2nd Exam

October 04 2022

General indications

- This is an **individual** exam with duration **180 minutes: from 9:00 a.m. to 12:00 m.**.
- Cellphones must be turned off during the exam.
- Only the use of a single sheet with notes is allowed for the 1st part of the exam. The use of books and/or other analog resources is forbidden as well as the use of electronic calculators.
- Any form of noncompliance with the rules above will result in annulment.
- Answers must be fully justified.
- When you are finished you can upload your practical solution to eaulas as a single .zip file.
- Good luck!

Part 1. Theoretical exercises:

1. **(5 pts)** Let $\Phi(t,\tau)$ be the state transition matrix corresponding to the linear system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t).$$

Prove that:

$$\frac{d}{d\tau}\Phi(t,\tau) = -\Phi(t,\tau)\mathbf{A}(\tau).$$

You may assume that the Product Rule for derivatives of 1D functions holds for matrices of functions as well.

Hint: Prove first that:

$$\Phi(t,\tau)^{-1} = \Phi(\tau,t).$$

2. **(5 pts)** Consider the continuous-time linear system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t).$$

We define the adjoint system as:

$$\dot{\mathbf{p}}(t) = -\mathbf{A}^T\mathbf{p}(t).$$

Show that if $\Phi(t,\tau)$ is the state-transition matrix of the original system, then the state transition matrix of the adjoint system is:

$$\Phi(\tau,t)^T.$$

3. (**5 pts**) Determine a closed-form expression (a matrix with explicit functions of t) for $e^{\mathbf{A}t}$ where:

$$\mathbf{A} = \begin{bmatrix} -1 & -2 \\ -3 & -6 \end{bmatrix}$$

4. (**10 pts**) Find the state transition matrix, $\Phi(k, 0)$, of the following discrete dynamical system:

$$\mathbf{x}(k+1) = \begin{bmatrix} \cos(\theta) & -sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \mathbf{x}(k), \qquad k \in \mathbb{N}^*$$

Where the system matrix above corresponds to a counter-clockwise rotation of an angle θ. Prove your answer if it was not derived analytically.

Part 2. Practical exercise:

1. (**25 pts**) Recall the equation describing the motion of a pendulum of length L (with damping due to friction and no other external forces):

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} sin(\theta) + \mu \frac{d\theta}{dt} = 0,$$

where θ is the angle with respect to the the downward vertical and μ is the damping coefficient -representing the strength of the drag force (the force exerted by the medium in opposite direction to the pendulum's movement).

Convert the system to state-space form and simulate the obtained system using Simulink or Matlab for $L = 1$ when taking an initial condition corresponding to a free fall of the pendulum and when the initial velocity is different from zero (use $\mu = 0.02$, $\mu = 0.1$). What dynamics do you observe? Describe and discuss your results.

$x_1 = \theta$ $\qquad \dot{x}_1 = \dot{\theta} = x_2$

$x_2 = \dot{\theta}$ $\qquad \dot{x}_2 = \ddot{\theta} = -\frac{g}{L} \sin(\theta) + \mu \dot{\theta}$

$$= -\frac{g}{L} \sin(x_1) - \mu x_2$$