
Computational and Differential Geometry
Homework 4

Professor: Nicolás Avilán Vargas, Ph.D.

Indicaciones

1. Fecha de entrega: 27 de noviembre de 2023 hasta las 11:55 pm.
2. Único medio de entrega e-aulas.
3. Formato de entrega: Un único archivo **.ipynb** con códigos en python, descripciones de códigos y procesos, gráficas y respuestas a las preguntas.
4. Solo es permitido el uso de librerías "básicas" (numpy, matplotlib, seaborn, pandas, sympy, etc). En ningún caso será válida la solución lograda, total o parcialmente, por el uso de una librería especializada para resolver problemas de geometría computacional.
5. La tarea **debe** realizarse **individualmente**.
6. Cualquier tipo de fraude o plagio es causa de anulación directa de la evaluación y correspondiente proceso disciplinario.
7. Las entregas están sujetas a herramientas automatizadas de detección de plagio en códigos.
8. Las tareas no entregadas antes de la hora indicada tendrán calificación de 0.0.

Support each piece of code with a thorough explanation of its methods, techniques, functions, and tricks. Reference your search source (papers, books, tutorials, websites, etc.). Add any necessary bibliographical references or links.

1. (1.5 points) One way to define a system of coordinates for the sphere S^2, given by $x^2 + y^2 + z^2 = 1$, is to consider the so-called **stereographic projection** $\pi : S^2 \backslash \{N\} \to \mathbb{R}^2$ which carries a point $p = (x, y, z)$ of the sphere S^2 minus the north pole $\{N\} = (0, 0, 1)$ onto the intersection of the xy plane with the straight line which connects N to p.

   Let $(X, Y) = \pi(x, y, z)$, where $(x, y, z) \in S^2 \backslash \{N\}$ and $(X, Y) \in xy$ plane (see the figure).

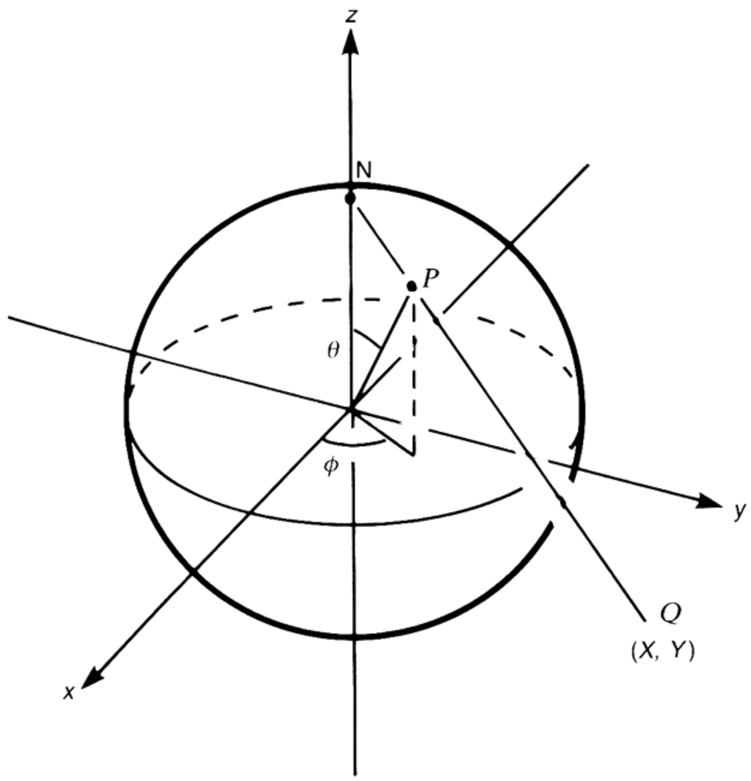
   - Show that $\pi : S^2 \backslash \{N\} \to \mathbb{R}^2$ is given by

   $$X = \frac{x}{1-z}, \qquad Y = \frac{y}{1-z}$$

   - Show that the inverse map $\mathbb{R}^2 \to S^2 \backslash \{N\}$ is given by

   $$\vec{X}(X, Y) = (x, y, z) = \left(\frac{2X}{1 + X^2 + Y^2}, \frac{2Y}{1 + X^2 + Y^2}, \frac{-1 + X^2 + Y^2}{1 + X^2 + Y^2} \right)$$

   - Proof that $S^2 \backslash \{N\}$ is a regular surface using the previous map $\vec{X}$.



2. (2 points) Consider the parametrization for the Torus T given by

$$\vec{X}(u,v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u)$$

$$U = \{(\theta, \phi) \in \mathbb{R}^2 | 0 < u < 2\pi, \quad 0 < v < 2\pi\}$$

with $0 < r < a$.

- Is T a regular surface? Justify your answer completely.
- Evaluate the tangent vectors on T.
- Evaluate the coeficients of the first fundamental form.
- Which is the metric tensor for the Torus?
- Which is the area element for the surface?
- Evaluate the area of the surface.
- Evaluate the second fundamental form for the surface.
- Evaluate and plot the principal curvatures of the surface.
- Evaluate and plot Gaussian curvature of the surface.

3. (1.5 points) Consider the parametrization for the Möbius strip

$$\vec{x}(\theta, t) = \left(\left(1 + \frac{1}{2}t\cos\left(\frac{\theta}{2}\right)\right)\cos(\theta), \left(1 + \frac{1}{2}t\cos\left(\frac{\theta}{2}\right)\right)\sin(\theta), \frac{1}{2}t\sin\left(\frac{\theta}{2}\right)\right)$$

for

$$U = \{(\theta, t) \in \mathbb{R}^2 | -0.25 < t < 0.25, \quad 0 < \theta < 2\pi\}$$

- Evaluate the first fundamental form for the Möbius strip .
- Evaluate the metric tensor for the Möbius strip.

- Which is the area element $|\vec{X}_\theta \times \vec{X}_t|$ for the Möbius strip ?
- Evaluate the total area for the ellipsoid as

$$\iint_Q |\vec{X}_\theta \times \vec{X}_t| d\theta dt$$

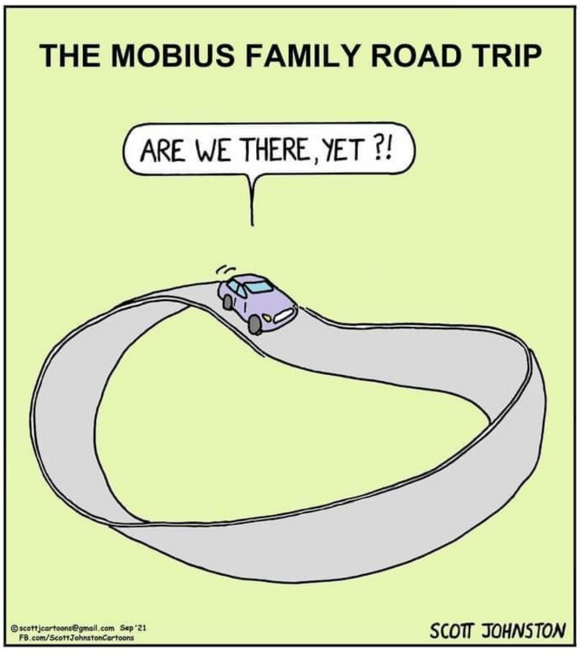
  If the analytical calculation is not possible, implement some numerical method. Describe the procedure you implemented.
- Obtain the normal vector as a function of the coordinates (θ, t)

$$\hat{N}(\theta, t) = \frac{\vec{X}_\theta \times \vec{X}_t}{|\vec{X}_\theta \times \vec{X}_t|}$$

- Evaluate the normal vector at $t = 0$ and obtain the limits

$$\lim_{\theta \to 0} \hat{N}(\theta, 0) \quad \text{and} \quad \lim_{\theta \to 2\pi} \hat{N}(\theta, 0)$$

  What do you conclude from your calculations? Is the surface orientable?



Additonal tips

- This code plots the Möbius strip

```
fig = plt.figure(figsize =(9, 5))
ax = plt.axes(projection ='3d')

a = np.linspace(0, 2.0 * np.pi,30)
b = np.linspace(-0.25, 0.25, 5)
a, b = np.meshgrid(a, b)

x = (1 + 0.5 * b * np.cos(a / 2.0)) * np.cos(a)
y = (1 + 0.5 * b * np.cos(a / 2.0)) * np.sin(a)
z = 0.5 * b * np.sin(a / 2.0)

ax.plot_surface(x, y, z, linewidth=2)
ax.set_title("Mobius strip", weight = "bold")
ax.set_zlim([-0.3,0.3])
plt.show()
```

- Check out these links for better intuition with the Möbius strip:

  1. A Möbius strip is not orientable
  2. mobiusescher.gif
  3. Arrow on Möbius strip.

Submit:
Upload to the platform an **.ipynb** file with answers, codes, descriptions and plots.