Work Sheet 1

Rules

- You can submit this assignment in pairs or individually.
- All answers must be thoroughly justified.
- All your answers must be submitted in a single .pdf file.
- Due date: 2 March 2023.
- **Any form of plagiarism will be reported.**

Part 1. Lyapunov Exponents:
Given a difference equation of the form

$$x_{n+1} = f(x_n),$$

with initial value x_0, the Lyapunov exponent is defined as

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |f'(x_j)|.$$

Lyapunov exponents are a measure of chaos; they measure the sensitivity of the system to small initial disturbances.
The discrete logistic equation is given by

$$x_{n+1} = f_\mu(x_n) = \mu x_n(1 - x_n),$$

where $\mu \in [0, 4]$.

1. (**6 pts**) Compute the Lyapunov exponent λ for different values of μ and plot your results. The starting points are:
   (a) $x_0 = 0$.
   (b) $x_0 = \frac{\mu-1}{\mu}$.
   (c) $x_0 = \frac{(1+\mu)-\sqrt{(\mu-3)(\mu+1)}}{2\mu}$.
2. (**4 pts**) What is the relation between stability and the Lyapunov exponent, for cases (a), (b) and (c)? Justify your answer.
3. (**6 pts**) Compute the Lyapunov exponent λ for different values of μ and plot your results. As a starting point take $x_0 = 1/2$.
4. (**4 pts**) At which point does $\lambda(1/2)$ become positive?
5. (**4 pts**) Is there any relation between the Lyapunov exponent $\lambda(1/2)$ and the 'chaotic' behavior of the system? Justify your answer.

Part 2. Newton's Method:
Consider the difference equation

$$z_{n+1} = g(z_n) = z_n - \frac{p(z_n)}{p'(z_n)},$$

where $p(z)$ is a polynomial in the complex plane $\mathbb{C}$. This iteration comes from the famous root-finding algorithm known as the Newton method.
Let $\alpha \in \mathbb{C}$ be a zero of p, i.e. $p(\alpha) = 0$. The basin of attraction of α is defined as

$$B(\alpha) = \{z \in \mathbb{C} \mid \lim_{n\to\infty} g^n(z) = \alpha, \text{ where } g^n \text{ stands for the n}^{\text{th}}\text{-iteration of the function } g\}.$$

Let $p(z) = z^3 - 1$.

1. (**2 pts**) Compute the three roots of p.
2. (**8 pts**) Plot the basin of attraction of each root. Use a different color for each region.
3. (**6 pts**) From the plot, what can you say about the sensitivity of the Newton's method to the choice of a starting point? Justify your answer.