1st Exam

August 24 2022

General indications

- This is an **individual** exam with duration **180 minutes: from 9:00 a.m. to 12:00 m.**.
- Cellphones must be turned off during the exam. Communication with your classmates is forbidden during the entire time.
- Only the use of a single sheet with notes is allowed for the 1st part of the exam. The use of books and/or other analog resources as well as the use of electronic calculators is forbidden during part of the exam and allowed during the second.
- Any form of noncompliance with the rules above will result in annulment.
- Answers must be fully justified.
- When you are finished you can upload your practical solution to eaulas as a single .zip file.
- Good luck!

Part 1. Theoretical exercises:

1. (**10 pts**) Consider the following difference equation:

$$x(k+1) = \frac{k+2}{k+1}x(k), \quad k \in \mathbb{N}.$$

   a) Write the difference equation in terms of the differences $\Lambda^0(k),\ \Lambda^1(k), \ldots$. Keep in mind that an equation of order m must be expressed as an equation involving the mth difference.

   b) Find the general solution of the equation.

2. (**10 pts**) Prove the following theorem:

   Thm: Let $\bar{y}(t)$ be a particular solution of a linear differential equation of the form:

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0(t)y = g(t). \tag{1}$$

   Then, all solutions to equation (1) form the set of functions of the form: $y(t) = \bar{y}(t) + z(t)$, where $z(t)$ is one of the solutions to the corresponding homogeneous problem.

3. (**10 pts**) Suppose that a linear homogeneous difference equation with constant coefficients has order 2. Suppose also that the characteristic polynomial has a repeated root $a \neq 0$.

   a) Show that $x(k) = a^k$ and $x(k) = ka^k$ are solutions to the diff. equation.

b) Find the general solution of the difference equation.

Part 2. Practical exercise:

1. **(20 pts)** Suppose you are running an experiment to study a population of bacteria. Initially, every hour, each of your bacteria produce $(r-1)$ new bacteria, $r \geq 1$. The difference equation you derive that describes the process is the following:

$$x(k+1) = x(k) + (r-1)x(k) = rx(k), \quad k \in \mathbb{N}^*$$

   However, you notice that after some time, the solution of the difference equation does not match your empirical observations. You come up with a new equation of the form:

$$y(k+1) = r_2 y(k)\left(1 - y(k)\right), \quad k \in \mathbb{N}^*,$$

   where $y(k) = \frac{x(k)}{K}$ (the fraction of current individuals with respect to the capacity of the petri dish), with K is the maximum number of individuals which can steadily survive in your petri dish (the carrying capacity). Explain in your report:
   (i) what you think led you to the disagreement between your empirical observations and your model's solution,
   (ii) what you think the motivated you to derive the second model.
   (iii) Simulate both models. In model 1 use any $r \leq 4$, in model 2 use $r_2 = 1.5, 2.8, 3.1, 3.7$ and comment on your results. You can take $x(0) = 1$, $y(0) = 0.5$. What interesting things do you observe?