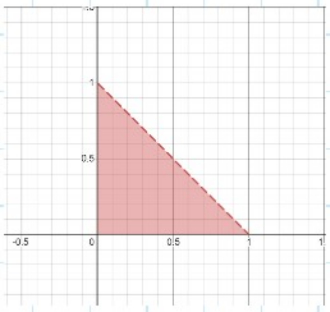
Final Viejo

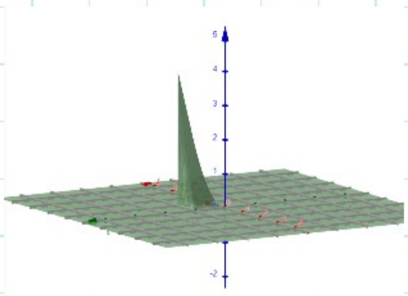
Tuesday, November 23, 2021 2:08 PM

1. (**20 pts**) Sean X y Y dos variables aleatorias con función de densidad conjunta:

$$f_{X,Y}(x,y) = \begin{cases} cxy, & 0<x<1, \quad 0<y<1, \quad 0<x+y<1, \\ 0 & \text{d.l.c} \end{cases}$$

- Determine el valor de la constante c.
- Determine la función de densidad condicional X dado que $Y=y$, donde $0<y<1$.
- Determine el valor esperado condicional de X dado que $Y=y$, donde $0<y<1$.
- Determine la covarianza de X e Y.
- ¿Son X y Y independientes? Justifique





• Para calcular C debemos buscar que $f_{X,Y}(x,y)$ sea legítima:

$$\int_x \int_y cxy \, dy \, dx = 1$$

$$\int_x cx \cdot \frac{y^2}{2}\Big|_0^1 = 1$$

$$\int_x \frac{c}{2} x = 1$$

$$\frac{c}{2} \cdot \frac{x^2}{2}\Big|_0^1 = \frac{c}{4} = 1$$

$$C = 4$$

$$f_{x,y}(x,y) = 4xy$$

• Para determinar la función de densidad condicional se puede empezar por obtener la marginal de Y:

$x+y=1 \rightsquigarrow (1-y)=x$

$$f(y) = \int_x f(x,y)\,dx = 4y\int_0^{1-y} x \, dx = (4y)\cdot \frac{x^2}{2}\Big|_0^{1-y} = (2y)(1-y)$$

Con la marginal obtenemos $f(x|y)$:

$$f(x|y) = \frac{f_{xy}(x,y)}{f(y)} = \frac{4x\cancel{y}}{(2\cancel{y})(1-y)} = \frac{2x}{(1-y)}$$

• Para hayar el valor esperado condicional $E(X|Y)$:

$$\int_{x=0}^{x+y=1} x \cdot f(x|y)\,dx = \int_0^{1-y} x \cdot \frac{2x}{(1-y)}\,dx = \frac{2}{(1-y)} \cdot \frac{x^3}{3}\Big|_0^{(1-y)} = \frac{2}{3}\cdot(1-y)^2$$

• Determine la $\text{cov}(x,y)$:

$$\text{Cov}(X,Y) = E\big((X - E(x))\cdot(Y - E(Y))\big)$$

Podemos calcular $E(Y)$:

$$* \; E(Y) = \int (2y)(1-y)\,dy = \int^1 2y - 2y^2 = y^2 - \frac{2}{3}y^3\Big|^1 = \frac{1}{3}$$

$E(Y) = \int_Y (2y)(1-y)\,dy = \int_0 2y - 2y^2 = y^2 - \frac{2}{3}y^3 \Big|_0 = \frac{1}{3}$

$E(X) = E(E(X|Y)) = E\left(\frac{2}{3}\cdot(1-y)^2\right)$

$= \frac{2}{3}\cdot\int_0^1 y^2 - 2y + 1\,dy$

$= \frac{2}{3}\cdot\left(\frac{y^3}{3} - y^2 + y\right)\Big|_0^1$

$= (2/3)\cdot\left(\frac{1}{3}\right) = 2/9$

$Cov(X,Y) = E((X - 2/9)(Y - 1/3))$

$= E\left(xy - \frac{2}{9}y - \frac{1}{3}x + \frac{2}{27}\right)$

$= E(XY) - \frac{2}{9}E(Y) - \frac{1}{3}E(X) + \frac{2}{27}$

$= E(XY) - \frac{4}{27} + \frac{2}{27} = E(XY) - \frac{2}{27}$

$= \int_x\int_y 4x^2y^2\,dy\,dx = \int_0^1 4x \int_0^{x+y=1} y^2\,dy\,dx = \int_0^1 4x\cdot\frac{y^3}{3}\Big|_0^{(1-x)}$

$= \int_0^1 4x\,\frac{(1-x)^3}{3}\,dx = \frac{4}{3}\left(\frac{1}{20}\right) = \frac{1}{15}$

$= \frac{1}{15} - \frac{2}{27} = \left(-\frac{1}{135}\right)$

Dado que su cov no es cero no son indep.

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2. (**10 pts**) La función generadora de momentos de X está dada por $\boxed{M_X(s) = e^{2e^s-2}}$ y la de Y por $\boxed{M_Y(s) = \left(\frac{3}{4}e^s + \frac{1}{4}\right)^{10}}$. Si X e Y son independientes, determine:

- $P(X+Y=1)$

$e^{2(e^s-1)}$ es poisson $\lambda = 2$

Binomial $p = \frac{3}{4}$, $n = 10$

$p_X(k) = \left\{ \frac{e^{-2}2^k}{k!} \right.$ $\quad p_Y(k) = \left\{ \binom{10}{k} p^k (1-p)^{10-k} \right.$

$Z = X+Y$, dado que son independientes podemos

Convolucionarlas así:

$$P(x+y=1) = \sum_{\{x,y \mid x+y=1\}} P_x(x)\cdot P_y(y) = \sum_x P_x(x)\cdot P_y(1-x)$$

$$= \sum_x \frac{e^{-2}\cdot 2^x}{x!}\cdot\binom{10}{1-x} p^{1-x}(1-p)^{10-(1-x)}$$

- $\mathbb{E}((X+Y)^2) = E(x^2+2xy+y^2) = E(x^2) + 2E(x)E(y) + E(y^2)$

$$\frac{dM_x(s)}{ds} = 2\cdot e^{s+2e^s-2}\Big|_0 = 2 = E(x)$$

$$\frac{d^2M_x(s)}{ds^2} = (e^{s+2e^s-2})\cdot(4e^s+2)\Big|_0 = 6 = E(x^2)$$

$$\frac{dM_y(s)}{ds} = \frac{15}{2}e^s\left(\frac{3e^s}{4}+\frac{1}{4}\right)^9\Big|_0 = \frac{15}{2}\left(\frac{3}{4}+\frac{1}{4}\right)^9 = \frac{15}{2} = E(y)$$

$$\frac{d^2M_y(s)}{d^2s} = \frac{15e^s(3e^s+1)^8(30e^s+1)}{524288}\Big|_0 = \frac{465}{8} = E(y^2)$$

$$\dots = E(x^2)+2E(x)E(y)+E(y^2) = 6 + 2\left(\frac{15}{2}\cdot 2\right) + \frac{465}{8}$$

$$= 94.125$$
