13. (a) Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u(0, y) = 0, \qquad u(a, y) = f(y), \qquad 0 < y < b,$$
$$u(x, 0) = 0, \qquad u_y(x, b) = 0, \qquad 0 \le x \le a.$$

Hint: Eventually, it will be necessary to expand $f(y)$ in a series that makes use of the functions $\sin(\pi y/2b), \sin(3\pi y/2b), \sin(5\pi y/2b), \ldots$ (see Problem 39 of Section 10.4).

(b) Find the solution if $f(y) = y(2b - y)$.

(c) Let $a = 3$ and $b = 2$; plot the solution in several ways.

14. (a) Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u_x(0, y) = 0, \qquad u_x(a, y) = 0, \qquad 0 < y < b,$$
$$u(x, 0) = 0, \qquad u(x, b) = g(x), \qquad 0 \le x \le a.$$

(b) Find the solution if $g(x) = 1 + x^2(x - a)^2$.

(c) Let $a = 3$ and $b = 2$; plot the solution in several ways.

15. By writing Laplace's equation in cylindrical coordinates r, θ, and z and then assuming that the solution is axially symmetric (no dependence on θ), we obtain the equation

$$u_{rr} + (1/r)u_r + u_{zz} = 0.$$

Assuming that $u(r, z) = R(r)Z(z)$, show that R and Z satisfy the equations

$$rR'' + R' + \lambda^2 rR = 0, \qquad Z'' - \lambda^2 Z = 0.$$

The equation for R is Bessel's equation of order zero with independent variable λr.

16. **Flow in an Aquifer.** Consider the flow of water in a porous medium, such as sand, in an aquifer. The flow is driven by the hydraulic head, a measure of the potential energy of the water above the aquifer. Let $R : 0 < x < a, 0 < z < b$ be a vertical section of an aquifer. In a uniform, homogeneous medium, the hydraulic head $u(x, z)$ satisfies Laplace's equation

$$u_{xx} + u_{zz} = 0 \qquad \text{in } R. \tag{i}$$

If water cannot flow through the sides and bottom of R, then the boundary conditions there are

$$u_x(0, z) = 0, \quad u_x(a, z) = 0, \qquad 0 \le z \le b \tag{ii}$$
$$u_z(x, 0) = 0, \qquad 0 \le x \le a. \tag{iii}$$

Finally, suppose that the boundary condition at $z = b$ is

$$u(x, b) = b + \alpha x, \qquad 0 \le x \le a, \tag{iv}$$

where α is the slope of the water table.

(a) Solve the given boundary value problem for $u(x, z)$.

(b) Let $a = 1000$, $b = 500$, and $\alpha = 0.1$. Draw a contour plot of the solution in R; that is, plot some level curves of $u(x, z)$.

(c) Water flows along paths in R that are orthogonal to the level curves of $u(x, z)$ in the direction of decreasing u. Plot some of the flow paths.

9. Show that Eq. (23) has periodic solutions only if λ is real.
Hint: Let $\lambda = -\mu^2$, where $\mu = \nu + i\sigma$ with ν and σ real.

10. Consider the problem of finding a solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a, 0 < y < b$, that satisfies the boundary conditions

$$u_x(0, y) = 0, \qquad u_x(a, y) = f(y), \qquad 0 < y < b,$$
$$u_y(x, 0) = 0, \qquad u_y(x, b) = 0, \qquad 0 \le x \le a.$$

This is an example of a Neumann problem.

(a) Show that Laplace's equation and the homogeneous boundary conditions determine the fundamental set of solutions

$$u_0(x, y) = c_0,$$
$$u_n(x, y) = c_n \cosh(n\pi x/b)\cos(n\pi y/b), \qquad n = 1, 2, 3, \ldots.$$

(b) By superposing the fundamental solutions of part (a), formally determine a function u satisfying the nonhomogeneous boundary condition $u_x(a, y) = f(y)$. Note that when $u_x(a, y)$ is calculated, the constant term in $u(x, y)$ is eliminated, and there is no condition from which to determine c_0. Furthermore, it must be possible to express f by means of a Fourier cosine series of period $2b$, which does not have a constant term. This means that

$$\int_0^b f(y)\, dy = 0$$

is a necessary condition for the given problem to be solvable. Finally, note that c_0 remains arbitrary, and hence the solution is determined only up to this additive constant. This is a property of all Neumann problems.

11. Find a solution $u(r, \theta)$ of Laplace's equation inside the circle $r = a$ that satisfies the boundary condition on the circle

$$u_r(a, \theta) = g(\theta), \qquad 0 \le \theta < 2\pi.$$

Note that this is a Neumann problem and that its solution is determined only up to an arbitrary additive constant. State a necessary condition on $g(\theta)$ for this problem to be solvable by the method of separation of variables (see Problem 10).

12. (a) Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u(0, y) = 0, \qquad u(a, y) = 0, \qquad 0 < y < b,$$
$$u_y(x, 0) = 0, \qquad u(x, b) = g(x), \qquad 0 \le x \le a.$$

Note that this is neither a Dirichlet nor a Neumann problem, but a mixed problem in which u is prescribed on part of the boundary and its normal derivative on the rest.

(b) Find the solution if

$$g(x) = \begin{cases} x, & 0 \le x \le a/2, \\ a - x, & a/2 \le x \le a. \end{cases}$$

(c) Let $a = 3$ and $b = 1$. By drawing suitable plots, compare this solution with the solution of Problem 1.

3. (a) Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$\begin{aligned} &u(0, y) = 0, \qquad u(a, y) = f(y), \qquad 0 < y < b, \\ &u(x, 0) = h(x), \qquad u(x, b) = 0, \qquad 0 \le x \le a. \end{aligned}$$

Hint: Consider the possibility of adding the solutions of two problems, one with homogeneous boundary conditions except for $u(a, y) = f(y)$, and the other with homogeneous boundary conditions except for $u(x, 0) = h(x)$.

(b) Find the solution if $h(x) = (x/a)^2$ and $f(y) = 1 - (y/b)$.

(c) Let $a = 2$ and $b = 2$. Plot the solution in several ways: u versus x, u versus y, u versus both x and y, and a contour plot.

4. Show how to find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$\begin{aligned} &u(0, y) = k(y), \qquad u(a, y) = f(y), \qquad 0 < y < b, \\ &u(x, 0) = h(x), \qquad u(x, b) = g(x), \qquad 0 \le x \le a. \end{aligned}$$

Hint: See Problem 3.

5. Find the solution $u(r, \theta)$ of Laplace's equation

$$u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta} = 0$$

outside the circle $r = a$, that satisfies the boundary condition

$$u(a, \theta) = f(\theta), \qquad 0 \le \theta < 2\pi,$$

on the circle. Assume that $u(r, \theta)$ is single-valued and bounded for $r > a$.

6. (a) Find the solution $u(r, \theta)$ of Laplace's equation in the semicircular region $r < a$, $0 < \theta < \pi$, that satisfies the boundary conditions

$$\begin{aligned} u(r, 0) = 0, \qquad u(r, \pi) = 0, \qquad 0 \le r < a, \\ u(a, \theta) = f(\theta), \qquad 0 \le \theta \le \pi. \end{aligned}$$

Assume that u is single-valued and bounded in the given region.

(b) Find the solution if $f(\theta) = \theta(\pi - \theta)$.

(c) Let $a = 2$ and plot the solution in several ways: u versus r, u versus θ, u versus both r and θ, and a contour plot.

7. Find the solution $u(r, \theta)$ of Laplace's equation in the circular sector $0 < r < a$, $0 < \theta < \alpha$, that satisfies the boundary conditions

$$\begin{aligned} u(r, 0) = 0, \qquad u(r, \alpha) = 0, \qquad 0 \le r < a, \\ u(a, \theta) = f(\theta), \qquad 0 \le \theta \le \alpha. \end{aligned}$$

Assume that u is single-valued and bounded in the sector and that $0 < \alpha < 2\pi$.

8. (a) Find the solution $u(x, y)$ of Laplace's equation in the semi-infinite strip $0 < x < a$, $y > 0$, that satisfies the boundary conditions

$$\begin{aligned} u(0, y) = 0, \qquad u(a, y) = 0, \qquad y > 0, \\ u(x, 0) = f(x), \qquad 0 \le x \le a \end{aligned}$$

and the additional condition that $u(x, y) \to 0$ as $y \to \infty$.

(b) Find the solution if $f(x) = x(a - x)$.

(c) Let $a = 5$. Find the smallest value of y_0 for which $u(x, y) \le 0.1$ for all $y \ge y_0$.

The boundary condition (18) then requires that

$$u(a,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n (c_n \cos n\theta + k_n \sin n\theta) = f(\theta) \tag{35}$$

for $0 \le \theta < 2\pi$. The function f may be extended outside this interval so that it is periodic with period 2π and therefore has a Fourier series of the form (35). Since the extended function has period 2π, we may compute its Fourier coefficients by integrating over any period of the function. In particular, it is convenient to use the original interval $(0, 2\pi)$; then

$$a^n c_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta, \qquad n = 0, 1, 2, \ldots; \tag{36}$$

$$a^n k_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta, \qquad n = 1, 2, \ldots. \tag{37}$$

With this choice of the coefficients, Eq. (34) represents the solution of the boundary value problem of Eqs. (18) and (19). Note that in this problem we needed both sine and cosine terms in the solution. This is because the boundary data were given on $0 \le \theta < 2\pi$ and have period 2π. As a consequence, the full Fourier series is required, rather than sine or cosine terms alone.

PROBLEMS

1. (a) Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a, 0 < y < b$, that satisfies the boundary conditions

$$\begin{aligned} u(0,y) &= 0, & u(a,y) &= 0, & 0 < y < b, \\ u(x,0) &= 0, & u(x,b) &= g(x), & 0 \le x \le a. \end{aligned}$$

(b) Find the solution if

$$g(x) = \begin{cases} x, & 0 \le x \le a/2, \\ a - x, & a/2 \le x \le a. \end{cases}$$

(c) For $a = 3$ and $b = 1$, plot u versus x for several values of y and also plot u versus y for several values of x.

(d) Plot u versus both x and y in three dimensions. Also draw a contour plot showing several level curves of $u(x, y)$ in the xy-plane.

2. Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a, 0 < y < b$, that satisfies the boundary conditions

$$\begin{aligned} u(0,y) &= 0, & u(a,y) &= 0, & 0 < y < b, \\ u(x,0) &= h(x), & u(x,b) &= 0, & 0 \le x \le a. \end{aligned}$$