Abelian Categories, Hochschild (Co)homology and Algebraic Deformations

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Year 4 Project
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17^{th} March 2023

Abstract

Homological Algebra has earned its place as part of the modern mathematical landscape. It has been applied extensively in algebraic topology and geometry, ring theory, group theory, number theory, and many other areas. Some example results are: the Serre intersection formula, the Weil conjectures as well as a reformulation of the Artin-Wedderburn theorem. Herein we present a down-to-earth introduction to homological algebra aimed at undergraduates. Homological algebra has a simple set of axioms that hold throughout many different areas of mathematics.

In Chapter 1 there is a brief introduction to category theory that will culminate in the description of abelian categories. Followed by a discussion of some of their basic properties and the statement of Mitchell's full embedding theorem. In Chapter 2 we study the structure of categories of modules over a unital associative ring. Their tensor product as well as the tensor product of algebras will be constructed. Chapter 3 proceeds with an abstract treatment of exact sequences, chain complexes, and homology. Attention to the categorical duals of the afformentioned notions will also be drawn. We begin with more concrete material in Chapter 4 where we introduce Hochschild (Co)homology. Here a number of the low dimensional homology and cohomology groups will be characterised, finished with examples of computations. Chapter 5 shows some applications of Hochschild cohomology to algebraic deformation theory. We conclude by presenting a method to produce associative deformations that circumvents the need for Hochschild cohomology entirely and pose some questions for future work. This method contributes to the understanding of the theory of associative deformations of algebras.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Dave Bowman*)

To Gerhard Hochschild and Israel Nathan Herstein.

I would like to thank my supervisor Agata Smoktunowicz, as well as Kenneth Hughes and Clark Barwick for their guidance along my journey. I would also like to thank Dora, Linden and Fred.

This project report is submitted in partial fulfilment of the requirements for the degree of *Bachelor of Science with Honours in Mathematics*.

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Chapter 1

Category Theory

1.1 Initial definition and examples

Modern homological algebra is the unification of many formerly disparate notions found in different areas of mathematics. This unification made use of a powerful language that could describe very general and vastly different ideas. This is the language of categories. They are introduced here. We shall draw on ideas from chapter 1.2 of [12] and chapter 1 of [8].

Definition 1. A *category* $\mathbf{C}$ consists of

- a class of objects $\mathrm{Ob}(\mathbf{C})$,
- for every two objects A, B a set $\mathrm{Hom}(A, B)$[1] of morphisms from A to B, often denoted by arrows $f : A \to B$, and identity morphisms 1_A for every object A,
- for every three objects A, B, C a composition rule

$$\begin{aligned} \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) &\to \mathrm{Hom}(A, C) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

  that is associative.

This definition is extremely general and manages to capture most algebraic structures with their appropriate homomorphisms.

Example 1. We shall first discuss some familiar examples.

- **Sets** which has sets as its objects and functions as its morphisms. This is a category because every set has an identity function and the composition of functions is associative.
- **Groups** the category of groups and group homomorphisms as well as **Ab** the category of abelian groups and group homomorphisms.

[1] This notation will be used when the category C is not ambiguous, otherwise the notation $\mathrm{Hom}_{\mathbf{C}}(A, B)$ will be used

- **Top** the category of topological spaces and continuous maps.
- $\mathbf{Vect}_k$ the category of vector spaces over a field k and linear transformations.
- ${}_R\mathbf{Mod}$ the category of left R-modules for a unital associative ring R and R-homomorphisms.
- **Rings** the category of unital associative rings and identity preserving ring homomorphisms as well as **ComRings** the category of commutative rings and identity preserving ring homomorphisms.

We notice that the category **Ab** appears to be a subcollection of **Groups** and indeed this is the case! We capture this formally as follows.

Definition 2. A category $\mathbf{D}$ is a *subcategory* of another category $\mathbf{C}$ if:

- the objects of $\mathbf{D}$ are a subcollection of the objects of $\mathbf{C}$,
- if the morphism $f : A \to B$ in $\mathbf{C}$ is in $\mathbf{D}$ then so are A, B,
- if two composable morphisms f, g in $\mathbf{C}$ are in $\mathbf{D}$ then so is their composition $g \circ f$,
- if an object $x \in \mathrm{Ob}(\mathbf{D})$ then $1_X : X \to X$ is also in $\mathbf{D}$.

We say $\mathbf{D}$ is a *full subcategory* of $\mathbf{C}$ if:

$$\mathrm{Hom}_{\mathbf{D}}(A, B) = \mathrm{Hom}_{\mathbf{C}}(A, B)$$

for every pair of objects $A, B \in \mathbf{D}$.

Example 2.
- **Ab** is a full subcategory of **Groups**,
- **ComRings** is a full subcategory of **Rings**.
- **Rings** is a subcategory of the category of rings and ring homomorphisms (not necessarily preserving the unit) and is not full.

For any category $\mathbf{C}$ we define its dual category $\mathbf{C}^{op}$ where the arrows go the other way, thus for objects A, B in $\mathbf{C}$:

$$\mathrm{Hom}_{\mathbf{C}^{op}}(A, B) = \mathrm{Hom}_{\mathbf{C}}(B, A).$$

While the preceding examples exclusively deal with algebraic structures and their homomorphisms, this need not be the case:

Definition 3. A *poset* is a set endowed with a relation " $\leq$ " that is reflexive, transitive and antisymmetric.

We can regard any poset $(P, \leq)$ as a category $\mathbf{P}$ as follows:

- the objects are the elements of P,

- $\mathrm{Hom}(a,b) = \begin{cases} \{t_a^b\}, & a \leq b \\ \emptyset, & \text{otherwise} \end{cases}$

where $t_a^b : a \to b$. Identity arrows are guaranteed by reflexivity of $\leq$ and composition is defined and is associative by the transitivity of $\leq$.

This leads us to:

Example 3.
- The set $\mathbb{Z}$ is a poset where we use the usual $\leq$ relation and is thus also a category. We will denote this category by $\mathbf{Z}$ and the category that arises by reversing the relation as $\mathbf{Z}^{op}$.
- As above $\mathbb{R}$ is also a poset under its usual ordering and we denote this by $\mathbf{R}$.
- Let X be a topological space and $\mathcal{U}$ be its topology. Then $\mathcal{U}$ is a poset where $x \leq y \iff x \subseteq y$. We can thus view $\mathcal{U}$ as a category where $\mathrm{Hom}(x,y) = \begin{cases} \{i_x^y\}, & x \subseteq y \\ \emptyset, & \text{otherwise} \end{cases}$ where $i_x^y : x \to y$ is the inclusion map.

Definition 4. An object I in a category $\mathbf{C}$ is *initial* if there exists a unique morphism from I into any object X in $\mathbf{C}$. This is to say

$$|\mathrm{Hom}(I,X)| = 1 \; \forall X \in \mathrm{Ob}(\mathbf{C}).$$

An object T is *terminal* if there exists a unique morphism from any object X into T. This is to say

$$|\mathrm{Hom}(X,T)| = 1 \; \forall X \in \mathrm{Ob}(\mathbf{C}).$$

An object is a *zero object* if it is both initial and terminal.

Initial and terminal objects are unique up to isomorphism if they exist. In **Sets** the empty set is initial since there is only one map from the empty set to any set. The one point set $*$ is terminal since for each set X the only map $f : X \to *$ that sends all elements of X to the one element in the set. The ring $\mathbb{Z}$ is initial in **ComRings** where:

$$\phi : \mathbb{Z} \to R$$
$$n \mapsto \sum_{i=1}^{n} 1_R$$

for all $R \in$ **ComRings**. The zero ring is terminal here by a similar argument to above. **Groups** has the trivial group as its zero object. The poset category formed with the set $(0,1)$ and the usual ordering has no initial or terminal object.

Category theory is very useful to generalise and organise mathematical concepts and it is very good at detecting that two seemingly disparate ideas are in fact the same. This comes at a cost, and that cost is that one must phrase all of their ideas only in the language of objects and morphisms. Where possible we attempt to generalise familiar concepts to categorical ones. Injective and surjective maps are concepts familiar to most and they give us an idea of how a given function behaves. We now give their categorical generalisation:

Definition 5. Let $\mathbf{C}$ be a category and $A, B, C \in \mathbf{C}$. Then a morphism $h : B \to C$ is a *monomorphism* if for any pair of maps $f, g : A \to B$:

$$h \circ f = h \circ g \implies f = g,$$

a morphism that is a monomorphism is said to be *monic*. We also define its dual: a morphism $f : A \to B$ is an *epimorphism* if for every pair of maps $g, h : B \to C$:

$$g \circ f = h \circ f \implies g = h,$$

a morphism that is an epimorphism is said to be epic.

We observe that in the category of sets that a monomorphism is exactly and injective map and that an epimorphism is exactly a surjective map. **Sets** also has the property that a map that is both a monomorphism and an epimorphism are isomorphisms, but this can fail in general:

Example 4. We consider $\mathbb{Z}, \mathbb{R} \in$ **ComRings**. Then $i : \mathbb{Z} \to \mathbb{R}$ the inclusion map of $\mathbb{Z}$ into $\mathbb{R}$ is an monomorphism and an epimorphism but it is not true that $\mathbb{Z} \cong \mathbb{R}$.

It does hold that an isomorphism is both a monomorphism and an epimorphism.

1.2 Functors

Functors give us a way to move information from one category to another in a structure preserving manner. A functor assigns to each object in a category an object in another while preserving morphisms and their composition, formally:

Definition 6. We follow [12], suppose $\mathbf{C}$ and $\mathbf{D}$ are categories then a *covariant functor* $T : \mathbf{C} \to \mathbf{D}$ assigns to each $A \in \mathrm{Ob}(\mathbf{C})$ an object $T(A) \in \mathrm{Ob}(\mathbf{D})$ and to each morphism $f : A \to A'$ in $\mathbf{C}$ a morphism $T(f) : T(A) \to T(A')$ in $\mathbf{D}$ such that:

- if $A \xrightarrow{f} A' \xrightarrow{g} A''$ in $\mathbf{C}$ then $T(A) \xrightarrow{T(f)} T(A') \xrightarrow{T(g)} T(A'')$ in $\mathbf{D}$ and $T(g \circ f) = T(g) \circ T(f)$,
- $T(1_A) = 1_{T(A)}$ for all objects A in $\mathbf{C}$.

A *contravariant functor* is a functor between categories that reverses arrows: let $f : A \to B$ be a morphism in a category $\mathbf{C}$ and let T be a contravariant functor, then $T(f) : T(B) \to T(A)$. Before the examples a brief justification will be given. Let A, B be objects in a category $\mathbf{C}$. Then A is isomorphic to B (or $A \cong B$) if there exist morphisms:

$$\begin{aligned} f : A \to B \quad &\text{and} \quad g : B \to A \\ \text{such that} \quad g \circ f = 1_A \quad &\text{and} \quad f \circ g = 1_B. \end{aligned}$$

Suppose we apply a functor $T : \mathbf{C} \to \mathbf{D}$ to the above:

$$\begin{aligned} T(f) : T(A) \to T(B) \quad &\text{and} \quad T(g) : T(B) \to T(A) \\ \text{such that} \quad T(g) \circ T(f) = 1_{T(A)} \quad &\text{and} \quad T(f) \circ T(g) = 1_{T(B)}, \end{aligned}$$

thus functors preserve isomorphisms. This fact can be used as follows, suppose we have two objects A, B in an unfamiliar category $\mathbf{C}$ and we wish to discern if $A \cong B$. Then we would consider a functor T from $\mathbf{C}$ into a category that is well known and then check if $T(A) \cong T(B)$, if this is not then $A \ncong B$.

Example 5.

- For a category $\mathbf{C}$ there is the identity functor

$$\begin{aligned} 1_{\mathbf{C}} : \mathbf{C} &\to \mathbf{C} \\ A &\mapsto A \quad \text{for all objects } A \text{ in } \mathbf{C} \\ f &\mapsto f \quad \text{for all morphisms } f \text{ in } \mathbf{C} \end{aligned}$$

- For objects A, B, C in a category $\mathbf{C}$ with a morphism $f : B \to C$ we define the hom functor $\mathrm{Hom}(A, \square)$:

$$\begin{aligned} T_A : \mathbf{C} &\to \mathbf{Sets} \\ B &\mapsto \mathrm{Hom}(A, B) \quad \text{for all objects } B \text{ in } \mathbf{C} \\ f &\mapsto T_A(f) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C) \end{aligned}$$

  where $T_A(f)$ is post composition with f. We unpack this and verify it is a functor. Suppose we have:

$$B \xrightarrow{f} C \xrightarrow{g} D \text{ in } \mathbf{C}$$

  then we get:

$$\mathrm{Hom}(A, B) \xrightarrow{T_A(f)} \mathrm{Hom}(A, C) \xrightarrow{T_A(g)} \mathrm{Hom}(A, D) \text{ in } \mathbf{Sets}$$

  where:

$$\begin{aligned} T_A(g) \circ T_A(f) : \mathrm{Hom}(A, B) &\to \mathrm{Hom}(A, D) \\ \phi &\mapsto g \circ f \circ \phi, \end{aligned}$$

  and thus composition is preserved. Next we consider the identity map

$1_B : B \to B$ then:

$$\begin{aligned} T_A(1_B) : \mathrm{Hom}(A, B) &\to \mathrm{Hom}(A, B) \\ \phi &\mapsto 1_B \circ \phi = \phi, \end{aligned}$$

and thus the identity maps are preserved.

- For objects A, B, C in a category $\mathbf{C}$ with a morphism $f : A \to B$ we define the contravariant hom functor $\mathrm{Hom}(\Box, C)$:

$$\begin{aligned} L_C : \mathbf{C} &\to \mathbf{Sets} \\ A &\mapsto \mathrm{Hom}(A, C) \quad \text{for all objects } A \text{ in } \mathbf{C} \\ f &\mapsto L_C(f) : \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C) \end{aligned}$$

  where $L_C(f)$ is precomposition with f.

- Consider the categories **Groups**, **Sets** and the functor that takes a group to its underlying set and sends group homomorphisms to set maps. This is a "forgetful functor".

- The inclusion from **Z** into **R**.

- The functor from **R** to **Z** that maps $x \mapsto \lceil x \rceil$ for all $x \in \mathbb{R}$. We recall that $x \leq y \implies \lceil x \rceil \leq \lceil y \rceil$, this fact is required for composition of morphisms to be preserved.

Definition 7. For two covariant functors $F, G : \mathbf{C} \to \mathbf{D}$ a *natural transformation* $\eta : F \to G$ is a family of morphisms in $\mathbf{D}$ that associates a morphism

$$\eta_A : F(A) \to G(A)$$

to each object A in $\mathbf{C}$ such that the following diagram commutes for $f : A \to B$ in $\mathbf{C}$

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

If η_A is an isomorphism for each A then η is a *natural isomorphism*.

Definition 8. As defined in [6], there exists an *equivalence* between categories $\mathbf{C}$ and $\mathbf{D}$ if there exist functors:

$$\begin{aligned} F &: \mathbf{C} \to \mathbf{D} \\ G &: \mathbf{D} \to \mathbf{C} \end{aligned}$$

and natural isomorphisms:

$$\eta : G \circ F \to 1_{\mathbf{C}}$$
$$\varepsilon : F \circ G \to 1_{\mathbf{D}}.$$

Proposition 1. There is an equivalence of categories between ${}_{\mathbb{Z}}\mathbf{Mod}$ and $\mathbf{Ab}$.

Proof. We first define the functor $F : {}_{\mathbb{Z}}\mathbf{Mod} \to \mathbf{Ab}$ that takes a left $\mathbb{Z}$-module to its underlying abelian group, forgetting the $\mathbb{Z}$-action, and behaves similarly for morphisms. Next we define $G : \mathbf{Ab} \to {}_{\mathbb{Z}}\mathbf{Mod}$ which endows an abelian group H with a $\mathbb{Z}$-action. We consider $(H, +)$ additively and then the Z-action is defined as follows:

$$\mathbb{Z} \times H \to H$$
$$(n, h) \mapsto \begin{cases} \sum_{i=1}^{n} h, & n > 0 \\ \sum_{i=1}^{-n} h, & n < 0 \\ 0, & \text{otherwise} \end{cases}$$

We observe that both $G \circ F = 1_{\mathbf{Ab}}$ and $F \circ G = 1_{{}_{\mathbb{Z}}\mathbf{Mod}}$. We note that equal functors are naturally isomorphic and thus conclude that ${}_{\mathbb{Z}}\mathbf{Mod} \cong \mathbf{Ab}$. □

1.3 Diagrams

We now introduce a piece of categorical formalism that is extremely useful. Diagrams are an excellent tool to make statements categorically. To the uninitiated it can seem both very strange and very difficult to make statements only in terms of objects and morphisms. Diagrams will make this process more intuitive and accessible.

Definition 9. A category $\mathbf{C}$ is *small* if $\mathbf{Ob(C)}$ is a set.[2]

Definition 10. Let $\mathbf{C}$ be a category and $\mathbf{I}$ be a small category. Then a *diagram in* $\mathbf{C}$ is a functor $F : \mathbf{I} \to \mathbf{C}$. The category $\mathbf{I}$ can be referred to as an "index category" and the resulting diagram an "$\mathbf{I}$-shaped diagram". The reader is welcome to think of the diagram as the image of the functor F.

When depicting a diagram the identity arrows and composition arrows are often omitted in order to make them easier to read. A diagram is said to commute if every path between every pair of objects is equivalent.

Example 6. We follow the convention of [2], the symbol "•" will be used as an anonymous placeholder and each one below should be considered a distinct object.

[2]Some authors also impose the condition that $\mathrm{Hom}(A, B)$ is a set but this follows from Definition 1

1.
$$\begin{tikzcd} \bullet \arrow[r] \arrow[dr] & \bullet \arrow[d] \\ & \bullet \end{tikzcd}$$

2.
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

3. Associativity can be restated in terms of a commutative diagram. It is equivalent to the following diagram commuting for all objects A, B, C, D and morphisms f, g, h in a category:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] & B \arrow[r, "g"] \arrow[rr, bend right, "h \circ g"'] & C \arrow[r, "h"] & D \end{tikzcd}$$

4. A **Z**-shaped diagram:

$$\begin{tikzcd} \dots \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \dots \end{tikzcd}$$

these will be discussed in later chapters.

Diagrams are extremely useful for reformulating definitions categorically. Consider the Cartesian product in **Sets**. For two sets X, Y we get maps

$$\begin{aligned} p_X : X \times Y &\to X \\ (x, y) &\mapsto x \\ p_Y : X \times Y &\to Y \\ (x, y) &\mapsto y \end{aligned}$$

with an interesting property. Suppose there is a set A, two maps $f_X : A \to X$ and $f_Y : A \to X$ then there is a unique map $p : A \to X \times Y$ such that $f_X = p_X \circ p$ and $f_Y = p_Y \circ p$. This is the defining property of a product and can be summarised in the following commutative diagram:

$$\begin{tikzcd}
& A \arrow[d, dotted, "\exists ! p"] \arrow[ddl, bend right, "f_X"'] \arrow[ddr, bend left, "f_Y"] & \\
& X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\
X & & Y
\end{tikzcd}$$

We can generalise this to a product of more than two objects, suppose we have a family of objects $\{X_i\}_{i\in I}$ and maps $\{f_i : A \to X_i\}_{i\in I}$ for some indexing set I and some object A. Then the following triangle commutes for all i simultaneously:

$$\begin{tikzcd}
A \arrow[d, dashed, "\exists ! p"] \arrow[dr, "f_i"] & \\
\prod\limits_{i\in I} X_i \arrow[r, "p_i"'] & X_i
\end{tikzcd}$$

A category $\mathbf{C}$ is said to have products if we can produce the product of any set of objects over any index set. Products are unique up to isomorphism.

It is often fruitful to examine a definition that stems from a diagram and take its dual, that is to reverse all the arrows. In this case we get the coproduct.

Suppose we have a family of objects $\{X_j\}_{j\in J}$ and maps $\{f_j : X_j \to A\}_{j\in J}$ for some indexing set J and some object A. Then the following triangle commutes for all j simultaneously:

$$\begin{tikzcd}
X_j \arrow[d, "i_j"'] \arrow[dr, "f_j"] & \\
\coprod\limits_{j\in J} X_j \arrow[r, dashed, "\exists ! i"'] & A
\end{tikzcd}$$

1.4 Limits and Colimits

The product of two objects is the special case of a categorical notion called a limit (and similarly the coproduct is an instance of a colimit). Limits and colimits will allow us to define some familiar concepts that arise in **Ab** categorically (that is using only the language of objects and morphisms). Limits combine two ideas: a universal object and a cone.

Suppose we have a category $\mathbf{C}$ we are interested in and a small index category $\mathbf{I}$ and we would like some understanding of the diagram that arises from a functor

$F : \mathbf{I} \to \mathbf{C}$. We can fix an object X in $\mathbf{C}$ and consider the constant functor $T : \mathbf{I} \to \mathbf{C}$ that maps all objects in $\mathbf{I}$ to X and all morphisms to 1_X. Finally we can consider a natural transformation $\eta : T \to F$. This gives us a tuple $(X, \{\eta_i\}_{i\in I})$ containing an object X and a family of morphisms $\eta_i : X \to F(i)$ for each $i \in \mathbf{I}$, such a tuple is a cone over the diagram given by F.

Example 7. Suppose we have the diagram

$$\bullet \qquad\qquad \bullet$$

then a cone over it is a tuple $(X, \{\eta_1, \eta_2\})$:

$$\begin{tikzcd}
 & X \arrow[dl, "\eta_1"'] \arrow[dr, "\eta_2"] & \\
\bullet & & \bullet
\end{tikzcd}$$

Thus a product is cone over the above diagram.

Definition 11. Consider a category $\mathbf{C}$ and a small category $\mathbf{I}$ and a functor $F : \mathbf{I} \to \mathbf{C}$. Then a *cone over the diagram arising from F* is a tuple $(X, \{\eta_i\}_{i\in I})$ where X is an object in $\mathbf{C}$ and the $\eta_i : X \to F(i)$ are morphisms in $\mathbf{C}$ for each $i \in \mathbf{I}$.

In a general category and for a general diagram cones need not exist, but it is also possible there are many as in the following diagram:

$$\begin{tikzcd}
 & & & & \bullet \\
\dots & C_3 \arrow[urrr, bend left] \arrow[drrr, bend right] & C_2 \arrow[urr, bend left=10] \arrow[drr, bend right=10] & C_1 \arrow[ur] \arrow[dr] & \\
 & & & & \bullet
\end{tikzcd}$$

Definition 12. Consider a category $\mathbf{C}$, a small category $\mathbf{I}$, a functor $F : \mathbf{I} \to \mathbf{C}$ and the diagram arising from F. The *limit* of the diagram is the universal cone. It is a cone $(X, \{\eta_i\}_{i\in I})$ such that for any other cone $(X', \{\eta'_i\}_{i\in I})$ there exists a unique morphism $\pi : X' \to X$ such that the following triangle commutes for all i simultaneously:

$$\begin{tikzcd}
X' \arrow[r, dashed, "\exists!\pi"] \arrow[dr, "\eta'_i"'] & X \arrow[d, "\eta_i"] \\
 & i
\end{tikzcd}$$

Thus the limit of the diagram

$$A \qquad\qquad B$$

is the product $A \prod B$. Before going over some examples we will define the dual notions of cocone and colimit.

Definition 13. Consider a category $\mathbf{C}$ and a small category $\mathbf{I}$ and a functor $F : \mathbf{I} \to \mathbf{C}$. Then a *cone under the diagram arising from* F or a *cocone* is a tuple $(X, \{\eta_i\}_{i \in I})$ where X is an object in $\mathbf{C}$ and the $\eta_i : F(i) \to X$ are morphisms in $\mathbf{C}$ for each $i \in \mathbf{I}$.

Definition 14. Consider a category $\mathbf{C}$, a small category $\mathbf{I}$, a functor $F : \mathbf{I} \to \mathbf{C}$ and the diagram arising from F. The *colimit* of the diagram is the universal cocone. It is a cocone $(X, \{\eta_i\}_{i \in I})$ such that for any other cocone $(X', \{\eta'_i\}_{i \in I})$ there exists a unique morphism $\pi : X \to X'$ such that the following triangle commutes for all i simultaneously:

$$\begin{tikzcd} X' & X \arrow[l, dashed, "\exists!\pi"'] \\ & i \arrow[ul, "\eta'_i"] \arrow[u, "\eta_i"'] \end{tikzcd}$$

Proposition 2. Limits, if they exist, are unique up to isomorphism.

Proof. See section 1 of chapter 3 of [10] □

Limits and colimits show up throughout category theory and have many uses, but we have a very specific purpose for them. In category theory we are only supposed to discuss things in terms of objects and morphisms but for homological algebra we want to make statements about images and kernels of maps. Later a specific limit will come to our aid, called an equalizer.

Example 8. Consider two abelian groups A, B and two group homomorphisms resulting in the following diagram:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Then we want an object C that has the property that for any map $h : X \to A$ such that $f \circ h = g \circ h$ will factor uniquely through C. The limit of this diagram will be $C = \{a \in A |\ f(a) = g(a)\} \subseteq A$. This is clearly an abelian group since homomorphisms preserve the identity and:

$$\begin{aligned} f(a) = g(a) &\implies f(a^{-1}) = g(a^{-1}) \\ (f(a) = g(a)) \cap (f(b) = g(b)) &\implies f(ab) = g(ab) \end{aligned}$$

and the associativity is inherited. Thus the limit exists in $\mathbf{Ab}$. The colimit of the diagram requires a bit more unpacking. We want a pair (D, ϕ) such that

$\phi \circ f = \phi \circ g$ and is universal with this property. The following must commute:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "\phi"] \arrow[dr, "\phi'"'] & D \arrow[d, dashed, "\exists!\hat{\phi}"] \\ & & D' \end{tikzcd}$$

In D it must hold that $(\phi \circ f) - (\phi \circ g) = 1_D$. Thus:

$$D = B \big/ \mathrm{im}(f - g)$$

and ϕ is the canonical map.

Definition 15. Consider the diagram:

$$\bullet \rightrightarrows \bullet$$

Its limit is called an *equalizer* and its colimit is called a *coequalizer*.

Clever use of the equalizer and coequalizer will allow us to generalise notions that do not have an obvious categorical analogue.

1.5 Additive Categories

On our path to developing ideas from homological algebra we need to introduce the notion of an additive category and strengthen it to that of an abelian category. Our motivating example will be **Ab**, the prototypical abelian category.

Consider the category $\mathbf{Vect}_k$ for a fixed field k. For any two objects V, W we consider $\mathrm{Hom}(V, W)$ and notice:

$$(f + \alpha g) \in hom(V, W) \quad \forall\, f, g \in hom(V, W);\ \alpha \in k$$

and therefore $\mathrm{Hom}(V, W)$ is a vector space itself.

There are many examples where the hom-sets in a category are in fact objects in another category that has more structure than **Sets**.

Example 9.

- In the category **Ab** we consider $\mathrm{Hom}(A, B)$ for two objects A, B. Then we note $f + g \in hom(A, B)$ for $f, g \in hom(A, B)$. This addition will be associative and commutative since it inherits from the operation defined for B. The identity of $\mathrm{Hom}(A, B)$ will be the trivial homomorphism and for a homomorphism $f : A \to B$ its additive inverse in $\mathrm{Hom}(A, B)$ is:

  $$f^{-1} : A \to B \quad \text{such that} \quad f^{-1}(a) = f(-a) = -f(a),$$

  thus $\mathrm{Hom}(A, B)$ is an abelian group for all A, B in **Ab**.

- Fix a unital commutative ring R and consider the category ${}_R\mathbf{Mod}$ of left $R-$modules. We consider $\mathrm{Hom}(M, N)$ for two objects and note that set has the structure of an abelian group (similarly to above). An $R-$action can be defined:

$$\begin{aligned} R \times hom(M, N) &\to hom(M, N) \\ (r, \phi) &\mapsto r\phi \end{aligned}$$

Thus we conclude that $\mathrm{Hom}(M, N)$ is a left $R-$module. Sometimes it will be convenient to forget the $R-$module structure of $\mathrm{Hom}(M, N)$ and regard it simply as an abelian group.

Definition 16 ([3], Section 1.2)**.** A category $\mathbf{C}$ is *additive* if:

- the hom-sets of $\mathbf{C}$ are objects in $\mathbf{Ab}$,
- composition is bilinear, that is for $f, f' \in hom(A, B)$ and $g, g' \in hom(B, C)$

$$g \circ (f + f') = g \circ f + g \circ f' \quad \text{and} \quad (g + g') \circ f = g \circ f + g' \circ f,$$

- $\mathbf{C}$ has a zero object 0 such that $\mathrm{Hom}(0, 0) = \{0\}$ the trivial group,
- and for any objects X_1, X_2 there exists an object Y and morphisms $p_1 : Y \to X_1$, $p_2 : Y \to X_2$, $i_1 : X_1 \to Y$ and $i_2 : X_2 \to Y$ satisfying $p_1 \circ i_1 = 1_{X_1}$, $p_2 \circ i_2 = 1_{X_2}$ and $i_1 \circ p_1 + i_2 \circ p_2 = 1_Y$. This object Y is called the *biproduct* of X_1, X_2 and is both their product and coproduct.

We note that the biproduct is unique up to isomorphism as it is both a product and a coproduct by Proposition 2. The natural functor for an additive category is one that behaves like a group homomorphism between hom-sets. Hence:

Definition 17. Let $\mathbf{C}$ and $\mathbf{D}$ be additive categories. Then a functor $F : \mathbf{C} \to \mathbf{D}$ of any variance is an *additive functor* if for any pair of morphisms $f, g : A \to B$ in $\mathbf{C}$ it holds that:

$$F(f + g) = F(f) + F(g)$$

and thus F does induce a group homomorphism:

$$\begin{aligned} F_{AB} &: \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{D}}(F(A), F(B)) \quad \text{if } F \text{ is covariant} \\ F_{BA} &: \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{D}}(F(B), F(A)) \quad \text{if } F \text{ is contravariant.} \end{aligned}$$

Additive functors are compatible with the structure of additive categories and this will be explored more in a later chapter. We consider two objects A, B in some additive category $\mathbf{C}$. Their biproduct comes equipped with the following maps:

$$\begin{tikzcd}
& A \times_{\mathbf{C}} B \arrow[dl, bend left, "p_A"] \arrow[dr, bend right, "p_B"'] & \\
A \arrow[ur, bend left, "i_A"] & & B \arrow[ul, bend right, "i_B"']
\end{tikzcd}$$

such that:

$$p_A \circ i_A = 1_A, \tag{1.1}$$
$$p_B \circ i_B = 1_B, \tag{1.2}$$

and

$$i_A \circ p_A + i_B \circ p_B = 1_{A\times B}. \tag{1.3}$$

Let $\mathbf{D}$ be an additive category, then lets apply an additive functor $F : \mathbf{C} \to \mathbf{D}$ to the above diagram:

$$\begin{tikzcd}
& F(A \times_{\mathbf{C}} B) \arrow[dl, bend left, "F(p_A)"] \arrow[dr, bend right, "F(p_B)"'] & \\
F(A) \arrow[ur, bend left, "F(i_A)"] & & F(B) \arrow[ul, bend right, "F(i_B)"']
\end{tikzcd}$$

We note that F respects the equations (1.1), (1.2), (1.3) thus by the uniqueness of products (and coproducts) it holds that:

$$F(A \times_{\mathbf{C}} B) \cong F(A) \times_{\mathbf{D}} F(B).$$

We have just proved:

Proposition 3. Let $\mathbf{C}$, $\mathbf{D}$ be additive categories and $A, B \in \mathbf{C}$. Let $F : \mathbf{A} \to \mathbf{B}$ be an additive functor then:

$$F(A \times_{\mathbf{C}} B) \cong F(A) \times_{\mathbf{D}} F(B).$$

Let A and B be objects in $\mathbf{Ab}$ and $f : A \to B$. There are 4 objects in $\mathbf{Ab}$ that describe f:

- $\ker(f) = \{a \in A |\ f(a) = 1_B\}$
- $\mathrm{im}(f) = \{b \in B |\ \exists a \in A : b = f(a)\}$
- $\mathrm{coker}(f) = B \big/ \mathrm{im}(f)$
- $\mathrm{coim}(f) = A \big/ \ker(f)$

and we note that f can be factored as follows:

$$A \xrightarrow{\quad \phi \quad} \mathrm{coim}(f) \xrightarrow{\overline{f}} \mathrm{im}(f) \xrightarrow{\quad i \quad} B$$

where ϕ is the natural map, $\overline{f}$ is an isomorphism and i is the inclusion. We would like to be able to define analogous objects in a general additive category.

Let $\mathbf{C}$ be an additive category and consider its zero object 0. We consider the following diagram:

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \\ & 0 \arrow[ur, "g"'] & \end{tikzcd}$$

The map $h = g \circ f$ is the zero map from A to B, we shall abuse notation and write:

$$A \xrightarrow{0} B$$

The zero map is also the identity of the group $\mathrm{Hom}_{\mathbf{C}}(A, B)$ for an additive category $\mathbf{C}$ and thus is preserved by additive functors.

We return to $\mathbf{Ab}$ and consider $f : A \to B$ for objects A, B. Consider the following diagram:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \end{tikzcd}$$

then its equalizer is exactly the kernel of f! Similarly, its coequalizer is the cokernel of f:

$$\begin{tikzcd} \ker(f) \arrow[r, "i"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \arrow[r, "\phi"] & \mathrm{coker}(f) \end{tikzcd}$$

Now we consider:

$$\begin{tikzcd} B \arrow[r, shift left, "\phi"] \arrow[r, shift right, "0"'] & \mathrm{coker}(f) \end{tikzcd}$$

and we see that the kernel of ϕ is $\mathrm{im}(f)$. Finally, consider:

$$\begin{tikzcd} \ker(f) \arrow[r, shift left, "i"] \arrow[r, shift right, "0"'] & A \end{tikzcd}$$

and observe that the cokernel of i is $\mathrm{coim}(f)$. We recall that f is injective if $\ker(f)$ is trivial. The kernel measures how far from an injection f is. Dually, f is surjective if $\mathrm{coker}(f)$ is trivial, thus the cokernel of f measures how far f is from being a surjection. It appears we have found a purely categorical way to define our desired objects in a general additive category. Thus we shall proceed:

Definition 18. Let $\mathbf{C}$ be an additive category, $f : A \to B$ be a map between two objects and consider the diagram:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \end{tikzcd}$$

Then:

- If the limit exists, it is the kernel of f,
- if the colimit exists, it is the cokernel of f,

- if it exists, the kernel of the cokernel is the image of f,
- and if it exists, the cokernel of the kernel is the coimage of f.

The reader will notice that in general the above limits and colimits need not exist, and we do not wish to be hamstrung by such a predicament. This leads us to our next section.

1.6 Abelian Categories

These are categories that behave like **Ab** and will mimic many of their properties. The axioms of an abelian category ensure that the basic objects of homological algebra are well defined and exist. In Chapter 2 it will be seen that homologies and cohomologies are additive functors from a particular abelian category to the category of abelian groups.

Definition 19 (Section 1.4, [6])**.** Let **C** be an additive category. Then **C** is an *abelian category* if and only if:

Ab 1) Any morphism admits a kernel and a cokernel.

Ab 2) All morphisms $f : A \to B$ in **C** can be decomposed into:

$$A \xrightarrow{\quad\phi\quad} \mathrm{coim}(f) \xrightarrow{\overline{f}} \mathrm{im}(f) \xrightarrow{\quad i \quad} B$$

where ϕ is an epimorphism, $\overline{f}$ is an isomorphism and i is a monomorphism.

Immediately we see that Ab 1 dispatches our existence issues by assumption.

Example 10 (Example 1.1.3 in [3])**.** The reader is likely familiar with some examples of abelian categories:

- The category **Ab** is clearly abelian (given that the axioms aim to capture its properties),
- $\mathbf{Vect}_k$ for a field k,
- ${}_R\mathbf{Mod}$ for an associative unital ring R and more generally the category of modules over an associative k-algebra for a field k.

Definition 20. Let **C** be an abelian category and A an object. A *subobject* of A is a pair consisting of an object X and a monomorphism $i : X \to A$ that is its inclusion into A. Dually, a *quotient object* of A is a pair consisting of an object Y and an epimorphism $\phi : A \to Y$ called its canonical map.

Often we will abuse notation and refer to the object X as the subobject and Y as the quotient object.

Proposition 4. Let **C** be an abelian category and A an object. Then subobjects of A are in one-to-one correspondence with quotient objects of A.

Proof. Let X be a subobject of A. Then there exists an inclusion $i : X \to A$ and this admits a cokernel, thus each subobject yields a quotient object. Next suppose Y is a quotient object of A thus there exists a canonical map $\phi : A \to Y$ and this admits a kernel, thus each quotient object yields a subobject. □

Intuitively this result tells you that a valid quotient arises out of any subobject. This fact can be used constructively for example to produce the tensor product in **Ab** and to define homology groups. We conclude this chapter with a very powerful result.

1.7 Abelian categories are modules over some ring

Theorem 1 (Mitchell's embedding theorem; [3])**.** Every abelian category is equivalent to a full subcategory of the category of left modules over an associative unital ring R.

Proof. The proof of this theorem is beyond the scope of this report and we refer the reader to [5]. □

This theorem has both practical and theoretical value. Theoretically it allows us think of all abelian categories as categories of modules. Practically the theorem lets us exploit the categorical structure of modules for proofs. In concrete terms we will be able to make use of the elements of modules and perform diagram-chase like proofs, as in Lemma 8. This theorem does have limitations: for example let **C** be an arbitrary abelian category and **D** be the category of modules it embeds into, then the projective and injective objects in **C** do not necessarily correspond to projective and injective modules in **D**. There is also a set theoretic issue as the original statement requires our abelian category **C** to be small, however all of these issues are beyond the scope of this report and in fact all of our concrete examples will be the category of modules over some ring.

We benefit from Theorem 1 immediately. We would like to generalise the fact that vanishing of the kernel implies injectivity and vanishing of the cokernel implies surjectivity, we proceed:

Lemma 2. Let **C** be an abelian category then:

1. A morphism $f : A \to B$ is a monomorphism if and only if its kernel is trivial.
2. A morphism $g : A \to B$ is a epimorphism if and only if its cokernel is trivial.

Proof. We apply Theorem 1 and consider A, B to be left modules over a ring R. We recall that a module has an underlying abelian group and that an R-homomorphism is simply a group homomorphism that respects the R-action. Thus our proof will be similar to the approach used for groups:

1. We will prove that $f : A \to B$ is an injection if and only if $\ker(f)$ is trivial and this will yield the desired result.
   First we suppose $\ker(f)$ is trivial and $\exists x, y \in A : \; f(x) = f(y)$ then:
   $$0_B = f(x) - f(y) = f(x - y) = f(0_A) \implies x = y$$
   and thus f is injective.
   Next we suppose f is injective and consider $x \in \ker(f)$ Then:
   $$f(a) + f(x) = f(a + x) = f(a) \implies x = 0_A \implies \ker(f) = \{0\}$$
2. We shall proceed with a similar approach, proving $f : A \to B$ is a surjection if and only if $\mathrm{coker}(f)$ is trivial. We argue directly from the definition of $\mathrm{coker}(f) = B \big/ \mathrm{im}(f)$. Then we observe that:
   $$\mathrm{coker}(f) = \{0\} \iff \mathrm{im}(f) = B \iff f \text{ is surjective.}$$

□

Lemma 3. Let $f : A \to B$ be a morphism in an abelian category $\mathbf{C}$. Then f is an isomorphism if and only if it is a monomorphism and an epimorphism.

Proof. We apply Theorem 1 and consider A, B to be left modules over a ring R. We then note that in ${}_R\mathbf{Mod}$ monomorphisms are exactly injections and epimorphisms are exactly surjections. Then we observe that an isomorphism is exactly a bijective R-homomorphism. Thus f is an isomorphism if and only if it is both monic and epic. □

This concludes our study of what has colloquially been called "general abstract nonsense". In the next chapter we will define some more concrete structures as well as the basic ideas of homological algebra.

Chapter 2

Some Homological Algebra

We have developed enough theory to start discussing the tools of homological algebra. Homological algebra takes place inside abelian categories, and thus we shall restrict our attention to categories of modules over a ring R. We will begin with a brief study of modules, constructing their tensor product and discussing some of its properties. Then exact sequences, chain complexes and homologies will be defined and discussed. We finish with the duals of the above notions. Our treatment is inspired by chapter 2 of [12] and chapter 1 of [9].

2.1 Constructing the tensor product of modules

There are many questions that could lead one to discover the tensor product. First, suppose we have a ring R a subring $S \subseteq R$. One might be curious when a module ${}_SM$ can be regarded as a left R-module. One might also try to avoid biadditive (or multiadditive) maps and desire to only work with R-homomorphisms, thus one seeks an object Y such that:

$$\begin{tikzcd} A \oplus B \arrow[r, "\phi"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "\exists! \hat{f}"] \\ & G \end{tikzcd}$$

commutes. The tensor product is also important in a purely homological context where they are used in the Künneth formulas, however this is beyond our scope and we refer the reader to section 10.10 in [12]. Before we are able to answer the above questions some theory will need to be developed.

Definition 21. Let R be a ring and $A_R, {}_RB, {}_RM$ be modules. Then a map $f : A \oplus B \to M$ is called *R-biadditive* if for all $a, a' \in A, b, b' \in B$ and $r \in R$ it satisfies:

$$\begin{aligned} f(a+a',b) &= f(a,b)+f(a',b)\\ f(a,b+b') &= f(a,b)+f(a,b')\\ f(ar,b) &= f(a,rb). \end{aligned}$$

Definition 22. Let R be a ring and S a left R-module. The module S is *cyclic* if $\exists g \in S$ such that $\forall s \in S\ \exists r \in R : s = rg$. We write $S = \langle g \rangle = \{rg | r \in R\}$ and call $\langle g \rangle$ the *span* of g. Analogously for any subset X of a left R-module M for some ring R we can define:

$$\langle X \rangle = \left\{ \sum_{x_i \in X} r_i x_i | r_i \in R, \text{ all but finitely many of them equal to } 0_R \right\}$$

and we call $\langle X \rangle$ the *submodule generated by* X. We say that X *generates* M if $\langle X \rangle \cong M$.

We denote by $\oplus$ the biproduct of modules and we note that in the product of infinitely many modules we restrict our sums to be finite to avoid questions about convergence.

Definition 23. A left R-module F is a *free* left R-module if it is isomorphic to a product of copies of R, that is to say there is a (possibly infinite) indexing set B such that:

$$F \cong \bigoplus_{b \in B} R.$$

The set B is called a *basis* of F. Free right and bimodules can be defined analogously.

Example 11. For any field k all left k-modules (which are just vector spaces) are free.

We note that for any set B there exists a free module with basis B.

Lemma 4. Let R be a ring, F be a free left R-module with basis B and M a left R-module. Then for a function $f : B \to M$ there exists a unique R-homomorphism $\hat{f} : F \to M$ such that:

$$\begin{tikzcd} F \arrow[dr, dashed, "\exists ! \hat{f}"] & \\ B \arrow[u, "i"] \arrow[r, "f"'] & M \end{tikzcd}$$

commute where $i : B \to F$ is the inclusion.

Proof. Any element $v \in F$ has a unique expression of the form:

$$v = \sum_{b \in B} r_b b, \quad \text{for } r_b \in R \text{ and all but finitely many of them equal to } 0_R.$$

Then:

$$\hat{f}(v) = \sum_{b \in B} \hat{f}(r_b b) = \sum_{b \in B} r_b f(b)$$

is an R-homomorphism. It is unique since it is the only function that properly respects scalar multiplication by R. □

Theorem 5. Let R be a ring. Every left R-module M is a quotient of a free left R-module F.

Proof. Choose a generating set X for M and let F be the free left R-module with basis X. Then by Lemma 4 there exists an R-homomorphism $\hat{f} : F \to M$ that with $\hat{f}(x) = x \ \forall x \in X$. It holds that $\hat{f}$ is a surjection since $\mathrm{im}(\hat{f}) = \langle X \rangle \cong M$ since X generates M by assumption. □

This result is pleasant in the sense that we are able to uniquely specify any left R-module in terms of generators and relations. We will use this fact to construct the tensor product.

Definition 24. Let R be a ring and A_R, ${}_RB$ be modules. Then their *tensor product* is an R-module $A \otimes_R B$ and an R-biadditive map:

$$\phi : A \oplus B \to A \otimes_R B$$

such that for every R-module M and every R-biadditive map $f : A \oplus B \to M$ there exists a unique R-homomorphism $\hat{f}$ such that the following diagram commutes:

$$\begin{tikzcd} A \oplus B \arrow[r, "\phi"] \arrow[dr, "f"'] & A \otimes_R B \arrow[d, dashed, "\exists! \hat{f}"] \\ & M \end{tikzcd} \tag{2.1}$$

Now we prove their uniqueness:

Theorem 6. Let R be a ring and $A_R, {}_RB$ be modules and suppose there are two tensor products T, T' of A and B. Then $T \cong T'$.

Proof. We proceed as in [8]: first we note that given an R-biadditve map $f : A \oplus B \to M$ there exists a unique R-homomorphism $\hat{f}$ such that the diagram 2.1 commutes. Now we consider the commutative diagram:

$$\begin{tikzcd} & A \oplus B \arrow[dl, "\phi"'] \arrow[d, "\phi'"] \arrow[dr, "\phi"] & \\ T \arrow[r, "j"'] & T' \arrow[r, "j'"'] & T \end{tikzcd}$$

and since ϕ' is biadditive we get a unique R-homomorphism $j : T \to T'$, similarly we also get a unique R-homomorphism $j' : T' \to T$. Since the the diagram commutes we have that $\phi = (j' \circ j) \circ \phi$. By the uniqueness of ϕ we have that $(j' \circ j) = 1_T$, a similar diagram shows that $(j \circ j') = 1_{T'}$ and hence $T \cong T'$. We conclude the the tensor product is unique up to isomorphism. $\square$

We have now earned the right to speak of *the* tensor product of two modules. Now we need to prove that it exists:

Theorem 7. Let R be a ring and A_R, ${}_RB$ be modules. Then their tensor product exists.

Proof. We consider the free R-module F with basis $A \times B$ - the set. We produce the submodule N generated by elements of the form:

$$(a + a', b) - (a, b) - (a', b) \tag{2.2}$$
$$(a, b + b') - (a, b) - (a, b') \tag{2.3}$$
$$(ar, b) - (a, rb) \tag{2.4}$$

for $a, a' \in A$, $b, b' \in B$ and $r \in R$. Then the quotient ${}^F/_N$ is the tensor product. We have the following commutative triangle:

$$\begin{tikzcd}
F \arrow[d, "f"'] \arrow[dr, "\pi"] & \\
A \oplus B \arrow[r, "\phi"'] & {}^F/_N
\end{tikzcd}$$

where:

- $\pi : F \to {}^F/_N$ is the natural map,
- $f : F \to A \oplus B$ is the unique extension of the map that sends $(a, 0_B) \mapsto (a, 0_B)$ for all $a \in A$ and $(0_A, b) \mapsto (0_A, b)$ for all $b \in B$,
- $\phi : A \oplus B \to {}^F/_N = A \otimes_R B$ that maps $(a, b) \mapsto a \otimes b$ and is R-biadditive. This map is a surjection, and it is unique by the uniqueness of f and π.

The quotient satisfies the required properties of the tensor product because of the relations imposed by the generators of N. $\square$

Let us unpack this. First we note that no checks had to be made to form the quotient, this is due to Proposition 4. The simplest example concerns vector spaces, thus let k be a field and consider two vector spaces V, W over k with bases $\{e_i\}_{i=1}^n$ and $\{a_j\}_{j=1}^m$ respectively. Then $V \otimes_k W$ is a k-vector space of dimension $n \times m$, we denote it:

$$V \otimes_k W = \left\{ \sum_{i,j} r_{i,j} e_i \otimes a_j | r_{i,j} \in \mathbb{R} \text{ all but finitely many of them equal to } 0_k \right\}.$$

For $v, v' \in V, w, w' \in W$ and $r \in k$ we note the following:

$$(v + v')\otimes w = v\otimes w + v'\otimes w \quad \text{by } 2.2,$$
$$v\otimes(w + w') = v\otimes w + v\otimes w' \quad \text{by } 2.3,$$
$$(vr)\otimes w = v\otimes(rw) \quad \text{by } 2.4.$$

For a module over an arbitrary ring the above relations hold by definition but we cannot make a statement about their basis since in general only free modules have bases. The subscript k of the tensor product denotes that we can move scalars from k across the product. The relations ensure that R-homomorphisms out of the tensor product behave like R-biadditive maps out of the direct product. Something interesting has happened here, we have managed to "hide" the complexity of biadditive maps inside of the tensor product of the modules involved. It turns out it is often more convenient to have complicated objects and simple morphisms, as it is often the morphisms that are being attended to. We are now equipped to answer our question about modules over subrings. Let R be a ring and $S \subseteq R$ a subring. Then suppose we have modules ${}_RA_S, {}_SB$ and we wish to regard ${}_SB$ as a left R-module. We form the tensor product:

$${}_RA \otimes_S B$$

and note that this is equipped with a left R-action. Then we recall that R can be regarded as an (R, S)-bimodule and thus we can always form:

$$R \otimes_S B$$

and this will have a left R-module structure.

Next we do some housekeeping:

Proposition 5. Let R be a ring $A_R, {}_RB_R, {}_RC$ be modules. Then:

$$(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C),$$

that is to say the tensor product is associative up to isomorphism.

Proof. See proposition 2.57 in Section 2.2 of [12]. □

Example 12. The tensor product can be a bit more complex than initially meets the eye.

- Let R be a ring and ${}_RM$ a module, then $R \otimes_R M \cong M$, this is due to the fact that:

  $$r\otimes m = 1\otimes rm$$

  thus we produce the isomorphism:

  $$\phi : R \otimes_R M \to M$$
  $$r\otimes m \mapsto rm.$$

- We consider the left $\mathbb{Z}$-modules ${}^{\mathbb{Z}}/_{2\mathbb{Z}}$ and ${}^{\mathbb{Z}}/_{3\mathbb{Z}}$. We note that multiplication by 2 produces an automorphism of ${}^{\mathbb{Z}}/_{3\mathbb{Z}}$ and consider:

  $$ {}^{\mathbb{Z}}/_{2\mathbb{Z}} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}/_{3\mathbb{Z}} = {}^{\mathbb{Z}}/_{2\mathbb{Z}} \otimes_{\mathbb{Z}} 2\left({}^{\mathbb{Z}}/_{3\mathbb{Z}}\right) = \left({}^{\mathbb{Z}}/_{2\mathbb{Z}}\right) 2 \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}/_{3\mathbb{Z}} = \{0\} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}/_{3\mathbb{Z}} = \{0\} $$

  where $\{0\}$ denotes the trivial group.

We recall that by Proposition 1 that our definition applies to abelian groups. We shall now introduce a generalisation of the tensor product of modules that will be necessary when we study Hochschild (co)homology in Chapter 3.

Definition 25. Let R be a commutative ring. A left R-module A is a *unital algebra* over R if it is a unital (possibly non-associative) ring in its own right and its multiplication is compatible with the R-action. That is to say for $r \in R, a, b \in A$ we have:

$$ r(ab) = (ra)b = a(rb). $$

All algebras in this report shall be unital and thus we shall drop the term. It is notable that the definition does not demand associativity and indeed in Chapter 4 we will see a non-associative algebra. We observe that the definition of an algebra can be shown to imply that $R \subseteq A$. The challenge is how to define (ar) for $r \in R$ and $a \in A$ since A is a left R-module. We say that $(ar) = (ra)$ and note that multiplication in R is compatible with multiplication in A by definition. By the axioms of a ring we have that R is closed under multiplication and subtraction. We conclude that not only is R a subring of A but it is actually a subring of $Z(A)$ the centre of A. This gives us the right to consider the ring homomorphism $i_R^{Z(A)} : R \to A$ which is the inclusion of R into $Z(A)$.

Example 13. Let R be a commutative ring:

- the polynomial ring $R[x]$ is an R-algebra. For any R-algebra A polynomials over A also form an R-algebra.

- left free R-modules can be equipped with element-wise multiplication and then can be regarded as an R-algebra.

- $n \times n$ matrices with entries in R have a ring structure compatible with an action of R and thus also can be regarded as an R-algebra.

We now extend the tensor product to algebras:

Definition 26. Let R be a commutative ring and A, B be R algebras. Their tensor product of modules $A \otimes_R B$ can be endowed with the multiplication rule:

$$ (a \otimes b)(a' \otimes b) = (aa') \otimes (bb') $$

and thus can be regarded as a module. This new tensor product is the *tensor product of algebras*.

In Chapter 3 we shall be concerned with tensor powers of algebras. We fix a ring R and consider an R-algebra A. Then:

$$A^{\otimes_R n} = A \otimes_R A \otimes_R \cdots \otimes_R A \text{ with } n \text{ factors.}$$

We also note that any R-algebra A raised to the tensor power of 0 returns R.

2.2 Exact sequences and chain complexes

The following all applies in a general abelian category but this report is only concerned with categories of modules and thus we fix a ring R and consider ${}_R\mathbf{Mod}$. Here we introduce the primary tools of homological algebra.

Definition 27. Let $\{C_n\}_{n\in\mathbb{Z}}$ be a set of modules in ${}_R\mathbf{Mod}$. The sequence bellow is *exact* if $\ker(\partial_n) = \operatorname{im}(\partial_{n+1})$

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

for all $n \in \mathbb{Z}$. Let A, B, C be 3 objects in ${}_R\mathbf{Mod}$. The following exact sequence is called a *short exact sequence*:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

The object B is called an *extension* of A by C.

It is immediate that $\partial_n \circ \partial_{n+1} = 0$ for all n. This condition will turn out to be extraordinarily important.

Proposition 6. We consider objects A, B in an abelian category $\mathbf{C}$:

1. A sequence $0 \to A \xrightarrow{f} B$ is exact if and only if f is a monomorphism.
2. A sequence $A \xrightarrow{g} B \to 0$ is exact if and only if g is an epimorphism.
3. A sequence $0 \to A \xrightarrow{f} B \to 0$ is exact if and only if f is an isomorphism

Proof. We shall appeal to Theorem 1 and regard A, B as left modules of some ring R.

1. We note exactness implies that $\ker(f) = \{0\}$ and thus by Lemma 2 f is a monomorphism.
2. In this case exactness implies that $\operatorname{im}(g) = \ker(0) = B$ and thus by Lemma 2 g is an epimorphism.
3. All subsequences of exact sequences are exact, thus f is monic by part 1, epic by part 2 and thus by Lemma 3 f is an isomorphism.

□

Example 14. In **Ab** we can form the following sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\phi} {}^{\mathbb{Z}}\!\big/_{2\mathbb{Z}} \longrightarrow 0$$

We check its exactness. There is only one map $0 \to \mathbb{Z}$ and its image $0 \in \mathbb{Z}$ must be the kernel of the map $\mathbb{Z} \to \mathbb{Z}$. Thus we must expect the map $\mathbb{Z} \to \mathbb{Z}$ to be an injection (which it is). Similarly there is only one map ${}^{\mathbb{Z}}\!/_{2\mathbb{Z}} \to 0$ and its kernel must be the image of ϕ. Thus we expect ϕ to be a surjection, which it is since it is the natural map $\mathbb{Z} \to {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$. We also note the kernel of ϕ is exactly $2\mathbb{Z}$.

Definition 28. Consider a tuple $C_\bullet := (\{C_n\}_{n\in\mathbb{Z}}, \{\partial_n\})$ where $\{C_n\}_{n\in I}$ is a set of modules in ${}_R\mathbf{Mod}$ and each $\partial_n : C_n \to C_{n-1}$ is a morphism in ${}_R\mathbf{Mod}$. Then $C_\bullet$ is a *chain complex* in ${}_R\mathbf{Mod}$ if:

$$\partial_n \circ \partial_{n+1} = 0$$

for all $n \in I$. The morphisms $\{\partial_n\}$ are called *boundary maps.*[1] We call a chain complex *non-negative* if $C_n = 0$ for $n < 0$.

It is again immediate that $\mathrm{im}(\partial_{n-1}) \subseteq \ker(\partial_n)$. A chain complex can be viewed as a **Z**-shaped diagram in an abelian category with the added condition that $\partial_n \circ \partial_{n+1} = 0$ for all n. We notice that this is a generalisation of an exact sequence.

We can measure how far from exact a sequence is:

Definition 29. Let $C_\bullet$ be a chain complex in ${}_R\mathbf{Mod}$. The n^{th} *homology* of $C_\bullet$ is the quotient:

$$H_n(C_\bullet) := {}^{\ker(\partial_n)}\Big/_{\mathrm{im}(\partial_{n+1})} \text{ regarded as an abelian group.}$$

We introduce some terminology:

- the elements of $\ker(\partial_n) \subseteq C_n$ are called *n-cycles* and are denoted Z_n,
- the elements of $\mathrm{im}(\partial_{n+1}) \subseteq Z_n \subseteq C_n$ are called *n-boundaries* and are denoted B_n.

We can thus also write:

$$H_n(C_\bullet) = {}^{Z_n}\big/_{B_n},$$

and denote $h \in H_n(C_\bullet)$ as $z + B_n$ for some $z \in Z_n$.

Definition 30. Let $C_\bullet$, $D_\bullet$ be chain complexes in **C**. Then a *chain map*

$$f : C_\bullet \to D_\bullet$$

[1] This name has a topological origin, being a map from a topological space to its boundary.

is a family of maps $f_n : C_n \to D_n$ such that the following square commutes

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "\partial'_n"'] & D_{n-1} \end{tikzcd}$$

for all $n \in \mathbb{Z}$.

We note that the commutativity of the square implies that cycles in C_n get mapped to cycles in D_n. This can be seen by considering a cycle $x \in C_n$ and noting that $\partial_n(x) = 0_{C_{n-1}}$ and thus $f_{n-1}(\partial_n(x)) = 0_{D_{n-1}}$ since f_{n-1} is an R-homomorphism. Then commutativity implies

$$\partial'_n(f_n(x)) = 0_{D_n}$$

and thus $f_n(x)$ is a cycle in D_n. We note a similar argument holds for boundaries in C_n.

Definition 31. For ${}_R\mathbf{Mod}$ we can produce the category $\mathrm{Ch}_\bullet({}_R\mathbf{Mod})$ of *chain complexes in* $\boldsymbol{C}$. Its objects are chain complexes in ${}_R\mathbf{Mod}$ and its morphisms are chain maps. Since chain maps are just families of morphisms in ${}_R\mathbf{Mod}$ it follows that composition is well defined and associative.

Proposition 7 ([12])**.** The category $\mathrm{Ch}_\bullet({}_R\mathbf{Mod})$ is abelian.

Proof. This is Proposition 5.100. in Section 5.5 of [12]. □

Since $\mathrm{Ch}_\bullet({}_R\mathbf{Mod})$ is abelian it must have a zero object, this is the complex $(\{0\}, \{0_n : 0 \to 0\}_{n\in\mathbb{Z}})$ where 0 is the trivial R-module.

Proposition 8. A chain map $f : C_\bullet \to D_\bullet$ induces maps:

$$f_n^* : H_n(C_\bullet) \to H_n(D_\bullet).$$

Proof. We denote by ϕ_{C_n} the natural map $C_n \to H_n(C_\bullet)$ and similarly we have $\phi_{D_n} : D_n \to H_n(D_\bullet)$ then we wish to produce a map f_n^* such that the diagram commutes:

$$\begin{tikzcd} C_n \arrow[r, "\phi_{C_n}"] \arrow[d, "f_n"'] & H_n(C_\bullet) \arrow[d, "f_n^*"] \\ D_n \arrow[r, "\phi_{D_n}"'] & H_n(D_\bullet) \end{tikzcd}$$

this can be achieved by defining:

$$\begin{aligned} f_n^* : H_n(C_\bullet) &\to H_n(D_\bullet) \\ z &\mapsto f(z). \end{aligned}$$

This is equivalent to:

$$\begin{aligned} f_n^* : H_n(C_\bullet) &\to H_n(D_\bullet) \\ \phi_{C_n}(c) &\mapsto \phi_{D_n}(f_n(c)) \text{ for } c \in C_n. \end{aligned}$$

We note that for a cycle $z \in C_n$ its image $f_n(z)$ is also a cycle, furthermore $f_n^*(z)$ is independent of choice of representative. Thus the map f_n^* is well defined and a group homomorphism.

□

Definition 32. Let $f : C_\bullet \to D_\bullet$ be a chain map. Then f is called a *quasi-isomorphism* if each f_n^* is an isomorphism.

Definition 33. Let $f, g : C_\bullet \to D_\bullet$ be chain maps. Then we define a *chain homotopy*

$$h : f \to g$$

as the family of morphisms:

$$h_n : C_n \to D_{n+1}$$

such that:

$$\partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n = f - g$$

for all $n \in \mathbb{Z}$. Diagrammatically:

$$\begin{tikzcd} C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] \arrow[dl, "h_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \arrow[dl, "h_{n-1}"'] \\ D_{n+1} \arrow[r, "\partial'_{n+1}"] & D_n \arrow[r, "\partial'_n"'] & D_{n-1} \end{tikzcd} \tag{2.5}$$

We say f is *chain homotopic* to g or $f \simeq g$ if there exists a chain homotopy $h : f \to g$.

We note that a chain homotopy h does not make diagram 2.5 commute.
Chain homotopies have a very useful property:

Lemma 8. Let $f, g : C_\bullet \to D_\bullet$ be chain maps. If $f \simeq g$ then the induced maps

$$f_n^* = g_n^*$$

for all $n \in \mathbb{Z}$.

Proof. We consider diagram 2.5 and the module C_n. Since our statement regards the induced maps it suffices for us to only consider the n-cycles and n-boundaries. Let x be an n-cycle then:

$$\begin{aligned} f_n(x) - g_n(x) &= (h_{n-1} \circ \partial_n)(x) + (\partial_{n+1} \circ h_n)(x) \\ &= (\partial_{n+1} \circ h_n)(x) \end{aligned}$$

and we observe $h_n(x)$ is either in $\ker(\partial_{n+1})$ or not. Thus $(\partial_{n+1} \circ h_n)(x)$ is either 0 or an n-cycle. We notice that f_n and g_n differ by at most a boundary and recall that

$$H_n(C_\bullet) = {Z_n}/{B_n}$$

hence f_n and g_n agree on all equivalence classes of cycles and therefore $f_n^* = g_n^*$. Since our argument places no significance on the chosen n the result has been proved. □

Lemma 9. Let $C_\bullet$ be a chain complex, if $1_{C_\bullet} \simeq 0$ then $C_\bullet$ is exact.

Proof. Since $1_{C_\bullet} \simeq 0$ by Lemma 8 we have that:

$$0_n^* = 1_n^*$$

as maps:

$$0_n^*, 1_n^* : H_n(C_\bullet) \to H_n(C_\bullet).$$

It is only possible for the zero map to agree with the identity map if the object itself is the zero object. Thus all the homologies vanish and $C_\bullet$ is exact. □

Proposition 9. We can regard $H_n(\square)$ as an additive functor. We write its signature

$$H_n(\square) : \mathrm{Ch}_\bullet({}_R\mathbf{Mod}) \to \mathbf{Ab}.$$

Proof. Due to the definition of $H_n(\square)$ we only need to extend the definition to morphisms and verify functoriality. From Proposition 8 there exists a morphism $f_n^* : H_n(C_\bullet) \to H_n(D_\bullet)$ and thus we can make the assignment $H_n(f) = f_n^*$. The composition of these morphisms will inherit from composition in $\mathbf{Ab}$ and thus be associative. Finally if $f = 1_{C_\bullet}$ then it is clear that $H_n(1_{C_\bullet}) = 1_{H_n(C_\bullet)}$. Thus H_n is a functor. In order to prove additivity we must show that for any two chain complexes $C_\bullet, D_\bullet$ that the map induced by $H_n(\square)$:

$$\mathrm{Hom}_{\mathrm{Ch}_\bullet({}_R\mathbf{Mod})}(C_\bullet, D_\bullet) \to \mathrm{Hom}_{\mathbf{Ab}}(H_n(C_\bullet), H_n(D_\bullet))$$

is a group homomorphism. We consider the module C_n and the nth component of two chain maps $f, g : C_\bullet \to D_\bullet$. Then for all $c \in C_n$:

$$H_n(f+g)(c) = (f_n^* + g_n^*)(c) = f_n^*(c) + g_n^*(c) = H_n(f)(c) + H_n(g)(c)$$

and we conclude that $H_n(\square)$ is an additive functor. □

2.3 Duality

We now define some notions that are dual to that of our previous section.

Definition 34. Consider a tuple $C^\bullet := (\{C^n\}_{n\in\mathbb{Z}}, \{\partial^n\})$ where $\{C^n\}_{n\in I}$ is a set of modules in ${}_R\mathbf{Mod}$ and each $\partial^n : C^n \to C^{n+1}$ are morphisms in ${}_R\mathbf{Mod}$. Then $C^\bullet$ is a *cochain complex* in ${}_R\mathbf{Mod}$ if:

$$\partial^{n+1} \circ \partial^n = 0$$

for all $n \in I$. The morphisms $\{\partial^n\}$ are called *coboundary maps.*

Diagrammatically a cochain complex is depicted:

$$\dots \xrightarrow{\partial^{n-2}} C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \xrightarrow{\partial^{n+1}} \dots$$

The reader will notice the contrast between a cochain complex and a chain complex. In a cochain complex the index counts up and is denoted as a superscript, whereas in a chain complex the index counts down and is denoted as a subscript. The notation is designed to keep the reader informed and we shall to conform to this convention. It is also important to note that the cochain complex appears to simply have it arrows "going the other way". This is true and in fact there is no concrete difference between Definition 28 and a cochain complex. However once we fill in some details and work with specific complexes they will be different. This will be demonstrated in Chapter 3.

Definition 35. Let $C^\bullet$ be a cochain complex in ${}_R\mathbf{Mod}$. The n^{th} *cohomology* of $C^\bullet$ is the quotient:

$$H^n(C^\bullet) := {\ker(\partial^n)} \big/ {\mathrm{im}(\partial^{n-1})} \text{ regarded as an abelian group.}$$

We introduce some terminology:

- the elements of $\ker(\partial^n) \subseteq C^n$ are called *n-cocycles* and are denoted Z^n,
- the elements of $\mathrm{im}(\partial^{n-1}) \subseteq Z^n \subseteq C^n$ are called *n-coboundaries* and are denoted B^n.

We can thus also write:

$$H^n(C^\bullet) = Z^n / B^n,$$

and denote $h \in H^n(C^\bullet)$ as $z + B^n$ for some $z \in Z^n$.

The notation again serves to inform the reader. We notice this definition does not differ very much from that of the homologies of a chain complex and that all of the theory developed in the previous section regarding chain maps and chain homotopies are also true for cochain complexes.

Chapter 3

Hochschild (Co)homology

In the previous chapter we defined the homology of a complex - this is a very general idea. In this chapter we present a *particular* homology theory. Hochschild (co)homology is concerned with associative algebras (Definition 25). The idea is that to each algebra we can canonically assign a complex and from there we can calculate homology and cohomology groups. These groups will contain important and useful information about the algebra. We will see them to be particularly useful in Chapter 4. We will follow the treatment of Hochschild (co)homology given in chapter 1 of [9]. We begin by defining the Hochschild complex and cocomplex of an algebra. Then we will characterise the lower homology and cohomology groups and present some concrete calculations.

3.1 Hochschild complexes

Throughout this section we shall fix a commutative ring k and a unital associative algebra A over k. Henceforth $\otimes$ will be used to denote $\otimes_k$. We consider an A-bimodule M. First we note that M is also a k-bimodule. This is due to the ring homomorphism $i_k^{Z(A)} : k \to A$. Since A acts on M we can define an action of k as follows:

$$\begin{aligned} k \times M &\to M \\ (\alpha, m) &\mapsto i_k^{Z(A)}(\alpha)m \end{aligned}$$

and similarly for a right action. Now we define:

$$C_n(A, M) := M \otimes A^{\otimes n} \tag{3.1}$$

Each $C_n(A, M)$ is an A-bimodule. For the sake of notation was shall denote an element $m \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_n \in C_n(A, M)$ as $(m, a_1, \ldots, a_n)$ following the convention used in [9]. We now define n^{th} the Hochschild boundary map, let:

$$\begin{aligned} d_0(m, a_1, \ldots, a_n) &:= (ma_1, a_2, \ldots, a_n) \\ d_i(m, a_1, \ldots, a_n) &:= (m, \ldots, a_i a_{i+1}, \ldots, a_n) \quad 1 \le i < n \\ d_n(m, a_1, \ldots, a_n) &:= (a_n m, a_1, \ldots, a_{n-1}) \end{aligned}$$

and note that each d_i is a map $C_n(A, M) \to C_{n-1}(A, M)$. Then the n^{th} the Hochschild boundary map is given by:

$$\begin{aligned} \partial_n : C_n(A, M) &\to C_{n-1}(A, M) \\ (m, a_1, \ldots, a_n) &\mapsto \sum_{j=0}^{n} (-1)^j d_j(m, a_1, \ldots, a_n). \end{aligned}$$

Example 15. We show some computations:

- We start with $\partial_1 : C_1(A, M) \to C_0(A, M)$. For $(m, a) \in M$ we compute:

  $$\partial_1(m, a) = ma - am$$

  and thus $\mathrm{im}(\partial_1) = \langle ma - am |\ a \in A,\ m \in M \rangle \subseteq M$.

- $\partial_2 : C_2(A, M) \to C_1(A, M)$. For $(m, a_1, a_2) \in C_2(A, M)$ we compute:

  $$\partial_2(m, a_1, a_2) = (ma_1, a_2) - (m, a_1 a_2) + (a_2 m, a_1)$$

  and thus

  $\mathrm{im}(\partial_2) = \langle (ma_1, a_2) - (m, a_1 a_2) + (a_2 m, a_1) |\ a_1, a_2 \in A,\ m \in M \rangle \subseteq M \otimes A.$

- $\partial_3 : C_3(A, M) \to C_2(A, M)$. For $(m, a_1, a_2, a_3) \in C_3(A, M)$ we compute:

  $\partial_3(m, a_1, a_2, a_3) = (ma_1, a_2, a_3) - (m, a_1 a_2, a_3) + (m, a_1, a_2 a_3) - (a_3 m, a_1, a_2).$

Proposition 10. Let $\{\partial_n\}$ be a set of Hochschild boundaries, then $\partial_n \circ \partial_{n+1} = 0$ for all n.

Proof. We fix an n and verify that:

$$d_i \circ d_j = d_{j-1} \circ d_i \text{ for } 0 \le i < j \le n. \tag{3.2}$$

We compute:

$$d_i \circ d_j(m, a_1, \ldots, a_n) = \begin{cases} d_i(m, a_1a_2, \ldots, a_n) & j = 1 \\ d_i(m, \ldots, a_ja_{j+1}, \ldots, a_n) & 1 < j < n \\ d_i(a_nm, a_1, \ldots, a_{n-1}) & j = n \end{cases}$$

$$= \begin{cases} (ma_1a_2, a_3, \ldots, a_n) & i = 0, j = 1 \\ (ma_1, \ldots, a_ja_{j+1}, \ldots a_n) & i = 0, 1 < j < n \\ (a_nma_1, a_2, \ldots, a_{n-1}) & i = 0, j = n \\ (m, \ldots, a_{j-1}a_ja_{j+1}, \ldots a_n) & 0 < i = j-1, 1 < j < n \\ (m, \ldots, a_ia_{i+1}, \ldots, a_ja_{j+1}, \ldots a_n) & 0 < i < j-1, 1 < j < n \\ (a_nm, \ldots, a_ia_{i+1}, \ldots, a_{n-1}) & 0 < i < n, j = n \end{cases} \tag{3.3}$$

and

$$d_{j-1} \circ d_i(m, a_1, \ldots, a_n) = \begin{cases} d_{j-1}(ma_1, a_2, \ldots, a_n) & i = 0 \\ d_{j-1}(m, \ldots, a_ia_{i+1}, \ldots, a_n) & 0 < i < n \end{cases}$$

$$= \begin{cases} (ma_1a_2, a_3, \ldots, a_n) & i = 0, j = 2 \\ (ma_1, \ldots, a_{j-1}a_j, \ldots a_n) & i = 0, 2 < j < n \\ (a_nma_1, a_2, \ldots, a_{n-1}) & i = 0, j = n \\ (m, \ldots, a_{j-2}a_{j-1}a_j, \ldots a_n) & 0 < i = j-1, 1 < j < n \\ (m, \ldots, a_ia_{i+1}, \ldots, a_{j-1}a_j, \ldots a_n) & 0 < i < j-1, 2 < j < n \\ (a_nm, \ldots, a_ia_{i+1}, \ldots, a_{n-1}) & 0 < i < n, j = n. \end{cases} \tag{3.4}$$

We first compare (3.3) and (3.4) for the cases dependent on j. We note that in (3.4) the value of j is reduced by 1 but it is quantified over a range that begins at a value 1 greater than that of (3.3), thus both compositions agree in the cases with a dependence on j. In the cases where there is no dependence on j both compositions also agree, thus we conclude that $d_i \circ d_j = d_{j-1} \circ d_i$ for $0 \leq i < j \leq n$. We observe that the sum:

$$\partial_n \circ \partial_{n+1} = \sum_{i=0}^{n} d_i \circ \sum_{j=0}^{n+1} d_j = \sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq n+1}} (-1)^{i+j} d_i \circ d_j$$

splits into two parts depending on whether $i < j$ or $i \geq j$. Thus:

$$\begin{aligned}\sum_{\substack{0\leq i\leq n\\0\leq j\leq n+1}}(-1)^{i+j}d_i\circ d_j &= \sum_{0\leq i<j\leq n+1}(-1)^{i+j}d_i\circ d_j+\sum_{n\geq i\geq j}(-1)^{j-1+i}d_{j-1}\circ d_i\\ &= \sum_{0\leq i<j\leq n+1}(-1)^{i+j}d_{j-1}\circ d_i-\sum_{n\geq i\geq j}(-1)^{i+j}d_{j-1}\circ d_i\\ &= 0.\end{aligned}$$

Since our argument was independent of n the claim has been proved. □

We are ready to define the Hochschild complex as follows:

Definition 36 (Section 1.1 in [9])**.** We require the data of a k-algebra A and an A-bimodule M. Then the *Hochschild complex of A with coefficients in M* is non-negative chain complex:

$$C_\bullet(A,M)=(\{C_n(A,M)=M\otimes A^{\otimes n}\},\{\partial_n : C_{n+1}(A,M)\to C_n(A,M)\})$$

where each ∂_n is the nth Hochschild boundary.

$$\dots \xrightarrow{\partial_3} M\otimes A^{\otimes 2}\xrightarrow{\partial_2} M\otimes A\xrightarrow{\partial_1} M\xrightarrow{\partial_0} 0$$

We are particularly interested in the case where $M = A$ and then obtain the complex:

$$\dots \xrightarrow{\partial_3} A^{\otimes 3}\xrightarrow{\partial_2} A^{\otimes 2}\xrightarrow{\partial_1} A\xrightarrow{\partial_0} 0$$

which we shall denote as $C_\bullet(A) = C_\bullet(A, A)$.

Definition 37. The n^{th} *Hochschild homology group of A with coefficients in M*, denoted $HH_n(C_\bullet(A, M))$ is the nth homology group of the complex $C_\bullet(A, M)$. That is to say

$$HH_n(C_\bullet(A,M))=\ker(\partial_n)\big/\mathrm{im}(\partial_{n+1}).$$

In the case where $A = M$ we write $HH_n(A) = HH_n(C_\bullet(A))$.

We recall Proposition 9, it implies that $HH_n(\square)$ is an additive functor from the category of Hochschild complexes and chain maps to the category of abelian groups. We attend to the case where $M = A$ because it provides for clear interpretations of the Hochschild homology groups.

From the Hochschild complex we can construct the Hochschild cocomplex:

Definition 38 (Section 1.5 in [9])**.** We require the data of a k-algebra A and an A-bimodule M. We define $C^n(A, M) := \mathrm{Hom}(A^{\otimes n}, M)$ in the category of A-bimodules. The coboundary map

$$\partial^n : C^n(A,M)\to C^{n+1}(A,M)$$

is defined as follows, for:

$$f : A^{\otimes n}\to M$$

we define

$$\partial^n(f) := -(-1)^n f \circ \partial_n.$$

The *Hochschild cocomplex of A with coefficients in M* is the non-negative cochain complex $C^\bullet(A, M) = (\{C^n(A, M) = \mathrm{Hom}(A^{\otimes n}, M)\}, \{\partial^n\})$ where each ∂^n is the nth Hochschild coboundary.

$$0 \longrightarrow C^0(A, M) \xrightarrow{\quad \partial^0 \quad} C^1(A, M) \xrightarrow{\quad \partial^1 \quad} C^2(A, M) \xrightarrow{\quad \partial^2 \quad} \dots$$

Again we are particularly interested in the case where $M = A$ and then obtain the complex:

$$0 \longrightarrow \mathrm{Hom}(k, A) \xrightarrow{\quad \partial^0 \quad} \mathrm{Hom}(A, A) \xrightarrow{\quad \partial^1 \quad} \mathrm{Hom}(A^{\otimes 2}, A) \xrightarrow{\quad \partial^2 \quad} \dots$$

which we shall denote as $C^\bullet(A, A) = C^\bullet(A)$.

We shall unpack the definition, beginning with how the Hochschild coboundary is defined. Suppose we have an A-homomorphism $f : A^{\otimes n} \to M$ and we wish to produce a new A-homomorphism $A^{\otimes n+1} \to M$. Then we can precompose with the Hochschild boundary to get an element of $A^{\otimes n}$ and then apply f since that is its domain. This precomposition may be reminiscent of the contravariant hom functor from Example 5 and indeed this is because of the definition of $C^n(A, M)$. The fact that $C^\bullet(A)$ is a complex is due to $C_\bullet(A)$ being a complex, in particular that $\partial_n \circ \partial_{n+1} = 0$.

Example 16. We show some computations of ∂^n:

- We start with $\partial^0 : C^0(A, M) \to C^1(A, M)$. We observe that $C^0(A, M) = \mathrm{Hom}(k, M) \cong M$. This is because A-homomorphisms out of k are determined by where $1 \in k$ is sent, we get a choice for each $m \in M$ and hence $\mathrm{Hom}(k, M) \cong M$. Thus ∂^0 takes an element $m \in M$ as input and returns a morphism $(\partial^0 m) : A \to M$. It is defined as follows:

$$(\partial^0 m)(a) = am - ma.$$

- Next we compute $\partial^1 : C^1(A, M) \to C^2(A, M)$ which takes a morphism $f : A \to M$ and returns a morphism $(\partial^1 f) : A^{\otimes 2} \to M$ defined as follows:

$$(\partial^1 f)(a_1 a_2) = a_1 f(a_2) - f(a_1 a_2) + f(a_1) a_2.$$

- Finally we show $\partial^2 : C^2(A, M) \to C^3(A, M)$ which extends a morphism $f : A^{\otimes 2} \to M$ to a morphism $(\partial^2 f) : A^{\otimes 3} \to M$. It is defined as:

$$(\partial^2 f)(a_1, a_2, a_3) = a_1 f(a_2, a_3) - f(a_1 a_2, a_3) + f(a_1, a_2 a_3) - f(a_1, a_2) a_3.$$

Definition 39. The n^{th} *Hochschild cohomology group of A with coefficients in M*, denoted $HH^n(C^\bullet(A, M))$ is the nth cohomology group of the complex $C^\bullet(A, M)$. That is to say

$$HH^n(C^\bullet(A, M)) = {\ker(\partial^n)} \Big/ {\mathrm{im}(\partial_{n-1})}.$$

In the case where $A = M$ we write $HH^n(A) = HH^n(C_\bullet(A))$.

We conclude this section with the statement of a powerful result.

Theorem 10 (Morita Invariance, section 1.5.6 in [9])**.** Let k be a commutative ring and A a k-algebra. For an A-bimodule M these exists an isomorphism:

$$HH^n(C^\bullet(A, M)) \to HH^n(C^\bullet(\mathcal{M}_r(A), \mathcal{M}_r(M))$$

for each $r \in \mathbb{N}$ and each $n \in \{0\} \cup \mathbb{N}$ where $\mathcal{M}_r(A)$ denotes the algebra of $r \times r$ matrices over A.

The proof of this theorem is beyond the scope of this report.

3.2 Derivations

We will now have a brief interlude introducing derivations of rings. We will follow the treatment in section 1 of [7].

Definition 40. Let R be a ring. A map $d : R \to R$ is a *derivation* if d is additive and satisfies Leibnitz' rule. That is to say for $a, b \in R$ d satisfies:

- $d(a + b) = d(a) + d(b)$,
- $d(ab) = ad(b) + d(a)b$.

Example 17. The motivating example is the differential operator on a polynomial algebra. We consider the algebra $R[x]$ and note its elements are of the form

$$\sum_{i=0}^{\infty} a_i x^i$$

where all but finitely many of the coefficients $a_i = 0$. We define the derivation:

$$\begin{aligned} d : R[x] &\to R[x] \\ \sum_{i=0}^{\infty} a_i x^i &\mapsto \sum_{i=0}^{\infty} i a_i x^{i-1}. \end{aligned}$$

Definition 41. Let R be a ring. We define the *commutator* of two elements $a, b \in R$ to be $[a, b] := ab - ba$.

Proposition 11. Let R be a ring. For $a, b, c \in R$ it holds that

$$[ab, c] = a[b, c] + [a, c]b.$$

Proof. We proceed by direct computation:

$$a[b, c] + [a, c]b = a(bc - cb) + (ac - ca)b = abc - cab = [ab, c].$$

□

In non-commutative rings there is another class of examples of derivations.

Example 18. Let R be a non-commutative ring and let $a \in R$. We define:

$$\begin{aligned} d_a : R &\to R \\ r &\to [r, a] \end{aligned}$$

and we verify this is a derivation. First we check additivity:

$$d_a(r+s) = [r+s, a] = (r+s)a - a(r+s) = (ra-ar)+(sa-as) = [r, a]+[s+a] = d_a(r)+d_a(s).$$

Next we check Leibnitz' rule:

$$d_a(rs) = [rs, a] = r[s, a] + [r, a]s = rd_a(s) + d_a(r)s.$$

We will call derivations of the form d_a for some $a \in d$ *inner derivations* of R.

Proposition 12. Let R be a ring, then the sum of two derivations will be a derivation.

Proof. Suppose we have $f, g : R \to R$ are derivations and let $r, s \in R$. We show $f + g$ is additive:

$$(f+g)(r+s) = f(r+s)+g(r+s) = f(r)+g(r)+f(s)+g(s) = (f+g)(r)+(f+g)(s).$$

We verify that Leibnitz' rule is satisfied:

$$\begin{aligned} (f+g)(rs) = f(rs) + g(rs) &= rf(s) + f(r)s + rg(s) + g(r)s \\ &= r(f(s) + g(s)) + (f(r) + g(r))s \\ &= r(f+g)(s) = (f+g)(r)s. \end{aligned}$$

□

For a ring R we will make use of the following notation:

- We denote the set of inner derivations $R \to R$ by $Inn(R)$. Due to Proposition 12 this set can be endowed with the structure of an abelian group with the addition inheriting from that of R and the identity being the zero map $R \to R$.
- We denote the set of derivations $R \to R$ by $Der(R)$. Similar to above this set can also be regarded as an abelian group.

3.3 Characterisations of the lower Hochschild (co)homology groups

3.3.1 Homology groups

Let A be a k-algebra. Then we consider the Hochschild chain complex $C_\bullet(A)$:

$$\cdots \xrightarrow{\partial_3} A^{\otimes 3} \xrightarrow{\partial_2} A^{\otimes 2} \xrightarrow{\partial_1} A \xrightarrow{\partial_0} 0$$

and we wish to calculate its homologies.

- $HH_0(A)$: We only need to compute $\text{im}(\partial_1)$ since $\ker(\partial_0) = A$. We recall

$$\partial_1(a_1, a_2) = a_1a_2 - a_2a_1$$

thus

$$\text{im}(\partial_1) = \{[a_1, a_2] |\ a_1, a_2 \in A\} = [A, A]$$

so we conclude that

$$HH_0(A) = A \big/ [A, A]$$

the abelianization of A.

- For a commutative algebra A it can be shown that $HH_1(A) = \Omega^1_{A|K}$ the A-module of Kähler differentials. This is both not the focus and beyond the scope of this report and is discussed in Proposition 1.1.10 in [9].

3.3.2 Cohomology groups

Let A be a k-algebra. Then we consider the Hochschild cochain complex $C^\bullet(A)$:

$$0 \longrightarrow \text{Hom}(k, A) \xrightarrow{\partial^0} \text{Hom}(A, A) \xrightarrow{\partial_1} \text{Hom}(A^{\otimes 2}, A) \xrightarrow{\partial_2} \dots$$

and we wish to calculate its cohomologies.

- $HH^0(A)$: We only need to compute $\ker(\partial^0)$. We observe:

$$(\partial_0 a_1)(a_2) = a_2a_1 - a_1a_2$$

thus

$$\ker(\partial_0) = Z(A)$$

so we conclude that

$$HH^0(A) = Z(A).$$

- $HH^1(A)$: We will need to calculate both $\text{im}(\partial^0)$ and $\ker(\partial^1)$. We note from the previous case that

$$\text{im}(\partial^0) = Inn(A).$$

Now we recall that ∂^1 extends a map from A to a map from $A \otimes A$. We compute:

$$(\partial^1 f)(a_1, a_2) = a_1 f(a_2) - f(a_1a_2) + f(a_1)a_2$$

hence:

$$\ker(\partial^1) = \{f \in \text{Hom}(A, A) |\ f(a_1a_2) = a_1 f(a_2) + f(a_2)a_1 \forall\ a_1, a_2 \in A\}$$

thus $\ker(\partial^1) = Der(A)$. We conclude that:

$$HH^1(A) = Der(A) \big/ Inn(A).$$

In Chapter 4 we will see the influence of $HH^2(A)$ and $HH^3(A)$ on deformation theory.

3.4 Explicit Calculations of Cohomology

In this section we will calculate some Hochschild cohomology groups of algebras.

3.4.1 Field

We fix a field k and will consider the case $A = M = k$. This case is degenerate however it is a good baseline and will be used for later computations. We will consider the complex:

$$0 \longrightarrow \mathrm{Hom}(k,k) \xrightarrow{\partial^0} \mathrm{Hom}(k,k) \xrightarrow{\partial_1} \mathrm{Hom}(k^{\otimes 2},k) \xrightarrow{\partial_2} \dots$$

- $HH^0(k)$: The kernel of ∂^0 is k since fields are commutative and thus $HH^0(k) = k$.

- $HH^1(k)$: We note that the image of $\mathrm{im}(\partial^0)$ is trivial. We consider $f : k \to k$ and compute:

  $$(\partial^1 f)(k_1,k_2) = k_2 f(k_1) - f(k_1k_2) + f(k_1)k_2.$$

  We note that $(\partial^1 f) \in \mathrm{Hom}(k^{\otimes 2},k)$ is bilinear in its 2 arguments. Since k is a field it is sufficient to consider:

  $$(\partial^1 f)(1,1) = f(1) - f(1) + f(1) = f(1) \tag{3.5}$$

  because:

  $$(\partial^1 f)(k_1,k_2) = k_1k_2(\partial^1 f)(1,1). \tag{3.6}$$

  We note that $f(1) \neq 0$ and therefore (3.5) never vanishes, thus the kernel of ∂^1 is trivial. We conclude that

  $$HH^1(k) = \{0\} \text{ the trivial group.}$$

- $HH^2(k)$: The image of ∂^1 is k by a similar argument used in Example 16. We consider $f : A^{\otimes 2} \to A$ and again only consider where the unit is mapped:

  $$(\partial^2 f)(1,1,1) = f(1,1) - f(1,1) + f(1,1) - f(1,1) = 0$$

  hence all maps are in the kernel of ∂^2. Thus:

  $$HH^2(k) = \mathrm{ker}(\partial^2) \Big/ \mathrm{im}(\partial^1) = k/k = \{0\}.$$

The complex we started with is isomorphic (as a chain complex) to:

$$0 \longrightarrow k \xrightarrow{\partial^0} k \xrightarrow{\partial_1} k \xrightarrow{\partial_2} \dots$$

since $M \otimes_k k \cong M$ for k-modules as shown in Example 12 and thus is a non-standard example.

3.4.2 2×2 matrices over a field

We will fix a field k and will consider the case $A = M = \mathcal{M}_2(k)$. We will apply Theorem 10 and it immediately yields:

- $HH^0(\mathcal{M}_2(k)) \cong k$,
- $HH^1(\mathcal{M}_2(k)) \cong \{0\}$,
- $HH^2(\mathcal{M}_2(k)) \cong \{0\}$.

3.4.3 The group algebra $\mathbb{C}[C_3]$

We the cyclic group of order 3 as $C_3 = \{e, g, g^2\}$ where e denotes the identity. Then we recall the group algebra is the algebra of formal sums of elements of C_3:

$$\mathbb{C}[C_3] = \{c_0 e + c_1 h + c_2 g^2 |\ c_0, c_1, c_2 \in \mathbb{C}\}$$

with additions defined element wise and multiplication defined as:

$$\begin{aligned}(a_0 e + a_1 g + a_2 g^2)(b_0 e + b_1 g + b_2 g^2) =& (a_0 b_0 + a_1 b_2 + a_2 b_0) e \\ &+ (a_0 b_1 + a_1 b_0 + a_2 b_2) g \\ &+ (a_0 b_2 + a_1 b_1 + a_2 b_0) g^2.\end{aligned}$$

We note we only need to compute the values of $(\partial^n f)$ on the basis elements e, g, g^2 because of linearity and we note that e is the multiplicative identity. We consider the complex:

$$0 \longrightarrow \mathrm{Hom}(\mathbb{C}, \mathbb{C}[C_3]) \xrightarrow{\partial^0} \mathrm{Hom}(\mathbb{C}[C_3], \mathbb{C}[C_3]) \xrightarrow{\partial^1} \dots$$

- $HH^0(\mathbb{C}[C_3]) = \mathbb{C}[C_3]$ since it is a commutative algebra.
- $HH^1(\mathbb{C}[C_3])$: We note that the image of ∂^0 is trivial thus we only have to compute $\ker(\partial^1)$. We consider $f : \mathbb{C}[C_3] \to \mathbb{C}[C_3]$ and will determine the conditions implying $f \in \ker(\partial^1)$. We consider f in $\ker(\partial^1)$ and compute:

  $$\begin{aligned}(\partial^1 f)(e, g) &= e f(g) - f(g) + f(e) g \\ &= e f(g) - e f(g) + f(e) g \\ &= f(e) g\end{aligned}$$

  which yields the relation:

  $$f(e) g = 0. \tag{R$_1$}$$

We continue:

$$\begin{aligned}(\partial^1 f)(g,g) &= gf(g) - f(g^2) + f(g)g \\ &= 2gf(g) - f(g^2)\end{aligned}$$

which yields the relation:

$$2gf(g) = f(g^2). \tag{R$_2$}$$

We compute:

$$\begin{aligned}(\partial^1 f)(g,g^2) &= gf(g^2) - f(e) + f(g)g^2 \\ &= 3g^2 f(g) - f(e) \quad \text{using } (R_2)\end{aligned}$$

which yields the relation:

$$3g^2 f(g) = f(e). \tag{R$_3$}$$

We substitute (R_3) into (R_1) we obtain:

$$\begin{aligned}&3g^2 f(g)g = 0 \\ \iff\ &3ef(g) = 0\end{aligned}$$

and this yields that $ef(g) = 0$. Since e is the multiplicative identity in $\mathbb{C}[C_3]$ we see that $f(g) = 0$. By (R_2) and (R_3) we have that $f(e) = f(g^2) = 0$. Thus $\ker(\partial^1) = \{0\}$. Finally we observe that

$$HH^1(\mathbb{C}[C_3]) = \{0\} \big/ \{0\} = \{0\}.$$

From this we can conclude that all derivations of $\mathbb{C}[C_3]$ are inner.

Chapter 4

Algebraic Deformation Theory

4.1 Introduction

In this chapter we will present a brief account of the theory of deformations of algebras. We will highlight the influence of Hochschild cohomology on the subject. We shall culminate with the presentation of a novel method to produce complex nilpotent algebra that deforms into a given finite dimensional unital algebra. We shall follow the treatment in [4]. All algebras will be over some commutative ring k that will not change and thus all tensor products will be tensor over k and all hom-sets will be the set of k-module homomorphisms between two k-algebras, which we shall denote Hom_k. Furthermore for any algebra A we will only consider the Hochschild cocomplex $C^{\bullet}(A)$, that is the Hochschild cocomplex of A with coefficients in A.

4.2 Ring multiplication as a bilinear map

Let $(R, +, \times)$ be a ring. Then it holds that $\times$ is given by some map

$$F : R \otimes_R R \to R.$$

We recall that the ring axioms demand that for $r, b, c \in R$:

$$\begin{aligned} r \times (a + b) &= r \times a + r \times b \\ (a + b) \times r &= a \times r + b \times r \end{aligned}$$

and we notice that F also satisfies these conditions:

$$\begin{aligned} F(r, a + b) &= F(r, a) + F(r, b) \\ F(a + b, r) &= F(a, r) + F(b, r). \end{aligned}$$

Ring multiplication has the added condition of associativity. This can be rewritten as:

$$F(F(a, b), c) - F(a, F(b, c)) = 0. \tag{4.1}$$

Now suppose we have an algebra $(A, +, \times)$ over a commutative ring k and we would like to perturb its multiplication. A candidate would be to find an $f \in \mathrm{Hom}_k(A \otimes A, A)$ that satisfies (4.1). We can then attempt produce a new algebra $A' = (A, +, \times + f)$. With multiplication is defined as:

$$ab = a \times b + f(a, b)$$

for $a, b \in A$. We observe that for all $a, b, c \in A$:

$$a(b+c) = a \times (b+c) + f(a, b+c) = a \times b + a \times c + f(a, b) + f(a, c) = ab + ac$$
$$(a+b)c = (a+b) \times c + f(a+b, c) = a \times c + b \times c + f(a, c) + f(b, c) = ac + bc$$

and

$$\begin{aligned}
a(bc) - (ab)c &= a \times (b \times c + f(b, c)) + f(a, (b \times c + f(b, c))) \\
&- (a \times b + f(a, b)) \times c - f((a \times b + f(a, b)), c) \\
&= a \times (b \times c) + a \times f(b, c) + f(a, b \times c) + f(a, f(b, c)) \\
&- (a \times b) \times c - f(a, b) \times c - f(a \times b, c) - f(f(a, b), c) \\
&= a \times (b \times c) - (a \times b) \times c + f(a, f(b, c)) - f(f(a, b), c) \\
&+ a \times f(b, c) - f(a \times b, c) + f(a, b \times c) - f(a, b) \times c \\
&= a \times f(b, c) - f(a \times b, c) + f(a, b \times c) - f(a, b) \times c \\
&= 0 \iff \partial^2(f) = 0 \text{ recall Example 16.}
\end{aligned}$$

Thus we conclude that A' is an associative algebra only if f is associative and a Hochschild 2-cocycle.

4.3 Deformations

We shall begin with some examples:

Example 19. We consider consider the $\mathbb{C}$-algebra $A = \mathbb{C}[x, y]$ which is a polynomial algebra in two commuting variables. Then we can define:

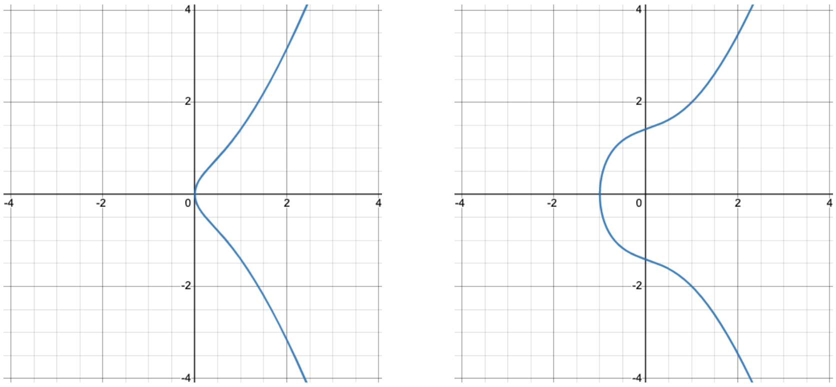
$$A_t = \mathbb{C}[x, y, t] \Big/ \langle xy - (t+1)yx \rangle$$

and consider how the structure of A_t varies as t moves around in $[0, \infty)$. We observe that $A_0 \cong A$ and thus for $t = 0$ the variables commute. For non-zero t values we obtain the relation:

$$xy = (t+1)yx$$

and thus the algebra is no longer commutative.

We unpack this example, we have introduced a new multiplication on the algebra A that has a dependence on a parameter t which we can perturb to



(a) $t = 0$ (b) $t = 2$

Figure 4.1: Graph of the curve represented by $A_t = \mathbb{C}[x, y, t] \Big/ \langle y^2 - x^3 - x - t\rangle$ for varying t values.

obtain different algebras. It also important to note that our new multiplication agrees with that of A for $t = 0$.

Example 20. There is a geometric interpretation of a deformation. The *co-ordinate ring* of a curve C over a field k is in the quotient $k[x, y] \Big/ \langle f\rangle$ where $C : f(x, y) = 0$, more on this can be seen on page 66 of [11]. Suppose that A is the coordinate ring of some curve then a deformation of A will correspond to a geometric deformation of the curve. Consider

$$A = \mathbb{C}[x, y] \Big/ \langle y^2 - x^3 - x\rangle$$

as a $\mathbb{C}$-algebra. We then define:

$$A_t = \mathbb{C}[x, y, t] \Big/ \langle y^2 - x^3 - x - t\rangle .$$

In Figure 4.1 we have plotted the curve corresponding to A_t for different t values.

Definition 42 (1.2 in [4])**.** A *one-parameter formal deformation* or a *deformation* of a k-algebra A is a formal power series

$$F = \sum_{n=0}^{\infty} f_n t^n$$

with coefficients in $\mathrm{Hom}_k(A \otimes A, A)$ such that

$$f_0 : A \otimes A \to A$$

is multiplication in A. We call the first f_i that is non-zero the *infinitesimal* of F.

We note this definition gives rise to a family of algebras $\mathcal{A} := \{A_t\}_{t\in[0,\infty)}$. We shall refer to both F and $\mathcal{A}$ as a deformation of A.

Example 21 ([4])**.** We shall present a non-associative example of a deformation. We consider the algebra $A = k[x]$ and denote by $f_0 : A \otimes A \to A$ the normal multiplication in A. We define:

$$\begin{aligned} f_1 : A \otimes A &\to A \\ (x^n, x^m) &\mapsto (mn)x^{m+n}, \end{aligned}$$

and then write $F = f_0 + f_1 t$. We compute:

$$\begin{aligned} F(x^2, F(x,x)) &= F(x^2, x^2 + tx^2) \\ &= x^4 + tx^4 + 4tx^4 + 4t^2x^4 \\ &= x^4 + 5tx^4 + 4t^2x^4, \end{aligned}$$

and

$$\begin{aligned} F(F(x^2, x), x) &= F(x^3 + 2tx^3, x) \\ &= x^4 + 2tx^4 + 3tx^4 + 6t^2x^4 \\ &= x^4 + 5tx^4 + 6t^2x^4 \neq x^4 + 5tx^4 + 4t^2x^4. \end{aligned}$$

Thus F is a non-associative deformation.

4.4 Associative deformations

We shall describe some necessary conditions for a deformation of a k-algebra A to be associative. This will lead us to more discussions Hochschild cohomology.

We consider a deformation $F = \sum\limits_{i=0}^{\infty} f_i t^i$ of A. Then we observe:

$$\begin{aligned} & F(F(a,b),c) = F(a, F(b,c)) \qquad (4.2) \\ \iff & \sum_{i=0}^{\infty} f_i \left(\sum_{j=0}^{\infty} f_j(a,b)t^j, c \right) t^i = \sum_{i=0}^{\infty} f_i \left(a, \sum_{j=0}^{\infty} f_j(b,c)t^j \right) t^i \end{aligned}$$

We note that for any f_i, f_j that $f_i(f_j(a,b)t^m, a)t_n$ will have degree $n+m$. Thus we can compare the coefficients of t^n to obtain the equation:

$$\sum_{i=0}^{n} f_i(f_{n-i}(a,b),c) = \sum_{i=0}^{n} f_i(a, f_{n-i}(b,c)). \qquad (4.3)$$

Suppose the infinitesimal of F is f_m then for $n = m$ equation (4.2) reads:

$$\begin{aligned} & f_0(f_m(a,b),c) + f_m(f_0(a,b),c) = f_0(a, f_m(b,c)) + f_m(a, f_0(b,c)) \\ \iff & f_m(a,b)c + f_m(ab,c) = af_m(b,c) + f_m(a,bc) \\ \iff & 0 = af_m(b,c) - f_m(ab,c) + f_m(a,bc) - f_m(a,b)c \\ \iff & \partial^2(f) = 0. \end{aligned}$$

We have just proved:

Proposition 13. If F is an associative deformation of A then the infinitesimal of F is a Hochschild 2-cocycle.

We wish to find a sufficient condition for associativity. This is:

Theorem 11. If $HH^3(A)$ is trivial then any 2-cocycle of may be extended to an associative deformation of A.

Proof. This is beyond the scope of this report and we refer the reader to section 9 of [4]. □

4.5 Some associative deformations of complex algebras

We consider a finite dimensional $\mathbb{C}$-algebra A that is generated by 2 elements. In this section we shall present a method to produce many $\mathbb{C}$-algebras that have A as an associative deformation. This method was invented during project discussions with Agata Smoktunowicz, and later extended and investigated further in collaboration with Dora Puljić in a soon-to-be released preprint. We refer the reader to Appendix C for the proofs concerning Method 1.

Definition 43. Let S be a subset of a k-algebra A. Then S is a *generating set* if all elements of A can be written as a sum of products of elements of S.

Definition 44. A k-algebra A is a *deformation of* a k-algebra N if there exists a deformation of N such that $A \in \{N_t\}_{t\in[0,\infty)}$. We say A is an *associative deformation of* N if the deformation is associative.

We shall present the method here and then do some examples.

4.5.1 The Method

We first develop some notation, by $\mathbb{C}[x, y][t]$ the algebra where x, y do not commute with each other but do commute with t.

Method 1. We fix a finite dimensional $\mathbb{C}$-algebra A with two generators a, b. Then:

1. We consider $A[t]$, the polynomial algebra over A. We will use t as the parameter of our deformation. We define:

$$x := ta$$
$$y := tb$$

2. We calculate:

$$\begin{aligned} x^2 &= t^2a^2 \\ xy &= t^2ab \\ yx &= t^2ba \\ y^2 &= t^2b^2 \end{aligned}$$

and continue to calculate larger products of x, y. We shall then proceed with a Diamond Lemma[1] like decomposition of large products of a, b in terms of smaller products of a, b. This will cause all elements of sufficiently large length to have a power of t as a factor. In doing so we will obtain relations on x, y and products thereof, we terminate this process when we have enough relations to reduce an arbitrary product of x, y to one of a finite list multiplied by some power of t. Our relations will be given by polynomials $p_1, \ldots, p_m$.

3. We present the algebra:

$$N := \mathbb{C}[x,y][t] \Big/ \langle p_1, \ldots, p_m \rangle .$$

We note that sufficiently large products of x, y will obtain a factor of t. Now we evaluate t at various values in $[0, \infty)$. We denote by N_s the algebra that arises from N by evaluating t at s. This algebra N and the family of algebras $\{N_s\}$ is the output of this method.

Proposition 14. Let A be a finite dimensional $\mathbb{C}$-algebra. Let N be the output of Method 1 applied to A. Then $N_1 \cong A$.

Proof. We refer the reader to Proposition 20 in Appendix C.1. □

Proposition 15. Let A be a finite dimensional $\mathbb{C}$-algebra. Let N be the output of Method 1 applied to A. Then A is an associative deformation of N_0.

Proof. We refer the reader to Appendix C.2. □

Changing the chosen generators a, b can result in different algebras N_0 that have A as an associative deformation.

4.5.2 Notes on the method

We give some notes before the examples:

- In step (1) we are trying to capture the behaviour of the elements in A but in a way that is controlled by t.
- We note that since a, b generate A some linear combination of products of a, b will be the identity. In step (2) it is important we find a relation on x, y, t and $1 \in A$.

[1] See section 1 of [1]

- In step (3) we discard A entirely however because of the relations p_i in some sense x and y "remember" that they came from A and that they form a generating set. This step results in something subtle: x, y themselves are no longer dependent on t (and thus do not vanish when we set $t = 0$) but their products maintain their dependence on t. We have loosened our grip on x, y just enough for the deformation to take place.

- We note that our deformation will be associative since the multiplication of N_t will inherit from the associative algebras N.

4.5.3 Examples

Practically it can be useful to fix a basis for the algebra A as this helps find the decompositions in step (2).

Example 22. We consider $A = \mathbb{C} \oplus \mathbb{C}$ and fix $a = (i, 0),\ b = (0, 1)$. Clearly $\mathbb{C}$-linear combinations of a and b span A so our basis is just $\{a, b\}$. We denote by $\mathbb{1}$ the multiplicative unit in A. Now we consider $A[t]$ and define:

$$\begin{aligned} x &:= ta = (ti, 0) \\ y &:= tb = (0, t). \end{aligned}$$

We calculate:

$$\begin{aligned} x^2 &= t^2a^2 = (-t^2, 0) = tix \\ xy &= t^2ab = (0, 0) = 0 \\ yx &= t^2ab = (0, 0) = 0 \\ y^2 &= t^2b^2 = (0, t^2) = ty \\ t\mathbb{1} &= -ix + y. \end{aligned}$$

Thus we have relations:

$$\begin{aligned} p_1 &= x^2 - tix = 0 \\ p_2 &= xy = 0 \\ p_3 &= yx = 0 \\ p_4 &= y^2 - ty = 0 \\ p_5 &= t\mathbb{1} - (-ix + y). \end{aligned}$$

Since $xy = yx = 0$ and squares of x, y can be reduced we have enough relations to decompose any large product into only multiples of x, y and powers of t.

Thus we present:

$$N := \mathbb{C}[x, y][t] \Big/ \langle p_1, p_2, p_3, p_4, p_5 \rangle \, .$$

We observe:

$$N_0 = \mathbb{C}[x, y] \Big/ \langle x^2, xy, yx, y^2, -ix + y \rangle \, ,$$

and
$$N_1 = \mathbb{C}[x,y] \Big/ \langle x^2 - ix, xy, yx, y^2 - y, 1 + ix - y \rangle \,.$$
We see that x, y behave exactly as $a, b \in A$ and by Proposition 14. The reader may check the isomorphism is given by:

$$\begin{aligned} \phi : N_1 &\to A \\ 1 &\mapsto \mathbb{1} \\ x &\mapsto (i, 0) \\ y &\mapsto (0, 1). \end{aligned}$$

Thus we have produced an algebra N_0 that has $A = \mathbb{C} \oplus \mathbb{C}$ as a deformation.

Example 23. We consider $A = \mathcal{M}_2(\mathbb{C})$ and fix:

$$a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We denote by $\mathbb{1}$ the multiplicative unit in A. Next we compute:

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= \frac{1}{2}(a + b^2) = e_1 \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= \frac{1}{2}(ab + b) = e_2 \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &= \frac{1}{2}(b - ab) = e_3 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &= \frac{1}{2}(b^2 - a) = e_4 \end{aligned}$$

and thus a, b generate A as an algebra. Now we consider $A[t]$ and define:

$$\begin{aligned} x &:= ta \\ y &:= tb. \end{aligned}$$

We compute:

$$\begin{aligned} x^2 &= t^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = t^2 \mathbb{1} \\ xy &= t^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = t^2(e_2 - e_3) \\ yx &= t^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = t^2(e_3 - e_2) \\ y^2 &= t^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = t^2 \mathbb{1} \end{aligned}$$

We have obtained the relations:

$$\begin{aligned} p_1 &= x^2 - t^2\mathbb{1} \\ p_2 &= x^2 - y^2 = 0 \\ p_3 &= xy + yx = 0 \\ p_4 &= y^2 - t^2\mathbb{1}. \end{aligned}$$

Now we consider larger products to produce more relations:

$$x^3 = t^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = t^2 x$$
$$y^3 = t^3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = t^2 y$$

hence we have also obtained:

$$\begin{aligned} p_5 &= x^3 - t^2 x = 0 \\ p_6 &= y^3 - t^2 y = 0. \end{aligned}$$

These allow us to derive the following:

$$\begin{aligned} x^2y = y^3 = yx^2 = y^3 = t^2 y \\ y^2x = x^3 = xy^2 = x^3 = t^2 x, \end{aligned}$$

which are enough to reduce an arbitrary product of x, y. Thus we present:

$$N := \mathbb{C}[x,y][t] \Big/ \langle p_1, p_2, p_3, p_4, p_5, p_6 \rangle \, .$$

We observe:

$$N_0 = \mathbb{C}[x,y] \Big/ \langle x^2, y^2, xy + yx \rangle \, ,$$

and

$$N_1 = \mathbb{C}[x,y] \Big/ \langle x^2 - 1, y^2 - 1, xy + yx, x^3 - x, y^3 - y \rangle \, .$$

By Proposition 14 we have $N_1 \cong A$. The reader may check the isomorphism is given by:

$$\begin{aligned} \phi : N_1 &\to A \\ 1 &\mapsto \mathbb{1} \\ x &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ y &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Thus we have produced an algebra N_0 that has $A = \mathcal{M}_2(\mathbb{C})$ as a deformation.

4.5.4 Future Work

In this section we pose some questions about Method 1 and suggest some generalisations.

Question 1. Let A be a $\mathbb{C}$-algebra and suppose $G_1 = \{a, b\}$ and $G_2 = \{a', b'\}$ are two distinct generating sets of A. Under what conditions does Method 1 using A, G_1 and A, G_2 result in the same algebra N_0 that has A as a deformation?

Question 2. Let A be a $\mathbb{C}$-algebra. Under what conditions does Method 1 only produce one algebra N_0 that has A as a deformation.

Question 3. Let A be a $\mathbb{C}$-algebra. Does their exist a finitely terminating algorithm that will produce all the algebras N_0 arising from Method 1 that have A as a deformation?

Question 4. Let A be a $\mathbb{C}$-algebra. Do ring theoretic properties of the generators a, b (for example being idempotent, being irreducible idempotent, prime) determine any properties of the algebra N_0 arising from Method 1 using $A, \{a, b\}$?

Question 5. Let A be a $\mathbb{C}$-algebra. Do ring theoretic properties of A (for example being semi-simple, local) determine any properties of the algebra N_0 arising from Method 1 using A?

Question 6. Can Method 1 be generalised to be used for algebras over a general field or a general commutative ring?

The technique described in Method 1 yields some natural generalisations. Firstly we note that defining x, y as described is quite a strong restriction. At the cost of more complicated computations one could investigate what happens when x, y are higher degree polynomials in $A[t]$, for example:

$$x := at^2$$
$$y := t^3 b.$$

Question 7. We consider Method 1 but in step (1) we instead define:

$$x := t^{n_a} a$$
$$y := t^{n_b} b$$

for $n_a, n_b \in \mathbb{N}$. Will this allow us to produce a more general class of algebras N_0 that have A as a deformation?

We note that Method 1 requires A to be generated by 2 elements. We could extend this to 3 generators as follows: consider a $\mathbb{C}$-algebra A generated by a, b, c. We pass to $A[t]$ and define:

$$x := ta$$
$$y := ty$$
$$z := tc$$

and then proceed as in Method 1.

Question 8. Does there exist a generalisation of Method 1 for $\mathbb{C}$-algebras A that are not finitely generated?

Chapter 5

Conclusion

In Chapter one we developed enough theory to describe abelian categories as they axiomatize the setting for homological algebra. A reader who enjoyed this section is encouraged to read [5]. Empowered by Mitchell's embedding theorem we were granted the right to restrict our attention to the concrete categories of modules over associative unital rings. Some of their theory was expounded, and their tensor product was constructed. We then proceeded with the basic components of homological algebra, defining chain complexes, homologies and their duals. Had there been more time a discussion of derived functors would have been included. The interested reader is encouraged to read *Introduction to Homological Algebra* by Charles A. Weibel. The next section treated the practicalities of a particular homology theory - that of associative algebras. Characterisations of the arising homology and cohomology groups were provided and some calculations were presented. The reader is encouraged to read [9] for more details. In our final chapter we introduced algebraic deformation theory and applied our knowledge of Hochschild cohomology. We closed by presenting a novel method to produce associative deformations with a specified target algebra. The reader may want to read some of the works of Murray Gerstenhaber on the subject.

The reader is now equipped to engage in further study of homological algebra. Some recommended topics are group cohomology, galois cohomology and the theory of derived categories. They been exposed to the theory of algebraic deformations. The physically minded reader is encouraged to read about their applications to mathematical physics in *Noncommutative Deformation Theory* by Eivind Eriksen, Olav Arnfinn Laudal, Arvid Siqveland.

Appendix A

Modules

Definition 45. Let R be an associative unital ring, then a *left R-module* is an abelian group M equipped with an R action:

$$\begin{aligned} R \times M &\to M \\ (r, m) &\mapsto rm \end{aligned}$$

such that for all $r, r' \in R$ and $m, m' \in M$

$$r(m + m') = rm + rm' \tag{A.1}$$
$$(r + r')m = rm + r'm \tag{A.2}$$
$$r(r'm) = (rr')m \tag{A.3}$$
$$1_R m = m \tag{A.4}$$

We can define a *right R-module* analogously, it is an abelian group M equipped with an R action:

$$\begin{aligned} M \times R &\to M \\ (m, r) &\mapsto mr \end{aligned}$$

such that for all $r, r' \in R$ and $m, m' \in M$

$$(m + m')r = mr + m'r \tag{A.5}$$
$$m(r + r') = mr + mr' \tag{A.6}$$
$$(mr)r' = m(rr') \tag{A.7}$$
$$m1_R = m. \tag{A.8}$$

Let S be another associative unital ring then we can define a *(R, S)-bimodule* as an abelian group M that is a left R-module and a right S-module.

A left R-module M may be denoted ${}_RM$, and similarly R may appear on the right or both sides to denote a right or bimodule respectively.

Example 24. These are quite common:

- Let k be a field then left k-modules are precisely vector spaces over k. We note that a left k-module is also a right k-module and a (k, k)-bimodule due to the commutativity of k (hence forth we shall write k-bimodule when both rings are the same).

- Let R be an associative unital ring, then R itself is clearly an R-bimodule. One sided ideals of R correspond to one sided modules over R and two sided ideals correspond to bimodules.

- We again consider a ring R and consider $n \times n$ matrices with entries in R for fixed n, this too gives an R-bimodule.

- We note that any bimodule will trivially give rise to a left or right module by simply forgetting the action on the second side.

Next we define a homomorphism of R-modules:

Definition 46. Let R be an associative unital ring and M, N be left R-modules. Then an *R-homomorphism* is a function $f : M \to N$ such that for $m, m' \in M$ and $r \in R$:

- $f(m + m') = f(m) + f(m')$

- $f(rm) = rf(m)$.

A bijective R-homomorphism is called an *R-isomorphism*.

For a ring R its left modules and their homomorphisms form a category using the usual compositions and identities.

The ring $\mathbb{Z}$ has a special property where modules are concerned:

Proposition 16. Let R be a ring and we consider a module ${}_RM$. Then M is also a left $\mathbb{Z}$-module.

Proof. We consider the unique ring homomorphism $\phi : \mathbb{Z} \to R$. Then we define an action on M:

$$\begin{aligned} \mathbb{Z} \times M &\to M \\ (z, m) &\mapsto \phi(z)m \end{aligned}$$

and thus ${}_RM$ can be regarded as ${}_{\mathbb{Z}}M$. □

Definition 47. Let R be a ring and M a left R-module then a *submodule* $N \subseteq M$ is a subgroup of the underlying group of M that is closed under left scalar multiplication. Submodules for right and bimodules can be defined analogously.

Example 25.

- For a module M both $\{1\}$ and M are submodules of M,

- a submodule of a vector space is a vector subspace,

- for a ring R we can view it as a left R-module and then its left ideals are submodules.

Definition 48. Let R be a ring and M a left R-module then a *quotient module* of M by a submodule N is the quotient group ${}^{M}/_{N}$ equipped with the scalar multiplication:

$$R \times {}^{M}/_{N} \to {}^{M}/_{N}$$
$$(r, m + N) \mapsto rm + N.$$

Then the natural map $\pi : M \to {}^{M}/_{N}$ is a surjection as well as a left R-module homomorphism. Quotient module s for right and bimodules can be defined analogously.

Appendix B

Homological Algebra

B.1 Functoriality of hom and tensor for modules

In this section we will always regard R to be a unital associative ring and we consider the category ${}_R\mathbf{Mod}$. We will describe tensor as a functor and prove that both hom and tensor are additive.

B.1.1 Hom

We recall from Example 5 that we can regard hom as a functor and we can write its signature:

$$\mathrm{Hom}(\square, \square) : {}_R\mathbf{Mod}^{\mathrm{op}} \times {}_R\mathbf{Mod} \to \mathbf{Ab}.$$

Proposition 17. We recall the notation used in Example **??** and fix an $A \in \mathrm{Ob}({}_R\mathbf{Mod})$ and let $T_A := \mathrm{Hom}_{{}_R\mathbf{Mod}}(A, \square)$. We similarly define $L_C := \mathrm{Hom}_{{}_R\mathbf{Mod}}(\square, C)$. Then T_A is an additive functor for all A and L_C is an additive functor for all C.

Proof. We only provide the proof that T_A is additive since the proof that L_C is additive will be identical but with the arrows reversed. We want to show that for all $A, B, C \in \mathrm{Ob}({}_R\mathbf{Mod})$ the map induced by T_A:

$$\phi : \mathrm{Hom}_{{}_R\mathbf{Mod}}(B, C) \to \mathrm{Hom}_{\mathbf{Ab}}(T_A(B), T_A(C))$$

is a group homomorphism. We recall that both ${}_R\mathbf{Mod}$ and $\mathbf{Ab}$ are additive categories and hence their hom-sets are always abelian groups. Next we observe that for each $f : B \to C$ we obtain an induced map

$$T_A(B) \to T_A(C)$$

that is compatible with their abelian group structures. Thus it holds that:

$$\mathrm{Hom}_{{}_R\mathbf{Mod}}(B, C) \subseteq \mathrm{Hom}_{\mathbf{Ab}}(T_A(B), T_A(C)) \text{ as groups.}$$

Thus the map ϕ is simply the inclusion:

$$\mathrm{Hom}_{{}_R\mathbf{Mod}}(B, C) \hookrightarrow \mathrm{Hom}_{\mathbf{Ab}}(T_A(B), T_A(C))$$

and is therefore a group homomorphism. Since our argument does not rely on a choice of A, B, C it holds that T_A is an additive functor for all $A \in \mathrm{Ob}({}_R\mathbf{Mod})$. $\square$

B.1.2 Tensor

First we show how tensor is a functor and then we shall prove it is additive:

Proposition 18 ([12])**.** Let $f : A_R \to A'_R$ and $g : {}_RB \to {}_RB'$ be maps of R-modules. Then there exists a unique R-homomorphism:

$$\begin{aligned} f \otimes g : A \otimes_R B &\to A' \otimes_R B' \\ (a \otimes b) &\mapsto (f(a) \otimes g(b)). \end{aligned}$$

Proof. We consider the R-biadditive map:

$$\begin{aligned} h : A \oplus B &\to A' \otimes_R B' \\ (a, b) &\mapsto f(a) \otimes g(b). \end{aligned}$$

By the definition of the tensor product there exists a unique R-homomorphism that makes:

$$\begin{tikzcd} A \oplus B \arrow[r, "\phi"] \arrow[dr, "h"'] & A \otimes_R B \arrow[d, "\hat{h}"] \\ & A' \otimes_R B' \end{tikzcd}$$

commute. Using the commutativity we observe:

$$h(a, b) = f(a) \otimes g(b) = \hat{h}(\phi(a, b)) = \hat{h}(a \otimes b)$$

and thus $\hat{h}(a \otimes b) = f(a) \otimes g(b)$ as required. $\square$

Corollary 1. Given maps of right R-modules $A \xrightarrow{f} A' \xrightarrow{f'} A''$ and maps of left R-modules $B \xrightarrow{g} B' \xrightarrow{g'} B''$, then:

$$(f \otimes g) \circ (f' \otimes g') = (f' \circ f) \otimes (g' \circ g).$$

Proof. Both maps send $a \otimes b \mapsto f'(f(a)) \otimes g'(g(b))$ for all $a \in A$ and $b \in B$. $\square$

We are now ready to prove:

Theorem 12. Let $A_R, {}_RB$ be R-modules then there are additive functors $F_A : {}_R\mathbf{Mod} \to \mathbf{Ab}$ and $G_B : \mathbf{Mod}_R \to \mathbf{Ab}$ defined as:

$$F_A(M) = A \otimes_R M \quad \text{and} \quad F_A(f) = 1_A \otimes f$$

for $f : M \to M'$, and

$$G_B(M) = M \otimes_R B \quad \text{and} \quad G_B(g) = g \otimes 1_B$$

for $g : M \to M'$.

Proof. We only prove the functoriality and additivity of F_A since the proof is identical for G_B.

- First we show functoriality. By Corollary 1 F_A preserves composition and it preserves identities because:

$$F_A(1_M) = 1_A \otimes 1_M = 1_{A \otimes_R M}$$

since it fixes every $a \otimes m$ for $a \in A$ and $m \in M$.

- Now we show additivity. We consider maps $f, g : M \to M'$ and proceed:

$$F_A(f + g) = 1_A \otimes (f + g) = 1_A \otimes f + 1_A \otimes g$$

where the last equality follows from the relation 2.3.

Thus F_A is an additive functor. $\square$

Appendix C

Proofs for Method 1

In this section the proofs of Proposition 14 and Proposition 15 are presented. They appear here as they shall appear in the upcoming preprint. The author does not claim them to be their own work and in fact they are the result of joint work with Agata Smoktunowicz and Dora Puljić.

C.1 Proposition 14

We recall $\mathbb{C}[x, y][t]$ denotes the algebra where x, y do not commute with each other but do commute with t and consider the following diagram.

$$\begin{tikzcd}
& \mathbb{C}[x,y][t] \arrow[dl, "can"'] \arrow[d, "f"] \arrow[dr, "\xi"] & \\
\mathbb{C}[x,y][t] \big/ I \arrow[r, "i"'] \arrow[dr, "h"'] & A[t] \arrow[r, "g"'] & A \\
& \mathbb{C}[x,y][t] \Big/ I + \langle t-1 \rangle &
\end{tikzcd}$$

where:

$$
\begin{aligned}
f : \mathbb{C}[x,y][t] &\to A[t]\\
x &\mapsto at\\
y &\mapsto bt\\
t &\mapsto t\\
g : A[t] &\to A\\
ta &\mapsto a\\
tb &\mapsto b\\
t &\mapsto 1\\
h : \mathbb{C}[x,y][t]\big/ I &\to \mathbb{C}[x,y][t]\Big/ I + \langle t-1\rangle\\
x &\mapsto x\\
y &\mapsto y\\
t &\mapsto 1\\
\xi : \mathbb{C}[x,y][t] &\to A\\
x &\mapsto a\\
y &\mapsto b\\
1 &\mapsto 1
\end{aligned}
$$

$\ker(f) = I$ and $i : \mathbb{C}[x,y][t]\big/ I \cong \operatorname{im}(f) \subseteq A[t]$. Next we establish some notation. For a monomial p_i in $\mathbb{C}[x,y]$ we will denote by p_i the same product as p_i, but with all instances of x replaced by ta, and all instances of y replaced by tb. We will denote by p_i the same product as p_i, but with all instances of x replaced by a, and all instances of y replaced by b.

Proposition 19. Let $\xi : \mathbb{C}[x,y][t] \to A$ be such that

$$\xi(x) = \sum_{i=0}^{n} a_i, \quad \xi(y) = \sum_{i=0}^{n} b_i, \quad \xi(t) = 1.$$

Then $\ker \xi = I + \langle t-1\rangle$.

Proof. We let $e = \sum_i \alpha_i p_i t^{\beta_i} \in \ker\xi$ where p_i are monomials in $\mathbb{C}[x,y]$, $\alpha_i \in \mathbb{C}$ and $\beta_i \in \mathbb{N}$. We will show there exists $\gamma_i \in \mathbb{N}$ such that $\hat{e} := \sum_i \alpha_i p_i t^{\gamma_i} \in I$, so that $e \in I + \langle t-1\rangle$ as $e - \hat{e} \in \langle t-1\rangle$. This follows since

$$e - \hat{e} = \sum_i \alpha_i p_i t^{\beta_i} - \sum_i \alpha_i p_i t^{\gamma_i} = \sum_i \alpha_i p_i (t^{\beta_i} - t^{\gamma_i}) = \sum_i \alpha_i p_i t^m (t^n - 1) \in \langle t-1\rangle$$

for some $m, n \in \mathbb{N}$.

Note that $\xi(e) = \sum_i \alpha_i p_i = 0$. Hence for large enough $k \in \mathbb{N}$ we have

$$t^k \sum_i \alpha_i p_i = \sum_i \alpha_i p_i t^{k - \operatorname{len} p_i} = 0.$$

Notice that by assumption there exist $j_i \in \mathbb{N}$ such that $p_i t^{j_i} \in \operatorname{im} f$. We let $f(c_i) = p_i t^{j_i}$ for some $c_i \in \mathbb{C}[x, y][t]$. It follows that

$$f(\sum_i \alpha_i c_i t^{k-j_i-\operatorname{len} p_i}) = \sum_i \alpha_i p_i t^{k-\operatorname{len} p_i} = 0.$$

□

Proposition 20. We have

$$\frac{\mathbb{C}[x, y][t]/I}{\langle t-1+I\rangle} \cong A.$$

Proof. Note that the ideal $\langle t-1+I\rangle$ of $\mathbb{C}[x, y, t]/I$ equals the set

$$\frac{\mathbb{C}[x, y][t](t-1)+I}{I} = \{g(t-1)+I \mid g \in \mathbb{C}[x, y, t].\}$$

By the third isomorphism theorem we have

$$\frac{\mathbb{C}[x, y[t]/I}{(\mathbb{C}[x, y][t](t-1)+I)/I} \cong \frac{\mathbb{C}[x, y, t]}{\mathbb{C}[x, y][t](t-1)+I}.$$

Now notice that $\mathbb{C}[x, y][t](t-1)+I = \ker \xi$ and as ξ is onto,

$$\frac{\mathbb{C}[x, y][t]}{\ker \xi} \cong A$$

by the first isomorphism theorem. □

C.2 Proposition 15

Recall that the algebra

$$\mathbb{C}[x, y, t]/I \,/ < t-1+I >$$

as in Proposition 19. Moreover, as showed in the proof of Proposition 20, this algebra is isomorphic to the algebra $\mathbb{C}[x, y][t]/I)/ < I, t-1 >$. We need to show that this algebra is isomorphic to the algebra:

$$\mathbb{C}[d_1, d_2, \ldots d_n]I'$$

where I' is the ideal generated in $\mathbb{C}[d_1, d_2, \ldots d_n]$ by elements:

$$d_k * d_m - \sum_{i=1}^{n} (\zeta_{i,k,m} d_k + \xi_{i,k,m}(1) d_k).$$

Notice that these relations are the specification of the deformation relations $d_k * d_m - \sum_{i=1}^{n} (\zeta_{i,k,m} d_k + t\xi_{i,k,m}(t) d_k)$ at $t = 1$.

Consider the following map:

$$\phi : \mathbb{C}[d_1, \ldots, d_n] \to \mathbb{C}[x, y, t] \,/ < I, t-1 >,$$

given by

$$\xi(d_i) = q_i + < I, t-1 >$$

for $i = 1, 2, \ldots, n$.

Observe that $I' \in ker(\xi)$ since

$$\begin{aligned}
&\xi\left(d_k * d_m - \sum_{i=1}^{n}(\zeta_{i,k,m} d_k + \xi_{i,k,m}(1) d_k\right) \\
=&c_k * c_m - \sum_{i=1}^{n}(\zeta_{i,k,m} c_k + \xi_{i,k,m}(1) c_k) + < I, t-1 >,
\end{aligned}$$

since $c_k * c_m - \sum_{i=1}^{n}(\zeta_{i,k,m} c_k + \xi_{i,k,m}(1) c_k) \in I + < t-1 >$.

Therefore

$$I' \subseteq ker(\xi).$$

Therefore the dimension of $\mathbb{C}[d_1, \ldots, d_n] \Big/ ker(\xi)$ does not exceed the dimension of $\mathbb{C}[d_1, \ldots, d_n] / I'$. Notice that $\mathbb{C}[d_1, \ldots, d_n] / I'$ is spanned as linear space by elements $d_i + I$, and hence has dimension at most n. On the other hand, the the first Isomorphism theorem for rings, $\mathbb{C}[d_1, \ldots, d_n] \Big/ ker(\xi)$ is isomorphic to $\mathrm{im}(\xi) = \mathbb{C}[x, y, t] / < I, t-1 >$, which in turn is isomorphic to A by Proposition 20, and hence has dimension n. It follows that $I' = ker(\xi)$ and hence $\mathbb{C}[d_1, \ldots, d_n] \Big/ ker(\xi)$ is isomorphic to $\mathbb{C}[d_1, \ldots, d_n] / I'$.

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