Product topology on $X \times Y$

Given X, Y topological spaces we call the product topology on $X \times Y$ to the topology generated by the basis

$\mathcal{B}$ = collection of all sets of the form $U \times V$

where U is an open subset of X.
and V is an open subset of Y.

Theorem If $\mathcal{B}$ is a basis for the topology of X, and $\mathcal{G}$ is a basis for the topology of Y.

$$\mathcal{D} = \{ B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{G} \}$$

then $\mathcal{D}$ is a basis for the topology of $X \times Y$

Now we can need a way to formally say I want the "x" or the "y" from (x, y)

this is a projection:

$$\pi_1(x, y) = x$$
$$\pi_2(x, y) = y$$

And the inverse of the projection:

$$\pi_2^{-1}(V) = X \times V$$

- $V \subseteq Y$
- $U \subseteq X$

$$\pi_1^{-1}(U) = U \times Y$$

Theorem the collection

$$S = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.