DAVE BOWMAN

A STORY ABOUT SCHEMES

Contents

Introduction

The purpose of these notes is to give a short exposition of the necessary theory required to state the definition of a scheme. We shall begin with a discussion of some "classical" ideas in algebraic geometry. Beginning with affine space and culminating with the Nullstellensatz[1]. Following this we shall discuss some basic point set topology and define the Zariski topology on affine space. Next we will define sheaves and give many examples in order to help the reader ingest this "tricky" subject. Finally we shall define the spectrum of a ring...[2] Knowledge of category theory shall be assumed.

[1] `https://en.wikipedia.org/wiki/Hilbert%27s_Nullstellensatz`

[2] rewrite this, maybe use chatgpt.

Tag some things (eg: stalk) as a construction.

Notation

$\langle S \rangle$	The ideal generated by elements of the subset S.
$\trianglelefteq$	Indicates an ideal of a ring.
1_X	The identity morphism of an object X in its appropriate category.
$\mathcal{C}(X, Y)$	The collection of morphisms from an object X to an object Y in the category $\mathcal{C}$.
Set	The category of sets and functions.
Top	The category of Topological spaces and continuous maps.
S^δ	The set S regarded as a topological space with the discrete topology.
CRing	The category of commutative unital rings and unit preserving ring homomorphisms.
Ab	The category of abelian groups and group homomorphisms.
$*$	The one point set.

Classical Algebraic Geometry

Let k be an algebraically closed field[3]. We shall consider the ring $k[x_1, \ldots, x_n]$ which is the polynomial ring in n commuting variables $x_1, \ldots, x_n$. Each $f \in k[x_1, \ldots, x_n]$ induces a function $k^n \to k$ by substituting an n-tuple $(\alpha_1, \ldots, \alpha_n)$ into the variables in f. This space k^n we call $\mathbb{A}^n_k$ or n-dimensional affine space over k. For a polynomial $f \in k[x_1, \ldots, x_n]$ we define its vanishing set $\mathbb{V}(f)$. That is points in $\mathbb{A}^n_k$ (which we shall now only call $\mathbb{A}^n$) at which f evaluates to 0. Formally:

[3] The reader is welcome to consider $\mathbb{C}$.

$$\mathbb{V}(f) := \{\alpha \in \mathbb{A}^n | \, f(\alpha) = 0\}.$$

Of course any polynomial divisible by f will also vanish on $\mathbb{V}(f)$ and thus $\mathbb{V}(f) = \mathbb{V}(\langle f \rangle)$. We shall see soon that is it "more natural" to think of the vanishing set of an ideal in $k[x_1, \ldots, x_n]$ than just a single polynomial.

We define an algebraic subset of $\mathbb{A}^n$ as a subset $\Lambda \subseteq \mathbb{A}^n$ such that $\Lambda = \mathbb{V}(\mathfrak{a})$ for some ideal $\mathfrak{a} \trianglelefteq k[x_1, \ldots, x_n]$. Here we note some properties of V that shall be important later:

(a) For ideals $\mathfrak{a}$, $\mathfrak{b} \trianglelefteq k[x_1, \ldots, x_n]$ we have: $\mathbb{V}(\mathfrak{a}) \cup \mathbb{V}(\mathfrak{b}) = \mathbb{V}(\mathfrak{a}b)$. That is: "the finite unions of algebraic subsets is an algebraic subset".

(b) For any set of ideals $\{\mathfrak{a}_i\} \trianglelefteq k[x_1, \ldots, x_n]$ we have

$$\bigcap \mathbb{V}(\mathfrak{a}_i) = V\left(\sum \mathfrak{a}_i\right).$$

That is: "the arbitrary intersection of algebraic subsets is is an algebraic subset".

(c) $\mathbb{V}(0) = \mathbb{A}^n$ and $\mathbb{V}(k[x_1, \ldots, x_n]) = \emptyset$, that is: "all of affine space as well as the empty set is an algebraic subset".[4]

[4] Do (a), (b), and (c) remind you of anything?

A few things about commutative rings must be said before we continue. An ideal $\mathfrak{a} \trianglelefteq R$ is radical[5] if and only if

[5] I must apologise to non-commutative ring theorists but I am afraid there are only so many words.

$$f^n \in \mathfrak{a} \implies f \in \mathfrak{a} \, \forall f \in R, \text{ for } n \in \mathbb{N}.$$

For some ideal $\mathfrak{a} \trianglelefteq R$ we define the radical ideal completion of $\mathfrak{a}$:

$$\sqrt{\mathfrak{a}} := \bigcap_{\text{radical ideals } I \supseteq \mathfrak{a}} I.$$

Next we say a ring is reduced it has no nilpotent elements[6].

[6] So if the ring is artinian this is equivalent to saying it has trivial Jacobson radical.

We take a moment to discuss another property of $\mathbb{V}$. For ideals $\mathfrak{a},\ \mathfrak{b} \trianglelefteq k[x_1, \ldots, x_n]$ it holds that $\mathbb{V}(\mathfrak{a}) \subseteq \mathbb{V}(\mathfrak{b}) \iff \sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$. Intuitively this should make sense since if we want more polynomials to vanish clearly this will be satisfied by fewer points[7].

For an algebraic subset $\Lambda \subseteq \mathbb{A}^n$ we define the vanishing ideal of Λ as:

$$\mathbb{I}(\Lambda) := \{f \in k[x_1, \ldots, x_n] |\ f(\lambda) = 0\ \forall \lambda \in \Lambda\}.$$

For an algebraic subset $\Lambda \subseteq \mathbb{A}^n$ one may wonder what $\mathbb{V}(\mathbb{I}(\Lambda))$ is. We note $\Lambda \subseteq \mathbb{V}(\mathbb{I}(\Lambda))$ but it is not obvious what this is. One may also wonder $\mathbb{I}(\mathbb{V}(\mathfrak{a}))$ is for $\mathfrak{a} \trianglelefteq k[x_1, \ldots, x_n]$. We observe that $\sqrt{\mathfrak{a}} \subseteq \mathbb{I}(\mathbb{V}(\mathfrak{a}))$ since $k[x_1, \ldots, x_n]$ is reduced. Hilbert's Nullstellensatz says that in an algebraically closed field that $\sqrt{\mathfrak{a}} = \mathbb{I}(\mathbb{V}(\mathfrak{a}))$[8]. This yields a 1-1 correspondence between radical ideals of $k[x_1, \ldots, x_n]$ and algebraic subsets of $\mathbb{A}^n$.

An algebraic subset $\Lambda \subseteq \mathbb{A}^n$ is irreducible if it is not the union of two distinct algebraic subsets - that is to say it is the vanishing set of a prime ideal. The Nullstellensatz further yields a 1-1 correspondence[9] between prime ideals of $k[x_1, \ldots, x_n]$ and irreducible algebraic subsets of $\mathbb{A}^n$.

Let $X \subseteq \mathbb{A}^n$ be an algebraic subset - we may wish to understand all polynomial functions $X \to k$. An immediate issue is that unequal polynomials $p, q \in k[x_1, \ldots, x_n]$ may induce the same function $X \to k$. For example suppose $f \in k[x_1, \ldots, x_n]$ and $0 \neq g \in \mathbb{I}(X)$, then $f : X \to k$ and $(f + g) : X \to k$ are the same function (as they agree on all inputs). To avoid this we define the coordinate ring[10] of X:

$$k[X] := {}^{k[x_1, \ldots, x_n]}\!\big/_{\mathbb{I}(X)}.$$

This ring can be thought of as "equivalence classes of polynomial maps $X \to k$". Thus we can assign to each algebraic subset X a ring[11] $k[X]$. The Nullstellensatz yields a 1-1 correspondence between radical ideals of $k[x_1, \ldots, x_n]$ and finitely generated reduced k-algebras. It also yields a 1-1 correspondence between prime ideals of $k[x_1, \ldots, x_n]$ and finitely generated k-algebras which are integral domains.

We now take a moment to look back at the above story. We considered a polynomial ring in finitely many variables over an algebraically closed field. The fact that such rings are noetherian and poses no zero-divisors made the theory particularly pleasant however we have relatively few examples. In what is to come we shall generalise this story to arbitrary commutative rings.

[7] This relation may lead one to conclude that $\mathbb{V}$ is a contravariant functor from the category of ideals of $k[x_1, \ldots, x_n]$ and the category of algebraic subsets of $\mathbb{A}^n$. This is true.

[8] This can be interpreted as saying that $\mathbb{I}$ and $\mathbb{V}$ give an adjunction between the category of ideals of $k[x_1, \ldots, x_n]$ and the category of algebraic subsets of $\mathbb{A}^n$. Verify this!!!!

[9] Both of these so called "1-1 correspondences" are in fact equivalences of categories.

[10] These have the following nice property. Two algebraic subsets X, Y are isomorphic (whatever that may mean) if and only if $k[X] \cong k[Y]$.

[11] In fact a k-algebra.

Some Point Set Topology

We define a topological space as a set equipped with a topology. A topology on a set X is a subset $\mathrm{Top}(X) \subseteq \mathbb{P}(X)$ such that:

(a) For a finite subset $\{\mathcal{U}_i\} \subseteq \mathrm{Top}(X)$ the intersection

$$\bigcap \mathcal{U}_i \in \mathrm{Top}(X).$$

That is to say "the intersection of finitely many open sets is open".

(b) For an arbitrary subset[12] $\{\mathcal{U}_i\}_{i\in I} \subseteq \mathrm{Top}(X)$ the union

$$\bigcup \mathcal{U}_i \in \mathrm{Top}(X).$$

That is to say "the arbitrary union of open sets is open".

(c) Both X and $\emptyset$ are in $\mathrm{Top}(X)$.

[12] Where I is some index set.

For a topological space we call an element $\mathcal{U} \in \mathrm{Top}(X)$ an open set. A subset $\mathcal{C} \subseteq X$ is closed if its the complement $X \setminus \mathcal{C}$ is open. Hence we can dualise the axioms for open sets to obtain axioms for closed sets. A subset $\mathrm{Pot}(X) \subset \mathbb{P}(X)$ is the set of closed sets of a topological space if and only if:

(a′) For a finite subset $\{\mathcal{C}_i\} \subseteq \mathrm{Pot}(X)$ the union

$$\bigcup \mathcal{C}_i \in \mathrm{Pot}(X).$$

That is to say "the union of finitely many closed sets is closed".

(b′) For an arbitrary subset $\{\mathcal{C}_i\}_{i\in I} \subseteq \mathrm{Pot}(X)$ the intersection

$$\bigcap \mathcal{C}_i \in \mathrm{Pot}(X).$$

That is to say "the arbitrary intersection of closed sets is closed".[13]

(c′) Both $\emptyset$ and X are in $\mathrm{Pot}(X)$.

[13] We obtained (a′) and (b′) by applying De Morgan's laws.

Suppose X is a set and $\mathrm{Pot}(X) \subseteq \mathbb{P}(X)$ satisfies (a′), (b′) and (c′) then $\mathrm{Pot}(X)$ uniquely determines a topology

$$\mathrm{Top}(X) = \{X \setminus \mathcal{C} | \ \mathcal{C} \in \mathrm{Pot}(X)\}$$

on X.

We return for a moment to affine space. Recall that the the algebraic subsets of $\mathbb{A}^n$ satisfy (a′), (b′) and (c′). The topology that arises from treating the algebraic subsets as closed sets is called the Zariski Topology.

Let X be a topological space, then for a subset $S \subseteq X$ we define its closure to be:

$$\overline{S} := \bigcap_{\text{closed sets } \mathcal{C} \supseteq S} \mathcal{C}.$$

Topology is in some sense an axiomatisation of "closeness". We say a point $x \in X$ is "close" to a subset $S \subseteq X$ if and only if $x \in \overline{S}$. What are the natural morphisms that would preserve this structure? Continuous maps!

Let X, Y be topological spaces. The following are equivalent for a set map $f : X \to Y$:

i The map f is continuous.

ii For any subset $T \subseteq Y$ one has $\overline{f^{-1}(T)} \subseteq f^{-1}(\overline{T})$.[14]

iii For any closed subset $\mathcal{C} \subseteq Y$, the preimage $f^{-1}(\mathcal{C})$ is closed.

iv For any open set $\mathcal{U} \subseteq Y$, the preimage $f^{-1}(\mathcal{U})$ is open.[15]

[14] Where the closure of T is taken in Y and the closure of $f^{-1}(T)$ is taken in X.

[15] This characterisation is most often used.

We observe that (ii) implies that continuous maps take "close points" to "close points". That is suppose $S \subseteq X$ and $x \in \overline{S}$ then $f(x) \in \overline{f(S)}$.

For each topological space X we can endow $\text{Top}(X)$ with the structure of a category as follows. For $\mathcal{U}, \mathcal{V} \in \text{Top}(X)$ we define:

$$\text{Hom}(\mathcal{U}, \mathcal{V}) := \begin{cases} \{i_{\mathcal{U}}^{\mathcal{V}}\}, & \text{if } \mathcal{U} \subseteq \mathcal{V}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

That is to say if $\mathcal{U} \subseteq \mathcal{V}$ then $\text{Hom}(\mathcal{U}, \mathcal{V})$ is the inclusion map $\mathcal{U} \hookrightarrow \mathcal{V}$ and $\text{Hom}(\mathcal{U}, \mathcal{V})$ is empty otherwise. Since $\forall \mathcal{U} \in \text{Top}(X)$ it holds that $\mathcal{U} \subseteq \mathcal{U}$ there are identity arrows. Composition inherits its associativity from the normal composition of continuous functions[16].

[16] Which in turn inherits *its* associativity from the normal composition of set maps.

Some Sheaf Theory

As is customary in this subject we begin with presheaves. Let X be a topological space, then a presheaf taking values in a category $\mathcal{C}$[17] is a contravariant functor from $\mathrm{Top}(X)$ to $\mathcal{C}$. Suppose $\mathcal{F}$ is a presheaf on X, then its signature is

$$\mathcal{F} : \mathrm{Top}(X)^{\mathrm{op}} \to \mathcal{C}.$$

[17] There are some requirements of the category $\mathcal{C}$ - for example it must have a terminal object. In these notes we shall only consider sheaves of sets, abelian groups, modules or rings.

We shall unpack this definition, the functoriality of $\mathcal{F}$ implies the following. Suppose $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{W}$ are open sets in X, then there exist "restriction maps" $\rho^{\mathcal{V}}_{\mathcal{U}} : \mathcal{F}(\mathcal{V}) \to \mathcal{F}(\mathcal{U})$, $\rho^{\mathcal{W}}_{\mathcal{V}} : \mathcal{F}(\mathcal{W}) \to \mathcal{F}(\mathcal{V})$, and $\rho^{\mathcal{W}}_{\mathcal{U}} : \mathcal{F}(\mathcal{W}) \to \mathcal{F}(\mathcal{U})$ such that $\rho^{\mathcal{V}}_{\mathcal{U}} \circ \rho^{\mathcal{W}}_{\mathcal{V}} = \rho^{\mathcal{W}}_{\mathcal{U}}$. Furthermore for any $\mathcal{U} \in \mathrm{Top}(X)$ we have $\rho^{\mathcal{U}}_{\mathcal{U}} = 1_{\mathcal{U}}$.

Example 1. Let $X = \mathbb{R}$, then we shall define a presheaf of sets as follows.

$$\text{We write:} \quad \mathcal{F} : \mathrm{Top}(X)^{\mathrm{op}} \to \mathbf{Set}$$
$$\mathcal{U} \mapsto \{\text{bounded functions } \mathcal{U} \to \mathbb{R}\},$$

and the restriction maps are the usual restriction of functions[18]. We notice for each open set $\mathcal{U}$ and each $\sigma \in \mathcal{F}(\mathcal{U})$ that $\rho^{\mathcal{U}}_{\mathcal{U}}(\sigma) = \sigma$. Suppose $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{W}$ are open sets in X. From the definition of $\mathcal{F}$ we note that:

[18] Hence this example motivates the term "restriction map".

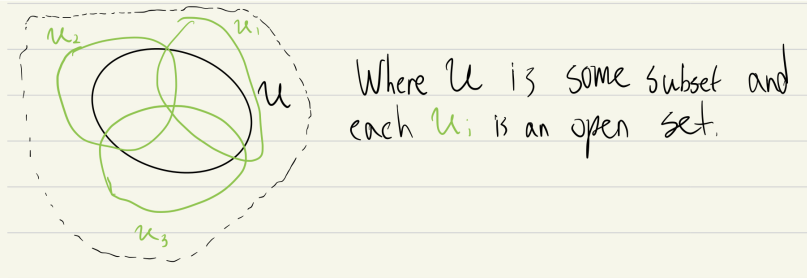
$$\rho^{\mathcal{W}}_{\mathcal{U}} : \mathcal{F}(\mathcal{W}) \to \mathcal{F}(\mathcal{U})$$
$$\sigma \mapsto \sigma|_{\mathcal{U}}.$$

Then for any $\sigma \in \mathcal{F}(\mathcal{W})$ we have that $\sigma|_{\mathcal{V}}|_{\mathcal{U}} = \sigma|_{\mathcal{U}}$ and hence $\rho^{\mathcal{V}}_{\mathcal{U}} \circ \rho^{\mathcal{W}}_{\mathcal{V}} = \rho^{\mathcal{W}}_{\mathcal{U}}$.

Soon we shall see that sheaves allow us to aggregate "local information" in order to obtain "global information". For this we need the notion of an open cover. Let X be a topological space, then a family of open sets $\{\mathcal{U}_i\}$ is an open cover of a subset $\mathcal{U}$ if and only if:

$$\mathcal{U} \subseteq \bigcup \mathcal{U}_i.$$

For the purpose of working with sheaves we will want it to be the case that $\mathcal{U} = \bigcup \mathcal{U}_i$ but this is not a problem. From any open cover $\{\mathcal{U}_i\}$ we can extract an "exact" open cover by defining $\mathcal{U}'_i := \mathcal{U}_i \cap U$ and using the set $\{\mathcal{U}'_i\}$. The picture is:



We are now ready to define a sheaf! A contravariant functor $\mathcal{F}$: $\text{Top}(X)^{\text{op}} \to \mathcal{C}$ is a sheaf if and only if:

(a) Suppose $\{\mathcal{U}_i\}_{i\in I}$ is an open cover of some open set $\mathcal{U} \subseteq X$. Further suppose we have sections[19] $\sigma_i \in \mathcal{F}(\mathcal{U}_i)$ such that:

$$\sigma_i|_{\mathcal{U}_i\cap\mathcal{U}_j} = \sigma_j|_{\mathcal{U}_i\cap\mathcal{U}_j} \ \forall i,j \in I,$$

then $\exists\sigma \in \mathcal{F}(\mathcal{U}) : \ \sigma|_{\mathcal{U}_i} = \sigma_i \ \forall i \in I$.

[19] The elements of $\mathcal{F}(\mathcal{U})$ for some open set $\mathcal{U}$ are referred to as "sections of $\mathcal{F}$ over $\mathcal{U}$". This nomenclature shall be justified in an example below.

(b) Suppose $\{\mathcal{U}_i\}_{i\in I}$ is an open cover of some open set $\mathcal{U} \subseteq X$; $\sigma, \tau \in \mathcal{F}(\mathcal{U})$ such that $\sigma|_{\mathcal{U}_i} = \tau|_{\mathcal{U}_i} \ \forall i \in I$ then $\sigma = \tau$.

More concisely we can say $\mathcal{F}$ satisfies the following condition.

(a′) Suppose $\{\mathcal{U}_i\}_{i\in I}$ is an open cover of some open set $\mathcal{U} \subseteq X$. Further suppose we have sections $\sigma_i \in \mathcal{F}(\mathcal{U}_i)$ such that:

$$\sigma_i|_{\mathcal{U}_i\cap\mathcal{U}_j} = \sigma_j|_{\mathcal{U}_i\cap\mathcal{U}_j} \ \forall i,j \in I,$$

then $\exists!\sigma \in \mathcal{F}(\mathcal{U}) : \ \sigma|_{\mathcal{U}_i} = \sigma_i \ \forall i \in I$.

The point here is that when we have a bunch of sections $\{\sigma_i\}$ that agree on overlaps then we can "glue" them together to produce a unique section σ over $\mathcal{U}$. In this way we are able to assign algebraic information to topological spaces in a way that respects their structure. Now we shall discuss some examples.

Example 2 (Sheaf of continuous functions)**.** Let $X = \mathbb{C}$, then[20] we define:

[20] We could use any topological space for this example.

$$\begin{aligned} \mathcal{F} : \text{Top}(X)^{\text{op}} &\to \textbf{CRing} \\ \mathcal{U} &\mapsto \{\text{continuous functions } \mathcal{U} \to \mathbb{R}\}^{21} \end{aligned}$$

We shall prove this is a sheaf. Let $\mathcal{U} \in \text{Top}(X)$ and $\{\mathcal{U}_i\}_{i\in I}$ be an open cover of $\mathcal{U}$. Further suppose:

[21] We could also write this as $\mathcal{U} \mapsto \textbf{Top}(\mathcal{U}, \mathbb{R})$.

$$\exists\sigma_i \in \mathcal{F}(\mathcal{U}_i) : \ \sigma_i|_{\mathcal{U}_i\cap\mathcal{U}_j} = \sigma_j|_{\mathcal{U}_i\cap\mathcal{U}_j} \ \forall i,j \in I.$$

We must prove $\exists!\sigma \in \mathcal{F}(\mathcal{U}) : \ \sigma|_{\mathcal{U}_i} = \sigma_i \ \forall i \in I$.
Existence of gluing: We define:

$$\begin{aligned} \overline{\sigma} : \mathcal{U} &\to \mathbb{R} \\ u &\mapsto \sigma_i(u) \text{ for } u \in \mathcal{U}_i. \end{aligned}$$

First we note this map is well defined precisely because the sections $\{\sigma_i\}$ agree on intersections. Next we note that that $\overline{\sigma}$ is continuous since each σ_i is continuous. Thus $\overline{\sigma} \in \mathcal{F}(\mathcal{U})$ and satisfies

$$\overline{\sigma}|_{\mathcal{U}_i} = \sigma_i \; \forall i \in I.$$

Uniqueness of gluing: Suppose $\exists \tau \in \mathcal{F}(\mathcal{U})$ such that $\tau|_{\mathcal{U}_i} = \sigma_i \; \forall i \in I$. Then for each $u \in \mathcal{U}$ we observe:

$$\overline{\sigma}(u) = \sigma_i(u) = \tau(u)$$

for an appropriate i and thus we conclude $\tau = \overline{\sigma}$.

Example 3 (Sheaf of continuous sections)**.** Let $X = \mathbb{R}$, $Z = \{1, 2, 3\}$, and

$$\begin{aligned} p : X \times Z^\delta &\to X \\ (x, z) &\mapsto x. \end{aligned}$$

For a subset $S \subseteq X$ a "section of p over $\mathcal{U}$" is a continuous map $\sigma : S \to X \times Z^\delta$ such that $p \circ \sigma = 1_S$. We define:

$$\begin{aligned} \mathcal{F} : \mathrm{Top}(X)^{\mathrm{op}} &\to \mathbf{Set} \\ \mathcal{U} &\mapsto \{\text{sections of } p \text{ over } \mathcal{U}\}, \end{aligned}$$

we shall prove this is a sheaf. It is a presheaf since the restriction of a function to its domain is itself and that the composition of iterated restrictions to smaller subsets of the domain is the same as just restricting to the smallest subset. Let $\mathcal{U} \in \mathrm{Top}(X)$ and $\{\mathcal{U}_i\}_{i \in I}$ be an open cover of $\mathcal{U}$. Further suppose:

$$\exists \sigma_i \in \mathcal{F}(\mathcal{U}_i) : \; \sigma_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = \sigma_j|_{\mathcal{U}_i \cap \mathcal{U}_j} \; \forall i, j \in I.$$

We must prove $\exists! \sigma \in \mathcal{F}(\mathcal{U}) : \; \sigma|_{\mathcal{U}_i} = \sigma_i \; \forall i \in I$.
Existence of gluing: We define:

$$\begin{aligned} \overline{\sigma} : \mathcal{U} &\to \mathbb{R} \\ u &\mapsto \sigma_i(u) \text{ for } u \in \mathcal{U}_i. \end{aligned}$$

First we note this map is well defined precisely because the sections $\{\sigma_i\}$ agree on intersections. Next we note that that $\overline{\sigma}$ is a section of p over $\mathcal{U}$ since each σ_i is.
Uniqueness of gluing: Suppose $\exists \tau \in \mathcal{F}(\mathcal{U})$ such that $\tau|_{\mathcal{U}_i} = \sigma_i \; \forall i \in I$. Then for each $u \in \mathcal{U}$ we observe:

$$\overline{\sigma}(u) = \sigma_i(u) = \tau(u)$$

for an appropriate i and thus we conclude $\tau = \overline{\sigma}$.

Let $\mathcal{U} := (-1, 1)$ then $\mathcal{F}(\mathcal{U}) = \{\sigma_1, \sigma_2, \sigma_3\}$ where:

$$\begin{aligned} \sigma_i : \mathcal{U} &\to X \times Z^\delta \\ u &\mapsto (u, i) \text{ for } i \in Z. \end{aligned}$$

However if we consider $(-1,1) \setminus \{0\} =: \mathcal{V}$. Then $|\mathcal{F}(\mathcal{V})| = 9$ where all sections are of the form:

$$\begin{aligned} \sigma_{i,j} &: \mathcal{V} \to X \times Z^\delta \\ v &\mapsto \begin{cases} \sigma_i(v), & v \in (0,1) \\ \sigma_j(v), & v \in (-1,0) \end{cases} \text{ for } i,j \in Z. \end{aligned}$$

It is notable that we were able to "detect" that $\mathcal{V}$ is not connected by simply counting the sections of $\mathcal{F}$ over it.

There are many similar examples. For any continuous map $f : X \to Y$ we can always define the sheaf[22] of continuous sections of f on $\text{Top}(Y)$.

[22] The sheaf of sections of a covering space $p : \tilde{X} \to X$ is often interesting.

Example 4 (Skyscraper sheaf)**.** Let X be a topological space, S a set and x be a point in X. We define:

$$\begin{aligned} \mathcal{F} &: \text{Top}(X)^{\text{op}} \to \mathbf{Set} \\ \mathcal{U} &\mapsto \begin{cases} S, & x \in \mathcal{U} \\ *, & \text{otherwise.} \end{cases} \end{aligned}$$

In order to ease the proof that this is a sheaf we shall give it an alternate description. Let s be a fixed element of S, then we write:

$$\begin{aligned} \mathcal{F} &: \text{Top}(X)^{\text{op}} \to \mathbf{Set} \\ \mathcal{U} &\mapsto \{\text{set maps } \mathcal{U} \to S \text{ sending everything in } \mathcal{U} \setminus \{x\} \text{ to } s\}.^{23} \end{aligned}$$

A moment of reflection will cause one to conclude that $|\mathcal{F}(\mathcal{U})| = |S|$ if $x \in \mathcal{U}$ and $|\mathcal{F}(\mathcal{U})| = 1$ otherwise. Thus our new definition is "isomorphic[24]" to the old one. Let $\mathcal{U} \in \text{Top}(X)$ and $\{\mathcal{U}_i\}_{i \in I}$ be an open cover of $\mathcal{U}$. Further suppose:

[23] It is often easier to work with sheaves of functions than any alternative. Thus it could be worth exercising some creativity to define one's sheaves to be sheaves of functions.

[24] We shall give a precise definition of a morphism of sheaves soon.

$$\exists \sigma_i \in \mathcal{F}(\mathcal{U}_i) : \; \sigma_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = \sigma_j|_{\mathcal{U}_i \cap \mathcal{U}_j} \; \forall i,j \in I.$$

We must prove $\exists! \sigma \in \mathcal{F}(\mathcal{U}) : \; \sigma|_{\mathcal{U}_i} = \sigma_i \; \forall i \in I$.
Existence of gluing: We define:

$$\begin{aligned} \overline{\sigma} &: \mathcal{U} \to S \\ u &\mapsto \sigma_i(u) \text{ for } u \in \mathcal{U}_i. \end{aligned}$$

First we note this map is well defined precisely because the sections $\{\sigma_i\}$ agree on intersections. Next we note that that $\overline{\sigma} \in \mathcal{F}(\mathcal{U})$ since each σ_i sends everything in $\mathcal{U}_i \setminus \{x\}$ to s.
Uniqueness of gluing: Suppose $\exists \tau \in \mathcal{F}(\mathcal{U})$ such that $\tau|_{\mathcal{U}_i} = \sigma_i \; \forall i \in I$. Then for each $u \in \mathcal{U}$ we observe:

$$\overline{\sigma}(u) = \sigma_i(u) = \tau(u)$$

for an appropriate i and thus we conclude $\tau = \overline{\sigma}$.

We shall now explain why **Example 1** fails to be a sheaf. Let $\{\mathcal{U}_i\}_{i \in I}$ be any open cover of $\mathbb{R}$ by bounded subsets. For each i we consider the section

$$\begin{aligned} \sigma_i : \mathcal{U}_i &\to \mathbb{R} \\ u &\mapsto u. \end{aligned}$$

These sections agree on intersections and if this construction was a sheaf we would be entitled to glue them. However the identity map on $\mathbb{R}$ is not a section over $\mathbb{R}$ since it is not bounded, thus it fails the existence axiom.

Since sheaves and presheaves are functors their morphisms are nothing but natural transformations. Let X be a topological space and suppose $\mathcal{F}, \mathcal{G} : \mathrm{Top}(X)^{\mathrm{op}} \to \mathcal{C}$ are (pre)sheaves. Then a morphism of (pre)sheaves $\eta : \mathcal{F} \to \mathcal{G}$ is a family of morphisms $\eta_{\mathcal{U}} : \mathcal{F}(\mathcal{U}) \to \mathcal{G}(\mathcal{U})$ in $\mathcal{C}$ indexed by the open sets of X such that the following square commutes for any open sets $\mathcal{U} \subseteq \mathcal{V}$.

$$\begin{tikzcd} \mathcal{F}(\mathcal{V}) \arrow[r, "\eta_{\mathcal{V}}"] \arrow[d, "\rho^{\mathcal{V}}_{\mathcal{U}}"'] & \mathcal{G}(\mathcal{V}) \arrow[d, "\varrho^{\mathcal{V}}_{\mathcal{U}}"] \\ \mathcal{F}(\mathcal{V}) \arrow[r, "\eta_{\mathcal{U}}"] & \mathcal{G}(\mathcal{V}) \end{tikzcd}$$

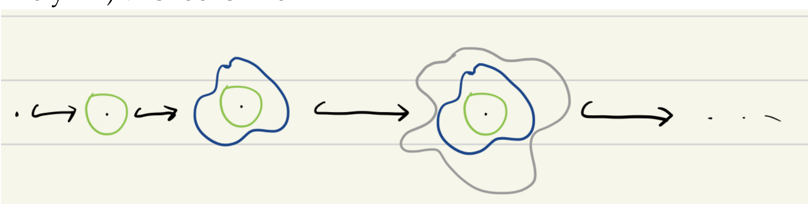
Suppose further that $\mathcal{F}, \mathcal{G}$ are sheaves of abelian groups. Then one may be interested[25] the kernel, cokernel and image of η. That is presheaves $\mathcal{U} \mapsto \ker(\eta_{\mathcal{U}})$, $\mathcal{U} \mapsto \operatorname{coker}(\eta_{\mathcal{U}})$, and $\mathcal{U} \mapsto \operatorname{im}(\eta_{\mathcal{U}})$. While it is always the case that $\ker(\eta)$ is always a sheaf, this need not be the case for $\operatorname{coker}(\eta)$ and $\operatorname{im}(\eta)$ - in general they are only sheaves. We shall now describe a method to canonically assign a sheaf (that is unique up to isomorphism) to any presheaf.

[25] For example when making use of techniques from homological algebra.

Construction 1 (Stalk)**.** Suppose that $\mathcal{F}$ is a (pre)sheaf on a topological space X and x is a point in X. Further suppose we would like to evaluate $\mathcal{F}$ at x. In general this is not possible since points need not be open. Another consideration that doesn't work would be to evaluate $\mathcal{F}$ on $\bigcap_{\text{open sets } \mathcal{U} \ni x} \mathcal{U}$ but this also need not be open. Luckily for us category theory provides an answer. We consider the diagram of all the open sets containing x. That is:

$$\ldots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \ldots$$

where each $\bullet$ is an open set of X containing X. In Euclidean space (namely $\mathbb{R}^2$) this looks like:



We define the stalk of $\mathcal{F}$ at x to be the colimit[26] of the diagram:

[26] This has a pleasant implication. Suppose $\mathcal{F}$ is a sheaf on a topological space X and $\mathcal{U}$ is open in X. Then for a point $x \in \mathcal{U}$ the stalk of F at x is the same as the stalk of $\mathcal{F}|_{\mathcal{U}}$ at x. In this sense the stalk is a purely local construction.

$$\dots \longleftarrow \mathcal{F}(\bullet) \longleftarrow \mathcal{F}(\bullet) \longleftarrow \mathcal{F}(\bullet) \longleftarrow \dots$$

where the $\bullet$'s are all the open sets containing x. This can be characterised as:

$$\mathcal{F}_x := {\{(\mathcal{U},\sigma)|\ x \in \mathcal{U} \in \mathrm{Top}(X),\ \sigma \in \mathcal{F}(\mathcal{U})\}}\big/{\sim}$$

where:

$$(\mathcal{U},\sigma) \sim (\mathcal{V},\tau) \iff \exists \text{ open } \mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V} \text{ with } x \in \mathcal{W} :\ \sigma|_{\mathcal{W}} = \tau|_{\mathcal{W}}.$$

Let $\mathcal{F}, \mathcal{G}$ be sheaves on a topological space X and suppose $\eta : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. Then η induces morphisms on the stalks in the following way:

$$\begin{aligned} \eta_x : \mathcal{F}_x &\to \mathcal{G}_x \\ [(\mathcal{U},\sigma)] &\mapsto [(\mathcal{U},\eta_{\mathcal{U}}(\sigma)]. \end{aligned}$$

Example 5. We shall solve a simple problem[27] involving in stalks in order to expose some of their properties. We let $X = \mathbb{R}$ and consider the sheaf of continuous functions $\mathcal{F}$ on X. We define the evaluation map:

[27] This is a variant of an exercise given to me Clark Barwick in a course he and Jeff Hicks taught about sheaf theory in Edinburgh.

$$\begin{aligned} e : \mathcal{F}_x &\to \mathbb{R} \\ [(\mathcal{U},\sigma)] &\mapsto \sigma(x). \end{aligned}$$

The exercise is to prove this is not a bijection. First we observe that e is a surjection since for any $x \in \mathbb{R}$, open set $\mathcal{U}$ containing x and any $r \in \mathbb{R}$ the map

$$\begin{aligned} \mathcal{U} &\to \mathbb{R} \\ u &\mapsto r \end{aligned}$$

is in $\mathcal{F}_x$. We shall explain how e fails to be an injection. Let $x \in \mathbb{R}$ then as described above the constant map to 0 is in $\mathcal{F}_x$. Let $\mathcal{U}$ be an open set containing x, we define the map

$$\begin{aligned} \sigma : \mathcal{U} &\to \mathbb{R} \\ u &\mapsto u - x \end{aligned}$$

and we claim $[(\mathcal{U},0)] \neq [(\mathcal{U},\sigma)]$ in $\mathcal{F}_x$. We observe that

$$\{r \in \mathbb{R}|\ 0(r) = \sigma(r)\} = \{x\}$$

which is not open. Thus we conclude e is not injective.

We shall now define sheafification. Let $\mathcal{F}$ be a presheaf on a topological space X. We define the étale space of $\mathcal{F}$ to be the set

$$\text{Ét}(\mathcal{F}) := \coprod_{x \in X} \mathcal{F}_x$$

endowed with the final topology such that

$$\begin{aligned} p : \text{Ét}(\mathcal{F}) &\to X \\ e &\mapsto x \text{ for } e \in \mathcal{F}_x \end{aligned}$$

is continuous. We define the sheafification of $\mathcal{F}$ to be the sheaf of sections of p, which is denoted $\mathcal{F}^+$. For any sheaf $\mathcal{G}$ we have that $\mathcal{G}^+ \cong \mathcal{G}$.

From this we observe that a sheaf is uniquely determined by its stalks, we record this in the following result.

Proposition 1. Let $\mathcal{F}, \mathcal{G}$ be sheaves on a topological space X and suppose $\eta : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. Then η is an isomorphism if and only if each induced map $\eta_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism.

We shall discuss two important constructions and then begin defining schemes.

Construction 2 (Pushforward Sheaf)**.** Let $\mathcal{F}$ be a sheaf taking values in a category $\mathcal{C}$ on a topological space X and let $f : X \to Y$ be a continuous map of topological spaces. Then we can define the presheaf:

$$\begin{aligned} f_*\mathcal{F} : \mathrm{Top}(Y)^{\mathrm{op}} &\to \mathcal{C} \\ \mathcal{U} &\mapsto \mathcal{F}(f^{-1}(\mathcal{U})). \end{aligned}$$

We shall prove this is a sheaf. Let $\mathcal{U} \in \mathrm{Top}(Y)$ and $\{\mathcal{U}_i\}_{i\in I}$ be an open cover of $\mathcal{U}$. Further suppose:

$$\exists \sigma_i \in f_*\mathcal{F}(\mathcal{U}_i) : \ \sigma_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = \sigma_j|_{\mathcal{U}_i \cap \mathcal{U}_j} \ \forall i, j \in I. \qquad (\star)$$

We write $\mathcal{V} := f^{-1}(\mathcal{U})$ and $\mathcal{V}_i := f^{-1}(\mathcal{U}_i)$ and observe that $\{\mathcal{V}_i\}_{i\in I}$ is an open cover of $\mathcal{V}$ since the inverse image respects unions. Then we notice $(\star)$ is equivalent to the existence of sections

$$\tau_i \in \mathcal{F}(\mathcal{V}_i) : \ \tau_i|_{\mathcal{V}_i \cap \mathcal{V}_j} = \tau_j|_{\mathcal{V}_i \cap \mathcal{V}_j} \ \forall i, j \in I,$$

and thus $\exists! \tau \in \mathcal{F}(\mathcal{V}) : \ \tau|_{\mathcal{V}_i} = \tau_i \ \forall i \in I$ since $\mathcal{F}$ is a sheaf. Hence there exists a unique section $\sigma \in f_*\mathcal{F}(\mathcal{U}) : \ \sigma|_{\mathcal{U}_i} = \sigma_i \ \forall i \in I$.

Construction 3 (Pullback sheaf)**.** Let $\mathcal{G}$ be a sheaf taking values in a category $\mathcal{C}$ on a topological space Y and let $f : X \to Y$ be a continuous map of topological spaces. We consider the space $\tilde{X} := \coprod_{x\in X} \mathcal{G}_{fx}$ endowed with the final topology such that

$$\begin{aligned} p : \tilde{X} &\to X \\ e &\mapsto x \text{ for } e \in \mathcal{G}_{fx} \end{aligned}$$

is continuous. Then we say the sheaf $f^{-1}\mathcal{G}$ is the sheaf of sections of p.

Localisation For Commutative Rings

Let $R \in \mathbf{CRing}$, then a subset $S \subseteq R$ is a multiplicative set if and only if:

$$x, y \in S \implies xy \in S.$$

Let $f \in R$ then we observe that $\{f^n \mid n \in \mathbb{N}\}$ is a multiplicative set.

Recall that an ideal $\mathfrak{p} \trianglelefteq R$ is prime if and only if:

$$ab \in \mathfrak{p} \implies (a \in \mathfrak{p}) \cup (b \in \mathfrak{p}).$$

Let $\mathfrak{p} \trianglelefteq R$ be a prime ideal, then observe $R \setminus \mathfrak{p}$ is a multiplicative set.

Construction 4. Let $S \subseteq R$ be a multiplicative subset of a commutative ring R that contains no zero divisors[28]. Then we consider the polynomial ring in $|S|$ variables over R. We can write this as

$$R[S] := \bigotimes_{s \in S} R[x_s].$$

[28] This hypothesis is always satisfied if R is an integral domain. This is the easy case.

We wish to enforce the relation that $sx_s = 1$ and this is achieved by quotienting by the ideal $\langle \{sx_s - 1 \mid s \in S\} \rangle$. We define

$$S^{-1}R := {R[S]}\big/{\langle \{sx_s - 1 \mid s \in S\} \rangle}.$$

This ring has the universal property that any ring homomorphism $\phi : R \to L$ that carries each $s \in S$ to an invertible element factors through $S^{-1}R$. That is to say:

$$\begin{tikzcd}
R \arrow[r, hook, "\iota"] \arrow[dr, "\phi"'] & S^{-1}R \arrow[d, "\exists! \psi"] \\
& L
\end{tikzcd}$$

commutes.

Example 6. Let $R = {\mathbb{Z}}\big/{10\mathbb{Z}}$ and consider the multiplicative set $S = \{2, 4, 8, 6\}$. Then if we follow **Construction 4** we obtain the ring

$$S^{-1}R = \{\frac{r}{2^n} \mid r \in R, n \in \mathbb{N}\}$$

where

$$\frac{a}{2^n}\frac{b}{2^m} = \frac{ab}{2^{mn \pmod{10}}}.$$

Start with the general definition from eisenbudd