Algoritmos

Thursday, October 28, 2021 4:34 PM

1. Métodos primales
   - Método de direcciones factibles
   - Método de conjuntos activos
   - Método del gradiente proyectado

Direcciones factibles

$$\bar{x} - x_k = d_k$$

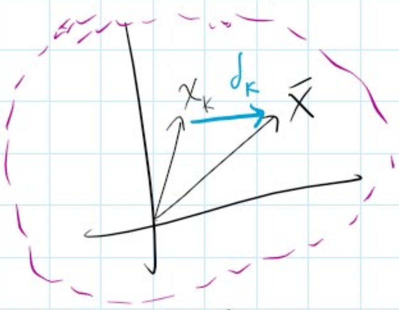
$$\min f(x)$$
$$\text{s.a. } x \in \Omega$$

$$x \xrightarrow[\alpha]{d_k} x_{k+1}$$

d_k → dirección de descenso

→ no deben salirse de su región factible

α → cantidad avanzada en esa dirección



¿Cómo sé que una dirección es de descenso?

$$\nabla f' \vec{d} < 0$$

Secuencia de puntos factibles:

$$\vec{x}_{k+1} = \vec{x}_k + \alpha_k \cdot \vec{d}_k$$

$\vec{d}_k$ → dir. factible

Teorema

Sea $\{x_k\}$ una sucesión generada por el método de direcciones factibles $x_{k+1} = x_k + \alpha_k d_k$, tal que la sucesión de direcciones $\{d_k\}_{k\in K}$ verifica:

1. $\lim_{k\to\infty} \sup_{k\in K} \nabla f(x_k)' d_k < 0$

La idea será minimizar el gradiente así $\nabla f(x) = 0$ y entonces se debe escoger una dirección tal que logre ese objetivo:

$$\min \nabla f(x_k)' \vec{d}_k \rightarrow \bar{x} - x_k$$

$x \in$ región factible.

Ej

$$\min \quad x_1^2 + x_2^2 - 4x_1 - 4x_2 + 8$$
$$\text{s.a.} \quad x_1 + 2x_2 - 4 \leq 0$$
$$x_1, x_2 \geq 0$$

$$x_0 = \begin{bmatrix} 4/3 \\ 4/3 \end{bmatrix} \xrightarrow{\text{buscar}} (\bar{x} - \vec{x}_k) = \vec{d}_k$$

① $\min \nabla \vec{f}(x_k)' (\bar{x} - x_k) \longrightarrow \vec{d}_k = \begin{bmatrix} 8/3 \\ -4/3 \end{bmatrix}$

② $\| \nabla \vec{f}(x_k)' \vec{d} \| \geq \varepsilon$

③ Calcular nuevo punto $\vec{x}_{k+1} = \vec{x}_k + \alpha \vec{d}_k$

$0 \leq \alpha \leq 1 \rightsquigarrow x_{k+1} = \begin{bmatrix} 4/3 + \alpha\, 8/3 \\ \ldots \end{bmatrix}$

$0 \leq \alpha \leq 1 \rightsquigarrow x_{k+1} = \begin{bmatrix} 4/3 + \alpha\, 8/3 \\ 4/3 - 4/3\alpha \end{bmatrix}$

encontrar α

$$\min f(x_k + 1)$$
$$\text{s.a.} \quad 0 \leq \alpha \leq 1$$

$$x_{k+1} = \begin{bmatrix} 8/5 \\ 6/5 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \nabla \vec{f}(x) = \begin{bmatrix} 2x_1 - 4 \\ 2x_2 - 4 \end{bmatrix}$$

① $\min \nabla f(x_0)' \vec{d}_0$ $\quad \nabla f(x_0) \to \begin{bmatrix} -4 \\ -4 \end{bmatrix}, \quad \vec{d}_0 \to \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$

$$\boxed{\min -4\bar{x}_1 - 4\bar{x}_2}$$
$$\text{s.a.} \quad \bar{x}_1 + 2\bar{x}_2 - 4 \leq 0$$
$$\bar{x}_1, \bar{x}_2 \geq 0$$

simplex $\rightsquigarrow \bar{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \longrightarrow d_0 = \bar{x} - \bar{x}_k = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \vec{0} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

② $\left\| \begin{bmatrix} -4 & -4 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \right\| \leq \varepsilon$

$$\|-16\| \nleq \varepsilon$$

③ $x_{k+1} = x_k + \alpha \vec{\delta}_k$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4\alpha \\ 0 \end{bmatrix}$$

$$\min f(\vec{x}_{k+1}) = (4\alpha)^2 - 4(4\alpha) + 8$$
$$= 16\alpha^2 - 16\alpha + 8$$

$$\text{s.a.} \quad 0 \leq \alpha \leq 1$$

$$\longrightarrow x_{k+1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Caso no lineal

Ej

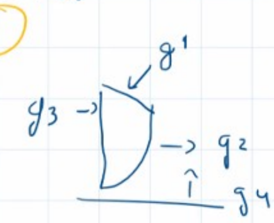
$$\min 2x_1^2 + 2x_2^2 - 2x_1x_2 - 4x_1 - 6x_2$$

s.a.
$x_1 + 5x_2 - 5 \leq 0 \quad g_1$

$2x_1^2 - x_2 \leq 0 \leftarrow g_2$

$-x_1 \leq 0 \quad g_3$

$-x_2 \leq 0. \quad g_4$



$x_0 = \begin{bmatrix} 0 \\ 3/4 \end{bmatrix} \quad J = \{3\}$ $\qquad \nabla f = \begin{bmatrix} 4x_1 - 2x_2 - 4 \\ 4x_2 - 2x_1 - 6 \end{bmatrix}$

① $\min z$

s.a. $\nabla f(x_0)' d + z \leq 0$

$\nabla g_3(x_0)' d + z \leq 0$

$$\nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$\nabla g_3(x_0)^T d + z \le 0$

① $\nabla f(x_0)^T \vec{d} = \begin{bmatrix} -\frac{11}{2} & 3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + z \le 0$

$= -\frac{11}{2} d_1 + 3 d_2 + z \le 0$

② $\begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + z \le 0$

$= -d_1 + z \le 0$

todo junto y mejor asunto:

$$\left.\begin{aligned} &\min \ z \\ -\tfrac{11}{2} d_1 + 3d_2 + z &\le 0 \\ -d_1 + z &\le 0 \\ -1 \le d_1 &\le 1 \\ -1 \le d_2 &\le 1 \end{aligned}\right]$$

esta parte va siempre a ser lineal

uso simplex

$d = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad z = -1$

②

$$x_{k+1} = x_k + \alpha d_k$$
$$= \begin{bmatrix} 0 \\ 3/4 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} \alpha \\ 3/4 - \alpha \end{bmatrix}$$

$$\min \ f(x_{k+1}) = 2\alpha^2 + 2\left(\frac{3}{4} - \alpha\right)^2 - 2\alpha\left(\frac{3}{4} - \alpha\right) - 4\alpha - 6\left(\frac{3}{4} - \alpha\right)$$
$$= 4\alpha^2 + \frac{9}{8} - 3\alpha + 2\alpha^2 + \frac{1}{2}\alpha - 9$$
$$= 6\alpha^2 - \frac{5}{2}\alpha - \frac{27}{8}$$

$$\min \ 6\alpha^2 - \frac{5}{2}\alpha - \frac{27}{8}$$

$$\text{s.a.} \quad \alpha + 5\left(\frac{3}{4} - \alpha\right) - 5 \le 0$$
$$2\alpha^2 - \left(\frac{3}{4} - \alpha\right) \le 0$$
$$-\alpha \le 0$$
$$-\frac{3}{4} \le 0$$
$$0 \le \alpha \le 1$$

$$\frac{f(x_{k+1})}{d\alpha} = 12\alpha - \frac{5}{2} = 0$$
$$\alpha = \frac{5}{24} \approx 0,2084 \ \le 0,4 \ ✓$$

Métodos de conjuntos activos

Integrantes:
- David Alsina
- Alejandra Archbold
- Andrey Lizarazo
- Camilo Silva
- Rodrigo

Ejemplo

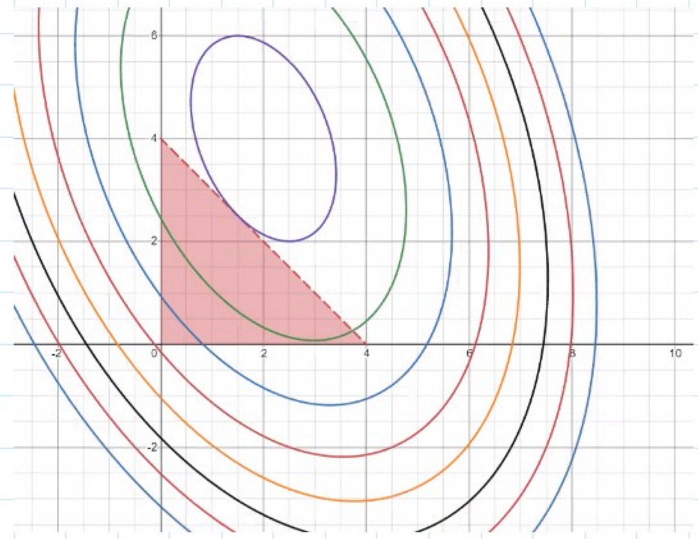
$$\min\ 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2$$

s.a. $x_1 + x_2 - 4 \le 0 \to g_1$

$x_1, x_2 \ge 0$

$g_2 \to -x_1 \le 0$

$g_3 \to -x_2 \le 0$



CNPO:

$$Df(x^*) + \lambda^{*\prime} \underbrace{Dh(x^*)}_{\to 0} + \mu^{*\prime} Dg(x^*) = 0$$

$$\nabla f(x^*) = \begin{bmatrix} 4x_1 + x_2 - 12 \\ x_1 + 2x_2 - 10 \end{bmatrix}$$

$$\nabla g_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\nabla f(x^*) = \begin{bmatrix} 4x_1 + x_2 - 12 \\ x_1 + 2x_2 - 10 \end{bmatrix} \qquad \nabla g_1(x^*) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \nabla g_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4x_1 + x_2 - 12 \\ x_1 + 2x_2 - 10 \end{bmatrix} + [\mu_1\ \ \mu_2\ \ \mu_3] \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = 0$$

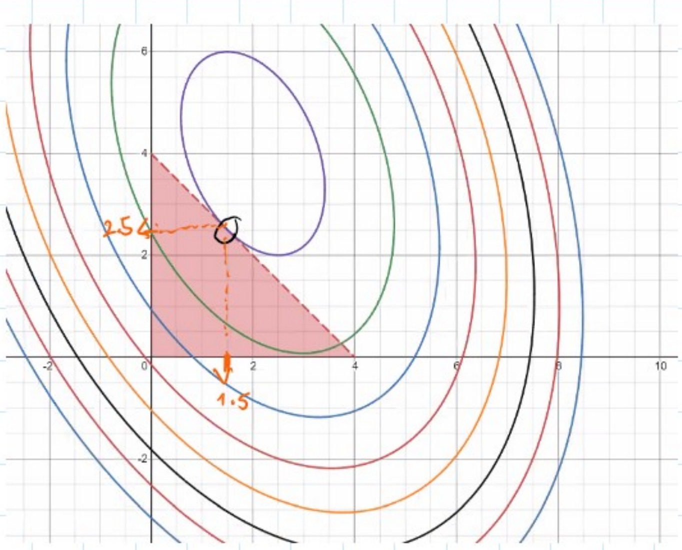
$$= \begin{bmatrix} 4x_1 + x_2 - 12 \\ x_1 + 2x_2 - 10 \end{bmatrix} + \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_1 - \mu_3 \end{bmatrix} = \begin{matrix} 4x_1 + x_2 - 12 + (\mu_1 - \mu_2) = 0 \\ x_1 + 2x_2 - 10 + (\mu_1 - \mu_3) = 0 \end{matrix}$$

- $\mu^{*\prime} \cdot g(x^*) = 0$
- $\mu^* \ge 0$

Sistema:

$$\begin{cases}
(1)\ x_1 + x_2 - 4 \le 0 \\
(2)\ -x_1 \le 0 \\
(3)\ -x_2 \le 0 \\
(4)\ 4x_1 + x_2 - 12 + (\mu_1 - \mu_2) = 0 \\
(5)\ x_1 + 2x_2 - 10 + (\mu_1 - \mu_3) = 0 \\
(6)\ \mu_1(x_1 + x_2 - 4) = 0 \\
(7)\ -\mu_2 x_1 = 0 \\
(8)\ -\mu_3 x_2 = 0 \\
(9)\ \mu_1 \ge 0 \\
(10)\ \mu_2 \ge 0 \\
(11)\ \mu_3 \ge 0
\end{cases}$$

✓



Casos:

(*)

$g_1 = 0 \quad \mu_1 \ne 0$

$g_2 = 0 \quad \mu_2 \ne 0$

$g_3 \ne 0 \quad \mu_3 = 0$

$\to x_1 = 0$

(4) $\cancel{4x_1}^{\to 0} + x_2 - 12 + (\mu_1 - \mu_2) = 0$

(5) $x_1^{\to 0} + 2x_2 - 10 + (\mu_1 - \mu_{\cancel{3}}) = 0$

④ $\cancel{4x_1} + x_2 - 12 + (\mu_1 - \mu_2) = 0$

⑤ $\cancel{x_1}^{\to 0} + 2x_2 - 10 + (\mu_1 - \cancel{\mu_3}^{\to 0}) = 0$

⑥ $\mu_1(\cancel{x_1}^{\to 0} + x_2 - 4) = 0$

$$\mu_1 x_2 = \mu_1 4$$

$$x_2 = 4$$

Ahora con ⑤:

$$2(4) - 10 + \mu_1 = 0$$

$$\mu_1 = 10 - 8$$

$$\mu_1 = 2$$

Ahora en ④:

$$(4) - 12 + 2 - \mu_2 = 0$$

$$\mu_2 = -6 \longrightarrow \mu_2 \not\geq 0$$

⊛ Solo g_1 actua, $g_1 = 0$, $\mu_1 \neq 0$
$g_2 \neq 0$, $\mu_2 = 0$
$g_3 \neq 0$, $\mu_3 = 0$

① $x_1 + x_2 - 4 = 0 \longrightarrow x_1 = 4 - x_2$

④ $4x_1 + x_2 - 12 + (\mu_1 - \cancel{\mu_2}^{\to 0}) = 0$

⑤ $x_1 + 2x_2 - 10 + (\mu_1 - \cancel{\mu_3}^{\to 0}) = 0$

⑥ $\mu_1(x_1 + x_2 - 4) = 0$

evaluando $x_1 = 4 - x_2$ en ⑤:

$$(4 - x_2) + 2x_2 - 10 + \mu_1 = 0$$

$$x_2 - 6 + \mu_1 = 0$$

$$\mu_1 = 6 - x_2$$

μ_1 y x_1 en ④:

$$4(4 - x_2) + \cancel{x_2}^{\to 0} - 12 + (6 - \cancel{x_2})^{\to 0} = 0$$

$$16 - 4x_2 - 12 + 6 = 0$$

$$-4x_2 = -10$$

$$x_2 = 10/4 = 5/2$$

Con $x_2 = 5/2$ en $x_1 = 4 - x_2$

$$x_1 = \left(4 - \frac{5}{2}\right)$$

$x_1 = (4 - \frac{5}{2})$

$= 1.5 = \frac{3}{2}$

$\Rightarrow \mu_1 = 6 - x_2$

$= 6 - 5/2 \geq 0$ ✓

$= 3.5 = 7/2$

CNSO: $y' L(x, \lambda, \mu) y \geq 0$

Para verificar la CNSO Necesitamos el plano tangente a x^*

$x^* = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \cdot 1/2$, $T(x^*) = \{ y : Dg_j(x^*) \cdot y = 0, \ j \in J(g) \}$

$\nabla g_1(x^*) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$T(x^*) = \left\{ y : [1 \ \ 1] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0 \right\}$

$\nabla f(x^*) = \begin{bmatrix} 4x_1 + x_2 - 12 \\ x_1 + 2x_2 - 10 \end{bmatrix}$

$G_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\nabla g_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\nabla g_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$y_1 + y_2 = 0$

$y_1 = -y_2$

$\vec{y} = \begin{bmatrix} y_1 \\ -y_1 \end{bmatrix} = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}$

$$L(x^*, \lambda, \mu) = \underbrace{H(x)}_{\text{hessiana de } f_x} + \cancel{\sum_{i=1}^{m} \lambda_i \underbrace{H_i(x)}_{\text{hessiana de } h_i}}^{0} + \sum_{j=1}^{p} \mu_j \underbrace{G_j(x)}_{\text{hessiana de } g_j}$$

$$= \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$y' L y = [\alpha \ \ -\alpha] \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}$$

$$= [\alpha \ \ -\alpha] \begin{bmatrix} 3\alpha \\ -\alpha \end{bmatrix}$$

$$= 3\alpha^2 + \alpha^2 > 0$$