3.7 THE TRACE-DETERMINANT PLANE

In the previous sections, we have encountered a number of different types of linear systems of differential equations. At this point, it may seem that there are many different possibilities for these systems, each with its own characteristics. In order to put all of these examples in perspective, it is useful to pause and review the big picture.

One way to summarize everything that we have done so far is to make a table. As we have seen, the behavior of a linear system is governed by the eigenvalues and eigenvectors of the system, so our table should contain the following:

1. The name of the system (spiral sink, saddle, source, . . .)

2. The eigenvalue conditions

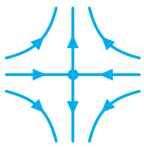
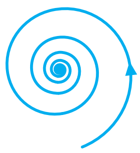
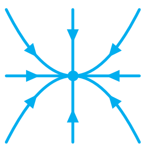
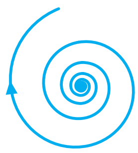
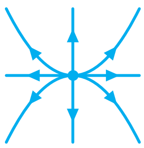
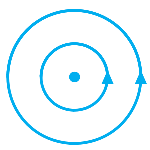
3. One or two representative phase portraits

For example, we could begin to construct this table as in Table 3.1.

This list is by no means complete. In fact, one exercise at the end of this section is to compile a complete table (see Exercise 1). There are eight other entries.

As is so often the case in mathematics, it is helpful to view information in several different ways. Since we are looking for "the big picture," why not try to summarize the different behaviors for linear systems in a picture rather than a table? One such picture is called the *trace-determinant plane*.

Table 3.1
Partial table of linear systems.

Type	Eigenvalues	Phase Plane	Type	Eigenvalues	Phase Plane
Saddle	$\lambda_1 < 0 < \lambda_2$		Spiral Sink	$\lambda = a \pm ib$ $a < 0, b \neq 0$	
Sink	$\lambda_1 < \lambda_2 < 0$		Spiral Source	$\lambda = a \pm ib$ $a > 0, b \neq 0$	
Source	$0 < \lambda_1 < \lambda_2$		Center	$\lambda = \pm ib$ $b \neq 0$	

Trace and Determinant

Suppose we begin with the linear system $d\mathbf{Y}/dt = \mathbf{AY}$, where $\mathbf{A}$ is the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The characteristic polynomial for $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda I) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ad - bc.$$

The quantity $a + d$ is called the **trace** of the matrix $\mathbf{A}$ and, as we know, the quantity $ad - bc$ is the determinant of $\mathbf{A}$. So the characteristic polynomial of $\mathbf{A}$ can be written more succinctly as

$$\lambda^2 - T\lambda + D,$$

where $T = a + d$ is the trace of $\mathbf{A}$ and $D = ad - bc$ is the determinant of $\mathbf{A}$. For example, if

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

then the characteristic polynomial is $\lambda^2 - 5\lambda - 2$, since $T = 5$ and $D = 4 - 6 = -2$. (Remember that the coefficient of the λ-term is $-T$. It is a common mistake to put this minus sign in the wrong place or even to forget it entirely.)

Since the characteristic polynomial of $\mathbf{A}$ depends only on T and D, it follows that the eigenvalues of $\mathbf{A}$ also depend only on T and D. If we solve the characteristic polynomial $\lambda^2 - T\lambda + D = 0$, we obtain the eigenvalues

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

From this formula we see immediately that the eigenvalues of $\mathbf{A}$ are complex if $T^2 - 4D < 0$, they are repeated if $T^2 - 4D = 0$, and they are real and distinct if $T^2 - 4D > 0$.

The Trace-Determinant Plane

We can now begin to paint the big picture for linear systems by examining the *trace-determinant* plane. We draw the T-axis horizontally and the D-axis vertically. Then the curve $T^2 - 4D = 0$, or equivalently $D = T^2/4$, is a parabola opening upward in this plane. We call it the *repeated-root* parabola. Above this parabola $T^2 - 4D < 0$, and below it $T^2 - 4D > 0$.

To use this picture, we first compute T and D for a given matrix and then locate the point (T, D) in this plane. Then we can immediately read off whether the eigenvalues are real, repeated, or complex, depending on the location of (T, D) relative to the repeated-root parabola (see Figure 3.46). For example, if

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix},$$

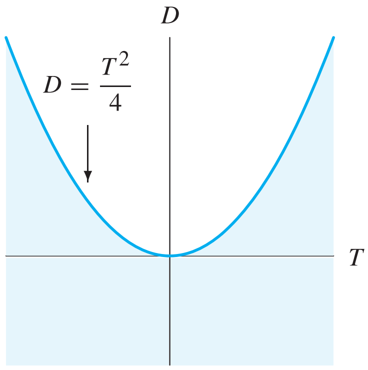


Figure 3.46
The shaded region corresponds to $T^2 - 4D > 0$.

then $(T, D) = (4, 1)$, and the point (4,1) lies below the curve $T^2 - 4D = 0$ (in this case, $T^2 - 4D = 12 > 0$), so the eigenvalues of A are real and distinct.

Refining the Big Picture

We can actually do much more with the trace-determinant plane. For example, if

$$T^2 - 4D < 0,$$

(the point (T, D) lies above the repeated-root parabola), then we know that the eigenvalues are complex and their real part is $T/2$. We have a spiral sink if $T < 0$, a spiral source if $T > 0$, and a center if $T = 0$. In the trace-determinant plane, the point (T, D) is located above the repeated-root parabola. If (T, D) lies to the left of the D-axis, the corresponding system has a spiral sink. If (T, D) lies to the right of the D-axis, the system has a spiral source. If (T, D) lies on the D-axis, then the system has a center. So our refined picture can be drawn this way (see Figure 3.47).

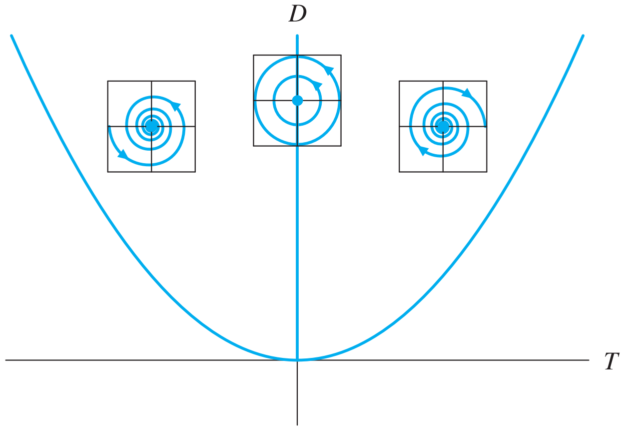


Figure 3.47
Above the repeated-root parabola, we have centers along the D-axis, spiral sources to the right, and spiral sinks to the left.

Real eigenvalues

We can also distinguish different regions in the trace-determinant plane where the linear system has real and distinct eigenvalues. In this case (T, D) lies below the repeated-root parabola. If $T^2 - 4D > 0$, the real eigenvalues are

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

If $T > 0$, the eigenvalue

$$\frac{T + \sqrt{T^2 - 4D}}{2}$$

is the sum of two positive terms and therefore is positive. Thus we only have to determine the sign of the other eigenvalue

$$\frac{T - \sqrt{T^2 - 4D}}{2}$$

to determine the type of the system.

If $D = 0$, then this eigenvalue is 0, so our matrix has one positive and one zero eigenvalue. If $D > 0$, then

$$T^2 - 4D < T^2.$$

Since we are considering the case where $T > 0$, we have

$$\sqrt{T^2 - 4D} < T$$

and

$$\frac{T - \sqrt{T^2 - 4D}}{2} > 0.$$

In this case both eigenvalues are positive, so the origin is a source.

On the other hand, if $T > 0$ but $D < 0$, then

$$T^2 - 4D > T^2,$$

so that

$$\sqrt{T^2 - 4D} > T$$

and

$$\frac{T - \sqrt{T^2 - 4D}}{2} < 0.$$

In this case the system has one positive and one negative eigenvalue, so the origin is a saddle.

In case $T < 0$ and $T^2 - 4D > 0$, we have

- two negative eigenvalues if $D > 0$,
- one negative and one positive eigenvalue if $D < 0$, or
- one negative eigenvalue and one zero eigenvalue if $D = 0$.

Finally, along the repeated-root parabola we have repeated eigenvalues. If $T < 0$, both eigenvalues are negative; if $T > 0$, both are positive; and if $T = 0$, both are zero.

The full picture is displayed in Figure 3.48. Note that this picture gives us some of the same information that we compiled in our table earlier in this section.

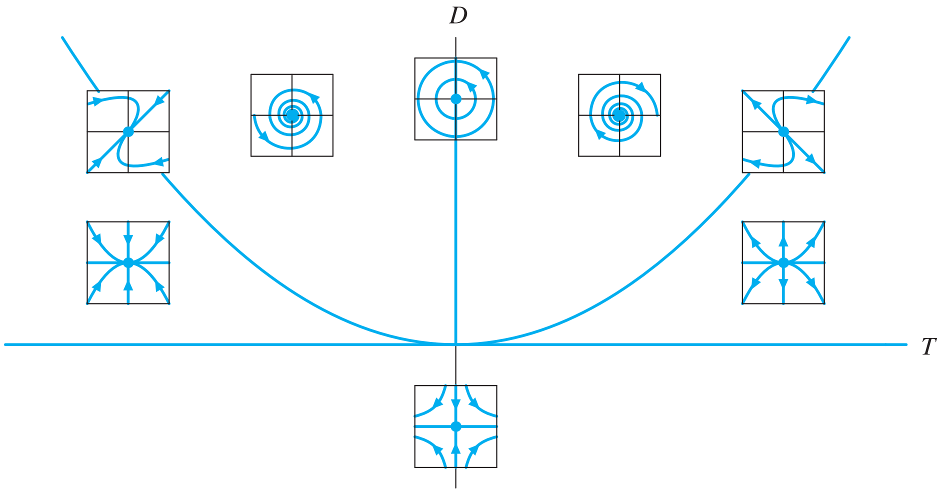


Figure 3.48
The big picture.

The Parameter Plane

The trace-determinant plane is an example of a *parameter plane*. The entries of the matrix **A** are parameters that we can adjust. When these entries change, the trace and determinant of the matrix also change, and our point (T, D) moves around in the parameter plane. As this point enters the various regions in the trace-determinant plane, we should envision the corresponding phase portraits changing accordingly. The trace-determinant plane is very much different from previous pictures we have drawn. It is a picture of a classification scheme of the behavior of all possible solutions to linear systems.

We must emphasize that the trace-determinant plane does not give complete information about the linear system at hand. For example, along the repeated-root parabola we have repeated eigenvalues, but we cannot determine whether we have one or many linearly independent eigenvectors. In order to make that distinction, we must actually calculate the eigenvectors. Similarly, we cannot determine the direction in which solutions wind about the origin if $T^2 - 4D < 0$. For example, both of the matrices

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

have trace 0 and determinant 1, but solutions of the system $d\mathbf{Y}/dt = \mathbf{AY}$ wind around the origin in the clockwise direction, whereas solutions of $d\mathbf{Y}/dt = \mathbf{BY}$ travel in the opposite direction.

The Harmonic Oscillator

We can also paint the same picture for the harmonic oscillator. Recall that this second-order equation is given by

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0,$$

where $m > 0$ is the mass, $k > 0$ is the spring constant, and $b \geq 0$ is the damping coefficient. As a system we have

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -k/m & -b/m \end{pmatrix}\mathbf{Y},$$

so the trace $T = -b/m$ and the determinant $D = k/m$. We plot $T = -b/m$ on the horizontal axis and $D = k/m$ on the vertical axis as before.

Since m and k are positive and b is nonnegative, we are restricted to one-quarter of the picture for general linear systems, namely the second quadrant of the TD-plane. The picture is shown in Figure 3.49.

The repeated-root parabola in this case is $T^2 - 4D = b^2 - 4km = 0$. Above this parabola we have a spiral sink (if $b \neq 0$) or a center (if $b = 0$). Below the repeated-root parabola we have a sink with real distinct eigenvalues. On the parabola, we have repeated negative eigenvalues.

In the language of oscillators introduced in the previous section, if $(-b/m, k/m)$ lies above the repeated-root parabola and $b > 0$, we have an underdamped oscillator, or if $b = 0$, we have an undamped oscillator. If $(-b/m, k/m)$ lies on the repeated-root parabola, the oscillator is critically damped. Below the parabola, the oscillator is overdamped.

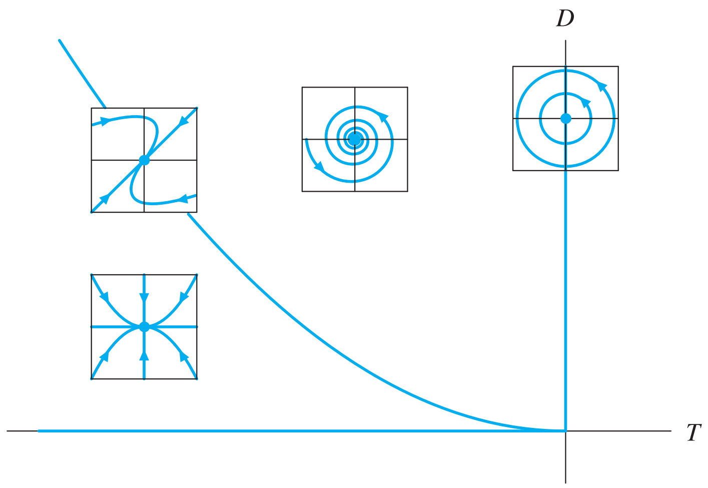


Figure 3.49
The trace-determinant plane for the harmonic oscillator.

Kathleen Alligood (1947–) received her Ph.D. in mathematics at the University of Maryland. She taught at the College of Charleston and at Michigan State University before assuming her current position as Professor of Mathematical Sciences at George Mason University.

Alligood's research centers on the behavior of nonlinear systems but encompasses many of the topics described in this chapter. Nonlinear systems of differential equations may possess sinks, just as linear systems do. However, it need not be the case that all solutions tend to the sink as in the linear case. Often the boundary of the set of solutions that tend to the sink is an extremely complicated mathematical object that contains infinitely many saddle points and their stable curves. Using techniques from topology, fractal geometry, and dynamical systems, Alligood and her coworkers were among the first to analyze the structure of these "fractal basin boundaries."

Navigating the Trace-Determinant Plane

One of the best uses of the trace-determinant plane is in the study of linear systems that depend on parameters. As the parameters change, so do the trace and determinant of the matrix. Consequently, the phase portrait for the system also changes.

Usually, small changes in the parameters do not affect the qualitative behavior of the linear system very much. For example, a spiral sink remains a spiral sink and a saddle remains a saddle. Of course the eigenvalues and eigenvectors change as we vary the parameters, but the basic behavior of solutions remains more or less the same.

The critical loci

There are, however, certain exceptions to this scenario. For example, suppose that a change in parameters forces the point (T, D) to cross the positive D-axis from left to right. The corresponding linear system has changed from a spiral sink to a center and then immediately thereafter to a spiral source. Instead of all solutions tending to the equilibrium point at $(0, 0)$, suddenly we have a center, and then all of the nonequilibrium solutions tend to infinity. That is, the family of linear systems has encountered a bifurcation at the moment the point (T, D) crosses the D-axis.

The trace-determinant plane provides us with a chart of those locations where we can expect significant changes in the phase portrait. There are three such *critical loci*.

The first critical locus is the positive D-axis, as we saw above. A second critical line is the T- axis. If (T, D) crosses this line as our parameters vary, our system moves from a saddle to a sink, a source, or a center (or vice versa). The third critical locus is the repeated-root parabola where spirals turn into real sinks or sources.