The product topology

now we generalize the idea to this products:

$$(X_1 \times \dots \times X_n) \text{ and } (X_1 \times X_2 \times \dots)$$

Box topology

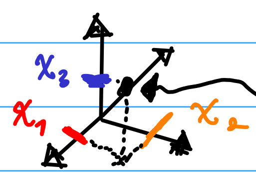
if we take as basis all sets of the form $U_1 \times \dots \times U_n$ or $U_1 \times U_2 \times \dots$, where U_i is an open set of X_i $\forall i$.

we are left with the basis for the box topology.

Def (J-tuple of X): is a function $f : J \to X$

Notation: given $\alpha \in J$ the corresponding value at X is noted as x_α, it is also called the αth coordinate of x.

something like a vector.



$(x_\alpha)_{\alpha \in J}$ tuple Notation

Def (cartesian product): given $X = \bigcup_{\alpha \in J} A_\alpha$,

we denote the cartesian product by:

$$\prod_{\alpha \in J} A_\alpha$$

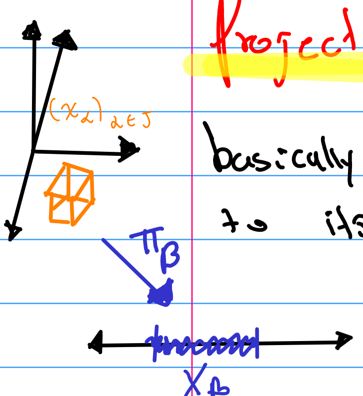
that is the set of all J-tuples $(x_\alpha)_{\alpha \in J}$, $x_\alpha \in A_\alpha$ for each $\alpha \in J$.

Formal Def (box topology):

$\{X_\alpha\}_{\alpha \in J}$ is an indexed family of topo. spaces.

The basis of $\prod_{\alpha \in J} X_\alpha$ is the collection of all sets of the form $\prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α, $\forall \alpha \in J$.

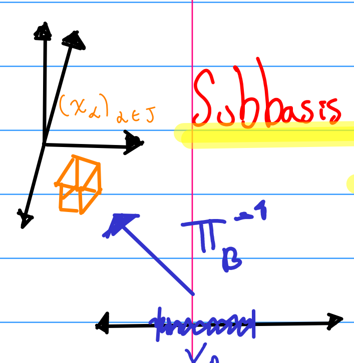
The topology generated by this basis is called the box topology.



Projection mapping: is a function $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to B_\alpha$

basically assigns each element from the product space to its coodinate:

$$\pi_\beta\left((x_\alpha)_{\alpha \in J}\right) = x_\beta$$



Subbasis of the product topology: product topology is different from box topology.

$$S_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}$$

$S = \bigcup_{\beta \in J} S_\beta$, S is the subbase of the product topology.

Theorem 19.1 (Comparison of the box and product topologies). The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.

Two things are immediately clear. First, for finite products $\prod_{\alpha=1}^{n} X_\alpha$ the two topologies are precisely the same. Second, the box topology is in general finer than the product topology.

this one is really important.

the box topology should be used mainly to propose counter examples. the majority of proofs are done with the product topology.

Whenever we consider the product $\prod X_\alpha$, we shall assume it is given the product topology unless we specifically state otherwise.

Theorem 19.2 (basis for both spaces):

suppose for each space X_α there is a basis $\mathcal{B}_\alpha$.

the collection:

$$\prod_{\alpha \in J} B_\alpha$$

is basis for the box and product topology when:

prod.	box.
where $B_\alpha \in \mathcal{B}_\alpha$, for finitely many indices α, and $B_\alpha = X_\alpha$ for all the remaining indices.	where $B_\alpha \in \mathcal{B}_\alpha$, $\forall \alpha$

Theorem 19.3. *Let A_α be a subspace of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.*

Theorem 19.4. *If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.*

Theorem 19.5. *Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology, then*

$$\prod \bar{A}_\alpha = \overline{\prod A_\alpha}.$$

Theorem 19.6. Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J},$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

esto es básicamente una función paramétrica.