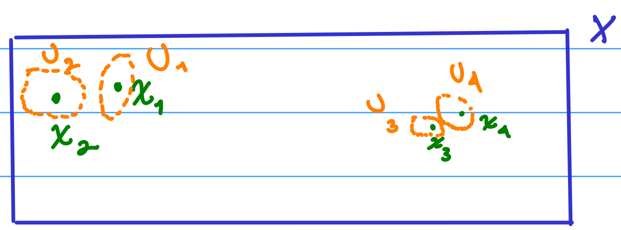
Hausdorff Space

if we consider our experience with $\mathbb{R}$ or $\mathbb{R}^2$ one can find it misleading because there can be more abstract spaces where we can converge to more than one point or find that the open sets $\{x_1\}$ could not be closed, in contrast with $\mathbb{R}$ and $\mathbb{R}^2$.

↳ these abstract topologies where this strange stuff happens are not that interesting.

So Hausdorff created a condition to limit to the interesting cases:

Def: A topological space is Hausdorff space if for any pair x_1, x_2 of distinct points on X, there exists neighborhoods U_1 for x_1, U_2 for x_2 which are disjoint.



Theorem 17.8 Every finite point set in a Hausdorff space is closed.

proof: It will be sufficient to proof that any single point set is closed.

remember theorem 17.5 which defines closure as: $\bar{A} = A \cup A'$, where A' is the set of limit points of A.

if we have $\{x_0\} = A$, in a housedorff space and we

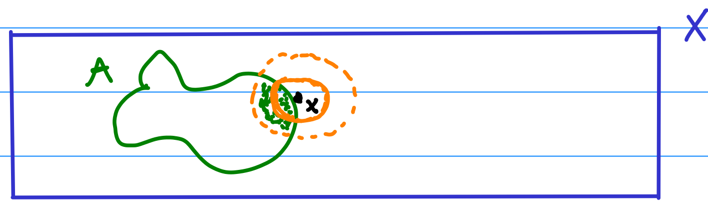
Consider any $x \neq x_0$ s.t. $x \in A$ remember that there is a neighborhood U_x which is disjoint with U_{x_0}, and as x is arbitrary there are no limit points, $A' = \{\}$ so $\bar{A} = A$.

T_1 axiom ⟶ this is the condition that finite point sets are closed.

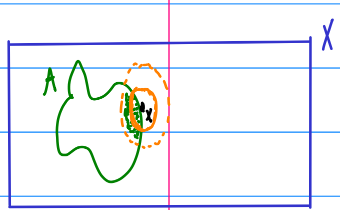
→ this axiom is not so important.

Theorem 17.9 let X be a space satisfying the T_1 axiom, let A be a subset of X.

then the point x is a limit point of A. $\iff$ every neighborhood of x contains infinitely many points of A.



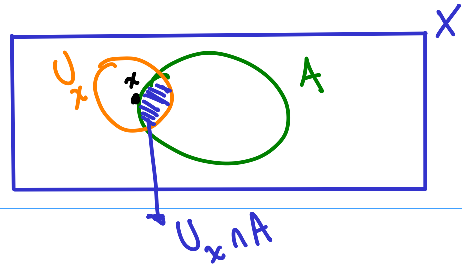
proof: $(\Leftarrow)$ if every neighborhood of x contains infinitely many points of A. then by the definition of limit point there is at least one element of A inside U_x. so x is a limit point.



$(\Rightarrow)$ if x is a limit point of A, supose that there is a neighborhood that intersects a finite set of points of A.

$\exists U_x, |U_x \cap A| < \infty$, so there is $\{x_1, \dots, x_n\} = U_x \cap (A - \{x\})$

$X - \{x_1, \dots, x_n\}$ is open, if we try $U_x \cap (X - \{x_1, \dots, x_n\})$ we see it is too an open set of x,

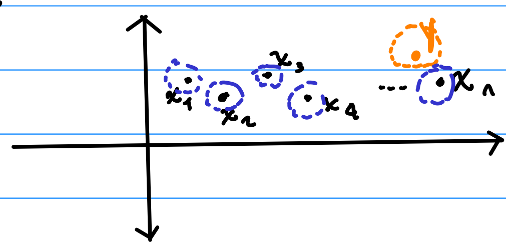


$U_x \cap \overline{\{x_1, \dots, x_n\}}$ is an open set.

but $U_x \cap \overline{\{x_1, \dots, x_n\}}$ is an openset of x which does not intersect A. so it contradicts the point that x is a limit point.

Theorem 17.10 if X is a hausdorff space, then a sequence of points of X, converges to at most one point of X.

Proof: Suppose that x_n is a sequence of points which converges to $x \neq y$, for every point of x_n I can create a open set U_{x_n} which contains x_n or any previous points of the sequence, and thus notice that U_y can't contain these points because it is a Hausdorff space. So x_n can't converge to y.



Theorem 17.11 Every simply ordered set is a Hausdorff space in the order topology.

→ The product of two Hausdorff spaces is a Hausdorff space.

→ A subspace of a Hausdorff space is a Hausdorff space.