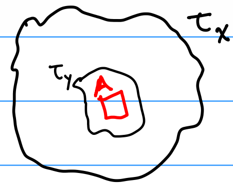
Closure and interior of a Set

Def: Given "A" subset of a topological space X:

Interior ⇝ the interior of A is defined as the union of all open sets contained in A.

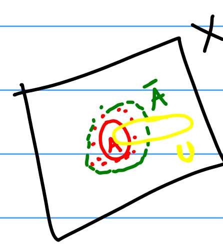
Closure ⇝ is defined as the intersection of all sets containing A.

Theorem 17.4. Let Y be a subspace of X, let A be a subset of Y, let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.



Notice that the def. of closure does not make it easy to find closures on a set, so here is how to find them using a basis:

Theorem let A be a subset of (X, τ):



① then $x \in \bar{A}$ (closure of A in X) iif every open set U with $x \in U$ also has $U \cap A \neq \emptyset$.

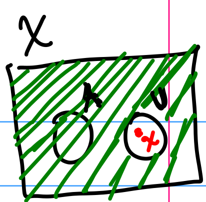
② suposing τ_X is generated by a basis then $x \in \bar{A}$ iif every $B \in \mathcal{B}$ s.t. $x \in B$ also has $B \cap A \neq \emptyset$.

Proof: to make the proof more easy on (1) let's consider the contra positive of (1):

$$P \Leftrightarrow Q \equiv \neg P \Leftrightarrow \neg Q$$

which is:

$$x \notin \bar{A} \text{ iif exists an open set } x \in U \text{ with } U \cap A = \emptyset.$$



($\Rightarrow$) given that $x \notin \bar{A}$ we can find a openset U such that $x \in U$ and $U = X - A$, with $U \cap A = \emptyset$.

($\Leftarrow$) given that there is U s.t. $x \in U$ and $U \cap A = \emptyset$ notice that $X - U$ is a closed set containing A and by definition of closure $\bar{A} \subset X - U$, and because $x \in U$ and $A \subset X - U$ we have that $x \notin \bar{A}$

For the second proof (2):

($\Rightarrow$) if T_x is gen. by a basis then $x \in \bar{A}$ iif every $B \in \mathcal{B}$ s.t. $x \in B$ and $B \cap A \neq \emptyset$.

Considering the previous demonstration if $x \in \bar{A}$ then every open set U with $x \in U$ also has $U \cap A \neq \emptyset$ now notice that B is also an openset.

($\Leftarrow$) Similar to the ($\Rightarrow$) we have that if for every $B \in \mathcal{B}$ s.t. $x \in B$, $B \cap A \neq \emptyset$, and there is always a openset $U \subset B$, therefore $x \in \bar{A}$.

Def: "U is a <u>neighborhood</u> of x" means U is an open set containing x.