HOMOMORPHISMS FROM THE COMPLEX NUMBERS TO THE REAL NUMBERS

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1. Introduction

In this article we shall investigate the the existence of both group and ring homomorphisms from $\mathbb{C}$ to $\mathbb{R}$. We shall note now that no isomorphism from $\mathbb{C}$ to $\mathbb{R}$ exists and thus this shall not be considered.

2. Groups Homomorphism

We shall consider the groups $(\mathbb{C}, +)$ and $(\mathbb{R}, +)$. A group homomorphism ϕ can be constructed as follows:
Let $\phi : \mathbb{C} \to \mathbb{R}$ such that:

- $\phi(0) = 0$
- $\phi(1) = 1$
- $\phi(i) = 1$

We shall now check that ϕ satisfies the necessary properties. Let $a, b, c, d \in \mathbb{R}$ such that $(a + bi), (c + di) \in \mathbb{C}$. Then:

$$\phi(a + bi) + \phi(c + di) = (a + b) + (c + d) = (a + c) + (b + d) = \phi((a + bi) + (c + di))$$

It is also true that:

$$\phi(-(a + bi)) = -a - b = -\phi(a + bi)$$

Thus ϕ is a group homomorphism from $\mathbb{C}$ to $\mathbb{R}$. We can therefore conclude there do exist homomorphisms from the additive group of $\mathbb{C}$ to the additive group of $\mathbb{R}$.

We shall consider the groups $(\mathbb{C}, \times)$ and $(\mathbb{R}, \times)$. Let $\phi : \mathbb{C} \to \mathbb{R}$ such that for $z \in \mathbb{C}$ we have $\phi(z) = |z|$. We shall now check if ϕ satisfies the necessary properties:

- $\phi(1) = 1$
- $\phi(zw) = |zw| = |z||w| = \phi(z)\phi(w)$
- $\phi(\frac{1}{z}) = \left|\frac{1}{z}\right| = \frac{1}{|z|} = \phi^{-1}(z)$.

Thus ϕ is a group homomorphism. We can therefore conclude there do exist homomorphisms from the multiplicative group of $\mathbb{C}$ to the multiplicative group of $\mathbb{R}$.

3. Ring Homomorphism

We now consider the rings $(\mathbb{C}, +, \times)$ and $(\mathbb{R}, +, \times)$. We claim that there exists no ring homomorphism ϕ from $\mathbb{C}$ to $\mathbb{R}$. Suppose there does exist a ring homomorphism $\phi : \mathbb{C} \to \mathbb{R}$:

$$\begin{aligned} -\phi(1) = \phi(-1) = \phi(i \times i) = \phi(i)\phi(i) = \phi(i)^2 &= -1 \\ \implies \exists \phi(i) \in \mathbb{R} : (\phi(i))^2 &= -1 \end{aligned}$$

But this is of course a contradiction.