will see that all the geometric properties of the $\mathbf{H}^n$ defined in the previous chapter are still valid for $\mathbf{H}^\infty$. However, as one would expect, some differences of topological nature appear in infinite dimension.

1 The infinite-dimensional hyperbolic space $\mathbf{H}^\infty$

The infinite-dimensional hyperbolic space will be defined using a quadratic form on an infinite-dimensional Hilbert space. We then start by presenting some general definitions that will also be useful to deduce some basic properties for our hyperbolic space.

1.1 Quadratic forms of finite index

The material of this part can be found in [BIM05, Section 2] where more details are exposed.

Definition 1.1. Let Q be a quadratic form on a real Hilbert space $\mathcal{H}$. We define

- its *index of positivity or negativity* by

$$i_\pm(Q) = \sup\left\{\dim W \mid W \text{ is a subspace of } \mathcal{H} \text{ such that } Q_{|W} \text{ is positive/negative definite}\right\}$$

- and its *index* by

$$i(Q) = \sup\left\{\dim W \mid W \text{ is a subspace of } \mathcal{H} \text{ such that } Q_{|W} = 0\right\}.$$

The pair $(\mathcal{H}, Q)$ will be called a *quadratic space*. A subspace $W \subset \mathcal{H}$ is *positive/negative definite* if the restriction $Q|_W$ is positive/negative definite.

Let $B : \mathcal{H} \times \mathcal{H} \to \mathbf{R}$ be the bilinear form associated to Q. The quadratic form Q is *non-degenerate* if $\mathcal{H}^\perp = \{0\}$, where $\mathcal{H}^\perp$ is the orthogonal of $\mathcal{H}$ with respect to Q (or B),

$$\mathcal{H}^\perp = \{x \in \mathcal{H} \mid \forall y \in \mathcal{H} \quad B(x, y) = 0\}.$$

Proposition 1.2 ([BIM05], Proposition 2.1)**.** Let $(\mathcal{H}, Q)$ be a quadratic space where Q is non-degenerate and has finite index. Then

1. $i(Q) = \min\{i_-(Q), i_+(Q)\}$.

Moreover, if we assume that $i(Q) = i_-(Q)$, then we have the following.

2. If $W_- \subset \mathcal{H}$ is a negative definite subspace with $\dim W_- = i(Q)$, then $W_+ = W_-^\perp$ is positive definite and $\mathcal{H} = W_- \oplus W_+$.
3. If $\mathcal{H} = W'_- \oplus W'_+$ is an orthogonal direct sum with $W'_\pm$ positive/negative definite, then $\dim W'_- = i(Q)$.

In particular, if $i(Q) = i_-(Q)$ is finite, then any negative definite subspace W that is maximal (with respect to the inclusion) has same dimension $\dim W = i(Q)$. This infinite-dimensional version of Sylverster's law of inertia was also observed in [Mad85, Lemma 3].

Definition 1.3. A $\pm$-*decomposition* of the quadratic space $(\mathcal{H}, Q)$ is any orthogonal direct sum decomposition $\mathcal{H} = W_- \oplus W_+$ (with respect to Q) where W_- is negative definite and W_+ is positive definite.

To such a decomposition, we associate the scalar product $\langle \cdot, \cdot \rangle_\pm$ defined for $x, y \in \mathcal{H}$ by

$$\langle x, y \rangle_\pm = B(x_+, y_+) - B(x_-, y_-),$$

where $x = x_- + x_+$ and $y = y_- + y_+$ are the corresponding decompositions of x and y.

Lemma 1.4 ([BIM05], Lemma 2.4)**.** Let $(\mathcal{H}, Q)$ be a quadratic space and suppose that Q is non-degenerate and of finite index. Let $\mathcal{H} = W_- \oplus W_+ = W'_- \oplus W'_+$ be two $\pm$-decompositions. Then $(\mathcal{H}, \langle \cdot, \cdot \rangle_\pm)$ is a Hilbert space if and only if $(\mathcal{H}, \langle \cdot, \cdot \rangle'_\pm)$ is a Hilbert space, in which case the two scalar products are equivalent.

Definition 1.5. Let $(\mathcal{H}, Q)$ be a quadratic space where Q is non-degenerate and of finite index. We say that Q is *strongly non-degenerate* if for some (and hence any) $\pm$-decomposition $\mathcal{H} = W_- \oplus W_+$, the space $(\mathcal{H}, \langle \cdot, \cdot \rangle_\pm)$ is a Hilbert space.

If $\mathcal{H} = W_- \oplus W_+$, the *signature* of Q is $(\dim W_+, \dim W_-)$. This is independent of the $\pm$-decomposition of $\mathcal{H}$.

Proposition 1.6 ([BIM05], Proposition 2.7)**.** For strongly non-degenerate quadratic forms of finite index, the signature is a complete invariant of isomorphism.

Proposition 1.7 ([BIM05], Proposition 2.8)**.** Let $(\mathcal{H}, Q)$ be a quadratic space where Q is strongly non-degenerate and of finite index. Let $W \subset \mathcal{H}$ be a closed subspace such that $Q|_W$ is non-degenerate. Then $Q|_W$ is strongly non-degenerate and $\mathcal{H} = W \oplus W^\perp$.

1.2 The infinite-dimensional hyperbolic space

Similarly to the classical (finite-dimensional) hyperbolic spaces, the hyperbolic space of infinite dimension can be defined in several equivalent ways, called *models*.

We will define the *hyperboloid model*, the *Poincaré ball model*, the *Poincaré half-space model* and the *Klein (or projective) model*. Having these different points of view offers several advantages. For example, the Poincaré ball or half-space models are conformal, which allows to visualise the angles between geodesic lines in the picture, while the Klein model can be identified to the unit ball of some Banach space, which allows to use properties for affine or convex spaces (the geodesics are also easier to draw since they are straight lines in this model).

The hyperboloid model. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a real separable Hilbert space and denote by $(e_i)_{i\in\mathbf{N}}$ a Hilbert basis of $\mathcal{H}$.

Definition 1.8. The *Lorentzian scalar product* on $\mathcal{H}$ is the bilinear form B defined by

$$B(x,y) = -x_0y_0 + \sum_{i=1}^{\infty} x_iy_i$$

where the x_i's and y_i's are the coordinates of x and y in the given basis. Denote by Q the corresponding quadratic form defined by $Q(x) = B(x,x)$ for $x \in \mathcal{H}$.

Proposition 1.9. The quadratic form Q is strongly non-degenerate, of finite index $i(Q) = i_-(Q) = 1$ and has signature $(\infty, 1)$.

Proof. We have $B(x,y) = 0$ for all $y \in \mathcal{H}$ if and only if $x = 0$, so Q is non-degenerate. Moreover, $\mathrm{Span}(e_0)$ is a maximal negative definite subspace, so $i(Q) = i_-(Q) = 1$ by Proposition 1.2, therefore the signature of Q is $(\infty, 1)$. Finally, $\mathcal{H} = \mathrm{Span}(e_0) \oplus \mathrm{Span}(e_n, n \geqslant 1)$ is a $\pm$-decomposition of $\mathcal{H}$ whose associated scalar product is $\langle\cdot,\cdot\rangle_\pm = \langle\cdot,\cdot\rangle$, so $(\mathcal{H}, \langle\cdot,\cdot\rangle_\pm)$ is a Hilbert space. □

Definition 1.10. The (separable) infinite-dimensional real hyperbolic space is

$$\mathbf{H}^\infty_\mathbf{R} = \{x \in \mathcal{H} \mid Q(x) = -1, x_0 > 0\}.$$

Remark 1.11. If $\mathcal{H}$ is a Hilbert space over the field of complex (respectively quaternionic) numbers, one obtains the complex (resp. quaternionic) infinite-dimensional hyperbolic space, denoted by $\mathbf{H}^\infty_{\mathbf{C}}$ (resp. $\mathbf{H}^\infty_{\mathbf{Q}}$).

In this thesis, we will focus on the real hyperbolic space, and we thus denote the real hyperbolic spaces without $\mathbf{R}$ as a subscript, $\mathbf{H}^\infty = \mathbf{H}^\infty_{\mathbf{R}}$.

One can also define non-separable infinite-dimensional hyperbolic spaces by replacing $\mathcal{H}$ by a non-separable Hilbert space in the definition. The dimension can be prescribed to be of any cardinality.

Definition 1.12. A vector $x \in \mathcal{H}$ is called

- *space-like* if $Q(x) > 0$;
- *light-like* if $Q(x) = 0$ and
- *time-like* if $Q(x) < 0$.

The set $\{x \in \mathcal{H} \mid Q(x) = -1\}$ is a hyperboloid with two sheets consisting of time-like vectors that are "unitary" with respect to Q, and $\mathbf{H}^\infty_{\mathbf{R}}$ is one of its two connected components. Thus, we may also define $\mathbf{H}^\infty$ as

$$\mathbf{H}^\infty = \{[x] \in \mathrm{P}(\mathcal{H}) \mid Q(x) < 0\},$$

where $\mathrm{P}(\mathcal{H})$ is the the *projectivisation* of $\mathcal{H}$, *i.e.* the quotient of $\mathcal{H} \setminus \{0\}$ under the relation $x \sim \lambda x$, for all $x \in \mathcal{H}$ and $\lambda \in \mathbf{R}^*$.

The set of light-like vectors forms a real cone called the *light cone*.

Let $d : \mathbf{H}^\infty \to \mathbf{H}^\infty$ be the map defined by the formula

$$\cosh(d(x,y)) = -B(x,y) = -\langle x, Jy \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathcal{H}$ and J is the linear map on $\mathcal{H}$ such that $Je_0 = -e_0$ and $Je_i = e_i$ for all $i \geqslant 1$.

Proposition 1.13 ([DSU17], Proposition 2.2.2)**.** The map d is a metric on $\mathbf{H}^\infty$ which is compatible with the topology on $\mathbf{H}^\infty$ viewed as a subspace of $\mathcal{H}$ (or as a subspace of $\mathrm{P}(\mathcal{H})$, depending on the definition). This metric will be called the *hyperbolic metric* or *hyperbolic distance*.

See also [BIM05, Proposition 3.2], [BH99, Proposition I.2.6] or [Rat06, Theorem 3.2.2] (the two latter ones only show that d is a metric on $\mathbf{H}^\infty$).

Proposition 1.14 ([BIM05], Theorems 3.2 and 3.3)**.** With this hyperbolic distance, $\mathbf{H}^\infty$ is a complete infinite-dimensional geodesic Riemannian manifold which is $\mathrm{CAT}(-1)$.

Infinite-dimensional Riemannian manifold means that it is a differential manifold modelled on an infinite-dimensional (separable) Hilbert space (open sets are diffeomorphic to a open ball of the Hilbert space). In particular, we have the following.

Proposition 1.15. The infinite-dimensional hyperbolic space is not locally compact. It is therefore not proper, *i.e.* there are closed and bounded subsets that are not compact.

The boundary of $\mathbf{H}^\infty$

It follows from Propositions 1.14 and I.2.47 that $(\mathbf{H}^\infty, d)$ is a Gromov-hyperbolic metric space. It admits a Gromov boundary, $\partial_\infty \mathbf{H}^\infty$. The Gromov product on $\mathbf{H}^\infty$ can be extended continuously to $\partial_\infty \mathbf{H}^\infty$ as in Definition I.2.27. Denote by $\mathcal{T}_\infty$ the topology on $\partial_\infty \mathbf{H}^\infty \cup \mathbf{H}^\infty = \overline{\mathbf{H}^\infty}$. Let $o \in \mathbf{H}^\infty$ be some fixed base point and define the map

$$\begin{array}{rccc} D_o : & \overline{\mathbf{H}^\infty} \times \overline{\mathbf{H}^\infty} & \to & [0, \infty) \\ & (x, y) & \mapsto & e^{-(x \cdot y)_o}. \end{array}$$

This map is a *metametric* on $\overline{\mathbf{H}^\infty}$, *i.e.* a map which satisfies all the axioms of a metric (see Remark I.2.7 for the triangle inequality) except reflexivity. Indeed, for any $x \in \mathbf{H}^\infty$, we have $(x \cdot x)_o = d(o, x)$, so $D_o(x, x) = e^{-d(o,x)} > 0$. The term "metametric" was introduced by Väisälä in [Vä05].

The set $Z_{\text{refl}} = \{x \in \overline{\mathbf{H}^\infty} \mid D_o(x, x) = 0\}$ is the *domain of reflexivity of* D_o (it was called "set of small points" in [Vä05]). The restriction of D_o to Z_{refl} is a metric. Observe that $Z_{\text{refl}} \subset \partial_\infty \mathbf{H}^\infty$.

Cauchy sequences and *convergent sequences* for the metametric D_o can be defined as in metric spaces.

Definition 1.16. We say that D_o is a *complete* metametric if every Cauchy sequence is convergent for D_o. A topology $\mathcal{T}$ on $\overline{\mathbf{H}^\infty}$ is *compatible* with the metametric D_o if for every $\xi \in Z_{\text{refl}}$, the sets

$$\{x \in \overline{\mathbf{H}^\infty} \mid D_o(x, \xi) < R\}$$

for $R > 0$ form a basis of neighbourhoods for $\mathcal{T}$ at ξ.

Proposition 1.17. For every $o \in \mathbf{H}^\infty$, the metametric $D_o : \overline{\mathbf{H}^\infty} \times \overline{\mathbf{H}^\infty} \to [0, \infty)$ is complete and compatible with the topology $\mathcal{T}_\infty$ on $\overline{\mathbf{H}^\infty}$. Moreover, the domain of reflexivity of D_o is $\partial_\infty \mathbf{H}^\infty$.

Using the metametric D_o for some fixed $o \in \mathbf{H}^\infty$, one can define a metric on $\overline{\mathbf{H}^\infty}$ which agrees on $\partial_\infty \mathbf{H}^\infty$ with D_o. Such metrics are called *extended visual metrics* in [DSU17].

Proposition 1.18 ([DSU17], Proposition 3.6.13)**.** For all $x, y \in \overline{\mathbf{H}^\infty}$, let

$$\overline{D_o}(x, y) = \min(d(x, y), D_o(x, y)).$$

Then $\overline{D_o}$ is a complete metric on $\overline{\mathbf{H}^\infty}$ which induces the topology $\mathcal{T}_\infty$ and such that $\overline{D_o} = D_o$ when restricted to the Gromov boundary $\partial_\infty \mathbf{H}^\infty$.

Notation. We will denote the extended visual metrics by $\overline{D}$, without specifying the base point o when it is not needed or implicit.

Observe that $\overline{\mathbf{H}^\infty}$ is bounded for the extended visual metric based at any point. For all $\xi, \eta \in \partial_\infty \mathbf{H}^\infty$, we have $\overline{D}(\xi, \eta) = D(\xi, \eta) = e^{-(\xi \cdot \eta)_o} \leqslant 1$.

In a CAT(-1) space, we already know by Proposition I.2.45 that the geodesic segments that connect two points are unique. This property can be extended to the boundary of this space thanks to the following proposition.

Proposition 1.19 ([DSU17], Proposition 4.4.4)**.** Suppose that X is a complete CAT(-1) space.

1. For any two distinct points $x, y \in \overline{X}$, there is a unique geodesic $[x, y]$ connecting them.
2. Let $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ be two sequences in $\overline{X}$ that converge to two points $x, y \in \overline{X}$. Then $[x_n, y_n] \underset{n\to\infty}{\longrightarrow} [x, y]$ for the Hausdorff metric on $(\overline{X}, \overline{D})$. If $x = y$, then $[x, y] = \{x\}$ and $[x_n, y_n] \underset{n\to\infty}{\longrightarrow} \{x\}$.

Definition 1.20. A hyperbolic space which satisfies the conclusion of Proposition 1.19 is said to be *regularly geodesic*.

As a complete CAT(-1) space, the infinite-dimensional hyperbolic space $\mathbf{H}^\infty$ is regularly geodesic. This property implies that its Gromov boundary $\partial_\infty \mathbf{H}^\infty$ and its visual boundary $\partial_{\mathrm{vis}}(\mathbf{H}^\infty, o)$ relative to the base point $o \in \mathbf{H}^\infty$ must be in bijection.

Observe that the uniqueness in Proposition 1.19 implies that $\partial_{\mathrm{vis}}(\mathbf{H}^\infty, o)$ is the set of all the geodesic rays starting at $o \in \mathbf{H}^\infty$ (the equivalence relation on the geodesic rays is trivial after fixing the base point).

Proposition 1.21. The map $\varphi_o : \partial_{\mathrm{vis}}(\mathbf{H}^\infty, o) \to \partial_\infty \mathbf{H}^\infty$ defined in Proposition I.2.31 is a bijection.

Proof. The first part of Proposition I.2.31 states that φ_o is injective. The second part does not yield surjectivity since $\mathbf{H}^\infty$ is not proper.

Let $\xi \in \partial_\infty \mathbf{H}^\infty$. By Proposition 1.19, there exists a unique geodesic $\rho : \mathbf{R}_+ \cup \{\infty\} \to \overline{\mathbf{H}^\infty}$ connecting o to ξ. This geodesic then defines the point $[\rho]$ in $\partial_{\mathrm{vis}}(\mathbf{H}^\infty, o)$ satisfying $\varphi_o([\rho]) = \xi$. Therefore φ_o is surjective. □

In the classical setting, $\mathbf{H}^n$ can be compactified by taking its closure in the topological space $\mathrm{P}(\mathbf{R}^{n+1})$. However, for $\mathbf{H}^\infty$, the closure inside $\mathrm{P}(\mathcal{H})$ is not compact. Thus, we use the terminology *bordification* instead of *compactification*.

Definition 1.22. The *bordification* of $\mathbf{H}^\infty$, denoted by $\overline{\mathbf{H}^\infty}$, is its closure relative to the topological space $\mathrm{P}(\mathcal{H})$, *i.e.*

$$\overline{\mathbf{H}^\infty} = \{[x] \in \mathrm{P}(\mathcal{H}) \mid Q(x) \leqslant 0\}.$$

The *boundary* of $\mathbf{H}^\infty$ is its topological boundary relative to $\mathrm{P}(\mathcal{H})$. It is the projectivisation of the light cone,

$$\partial\mathbf{H}^\infty = \overline{\mathbf{H}^\infty} \setminus \mathbf{H}^\infty = \{[x] \in \mathrm{P}(\mathcal{H}) \mid Q(x) = 0\}.$$

As a consequence of Propositions 1.14 and I.2.47, $(\mathbf{H}^\infty, d)$ is a Gromov-hyperbolic metric space. It admits a Gromov boundary, $\partial_\infty\mathbf{H}^\infty$.

The notation $\overline{\mathbf{H}^\infty}$ for $\mathbf{H}^\infty \cup \partial_\infty\mathbf{H}^\infty$ is not in conflict with the bordification thanks to the following statement.

Proposition 1.23 ([DSU17], Proposition 3.5.3)**.** The identity map $\mathrm{id} : \mathbf{H}^\infty \to \mathbf{H}^\infty$ extends uniquely to a homeomorphism $\widehat{\mathrm{id}} : \mathbf{H}^\infty \cup \partial_\infty\mathbf{H}^\infty \to \mathbf{H}^\infty \cup \partial\mathbf{H}^\infty$.

Notation. With this proposition, these two boundaries are the "same" and from now on, we will keep the notation $\partial\mathbf{H}^\infty$ instead of $\partial_\infty\mathbf{H}^\infty$ for the boundary of $\mathbf{H}^\infty$. The topology on $\overline{\mathbf{H}^\infty} = \mathbf{H}^\infty \cup \partial\mathbf{H}^\infty$ will be denoted by $\mathcal{T}$ instead of $\mathcal{T}_\infty$ and will be referred to as the *cone topology* (see [BH99, Chapter II.8]).

Remark 1.24. One can use Busemann functions along with the first point of Proposition I.2.37 to define the *horicompactification* of a hyperbolic space (which is compact even when the space we started with is not proper). The horicompactification of $\mathbf{H}^\infty$ was studied in [Duc23], but we will not need this object in the remainder of the thesis.

There are other ways to define a notion of boundary for $\mathbf{H}^\infty$ (or Gromov-hyperbolic or CAT(0) spaces in general). One can define another version of the visual boundary using quasi-geodesics instead of geodesics. This definition is more flexible since quasi-geodesics may exist even when geodesics do not. If we denote this boundary by $\partial_q\mathbf{H}^\infty$, it can be shown that $\partial_\infty\mathbf{H}^\infty$ can be identified as a topological space with $\partial_q\mathbf{H}^\infty$ (see [Vä05, Has22, LT24]).

Hyperbolic subspaces of $\mathbf{H}^\infty$

Subspaces of $\mathbf{H}^\infty$ have a very nice description in terms of linear subspaces of the ambient Hilbert space.

Remark 1.25 ([BIM05], Remark 3.1)**.** If S is a finite set of points in $\mathbf{H}^\infty$, let $\mathcal{H}_S = \mathrm{Span}(S)$ be the vector space spanned by S in the ambient Hilbert space $\mathcal{H}$. The restriction of the quadratic form Q to $\mathcal{H}_S$ is still a non-degenerate quadratic form and its signature is $(\dim \mathcal{H}_S - 1, 1)$. Therefore it defines an embedding of a hyperbolic space of finite dimension $\dim \mathcal{H}_S - 1$ into $\mathbf{H}^\infty$.

If S is not finite, the same is true, except that $Q|_{\mathcal{H}_S}$ has signature $(\infty, 1)$ and define a hyperbolic space of infinite dimension.

Definition 1.26. A *totally geodesic subspace* of $\mathbf{H}^\infty$ is a set of the form $\mathrm{Span}(S) \cap \mathbf{H}^\infty$ as in the previous remark.

This terminology comes from Riemannian geometry where a totally geodesic subspace Y of a Riemmanian manifold X is a subspace such that every geodesic line of Y is also a geodesic line of X.

Remark 1.27. For each $n \in \mathbf{N}$, the subspace of $\mathcal{H}$ spanned by the first n vectors of a Hilbert basis defines an embedding $\mathbf{H}^n \hookrightarrow \mathbf{H}^\infty$. We thus have an increasing sequence of finite-dimensional hyperbolic spaces in $\mathbf{H}^\infty$,

$$\mathbf{H}^1 \subset \mathbf{H}^2 \subset \cdots \subset \mathbf{H}^\infty.$$

Since the Hilbert basis is total in $\mathcal{H}$, the increasing union $\bigcup_{n\in\mathbf{N}} \mathbf{H}^n$ is dense in $\mathbf{H}^\infty$ with respect to the topology induced by the Euclidean topology on $\mathcal{H}$, which coincides with the topology of the hyperbolic metric on $\mathbf{H}^\infty$ by 1.13.

Geodesics in $\mathbf{H}^\infty$

We may now describe geodesics in $\mathbf{H}^\infty$ as one-dimensional subspaces of $\mathbf{H}^\infty$.

Definition 1.28. Let V be a vector subspace of $\mathcal{H}$. We say that V is
- *time-like* if it contains a time-like vector;
- *space-like* if all the non-zero vectors are space-like;
- *light-like* otherwise (it has non-zero light-like vectors, but no time-like vectors).

Definition 1.29. A *hyperbolic line* of $\mathbf{H}^\infty$ is the intersection of $\mathbf{H}^\infty$ with a 2-dimensional time-like vector subspace of $\mathcal{H}$.

Let $x, y \in \mathbf{H}^\infty$ be two distinct points. Viewed as points in the ambient Hilbert space $\mathcal{H}$, they span a 2-dimensional time-like subspace of $\mathcal{H}$. By Remark 1.25, the intersection of $\mathrm{Span}(x, y)$ with $\mathbf{H}^\infty$ is a hyperbolic subspace of $\mathbf{H}^\infty$ of dimension 1, which is the unique hyperbolic line containing x and y.

Definition 1.30. A *geodesic* (or *geodesic line*) of $\mathbf{H}^\infty$ is an isometric embedding of $\mathbf{R}$ into $\mathbf{H}^\infty$, *i.e.* a map $c : \mathbf{R} \to \mathbf{H}^\infty$ such that $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in \mathbf{R}$.

A *geodesic ray* (respectively *geodesic segment*) is an isomorphic embedding of $[0, +\infty)$ (resp. a segment $[a, b]$) into $\mathbf{H}^\infty$.

For convenience, we will not distinguish the map c from its image in $\mathbf{H}^\infty$ and call both of them a geodesic line, ray or segment.

Theorem 1.31. The geodesics in $\mathbf{H}^\infty$ are its hyperbolic lines. Moreover we have the following description, $c : \mathbf{R} \to \mathbf{H}^\infty$ is a geodesic if and only if there exists $a, b \in \mathcal{H}$ with $Q(a) = -1$, $Q(b) = 1$ and $B(a, b) = 0$ such that for all $t \in \mathbf{R}$,

$$c(t) = \cosh(t)a + \sinh(t)b.$$

The proof of this theorem is similar to the finite-dimensional case since the restriction to a subspace containing a finite number of points in $\mathbf{H}^\infty$ gives a finite-dimensional hyperbolic subspace by Remark 1.25. We refer to [Rat06, Theorem 3.2.5] for example for a proof in finite dimension.

Finite-dimensional hyperbolic spaces $\mathbf{H}^n$ have several other models which can also be extended for $\mathbf{H}^\infty$. The description of these models as well as proofs in finite dimension can be found in [Mar23, Chapter 2]. The proofs are similar for the infinite-dimensional case.

Notation. We will use the notations $\mathbb{I}$ for the hyperboloid model, $\mathbb{H}$ for the upper half-space model, $\mathbb{D}$ for the Poincaré ball model and $\mathbb{B}$ for the Klein model. As usual, we add a superscript to indicate the dimension. Denote by $d_{\mathbb{I}}$ the hyperbolic distance on the hyperboloid model.

When we do not need to distinguish between these models, we will keep the notation $\mathbf{H}$.

In order to define these models, we first introduce the subspace $\mathcal{H}'_0 = \{x \in \mathcal{H} \mid x_0 = 0\} \subset \mathcal{H}$. This is the completion (for the Euclidean metric) of the subspace of $\mathcal{H}$ spanned by the vectors e_n starting from $n = 1$ instead of $n = 0$. We have $\mathcal{H} = \mathbf{R}\cdot e_0 \oplus \mathcal{H}'_0$ and write $\mathcal{H}'_0 = \mathcal{H} \ominus \mathbf{R}\cdot e_0$.

Notation. For convenience, if a vector is in $\mathcal{H}'_0$, we remove its first vanishing coordinate. A vector $x = (x_0, x_1, x_2, \dots) = (0, x_1, x_2, \dots) \in \mathcal{H}'_0$ will then be denoted by $x = (x_1, x_2, \dots)$. We shall enumerate the indices of the coordinates by starting from 1 to avoid confusions with the coordinates in $\mathcal{H}$.

The Poincaré ball model. The *Poincaré ball model* is

$$\mathbb{D}^\infty = \{x \in \mathcal{H}'_0 \mid \|x\| < 1\},$$

the (open) unit ball of $\mathcal{H}'_0$ endowed with the Euclidean topology (inherited by $\mathcal{H}$).

The boundary and bordification of $\mathbb{D}^\infty$ with respect to the Euclidean topology are

$$\partial\mathbb{D}^\infty = \{x \in \mathcal{H}'_0 \mid \|x\| = 1\};$$
$$\overline{\mathbb{D}^\infty} = \{x \in \mathcal{H}'_0 \mid \|x\| \leqslant 1\}.$$

In the Poincaré ball model, we define the following distance between two points $x, y \in \mathbb{D}^\infty$:

$$\cosh(d_{\mathbb{D}}(x,y)) = 1 + \frac{\|x-y\|^2}{(1-\|x\|^2)(1-\|y\|^2)}.$$

Proposition 1.32 ([Mar23], Section 2.1.6 and [Rat06], Theorem 4.5.1)**.** The Poincaré ball model $\mathbb{D}^\infty$ and the hyperboloid model $\mathbb{I}^\infty$ are isometric, and therefore homeomorphic. The isometry is given by the stereographic projection from the hyperboloid $\mathbb{I}^\infty$ on the ball $\mathbb{D}^\infty$ based at the point $(-1, 0, \dots) \in \mathcal{H}$:

$$\begin{array}{ccc} \mathbb{I}^\infty & \to & \mathbb{D}^\infty \\ (x_0, x_1, x_2, \dots) & \mapsto & \dfrac{1}{x_0+1}(x_1, x_2, x_3, \dots). \end{array}$$

Proposition 1.33 ([Mar23], Propositions 2.1.16 and 2.1.17)**.**
1. The Poincaré ball model is a conformal model of $\mathbf{H}^\infty$, *i.e.* the angles formed by two vectors in $\mathbb{D}^\infty \subset \mathcal{H}'_0$ coincide with the Euclidean ones in $\mathcal{H}'_0$.
2. The n-dimensional subspaces of $\mathbb{D}^\infty$ are intersections of $\mathbb{D}^\infty$ with n-spheres or n-linear subspaces in $\mathcal{H}'_0$ that are orthogonal to $\partial\mathbb{D}^\infty$.

The Poincaré half-space model. The *Poincaré half-space model* is

$$\mathbb{H}^\infty = \left\{x = (x_i)_{i\geqslant 1} \in \mathcal{H}'_0 \mid x_1 > 0\right\},$$

the "upper" half-space of $\mathcal{H}'_0$. We view $\mathbb{H}^\infty$ as lying inside $\widehat{\mathcal{H}'_0} = \mathcal{H}'_0 \cup \{\infty\}$.

The topology on $\widehat{\mathcal{H}'_0}$ is defined as follows: a subset $U \subset \widehat{\mathcal{H}'_0}$ is open if and only if $U \cap \mathcal{H}'_0$ is open and ($\infty \in U \Rightarrow \mathcal{H}'_0 \setminus U$ is bounded).

According to this topology, we have

$$\partial\mathbb{H}^\infty = \{x \in \mathcal{H}'_0 \mid x_1 = 0\} \cup \{\infty\};$$
$$\overline{\mathbb{H}^\infty} = \{x \in \mathcal{H}'_0 \mid x_1 \geqslant 0\} \cup \{\infty\}.$$

On the half-space model, we define the following distance between two points $x, y \in \mathbb{H}^\infty$:

$$\cosh(d_{\mathbb{H}}(x,y)) = 1 + \frac{\|x-y\|^2}{2x_1y_1}.$$

The coordinate x_1 of $x \in \mathbb{H}^\infty$ can be thought of as the "height" of x. In this model, the horospheres centred at $\infty \in \partial\mathbb{H}^\infty$ are affine subspaces of $\mathcal{H}_0'$ which are parallel to $\{x_1 = 0\}$.

Proposition 1.34 ([Mar23], Section 2.1.8 and [Rat06], Theorem 4.6.1)**.** The half-space model $\mathbb{H}^\infty$ and the Poincaré ball model $\mathbb{D}^\infty$ are isometric. It can be established using an inversion along the sphere $\mathbf{S}(y, \sqrt{2})$ with $y = (-1, 0, 0, \dots) \in \partial\mathbb{D}^\infty$ (recall that $y \in \mathcal{H}_0'$ is actually the point $(0, -1, 0, \dots) \in \mathcal{H}$ where the first coordinate is removed). The isometry is then

$$\begin{array}{ccc} \mathbb{D}^\infty & \to & \mathbb{H}^\infty \\ x & \mapsto & y + 2\dfrac{x-y}{\|x-y\|^2}. \end{array}$$

The point $\infty \in \partial\mathbb{H}^\infty$ corresponds to the point $(-1, 0, 0, \dots) \in \partial\mathbb{D}^\infty \subset \mathcal{H}_0'$.

Proposition 1.35 ([Mar23], Proposition 2.1.22)**.**
1. The half-space model $\mathbb{H}^\infty$ is also a conformal model for $\mathbf{H}^\infty$.
2. The n-dimensional subspaces of $\mathbb{H}^\infty$ are the intersections of $\mathbb{H}^\infty$ with n-spheres or n-linear subspaces in $\mathcal{H}_0'$ that are orthogonal to $\partial\mathbb{H}^\infty$.

The Klein (or projective) model. To define this model, we will need to consider vectors in $\mathcal{H}_0' = \{x \in \mathcal{H} \mid x_0 = 0\}$ with their coordinates along e_0. Define now $\mathcal{H}_1' = \{x \in \mathcal{H} \mid x_0 = 1\}$. Then $\mathcal{H}_1'$ is the affine vector space parallel to $\mathcal{H}_0'$ and containing e_0,

$$\mathcal{H}_1' = e_0 + \mathcal{H}' = \{e_0 + x \mid x \in \mathcal{H}'\}.$$

We define the *Klein model* as

$$\mathbb{B}^\infty = \{x \in \mathcal{H}_1' \mid \|x - e_0\| < 1\} = e_0 + \mathbb{D}^\infty,$$

the (open) unit ball inside $\mathcal{H}_1'$ endowed again with the topology inherited from the ambient Hilbert space $\mathcal{H}$.

In the Klein model, we define the following distance between two points $x, y \in \mathbb{B}^\infty$:

$$\cosh(d_{\mathbb{B}}(x,y)) = \frac{|1 - \langle x, y\rangle|}{\sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}}.$$

Proposition 1.36 ([Mar23], Section 2.1.12 and [Rat06], Theorem 6.1.1)**.** The Klein model $\mathbb{B}^\infty$ and the hyperboloid model $\mathbb{I}^\infty$ are isometric. The isometry is given by the stereographic projection from $\mathbb{I}^\infty$ on $\mathcal{H}_1'$ based at the origin of $\mathcal{H}$, $(0, 0, \dots) \in \mathcal{H}$. This isometry is

$$\begin{array}{ccc} \mathbb{I}^\infty & \to & \mathbb{B}^\infty \\ x = (x_0, x_1, x_2, \dots) & \mapsto & \dfrac{x}{x_0} = \left(1, \dfrac{x_1}{x_0}, \dfrac{x_2}{x_0}, \dots\right). \end{array}$$

We may also view this Klein model $\mathbb{B}^\infty$ as the projectivisation of $\mathbb{I}^\infty$ in $\mathrm{P}(\mathcal{H})$. The Klein model is also known as the *projective model*.

Proposition 1.37 ([Mar23], Section 2.1.12)**.** 1. The Klein model is not conformal, hyperbolic spheres are ellipsoids in $\mathbb{B}^\infty$.

2. In $\mathbb{B}^\infty$, geodesics are straight lines, *i.e.* Euclidean lines in $\mathcal{H}'_1$.

We also have a one-to-one correspondence between hyperplanes in $\mathbb{B}^\infty$ and points in $\mathcal{H}'_1$ that lie outside $\mathbb{B}^\infty$, which gives a useful viewpoint for working with reflections of the hyperbolic space. This model will also be particularly useful when dealing with convex subsets of the hyperbolic space in the proof of Theorem III.1.7 for example.

1.3 The weak topology

Let (X, d) be a metric space. Denote by $\mathcal{T}$ the topology on X defined by the metric d. One can define a weaker topology $\mathcal{T}_W$ by letting $\mathcal{T}_W$ be the weakest topology on X such that for all $y, z \in X$, the Busemann function defined in Definition I.2.36, $x \mapsto \beta_x(y,z) = d(x,y) - d(x,z)$ is continuous. This topology is always Hausdorff.

Definition 1.38. Let X be a metric space. We define the topology $\mathcal{T}_c$ to be the weakest topology on X for which all $\mathcal{T}$-closed convex sets are $\mathcal{T}_c$-closed.

Theorem 1.39 ([Mon05], Theorem 14)**.** Let X be a complete $\mathrm{CAT}(0)$ space and $C \subset X$ a bounded closed convex subset. Then C is compact for the topology $\mathcal{T}_c$.

Lemma 1.40 ([Mon05], Lemma 16)**.** For any complete $\mathrm{CAT}(0)$ space X, we have $\mathcal{T}_c \subset \mathcal{T}_W \subset \mathcal{T}$.

Remark 1.41. If $X = \mathcal{H}$ is a Hilbert space, then $\mathcal{T}_W$ and $\mathcal{T}_c$ both coincide with the weak topology.

If X is the infinite-dimensional hyperbolic space, $X = \mathbf{H}^\infty$, then $\mathcal{T}_W$ and $\mathcal{T}_c$ coincide. This topology can also be described as the coarsest topology in which the (open) half-spaces defined by hyperbolic hyperplanes are open (see [Duc23, Lemma 2.2]), where a half-space of $\overline{\mathbf{H}^\infty}$ is the intersection of $\overline{\mathbf{H}^\infty}$ with a linear half-space of the ambient Hilbert space. In the Klein model $\overline{\mathbb{B}^\infty}$, this coincides with the weak topology on $\mathcal{H}$. We will call it the *weak topology* on $\overline{\mathbf{H}^\infty}$.

In particular, by Banach-Alaoglu's theorem, Hilbert spaces being reflexive, we get the following:

Proposition 1.42. Endowed with the weak topology $\mathcal{T}_W$, $\overline{\mathbf{H}^\infty}$ is compact.

Proposition 1.43 ([Duc23], Lemma 2.4)**.** The restriction of the weak topology on $\partial\mathbf{H}^\infty$

coincides with the cone topology.

Proposition 1.44 ([Duc23], Proposition 6.2)**.** The group $\mathrm{Isom}(\mathbf{H}^\infty)$ with the topology of pointwise convergence acts continuously on $\overline{\mathbf{H}^\infty}$ with the weak topology.

2 The group of isometries

2.1 Isometries

Definition 2.1. An *isometry* of $\mathbf{H}^\infty$ is a diffeomorphism $f : \mathbf{H}^\infty \to \mathbf{H}^\infty$ such that

$$\forall x, y \in \mathbf{H}^\infty \quad d(f(x), f(y)) = d(x, y).$$

Let $\mathcal{H}$ be a Hilbert space and Q is a quadratic form on $\mathcal{H}$, denote by $\mathrm{O}(\mathcal{H}, Q)$ the group of linear automorphisms of $\mathcal{H}$ that preserve Q, *i.e.* the group of automorphisms $T : \mathcal{H} \to \mathcal{H}$ such that $Q \circ T = Q$.

The projectivisation of this group is

$$\mathrm{PO}(\mathcal{H}, Q) = \mathrm{O}(\mathcal{H}, Q)/\{\pm \mathrm{id}\}.$$

Theorem 2.2 ([DSU17], Theorem 2.3.3)**.** The group of isometries of $\mathbf{H}^\infty$ is $\mathrm{Isom}(\mathbf{H}^\infty) = \mathrm{PO}(\mathcal{H}, Q)$, the projectivisation of the group of linear automorphisms of $\mathcal{H}$ that preserve Q.

By Proposition 1.6, two non-degenerate quadratic forms having same signature are isomorphic, so are the associated orthogonal groups.

Notation. If (α, β) denotes the signature of Q, we will use the notation $\mathrm{O}(\alpha, \beta)$ and $\mathrm{PO}(\alpha, \beta)$ from now on when there is no ambiguity on the Hilbert space or the quadratic form.

Theorem 2.3 ([DSU17], Theorem 2.3.3)**.** With these notations, we have

$$\mathrm{Isom}(\mathbf{H}^\infty) = \mathrm{PO}(\infty, 1).$$

See also [Duc13, Corollary 1.6] for a more general statement on symmetric spaces of infinite dimension and finite rank.

Corollary 2.4 ([DSU17], Corollary 2.3.13)**.** Every isometry of $\mathrm{Isom}(\mathbf{H}^\infty)$ extends uniquely to a homeomorphism of $\overline{\mathbf{H}^\infty}$.

Proposition 2.5 ([DSU17], Observation 2.3.2)**.** The group $\mathrm{Isom}(\mathbf{H}^\infty)$ acts transitively on $\mathbf{H}^\infty$.

In the upper half-space model $\mathbb{H}^\infty$, we have the following description of the isometries fixing the particular point $\infty \in \partial\mathbb{H}^\infty$:

Recall that in this model, $\partial\mathbb{H}^\infty = \{x \in \mathcal{H}'_0 \mid x_1 = 0\} \cup \{\infty\}$. Then $\partial\mathbb{H}^\infty \setminus \{\infty\}$ is a Hilbert space. A *similarity* $g : \partial\mathbb{H}^\infty \setminus \{\infty\} \to \partial\mathbb{H}^\infty \setminus \{\infty\}$ is a map of the form

$$g(x) = \lambda T x + b,$$

where $\lambda > 0$, T is an orthogonal operator on $\{x \in \mathcal{H}'_0 \mid x_1 = 0\}$ with respect to the Euclidean norm on $\mathcal{H}$, and $b \in \partial\mathbb{H}^\infty \setminus \{\infty\}$.

Proposition 2.6 ([DSU17], Observation 2.5.6)**.** Let $g : \partial\mathbb{H}^\infty \setminus \{\infty\} \to \partial\mathbb{H}^\infty \setminus \{\infty\}$ be a similarity. Then g induces an isomorphism of $\overline{\mathbb{H}^\infty}$, $\widehat{g} : \overline{\mathbb{H}^\infty} \to \overline{\mathbb{H}^\infty}$ defined by

$$\widehat{g}(x) := \begin{cases} (\lambda x_1, g(\pi(x))) & \text{if } x \neq \infty \\ \infty & \text{if } x = \infty \end{cases}$$

where π denotes the orthogonal projection onto $\partial\mathbb{H}^\infty \setminus \{\infty\}$. In particular, the restriction of $\widehat{g}$ to $\mathbb{H}^\infty$ is an isometry of $\mathbb{H}^\infty$.

The isometry $\widehat{g}$ is called the *Poincaré extension* of g.

Moreover, any isometry $g \in \mathrm{Isom}(\mathbb{H}^\infty)$ fixing ∞ actually is the Poincaré extension of some similarity on $\partial\mathbb{H}^\infty \setminus \{\infty\}$.

Proposition 2.7 ([DSU17], Proposition 2.5.8)**.** For all $g \in \mathrm{Isom}(\mathbb{H}^\infty)$ such that $g(\infty) = \infty$, there exists a similarity $h : \partial\mathbb{H}^\infty \setminus \{\infty\} \to \partial\mathbb{H}^\infty \setminus \{\infty\}$ such that $g = \widehat{h}$.

2.2 Classification of the isometries

Translation length

As in finite dimension, the isometries of $\mathbf{H}^\infty$ can be classified into the three categories: *loxodromic*, *parabolic* and *elliptic*. One can define them using the *translation length* of the isometries.

Let (X, d) be a metric space and $\mathrm{Isom}(X)$ be its group of isometries.

Definition 2.8. For $g \in \mathrm{Isom}(X)$, the *translation length* (or *minimal displacement function*) of g is

$$\ell(g) := \inf_{x \in X} d(x, g(x)).$$

This notion allows us to classify the isometries of X.

Definition 2.9. Let $g \in \mathrm{Isom}(X)$, we say that g is

- *semisimple* if the set $\mathrm{Min}(g) = \{x \in X \mid d(x, g(x)) = \ell(g)\}$ is non-empty;

- *neutral* if $\ell(g) = 0$.

Definition 2.10. Let $g \in \mathrm{Isom}(X)$, we say that g is
- *elliptic* if g is neutral and semisimple;
- *parabolic* if g is neutral but not semisimple;
- *loxodromic* if g is not neutral.

Remark 2.11.
- Loxodromic isometries are also called *hyperbolic*.
- If X is a CAT(0) space, then any elliptic isometry necessarily has a fixed point in X by Proposition I.2.51 (see also [DSU17, Theorem 6.2.5]).

Theorem 2.12 ([DSU17], Theorem 6.1.4)**.** The categories of elliptic, parabolic and loxodromic isometries are mutually exclusive and any isometry $g \in \mathrm{Isom}(\mathbf{H}^\infty)$ is either elliptic, parabolic or loxodromic.

Lemma 2.13 ([Duc23], Lemma 9.1)**.** The translation length on $\mathrm{Isom}(X)$ is upper semi-continuous for the pointwise convergence topology.

Proof. Let $(g_n)_{n\in\mathbf{N}} \in \mathrm{Isom}(X)^{\mathbf{N}}$ be a sequence converging to g in $\mathrm{Isom}(X)$. By continuity of the distance function, for any $\epsilon > 0$, there exist $x \in X$ such that $d(gx, x) \leqslant \ell(g) + \epsilon$ and $N \in \mathbf{N}$ such that $\forall n \geqslant N$ $d(g_n x, gx) \leqslant \epsilon$. By triangle inequality we get, for $n \geqslant N$, $d(g_n x, x) \leqslant d(g_n x, gx) + d(gx, x) \leqslant \ell(g) + 2\epsilon$. Thus

$$\limsup_{n\to\infty} \ell(g_n) \leqslant \limsup_{n\to\infty} d(g_n x, x) \leqslant \ell(g) + 2\epsilon,$$

and $\limsup_{n\to\infty} \ell(g_n) \leqslant \ell(g)$. □

The following lemma is part of the proof of [Duc23, Theorem 9.7] which states that $\mathrm{Isom}(\mathbf{H}^\infty)$ as no dense conjugacy classes.

The proof of Duchesne relies on the observation that in the hyperbolic spaces, a "limit" of horocycles is not a geodesic line (or more generally, a "limit" of horospheres is not a totally geodesic subspace). We present here a different proof using the continuity of the action $\mathrm{Isom}(\mathbf{H}^\infty) \curvearrowright \overline{\mathbf{H}^\infty}$, where $\mathrm{Isom}(\mathbf{H}^\infty)$ is endowed with the pointwise convergence topology and $\overline{\mathbf{H}^\infty}$ with the weak topology (see Proposition 1.44).

Lemma 2.14 ([Duc23], Theorem 9.7)**.** If $X = \mathbf{H}^\infty$, and $(g_n)_{n\in\mathbf{N}}$ is a sequence of neutral isometries of X such that $g_n \xrightarrow[n\to\infty]{} g \in \mathrm{Isom}(X)$ for the pointwise convergence topology, then g is neutral.

Proof. We may assume, up to extracting a subsequence, that the g_n are either all elliptic or all parabolic.

If the g_n are parabolic, let $\xi_n \in \partial\mathbf{H}^\infty$ be the unique fixed point of g_n. By compactness of $\overline{\mathbf{H}^\infty}$ for the weak topology, let $(\xi_{\phi(n)})_n$ be a converging subsequence of $(\xi_n)_n$ and denote by $\eta \in \overline{\mathbf{H}^\infty}$ its weak limit. Then

$$\begin{array}{ccc} g_{\phi(n)}\xi_{\phi(n)} & \underset{n\to\infty}{\rightharpoonup} & g\eta \\ \| & & \\ \xi_{\phi(n)} & \underset{n\to\infty}{\rightharpoonup} & \eta \end{array}$$

therefore $g\eta = \eta$. If $\eta \in \mathbf{H}^\infty$, then g is elliptic and $\ell(g) = 0$. Otherwise, $\eta \in \partial\mathbf{H}^\infty$ and $(\xi_{\phi(n)})_n$ converges then strongly to η, $\xi_{\phi(n)} \xrightarrow[n\to\infty]{} \eta$. So, by continuity of the Busemann functions for the strong topology, $0 = \ell(g_{\phi(n)}) = \beta_{\xi_{\phi(n)}}(o, g_{\phi_n}o) \xrightarrow[n\to\infty]{} \beta_\eta(o, go)$, thus $\beta_\eta(o, go) = 0$ and g is then either parabolic or elliptic, *i.e.* g is neutral.

Now if the g_n are elliptic, let $x_n \in \mathbf{H}^\infty$ be a fixed point of g_n. As above, we may find a subsequence $x_{\phi(n)}$ converging weakly to a point $x \in \overline{\mathbf{H}^\infty}$. The continuity of the action for the weak topology again yields $gx = x$. If $x \in \mathbf{H}^\infty$, then g is elliptic. And if $x \in \partial\mathbf{H}^\infty$, we have

$$1 = \|x\| \leqslant \liminf_n \|x_{\phi(n)}\| \leqslant \limsup_n \|x_{\phi(n)}\| \leqslant 1$$

in the Klein model $\mathbb{B}^\infty$ of $\mathbf{H}^\infty$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{B}^\infty$. So the sequence of norms $\|x_{\phi(n)}\|$ converges to $\|x\|$, and thus $x_{\phi(n)}$ converges strongly to x. We then have

$$\begin{aligned} \beta_{x_{\phi(n)}}(o, g_{\phi(n)}o) &= d(o, x_{\phi(n)}) - d(g_{\phi(n)}o, x_{\phi(n)}) \\ &= d(o, x_{\phi(n)}) - d(o, g_{\phi(n)}^{-1}x_{\phi(n)}) \\ &= d(o, x_{\phi(n)}) - d(o, x_{\phi(n)}) = 0 \end{aligned}$$

and $\beta_{x_{\phi(n)}}(o, g_{\phi(n)}o) \xrightarrow[n\to\infty]{} \beta_x(o, go)$, thus $\beta_x(o, go) = 0$ and we conclude that g is neutral. □

Lemma 2.15. If $(g_n)_{n\in\mathbf{N}}$ and g are hyperbolic isometries of $\mathbf{H}^\infty$. Denote by $\xi_n^\pm$ the attractive and repulsive fixed points of g_n and $\xi^\pm$ those of g. If $g_n \longrightarrow g$ for the pointwise convergence topology, then $\ell(g_n) \longrightarrow \ell(g)$. Moreover, $\xi_n^+ \longrightarrow \xi^+$ and $\xi_n^- \longrightarrow \xi^-$ when n goes to ∞.

Proof. Consider the sequence $(\xi_n^+)_{n\in\mathbf{N}} \in (\partial\mathbf{H}^\infty)^\mathbf{N}$ of attractive fixed points of the isometries g_n. The bordification $\overline{\mathbf{H}^\infty}$ endowed with the topology generated by the closed and convex subsets can be identified to the closed unit ball of a Hilbert space with its weak topology. It is then compact since Hilbert spaces are reflexive. Let $(\xi_{\phi(n)}^+)_n$ be a convergent subsequence of $(\xi_n^+)_n$ and denote by $\eta \in \overline{\mathbf{H}^\infty}$ its weak limit, $\xi_{\phi(n)}^+ \underset{n\to\infty}{\rightharpoonup} \eta$. The action of $\mathrm{Isom}(\mathbf{H}^\infty)$ on $\overline{\mathbf{H}^\infty}$ is continuous for the weak topology on $\overline{\mathbf{H}^\infty}$ (see [Duc23], Proposition 6.2) and since $g^{-1}g_{\phi(n)} \longrightarrow \mathrm{id} \in \mathrm{Isom}(\mathbf{H}^\infty)$, we have

$$\begin{array}{ccc} g_{\phi(n)}\xi^+_{\phi(n)} & \underset{n\to\infty}{\rightharpoonup} & g\eta \\ \| & & \\ \xi^+_{\phi(n)} & \underset{n\to\infty}{\rightharpoonup} & \eta \end{array}$$

therefore $g\eta = \eta$, η must be a fixed point of g, which means that either $\eta = \xi^+$ or $\eta = \xi^-$.

We show that η must be ξ^+. The limit η belongs to $\partial\mathbf{H}^\infty$, thus $(\xi^+_{\phi(n)})_n$ converges strongly to η. The Busemann function $\beta_{\xi^+_{\phi(n)}}$ computes the distance between two horospheres centered at $\xi^+_{\phi(n)}$. For $n \in \mathbf{N}$, we have $\beta_{\xi^+_{\phi(n)}}(g_{\phi(n)}o, o) = \ell(g_{\phi(n)}) > 0$. By continuity of the Busemann functions for the cone topology on $\overline{\mathbf{H}^\infty}$ (Proposition I.2.40),

$$\beta_{\xi^+_{\phi(n)}}(g_{\phi(n)}o, o) \underset{n\to\infty}{\longrightarrow} \beta_\eta(go, o).$$

Thus $\beta_\eta(go, o)$ is non negative. It follows from $\beta_{\xi^-}(go, o) = -\ell(g) < 0$ that $\eta = \xi^+$.

Finally, the sequence $(\xi^+_n)_n$ in the compact set $\overline{\mathbf{H}}$ (with the weak topology) has a unique accumulation point ξ^+. It thus converges (weakly) to this accumulation point and since the limit ξ^+ lies in $\partial\mathbf{H}^\infty$, $(\xi^+_n)_n$ actually converges for the cone topology on $\overline{\mathbf{H}^\infty}$, $\xi^+_n \underset{n\to\infty}{\longrightarrow} \xi^+$. The same goes for the repulsive fixed points, $\xi^-_n \underset{n\to\infty}{\longrightarrow} \xi^-$. □

Proposition 2.16. For $X = \mathbf{H}^\infty$, the translation length is a continuous function on $\mathrm{Isom}(X)$ for the pointwise convergence topology.

Proof. Suppose that g_n converges to g in $\mathrm{Isom}(\mathbf{H}^\infty)$. If $\ell(g) = 0$, by Lemma 2.13, we have $0 \leqslant \limsup_n \ell(g_n) \leqslant \ell(g) = 0$. So $(\ell(g_n))_n$ converges and its limit is $\ell(g) = 0$.
If $\ell(g) \neq 0$, *i.e.* g is hyperbolic, we may assume that all the g_n are hyperbolic by Lemma 2.14. And by Lemma 2.15, we get $\ell(g_n) \longrightarrow \ell(g)$. □

Remark 2.17. This is not true for Euclidean spaces, even in dimension 2. Consider for example a sequence $g_n \in \mathrm{Isom}(\mathbf{R}^2)$ of rotations centered at the points $(n, 0)$. As rotations, they are elliptic isometries and their translation lengths vanish, $\ell(g_n) = 0$. They converge to the translation along the y-axis which has positive translation length.

One may also prove Proposition 2.16 by using the fact that the hyperbolic space $\mathbf{H}^\infty$ is regularly geodesic (Proposition 1.19) and considering the following property of projections onto geodesics.

Lemma 2.18. Suppose that $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ are sequences in $\overline{\mathbf{H}^\infty}$ that converge to x and y in $\overline{\mathbf{H}}$ respectively. Let $z \in [x, y]$, then $\pi_n(z) \longrightarrow z$ where π_n is the (nearest point) projection onto $[x_n, y_n]$.

Proof. By Proposition 1.19, $[x_n, y_n]$ converges to $[x, y]$ for the Hausdorff metric on $(\overline{\mathbf{H}^\infty}, \overline{D})$, *i.e.*

$$H_{\overline{D}}([x,y],[x_n,y_n]) \longrightarrow 0.$$

We have

$$\begin{aligned} H_{\overline{D}}([x,y],[x_n,y_n]) &= \max\left\{\sup_{a\in[x,y]} \overline{D}(a,[x_n,y_n]), \sup_{b\in[x_n,y_n]} \overline{D}(b,[x,y])\right\} \\ &\geqslant \overline{D}(z,[x_n,y_n]). \end{aligned}$$

Moreover, for any $z' \in \overline{\mathbf{H}}$, $(z\cdot z')_o = \dfrac{1}{2}(d(o,z)+d(o,z')-d(z,z')) \leqslant d(o,z)$ by triangle inequality, so $e^{-(z\cdot z')_o} \geqslant e^{-d(o,z)}$. Let $0<\epsilon<e^{-d(o,z)}$. For all $n\in\mathbf{N}$ large enough, there is $z_n \in [x_n, y_n]$ such that

$$\overline{D}(z,z_n) \leqslant \epsilon < e^{-d(o,z)}.$$

Hence $\overline{D}(z,z_n) = \min(d(z,z_n), e^{-(z\cdot z_n)_o}) = d(z,z_n)$ and $d(z,\pi_n(z)) \leqslant d(z,z_n) \leqslant \epsilon$. □

Fixed points

Let $g \in \mathrm{Isom}(\mathbf{H}^\infty)$, the set of fixed points of g will be denoted by

$$\mathrm{Fix}(g) = \{x \in \overline{\mathbf{H}^\infty} \mid g(x) = x\}.$$

One can obtain the same classification of the isometries (into loxodromic, parabolic and elliptic) by looking at their fixed points sets. Here is another equivalent definition for the different categories of isometries of $\mathbf{H}^\infty$.

Definition 2.19. Let $g \in \mathrm{Isom}(\mathbf{H}^\infty)$. Then g is called

- *elliptic* if for any (hence for all) $x \in \mathbf{H}^\infty$, the orbit $\{g^n(x) \mid n \in \mathbf{N}\}$ is bounded;
- *parabolic* if it is not elliptic and has a unique fixed point in $\partial\mathbf{H}^\infty$, *i.e.* $\#\mathrm{Fix}(g) = 1$ and $\mathrm{Fix}(g) \cap \mathbf{H}^\infty = \emptyset$;
- *loxodromic* if it has exactly two fixed points in $\partial\mathbf{H}^\infty$ (one attractive and one repulsive), *i.e.* $\#\mathrm{Fix}(g) = 2$ and $\mathrm{Fix}(g) \cap \mathbf{H}^\infty = \emptyset$.

Remark 2.20. With this definition, we see thanks to the existence of fixed points in CAT(0) spaces (Proposition I.2.51), that an elliptic isometry necessarily has a fixed point in $\mathbf{H}^\infty$ (see also [DSU17, Theorem 6.2.5]). Then, an isometry g is elliptic if and only if $\mathrm{Fix}(g) \cap \mathbf{H}^\infty \neq \emptyset$.

2.3 A matrix representation of the isometries: Clifford matrices

For the hyperbolic plane $\mathbf{H}^2$ and the hyperbolic 3-space $\mathbf{H}^3$, the orientation-preserving isometries can be represented as 2×2 matrices by the following very well-known isomorphisms:

$$\begin{aligned} &\mathrm{Isom}^+(\mathbf{H}^2) \simeq \mathrm{PSL}(2, \mathbf{R}) \\ \text{and} \quad &\mathrm{Isom}^+(\mathbf{H}^3) \simeq \mathrm{PSL}(2, \mathbf{C}). \end{aligned}$$

For any $n \in \mathbf{N}$, one can actually show that $\mathrm{Isom}^+(\mathbf{H}^n)$ is isomorphic to some group of 2×2 matrices whose coefficients lie in some group G^n, $\mathrm{Isom}^+(\mathbf{H}^n) \simeq \mathrm{PSL}_2(G^n)$. These groups G^n are called *Clifford groups*, and we have $G^2 = \mathbf{R}$ and $G^3 = \mathbf{C}$.

Notation. In order to avoid confusions when dealing with 2×2 matrices with coefficients in a Clifford group, we will use the notations GL_2, SL_2, PSL_2 with the 2 as a subscript instead of $\mathrm{SL}(2, \mathbf{R})$ for example.

For finite dimension, the theory of Clifford algebras and Clifford groups to describe Möbius transformations in $\mathbf{R}^n$ has been well studied in [Ahl85, Wat93]. The case of Clifford algebras of countable infinite dimension has also been considered, see [Fru91]. Several properties of Clifford matrices have been established in the series of papers including [Fru91, LW05, Li09, Li11, Li12, Fu13, Li13, Gon18].

We expose here the approach of [Gug17] which holds more generally for any dimension (finite, countable and also uncountable), although we are only interested in the case of separable Hilbert and hyperbolic spaces. Although Clifford matrices will not be used in the main sections of this thesis, we wish to present some results about the topology of this group. In particular, discreteness properties of a subgroup of Clifford matrices have been investigated in [Li11, Li13, Gon18] and those can be related to the notions that we define in the next section.

Clifford algebras and Clifford groups

Let (V, Q) be a quadratic space (V is a real Hilbert space and Q a quadratic form on V). All the following can be defined for a quadratic space over any field $\mathbf{K}$ (of any characteristic), however we will only be interested in the case $\mathbf{K} = \mathbf{R}$. The algebras will be taken over the real numbers.

Definition 2.21. A *Clifford algebra* associated to (V, Q) is a unitary associative algebra over $\mathbf{R}$, denoted by $\mathrm{Cl}(V, Q)$, together with a $\mathbf{R}$-linear map $i : V \to \mathrm{Cl}(V, Q)$ satisfying the two following conditions:

1. for all $v \in V$, $i(v)^2 = -Q(v) \cdot 1_{\mathrm{Cl}(V,Q)}$ (*Clifford identity*);

2. if A is another unitary associative algebra over $\mathbf{R}$ with a map $i_A : V \to A$ satisfying the Clifford identity, then there exists a unique morphisms of algebras $\psi : \mathrm{Cl}(V,Q) \to A$ such that the diagram commutes (*universal property of Clifford*

$$\begin{tikzcd} V \arrow[r, "i_A"] \arrow[d, "i"'] & A \\ \mathrm{Cl}(V,Q) \arrow[ur, "\psi"'] & \end{tikzcd}$$

algebras).

The universal property implies that if the Clifford algebra exists, then it is unique up to isomorphism. There is an explicit construction for the Clifford algebra associated to (V,Q):

Let $T(V) = \bigoplus_{n\in\mathbf{N}} V^{\otimes n}$ be the tensor algebra of V, where $V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$ (n times) and $V^{\otimes 0} = \mathbf{R}$. Consider the ideal I of $T(V)$ generated by all the elements of the form $v^2 + Q(v)$, for $v \in V$. Then the quotient $T(V)/I$ satisfies the required universal property, therefore $\mathrm{Cl}(V,Q) \simeq T(V)/I$. Moreover, the map $i : V \hookrightarrow T(V) \twoheadrightarrow T(V)/I$ is injective, we will identify the vectors $v \in V$ with their images $i(v) \in \mathrm{Cl}(V,Q)$ and we will denote $i(v)$ by v.

It follows from the universal property that any morphism between quadratic spaces (linear morphism preserving the quadratic form) gives rise to a morphism of algebras between the associated Clifford algebras (in fact, Cl is a functor from the category of quadratic spaces over some field $\mathbf{K}$ to the category of unitary associative algebras over $\mathbf{K}$).

One can define three involutions on $\mathrm{Cl}(V,Q)$ as follows:

- The automorphism of (V,Q) sending a vector v to $-v$ induces an automorphism $' : \mathrm{Cl}(V,Q) \to \mathrm{Cl}(V,Q)$ such that for any element of the form $v_1 \cdots v_m$,
$$(v_1 \cdots v_m)' = (-1)^m v_1 \cdots v_m.$$
- The inclusion of $\mathrm{Cl}(V,Q)$ in its opposite algebra $\mathrm{Cl}(V,Q)^{\mathrm{op}}$ induces an anti-automorphism $^* : \mathrm{Cl}(V,Q) \to \mathrm{Cl}(V,Q)$ such that
$$(v_1 \cdots v_m)^* = v_m \cdots v_1.$$
- Observing that the two previous maps commute, the last involution is their composition, $\bar{\cdot} : \mathrm{Cl}(V,Q) \to \mathrm{Cl}(V,Q)$, defined by
$$\bar{a} = (a')^* = (a^*)',$$
for all $a \in \mathrm{Cl}(V,Q)$. It is also an anti-automorphism.

Example 2.22 (Standard Clifford algebras)**.** For $n \in \mathbf{N}$, let $V_n = \mathbf{R}^n$ with the usual scalar product $\langle \cdot, \cdot \rangle$. In an orthonormal basis $\{e_1, \dots, e_n\}$ of V_n, we can write the quadratic form $Q_n = \| \cdot \|^2$ associated to $\langle \cdot, \cdot \rangle$ as the identity matrix

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

Since $Q_n(e_i) = 1$ for all $1 \leqslant i \leqslant n$ and $\langle e_i, e_j \rangle = 0$ when $i \neq j$, we have, $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ ($i \neq j$), in $\mathrm{Cl}(V_n, Q_n)$. A basis of $\mathrm{Cl}(V_n, Q_n)$ is given by the collection

$$\{e_{i_1} \cdots e_{i_k} \mid k \in [\![0, n]\!],\ 1 \leqslant i_1 < \cdots < i_k \leqslant n\},$$

therefore $\dim_{\mathbf{R}}(\mathrm{Cl}(V_n, Q_n)) = 2^{\dim_{\mathbf{R}}(V_n)} = 2^n$. We denote this Clifford algebra by Cl_n and for $n = 0, 1$ and 2, we see that $\mathrm{Cl}_0 = \mathbf{R}$, $\mathrm{Cl}_1 = \mathbf{C}$ and Cl_2 is the field of quaternionic numbers, with their usual basis $\{1\}$, $\{1, i\}$ and $\{1, i, j, ij\}$. Moreover, the anti-automorphism $\bar{\cdot}$ coincide with the conjugation for the complex and quaternionic numbers.

These examples are called the *standard Clifford algebras*.

The quadratic form Q on V can be extended to the Clifford algebra $\mathrm{Cl}(V, Q)$ as follows: denote by $\mathrm{Re} : \mathrm{Cl}(V, Q) \to \mathbf{R}$ the linear map induced by the projection $T(V) = \bigoplus_{n \in \mathbf{N}} V^{\otimes n} \to V^{\otimes 0} = \mathbf{R}$. For $x \in \mathrm{Cl}(V, Q)$, $\mathrm{Re}(x)$ will be called the *real part* of x (and $\mathrm{Im}(x) = x - \mathrm{Re}(x)$ will be called the *imaginary part* of x). This map defines a quadratic form on $\mathrm{Cl}(V, Q)$, denoted by Q as well:

$$\begin{array}{rccc} Q: & \mathrm{Cl}(V, Q) & \to & \mathbf{R} \\ & x & \mapsto & \mathrm{Re}(\overline{x} x). \end{array}$$

When restricted to V, the quadratic form coincides with the initial one, so the notation is unambiguous. Notice that the maps Re and Im coincide with the "real" and "imaginary" parts of a complex or quaternionic number, when $\mathrm{Cl}(V, Q) = \mathrm{Cl}_1$ or Cl_2.

The associated bilinear form is

$$\begin{array}{rccc} B: & \mathrm{Cl}(V, Q) \times \mathrm{Cl}(V, Q) & \to & \mathbf{R} \\ & (x, y) & \mapsto & \frac{1}{2}\mathrm{Re}(\overline{x} y + \overline{y} x) = \mathrm{Re}(\overline{x} y). \end{array}$$

Remark 2.23. This quadratic form naturally endows $\mathrm{Cl}(V, Q)$ with a norm $| \cdot |$ defined by $|x|^2 = Q(x)$.

Definition 2.24. The elements of the subspace $V_{ext} = \mathbf{R} \oplus V \subset \mathrm{Cl}(V, Q)$ are called *vectors*.

If $x \in \mathrm{Cl}(V, Q)$ is a vector, then $x^* = x$ and $x' = \overline{x}$.

Proposition 2.25 ([Gug17], Propositions 8.1.4 and 8.1.7)**.**
1. If $x, y \in V_{ext} = \mathbf{R} \oplus V$, then $x\overline{y} + y\overline{x} = B(x,y)$, and $x\overline{x} = Q(x)$.
2. The quadratic form on $\mathrm{Cl}(V,Q)$ is multiplicative when restricted to the vectors, *i.e.* for all $x_1, \ldots, x_k \in V_{ext}$,
$$Q(x_1 \cdots x_k) = Q(x_1) \cdots Q(x_k).$$
3. If $x \in \mathrm{Cl}(V,Q)$ is a product of vectors, then x is invertible if and only if $Q(x) \neq 0$. Its inverse is then given by
$$x^{-1} = \frac{1}{Q(x)}\overline{x}.$$
In particular, the inverse of a product of vectors is also a product of vectors.

It follows from the third point of Proposition 2.25 that the invertible vectors in a Clifford algebra generate a subgroup of $\mathrm{Cl}(V,Q)$.

Definition 2.26. The group consisting of all products of invertible vectors is called the *Clifford group* of $\mathrm{Cl}(V,Q)$. We denote it by $G_{\mathrm{Cl}(V,Q)}$.

Example 2.27. For the standard Clifford algebras of Example 2.22, the quadratic form is anisotropic (there is no non-zero element $x \in \mathrm{Cl}(V,Q)$ such that $Q(x) = 0$), so the Clifford group consists of all the products of non-zero vectors. We will denote these Clifford groups by G_n.

We have $G_0 = \mathbf{R}^*$, $G_1 = \mathbf{C}^*$ and G_2 consists of the non-zero quaternionic numbers. However for $n \geqslant 3$, it is not true that $G_n = \mathrm{Cl}_n \setminus \{0\}$.

Clifford matrices

Let $\mathcal{M}_2\left(G_{\mathrm{Cl}(V,Q)} \cup \{0\}\right)$ be the set of 2×2 matrices with coefficients in $G_{\mathrm{Cl}(V,Q)} \cup \{0\}$. Note that $\mathcal{M}_2\left(G_{\mathrm{Cl}(V,Q)} \cup \{0\}\right)$ is not a group since $G_{\mathrm{Cl}(V,Q)} \cup \{0\}$ is not stable under addition. We will define the *Clifford matrices* as a subset of $\mathcal{M}_2\left(G_{\mathrm{Cl}(V,Q)} \cup \{0\}\right)$ that will indeed be a group. An element $g \in \mathcal{M}_2\left(G_{\mathrm{Cl}(V,Q)} \cup \{0\}\right)$ is of the form $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This matrix is made to act on the vectors $V_{ext} = \mathbf{R} \oplus V$ according to the following formula which is reminiscent of the Möbius transformations of the complex plane:
$$g(x) = (ax+b)(cx+d)^{-1}.$$
However, for such a definition to make sense, we need to extend the set of vectors by adding an element denoted by ∞ as for the case of the complex plane. In particular, when

the quadratic form Q on V is positive-definite as for the standard Clifford algebras, the term $cx+d$ is invertible whenever it is non-zero by the third point of Proposition 2.25 and if $cx+d=0$, $(ax+b)(cx+d)^{-1}$ should be equal to ∞. Following the notations in [Gug17], we thus introduce $\widehat{V}_{ext} = V_{ext} \cup \{\infty\} = (\mathbf{R} \oplus V) \cup \{\infty\}$ whose elements are called *extended vectors* and we look for matrices g that induce a bijective map on $\widehat{V}_{ext}$. Observe that if $g_1 = \lambda g_2$ where $\lambda \in \mathbf{R}^*$, then g_1 and g_2 induce the same map on $\widehat{V}_{ext}$.

Proposition 2.28 ([Gug17], Proposition 8.2.4)**.** Let $g, h \in \mathcal{M}_2\left(G_{\mathrm{Cl}(V,Q)} \cup \{0\}\right)$. The composition of the two corresponding maps on $\widehat{V}_{ext}$ is induced by the product gh of the two matrices.

Under certain conditions, g induces a bijection on the set of extended vectors $\widehat{V}_{ext} = V_{ext} \cup \{\infty\}$ (see for example [Gug17, Section 8.2] or [Ahl85, Section 2.2]). These conditions lead to the following definition.

Definition 2.29. Define the set of *Clifford matrices* to be the subset of $\mathcal{M}_2\left(G_{\mathrm{Cl}(V,Q)} \cup \{0\}\right)$ consisting of all the elements $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that:

- $\Delta(g) = ad^* - bc^* \in \mathbf{R}^*$, $\Delta(g)$ is called the *determinant* of g;
- $ab^*,\ cd^*,\ c^*a,\ d^*b \in V_{ext}$.

We denote the set of Clifford matrices by $\mathrm{GL}_2\left(G_{\mathrm{Cl}(V,Q)}\right)$.

Define also $\mathrm{SL}_2\left(G_{\mathrm{Cl}(V,Q)}\right) = \left\{g \in \mathrm{GL}_2\left(G_{\mathrm{Cl}(V,Q)}\right) \mid \Delta(g) = 1\right\}$, and

$$\mathrm{PSL}_2\left(G_{\mathrm{Cl}(V,Q)}\right) = {\mathrm{SL}_2\left(G_{\mathrm{Cl}(V,Q)}\right)}\big/{\{\pm\,\mathrm{id}\}}.$$

The *trace* of a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\mathrm{tr}(g) = a + d^*$. There is a natural norm on Clifford matrices given by the norm on Clifford algebras (see Remark 2.23). If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a Clifford matrix, we can define the norm of g by

$$\|g\|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2.$$

We call it the *Hilbert-Schmidt norm* by analogy with the usual norm on matrices.

Theorem 2.30 ([Gug17], Theorems 8.2.11 and 8.2.12)**.**
1. If Cl_n is a standard Clifford algebra as in Example 2.22, then $g \in \mathcal{M}_2(G_n \cup \{0\})$ induces a bijection on $\widehat{\mathbf{R}}^{n-1}_{ext}$ and on $\widehat{\mathbf{R}}^{n}_{ext}$ if and only if $g \in \mathrm{GL}_2(G_n)$, *i.e.* g is a Clifford matrix.

2. If Cl_∞ is a standard Clifford algebra associated to an infinite-dimensional Hilbert space $\mathcal{H}$, then $g \in \mathcal{M}_2\left(G_\infty \cup \{0\}\right)$ induces a bijection on $\widehat{\mathcal{H}}_{ext}$ if and only if $g \in \mathrm{GL}_2\left(G_\infty\right)$.

It was observed in [Gug17, Remark 8.1.8] that Theorem 2.30 implies that for standard Clifford algebras of finite or infinite dimension, the set of Clifford matrices is a group. This was proved in the finite-dimensional case in [Ahl85, Section 2.6].

If $\mathcal{H}$ is a (finite- or infinite-dimensional) Hilbert space, denote by $\mathcal{M}\ddot{o}b(\mathcal{H})$ the group of all Möbius transformations of $\mathcal{H}$ and by $\mathcal{M}\ddot{o}b^*(\mathcal{H})$ the group of transformations that can be written as a finite composition of reflections in spheres in $\mathcal{H}$.

Proposition 2.31 ([Gug17], proposition 8.2.17)**.** Let $\mathcal{H}$ be a Hilbert space (of finite or infinite dimension) and consider a codimension one subspace $\mathcal{H}'$ of $\mathcal{H}$. If we let G' be the Clifford group associated to $\mathrm{Cl}(\mathcal{H}', Q')$ where Q' is the restriction of $\|\cdot\|^2$ to $\mathcal{H}'$, then we get

$$\mathrm{GL}_2(G')\big/_{\mathbf{R}^*} \simeq \mathcal{M}\ddot{o}b^*(\mathcal{H}).$$

Note that for any finite-dimensional vector space $V \in \mathcal{V}$, we have $\mathcal{M}\ddot{o}b^*(V) = \mathcal{M}\ddot{o}b(V)$, *i.e.* any Möbius transformation decomposes as a composition of a finite number of reflections in spheres or hyperplanes.

Example 2.32.
- For $\mathcal{H} = \mathbf{R}$, we recover the fact that $\mathcal{M}\ddot{o}b(\mathbf{R}) \simeq \mathrm{GL}_2(\mathbf{R})/\mathbf{R}^* \simeq \mathrm{PSL}(2, \mathbf{R})$.
- Similarly, for $\mathcal{H} = \mathbf{R}^2$, we get $\mathcal{M}\ddot{o}b(\mathbf{R}^2) \simeq \mathrm{GL}_2(\mathbf{C})/\mathbf{R}^* \simeq \mathrm{PSL}(2, \mathbf{C})$.

Remark 2.33. The statements of Theorem 2.30 and Proposition 2.31 concerning finite-dimensional Hilbert spaces were first proved in [Ahl85, Theorems A and B].

Proposition 2.34 ([Gug17], Proposition 8.2.18)**.** Let $\mathcal{H}$ be a Hilbert space, denote by $\mathcal{V}$ the set of all finite-dimensional subspaces of $\mathcal{H}$. For every $V \in \mathcal{V}$, let $\mathrm{GL}_2(G_V)$ the group of Clifford matrices corresponding to the quadratic space $(V, \|\cdot\|^2)$. Then, we have

$$\varinjlim_{V \in \mathcal{V}} \mathrm{GL}_2(G_V) \simeq \mathrm{GL}_2(G),$$

where $\mathrm{GL}_2(G)$ is the group of Clifford matrices corresponding to $(\mathcal{H}, \|\cdot\|^2)$.

Action on the hyperbolic space

Let $\mathcal{H}$ be a Hilbert space of any dimension $\alpha \geqslant 2$ and decompose it as

$$\mathcal{H} = (\mathbf{R}\cdot u) \oplus V \oplus (\mathbf{R}\cdot x_0) \simeq (\mathbf{R}\cdot u) \oplus V_{ext},$$

where u and x_0 are vectors in $\mathcal{H}$. Let $G = G_{\mathrm{Cl}(V,\|\cdot\|^2)}$. Then all the elements of $\mathrm{GL}_2(G)$ induce bijections on $\widehat{V}_{ext} = V_{ext} \cup \{\infty\}$ by Theorem 2.30. Thus, they also act on $\widehat{\mathcal{H}} = \mathcal{H} \cup \{\infty\}$.

Consider the upper half-space model for $\mathbf{H}^\alpha$, $\mathbb{H}^\alpha = \{x \in \mathcal{H} \mid h_u(x) > 0\}$, where $h_u(x)$ denotes the component of x in u (the "height" of x with respect to u).

One can show that matrices in $\mathrm{GL}_2(G)$ with positive determinant preserve the upper half-space $\mathbb{H}^\alpha$.

Proposition 2.35 ([Gug17], Proposition 8.3.2 and Theorem 8.3.3)**.**

1. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(G)$ such that $\Delta(g) > 0$, then g preserve $\mathbb{H}^\alpha$.
2. The group $\mathrm{SL}_2(G)$ acts by isometries on $\mathbb{H}^\alpha$ via Poincaré extension (see Proposition 2.6).

Remark 2.36. By Proposition 2.34, we see that the isometries of $\mathbb{H}^\infty$ that one can write as a Clifford matrix are actually the ones coming from the Poincaré extension of some Clifford matrix in $\mathrm{PSL}_2(V)$, where V is finite-dimensional. Therefore this proposition shows that $\mathrm{PSL}_2(G)$ is equal to the increasing union $\bigcup_{n\in\mathbf{N}} \mathrm{Isom}^+(\mathbf{H}^n)$.

In particular, a Clifford matrix acting on $\mathbf{H}^\infty$ is actually an isometry preserving a finite-dimensional hyperbolic subspace $\mathbf{H}^n$ and acting trivially on the orthogonal of $\mathbf{H}^n$.

Similarly as in dimension 2 and 3, we have the following description, in any dimension, of loxodromic and parabolic isometries in terms of 2×2 matrices.

Proposition 2.37 ([Gug17], Proposition 8.3.5 and 8.3.6)**.** Let $g \in \mathrm{SL}_2(G)$ be a Clifford matrix. Then g is

1. loxodromic if and only if g is conjugate to a Clifford matrix of the type $\begin{pmatrix} ra & 0 \\ 0 & \frac{1}{r}a' \end{pmatrix}$, where $r \in \mathbf{R}_+^*$, $r \neq 1$ and $a \in G$ with $Q(a) = 1$;
2. parabolic if and only if g is conjugate to a Clifford matrix of the type $\begin{pmatrix} a & b \\ 0 & a' \end{pmatrix}$, where $a \in G$ with $Q(a) = 1$ and $b \in V_{ext} = V \oplus \mathbf{R}\cdot x_0$ is such that $ab = ba'$.

3 Discrete groups of isometries

There are several equivalent ways to define discreteness for subgroups of isometries of finite-dimensional hyperbolic spaces. However, they do not agree in general for subgroups in $\mathrm{Isom}(\mathbf{H}^\infty)$.

Let $\Gamma < \mathrm{Isom}(\mathbf{H}^\infty)$ be a subgroup. We can say that Γ is discrete

- either as a discrete subset of $\mathrm{Isom}(\mathbf{H}^\infty)$ endowed with some topology;
- or if it acts on $\mathbf{H}^\infty$ in a discrete fashion.

3.1 Topologies on $\mathrm{Isom}(\mathbf{H}^\infty)$

The most natural topology on $\mathrm{Isom}(\mathbf{H}^\infty)$ is the *compact-open topology*. This is the topology whose basis of open sets consists of all the sets

$$\{f \in \mathrm{Isom}(\mathbf{H}^\infty) \mid f(K) \subset U\}$$

where $K \subset \mathbf{H}^\infty$ is compact, and $U \subset \mathbf{H}^\infty$ is open.

Since $\mathrm{Isom}(\mathbf{H}^\infty)$ is a set of maps from $\mathbf{H}^\infty$ to itself, we can also define the *topology of pointwise convergence* in $\mathbf{H}^\infty$ (which was already used in Proposition 1.44 for example). In this topology, every isometry $f \in \mathrm{Isom}(\mathbf{H}^\infty)$ has a basis of neighbourhoods consisting of all the sets of the form

$$V(f, \epsilon, P) = \{g \in \mathrm{Isom}(X) \mid \forall x \in P \quad d(f(x), g(x)) < \epsilon\}$$

where P is a finite set of points in X and $\epsilon > 0$.

Moreover, since we have $\mathrm{Isom}(\mathbf{H}^\infty) = \mathrm{PO}(\infty, 1)$, this group also inherits topologies from $\mathrm{O}(\infty, 1)$ among which we will be interested in the two following.

- The *strong operator topology*, defined as the coarsest topology such that for all $x \in \mathcal{H}$, the evaluation map $T \mapsto T(x)$ is continuous.
  We may also define this topology by its converging sequences:

$$T_n \underset{n\to\infty}{\longrightarrow} T \Leftrightarrow \forall x \in \mathcal{H} \quad T_n(x) \underset{n\to\infty}{\longrightarrow} T(x).$$

- The *uniform operator topology* which is the topology of the operator norm,

$$\|T\|_{\mathrm{op}} = \sup\{\|T(x)\| \mid x \in \mathcal{H}, \|x\| = 1\}.$$

  Let us denote the induced topology on $\mathrm{Isom}(\mathbf{H}^\infty)$ by UOT.

Remark 3.1. The uniform operator topology is finer than the strong operator topology.

Proposition 3.2 ([DSU17], Proposition 5.1.2)**.** On $\mathrm{Isom}(\mathbf{H}^\infty)$, the topology of pointwise convergence, the strong operator topology and the compact-open topology are the same.

We will refer to this topology as the compact-open topology and denote it by COT. We are now left with two topologies on $\mathrm{Isom}(\mathbf{H}^\infty)$, namely COT and UOT.

Topological properties of this group were studied by Duchesne in [Duc23].

Definition 3.3. A topological group G is *Polish* if it is separable and completely metrisable, *i.e.* there is a metric which is compatible with the topology of G and for which G is complete.

Endowed with the compact-open topology, $\mathrm{Isom}(\mathbf{H}^\infty)$ is a Polish group. Moreover, Duchesne proved the following.

Theorem 3.4 ([Duc23], Corollary 1.7)**.** The Polish topology (COT) is the unique separable and Hausdorff group topology on $\mathrm{Isom}(\mathbf{H}^\infty)$.

If $\Gamma < \mathrm{Isom}(\mathbf{H}^\infty)$ is a discrete subgroup for the compact-open topology on $\mathrm{Isom}(\mathbf{H}^\infty)$ (respectively the uniform operator toplogy), keeping the notations in [DSU17], we will denote this by COT-*discrete* or COTD (resp. UOT-*discrete* or UOTD).

The topology on Clifford matrices

Recall the notations used for defining the action of Clifford matrices on hyperbolic spaces: let $\mathcal{H}$ be a Hilbert space and decompose it as

$$\mathcal{H} = (\mathbf{R}\cdot u) \oplus V \oplus (\mathbf{R}\cdot x_0) \simeq (\mathbf{R}\cdot u) \oplus V_{ext},$$

where u and x_0 are vectors in $\mathcal{H}$. Let $\mathbb{H}^\infty = \{x \in \mathcal{H} \mid h_u(x) > 0\}$ be the half-space model lying in $\mathcal{H} = (\mathbf{R}\cdot u) \oplus V \oplus (\mathbf{R}\cdot x_0)$. Let G be the Clifford group associated to $\mathrm{Cl}(V, \|\cdot\|^2)$ as before, we have endowed the set of Clifford matrices $\mathrm{PSL}_2(G)$ with the topology of the Hilbert-Schmidt norm. We can view this group as a subgroup of $\mathrm{Isom}(\mathbf{H}^\infty)$ by Proposition 2.35. We now show that the topology induced by the COT on $\mathrm{PSL}(G)$ coincides with the Hilbert-Schmidt topology of Clifford algebras.

We start with some computational lemmas.

Lemma 3.5. For all $x \in V_{ext}$ and $y \in G$, we have

1. $\overline{xu} = -xu$;
2. $u\bar{y} = y^* u$;
3. $y^* y' = \bar{y} y = |y|^2$.

Proof. See [Gug17, Lemma 8.3.1] for the first two points. For the last point, observe that $(y^*y')' = \bar{y}y = |y|^2 \in \mathbf{R}$, so $y^*y' = (|y|^2)' = |y|^2$. □

Lemma 3.6. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(G)$, then

1. $1 + |a\bar{c} + b\bar{d}|^2 = \left(|a|^2 + |b|^2\right)\left(|c|^2 + |d|^2\right)$;
2. $|g(u)|^2 = \dfrac{|a|^2 + |b|^2}{|c|^2 + |d|^2}$.

Proof. Since $g \in \mathrm{SL}_2(G)$, we have $\Delta(g) = ad^* - bc^* = 1$. Therefore

$$\begin{aligned}
|a\bar{c} + b\bar{d}|^2 + |ad^* - bc^*|^2 &= \mathrm{Re}((a\bar{c} + b\bar{d})(c\bar{a} + d\bar{b})) + \mathrm{Re}((ad^* - bc^*)(d'\bar{a} - c'\bar{b})) \\
&= |a|^2|c|^2 + |b|^2|d|^2 + \mathrm{Re}(a\bar{c}d\bar{b} + b\bar{d}c\bar{a}) + |a|^2|d|^2 + |b|^2|c|^2 \\
&\qquad - \mathrm{Re}(ad^*c'\bar{b} + bc^*d'\bar{a}) \\
&= (|a|^2 + |b|^2)(|c|^2 + |d|^2) + \mathrm{Re}(a\bar{c}d\bar{b} - a(\bar{c}d)^*\bar{b}) \\
&\qquad + \mathrm{Re}(b\bar{d}c\bar{a} - b(\bar{d}c)^*\bar{a})
\end{aligned}$$

Since $\mathrm{Re}(\bar{c}d) = \mathrm{Re}((\bar{c}d)^*)$, we have $\mathrm{Re}(a\bar{c}d\bar{b} - a(\bar{c}d)^*\bar{b}) = 0 = \mathrm{Re}(b\bar{d}c\bar{a} - b(\bar{d}c)^*\bar{a})$, which concludes the first point.

For the second part, we compute

$$\begin{aligned}
g(u) &= (au + b)(cu + d)^{-1} = (au + b)(\overline{cu + d})(\overline{cu + d})^{-1}(cu + d)^{-1} \\
&= (au + b)(-u\bar{c} + \bar{d})((cu + d)(\overline{cu + d}))^{-1} \\
&= (-a\bar{c}u^2 + au\bar{d} - bu\bar{c} + b\bar{d})(|c|^2 + |d|^2)^{-1} \\
&= (a\bar{c} + b\bar{d} + (ad^* - bc^*)u)(|c|^2 + |d|^2)^{-1} \\
&= (a\bar{c} + b\bar{d} + u)(|c|^2 + |d|^2)^{-1} \\
&= v + wu,
\end{aligned}$$

where $v = (a\bar{c} + b\bar{d})(|c|^2 + |d|^2)^{-1}$ and $w = (|c|^2 + |d|^2)^{-1}$. We get

$$|g(u)|^2 = \frac{|a\bar{c} + b\bar{d}|^2}{(|c|^2 + |d|^2)^2} + \frac{1}{(|c|^2 + |d|^2)^2} = \frac{|a\bar{c} + b\bar{d}|^2 + |ad^* - bc^*|^2}{(|c|^2 + |d|^2)^2}.$$

It follows that

$$|g(u)|^2 = \frac{\left(|a|^2 + |b|^2\right)\left(|c|^2 + |d|^2\right)}{(|c|^2 + |d|^2)^2} = \frac{|a|^2 + |b|^2}{|c|^2 + |d|^2}.$$

□

Proposition 3.7. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(G)$, then $\|g\|^2 = 2\cosh d(u, g(u))$.

Proof. The computation of the previous lemma gives $g(u) = v + wu$, where $v = (a\bar{c} + b\bar{d})(|c|^2 + |d|^2)^{-1}$ and $w = (|c|^2 + |d|^2)^{-1}$.

By Proposition 1.34, the hyperbolic distance $d(u, g(u)) = d_{\mathbb{H}}(u, g(u))$ is

$$\cosh d(u, g(u)) = 1 + \frac{|v|^2 + |1 - w|^2}{2w}.$$

Therefore we have

$$
\begin{aligned}
\cosh d(u, g(u)) &= 1 + \frac{|c|^2 + |d|^2}{2\Delta(g)} \left(\frac{|a\bar{c} + b\bar{d}|^2 + (|c|^2 + |d|^2 - \Delta(g))^2}{(|c|^2 + |d|^2)^2} \right) \\
&= \frac{1}{2(|c|^2 + |d|^2)} (2(|c|^2 + |d|^2) + |a\bar{c} + b\bar{d}|^2 + (|c|^2 + |d|^2)^2 \\
&\qquad - 2(|c|^2 + |d|^2) + 1) \\
&= \frac{1}{2(|c|^2 + |d|^2)} (|a\bar{c} + b\bar{d}|^2 + (|c|^2 + |d|^2)^2 + 1)
\end{aligned}
$$

Using the identity of Lemma 3.6,

$$
\begin{aligned}
|a\bar{c} + b\bar{d}|^2 + 1 &= |a\bar{c} + b\bar{d}|^2 + |ad^* - bc^*|^2 \\
&= (|a|^2 + |b|^2)(|c|^2 + |d|^2)
\end{aligned}
$$

we get $\cosh d(u, g(u)) = \dfrac{(|c|^2 + |d|^2)(|a|^2 + |b|^2 + |c|^2 + |d|^2)}{2(|c|^2 + |d|^2)} = \dfrac{\|g\|^2}{2}$. □

Proposition 3.8. The norm topology on Clifford matrices coincides with the topology of pointwise convergence on $\mathcal{M}\ddot{o}b^*(\mathcal{H})$ inherited by $\mathrm{Isom}(\mathbf{H}^\infty)$.

Proof. Since both topologies are metric (the compact-open topology being Polish), it suffices to show that they share the same convergent sequences.

The topology on Clifford matrices is finer than that of simple convergence. Indeed, if $(g_n)_n$ is a converging sequence of Clifford matrices, then all the coefficients of g_n converge and the induced maps in $\mathcal{M}\ddot{o}b^*(\mathcal{H})$ converge as well.

Now let $g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \mathrm{PSL}_2(G)$ such that for all $x \in \mathcal{H}_{ext}$, $g_n x \underset{n\to\infty}{\longrightarrow} x$. By Proposition 3.7, we have $\|g_n\|^2 = |a_n|^2 + |b_n|^2 + |c_n|^2 + |d_n|^2 = 2\cosh d(u, g_n(u)) \longrightarrow 2$ as $n \to \infty$. Therefore, the sequences of real numbers $(|a_n|)_n$, $(|b_n|)_n$, $(|c_n|)_n$ and $(|d_n|)_n$ are bounded.

Computing $g_n(\infty) = a_n c_n^{-1} \underset{n\to\infty}{\longrightarrow} \infty$ where $(a_n)_n$ is bounded, we deduce that $c_n \underset{n\to\infty}{\longrightarrow} 0$. Similarly, looking at $g_n(0)$, we get $b_n \underset{n\to\infty}{\longrightarrow} 0$.

Since $\Delta(g_n) = a_n d_n^* - b_n c_n^* = 1$ with $b_n c_n^* \underset{n\to\infty}{\longrightarrow} 0$, we may assume that $a_n \neq 0$ and $d_n \neq 0$ for all $n \in \mathbf{N}$, and we may also suppose that $a_n = 1$. Then, computing again $g_n(x)$ for any $x \in \mathbb{H}^\infty$, we get $d_n \underset{n\to\infty}{\longrightarrow} 1$, *i.e.* $g_n \underset{n\to\infty}{\longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. □

While the theory of Clifford matrices acting on hyperbolic spaces by Poincaré extension seems promising as all the formulas are similar to the 2- and 3-dimensional cases, computations in Clifford algebras are rather difficult because of the lack of commutativity.

3.2 Discrete group actions

For any metric space X, using the action of $\mathrm{Isom}(X)$ on X, one has other ways to define discreteness. We will again use the same terminology and notations as in [DSU17].

Definition 3.9 ([DSU17], 5.2.1 and 5.2.3)**.** Let X be a metric space and $\Gamma < \mathrm{Isom}(X)$.

- We say that Γ is *strongly discrete* (SD) if for every bounded set $B \subset X$, we have

$$\#\{\gamma \in \Gamma \mid \gamma \cdot B \cap B \neq \emptyset\} < \infty.$$

  Equivalently, Γ is strongly discrete if for every $x \in X$ and $R > 0$,

$$\#\{\gamma \in \Gamma \mid d(x, \gamma \cdot x) \leqslant R\} < \infty.$$

- We say that Γ is *moderately discrete* (MD) if for every $x \in X$, there exists an open set $U \subset X$ containing x such that

$$\#\{\gamma \in \Gamma \mid \gamma \cdot U \cap U \neq \emptyset\} < \infty.$$

  Equivalently, Γ is moderately discrete if for every $x \in X$, there exists $R > 0$ such that

$$\#\{\gamma \in \Gamma \mid d(x, \gamma \cdot x) \leqslant R\} < \infty.$$

- We say that Γ is *weakly discrete* (WD) if for every $x \in X$, there exists $\epsilon > 0$ such that

$$\Gamma \cdot x \cap B(x, \epsilon) = \{x\}.$$

Remark 3.10. Note that (SD) is also known as *metrically proper*, (MD) as *wandering* and that (WD) means having discrete orbits.

Proposition 3.11 ([DSU17], Propositions 5.2.4 and 5.2.7)**.** For $X = \mathbf{H}^\infty$, let $\Gamma < \mathrm{Isom}(\mathbf{H}^\infty)$. We have the following chain of implications

$$\mathrm{SD} \Rightarrow \mathrm{MD} \Rightarrow \mathrm{WD} \Rightarrow \mathrm{COTD} \Rightarrow \mathrm{UOTD},$$

where none of the reverse implications holds.

More details can be found in [DSU17, Chapter 5]. In particular, the authors discuss these definitions for other spaces than merely $\mathbf{H}^\infty$. Counter-examples to all the missing implications in Proposition 3.11 are presented in [DSU17, Chapter 13].

3.3 A characterisation of discreteness

In dimension 3, it is well-known that two elements in $\mathrm{PSL}(2,\mathbf{C}) = \mathrm{Isom}^+(\mathbf{H}^3)$ generating a discrete subgroup satisfy the *Jørgensen inequality*.

Proposition 3.12 ([Jør76], Lemma 1)**.** If $f, g \in \mathrm{PSL}(2,\mathbf{C})$ are two Möbius transformations generating a non-elementary discrete group, then they satisfy the following inequality, known as the *Jørgensen inequality*.

$$|\operatorname{tr}^2(f) - 4| + |\operatorname{tr}([f,g]) - 2| \geqslant 1.$$

From this, the following characterisation was obtained.

Proposition 3.13 ([Jør76], Proposition 2)**.** A non-elementary group of Möbius transformations in $\mathrm{PSL}(2,\mathbf{C})$ is discrete if each subgroup generated by two elements is discrete.

Martin obtained a similar inequality in any dimension $n \in \mathbf{N}$ and derived the following characterisation.

Theorem 3.14 ([Mar89], Theorem 5.6)**.** A finitely generated non-elementary subgroup of $\mathcal{M}\ddot{o}b(\mathbf{R}^n)$ is discrete if and only if every subgroup generated by two elements is discrete.

The same result was also achieved by Waterman (see the Corollary of [Wat93, Theorem 13]) using the formalism of Clifford algebras.

For the infinite-dimensional case, Li proved the Jørgensen inequality using Clifford matrices and Gongopadhyay refined the result by showing that the inequality is actually strict.

Let $\mathcal{H}$ be a Hilbert space, G the associated Clifford group and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(G)$ (see Definitions 2.26 and 2.29). The Clifford matrix g is called *vectorial* if $b^* = b$, $c^* = c$ and $\operatorname{tr}(g) \in \mathbf{R}$.

Theorem 3.15 ([Li11, Gon18])**.** Let $\mathcal{H}$ be a Hilbert space and G be the associated Clifford group. Let $f, g \in \mathrm{SL}_2(G)$ such that f is loxodromic and $[f,g]$ is vectorial. Suppose that the group generated by f and g, $\langle f, g\rangle$, is discrete and non-elementary. Then

$$|\operatorname{tr}^2(f) - 4| + |\operatorname{tr}([f,g]) - 2| > 1.$$

Proposition 3.16 ([Li13], Theorem 3.1)**.** Let $\Gamma \subset \mathrm{SL}_2(G)$ be a group of Clifford matrices. Then the following statements are equivalent.

1. Γ is discrete.
2. If $g, (g_n)_{n\in\mathbf{N}}$ are in $\mathrm{SL}_2(G)$ and $g_n \underset{n\to\infty}{\longrightarrow} g$, then $g_n = g$ for all n large enough.
3. $\inf\{\|g - \mathrm{id}\| \mid g \in \Gamma, g \neq \mathrm{id}\} > 0$.

The notion of discreteness in Theorem 3.15 and Proposition 3.16 is the one given by the topology of the Hilbert-Schmidt norm on the Clifford matrices (see Definition 2.29). By Proposition 3.8, it coincides with the compact-open topology COT. Thus, being discrete for the Hilbert-Schmidt norm on $\mathrm{SL}_2(G)$, *i.e.* COTD, does not imply any of the stronger notions of discreteness in Proposition 3.11. The fact that a discrete group of Clifford matrices is not strongly discrete (COTD $\not\Rightarrow$ SD) was already observed in [Li13, Example 3.2 and Remark 3.3].

Chapter III

Deformations of convex-cocompact representations

In this chapter, we study representations of surface groups into $\mathrm{Isom}(\mathbf{H}^\infty)$. Using ideas present in [MP14], we first prove that convex-cocompact representations of a finitely generated group Γ into $\mathrm{Isom}(\mathbf{H}^\infty)$ coincide with the representations whose orbit maps are quasi-isometric embeddings. It follows that the convex-cocompact representations $\Gamma \to \mathrm{Isom}(\mathbf{H}^\infty)$ form an open set inside the set of all representations $\mathrm{Hom}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$. Then, we show that the space of convex-cocompact representations of a cocompact lattice $\Gamma < \mathrm{Isom}(\mathbf{H}^2)$ into $\mathrm{Isom}(\mathbf{H}^\infty)$ contains at least an open subset of some infinite-dimensional Lie group.

Infinite-dimensional Lie groups (or *Banach-Lie groups*) are groups which are also manifolds modelled on infinite-dimensional Banach spaces, where the manifold structure is compatible with the group operations. The Lie algebra of an infinite-dimensional Lie group is well-defined as well as the exponential map, however some differences with the classical finite-dimensional case are

- a closed subgroup of an infinite-dimensional Lie group is not always a Lie group;
- and an infinite-dimensional Lie algebra is not necessarily the Lie algebra of some infinite-dimensional Lie group.

Harris and Kaup, in [HK77], studied infinite-dimensional Lie groups that are algebraic, *i.e.* defined by polynomials (possibly infinitely many, but with uniformly bounded degree). In this case, the Lie groups resemble those in finite dimension. The Lie group involved in our space of deformations is defined by only two polynomials. We are then able to compute its Lie algebra and show that is has infinite dimension. Some more material on infinite-dimensional Lie groups can be found in [dlH72, Kac85].

1 Stability of convex-cocompact representations into $\mathrm{Isom}(\mathbf{H}^\infty)$

The space $\mathbf{H}^\infty$ is an infinite-dimensional manifold, modelled on a Hilbert space. Thus it is not locally compact and not proper (closed balls are not compact), see Proposition II.1.15. It then becomes more difficult to find compact subsets inside $\mathbf{H}^\infty$ which do not lie inside a finite-dimensional hyperbolic space $\mathbf{H}^n$ embedded in $\mathbf{H}^\infty$. In this non-proper setting, it seems then more natural to look for bounded sets instead of compact ones. Therefore, we start by defining convex-cobounded actions. However, we will see that the finite-generation assumption of the group allows us to recover convex-cocompactness. At the end of this section, we emphasize that cocompactness and coboundedness are different in general with an example involving a regular tree of infinite valency.

1.1 Convex-cocompact representations and quasi-isometric embeddings

Definition 1.1. Let Γ be a group acting by isometries on a metric space X. We say that Γ is *cobounded* if there exists $\sigma > 0$ such that

$$\Gamma \cdot B(0, \sigma) = X.$$

Moreover, we say that Γ is *convex-cobounded* if there is a Γ-invariant convex subset $\mathcal{C} \subset X$ such that the restriction of Γ to $\mathcal{C}$ is cobounded.

Recall (Definition I.2.14) that a map $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called a (K, C)-*quasi-isometric embedding* if there exist $K \geqslant 1$ and $C \geqslant 0$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{K} d_Y(f(x_1), f(x_2)) - C \leqslant d_X(x_1, x_2) \leqslant K d_Y(f(x_1), f(x_2)) + C.$$

If, in addition, f is coarsely surjective, *i.e.* there exists a constant $L \geqslant 0$ such that every point of Y lies in the L–neighbourhood of the image of f, then f is called a (K, C)-*quasi-isometry*.

By Proposition I.2.35, a quasi-isometry between two Gromov-hyperbolic spaces X and Y extend to a homeomorphism between their Gromov-boundaries $\partial_\infty X \to \partial_\infty Y$.

If Γ is a group acting properly and cocompactly on some proper geodesic space X, then the Švarc-Milnor lemma states that Γ must be finitely generated and that for any choice of base point $x_0 \in X$, the orbit map $\gamma \mapsto \gamma \cdot x_0$ is a quasi-isometry between the Cayley graph of Γ (for some finite generating set S) and X, see Proposition I.2.22. Recall that two elements $\gamma_1, \gamma_2 \in \Gamma$ are connected by an edge in the Cayley graph $\mathrm{Cay}_S(\Gamma)$ whenever there is an element $s \in S \cup S^{-1}$ such that $\gamma_1 = \gamma_2 s$ and that the metric d_S on $\mathrm{Cay}_S(\Gamma)$ is the metric that assigns length 1 to every edge in $\mathrm{Cay}_S(\Gamma)$.

When the space is not proper, there is an analog of Švarc-Milnor's lemma for a convex-cobounded action instead of cocompact.

Theorem 1.2 ([DSU17], Theorem 12.2.12)**.** Let X be a CAT(-1) space, let $x_0 \in X$ be a base point. Suppose that $\Gamma < \mathrm{Isom}(X)$ is strongly discrete and convex-cobounded. Then Γ is finitely generated and the orbit map $\gamma \mapsto \gamma \cdot x_0$ is a quasi-isometric embedding.

Remark 1.3. The convex-coboundedness assumption cannot be replaced by only cobounded because of the following proposition showing that when X is the infinite-dimensional hyperbolic space, a subgroup $\Gamma < \mathrm{Isom}(\mathbf{H}^\infty)$ cannot be simultanuously cobounded (for its action by isometries on $\mathbf{H}^\infty$) and strongly discrete.

Proposition 1.4 ([DSU17], Proposition 12.2.2)**.** A strongly discrete subgroup of $\mathrm{Isom}(\mathbf{H}^\infty)$ is not cobounded.

Proposition 1.5. Let Γ be a finitely generated group and S be a finite generating set. If $\rho : \Gamma \to \mathrm{Isom}(\mathbf{H}^\infty)$ is a representation such that the orbit map is a quasi-isometric embedding, then $\rho(\Gamma)$ is strongly discrete. In particular, the representation ρ is almost faithful (*i.e.* the kernel of ρ is finite).

Proof. The proof does not differ from the finite dimension case. Let $x_0 \in \mathbf{H}^\infty$ and suppose that the orbit map $\tau_\rho : \Gamma \to \mathbf{H}^\infty, \gamma \mapsto \rho(\gamma)(x_0)$ is a (K, C)-quasi-isometric embedding. Let $R > 0$. Since τ_ρ is (K, C)-quasi-isometric, we have

$$\frac{1}{K} d_S(id, \gamma) - C \leqslant d(\tau_\rho(id), \tau_\rho(\gamma)) = d(x_0, \rho(\gamma)(x_0))$$

so $d(x_0, \rho(\gamma)(x_0)) \leqslant R$ implies that $d_S(id, \gamma) \leqslant K(R + C)$, thus

$$\#\{\gamma \in \Gamma \mid d(x_0, \rho(\gamma)(x_0)) \leqslant R\} \leqslant \#\{\gamma \in \Gamma \mid d_S(id, \gamma) \leqslant K(R + C)\}.$$

The right-hand side is finite since Γ is finitely generated, this shows that Γ is strongly discrete.

It follows that the kernel of ρ is finite, since

$$\ker \rho = \{\gamma \in \Gamma \mid \rho(\gamma) = id\} \subset \{\gamma \in \Gamma \mid d(x_0, \rho(\gamma)(x_0)) = 0\}.$$

□

The proof of the stability of convex-cocompact representations of Γ in $\mathbf{H}^n$ relies on the characterisation of convex-cocompact representations as those whose orbit maps are quasi-isometric embeddings. We aim to get a similar result for representations in $\mathbf{H}^\infty$. The assumption that Γ is finitely generated allows us to recover some compactness thanks to the following theorem from Mazur (see for example [Meg98, Theorem 2.8.15]). This fact was first observed in [MP14, Lemma 4.2].

Theorem 1.6 (Mazur's compactness theorem)**.** The closed convex hull of a compact subset of a Banach space is itself compact.

Using this idea, we may prove that in infinite dimension, it is also true that a group is convex-cocompact if and only if its orbit map is a quasi-isometric embedding. The direct implication of the following theorem is similar to the proof of [MP14, Proposition 4.3].

Theorem 1.7. Let Γ be a finitely generated group and $\rho : \Gamma \to \mathrm{Isom}(\mathbf{H}^\infty)$ be a representation. The two following assertions are equivalent.

1. For any $x_0 \in \mathbf{H}^\infty$, the orbit map $\tau_\rho : \gamma \mapsto \rho(\gamma)(x_0)$ is a Γ-equivariant quasi-isometric embedding from Γ into $\mathbf{H}^\infty$, where Γ is endowed with the word distance associated to a finite set of generators.
2. There is a closed, convex and locally compact Γ-invariant subset $\mathcal{C} \subset \mathbf{H}^\infty$ on which Γ acts cocompactly and $\rho(\Gamma)$ is strongly discrete in $\mathrm{Isom}(\mathbf{H}^\infty)$.

Proof. Suppose first that the orbit map is a quasi-isometric embedding from Γ to $\mathbf{H}^\infty$. It extends to a continuous boundary map $\partial\tau_\rho : \partial_\infty\Gamma \to \partial\mathbf{H}^\infty$ which is Γ-equivariant by Proposition I.2.35.

Fix a finite generating set S for Γ. The Cayley graph $\mathrm{Cay}_S(\Gamma)$ is locally finite and the boundary $\partial_\infty\Gamma$ is compact (Proposition I.2.34). Thus its image $\partial\tau_\rho(\partial_\infty\Gamma) \subset \partial\mathbf{H}^\infty$ is also compact for the cone topology on $\partial\mathbf{H}^\infty$. Denote by $\mathcal{C} \subset \mathbf{H}^\infty$ the closed convex hull of $\partial\tau_\rho(\partial_\infty\Gamma)$. With the Klein model of the hyperbolic space, $\mathbf{H}^\infty$ is identified with the unit ball $\mathbb{B}^\infty$ of a real Hilbert space. The convex hull $\mathcal{C}$ is identified with the intersection of $\mathbb{B}^\infty$ and the closed affine convex hull $\overline{\mathcal{C}}$ of $\partial\tau_\rho(\partial_\infty\Gamma)$ in $\overline{\mathbb{B}^\infty}$. Theorem 1.6 shows that $\overline{\mathcal{C}}$ is compact in $\overline{\mathbb{B}^\infty}$. Therefore, $\mathcal{C}$ is locally compact and closed in $\mathbb{B}^\infty$, so it is a closed and proper subspace of $\mathbf{H}^\infty$. Moreover, it follows from Proposition 1.5 that the action of Γ on $\mathcal{C}$ is strongly discrete.

In order to show the cocompactness of the action of Γ on $\mathcal{C}$, it is then sufficient to show coboundedness since $\mathcal{C}$ is proper. Let $x_0 \in \mathcal{C}$ and suppose for a contradiction that $\mathcal{C}$ contains a sequence $(y_i)_{i\in\mathbf{N}}$ such that the distance $d(y_i, \Gamma \cdot x_0)$ goes to infinity. By strong discreteness of the action, we can suppose that $d(y_i, \Gamma \cdot x_0) = d(y_i, x_0)$ up to replacing y_i by one of its translates by Γ. Since $\overline{\mathcal{C}}$ is compact, $(y_i)_{i\in\mathbf{N}}$ converges to a point $y \in \partial\mathcal{C} = \partial\tau_\rho(\partial_\infty\Gamma)$ up to picking a subsequence. Choose a sequence $(z_i)_{i\in\mathbf{N}}$ in $\mathrm{Cay}_S(\Gamma)$ such that $\tau_\rho(z_i)$ converges also to y. Thus the Gromov product

$$L_{ij} = (\tau_\rho(z_i) \cdot y_j)_{x_0} = \frac{1}{2}\left(d(\tau_\rho(z_i), x_0) + d(y_j, x_0) - d(\tau_\rho(z_i), y_j)\right)$$

tends to infinity as $i, j \to \infty$. Let m_{ij} be the point of the geodesic $[x_0, y_j]$ at distance L_{ij} from x_0. Since $d(y_j, \Gamma \cdot x_0) = d(y_j, x_0) \underset{j\to\infty}{\longrightarrow} +\infty$, we have $d(m_{ij}, \Gamma \cdot x_0) = d(m_{ij}, x_0) = L_{ij} \underset{i,j\to\infty}{\longrightarrow} +\infty$. By δ-hyperbolicity, the distance from m_{ij} to the geodesic $[x_0, \tau_\rho(z_i)]$ is bounded by some constant depending only on δ. Since τ_ρ is a quasi-isometric embedding, the image of the geodesic $[1, z_i]$ in $\mathrm{Cay}_S(\Gamma)$ by τ_ρ is a quasi-geodesic between x_0 and $\tau_\rho(z_i)$. Thus, by stability of quasi-geodesics (Proposition I.2.12 or [BH99, Theorem III.1.7]), the geodesic $[x_0, \tau_\rho(z_i)]$ is in the R-neighbourhood of $\tau_\rho(\mathrm{Cay}_S(\Gamma))$ for some fixed constant $R \geqslant 0$. Then the m_{ij} remain at bounded distance from $\tau_\rho(\mathrm{Cay}_S(\Gamma))$ by triangle inequality so they also remain at bounded distance from $\tau_\rho(\Gamma)$ which is the orbit of x_0. This is in contradiction with $d(m_{ij}, \Gamma \cdot x_0) \underset{i,j\to\infty}{\longrightarrow} +\infty$.

For the converse implication, let $\mathcal{C}$ be a closed, convex and locally compact Γ-invariant subset of $\mathbf{H}^\infty$ on which Γ acts properly and cocompactly. Then by Švarc-Milnor's lemma, Proposition I.2.22, the orbit map $\tau_\rho : \Gamma \to \mathcal{C}, \gamma \mapsto \gamma(x_0)$ is a Γ-equivariant quasi-isometric embedding. And since the embedding of $\mathcal{C}$ to $\mathbf{H}^\infty$ is isometric, we get a quasi-isometric embedding $\tau_\rho : \Gamma \to \mathbf{H}^\infty$. □

1.2 Stability of convex-cocompact representations

Let Γ be a finitely generated group. Let $S = \{\gamma_1, \dots, \gamma_m\}$ be a generating family of Γ. Denote by d_S the metric on the Cayley graph of Γ associated to S. Given a representation $\rho : \Gamma \to \mathrm{Isom}(\mathbf{H}^\infty)$ and a base point $x_0 \in \mathbf{H}^\infty$, the orbit map $\tau_\rho : \Gamma \to \mathbf{H}^\infty$ given by

$$\tau_\rho(\gamma) = \rho(\gamma)(x_0)$$

is Γ-equivariant. We will call ρ a *quasi-isometric representation* if its orbit map τ_ρ is a quasi-isometric embedding.

Since $\mathbf{H}^\infty$ is δ-hyperbolic, the group Γ is Gromov-hyperbolic when $\rho : \Gamma \to \mathrm{Isom}(\mathbf{H}^\infty)$ is a quasi-isometric representation.

Theorem 1.8 (Local-to-global principle, [CDP90] Theorem 3.1.4)**.** Given $K \geqslant 1$, $C \geqslant 0$ and $\delta \geqslant 0$, there exist $\widehat{K}$, $\widehat{C}$ and A so that if J is an interval in $\mathbf{R}$, X is a δ-hyperbolic geodesic space and $h : J \to X$ is a (K, C)-quasi-isometric embedding when restricted to any connected subsegment of J with length at most A, then h is a $(\widehat{K}, \widehat{C})$-quasi-isometric embedding.

Let X be a δ-hyperbolic geodesic metric space. The space of all representations $\mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$ can be embedded into $\mathrm{Isom}(X)^m$ by

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma, \mathrm{Isom}(X)) & \to & \mathrm{Isom}(X)^m \\ \rho & \mapsto & (\rho(\gamma_1), \dots, \rho(\gamma_m)). \end{array}$$

Endow $\mathrm{Isom}(X)$ with the pointwise convergence topology. If $f \in \mathrm{Isom}(X)$, a subbasis of neighbourhoods at f is given by the open sets of the form

$$V(f, \epsilon, P) = \{g \in \mathrm{Isom}(X) \mid \forall x \in P \quad d(f(x), g(x)) < \epsilon\}$$

where P is a finite set of points in X and $\epsilon > 0$. The topology on $\mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$ is then inherited from the product topology on $\mathrm{Isom}(X)^m$.

The following theorem is due to Marden in the case of representations into $\mathrm{Isom}(\mathbf{H}^3)$ and to Thurston for $\mathrm{Isom}(\mathbf{H}^n)$, $n \in \mathbf{N}$. The same proof remains valid for any δ-hyperbolic space. We follow the proof of [Can21, Theorem 11.4].

Theorem 1.9. Let X be a geodesic metric space. If Γ is a finitely generated group and $\rho : \Gamma \to \mathrm{Isom}(X)$ is a quasi-isometric representation, then there exists a neighbourhood U of ρ in $\mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$ such that if $\sigma \in U$, then σ is a quasi-isometric representation. Moreover, we may choose U so that all the representations $\sigma \in U$ have the same constants of quasi-isometry.

Proof. Let $x_0 \in X$ and suppose that the orbit map τ_ρ is a (K, C)-quasi-isometric embedding with respect to the finite generating set $S = \{\gamma_1, \dots, \gamma_m\}$. Let $A > 0$ to be defined later. Let U be an open neighbourhood of ρ in $\mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$ such that for all $\sigma \in U$ and $\gamma \in \Gamma$ with $d_S(1_\Gamma, \gamma) \leqslant A$, $d(\rho(\gamma)(x_0), \sigma(\gamma)(x_0)) < 1$. It suffices to choose

$$U = \prod_{i=1}^{m} V\left(\rho(\gamma_i), \frac{1}{A+1}, P\right)$$

where $P = \{\rho(\gamma)(x_0) \mid \gamma \in \Gamma \text{ and } d_S(1_\Gamma, \gamma) \leqslant A\}$ is finite since $\#\{\gamma \in \Gamma \mid d_S(1_\Gamma, \gamma) \leqslant A\} < \infty$.
Since τ_ρ is a (K, C)-quasi-isometric embedding, then for all $\gamma \in \Gamma$,

$$\frac{1}{K} d_S(1_\Gamma, \gamma) - C \leqslant d(x_0, \tau_\rho(\gamma)) \leqslant K d_S(1_\Gamma, \gamma) + C.$$

Let $\sigma \in U$, we see that if $d_S(1_\Gamma, \gamma) \leqslant A$, then

$$\frac{1}{K} d_S(1_\Gamma, \gamma) - C - 1 \leqslant d(x_0, \tau_\sigma(\gamma)) \leqslant K d_S(1_\Gamma, \gamma) + C + 1$$

and by Γ-equivariance of τ_σ,

$$\frac{1}{K} d_S(\alpha, \beta) - C - 1 \leqslant d(\tau_\sigma(\alpha), \tau_\sigma(\beta)) \leqslant K d_S(\alpha, \beta) + C + 1$$

for all $\alpha, \beta \in \Gamma$ with $d_S(\alpha, \beta) \leqslant A$.

If $\mathrm{Cay}_S(\Gamma)$ is the Cayley graph of Γ with respect to S, then we may extend τ_σ to $\mathrm{Cay}_S(\Gamma)$ by mapping all the edges to geodesics in X. The resulting map is a (K', C')-quasi-isometric embedding on all geodesic segments in $\mathrm{Cay}_S(\Gamma)$ of length at most A for some $K' \geqslant 1$ and $C' \geqslant 0$ depending only on K and C. We may then choose A according to Theorem 1.8. This theorem provides some $\widehat{K}$ and $\widehat{C}$ such that τ_σ is a $(\widehat{K}, \widehat{C})$-quasi-isometric embedding on all geodesic segments in $\mathrm{Cay}_S(\Gamma)$. Thus τ_σ is a $(\widehat{K}, \widehat{C})$-quasi-isometric embedding on the whole Cayley graph $\mathrm{Cay}_S(\Gamma)$. □

Denote by $\mathrm{QI}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$ the set of representations of Γ into $\mathrm{Isom}(\mathbf{H}^\infty)$ which have quasi-isometric orbit maps and let $\mathrm{CC}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$ be the set of convex-cocompact representations in $\mathrm{Hom}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$. We can deduce as a consequence of Theorems 1.7 and 1.9 that convex-cocompact representations of finitely generated groups into $\mathrm{Isom}(\mathbf{H}^\infty)$ form an open subset of the space of all the representations.

Corollary 1.10. If Γ is finitely generated, then $\mathrm{QI}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty)) = \mathrm{CC}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$. Moreover, this set is open in $\mathrm{Hom}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$.

1.3 Convex-coboundedness and convex-cocompactness

The argument of Theorem 1.7 to get a locally compact convex set $\mathcal{C} \subset \mathbf{H}^\infty$ on which the group Γ acts cocompactly fails when the Cayley graph of Γ is not locally finite. The assumption that Γ is finitely generated is then important for the proof to hold. To illustrate this, we will consider embeddings of trees into $\mathbf{H}^\infty$ introduced in [BIM05] to produce a representation of an infinitely generated group into $\mathrm{Isom}(\mathbf{H}^\infty)$ which is convex-cobounded but not convex-cocompact even though this embedding is quasi-isometric.

Theorem 1.11 ([BIM05], Theorems *A* and 8.1)**.** Let $\mathcal{T}$ be a simplicial tree, denote by V the set of vertices of $\mathcal{T}$ and by $\mathbf{H}$ the hyperbolic space of dimension $|V|-1$, $\mathbf{H} = \mathbf{H}^{|V|-1}$. Let $d_\mathcal{T}$ be the metric on $\mathcal{T}$ that assigns length 1 to all the edges of $\mathcal{T}$. Then for every $\lambda > 1$ there exists an embedding $\psi : \mathcal{T} \to \mathbf{H}$ and a representation $\rho : \mathrm{Aut}(\mathcal{T}) \to \mathrm{Isom}(\mathbf{H})$ such that

1. the map ψ is $\mathrm{Aut}(\mathcal{T})$-equivariant for ρ.
2. $\lambda^{d_\mathcal{T}(x,y)} = \cosh d(\psi(x), \psi(y))$ for any $x, y \in V$.
3. ψ extends to an equivariant boundary map $\partial\psi : \partial\mathcal{T} \to \partial\mathbf{H}$ which is a homeomorphism onto its image.
4. $\psi(V)$ is cobounded in the convex hull of the image of $\partial\psi$.

Remark 1.12. The second point of the theorem implies that ψ is a (K, C)-quasi-isometric embedding on the vertices of $\mathcal{T}$ for $K = \max\left\{\ln(\lambda), \dfrac{1}{\ln(\lambda)}\right\}$ and $C = \dfrac{\ln 2}{\ln \lambda}$.

Let $\mathcal{T} = (V, E)$ be the regular tree of countably infinite valency, let $d_\mathcal{T}$ be the combinatorial distance on $\mathcal{T}$ (every edge has length 1). The group $\mathrm{Aut}(\mathcal{T})$ acts cocompactly on $\mathcal{T}$ and a fundamental region is a half-edge of $\mathcal{T}$, *i.e.* an edge of the first barycentric subdivison of $\mathcal{T}$. Consider also $\mathbf{F}_\infty$, the free group on countably infinite generators. Let $S = \{s_1, s_2, \dots\}$ be a system of generators of $\mathbf{F}_\infty$. The tree $\mathcal{T}$ can be seen as the Cayley graph of $\mathbf{F}_\infty$ associated to S where all the edges around each vertex are labelled with one element of $S \cup S^{-1}$ (two edges starting from the same vertex cannot have the same label). The free group $\mathbf{F}_\infty$ acts on $\mathcal{T}$ by sending an edge onto another edge with the same label. This provides an embedding of $\mathbf{F}_\infty$ as a subgroup of $\mathrm{Aut}(\mathcal{T})$ and a fundamental region for the action $\mathbf{F}_\infty \curvearrowright \mathcal{T}$ is given by the tree $\mathcal{T}'$ corresponding to the ball of radius $\frac{1}{2}$ around a vertex, *i.e.* a vertex of $\mathcal{T}$ and all its adjacent edges in the first barycentric subdivision of $\mathcal{T}$.

Proposition 1.13. The action of $\mathbf{F}_\infty$ on $\mathcal{T}$ is cobounded but not cocompact.

Proof. This action is transitive on the vertices of $\mathcal{T}$, so the quotient $\widehat{\mathcal{T}} = \mathcal{T}/\mathbf{F}_\infty$ is isomorphic to a single vertex with infinitely many loops of length 1 around it, hence it is bounded. To show that it is not compact, let v be a vertex of $\mathcal{T}$ and for all $i \in \mathbf{N}$, let m_i be the point at distance $\frac{1}{2}$ from v on the edge labelled by s_i. The distance $d_{\mathcal{T}}$ on $\mathcal{T}$ induces a distance $d_{\widehat{\mathcal{T}}}$ on the quotient such that for all $i \neq j \in \mathbf{N}$, $d_{\widehat{\mathcal{T}}}(m_i, m_j) = 1$. Thus the sequence $(m_i)_{i\in\mathbf{N}}$ has no convergent subsequence and $\widehat{\mathcal{T}}$ is not compact. □

Fix $\lambda > 1$. By Theorem 1.11, there is a representation $\rho = \rho_\lambda : \mathbf{F}_\infty < \mathrm{Aut}(\mathcal{T}) \to \mathrm{Isom}(\mathbf{H}^\infty)$ and an $\mathrm{Aut}(\mathcal{T})$-equivariant quasi-isometric embedding $\psi = \psi_\lambda : \mathcal{T} \to \mathbf{H}^\infty$ which extends to the boundary $\partial\mathcal{T}$.

Denote by $\mathcal{C}_\lambda$ the closed convex hull of $\partial\psi(\partial\mathcal{T})$ in $\mathbf{H}^\infty$ and let $\mathcal{C}_0$ be the closed convex hull of $\psi(\mathcal{T})$. Both $\mathcal{C}_\lambda$ and $\mathcal{C}_0$ are $\mathrm{Aut}(\mathcal{T})$-invariant and $\mathcal{C}_0$ is the unique minimal invariant closed convex set in $\mathbf{H}^\infty$, thus we have $\mathcal{C}_0 \subset \mathcal{C}_\lambda$. However they are not equal in general, see [MP14, section 5.A].

Proposition 1.14. The group $\rho(\mathbf{F}_\infty) < \mathrm{Isom}(\mathbf{H}^\infty)$ is convex-cobounded.

Proof. The embedding $\psi : \mathcal{T} \to \mathbf{H}^\infty$ being $\mathrm{Aut}(\mathcal{T})$-equivariant, we have $\psi(\mathcal{T}) = \rho(\mathbf{F}_\infty) \cdot \psi(\mathcal{T}')$, where $\mathcal{T}'$ is a bounded fundamental region for $\mathbf{F}_\infty \curvearrowright \mathcal{T}$. By the fourth point of Theorem 1.11, $\psi(\mathcal{T})$ is cobounded in $\mathcal{C}_\lambda$, so $\rho(\mathbf{F}_\infty)$ acts coboundedly on $\mathcal{C}_\lambda$ since $\psi(\mathcal{T}')$ is bounded. □

Lemma 1.15. Let $G < \mathrm{Aut}(\mathcal{T})$. If $\rho(G)$ acts cocompactly on any invariant closed convex subset $\mathcal{C} \subset \mathbf{H}^\infty$, then the action of G on the minimal inviariant convex set $\mathcal{C}_0$ via ρ is also cocompact.

Proof. Suppose that $\mathcal{C} = \rho(G) \cdot A$, where $A \subset \mathbf{H}^\infty$ is compact. Let $x \in \mathcal{C}_0$. Since $\mathcal{C}_0 \subset \mathcal{C}$, there exists $g \in G$ such that $x \in \rho(g) \cdot A$. Then $x \in (\rho(g) \cdot A) \cap \mathcal{C}_0 = \rho(g) \cdot (A \cap \mathcal{C}_0)$ by invariance of $\mathcal{C}_0$. Therefore $\mathcal{C}_0 = \rho(G) \cdot (A \cap \mathcal{C}_0)$. Moreover, $A \cap \mathcal{C}_0$ is closed in A, so $A \cap \mathcal{C}_0$ is compact. □

This lemma implies that in order to prove that $\rho(\mathbf{F}_\infty) < \mathrm{Isom}(\mathbf{H}^\infty)$ is not convex-cocompact, it is sufficient to show this property for the action of $\mathbf{F}_\infty$ on $\mathcal{C}_0$ via ρ.

Proposition 1.16. The action of $\rho(\mathbf{F}_\infty)$ on $\psi(\mathcal{T})$ is not cocompact.

Proof. The proof is similar to that of Proposition 1.13. Suppose that $\psi(\mathcal{T}) \subset \rho(\mathbf{F}_\infty) \cdot K$ for $K \subset \mathbf{H}^\infty$. Let v be a vertex in $\mathcal{T}$ and enumerate the edges incident to v. For all $i \in \mathbf{N}$, denote by m_i the point on the i-th edge around v such that $d_{\mathcal{T}}(v, m_i) = \frac{1}{2}$. Then for all $i \in \mathbf{N}$, there exists m_i' in the orbit $\mathbf{F}_\infty \cdot m_i$ such that $\psi(m_i') \in K$. For $i \neq j$, we have $d_{\mathcal{T}}(m_i', m_j') \geqslant 1$ so by the second point of Theorem 1.11,

$$d(\psi(m_i'), \psi(m_j')) = \cosh^{-1}\left(\lambda^{d_{\mathcal{T}}(m_i', m_j')}\right) \geqslant \cosh^{-1}(\lambda) > 0.$$

The sequence $(\psi(m_i'))_{i \in \mathbf{N}} \in K^{\mathbf{N}}$ has no convergent subsequence, therefore K cannot be compact. □

Corollary 1.17. The group $\rho(\mathbf{F}_\infty) < \mathrm{Isom}(\mathbf{H}^\infty)$ is not convex-cocompact.

Proof. By Lemma 1.15, it suffices to prove that $\rho(\mathbf{F}_\infty)$ does not act cocompactly on $\mathcal{C}_0$. If $\mathcal{C}_0 \subset \rho(\mathbf{F}_\infty) \cdot K$ for some $K \subset \mathbf{H}^\infty$, then we may construct a sequence of points $m_i' \in \mathcal{T}$ such that $\psi(m_i') \in K$ and all the $d(\psi(m_i'), \psi(m_j'))$ are greater than a uniform positive constant for $i \neq j$, as in Proposition 1.16. Since $\psi(\mathcal{T}) \subset \mathcal{C}_0$, all the $\psi(m_i')$ also belong to $\mathcal{C}_0$. □

2 Embeddings of $\mathrm{Isom}(\mathbf{H}^2)$ into $\mathrm{Isom}(\mathbf{H}^\infty)$

Definition 2.1. Let $\rho : \Gamma \to \mathrm{Isom}(\mathbf{H}^\infty)$ be a representation of a group Γ. We say that ρ is *irreducible* if $\rho(\Gamma)$ has no fixed point in the boundary $\partial\mathbf{H}^\infty$ and does not preserve any non-trivial closed totally geodesic subspace of $\mathbf{H}^\infty$.

Remark 2.2. This definition coincides with the usual definition of irreducibility for the linear representation of Γ on the underlying Hilbert space $\mathcal{H}$: ρ is irreducible if and only if there is no non-trivial invariant closed linear subspaces in $\mathcal{H}$ (see Definition I.1.2). This was also referred to as *geometric Zariski density* in [MP14, section 5.B].

Recall that from the hyperboloid model $\mathbf{H}^\infty = \{x \in \mathcal{H} \mid Q(x) = -1, x_0 > 0\}$, we can obtain an embedding of $\mathbf{H}^n$ into $\mathbf{H}^\infty$ by restricting to vectors $x = (x_i)_{i \in \mathbf{N}} \in \mathcal{H}$ where all the coordinates vanish for $i > n+1$. This also induces an injection of the isometry group $\mathrm{Isom}(\mathbf{H}^n) \hookrightarrow \mathrm{Isom}(\mathbf{H}^\infty)$ (see Remark II.1.27). Monod and Py classified all the continuous representations from $\mathrm{Isom}(\mathbf{H}^n)$ to $\mathrm{Isom}(\mathbf{H}^\infty)$ in [MP14] that are irreducible.

Theorem 2.3 ([MP14], Theorem *B*)**.**

1. Let $\rho : \mathrm{Isom}(\mathbf{H}^n) \to \mathrm{Isom}(\mathbf{H}^\infty)$ be a continuous non-elementary action. There exists $t \in (0,1]$ such that $\ell_{\mathbf{H}^\infty}(\rho(g)) = t\ell_{\mathbf{H}^n}(g)$ for all $g \in \mathrm{Isom}(\mathbf{H}^n)$, where $\ell_{\mathbf{H}^\infty}$ and $\ell_{\mathbf{H}^n}$ denote the translation lengths in $\mathbf{H}^\infty$ and $\mathbf{H}^n$ respectively. Moreover, $t = 1$ if and only if ρ preserves an n-dimensional totally geodesic subspace of $\mathbf{H}^\infty$.

2. For each $t \in (0,1)$, there is, up to conjugacy in $\mathrm{Isom}(\mathbf{H}^\infty)$, exactly one irreducible continuous representation $\rho_t : \mathrm{Isom}(\mathbf{H}^n) \to \mathrm{Isom}(\mathbf{H}^\infty)$ such that for all $g \in \mathrm{Isom}(\mathbf{H}^n)$, $\ell_{\mathbf{H}^\infty}(\rho_t(g)) = t\ell_{\mathbf{H}^n}(g)$.

Theorem 2.4 ([MP14], Theorem C)**.** There exists a smooth harmonic ρ_t-equivariant map $f_t : \mathbf{H}^n \to \mathbf{H}^\infty$ which is asymptotically an isometric embedding after rescaling, *i.e.* there is $D \geqslant 0$ such that for all $x, y \in \mathbf{H}^n$,

$$|d_{\mathbf{H}^\infty}(f_t(x), f_t(y)) - t d_{\mathbf{H}^n}(x,y)| \leqslant D.$$

Moreover, f_t extends to a continuous map $f_t : \overline{\mathbf{H}^n} \to \overline{\mathbf{H}^\infty}$.

Notice that being asymptotically an isometric embedding after rescaling implies that f_t is a quasi-isometric embedding from $\mathbf{H}^n$ to $\mathbf{H}^\infty$.

For $t \in (0,1)$, since ρ_t is non-elementary, there is a unique minimal non-empty closed convex ρ_t-invariant subset $\mathcal{C}_t \subset \mathbf{H}^\infty$ (see [Mon05]).

Theorem 2.5 ([MP14], Theorem D and Proposition E)**.** For any $t \in (0,1]$, the $\mathrm{CAT}(-1)$ space $\mathcal{C}_t$ is proper, the action of $\mathrm{Isom}(\mathbf{H}^n)$ on $\mathcal{C}_t$ is cocompact and ρ_t induces an isomorphism $\mathrm{Isom}(\mathbf{H}^n) \simeq \mathrm{Isom}(\mathcal{C}_t)$.

Notice that if Γ is a cocompact lattice of $\mathrm{Isom}(\mathbf{H}^n)$ and if $\rho_t : \mathrm{Isom}(\mathbf{H}^n) \to \mathrm{Isom}(\mathbf{H}^\infty)$ is a representation for some $t \in (0,1]$ as in Theorem 2.3, the restriction of ρ_t to Γ is then convex-cocompact. Indeed, since $\mathrm{Isom}(\mathbf{H}^n) \curvearrowright \mathcal{C}_t$ is cocompact, there is a compact set $K_1 \subset \mathcal{C}_t$ such that $\mathcal{C}_t = \mathrm{Isom}(\mathbf{H}^n) \cdot K_1$. Moreover, since Γ is a cocompact lattice in $\mathrm{Isom}(\mathbf{H}^n)$, we have $\mathrm{Isom}(\mathbf{H}^n) = \Gamma \cdot K_2$ for some compact subset $K_2 \subset \mathrm{Isom}(\mathbf{H}^n)$. Thus, $\mathcal{C}_t = \mathrm{Isom}(\mathbf{H}^n) \cdot K_1 = (\Gamma \cdot K_2) \cdot K_1 = \Gamma \cdot K$, where $K = K_2 \cdot K_1$ is a compact subset of $\mathcal{C}_t$. Using Corollary 1.10, we can find a neighbourhood U of $\rho_{t|\Gamma}$ in $\mathrm{Hom}(\Gamma, \mathrm{Isom}(\mathbf{H}^\infty))$ consisting only of convex-cocompact representations of Γ.

2.1 Some properties of the exotic representations

Fix $t \in (0,1]$ and let $\rho_t : \mathrm{Isom}(\mathbf{H}^n) \to \mathrm{Isom}(\mathbf{H}^\infty)$ and $f_t : \mathbf{H}^n \to \mathbf{H}^\infty$ be a representation and its corresponding harmonic map given by Theorems 2.3 and 2.4 respectively.

Lemma 2.6 ([MP14], Lemma 4.1)**.** Let $\mathcal{C}_t$ be the closed convex hull of $f_t(\mathbf{H}^n)$ inside $\mathbf{H}^\infty$. The action of $\mathrm{Isom}(\mathbf{H}^n)$ on $\mathcal{C}_t$ is minimal, *i.e.* $\mathcal{C}_t$ admits no nontrivial $\mathrm{Isom}(\mathbf{H}^n)$-invariant closed convex subspace. Moreover, this is the unique minimal $\mathrm{Isom}(\mathbf{H}^n)$-invariant closed convex subset of $\mathbf{H}^\infty$.

We point out in the next lemma some properties of ρ_t and f_t that can be deduced from the results in [MP14].

In the following, a map $f : X \to \mathbf{R}$, where X is a complete geodesic metric space, will be called *analytic along geodesics* if for any geodesic line L (the image of an isometric embedding $\gamma : \mathbf{R} \to X$), the restriction $f|_L = f \circ \gamma : \mathbf{R} \to \mathbf{R}$ is an analytic function.

Lemma 2.7. Let $0 < t \leqslant 1$,

1. If $\Gamma < \mathrm{Isom}(\mathbf{H}^n)$ is a cocompact lattice, then $\rho_t(\Gamma)$ is cocompact in $\rho_t(\mathrm{Isom}(\mathbf{H}^n)) \simeq \mathrm{Isom}(\mathcal{C}_t)$.
2. If $\Gamma < \mathrm{Isom}(\mathbf{H}^n)$ has no fixed point in $\partial\mathbf{H}^n$, then $\rho_t(\Gamma)$ also has no fixed point in $\partial\mathcal{C}_t = f_t(\partial\mathbf{H}^n)$.
3. The harmonic map $f_t : \mathbf{H}^n \to \mathbf{H}^\infty$ is analytic and therefore, if Y is a totally geodesic subspace of $\mathbf{H}^\infty$, the map $x \mapsto d(f_t(x), Y)$ is analytic along geodesics.

Proof. The first point is a consequence of Theorem 2.5 and the proof of Theorem 1.7, where the invariant convex set is $\mathcal{C}_t$.

The second point follows from the equivariance of f_t and the fact that f_t is a quasi-isometric embedding which induces an injective map (actually a homeomorphism) from $\partial\mathbf{H}^n$ to $f_t(\partial\mathbf{H}^n)$ by Proposition I.2.35.

For the last point, the analyticity of the harmonic map f_t comes from its expression explicited in [MP14, section 3.A]. For $g \in \mathrm{Isom}(\mathbf{H}^n)$ and $b \in \partial\mathbf{H}^n$, f_t can be identified with the map from $\mathbf{H}^n$ to $\mathrm{L}^2(\partial\mathbf{H}^n, \mathbf{R})$ defined by

$$f_t(g \cdot o)(b) = |\,\mathrm{Jac}(g^{-1})(b)|^{1+\frac{t}{n-1}} = \left(\frac{B_n(o,b)}{B_n(g(o),b)}\right)^{t+n-1},$$

where B_n is the bilinear form on $\mathbf{R}^{n+1}$ associated to the quadratic form of signature $(n,1)$ defining $\mathbf{H}^n$ and $o \in \mathbf{H}^n$ is some base point. Define $\varphi : \mathbf{H}^\infty \to \mathbf{R}$ by $\varphi(x) = d(x, Y)$. If π_Y denotes the orthogonal projection onto Y, then $\varphi(x) = d(x, \pi_Y(x))$. Since $Y \subset \mathbf{H}^\infty$ is totally geodesic, it is a hyperbolic subspace of $\mathbf{H}^\infty \subset \mathcal{H}$ and we may choose a basis $(e_i)_{i\in\mathbf{N}}$ of the Hilbert space $\mathcal{H}$ such that

$$Y = \{x \in \mathbf{H}^\infty \mid \forall i \in I \quad x_i = 0\},$$

where I is some subset of $\mathbf{N}_{\geqslant 1} = \{1, 2, \ldots\}$ and the x_i's are the coordinates of x in this basis. Then for $x \in \mathbf{H}^\infty$, $\pi_Y(x) = \dfrac{y}{\sqrt{-Q(y)}}$ with $y = x - \sum_{i\in I} B(x, e_i)e_i = \sum_{i\notin I} x_i e_i$. We then get

$$\cosh d(x, Y) = \cosh d(x, \pi_Y(x)) = -B(x, \pi_Y(x)) = \left(x_0^2 - \sum_{i\in\mathbf{N}_{\geqslant 1}\setminus I} x_i^2\right)^{\frac{1}{2}}.$$

Let $\hat{J}$ denote the bounded linear map on $\mathcal{H}$ such that $\hat{J}e_0 = e_0$, $\hat{J}e_i = 0$ if $i \in I$ and $\hat{J}e_i = -e_i$ otherwise. Then

$$\varphi(x) = \cosh^{-1}\left(\langle x, \hat{J}x\rangle\right)^{\frac{1}{2}} = \cosh^{-1}\left({}^{\mathrm{t}}x\hat{J}x\right)^{\frac{1}{2}}.$$

For any hyperbolic geodesic $L \subset \mathbf{H}^n$, there exist $a, b \in \mathbf{R}^{n+1}$ such that L is the image of $\gamma : u \in \mathbf{R} \mapsto \cosh(u)a + \sinh(u)b \in \mathbf{H}^n$. The restriction of $\varphi \circ f_t$ to L gives

$$\begin{aligned}(\varphi \circ f_t)_{|L}(u) &= (\varphi \circ f_t)(\gamma(u))\\ &= \varphi(f_t(\gamma(u)))\\ &= \cosh^{-1}\left({}^{\mathrm{t}}f_t(\gamma(u))\hat{J}f_t(\gamma(u))\right)^{\frac{1}{2}}\end{aligned}$$

which is an analytic function in $u \in \mathbf{R}$. □

Caprace and Monod proved the following theorem which can be thought of as an analog of the Borel-density theorem for lattices acting a proper CAT(0) space.

Theorem 2.8 ([CM09], Theorem 2.4)**.** Let G be a locally compact group with a continuous isometric action on a proper CAT(0) space X without Euclidean factor. If G acts minimally on X and without a global fixed point in ∂X, then any closed subgroup with finite invariant covolume in G still has these properties.

In our context, we are interested in the case of the group $G = \rho_t(\mathrm{Isom}(\mathbf{H}^n))$ acting continuously by isometries on the space $X = \mathbf{H}^\infty$. Moreover, instead of a closed subgroup in $\mathrm{Isom}(\mathbf{H}^n)$ with finite invariant covolume, we consider a convex-cocompact subgroup $A < \mathrm{Isom}(\mathbf{H}^n)$ and its image in $\mathrm{Isom}(\mathbf{H}^\infty)$ by an irreducible representation ρ_t $(0 < t < 1)$. We could not show that for any irreducible convex-cocompact subgroup $A < \mathrm{Isom}(\mathbf{H}^n)$, its image $\rho_t(A)$ is still irreducible in $\mathrm{Isom}(\mathbf{H}^\infty)$. However, it remains at least true that $\rho_t(A)$ admits no global fixed point in the boundary of $\partial\mathbf{H}^\infty$ as soon as $A < \mathrm{Isom}(\mathbf{H}^n)$ has no fixed point in $\mathbf{H}^n \cup \partial\mathbf{H}^n$. A similar statement was proved by Caprace and Monod as a first step to obtain Theorem 2.8, see [CM09, Proposition 2.1]. It relies in particular on the following result asserting that the intersection of nested sequences of closed convex sets in hyperbolic spaces is either non-empty, or it converges to one point in the boundary.

Theorem 2.9 ([CL10], Theorem 1.1)**.** Let X be a complete CAT(0) space of finite telescopic dimension and $\{X_\alpha\}_{\alpha\in A}$ be a filtering family of closed convex subspaces. Then either the intersection $\bigcap_{\alpha\in A} X_\alpha$ is non-empty, or the intersection of the visual boundaries $\bigcap_{\alpha\in A} \partial X_\alpha$ is a non-empty subset of ∂X of intrinsic radius at most $\pi/2$.

See also [Duc13, Theorem 4.3] for a more geometric proof of this statement. In particular, when $X = \mathbf{H}^\infty$, if the intersection of the visual boundaries is non-empty, then it is reduced to a point $\{\xi\}$.

Proposition 2.10. Let $\Gamma < \mathrm{Isom}(\mathbf{H}^n)$ be a subgroup which has no fixed point in $\mathbf{H}^n \cup \partial\mathbf{H}^n$. Then $\rho_t(\Gamma) < \rho_t(\mathrm{Isom}(\mathbf{H}^n)) \simeq \mathrm{Isom}(\mathcal{C}_t)$ has no fixed point in $\partial\mathbf{H}^\infty$, where $\mathcal{C}_t$ is the convex hull of $f_t(\mathbf{H}^n)$ inside $\mathbf{H}^\infty$.

Proof. Suppose for a contradiction that $\rho_t(\Gamma)$ has a fixed point $\xi \in \partial\mathbf{H}^\infty$. Fix a base point $x_0 \in \mathcal{C}_t$. The Busemann function restricted to $\mathcal{C}_t$, $\beta_{\xi,x_0}|_{\mathcal{C}_t}$, is convex and does not have a minimum on $\mathcal{C}_t$.

Indeed, if the minimum set $\mathrm{Min}(\beta_{\xi,x_0}|_{\mathcal{C}_t}) \subset \mathcal{C}_t$ is non-empty, it should lie inside some horosphere of $\mathbf{H}^\infty$ centered at ξ. It is moreover bounded, otherwise its closure in $\overline{\mathbf{H}^\infty}$ would contain ξ, and we would have $\xi \in \partial\mathcal{C}_t = \partial(f_t(\mathbf{H}^n))$ which is in contradiction with the second point of Lemma 2.7. Since $\mathcal{C}_t$ is at finite distance from $f_t(\mathbf{H}^n)$, there exists $K \in \mathbf{R}$ such that the $\rho_t(\Gamma)$-invariant set

$$\{x \in f_t(\mathbf{H}^n) \mid d(x, \mathrm{Min}(\beta_{\xi,x_0}|_{\mathcal{C}_t})) \leqslant K\}$$

is non-empty. It is moreover bounded and its preimage by f_t is a bounded subset of $\mathbf{H}^n$ which is Γ-invariant by ρ_t-equivariance of f_t. It then has a circumcentre which is a fixed point for Γ in $\mathbf{H}^n$, this contradicts the hypothesis on Γ. So $\beta_{\xi,x_0}|_{\mathcal{C}_t}$ does not have a minimum.

For $a > \inf(\beta_{\xi,x_0}|_{\mathcal{C}_t})$, define

$$C_a = \beta_{\xi,x_0}|_{\mathcal{C}_t}^{-1}((-\infty, a)) \subset \mathcal{C}_t.$$

All the C_a, for $a > \inf(\beta_{\xi,x_0}|_{\mathcal{C}_t})$, are convex and thus $\bigcap_{a>\inf(\beta_{\xi,x_0}|_{\mathcal{C}_t})} C_a = \emptyset$. Since $\mathbf{H}^\infty$ is δ-hyperbolic, its telescopic dimension is finite (it is actually equal to 1), hence $\bigcap_{a>\inf(\beta_{\xi,x_0}|_{\mathcal{C}_t})} \partial C_a$ is non-empty (by Theorem 2.9) and must contain ξ, so we deduce again that $\xi \in \partial\mathcal{C}_t = \partial(f_t(\mathbf{H}^n))$, which is impossible. □

For $X = \mathbf{H}^n$ or $\mathbf{H}^\infty$, the *limit set* of a subgroup $\Gamma < \mathrm{Isom}(X)$ is $\Lambda(\Gamma) := \overline{\Gamma\cdot o} \cap \partial X$ for any base point $o \in X$. A subgroup $\Gamma < \mathrm{Isom}(X)$ is called *non-elementary* if it has no fixed point in $X \cup \partial X$ and fixes no geodesic in X. Equivalently, Γ is elementary if and only if its limit set is finite.

Corollary 2.11. For $0 < t \leqslant 1$, if $A < \mathrm{Isom}(\mathbf{H}^n)$ is a discrete and non-elementary subgroup, then $\rho_t(A)$ is also non-elementary in $\mathrm{Isom}(\mathbf{H}^\infty)$.

Proof. The image $\rho_t(A)$ has no global fixed point in $\mathbf{H}^\infty$, otherwise A would consist only of elliptic isometries since ρ_t preserves the types of the isometries (see [MP14, Proposition 2.1]). Moreover, by Proposition 2.10, $\rho_t(A)$ also has no fixed point in $\partial\mathbf{H}^\infty$. And if $\rho_t(A)$ preserves a geodesic in $\mathbf{H}^\infty$ with endpoints $\xi, \eta \in \partial\mathbf{H}^\infty$, then it leaves the pair $\{\xi, \eta\}$ invariant. The limit set of $\rho_t(A)$ is then contained in $\{\xi, \eta\}$. This contradicts the fact that A is non-elementary since the ρ_t-equivariant embedding f_t extends to an injective map from the limit set of A to that of $\rho_t(A)$. □

Remark 2.12. When $t \neq 1$, the representation ρ_t is irreducible by Theorem 2.3 so the image $\rho_t(\mathrm{Isom}(\mathbf{H}^n))$ is irreducible (or geometrically Zariski dense) in $\mathrm{Isom}(\mathbf{H}^\infty)$. In view of Proposition 2.10 and Corollary 2.11, as we are studying convex-cocompact subgroups $A < \mathrm{Isom}(\mathbf{H}^n)$, it becomes natural to wonder whether such a group A satisfies some kind of density property as the one given by Theorem 2.8 for subgroups with finite covolume: if $A < \mathrm{Isom}(\mathbf{H}^n)$ is irreducible and convex-cocompact and $0 < t < 1$, is $\rho_t(A)$ irreducible in $\mathrm{Isom}(\mathbf{H}^\infty)$?

In Subsection 2.3, we will see that under the assumption that $\rho_t(A)$ is indeed irreducible in $\mathrm{Isom}(\mathbf{H}^\infty)$, one can assert that the deformations we will construct cannot be conjugate to one another, thus providing a large space of deformations up to conjugation.

2.2 Spectral representation for real Hilbert spaces

Before discussing the deformations of convex-cocompact representations by bending, we first recall the spectral representation theorem for operators on a real Hilbert space that we will use in the next section to show that the spaces of deformations are infinite-dimensional. In this paragraph, $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ refers to a real Hilbert space and $\mathfrak{B}(\mathcal{H})$ is the set of bounded linear operators on $\mathcal{H}$.

Definition 2.13. Let $T \in \mathfrak{B}(\mathcal{H})$ be a bounded operator on $\mathcal{H}$. The *adjoint* operator of T, denoted by T^*, is the (unique) bounded linear operator on $\mathcal{H}$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x, y \in \mathcal{H}$. The operator T is said to be *normal* if $TT^* = T^*T$.

In the finite-dimensional situation, every normal operator on a real Hilbert space is othogonally equivalent to a block-diagonal matrix of the form

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & C_1 & & \\ & & & & C_2 & \\ & & & & & \ddots \end{pmatrix}$$

where the λ_k are its real eigenvalues (counted with their multiplicities) and the C_i are 2×2 matrices of the form $\begin{pmatrix} a_k & -b_k \\ b_k & a_k \end{pmatrix}$ corresponding to the complex eigenvalues $a_k \pm i b_k$. There is an analog of this decomposition in the infinite-dimensional situation.

In [Goo72], Goodrich obtained a similar description for operators on infinite-dimensional real Hilbert spaces where the blocks correspond to operators of the form given in the following example.

Example 2.14 ([Goo72])**.** Let $X \subset \mathbf{C}$ be a compact subset which is symmetric about the real axis. Let μ be a regular Borel measure on X and consider the real Hilbert space $\mathcal{H} = \mathrm{L}^2(X, \mu)$ of real-valued square-integrable functions on X with respect to the measure μ. Suppose that μ is symmetric about the real axis, *i.e.* $\mu(U) = \mu(U^*)$ where U^* is the reflection of U about the real axis. Let $\mathcal{H}_e$ be the set of functions in $\mathcal{H}$ that are symmetric about the real axis and $\mathcal{H}_o$ those that are anti-symmetric (the subscripts refer to "even" and "odd"). We have $\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_o$.

Define the operator T on $\mathcal{H}$ by

$$Tf = \begin{pmatrix} M_{\mathrm{Re}} & -M_{\mathrm{Im}} \\ M_{\mathrm{Im}} & M_{\mathrm{Re}} \end{pmatrix} \begin{pmatrix} f_e \\ f_o \end{pmatrix},$$

where $f = f_e + f_o \in \mathcal{H}$, $\mathrm{Re}, \mathrm{Im} : \mathbf{C} \to \mathbf{R}$ denote the real an imaginary parts of a complex number and $M_\varphi : \mathrm{L}^2(\mathbf{C}) \to \mathrm{L}^2(\mathbf{C})$ is the operator defined by

$$M_\varphi(f)(z) = \varphi(z) f(z)$$

for a function $\varphi : \mathbf{C} \to \mathbf{C}$. Then T is a normal operator on $\mathcal{H}$ and its adjoint is

$$T^* f = {}^{\mathrm{t}}T f = \begin{pmatrix} M_{\mathrm{Re}} & M_{\mathrm{Im}} \\ -M_{\mathrm{Im}} & M_{\mathrm{Re}} \end{pmatrix} \begin{pmatrix} f_e \\ f_o \end{pmatrix}.$$

The following theorem shows that every normal operator is orthogonally equivalent to a direct sum of operators of this form.

Theorem 2.15 ([Goo72], Theorem 3)**.** Every bounded normal operator T on a real Hilbert space $\mathcal{H}$ is orthogonally equivalent to an orthogonal sum $\bigoplus_{i \in I} T_i$ of operators on spaces $\mathrm{L}^2(\mathbf{C}, \mu_i)$ of the type defined in Example 2.14.

See also [BJ19, Section 4.1] for a different description of the spectral representation that distinguishes between real and complex spectral values.

Proposition 2.16. Let T be an orthogonal operator on a real Hilbert space $\mathcal{H}$. Denote by Z_T its centraliser in $\mathrm{O}(\mathcal{H})$, $Z_T = \mathcal{Z}(T) \cap \mathrm{O}(\mathcal{H})$. Then Z_T is an infinite-dimensional Lie group.

Proof. The groups $\mathcal{Z}(T)$ and $\mathrm{O}(\mathcal{H})$ are both infinite-dimensional algebraic groups defined respectively by the polynomials $p_1(x) = xT - Tx$ and $p_2(x) = {}^{\mathrm{t}}xx - I$. So

$$Z_T = \mathcal{Z}(T) \cap \mathrm{O}(\mathcal{H}) = \{S \in \mathrm{GL}(\mathcal{H}) \mid p_1(S) = 0, p_2(S) = 0\}$$

where $\mathrm{GL}(\mathcal{H})$ is the set of invertible bounded linear operators of $\mathcal{H}$, and its Lie algebra is

$$\begin{aligned} \mathrm{Lie}(Z) &= \{S \in \mathfrak{B}(\mathcal{H}) \mid \mathrm{d}p_1(I) \cdot S = 0,\ \mathrm{d}p_2(I) \cdot S = 0\} \\ &= \{S \in \mathfrak{B}(\mathcal{H}) \mid ST - TS = 0,\ {}^{\mathrm{t}}S + S = 0\} \end{aligned}$$

The dimension of Z_T as a Lie group is that of its Lie algebra. We show that this dimension is infinite.

Since $T \in \mathrm{O}(\mathcal{H})$ is normal, by Theorem 2.15, we can decompose $\mathcal{H}$ as an orthogonal sum $\mathcal{H} = \bigoplus_{\alpha \in I} \mathcal{H}_\alpha$ where each $\mathcal{H}_\alpha$ is a space $\mathrm{L}^2(\mathbf{C}, \mu_\alpha)$ for some measure μ_α on $\mathbf{C}$ with compact support and which is symmetric about the real axis. The operator T preserves this decomposition and can also be decomposed as an orthogonal sum $T = \bigoplus_{\alpha \in I} T_\alpha$. Since T is orthogonal, we get $\mathrm{supp}(\mu_\alpha) \subset \mathbf{S}^1$ for every $\alpha \in I$. For $\alpha \in I$, if $f = f_e + f_o \in \mathcal{H}_\alpha = \mathrm{L}^2(\mathbf{C}, \mu_\alpha)$ and $\lambda \in \mathrm{supp}(\mu_\alpha)$, we have

$$T_\alpha f(\lambda) = \begin{pmatrix} M_{\mathrm{Re}} & -M_{\mathrm{Im}} \\ M_{\mathrm{Im}} & M_{\mathrm{Re}} \end{pmatrix} \begin{pmatrix} f_e \\ f_o \end{pmatrix} (\lambda) = \begin{pmatrix} \mathrm{Re}(\lambda) f_e(\lambda) - \mathrm{Im}(\lambda) f_o(\lambda) \\ \mathrm{Im}(\lambda) f_e(\lambda) + \mathrm{Re}(\lambda) f_o(\lambda) \end{pmatrix}.$$

Now let $S \in \mathfrak{B}(\mathcal{H})$ be a bounded operator such that S decomposes along $\bigoplus_{\alpha \in I} \mathcal{H}_\alpha$ into the orthogonal sum $\bigoplus_{\alpha \in I} S_\alpha$ such that each S_α acts on $\mathcal{H}_\alpha$ as

$$S_\alpha f = \begin{pmatrix} 0 & -M_{g_\alpha} \\ M_{g_\alpha} & 0 \end{pmatrix} \begin{pmatrix} f_e \\ f_o \end{pmatrix},$$

where $g_\alpha : \mathrm{supp}(\mu_\alpha) \to \mathbf{R}$ is a continuous or measurable map. Then S_α is anti-symmetric and commutes with T_α. Thus the map $g_\alpha \mapsto S_\alpha$ provides an embedding of $\mathcal{C}(\mathrm{supp}(\mu_\alpha), \mathbf{R})$ into $\mathrm{Lie}(Z_T)$.

If there exists $\alpha \in I$ such that $\mathrm{supp}(\mu_\alpha)$ is infinite, then $\mathcal{C}(\mathrm{supp}(\mu_\alpha), \mathbf{R})$ is an infinite-dimensional vector space. Hence the dimension of $\mathrm{Lie}(Z_T)$ is infinite. Otherwise, if all the $\mathrm{supp}(\mu_\alpha)$ are finite, then I is infinite. This means that T has only eigenvalues and is then orthogonally equivalent to an infinite block-diagonal matrix where the blocks on the diagonal are rotation matrices $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ corresponding to complex eigenvalues $\cos(\theta) \pm i \sin(\theta)$. In this case, any anti-symmetric matrix $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ commutes with $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ and belong to $\mathrm{Lie}(Z_T)$.

□

Proposition 2.17. Let $\gamma \in \mathrm{Isom}(\mathbf{H}^n)$ be a loxodromic isometry and $\rho_t : \mathrm{Isom}(\mathbf{H}^n) \to \mathrm{Isom}(\mathbf{H}^\infty)$ be an irreducible representation from Theorem 2.3. Then the centraliser of $\rho_t(\gamma)$ in $\mathrm{Isom}(\mathbf{H}^\infty)$ is an infinite-dimensional Lie group consisting of elliptic isometries fixing the axis of $\rho_t(\gamma)$ pointwise, and loxodromic isometries which share the same axis with $\rho_t(\gamma)$.

Proof. The representation ρ_t preserves the type of the isometries, so $\rho_t(\gamma) \in \mathrm{Isom}(\mathbf{H}^\infty)$ is loxodromic. Denote by $\xi^\pm \in \partial\mathbf{H}^\infty$ the fixed points of $\rho_t(\gamma)$. They correspond to two independant vectors v^+ and v^- in the light cone $\{x \in \mathcal{H} \mid Q(x) = 0\}$. We may assume that $B(v^+, v^-) = -1$. Let $E = (\mathbf{R}\cdot v^+ \oplus \mathbf{R}\cdot v^-)^\perp$ be the orthogonal complement of $\mathbf{R}\cdot v^+ \oplus \mathbf{R}\cdot v^-$ in $\mathcal{H}$. The restriction of the bilinear form B to E is a scalar product, so E is a real Hilbert space. We can write the isometry $\rho_t(\gamma)$ in a matrix form according to the decomposition $\mathcal{H} = \mathbf{R}\cdot v^+ \oplus \mathbf{R}\cdot v^- \oplus E$,

$$\rho_t(\gamma) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & T \end{pmatrix}$$

where $\lambda > 1$ and $T \in \mathrm{O}(E)$ is an orthogonal transformation of E.

If $g \in \mathrm{Isom}(\mathbf{H}^\infty)$ commutes with $\rho_t(\gamma)$, then g stabilises the axis of $\rho_t(\gamma)$ and acts on it either as the identity or as a translation. In the first case, g is elliptic and can be written in the decomposition $\mathcal{H} = \mathbf{R}\cdot v^+ \oplus \mathbf{R}\cdot v^- \oplus E$ as

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & T' \end{pmatrix}$$

where $T' \in Z_T$, the centraliser of T in $\mathrm{O}(E)$. And in the second case, g has the form

$$g = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu^{-1} & 0 \\ 0 & 0 & T' \end{pmatrix}$$

with $\mu > 0$, $\mu \neq 1$ and $T' \in Z_T$, g is then loxodromic with translation length $\max(\mu, \mu^{-1})$.

Conversely, any operator of the form

$$\begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu^{-1} & 0 \\ 0 & 0 & T' \end{pmatrix}$$

for $\mu > 0$ and $T' \in Z_T$ do commute with $\rho_t(\gamma)$.

Therefore, the centraliser of $\rho_t(\gamma)$ in $\mathrm{Isom}(\mathbf{H}^\infty)$ can be identified with $\mathbf{R}_+^* \times Z_T$ where Z_T is an infinite-dimensional Lie group by Proposition 2.16. □

2.3 Bending irreducible representations from $\mathrm{Isom}(\mathbf{H}^2)$

The rigidity theorem of Mostow states that if Γ is a lattice in $\mathrm{PO}(n,1)$ with $n \geqslant 3$, and $\rho : \Gamma \to \mathrm{PO}(n,1)$ is a faithful representation such that $\rho(\Gamma)$ is a lattice, then Γ and $\rho(\Gamma)$ are conjugate in $\mathrm{PO}(n,1)$. In other words, there is a unique hyperbolic structure on an n-dimensional manifold of finite volume up to isometry. However, for some lattices Γ, there exist many faithful and discrete non-conjugate representations $\Gamma \to \mathrm{PO}(m,1)$, where $2 \leqslant n < m$. Bending is a way to construct examples of such non-conjugate representations.

In this section, we only consider representations ρ_t from $\mathrm{Isom}(\mathbf{H}^2) = \mathrm{PO}(2,1)$ to $\mathrm{Isom}(\mathbf{H}^\infty) = \mathrm{PO}(\infty,1)$ as in Theorem 2.3 for $n = 2$. Let $0 < t \leqslant 1$ and let $\Gamma < \mathrm{Isom}(\mathbf{H}^2)$ be the fundamental group of a hyperbolic closed surface (compact, connected, orientable and without boundary), $\Gamma = \pi_1(S)$. We are going to use this technique to deform $\rho_t(\Gamma)$ in $\mathrm{Isom}(\mathbf{H}^\infty)$. The deformation is defined depending on whether Γ is written as a free product with amalgamation or as an HNN extension. Since S is a closed surface of genus $g \geqslant 2$, we can always decompose Γ in both forms.

Suppose first that $\alpha \subset S$ is a simple closed geodesic which separates S into two connected components S_1 and S_2 that are non-contractible. Then by Van Kampen theorem, the fundamental group of S is an amalgamated product, $\pi_1(S) = \Gamma = A *_C B$, where $A = \pi_1(S_1)$, $B = \pi_1(S_2)$ and $C = \pi_1(\alpha)$. In particular, the group C preserves and acts cocompactly on a geodesic in $\mathbf{H}^2$ (a totally geodesic copy of $\mathbf{H}^1 \simeq \mathbf{R}$). Moreover, this group is cyclic, let $c \in C$ be a generator. If $\mathcal{Z}(\rho_t(c)) = \mathcal{Z}(\rho_t(C))$ is the centraliser of $\rho_t(C)$ in $\mathrm{Isom}(\mathbf{H}^\infty)$, then any $z \in \mathcal{Z}(\rho_t(c))$ provides a deformation of $\rho_t(\Gamma)$ by conjugating the elements of B:

$$\begin{array}{rccc} \sigma_{z,t}: & \Gamma = A *_C B & \to & \mathrm{Isom}(\mathbf{H}^\infty) \\ & a \in A & \mapsto & \rho_t(a) \\ & b \in B & \mapsto & z\rho_t(b)z^{-1}. \end{array}$$

Bending can also be defined when the cocompact lattice is an HNN extension. If α is now a non-separating simple closed geodesic of the surface S, then $\pi_1(S) = \Gamma = A*_C$ where $A = \pi_1(S \setminus \alpha)$ and $C = \pi_1(\alpha)$. The surface $S \setminus \alpha$ has a single connected component but two boundary components which induce two injections of C into A, denoted by f_1 and f_2. Let us denote by s the stable letter. We have

$$\Gamma = \langle A, s \mid \forall c \in C \; f_1(c) = s f_2(c) s^{-1} \rangle.$$

In a similar fashion, for $z \in \mathcal{Z}(\rho_t(c))$, we can deform $\rho_t(\Gamma)$:

$$\begin{array}{rccc} \sigma_{z,t}: & \Gamma = A*_C & \to & \mathrm{Isom}(\mathbf{H}^\infty) \\ & a \in A & \mapsto & \rho_t(a) \\ & s & \mapsto & z\rho_t(s). \end{array}$$

Lemma 2.18. Let $\Gamma < \mathrm{Isom}(\mathbf{H}^\infty)$ be any subgroup. If Γ has no fixed point on the boundary of $\mathbf{H}^\infty$ and leaves no proper totally geodesic subspace of $\mathbf{H}^\infty$ invariant, then the centraliser of Γ is trivial.

Proof. Let $\Gamma < \mathrm{Isom}(\mathbf{H}^\infty)$ and g be an element in the centraliser of Γ in $\mathrm{Isom}(\mathbf{H}^\infty)$.

- If g is elliptic, then the set of fixed points $\mathrm{Fix}(g) \subset \mathbf{H}^\infty$ is totally geodesic and Γ-invariant.
- If g is hyperbolic, the axis of g is a geodesic in $\mathbf{H}^\infty$ which is Γ-invariant.
- If g is parabolic, denote by $\xi \in \partial\mathbf{H}^\infty$ its fixed point in the boundary of $\mathbf{H}^\infty$. By Theorem 2.9, we have
$$\{\xi\} = \bigcap_{\epsilon>0} \overline{\{y \in \mathbf{H}^\infty \mid d(y, g(y)) \leqslant \epsilon\}},$$
where all the $\{y \in \mathbf{H}^\infty \mid d(y, g(y)) \leqslant \epsilon\}$ are closed, convex and Γ-invariant sets. Thus ξ must be a fixed point for Γ.

It follows that when Γ is irreducible, g must be elliptic and $\mathrm{Fix}(g) = \mathbf{H}^\infty$, so the centraliser of Γ is reduced to the identity element of $\mathrm{Isom}(\mathbf{H}^\infty)$. □

This statement also holds in finite dimension (see for example the second corollary of [Rat06, Theorem 12.2.6]).

Proposition 2.19. Let Γ be a cocompact lattice in $\mathrm{Isom}(\mathbf{H}^2)$ and $\rho_t : \mathrm{Isom}(\mathbf{H}^2) \to \mathrm{Isom}(\mathbf{H}^\infty)$ be an irreducible representation, $0 < t < 1$. Suppose that $\Gamma = A *_C B$ or $\Gamma = A*_C$ and let $\mathcal{Z}(\rho_t(c))$ be the centraliser of $\rho_t(c)$ in $\mathrm{Isom}(\mathbf{H}^\infty)$. If the image by ρ_t of an irreducible convex-cocompact group is irreducible, then for $z_1 \neq z_2 \in \mathcal{Z}(\rho_t(c))$, the deformations $\sigma_{z_1,t}$ and $\sigma_{z_2,t}$ are non-conjugate.

Proof. In either cases ($\Gamma = A *_C B$ or $\Gamma = A*_C$), suppose for a contradiction that $\sigma_{z_1,t}$ and $\sigma_{z_2,t}$ are conjugate by an element $g \in \mathrm{Isom}(\mathbf{H}^\infty)$, $\sigma_{z_1,t} = g\sigma_{z_2,t}g^{-1}$. Restricting the representations to $A < \Gamma$, we get
$$\rho_t|_A = \sigma_{z_1,t}|_A = g\sigma_{z_2,t}|_A g^{-1} = g\rho_t|_A g^{-1},$$
so $\rho_t|_A = g\rho_t|_A g^{-1}$. Since $\rho_t(A)$ is irreducible, its centraliser is trivial by Lemma 2.18, thus g must be the identity element of $\mathrm{Isom}(\mathbf{H}^\infty)$.

If $\Gamma = A *_C B$, then restricting to B, we now have $\sigma_{z_1,t}|_B = \sigma_{z_2,t}|_B$, so $z_1\rho_t|_B z_1^{-1} = z_2\rho_t|_B z_2^{-1}$ and $\rho_t|_B = (z_1^{-1}z_2)\rho_t|_B(z_1^{-1}z_2)^{-1}$. Since $\rho_t(B)$ is also irreducible, Lemma 2.18 implies that $z_1^{-1}z_2 = \mathrm{id}$, so $z_1 = z_2$.

If now $\Gamma = A*_C$, evaluating on the stable letter gives $z_1\rho_t(s) = z_2\rho_t(s)$, *i.e.* $\rho_t(s) = z_1^{-1}z_2\rho_t(s)$. Then the fixed point set $\mathrm{Fix}(z_1^{-1}z_2)$ of the isometry $z_1^{-1}z_2$ must contain the image of $\rho_t(s)$, so $\mathbf{H}^\infty \subset \mathrm{Fix}(z_1^{-1}z_2)$ and thus $z_1 = z_2$. □

Proposition 2.20. Let Γ be the fundamental group of a closed hyperbolic surface and $\rho_t : \mathrm{Isom}(\mathbf{H}^2) \to \mathrm{Isom}(\mathbf{H}^\infty)$ be an irreducible representation, $0 < t < 1$. If the image by ρ_t of an irreducible convex-cocompact group is irreducible, then the space of convex-cocompact representations of Γ into $\mathrm{Isom}(\mathbf{H}^\infty)$ up to conjugation has infinite dimension.

Proof. Since $\Gamma < \mathrm{Isom}(\mathbf{H}^2)$ is cocompact, the image $\rho_t(\Gamma)$ is convex-cocompact. Write Γ as a free product with amalgamation $A *_C B$ or as an HNN extension $A*_C$. Let $\mathcal{Z}(\rho_t(c))$ be the centraliser of $\rho_t(c)$ inside $\mathrm{Isom}(\mathbf{H}^\infty)$. Any distinct z_1 and z_2 in $\mathcal{Z}(\rho_t(c))$ provide non-conjugate representations of Γ into $\mathrm{Isom}(\mathbf{H}^\infty)$ by Proposition 2.19. Since the space of convex-cocompact representations of Γ into $\mathrm{Isom}(\mathbf{H}^\infty)$ is open by Corollary 1.10, it contains an open subset of $\mathcal{Z}(\rho_t(c))$ which is infinite-dimensional as a Lie group by Proposition 2.17. □

Lemma 2.21. Let Γ be the fundamental group of a closed hyperbolic surface and write $\Gamma = A *_C B$ or $\Gamma = A*_C$. Let $\mathcal{Z}(\rho_t(c))$ be the centraliser of $\rho_t(c)$, $z_1, z_2 \in \mathcal{Z}(\rho_t(c))$ and $0 < t_1, t_2 \leqslant 1$. If the deformations σ_{z_1,t_1} and σ_{z_2,t_2} are conjugate in $\mathrm{Isom}(\mathbf{H}^\infty)$, then $t_1 = t_2$.

Proof. Suppose that there exists $g \in \mathrm{Isom}(\mathbf{H}^\infty)$ such that $\sigma_{z_1,t_1} = g\sigma_{z_2,t_2}g^{-1}$ on Γ. For any loxodromic element $a \in A$, computing the translation lengths, we have on the one side

$$\ell_{\mathbf{H}^\infty}(\sigma_{z_1,t_1}(a)) = \ell_{\mathbf{H}^\infty}(\rho_{t_1}(a)) = t_1\ell_{\mathbf{H}^2}(a),$$

and on the other side

$$\ell_{\mathbf{H}^\infty}(g\sigma_{z_2,t_2}(a)g^{-1}) = \ell_{\mathbf{H}^\infty}(\rho_{t_2}(a)) = t_2\ell_{\mathbf{H}^2}(a).$$

Therefore, $t_1 = t_2$ since $\ell_{\mathbf{H}^2}(a) > 0$. □

Proposition 2.22. Let Γ be the fundamental group of a closed hyperbolic surface and let $0 < t < 1$. If the image by ρ_t of an irreducible convex-cocompact group is irreducible in $\mathrm{Isom}(\mathbf{H}^\infty)$, then the representation of Γ obtained by bending are not conjugate to the restriction of any irreducible representation $\mathrm{Isom}(\mathbf{H}^2) \to \mathrm{Isom}(\mathbf{H}^\infty)$.

Proof. Let $\mathcal{Z}(\rho_t(c))$ be the centraliser of $\rho_t(c)$ in $\mathrm{Isom}(\mathbf{H}^\infty)$. Suppose that $\sigma_{z,t}$ is conjugate to the restriction to Γ of some representation $\rho_{t'} : \mathrm{Isom}(\mathbf{H}^2) \to \mathrm{Isom}(\mathbf{H}^\infty)$ for $0 < t' < 1$. Then $\sigma_{z,t} = g\rho_{t'}|_\Gamma g^{-1}$ for some $g \in \mathrm{Isom}(\mathbf{H}^\infty)$. Since $\rho_{t'}|_\Gamma = \sigma_{\mathrm{id},t'}$, we get $t = t'$ by Lemma 2.21. Moreover, from $\sigma_{z,t} = g\rho_t|_\Gamma g^{-1} = g\sigma_{\mathrm{id},t}g^{-1}$, we deduce that z is the identity by Proposition 2.19. □

Remark 2.23. In Propositions 2.19, 2.20 and 2.22, we heavily use the assumption that the image by ρ_t of an irreducible convex-cocompact subgroup remains irreducible. Yet, we could not ascertain that this fact holds.

However, without this assumption, one can still show that there exists uncountably many deformations that are pairwise non-conjugate and that they do not arise as conjugates of restrictions of an exotic representation ρ_t.

In the following, we only consider the case where $\Gamma = A *_C B$, *i.e.* the geodesic α separates the surface S into two non-contractible components so that $\Gamma = \pi_1(S) < \mathrm{Isom}(\mathbf{H}^2)$ is decomposed as $\Gamma = A *_C B$, where $C = \langle c \rangle \simeq \mathbf{Z}$. Let $\mathcal{Z}(\rho_t(c))$ denote the centraliser of $\rho_t(c)$ in $\mathrm{Isom}(\mathbf{H}^\infty)$.

Lemma 2.24. Let $0 < t \leqslant 1$ and $A < \mathrm{Isom}(\mathbf{H}^2)$ be a non-elementary discrete subgroup. If for all $a \in A$, $\rho_t(a) = g\rho_t(a)g^{-1}$ for some $g \in \mathrm{Isom}(\mathbf{H}^\infty)$, then g is elliptic and fixes pointwise the axis of every loxodromic isometry $\rho_t(a)$ for $a \in A$.

Proof. If the isometry g is parabolic, it has a unique fixed point in the boundary of $\mathbf{H}^\infty$. Since $\rho_t(a)$ commutes with g for every $a \in A$, then $\rho_t(A)$ has a global fixed point in $\partial\mathbf{H}^\infty$, which is in contradiction with Proposition 2.10.

If g is now loxodromic, it has two fixed points $\xi, \eta \in \partial\mathbf{H}^\infty$ in the boundary of $\mathbf{H}^\infty$. Then, the fixed point sets of all the $\rho_t(a)$, for $a \in A$, are contained in $\{\xi, \eta\}$, which is in contradiction with Corollary 2.11.

Therefore g must be elliptic. Let $a \in A$ be a loxodromic isometry, its image $\rho_t(a)$ has the same type. Denote by L_a the axis of $\rho_t(a)$. Since $\rho_t(a) = g\rho_t(a)g^{-1}$, g preserves L_a. Restricted to this axis, g then acts either as the identity, or with exactly one fixed point, *i.e.* as a reflection. The second case is excluded since g commutes with $\rho_t(a)$ which acts by translation on L_a. □

Proposition 2.25. Let $0 < t \leqslant 1$. If $z_1, z_2 \in \mathcal{Z}(\rho_t(c))$ are such that $\ell_{\mathbf{H}^\infty}(z_1) \neq \ell_{\mathbf{H}^\infty}(z_2)$, then $\sigma_{z_1,t}$ and $\sigma_{z_2,t}$ are not conjugate in $\mathrm{Isom}(\mathbf{H}^\infty)$.

Proof. If $g \in \mathrm{Isom}(\mathbf{H}^\infty)$ is such that $\sigma_{z_1,t} = g\sigma_{z_2,t}g^{-1}$ on $\Gamma = A *_C B$, then for every element $a \in A$, we have $\rho_t(a) = g\rho_t(a)g^{-1}$, and for every $b \in B$, we get $z_1\rho_t(b)z_1^{-1} = gz_2\rho_t(b)z_2^{-1}g^{-1}$, so $\rho_t(b) = (z_1^{-1}gz_2)\rho_t(b)(z_1^{-1}gz_2)^{-1}$. Since A and B are both discrete and non-elementary, g and $z_1^{-1}gz_2$ must be elliptic by Lemma 2.24. If L_c is the axis of $\rho_t(c)$, we have moreover that g acts as the identity on L_c. For $i \in \{1, 2\}$, z_i acts on L_c as a translation by $\ell_{\mathbf{H}^\infty}(z_i)$ according to Proposition 2.17 (if one of the z_i is elliptic, then $\ell_{\mathbf{H}^\infty}(z_i) = 0$ and its action of L_c is trivial). It follows that $z_1^{-1}gz_2$ acts on L_c as a translation by $\ell_{\mathbf{H}^\infty}(z_2) - \ell_{\mathbf{H}^\infty}(z_1) \neq 0$, so $z_1^{-1}gz_2$ is a loxodromic isometry. □

Corollary 2.26. Let $0 < t \leqslant 1$ and $z \in \mathcal{Z}(\rho_t(c))$ be a loxodromic element. The representation $\sigma_{z,t} : \Gamma \to \mathrm{Isom}(\mathbf{H}^\infty)$ obtained by bending with z is not conjugate in $\mathrm{Isom}(\mathbf{H}^\infty)$ to the restriction of any irreducible representation $\rho_{t'} : \mathrm{Isom}(\mathbf{H}^2) \to \mathrm{Isom}(\mathbf{H}^\infty)$ for $0 < t' \leqslant 1$.

Proof. Suppose by contradiction that there exists $g \in \mathrm{Isom}(\mathbf{H}^\infty)$ such that $\sigma_{z,t} = g\rho_{t'}g^{-1}$ on $\Gamma = A *_C B$. Since $\rho_{t'}|_\Gamma = \sigma_{\mathrm{id},t'}$, we have $t = t'$ by Lemma 2.21. We are then left with $\sigma_{z,t} = g\sigma_{\mathrm{id},t}g^{-1}$. This is impossible due to Proposition 2.25. □

Recall from Proposition 2.17 that the centraliser $\mathcal{Z}(\rho_t(c))$ contains infinitely many loxodromic elements. The previous results show that there is at least a one-parameter family of elements in $\mathcal{Z}(\rho_t(c))$ providing non-conjugate deformations of the group Γ, none of which being conjugate to the restriction of any exotic representation.

Corollary 2.27. Let $0 < t \leqslant 1$ and Γ be the fundamental group of a closed hyperbolic surface. Then the space of deformations of $\rho_t|_\Gamma$, up to conjugation in $\mathrm{Isom}(\mathbf{H}^\infty)$, contains a one-parameter family of representations which are not pairwise conjugate, nor conjugate to the restriction of any $\rho_{t'}$, for $0 < t' \leqslant 1$.

Proof. Let $\mathcal{Z}(\rho_t(c))$ be the centraliser of $\rho_t(c)$ in $\mathrm{Isom}(\mathbf{H}^\infty)$. Write $\rho_t(c)$ in a suitable basis as in Proposition 2.17 in the form $\rho_t(c) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & T \end{pmatrix}$, with $\lambda > 1$. Fix any T' commuting with T. Then for every $\mu > 1$, the isometry

$$z_\mu = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu^{-1} & 0 \\ 0 & 0 & T' \end{pmatrix}$$

belongs to $\mathcal{Z}(\rho_t(c))$. Moreover, these isometries have distinct translation lengths μ, so they satisfy Proposition 2.25 and Corollary 2.26. □

Remark 2.28. In particular, the cocompact lattice $\Gamma < \mathrm{Isom}(\mathbf{H}^2)$ has more representations into $\mathrm{Isom}(\mathbf{H}^\infty)$ than the whole group $\mathrm{Isom}(\mathbf{H}^2)$, and its space of convex-cocompact representations is not reduced to the restrictions of the exotic representations ρ_t.

When $n \geqslant 3$, let $\rho : \mathrm{Isom}(\mathbf{H}^n) \to \mathrm{Isom}(\mathbf{H}^\infty)$ be an irreducible representation. The previous construction does not provide deformations of this representation ρ in $\mathrm{Isom}(\mathbf{H}^\infty)$. Indeed, let M be a compact hyperbolic n-manifold with fundamental group $\Gamma < \mathrm{Isom}(\mathbf{H}^n)$. If $N \subset M$ is a totally geodesic hypersurface, then $C := \pi_1(N)$ preserves and acts cocompactly on a totally geodesic copy of $\mathbf{H}^{n-1}$ inside $\mathbf{H}^n$ and $N \simeq \mathbf{H}^{n-1}/C$. As previously, if N is separates M into two connected components M_1 and M_2 which are non-contractible, we can split Γ into an amalgamated product $\Gamma = A *_C B$ with $A = \pi_1(M_1)$ and $\pi_1(M_2)$. And if N is non-separating, then Γ writes as the HNN extension $\Gamma = A*_C$ where $A = \pi_1(M \setminus N)$. The universal covers $\widetilde{M_1}$ and $\widetilde{M_2}$ are simply connected complete hyperbolic manifolds with totally geodesic boundaries, thus they are isometric to some intersection of half-spaces in $\mathbf{H}^n$ with disjoint boundaries ([Mar23, Theorem 3.5.2]). Their fundamental groups A and B are thus convex-cocompact in $\mathrm{Isom}(\mathbf{H}^n)$.

The group C is finitely generated as the fundamental group of a compact hyperbolic manifold, however it is not cyclic anymore. Let $S = \{c_1, \ldots, c_m\}$ be a finite generating set for C. The centraliser of $\rho(C)$ in $\mathrm{Isom}(\mathbf{H}^\infty)$ is the intersection of the $\mathcal{Z}(\rho(c_k))$, $1 \leqslant k \leqslant m$, and might be trivial even though each $\mathcal{Z}(\rho(c_k))$ is as described in Proposition 2.17.

Remark 2.29. Note that hyperbolic manifolds with totally geodesic hypersurfaces do exist in any dimension, see for example [JM87, chapter 7] and [Kap06, chapter 3] for constructions using arithmetic lattices. It was actually proved in [BFMS21] that a finite volume hyperbolic manifold with infinitely many totally geodesic hypersurfaces is necessarily arithmetic.

Chapter IV

Groups generated by reflections

In this chapter, we construct Coxeter groups acting on the infinite-dimensional hyperbolic space. These actions will be irreducible. However, we will see that the discreteness properties provided by the theory in finite dimension does not hold when the group is infinitely generated.

1 Some background on Coxeter groups

1.1 Coxeter groups

Definition 1.1. Let S be a set and let $M = (m_{s,t})_{s,t \in S}$ be a matrix such that

- for all $s, t \in S$, $m_{s,t} \in \mathbf{N} \cup \{\infty\}$;
- M is symmetric;
- for all $s \in S$, $m_{s,s} = 1$;
- for all $s \neq t \in S$, $m_{s,t} \geqslant 2$.

Such a matrix is called a *Coxeter matrix of type S*. The (abstract) *Coxeter group* associated to M is the group defined by the following presentation

$$W = \langle s \in S \mid \forall s, t \in S,\ (st)^{m_{s,t}} = 1 \rangle.$$

Note that every generator $s \in S$ has order 2. When $m_{s,t} = \infty$, then there is no relation between the two generators s and t. The pair (W, S) is called a *Coxeter system*. We will indifferently use either the Coxeter system (W, S) or the Coxeter group W in what follows.

To a Coxeter system (W, S), one can associate the $S \times S$ matrix $A = (a_{s,t})_{s,t \in S}$ where $a_{s,t} = -2\cos\left(\dfrac{\pi}{m_{s,t}}\right)$. This matrix is called the *Cartan matrix*. If $m_{s,t} = \infty$, we consider that $\dfrac{\pi}{m_{s,t}} = 0$, so that $a_{s,t} = -2$. The diagonal entries of A are equal to 2 and all the off-diagonal entries are non-positive.

Remark 1.2. Some authors prefer using the so-called *cosine matrix* C instead of the Cartan matrix A. These two matrices only differ by a factor 2, we have $A = 2C$.

Let (W, S) be a Coxeter system. One can also represent the Coxeter system as a diagram, the *Coxeter diagram*, defined as a graph such that

- the set of vertices is S;
- if $s, t \in S$, there is an edge between s and t if and only if $m_{s,t} \geqslant 3$;
- an edge between s and t is labelled by $m_{s,t}$ (when $m_{s,t} = 3$, the label is usually omitted).

Example 1.3.

- The Coxeter diagram of the dihedral group of order $2m$ is $\bullet\overset{m}{\text{——}}\bullet$. Its Cartan matrix is the 2×2 matrix $\begin{pmatrix} 2 & -2\cos\left(\dfrac{\pi}{m}\right) \\ -2\cos\left(\dfrac{\pi}{m}\right) & 2 \end{pmatrix}$.
- The symmetric group $\mathfrak{S}_n$ (group of permutations of the integers $\{1, \ldots, n\}$) is isomorphic to the Coxeter group

$$W = \langle s_1, \ldots, s_{n-1} \mid s_i^2 = 1,\ (s_i s_{i+1})^3 = 1 \rangle$$

whose Cartan matrix is of the form

$$\begin{pmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \ldots & -1 & 2 \end{pmatrix}$$

and whose Coxeter diagram is of the form $\bullet\text{——}\bullet\cdots\bullet\text{——}\bullet$. This Coxeter group is called Coxeter group of type A_{n-1}. The isomorphism with $\mathfrak{S}_n$ is given by $s_i \mapsto \tau_{i,i+1}$ where $\tau_{i,i+1}$ denotes the transposition permuting i and $i+1$, for $i \in \{1, \ldots, n-1\}$.

Definition 1.4. A Coxeter group is *irreducible* if its Coxeter diagram is connected.

Irreducibility of a Coxeter group can also be defined using its Cartan matrix.

Definition 1.5. A matrix $A \in \mathcal{M}_n(\mathbf{R})$ is *indecomposable* if it is not equivalent to a block-diagonal matrix of the form

$$A' = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right),$$

i.e. there exist no matrices $P, Q \in \mathrm{GL}(n, \mathbf{R})$ such that $P^{-1}AQ = A'$.

If $A = (a_{s,t})_{s,t \in S}$ is the Cartan matrix of a Coxeter group W. Let $A^* = (a^*_{s,t})_{s,t \in S}$ be the matrix defined by

$$a^*_{i,j} = \begin{cases} 1 & \text{if } a_{i,j} \neq 0 \text{ or } a_{j,i} \neq 0 \\ 0 & \text{if } a_{i,j} = 0 \text{ and } a_{j,i} = 0 \end{cases}.$$

Then A^* is the incidence matrix of the Coxeter diagram of W which is connected if and only if A^* is indecomposable. Therefore, a Coxeter group is irreducible if and only if its Cartan matrix is indecomposable.

In a Coxeter diagram, two vertices s and t are not joined by an edge if $m_{s,t} = 2$, *i.e.* s and t commute in the group. In particular, any Coxeter group W is the product of the groups corresponding to each connected component of its Coxeter diagram.

1.2 Representations of Coxeter groups

Groups generated by reflections

Let X be a Euclidean space, a sphere or a hyperbolic space (of any dimension, not necessarily finite).

Definition 1.6. A *reflection* of X is an isometry $\sigma : X \to X$ of order 2 ($\sigma^2 = \sigma \circ \sigma = \mathrm{id}$) such that the fixed points set $\mathrm{Fix}(\sigma)$ separates X into two connected components.

Remark 1.7. The fixed points set $\mathrm{Fix}(\sigma)$ is necessarily a codimension one totally geodesic subspace of X, *i.e.* a hyperplane.

For $n \in \mathbf{N}$, we can embed $\mathbf{S}^n$ and $\mathbf{H}^n$ into $\mathbf{R}^{n+1}$ as the unit sphere or with the hyperboloid model, respectively. In both cases, a hyperplane in $X = \mathbf{S}^n$ or $\mathbf{H}^n$ is the intersection $X \cap H$ of X with a hyperplane H in $\mathbf{R}^{n+1}$.

In the Euclidean space $\mathbf{R}^n$, a linear transformation σ is a reflection if $\sigma^2 = \mathrm{id}$ and σ has -1 as a simple eigenvalue. Every reflection $\sigma : \mathbf{R}^n \to \mathbf{R}^n$ can be written in the form

$$\sigma = \mathrm{id} - \alpha \otimes b,$$

where $\alpha \in (\mathbf{R}^n)^*$ is a linear form on $\mathbf{R}^n$ and $b \in \mathbf{R}^n$ is such that $\alpha(b) = 2$.

Example 1.8. In the Euclidean space $\mathbf{R}^n$, endowed with the scalar product $\langle \cdot, \cdot \rangle$, a linear reflection can be defined with one unitary vector $x_0 \in \mathbf{R}^n$ by

$$\sigma : v \mapsto v - 2\langle x_0, v \rangle x_0.$$

The corresponding hyperplane of reflection is $x_0^\perp = \{v \in \mathbf{R}^n \mid \langle x_0, v \rangle = 0\}$.

The geometric representation

A Coxeter system (W, S) has a natural representation as a group generated by reflections in a Euclidean space. Denote by $M = (m_{s,t})_{s,t\in S}$ the Coxeter matrix of (W, S).

Definition 1.9. Let $V = \mathbf{R}^{(S)}$ and let $(e_s)_{s\in S}$ be the canonical basis of V. We define the bilinear form B_M on V by

$$B_M(e_s, e_t) = \begin{cases} -\cos\left(\dfrac{\pi}{m_{s,t}}\right) & \text{if } s \neq t \\ 1 & \text{if } s = t \end{cases}.$$

The form B_M is symmetric since M is. It is called the *Tits form* of the Coxeter group W. For all $s, t \in S$, we have $B_M(e_s, e_s) = 1$ and $B_M(e_s, e_t) \leqslant 0$ if $s \neq t$. Actually, $2B_M(e_s, e_t)$ is the (s, t) entry of the Cartan matrix.

Let $s \in S$, and let α_s be the bilinear form: $x \mapsto 2B_M(e_s, x)$. We denote by σ_s the map

$$\sigma_s : x \mapsto x - \alpha_s(x)e_s = x - 2B_M(e_s, x)e_s.$$

Proposition 1.10 ([Bou02], Proposition V.4.3.3)**.** There exists a unique homomorphism $\sigma : W \to \mathrm{GL}(V)$ such that $\sigma(s) = \sigma_s$. Every linear map in the image $\sigma(W)$ preserve the bilinear form B_M. We call σ the *geometric representation* of W.

Remark 1.11. If A is the Cartan matrix of (W, S), observe that α_s can be identified with the "s-th" row of A as an element of the dual V^*.

For $s \in S$, since e_s is not isotropic for B_M (*i.e.* $B_M(e_s, e_s) \neq 0$), the space V is the direct sum of the line $\mathbf{R}\cdot e_s$ and the hyperplane H_s orthogonal to e_s with respect to B_M, $H_s = e_s^{\perp}$. Since σ_s is equal to $-\,\mathrm{id}$ on $\mathbf{R}\cdot e_s$ and to id on H_s, it follows that σ_s preserves the form B. The geometric representation is a linear representation into the orthogonal group of E for the bilinear form B_M, $\sigma : W \to \mathrm{O}(V, B_M)$.

The contragredient representation

Let V^* be the algebraic dual of V. Since W acts on V via σ, it also acts on V^*. The corresponding representation

$$\sigma^* : W \to \mathrm{GL}(V^*)$$

is called the *contragredient representation* of σ. It is defined, for all $w \in W$, by

$$\sigma^*(w) = {}^{\mathrm{t}}\sigma(w^{-1}).$$

If $\alpha \in V^*$, and $w \in W$, we denote by $w(\alpha)$ the image of α by $\sigma^*(w)$, $w(\alpha) = \sigma^*(w)(\alpha)$. Define

$$C = \bigcap_{s\in S} \{\alpha \in V^* \mid \alpha(e_s) > 0\}.$$

When S is finite, C is an open polyhedral cone in V^* (see [Bou02, Section V.4.4]).

Theorem 1.12 ([Bou02], Theorem V.4.1)**.** If $w \in W$ and $C \cap w(C) \neq \emptyset$, then $w = 1$.

Here are some consequences of this theorem. The two following corollaries can be found in [Bou02, Chapter V.4.4] as well as in [Hum90, Sections 5.4 and 6.2].

Corollary 1.13. The Coxeter group W acts simply-transitively on the collection $\{w(C) \mid w \in W\}$.

Proof. The action is clearly transitive. Let $w_1, w_2 \in W$ such that $w_1(C) = w_2(C)$. Since

$$w_1(C) \cap w_2(C) = w_1(C) \neq \emptyset,$$

we have $W \cap w_1^{-1}w_2(C) \neq \emptyset$. By Theorem 1.12, $w_1^{-1}w_2 = 1$, and thus $w_1 = w_2$. □

Corollary 1.14. The representations σ and σ^* are injective.

Proof. Let $w \in W$ such that $\sigma^*(w) = 1$, then $w(C) = C$, hence $w = 1$ by Theorem 1.12. So σ^* is injective. The injectivity of σ follows from that of σ^*. Indeed, if $\sigma(w) = 1$, then $\sigma^*(w^{-1}) = {}^{t}\sigma(w) = 1$, thus $w^{-1} = 1$ and $w = 1$. □

Corollary 1.15. If S is finite, the image $\sigma(W)$ is a discrete subgroup of $\mathrm{GL}(V)$ (endowed with the compact-open topology). Similarly, $\sigma^*(W)$ is a discrete subgroup of $\mathrm{GL}(V^*)$.

Proof. Let $\alpha \in C$. The set $U = \{g \in \mathrm{GL}(V^*) \mid g(\alpha) \in C\}$ is a neighbourhood of the identity in $\mathrm{GL}(V^*)$. It suffices to show that $\sigma^*(W) \cap U = \{1\}$. Let $g \in \sigma^*(W) \cap U$, then there exists $w \in W$ such that $g = \sigma^*(w)$, and $g(\alpha) = \sigma^*(w)(\alpha) \in C$. Therefore $g(\alpha) \in C \cap w(C)$ and by Theorem 1.12, this means that $w = 1$, *i.e.* $g = 1$. Thus $\sigma^*(W)$ is a discrete subgroup of $\mathrm{GL}(V^*)$. Then, by transposing, it follows that $\sigma(W)$ is also discrete in $\mathrm{GL}(V)$. □

Remark 1.16. Corollary 1.15 shows that $\sigma(W) < \mathrm{GL}(V)$ is discrete for the compact-open topology (or COT-discrete with the notation of Section II.3.1). One can define the same notions of discreteness for a subgroup of isometries of a Euclidean space. Under the assumption that S finite, the vector space V is finite-dimensional. Therefore, all the definitions of discreteness in II.3.11 are actually equivalent.

In the following, we will be interested in constructing "discrete" groups generated by reflections that act irreducibly on $\mathbf{H}^\infty$ via the geometric representation. In order to do so, the group should not be finitely generated, otherwise this geometric representation will not be irreducible since it preserves a vector space of dimension corresponding to the cardinality of the generating set. Therefore, the Corollary 1.15 cannot be used to deduce discreteness in the infinite-dimensional cases.

Irreducibility of the representation

Let (W, S) be a Coxeter system with the action on $V = \mathbf{R}^{(S)}$ via the geometric representation $\sigma : W \to \mathrm{GL}(V)$ as previously. Denote by B_M the Tits form.

Proposition 1.17 ([Bou02], Section V.4.7)**.** Suppose that (W, S) is irreducible.

1. If B_M is non-degenerate (as a bilinear form), then the action $W \curvearrowright V$ via σ is irreducible.
2. If B_M is degenerate, then the action of W on $V^\perp$, the orthogonal of V with respect to B_M, is trivial. Moreover, any proper subspace of V that is invariant under W is contained in $V^\perp$.

1.3 Spherical and affine Coxeter groups

It is easy to see, thanks to Corollaries 1.14 and 1.15, that if S is finite and B_M is positive-definite, then W must be a finite group. Indeed, under these assumptions W is isomorphic to $\sigma(W)$ and $\sigma(W)$ is a discrete subgroup of $\mathrm{O}(V, B_M)$ which is a compact subset of $\mathrm{GL}(V)$.

The fact that every finite Coxeter group is isomorphic to a group generated by reflections in some vector space V was proved by Coxeter in [Cox35]. These groups act on the unit sphere of V. Thus the finite Coxeter groups are called *spherical*. The irreducible ones are very well understood and were classified in [Cox35]. The complete list of the corresponding diagrams is given in Figure 1.

Remark 1.18. These Coxeter groups are of course finitely generated (since they are finite) but note that there are three families (denoted by A_n, B_n and D_n) of irreducible spherical Coxeter groups that may have an arbitrarily large number of generators.

In fact, there are also infinitely generated irreducible spherical Coxeter groups, they are obtained by extending the definition of Coxeter groups of type A_n, B_n and D_n to an infinite number of generators. It suffices to add relations of the type $(s_i s_{i+1})^3 = 1$. Their Coxeter diagrams are called A_∞, B_∞ and D_∞. One may obtain them by prolonging the diagrams of type A, B or D with the pattern:

$$\cdots \bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet \cdots$$

Note that one can extend the diagram of type A_n, by adding infinitely many vertices either on both sides or on one side of $\bullet\!\!-\!\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\!-\!\!\bullet$.

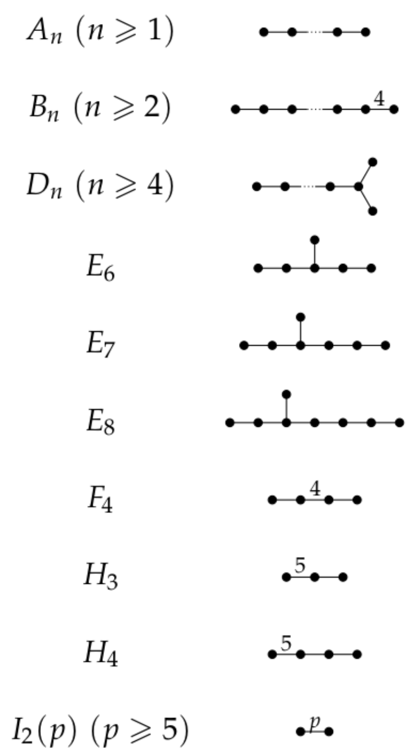


Figure 1 – Irreducible spherical Coxeter groups

Among the infinite Coxeter groups, another well-known family is that of *affine* Coxeter groups that can be defined as those whose Cartan matrix is positive but degenerate. These groups can be made to act on a Euclidean space by *affine* reflections, via the geometric representation σ (see for example [Hum90, Section 6.5]). Finitely generated irreducible affine Coxeter group are also classified. Their Coxeter diagrams are given in Figure 2. They can be obtained by adding one vertex (drawn in white) to a spherical diagram.

As for the spherical Coxeter groups, there are four families of irreducible affine Coxeter groups that can be defined with an arbitrarily large number of generators. We may wonder whether there exist infinitely generated irreducible affine Coxeter groups.

One can expect that it is impossible to extend these affine Coxeter diagrams to get an graph with infinitely many vertices.

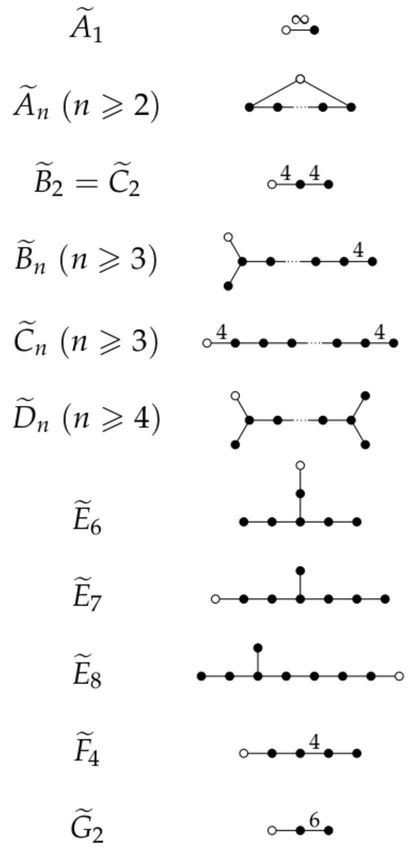
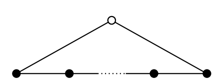
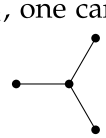


Figure 2 – Irreducible affine Coxeter groups

- The Coxeter diagram of type $\widetilde{A}_n$ is a cycle . Intuitively, extending it would result in an "infinite line" that does not seem to be closable since it has no starting or ending point.
- For the types $\widetilde{B}_n$, $\widetilde{C}_n$ and $\widetilde{D}_n$, one can observe that both ends of the corresponding diagrams are either  or . Adding one generator to the Coxeter group results in adding a vertex in the middle of the Coxeter diagram which locally looks like . Therefore, the same problem appears when trying to extend to infinity.

Remark 1.19. Observe that none of the affine diagrams in Figure 2 arises as a subdiagram of another irreducible affine diagram. In fact, a direct consequence of [Vin71, Proposition 11] is that an irreducible affine Coxeter group can never be embedded into a larger irreducible affine Coxeter group.

In the introduction of the unpublished note [dC12], de Cornulier mentioned in particular that affine Coxeter groups are finitely generated. In other words, there is no infinitely generated affine Coxeter group. We provide a proof of this statement in what follows.

Let $A = (a_{s,t})_{s,t\in S}$ be an $S \times S$ Cartan matrix and (W,S) be the corresponding Coxeter system. On the vector space $V = \mathbf{R}^{(S)}$ with canonical basis $(e_s)_{s\in S}$, the Tits form B_M is a symmetric bilinear form. When S is finite, let (p,q,r) be the signature of B_M with $p+q+r = |S|$.

Proposition 1.20 ([Bou02], Theorem V.4.8.2)**.** The Coxeter system is spherical if and only if $q = r = 0$, *i.e.* B_M is a scalar product (B_M is positive-definite).

Proposition 1.21 ([Bou02], Lemma V.4.9.2)**.** If the Coxeter system is irreducible and $q = 0$, then either the Coxeter system is spherical and $(p,q,r) = (p,0,0)$ or it is affine and $(p,q,r) = (p,0,1)$.

Let us then give the following definition of an infinitely generated affine Coxeter group.

Definition 1.22. If S is infinite, the Coxeter system (W,S) is an *irreducible affine Coxeter system* if the Coxeter matrix M (and the Cartan matrix A) is indecomposable and if the Tits form B_M is positive and has signature $(\infty,0,1)$, *i.e.* its kernel has dimension 1.

We prove that this definition is empty, as suggested in [dC12].

Proposition 1.23. There is no infinitely generated irreducible affine Coxeter group.

Proof. Let $W = \langle \sigma_i \mid i \geqslant 1\rangle < \mathrm{GL}(V)$ be an infinitely generated irreducible affine Coxeter group where $V = \mathbf{R}^{(\mathbf{N})}$ (we identify the group W with its image by the geometric representation). Let $W_n = \langle \sigma_i \mid i \in \{1,\dots,n\}\rangle$ be the group generated by the n first generators of W. All the W_n are Coxeter groups (this was proved independently with three different approaches in [Tit88, Deo89, Dye90]). Denote by M_n the Coxeter matrix of W_n and by B_{M_n} the associated Tits form.

Since B_M has signature $(\infty,0,1)$, there exists $x \in \mathbf{R}_+^{(\mathbf{N})} \setminus \{0\}$ such that $B_M(x,x) = 0$. For all $n > 0$, denote by x_n the vector in $\mathbf{R}^n$ consisting of the n first coordinates of x. Let $n_0 \in \mathbf{N}$ be the rank such that all the coordinates of x vanish after the n_0-th. We have $B_{M_{n_0}}(x_{n_0}) = 0$ where $x_{n_0} \neq 0$. Thus, W_{n_0} is an affine Coxeter group. Let $W_{n_0} = H_1 \times \cdots \times H_p$ be its decomposition into irreducible Coxeter groups where at least one of the factors is affine. Up to restricting to this factor, we may assume that W_{n_0} is irreducible.

The Cartan matrix of W_{n_0} has signature $(n_0 - 1, 0, 1)$. By irreducibility of W, there exists $\sigma \in W \setminus W_{n_0}$ of order 2 which does not commute with at least one of the σ_i, $1 \leqslant i \leqslant n_0$. Consider the Coxeter group G generated by $\{\sigma_1, \dots, \sigma_{n_0}, \sigma\}$. The signature of the associated Cartan matrix must then be one of the three following, obtained by adding 1 to one of the components:

1. $(n_0, 0, 1)$;
2. $(n_0 - 1, 1, 1)$;
3. $(n_0 - 1, 0, 2)$.

In the first case, G would be an irreducible affine Coxeter group containing W_{n_0} which is also irreducible and affine. This is impossible in view of Remark 1.19 and Figure 2. Both second and third cases are also impossible since either the index of negativity or the index of isotropy for the Cartan matrix of G would be greater than that of A, this would be in contradiction with Sylvester's law of inertia. □

1.4 Coxeter groups acting on hyperbolic spaces

Amongst the Coxeter groups that are neither spherical nor affine, we are interested in those acting irreducibly on a hyperbolic space $\mathbf{H}^n$ (by hyperbolic reflections).

Let $V = \mathbf{R}^{n+1}$, $W \subset \mathrm{GL}(V)$ be a Coxeter group with $n+1$ generators (we identify W with its image $\sigma(W)$ via the geometric representation) and A be the Cartan matrix of W. As in Remark 1.11, let $(b_i)_{1 \leqslant i \leqslant n+1}$ be the canonical basis of V and let $\alpha_i \in V^*$ denote the linear form corresponding to the i-th row of A, for $1 \leqslant i \leqslant n+1$. The geometric representation gives the linear reflections $\sigma_i = \mathrm{id} - \alpha_i \otimes b_i$.

Vinberg proved the following characterisation in terms of the Cartan matrix.

Theorem 1.24 ([Vin71], Proposition 24)**.** The following properties are equivalent:

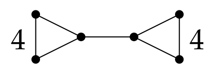
1. W is a group generated by reflections of the hyperbolic space $\mathbf{H}^n$, *i.e.* the linear reflections σ_i preserve $\mathbf{H}^n$, and no proper subspace of $\mathbf{H}^n$ nor any point on the boundary $\partial\mathbf{H}^n$ is W-invariant.
2. W operates irreducibly on V, and the Tits form B_M is a W-invariant non-degenerate bilinear form with index of negativity 1 such that for every i, $B_M(b_i, b_i) > 0$. Equivalently, the Cartan matrix A is indecomposable, non-degenerate, of rank $n+1$ and index of negativity 1.

Coxeter groups acting on hyperbolic spaces are sometimes called *hyperbolic*. However, we will not use this terminology to avoid any confusion with groups that are Gromov-hyperbolic.

Moussong proved a characterisation for a Coxeter group to be (Gromov-)hyperbolic, see [Mou88, Theorem 17.1] or [Dav08, Corollary 12.6.3]: a Coxeter group is hyperbolic if and only if it does not contain $\mathbf{Z} \times \mathbf{Z}$ as a subgroup. He also gave an equivalent condition depending on all the subgraphs of the Coxeter diagram.

It is not true that every hyperbolic Coxeter group acts on a hyperbolic space via the geometric representation, Moussong gave the following example.

Example 1.25 ([Dav08], Example 12.6.8)**.** Consider the Coxeter group W associated to the diagram



whose associated Cartan matrix is

$$A = \begin{pmatrix} 2 & -\sqrt{2} & -1 & 0 & 0 & 0 \\ -\sqrt{2} & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -\sqrt{2} \\ 0 & 0 & 0 & -1 & -\sqrt{2} & 2 \end{pmatrix}.$$

The Cartan matrix A is indecomposable and has signature $(4, 2, 0)$ (one can compute its eigenvalues for example), so it does not satisfy Theorem 1.24. However, one can easily check that W satisfies the criterion for (Gromov-)hyperbolicity given by Moussong's theorem.

1.5 Discrete groups of reflections

We have seen that any Coxeter group admits a representation as a group generated by linear reflections. Theorem 1.24 gives a necessary and sufficient condition for the group to preserve a hyperbolic space. Vinberg's theory actually also provides conditions for a reflection group to act discretely on a hyperbolic space $\mathbf{H}^n$.

Let K be a (closed) n-dimensional convex polyhedral cone in $V = \mathbf{R}^n$ defined as the intersection of a finite number of half-spaces bounded by hyperplanes H_i, $1 \leqslant i \leqslant m$.

Definition 1.26.

- A *face* of K is the intersection $F_i := H_i \cap K$ for some $1 \leqslant i \leqslant m$, such that $\dim(F_i) = n - 1$.
- Two faces F_i and F_j of K are *adjacent* if their intersection $F_i \cap F_j$ has dimension $n - 2$ (in particular, a face is not adjacent to itself).

Let α_i be a linear form on V such that $H_i = \ker(\alpha_i)$. One can choose the α_i in such a way that

$$K = \bigcap_{i=1}^{m} \{\alpha_i \geqslant 0\}.$$

Let $b_i \in V$ be vectors such that $\alpha_i(b_i) = 2$. Consider the group $W \subset \mathrm{GL}(V)$ generated by the linear reflections $\sigma_i = \mathrm{id} - \alpha_i \otimes b_i$ in the hyperplanes H_i. For any subset $I \subset \{1, \dots, m\}$, we denote by W_I the subgroup of W generated by the σ_i, $i \in I$, and we let K_I be the polyhedral cone defined by the inequalities $\{\alpha_i \geqslant 0\}$ for $i \in I$, $K_I = \bigcap_{i \in I} \{\alpha_i \geqslant 0\}$.

The cone K will be called a *fundamental chamber* for the group W if for all $w \in W \setminus \{1\}$, $w(\mathring{K}) \cap \mathring{K} = \varnothing$ (this implies that the group $W < \mathrm{GL}(V)$ is discrete as in Corollary 1.15).

Vinberg proved that the discreteness of W only depends on local conditions on pairs of adjacent faces of K. These conditions can be easily checked by looking at the Cartan matrix of the group.

Theorem 1.27 ([Vin71], Theorem 1)**.** Let $A = (a_{i,j})_{1 \leqslant i,j \leqslant m}$ be the matrix whose entries are $a_{i,j} = \alpha_i(b_j)$ for $1 \leqslant i, j \leqslant m$. Let W be the group generated by the reflections $\sigma_1, \dots, \sigma_m$. The polyhedral cone K is a fundamental chamber for W if and only if for every pair of adjacent faces F_i, F_j, the two following conditions are satisfied,

$(C1)$ $a_{i,j} \leqslant 0$ and $(a_{i,j} = 0 \Leftrightarrow a_{j,i} = 0)$;

$(C2)$ $a_{i,j} a_{j,i} \geqslant 4$ or $a_{i,j} a_{j,i} = 4 \cos^2 \left(\dfrac{\pi}{m_{i,j}} \right)$, with $m_{i,j} \in \mathbf{N}$ and $m_{i,j} \geqslant 2$.

2 Groups generated by reflections acting on $\mathrm{Isom}(\mathbf{H}^\infty)$

Definition 2.1. Let M be a matrix (of any size, potentially infinite). A *submatrix* of M is a matrix obtained by removing some collections of rows and columns from M. We define a *principal submatrix* as a finite-dimensional square submatrix obtained by keeping only the first n rows and n columns of M for some $n \in \mathbf{N}$.

If $M = (m_{i,j})_{1 \leqslant i,j \leqslant n_0}$ and $n \leqslant n_0$, we will denote by $M_n = (m_{i,j})_{1 \leqslant i,j \leqslant n}$ the n-th principal submatrix of M.

Lemma 2.2. Let $A = (a_{i,j})_{i,j \in \mathbf{N}}$ be an infinite symmetric matrix such that for all $i \in \mathbf{N}$, $a_{i,i} = 2$. Suppose that there is an integer $n_0 \in \mathbf{N}$ such that for all $n \geqslant n_0$, the principal submatrix A_n is non-degenerate and has signature $(n-1, 1)$. Then A has signature $(\infty, 1)$ as the matrix of a bilinear (or quadratic) form of $\mathbf{R}^{(\mathbf{N})}$.

Proof. Let $V = \mathbf{R}^{(\mathbf{N})} = \mathrm{Span}(\{e_i\}_{i \in \mathbf{N}})$, where $\{e_i\}_{i \in \mathbf{N}}$ is the canonical basis of V. For all $i \in \mathbf{N}$, let $b_i = e_i \in V$ and let $\alpha_i \in V^*$ be the linear form such that for all $j \in \mathbf{N}$,

$\alpha_i(b_j) = a_{i,j}$. For all $i \in \mathbf{N}$, let σ_i be the reflections associated to b_i and α_i defined by the formula

$$\sigma_i = \mathrm{id} - \alpha_i \otimes b_i.$$

The vector space V can be endowed with the bilinear form $(\cdot,\cdot)$ defined by

$$\forall v \in V \quad \alpha_i(v) = (b_i, v).$$

The reflections σ_i are orthogonal with respect to $(\cdot,\cdot)$. Notice that when restricted to a finite-dimensional subspace $V_n = \mathrm{Span}(e_1, \ldots, e_n)$, for $n \geqslant n_0$, $(\cdot,\cdot)$ coincides with the bilinear form $(\cdot,\cdot)_n$ defined be the principal submatrix $A_n = (a_{i,j})_{i,j\in\{1,\ldots,n\}}$ of A. By hypothesis, $(\cdot,\cdot)_n$ is non-degenerate and has signature $(n-1,1)$.

We first show that the bilinear form $(\cdot,\cdot)$ is non-degenerate. Suppose that $v \in V^\perp$, *i.e.* v is such that for all $w \in V$, $(v,w) = 0$. There exists $n \in \mathbf{N}$, with $n \geqslant n_0$, such that $v \in \mathrm{Span}(e_1, \ldots, e_n) = V_n$, $v = \sum_{i=1}^n v_i e_i$. So in particular, we have

$$\forall w \in V_n \quad (v,w) = (v,w)_n = 0.$$

Since $(\cdot,\cdot)_n$ is non-degenerate, the vector $v \in V_n \subset V$ must be 0, thus $(\cdot,\cdot)$ is non-degenerate.

We now show that $(\cdot,\cdot)$ has index of negativity 1. The principal submatrix A_{n_0} has signature $(n_0-1,1)$, so there exists a vector $x_0 \in V_{n_0}$ such that $(x_0,x_0) = (x_0,x_0)_{n_0} < 0$. This vector x_0 is time-like and the linear span $\mathbf{R}\cdot x_0$ is negative-definite (the quadratic form of $(\cdot,\cdot)$ restricted to $\mathbf{R}\cdot x_0$ is negative-definite). Suppose now that there exists $y \in V$, $y \notin \mathbf{R}\cdot x_0$ such that $\mathrm{Span}(x_0,y)$ is negative-definite. There is an integer $n \in \mathbf{N}$, $n \geqslant n_0$, such that $y \in V_n$. The line $\mathbf{R}\cdot x_0 \subset V_n$ is also negative-definite for $(\cdot,\cdot)_n$ and since $(\cdot,\cdot)_n$ has index of negativity 1 and $\dim(\mathrm{Span}(x_0,y)) = 2$, the subspace $\mathrm{Span}(x_0,y)$ cannot be negative-definite with respect to $(\cdot,\cdot)_n$. Thus, there is a vector $z \in \mathrm{Span}(x_0,y) \subset V_n$ such that $(z,z)_n = (z,z) \geqslant 0$ and $\mathrm{Span}(x_0,y)$ is not negative-definite for $(\cdot,\cdot)$. Hence $\mathbf{R}\cdot x_0$ is a maximal negative-definite subspace of V, and $(\cdot,\cdot)$ has signature $(\infty,1)$. □

Proposition 2.3. Let $A = (a_{i,j})_{i,j\in\mathbf{N}}$ be an infinite symmetric matrix such that for all $i \in \mathbf{N}$, $a_{i,i} = 2$. Suppose that there is an integer $n_0 \in \mathbf{N}$ such that for all $n \geqslant n_0$, the principal submatrix A_n satisfies Theorem 1.24. Then the group generated by reflections associated to A acts isometrically on the infinite-dimensional hyperbolic space $\mathbf{H}^\infty$.

Proof. As previously, let $V = \mathrm{Span}(\{e_i\}_{i\in\mathbf{N}})$, where $\{e_i\}_{i\in\mathbf{N}}$ is the canonical basis of V. For all $i \in \mathbf{N}$, let $b_i = e_i \in V$, $\alpha_i \in V^*$ such that for all $j \in \mathbf{N}$, $\alpha_i(b_j) = a_{i,j}$ and let σ_i be the reflections

$$\sigma_i = \mathrm{id} - \alpha_i \otimes b_i.$$

Denote by $W \subset \mathrm{GL}(V)$ the group generated by all those reflections σ_i.

By Lemma 2.2, the bilinear form associated to A, $(\cdot,\cdot)$, is non-degenerate, thus the vector space V can be decomposed as the orthogonal direct sum

$$V = \mathbf{R}\cdot x_0 \oplus \{x_0\}^{\perp}$$

where $\mathbf{R}\cdot x_0$ is negative-definite and $\{x_0\}^{\perp}$ is positive-definite. Let $\langle\cdot,\cdot\rangle_{\pm}$ be the scalar product associated to this decomposition as in Definition II.1.3. Considering the completion of V for $\langle\cdot,\cdot\rangle_{\pm}$, we get the Hilbert $\mathcal{H} = \overline{V}$. Denote by $\mathbf{H}^{\infty}$ the (separable) infinite-dimensional hyperbolic space associated to $\mathcal{H}$. Notice that if V is the direct sum $V = E \oplus F$ where E and F are orthogonal for $\langle\cdot,\cdot\rangle$, then $\mathcal{H} = \overline{E} \oplus \overline{F}$.

The group W then acts on $\mathbf{H}^{\infty}$. To prove this, it suffices to show that each reflection generating W extends to a continuous reflection on $\mathcal{H}$. Let $i \in \mathbf{N}$, we have $\sigma_i(b_i) = -b_i$, so b_i is an eigenvector of σ_i corresponding to the eigenvalue -1. Since σ_i is an orthogonal reflection in V for the bilinear form $(\cdot,\cdot)$, it acts as id on $H_i = \ker \alpha_i$ and we can decompose V as the direct sum

$$V = \mathbf{R}\cdot b_i \oplus H_i.$$

The vector b_i satisfies $(b_i, b_i) = (e_i, e_i) = 2 > 0$, so $\mathbf{R}\cdot b_i$ is a positive-definite subspace of V. Then the restriction of $(\cdot,\cdot)$ to H_i is non-degenerate and has signature $(\infty, 1)$. Thus, there exists $x_i \in H_i$ such that $\mathbf{R}\cdot x_i$ is negative-definite and $H_i = \mathbf{R}\cdot x_i \oplus \{x_i, b_i\}^{\perp}$. This provides another $\pm$-decomposition of V:

$$V = \mathbf{R}\cdot b_i \oplus \mathbf{R}\cdot x_i \oplus \{x_i, b_i\}^{\perp}.$$

Let $\langle\cdot,\cdot\rangle_i$ be the associated scalar product. The reflection σ_i acts as id on H_i and as $-\operatorname{id}$ on $\mathbf{R}\cdot b_i$, so σ_i is uniformly continuous on $(V, \langle\cdot,\cdot\rangle_i)$ (it is actually of norm 1). By Lemma II.1.4, since $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\pm})$ is complete, $(\mathcal{H}, \langle\cdot,\cdot\rangle_i)$ is also complete and the two scalar products $\langle\cdot,\cdot\rangle_{\pm}$ and $\langle\cdot,\cdot\rangle_i$ are equivalent. Therefore $\sigma_i : (V, \langle\cdot,\cdot\rangle_{\pm}) \to (\mathcal{H}, \langle\cdot,\cdot\rangle_{\pm})$ is uniformly continuous. Besides, V is dense in $\mathcal{H}$, so σ_i extends into a continuous reflection on $\mathcal{H}$. This holds for all the σ_i, $i \in \mathbf{N}$, which concludes the proof. It also follows that the linear forms α_i are continuous. □

Remark 2.4. Denote by $\overline{\sigma_i}$ and $\overline{\alpha_i}$ the extensions of σ_i and α_i to $\mathcal{H}$. We have $\ker \overline{\alpha_i} = \overline{\ker \alpha_i} = \overline{H_i}$. Since $V = \mathbf{R}\cdot b_i \oplus H_i$ where the two spaces $\mathbf{R}\cdot b_i$ and H_i are orthogonal, we get

$$\mathcal{H} = \overline{\mathbf{R}\cdot b_i} \oplus \overline{H_i} = \mathbf{R}\cdot b_i \oplus \overline{H_i}.$$

Proposition 2.5. Let A as in Proposition 2.3. The action of the group W generated by reflections on $\mathbf{H}^{\infty}$ is irreducible, *i.e.* it admits no invariant proper subspace, nor invariant point at infinity.

Proof. It suffices to show that there is no W-invariant proper subspace in $\mathcal{H}$. Indeed, any point at infinity, in $\partial\mathbf{H}^\infty$, corresponds to an isotropic line in the ambient Hilbert space (see Definition II.1.22 and Remark III.2.2).

Suppose that W stabilises a closed subspace $F \subset \mathcal{H}$. Let $I = \{i \in \mathbf{N} \mid b_i \in F\}$ and $J = \mathbf{N} \setminus I$. For every $z \in F$ and $j \in J$, we have $\sigma_j(z) = z - \alpha_j(z)b_j \in F$. Since $b_j \notin F$, this implies that $\alpha_j(z) = 0$. In particular, for all $i \in I$, we have $\alpha_j(b_i) = 0$. Suppose that both I and J are non-empty. Let $i_0 \in I$, $j_0 \in J$ and let $n \geqslant max\{i_0, j_0, n_0\}$ (n_0 being the dimension from which the submatrices A_n have index 1). Then we have $\{1, \dots, n\} = I_n \sqcup J_n$ where $I_n = \{1, \dots, n\} \cap I \neq \emptyset$ and $J_n = \{1, \dots, n\} \cap J \neq \emptyset$. So

$$\forall (i,j) \in I_n \times J_n \quad \alpha_j(b_i) = 0.$$

This is in contradiction with the indecomposability of A_n. Hence we have either $I = \emptyset$ or $J = \emptyset$.

If $J = \emptyset$, then $I = \mathbf{N}$ and F contains all the vectors b_i, $i \in \mathbf{N}$, and so $V \subset F$. As F is closed, it contains the closure of V which is $\mathcal{H}$. We therefore get $F = \mathcal{H}$.

Now if $I = \emptyset$, F contains none of the b_i. Let $i \in \mathbf{N}$, F is invariant by σ_i, so $F \subset \overline{H_i}$. Indeed, if F contains a vector $x \notin \overline{H_i}$, it can be written using the decomposition $\mathcal{H} = \mathbf{R} \cdot b_i \oplus \overline{H_i}$ as $x = \lambda b_i + x'$ with $\lambda \neq 0$. And $\sigma_i(x) = -\lambda b_i + x' \in F$. So $\dfrac{1}{2\lambda}(x - \sigma_i(x)) = b_i \in F$ which leads to a contradiction. Then $F \subset \overline{H_i}$ for every $i \in \mathbf{N}$, and

$$F \subset \bigcap_{i \in \mathbf{N}} \overline{H_i}.$$

Moreover, for all $i \in \mathbf{N}$, $b_i \in \left(\bigcap_{n \in \mathbf{N}} \overline{H_n}\right)^\perp$, hence $\left(\bigcap_{n \in \mathbf{N}} \overline{H_n}\right)^\perp$ is dense in $\mathcal{H}$. So its orthogonal must be trivial and

$$\bigcap_{n \in \mathbf{N}} \overline{H_n} \subset \left(\left(\bigcap_{n \in \mathbf{N}} \overline{H_n}\right)^\perp\right)^\perp = \{0\}.$$

Therefore $F = \{0\}$. □

3 Examples

3.1 Coxeter groups which are "eventually spherical"

Consider the Cartan matrix of the following form where $m \geqslant 7$ and $c_m = \cos\left(\frac{\pi}{m}\right)$,

$$A = \begin{pmatrix} 2 & -2c_m & 0 & 0 & \cdots \\ -2c_m & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & \ddots & \\ 0 & 0 & \ddots & \ddots & \\ \vdots & \vdots & & & 2 \end{pmatrix}.$$

Observing that the red part of the matrix has signature $(2,1)$ and the blue part is the Cartan matrix associated to an infinite spherical Coxeter group of type A_∞, it follows that each principal submatrix A_n of A is non-degenerate and has signature $(n-1,1)$. By Lemma 2.2, the signature of A is $(\infty, 1)$. Such a matrix satisfies Propositions 2.3 and 2.5 and gives rise to another family of Coxeter groups acting on $\mathbf{H}^\infty$.

Denote by $(e_i)_{i\in\mathbf{N}}$ the canonical basis of $\mathbf{R}^{(\mathbf{N})}$. Let B be the non-degenerate bilinear form on $\mathbf{R}^{(\mathbf{N})}$ whose matrix is A. Since B has signature $(\infty,1)$, there exists a vector $u = (u_n)_{n\in\mathbf{N}} \in \mathbf{R}^{(\mathbf{N})}$ whose coordinate u_n vanishes for all $n \geqslant 3$ and such that $B(u,u) = -1$. We can then decompose the vector space $\mathbf{R}^{(\mathbf{N})}$ into $\mathbf{R}\cdot u \oplus u^\perp$, where $\mathbf{R}\cdot u$ and $u^\perp$ are orthogonal with respect to B. In this decomposition, any vector $x \in \mathbf{R}^{(\mathbf{N})}$ can be uniquely written as $x = x_- + x_+$ where $x_- \in \mathbf{R}\cdot u$ and $x_+ \in u^\perp$. We actually have

$$x_- = B(x,u)u \quad \text{and} \quad x_+ = x - B(x,u)u.$$

This is a $\pm$-decomposition of $\mathbf{R}^{(\mathbf{N})}$ (see Definition II.1.3), and thus defines the following scalar product

$$\langle x,y\rangle_\pm = B(x_+,y_+) - B(x_-,y_-),$$

where $x = x_- + x_+$ and $y = y_- + y_+$ according to the aforementioned decomposition. Let $\|x\|_\pm^2 = \langle x,x\rangle_\pm$ be the associated norm.

Remark 3.1. Since $u = u_-$, we have $\|u\|_\pm^2 = \langle u,u\rangle_\pm = -B(u,u) = 1$.

Denote by $\mathcal{H}$ the completion of $\mathbf{R}^{(\mathbf{N})}$ with respect to the $\pm$-norm. Let $\alpha_i \in \left(\mathbf{R}^{(\mathbf{N})}\right)^*$ be the linear form represented by the i-th row of the matrix A. For example, if $x = (x_0, x_1, \dots) \in \mathbf{R}^{(\mathbf{N})}$ (x has finitely many non-vanishing coefficients), then

$$\alpha_0(x) = 2x_0 - 2c_m x_1.$$

The Coxeter group associated to A is the group $W < \mathrm{GL}(\mathbf{R}^{(\mathbf{N})})$ generated by the reflections

$$\sigma_i = \mathrm{id} - \alpha_i \otimes e_i.$$

By Propositions 2.3 and 2.5, the σ_i are uniformly continuous with respect to the $\pm$-norm and thus can be extended to $\mathcal{H}$, as well as the α_i. Moreover, all the σ_i act on $\mathcal{H}$ by preserving $\mathbf{H}^\infty \subset \mathcal{H}$, so $W < \mathrm{Isom}(\mathbf{H}^\infty)$.

Lemma 3.2. The topology induced on $\mathbf{H}^\infty \subset \mathcal{H}$ by the norm $\|\cdot\|_\pm$ coincides with the topology of the hyperbolic distance on $\mathbf{H}^\infty$.

Proof. This is actually Proposition II.1.13. $\square$

Lemma 3.3. If $\mathbf{H}^\infty$ is the infinite-dimensional hyperbolic space defined by the Cartan matrix A seen as a quadratic form, then $\overline{\mathbf{H}^\infty \cap \mathbf{R}^{(\mathbf{N})}}^{\pm} = \mathbf{H}^\infty$.

Proof. This follows from the fact that $\mathbf{H}^\infty$ is the completion of the inductive limit of all the finite-dimensional hyperbolic spaces, see Remark II.1.27. $\square$

Lemma 3.4. The operator norms of all the reflections σ_i are uniformly bounded for the norm $\|\cdot\|_\pm$.

Proof. Let $x \in \mathcal{H}$ be any vector in the Hilbert space. Recall that only the three first coordinates of u are non-vanishing. We show that all the reflections σ_i belong to the orthogonal group $\mathrm{O}(\mathcal{H}, \|\cdot\|_\pm^2)$ as soon as $i \geqslant 4$. In the (orthogonal) decomposition $\mathcal{H} = \mathbf{R}\cdot u \oplus u^\perp$, write x as $x = x_- + x_+$. Let $i \geqslant 4$. We have $\alpha_i(x_-) = 0$ and thus

$$\begin{aligned}
\|\sigma_i(x)\|_\pm^2 &= \|x - \alpha_i(x)e_i\|_\pm^2 \\
&= \|x_- + x_+ - \alpha_i(x_-)e_i - \alpha_i(x_+)e_i\|_\pm^2 \\
&= \|x_- + x_+ - \alpha_i(x_+)e_i\|_\pm^2 \\
&= \|x_-\|_\pm^2 + \|x_+ - \alpha_i(x_+)e_i\|_\pm^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|x_+ - \alpha_i(x_+)e_i\|_\pm^2 &= B(x_+ - \alpha_i(x_+)e_i, x_+ - \alpha_i(x_+)e_i) \\
&= \|x_+\|_\pm^2 - 2\alpha_i(x_+)\underbrace{B(x_+, e_i)}_{=\alpha_i(x_+)} + \alpha_i(x_+)^2\underbrace{B(e_i, e_i)}_{=2} \\
&= \|x_+\|_\pm^2.
\end{aligned}$$

Therefore $\|\sigma_i(x)\|_\pm = \|x\|_\pm$ for $i \geqslant 4$ and all the operator norms $\|\sigma_i\|_\pm$ are equal to 1 for $i \geqslant 4$. We may then conclude that $\sup\{\|\sigma_i\|_\pm \mid i \in \mathbf{N}\} < \infty$. $\square$

Remark 3.5. It is not surprising that the reflections σ_i are orthogonal operators with respect to $\|\cdot\|_\pm$ for i large enough since the Cartan matrix A contains a Cartan matrix of type A_∞ corresponding to a spherical Coxeter group acting by linear reflections on a sphere. However, observe that the first four reflections $\sigma_0, \ldots, \sigma_3$ do not belong to $\mathrm{O}(\mathcal{H}, \|\cdot\|_\pm)$.

Lemma 3.6. For all $x \in \mathbf{H}^\infty \cap \mathbf{R}^{(\mathbf{N})}$, the norm $\|\sigma_i(x) - x\|_\pm$ is equal to 0 for i large enough.

Proof. We have $\|\sigma_i(x) - x\|_\pm = \| - \alpha_i(x)e_i\|_\pm = |\alpha_i(x)|\|e_i\|_\pm$. Let $u = (u_n)_{n\in\mathbf{N}} \in \mathbf{R}^{(\mathbf{N})}$ be the vector used to define $\|\cdot\|_\pm$. Since the vector u has only three non-vanishing coefficients, we have $\|e_i\|_\pm = 2$ for all $i \geqslant 4$. Moreover, as $x \in \mathbf{R}^{(\mathbf{N})}$ has finitely many non-zero coefficients, we have

$$|\alpha_i(x)| = |-x_{i-1} + 2x_i - x_{i+1}| = 0$$

for i large enough. □

Proposition 3.7. The group $W < \mathrm{Isom}(\mathbf{H}^\infty)$ is not COTD.

Proof. By Lemma 3.2, it is sufficient to show that for all $x \in \mathbf{H}^\infty \subset \mathcal{H}$, $\|\sigma_i(x) - x\|_\pm \longrightarrow 0$ as $i \to \infty$. Let then $x \in \mathbf{H}^\infty$. Lemma 3.3 states we can approximate x by points in $\mathbf{H}^\infty$ which have only finitely many non-vanishing coordinates, *i.e.* for all $\epsilon > 0$, there exists some point $x' \in \mathbf{H}^\infty \cap \mathbf{R}^{(\mathbf{N})}$ such that $\|x - x'\|_\pm \leqslant \epsilon$. It follows from Proposition 3.4 that for every pair of points $x, x' \in \mathcal{H}$ and for all $i \in \mathbf{N}$, $\|\sigma_i(x) - \sigma_i(x')\|_\pm \leqslant \lambda\|x - x'\|_\pm$, for some fixed $\lambda \in \mathbf{R}$. therefore we have

$$\begin{aligned}
\|\sigma_i(x) - x\|_\pm &\leqslant \|\sigma_i(x) - \sigma_i(x')\|_\pm + \|\sigma_i(x') - x'\|_\pm + \|x' - x\|_\pm \\
&\leqslant (1+\lambda)\|x - x'\|_\pm + \|\sigma_i(x') - x'\|_\pm \\
&\leqslant (1+\lambda)\epsilon + \|\sigma_i(x') - x'\|_\pm.
\end{aligned}$$

Finally, since $x' \in \mathbf{H}^\infty \cap \mathbf{R}^{(\mathbf{N})}$, for i large enough, $\|\sigma_i(x') - x'\| = 0$ by Lemma 3.6. Hence, for i large enough,

$$\|\sigma_i(x) - x\|_\pm \leqslant (1+\lambda)\epsilon.$$

Thus, σ_i converges to id for the pointwise convergence topology on $\mathbf{H}^\infty$ and the group $W = \langle \sigma_i \mid i \in \mathbf{N} \rangle$ is not discrete with respect to this topology. □

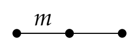
Corollary 3.8. The fundamental domain of Γ has empty interior.

Proof. For all $x \in \mathbf{H}^\infty$, there is a sequence $w_n \in W$ such that $w_n(x) \to x$ as $n \to \infty$. Therefore, the quotient $\mathbf{H}^\infty/W$ cannot contain any open ball. □

Remark 3.9. If W is not COTD, then it is not WD as well by Proposition II.3.11. It follows that W has at least one non-discrete orbit.

Remark 3.10. The example of this section can be further generalised. Indeed, the Cartan matrix A is built upon the hyperbolic triangle group

$$\begin{pmatrix} 2 & -2c_m & 0 \\ -2c_m & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$



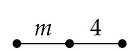
where the bottom-right part $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ consists in the Cartan matrix of a spherical Coxeter group of type A_2. We are then able to extend the matrix A by adding rows and columns in such a way that the remaining Cartan matrix obtained by removing the first row and first column is exactly that of the infinite spherical Coxeter group of type A_∞. There are only three families of irreducible spherical Coxeter groups (or irreducible Coxeter diagrams) that can be extended to an arbitrary number of generators (or vertices). The same construction may be conducted using groups of type B_n and D_n instead of A_n.

Example 3.11 (Spherical Coxeter groups of type B_n)**.** The Cartan matrix of a spherical Coxeter groups of type B_n is of the type

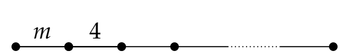
$$\begin{pmatrix} 2 & -\sqrt{2} & 0 & 0 & \cdots \\ -\sqrt{2} & 2 & -1 & 0 & \\ 0 & -1 & 2 & -1 & \\ 0 & 0 & -1 & 2 & \ddots \\ \vdots & & & \ddots & \ddots \end{pmatrix}.$$

They have signature $(n, 0)$ and one can add one row and one column to make its index of negativity equal to 1. For example, if $c_m = \cos\left(\dfrac{\pi}{m}\right)$, the 3×3 matrix

$$\begin{pmatrix} 2 & -2c_m & 0 \\ -2c_m & 2 & -\sqrt{2} \\ 0 & -\sqrt{2} & 2 \end{pmatrix}$$

has signature $(2, 1)$ whenever $m \geqslant 5$. Its Coxeter diagram is 

. Thus expanding it with 2's on the diagonal and -1's around it, we get a Cartan matrix satisfying again the hypothesis of Propositions 2.3 and 2.5. Therefore, the associated Coxeter group acts irreducibly on an infinite-dimensional hyperbolic space $\mathbf{H}^\infty$ by hyperbolic reflections.



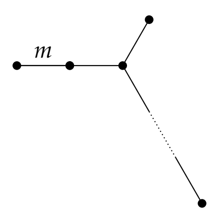
Example 3.12 (Spherical Coxeter groups of type D_n)**.** The Cartan matrix of a spherical Coxeter groups of type D_n is of the type

$$\begin{pmatrix} 2 & 0 & -1 & 0 & \cdots \\ 0 & 2 & -1 & 0 & \\ -1 & -1 & 2 & -1 & \\ 0 & 0 & -1 & 2 & \ddots \\ \vdots & & & \ddots & \ddots \end{pmatrix}.$$

They have signature $(n, 0)$ and we can again add one row and one column to make its index of negativity equal to 1. For example, for all $m \geqslant 6$, $c_m = \cos\left(\frac{\pi}{m}\right)$, the 4×4 matrix

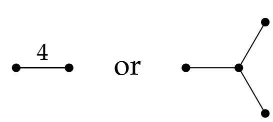
$$\begin{pmatrix} 2 & -2c_m & 0 & 0 \\ -2c_m & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

has signature $(3, 1)$. As before, expanding it with 2's on the diagonal and -1's around it, we get a Cartan matrix satisfying the hypothesis of Propositions 2.3 and 2.5. Therefore, the associated Coxeter group acts irreducibly on an infinite-dimensional hyperbolic space $\mathbf{H}^\infty$ by hyperbolic reflections.



Proposition 3.13. All the groups $W < \mathrm{Isom}(\mathbf{H}^\infty)$ of the type described in the examples 3.11 and 3.12 are not COTD.

Proof. Observe that for each of the three families of irreducible spherical Coxeter groups that are defined for an arbitrary number of generators, all the Coxeter diagrams are the same when we remove the subgraphs



for the type B or D, respectively. Therefore, for $i \in \mathbf{N}$ large enough, the linear form α_i corresponding to the i-row of the Cartan matrices only has three non-vanishing coefficients $(0, \ldots, 0, -1, 2, -1, 0, \ldots)$.

Then the proof of non-discreteness with respect to the compact-open topology is the same as for Proposition 3.7. □

These examples thus provide counter-example to Corollary 1.15 for Coxeter groups which are infinitely generated.

Remark 3.14. A natural question arising now is whether these groups are discrete for the weakest notion of discreteness, namely the discreteness for the topology induced by the uniform operator topology, UOTD. A first step to answer this is to notice that the family $\{\sigma_i \mid i \in \mathbf{N}\}$ is UOTD by considering the operator norm $\|\sigma_i - \sigma_j\|_\pm$ for $i \neq j$ and get a lower bound on this distance.

However, this does not imply that the group generated by the σ_i is still UOTD.

3.2 Regular polyhedron

For $n \in \mathbf{N}$, $n \geqslant 4$, let A_n be the n by n matrix

$$\begin{pmatrix} 2 & -1 & -1 & \dots & -1 \\ -1 & 2 & -1 & \dots & -1 \\ -1 & -1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ -1 & -1 & \dots & -1 & 2 \end{pmatrix}.$$

The matrix A_n is symmetric, indecomposable (all the entries are nonzero), of rank n with index of negativity 1. Indeed, A_n is diagonalisable with $-n+3 < 0$ as a simple eigenvalue, and 3 is an eigenvalue of multiplicity $n-1$. The associated Coxeter group W_n acts on the vector space $V = \mathbf{R}^n$ by the reflections $\sigma_i = \mathrm{id} - \alpha_i \otimes b_i$, where the $b_i = e_i$ are the vectors of the canonical basis of $\mathbf{R}^n$ and the linear forms $\alpha_i \in V^*$ are identified with the rows of the matrix A_n:

$$\begin{aligned} \alpha_1 &= (\; 2 \quad -1 \quad -1 \quad \dots \quad -1 \;) \\ \alpha_2 &= (\; -1 \quad 2 \quad -1 \quad \dots \quad -1 \;) \\ &\vdots \\ \alpha_n &= (\; -1 \quad -1 \quad \dots \quad -1 \quad 2 \;). \end{aligned}$$

For $i \neq j$, we have $(\sigma_i\sigma_j)^3 = \mathrm{id}$. Let $\Gamma_n = \langle \sigma_1, \dots, \sigma_n \rangle$. According to Theorem 1.24, W_n is a Coxeter group acting on $\mathbf{H}^{n-1}$ and has no hyperbolic subspace nor point at infinity that is W-invariant.

When $n = 4$, the reflections σ_i in $\mathbf{H}^3$, $i \in \{1, \dots, 4\}$, correspond to reflections through the faces of an ideal regular tetrahedral whose dihedral angles are all $\dfrac{\pi}{3}$.

Let $A = (a_{i,j})_{i,j\in\mathbf{N}}$ be the infinite matrix such that for all $i, j \in \mathbf{N}$, $a_{i,i} = 2$ and $a_{i,j} = -1$ if $i \neq j$. Propositions 2.3 and 2.5 imply the following.

Proposition 3.15. The infinitely generated Coxeter group Γ associated to A acts on a hyperbolic space without any Γ-invariant proper hyperbolic subspace nor any Γ-invariant point at infinity.

Remark 3.16. For all $m \in \mathbf{N}$, $m \geqslant 3$, let $c_m = \cos\left(\frac{\pi}{m}\right)$. The matrices of the form

$$\begin{pmatrix} 2 & -2c_m & -2c_m & \dots & -2c_m \\ -2c_m & 2 & -2c_m & \dots & -2c_m \\ -2c_m & -2c_m & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -2c_m \\ -2c_m & -2c_m & \dots & -2c_m & 2 \end{pmatrix}$$

also satisfy Theorem 1.24 whenever their size is larger or equal to 4 and Propositions 2.3 and 2.5 can be applied to the infinite matrix $A = (a_{i,j})_{i,j\in\mathbf{N}}$ where $a_{i,i} = 2$ and $a_{i,j} = -2c_m$ when $i \neq j$, providing again an irreducible action by reflections on $\mathbf{H}^\infty$.

As for the discreteness properties of the group Γ, the ideas of the examples from Section 3.1 do not work anymore since we are not able to do explicit computations with the reflections σ_i which involve now infinite sums containing potentially infinitely many negative and positive terms.

Bibliography

[AB60] Lars V. Ahlfors and Lipman Bers. Riemann's mapping theorem for variable metrics. *Annals of Mathematics*, 72(2):385–404, 1960.

[Ahl64] Lars V. Ahlfors. Finitely generated kleinian groups. *American Journal of Mathematics*, 86(2):413–429, 1964.

[Ahl85] Lars V. Ahlfors. Möbius transformations and Clifford numbers. In *Differential geometry and complex analysis*, pages 65–73. Springer, Berlin, 1985.

[Ber70] Lipman Bers. Spaces of kleinian groups. In John Horváth, editor, *Several Complex Variables I Maryland 1970*, pages 9–34, Berlin, Heidelberg, 1970. Springer Berlin Heidelberg.

[BFMS21] Uri Bader, David Fisher, Nicholas Miller, and Matthew Stover. Arithmeticity, superrigidity, and totally geodesic submanifolds. *Ann. of Math. (2)*, 193(3):837–861, 2021.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[BIM05] Marc Burger, Alessandra Iozzi, and Nicolas Monod. Equivariant embeddings of trees into hyperbolic spaces. *International Mathematics Research Notices*, (22), 2005.

[BJ19] B. V. Rajarama Bhat and Tiju Cherian John. Real normal operators and Williamson's normal form. *Acta Sci. Math. (Szeged)*, 85(3-4):507–518, 2019.

[Bou02] Nicolas Bourbaki. *Lie groups and Lie algebras Chapter 5*. Springer-Verlag Berlin Heidelberg, 2002.

[Can11] Serge Cantat. On the groups of birational transformations of surfaces. *Ann. Math. (2)*, 174(1):299–340, 2011.

[Can21] Richard Canary. Anosov representations: informal lecture notes. 2021.

[CDP90] Michel Coornaert, Thomas Delzant, and Athanase Papadopoulos. *Géométrie et théorie des groupes. Les groupes hyperboliques de Gromov. (Geometry and group theory. The hyperbolic groups of Gromov)*, volume 1441. Berlin etc.: Springer-Verlag, 1990.

[CL10] Pierre-Emmanuel Caprace and Alexander Lytchak. At infinity of finite-dimensional CAT(0) spaces. *Math. Ann.*, 346(1):1–21, 2010.

[CLdC13] Serge Cantat, Stéphane Lamy, and Yves de Cornulier. Normal subgroups in the Cremona group. *Acta Math.*, 210(1):31–94, 2013.

[CM09] Pierre-Emmanuel Caprace and Nicolas Monod. Isometry groups of non-positively curved spaces: discrete subgroups. *J. Topol.*, 2(4):701–746, 2009.

[Cox35] H. S. M. Coxeter. The Complete Enumeration of Finite Groups of the Form $R_i^2 = (R_iR_j)^{k_{ij}} = 1$. *J. London Math. Soc.*, 10(1):21–25, 1935.

[Dav08] Michael W. Davis. *The geometry and topology of Coxeter groups*, volume 32 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2008.

[dC12] Yves de Cornulier. Semisimple zariski closure of coxeter groups, 2012.

[Deo89] Vinay V. Deodhar. A note on subgroups generated by reflections in Coxeter groups. *Arch. Math. (Basel)*, 53(6):543–546, 1989.

[DH13] Marc Desgroseilliers and Frédéric Haglund. On some convex cocompact groups in real hyperbolic space. *Geom. Topol.*, 17(4):2431–2484, 2013.

[dlH72] Pierre de la Harpe. *Classical Banach-Lie algebras and Banach-Lie groups of operators in Hilbert space*, volume Vol. 285 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1972.

[DLP23] Bruno Duchesne, Jean Lécureux, and Maria Beatrice Pozzetti. Boundary maps and maximal representations on infinite-dimensional hermitian symmetric spaces. *Ergodic Theory and Dynamical Systems*, 43(1):140–189, 2023.

[DP12] Thomas Delzant and Pierre Py. Kähler groups, real hyperbolic spaces and the Cremona group. *Compos. Math.*, 148(1):153–184, 2012.

[DSU17] Tushar Das, David Simmons, and Mariusz Urbański. *Geometry and dynamics in Gromov hyperbolic metric spaces*, volume 218 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017. With an emphasis on non-proper settings.

[Duc13] Bruno Duchesne. Infinite-dimensional nonpositively curved symmetric spaces of finite rank. *Int. Math. Res. Not. IMRN*, (7):1578–1627, 2013.

[Duc15] Bruno Duchesne. Infinite dimensional Riemannian symmetric spaces with fixed-sign curvature operator. *Ann. Inst. Fourier (Grenoble)*, 65(1):211–244, 2015.

[Duc23] Bruno Duchesne. The Polish topology of the isometry group of the infinite-dimensional hyperbolic space. *Groups Geom. Dyn.*, 17(2):633–670, 2023.

[Dye90] Matthew Dyer. Reflection subgroups of coxeter systems. *Journal of Algebra*, 135(1):57–73, 1990.

[Fru91] Monica Frunză. Möbius transformations in infinite dimension. volume 36, pages 369–376. 1991. Analyse complexe (Bucharest, 1989).

[Fu13] Xi Fu. Inequalities for discrete Möbius groups in infinite dimension. *J. Complex Anal.*, pages Art. ID 515109, 4, 2013.

[GdlH90] Etienne Ghys and Pierre de la Harpe, editors. *Sur les groupes hyperboliques d'après Mikhael Gromov. (On the hyperbolic groups à la M. Gromov)*, volume 83. Boston, MA: Birkhäuser, 1990.

[Gon18] Krishnendu Gongopadhyay. On generalized Jørgensen inequality in infinite dimension. *New York J. Math.*, 24:865–869, 2018.

[Goo72] Robert Kent Goodrich. The spectral theorem for real Hilbert space. *Acta Sci. Math.*, 33:123–127, 1972.

[Gro87] Mikhael Gromov. *Hyperbolic Groups*, pages 75–263. Springer New York, New York, NY, 1987.

[Gro93] Mikhael Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[Gug17] Rafael Guglielmetti. *Hyperbolic isometries in (in-)finite dimensions and discrete reflection groups: theory and computations*. 2017. Thesis (Ph.D.)–University of Fribourg.

[Has22] Yo Hasegawa. Gromov Boundaries of Non-proper Hyperbolic Geodesic Spaces. *Tokyo Journal of Mathematics*, 45(2):319 – 331, 2022.

[HK77] L. A. Harris and W. Kaup. Linear algebraic groups in infinite dimensions. *Illinois J. Math.*, 21(3):666–674, 1977.

[Hum90] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[JM87] Dennis Johnson and John J. Millson. Deformation spaces associated to compact hyperbolic manifolds. In *Discrete Groups in Geometry and Analysis*, Progress in Mathematics, pages 48–106. Birkhäuser Boston, Boston, MA, 1987.

[Jør76] Troels Jørgensen. On discrete groups of Möbius transformations. *Amer. J. Math.*, 98(3):739–749, 1976.

[Kac85] Victor G. Kac. Constructing groups associated to infinite-dimensional lie algebras. In V. Kac, editor, *Infinite Dimensional Groups with Applications*, pages 167–216, New York, NY, 1985. Springer New York.

[Kap06] Michael Kapovich. Convex projective structures on gromov–thurston manifolds. *Geometry & topology*, 11(3):1777–1830, 2006.

[Li09] Liulan Li. Ball-preserving Möbius transformations in infinite dimension. *Complex Var. Elliptic Equ.*, 54(7):697–703, 2009.

[Li11] Liulan Li. A generalization of Jørgensen's inequality to infinite dimension. *New York J. Math.*, 17:41–49, 2011.

[Li12] Liu Lan Li. An inequality on parabolic Möbius groups in infinite dimension and their application. *Acta Math. Sci. Ser. A (Chinese Ed.)*, 32(2):349–355, 2012.

[Li13] Liulan Li. Discreteness of Möbius groups in infinite dimension. *Complex Var. Elliptic Equ.*, 58(1):109–112, 2013.

[Lon16] Anne Lonjou. Non-simplicity of the Cremona group over any field. *Ann. Inst. Fourier*, 66(5):2021–2046, 2016.

[Lon19a] Anne Lonjou. On the hyperbolicity of graphs associated to the Cremona group. *Épijournal de Géom. Algébr., EPIGA*, 3:29, 2019. Id/No 11.

[Lon19b] Anne Lonjou. Voronoi tessellations associated to the Cremona group. *Publ. Mat., Barc.*, 63(2):521–599, 2019.

[LT24] Yusen Long and Dong Tan. Connectedness of the gromov boundary of fine curve graphs, 2024.

[LU21] Anne Lonjou and Christian Urech. Actions of Cremona groups on CAT(0) cube complexes. *Duke Math. J.*, 170(17):3703–3743, 2021.

[LW05] Liu Lan Li and Xian Tao Wang. Möbius transformations in infinite dimension. *Heilongjiang Daxue Ziran Kexue Xuebao*, 22(4):497–500, 2005.

[Mad85] John H. Maddocks. Restricted quadratic forms and their application to bifurcation and stability in constrained variational principles. *SIAM Journal on Mathematical Analysis*, 16(1):47–68, 1985.

[Mar89] Gaven J. Martin. On discrete Möbius groups in all dimensions: a generalization of Jørgensen's inequality. *Acta Math.*, 163(3-4):253–289, 1989.

[Mar23] Bruno Martelli. An introduction to geometric topology. page 488, 2023.

[Meg98] Robert E. Megginson. *An introduction to Banach space theory*, volume 183. New York, NY: Springer, 1998.

[Mon05] Nicolas Monod. Superrigidity for irreducible lattices and geometric splitting. *C. R. Math. Acad. Sci. Paris*, 340(3):185–190, 2005.

[Mou88] Gabor Moussong. *Hyperbolic Coxeter groups*. ProQuest LLC, Ann Arbor, MI, 1988. Thesis (Ph.D.)–The Ohio State University.

[MP14] Nicolas Monod and Pierre Py. An exotic deformation of the hyperbolic space. *Amer. J. Math.*, 136(5):1249–1299, 2014.

[MP19] Nicolas Monod and Pierre Py. Self-representations of the Möbius group. *Annales Henri Lebesgue*, 2:259–280, 2019.

[Rat06] John G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.

[Thu22] William P. Thurston. *The geometry and topology of three-manifolds. Vol. IV*. American Mathematical Society, Providence, RI, [2022] ©2022. Edited and with a preface by Steven P. Kerckhoff and a chapter by J. W. Milnor.

[Tit88] Jacques Tits. Sur le groupe des automorphismes de certains groupes de coxeter. *Journal of Algebra*, 113(2):346–357, 1988.

[Vin71] Èrnest. B. Vinberg. Discrete linear groups generated by reflections. *Mathematics of the USSR-Izvestiya*, 5(5):1083, 1971.

[Vä05] Jussi Väisälä. Gromov hyperbolic spaces. *Expositiones Mathematicae*, 23(3):187–231, 2005.

[Wat93] P. L. Waterman. Möbius transformations in several dimensions. *Adv. Math.*, 101(1):87–113, 1993.