Midterm Assignment 3

April 20, 2021

Indicaciones generales

1. Fecha de publicación: 20 de abril de 2021.
2. Fecha de entrega: 26 de abril de 2020 hasta las 23:55.
3. Único medio de entrega: `https://e-aulas.urosario.edu.co`.
4. Formato de entrega: código en Python.
5. Importante: no use acentos ni deje espacios en los nombres de los archivos que cree.
6. La actividad **debe** realizarse **individualmente**.
7. Los grupos pueden consultar sus ideas con los profesores para recibir orientación; sin embargo, la solución y detalles del ejercicio debe realizarlos **individualmente**. Cualquier tipo de fraude o plagio es causa de anulación directa de la evaluación y correspondiente proceso disciplinario.
8. El grupo de trabajo debe indicar en su entrega de la solución a la actividad cualquier asistencia que haya recibido.
9. El grupo no debe consultar ninguna solución a la actividad que no sea la suya.
10. El grupo no debe intentar ocultar ningún código que no sea propio en la solución a la actividad (a excepción del que se encuentra en las plantillas).
11. Las entregas están sujetas a herramientas automatizadas de detección de plagio en códigos.
12. `e-aulas` se cerrará a la hora acordada para el final de la evaluación. La solución de la actividad debe ser subida antes de esta hora. El material entregado a través de `e-aulas` será calificado tal como está. Si ningún tipo de material es entregado por este medio, la nota de la evaluación será 0.0.

Problem set

1. As a consultant in the field of mathematical modeling and computer simulations, an engineering company has hired you to help them solve an optimization problem regarding loads on overhead wiring, hanging on the air, that transfers electricity on a power grid. An instance of such overhead power cables is shown in the Figure 1.

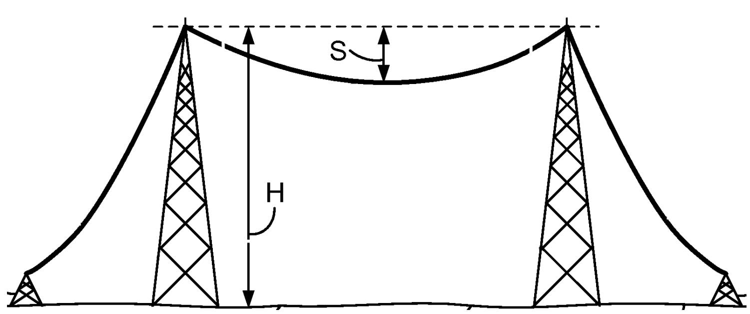

Figure 1: Overhead wiring. Cable suspended between to electric towers.

Based on the loads required by the cables, the engineers at the company have come up with two different materials for the cables, which give rise to two different models for the spatial distribution of a cable attached to two towers. Due to the particular characteristics of the cable's material, the models are represented as the parametrized plane curves $\alpha : (-\delta, \delta) \to \mathbb{R}^2$, $\delta > 0$. The curves are

$$\alpha_P(t) = \left(t, \frac{t^2}{2a} + b\right), \quad \alpha_C(t) = \left(t, a\cosh\frac{t}{a} - c\right), \quad \alpha_E(t) = \left(a\sin\frac{\pi t}{2}, 1 - c\cos\frac{t}{2}\right), \quad (1)$$

where $a > 0$ is a constant that is proportional to the weight (mass per unit length times the acceleration of gravity) of the cables. From now on we set $b = 0$ and $c = 1$.

Your task is to generate graphs that will aid the company with the solution of the optimization problem. The company is interested in making decisions on what material is the best to design the cables, based on aspects such as arc length and curvature. In order to carry out your job you have to write programs that calculate numerically and generate for each model the following plots:

(a) For $a = 1/2$, calculate the length of a cable as a function of the parameter δ in the range $[0, 1]$.

(b) For $\delta = 1$, calculate the length of a cable as a function of the physical parameter a in the range $[1/2, 2]$.

(c) For $a = 1$, calculate the curvature of a cable as a function of the time parameter t when $\delta = 4$.

(d) For $t = 1$, calculate the curvature of a cable as a function of the physical parameter a in the range $[-4, 4]$.

For each item produce only one plot displaying the results for models $\alpha_P(t)$, $\alpha_C(t)$, and $\alpha_E(t)$. With this information the company should be able to make an informed decision, depending on the loads on the wires, on which model is more appropriate to construct the overhead wiring of the power grid.

2. The robotics laboratory of the University wishes to develop software that helps stabilize mechanical components in a micro-machine that moves on microscopic scales. The team at the laboratory has measured, using a variation of spherometer and a torque tester, the behavior of the machine as it moves freely in a three-dimensional space. The spherometer and torque tester have provided data for curvature and torsion that the team has fitted to the following equations

$$\kappa_M(t) = a\frac{\sqrt{4 + 36t^2 + 36t^4}}{(1 + 4t^2 + 9t^4)^{3/2}}, \qquad \tau_M(t) = \frac{3b}{1 + 9t^2 + 9t^4}, \quad (2)$$

where a and b are known real numbers product from the fitting mentioned above. Notice that the values of these functions *are not* constant as a function of the time parameter t.

Now it's your turn to help the robotics laboratory by developing a numerical integrator that, given the functions for curvature $\kappa(t)$ and torsion $\tau(t)$, finds the velocity, normal, and binormal vectors. The software must also be able to find (reproduce) the trajectory or curve followed by the machine.

The relevant formulae involved in the successful development of the software are the Frenet-Serret equations, namely,

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} \tag{3}$$

where $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ are three mutually orthogonal unit vectors known as the tangent, normal, and binormal vectors, respectively. Remember that these three vector define a frame that describe the motion of the machine. The twisting and turning of the Frenet-Serret frame field can be measured by curvature and torsion. The values for curvature and torsion to be used in this case are, of course, those of $\kappa_M(t)$ and $\tau_M(t)$.

Finally, in order to obtain the trajectory of the machine you must make use of the fact that the tangent vector is the derivative of the space curve, so an extra numerical integration is necessary to secure such trajectory. This time you have to generate two plots

(a) The norm of the vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$ as a function of the parameter t. This plot serves as a benchmark on the numerical integration of the Frenet-Serret formulae.

(b) The numerically obtained trajectory followed by the machine within a given range of t. This plot corresponds to a three-dimensional curve.

(c) The speed of the machine's resulting trajectory as a function of time within the range used in the previous item. What graph do you anticipate? Is it a unit-speed curve?

To make the plots, choose $a = 1$ and $b = 1/2$. The numerical integration range should be $t \in [-3, 3]$, although smaller intervals should be considered during the construction and debugging of the numerical integrator of the Frenet-Serret equations.

References

M. P. do Carmo. *Differential Geometry of Curves and Surfaces*. Dover Publications, 2016.

J. Oprea. *Differential Geometry and Its Applications*. The Mathematical Association of America, 2007.