4
The Solow Model

Tools: Capital accumulation dynamics; Cobb-Douglas production function.

Key Words: Investment; saving; depreciation; steady state; convergence.

Big Ideas:

- The Solow model connects saving and investment with economic growth.
- In the Solow model without productivity (TFP) growth, capital accumulation does not generate long-run growth. The reason is diminishing returns to capital: the impact of additional capital declines the more you have. As a result, differences in saving rates have only modest effects on output per worker and none at all on its long-run growth rate.
- TFP growth generates long-run growth in output per worker.

We see large differences in saving and investment rates across countries, with (for example) the US investing 20 percent of GDP, China 40 percent, and India 30 percent in recent years (ratios of real investment to real GDP from the Penn World Tables). How important are these differences to the growth rates of countries? The answer: not important at all. Why? Because diminishing returns to capital means (in practice) that additional capital generates smaller and smaller additions to output. This insight comes from work by Robert Solow, who received the 1987 Nobel Prize in economics for his work. His model is also a useful tool for extrapolating current trends and pointing out the critical inputs to any such exercise.

4.1 The model

Solow's model has four relatively simple components. The first is our friend the production function:

$$Y_t \;=\; A_t F(K_t, L_t) \;=\; A_t K_t^{\alpha} L_t^{1-\alpha}. \tag{4.1}$$

Changes in output, therefore, come from changes in (total factor) productivity, capital, and/or labor. Recall that one of the properties of this production function is diminishing returns to capital — each additional unit of capital leads to a smaller addition to output. This is the critical ingredient in what follows. The second component is a link between investment and saving. You'll recall that the flow identity, $S = I + NX$, linked saving to investment and net exports. Solow ruled out the last one (we can put it back later if we like), giving us

$$S_t \;=\; I_t.$$

Lurking behind the scenes here is the expenditure identity, $Y = C + I$ in this case.

The third component is a description of saving behavior: people save a constant fraction s of their income,

$$S_t \;=\; sY_t,$$

where the saving rate s is a number between zero and one. This is a little simplistic — you might expect saving to depend on the rate of return and/or expectations of future income — but there is a lot to be said for simplicity. For our purposes, s is really the investment rate (the ratio of investment to GDP), but since saving and investment are the same here, we can call it the saving rate. Finally, the capital stock depreciates at a constant rate δ, so that

$$K_{t+1} \;=\; (1-\delta)K_t + I_t, \tag{4.2}$$

where the depreciation rate δ is a number between zero and one.

The model consists of these four equations. This seems kind of simple for a Nobel Prize, but they really are good equations. Now let's see where they lead.

4.2 Capital dynamics

Let's think about how the model behaves if the labor input L and productivity A are constant. Analysis of the model in this case consists of describing

Table 4.1: Output dynamics in the Solow model.

Date t	Capital Stock K	Output Y
0	250.0	135.7
1	252.1	136.1
2	254.2	136.5
3	256.0	136.8
4	257.8	137.1
5	259.4	137.4
6	261.0	137.7
7	262.4	137.9
8	263.8	138.2
9	265.0	138.4
10	266.2	138.6

how the capital stock evolves through time. Other variables follow from their relations to the capital stock. We can find output from the production function, saving (= investment) from output, and consumption (should we need it) from the expenditure identity ($C = Y - I$).

The key step is to describe how the capital stock changes from one period to the next. To do that, we add time subscripts to the equations that don't have them already. Then, with a little work, we see that the capital stock behaves like this:

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + I_t \\ &= (1-\delta)K_t + S_t \\ &= (1-\delta)K_t + sY_t \\ &= (1-\delta)K_t + sAK_t^{\alpha}L^{1-\alpha}. \end{aligned} \tag{4.3}$$

Note that each step follows from one of the components of the model. The result is a formula for computing K_{t+1} from K_t and some other stuff. If we have numerical values for the parameters (A, α, s, δ), we can do the computations in a spreadsheet or other program and see how K moves through time.

Example. A numerical example will show you how this works. Let $L = 100$, $A = 1$, $s = 0.2$, $\delta = 0.1$, and $\alpha = 1/3$. (We'll use the same parameters throughout.) If the initial capital stock is 250, we can compute future values of the capital stock by applying equation (4.3) repeatedly. We then compute output from the capital stock using the production function. The results for this case are summarized in Table 4.1. [Suggestion: Try to reproduce a few periods of the table to make sure you understand how it works. If you get stuck, read the last two pages again. The trick is to set up formulas that tie each period to the previous one.]

You can see in the table that capital and output both increase over time. Will they increase forever? The answer is no, but it takes a little work to show. (Alternatively, you could extend the simulation and see what happens.) This is an important conclusion, because it tells us saving and capital formation can't be the reason (in this model, anyway) that some countries grow faster than others. More on this soon.

The dynamics of the capital stock reflect a balance of two factors: (i) saving tends to increase the capital stock by financing new investment and (ii) depreciation tends to reduce it. A modest change to equation (4.3) makes this clear:

$$\Delta K_{t+1} \;\equiv\; K_{t+1} - K_t \;=\; sAK_t^{\alpha}L^{1-\alpha} - \delta K_t. \tag{4.4}$$

(The equal sign with three lines means that the equation defines the expression that comes before it, in this case ΔK_{t+1}.) You can see that the change is zero (the capital stock doesn't change) when

$$K_{ss} \;=\; \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)} L,$$

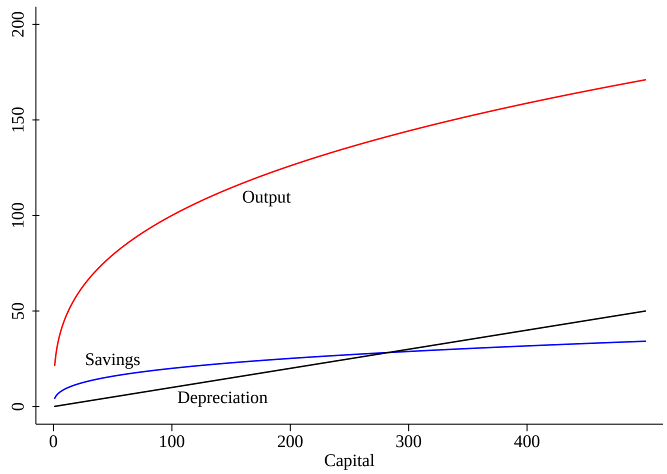
where K_{ss} is the "steady-state" capital stock. This is a little complicated, but remember: it's just a formula. In our example, $K_{ss} = 282.8$, so we have a ways to go before the model reaches its steady state.

What happens if we are above or below K_{ss}? You can get a sense of the dynamics from Figure 4.1. The top line is output, which is related to the capital stock through the production function. The next line is saving, a constant fraction of output and the first expression on the right side of equation (4.4): $sAK^{\alpha}L^{1-\alpha}$. The third line is depreciation, a constant fraction δ of the capital stock and the second object on the right side of equation (4.4): δK. Diminishing returns to capital gives the saving line its curvature. It leads to higher saving than depreciation at low values of the capital stock, so the capital stock is increasing. Similarly, saving is lower than depreciation at high values of the capital stock, so the capital stock falls. The crossing point is K_{ss}, where saving is just enough to make up for depreciation, leaving the capital stock unchanged.

4.3 Convergence

The central feature of the model is what we call the convergence property: If countries have the same parameters, they will eventually converge to the

Figure 4.1: The Solow model.



same level of output per worker. We haven't quite shown this yet, but the only thing missing is the "per worker" qualification.

Consider, then, a version of the model in per-worker terms. The first step is to divide both sides of (4.3) by L. If $k \equiv K/L$ is capital per worker (or the capital-labor ratio), the equation becomes

$$k_{t+1} \quad = \quad (1-\delta)k_t + sAk_t^{\alpha}$$

or

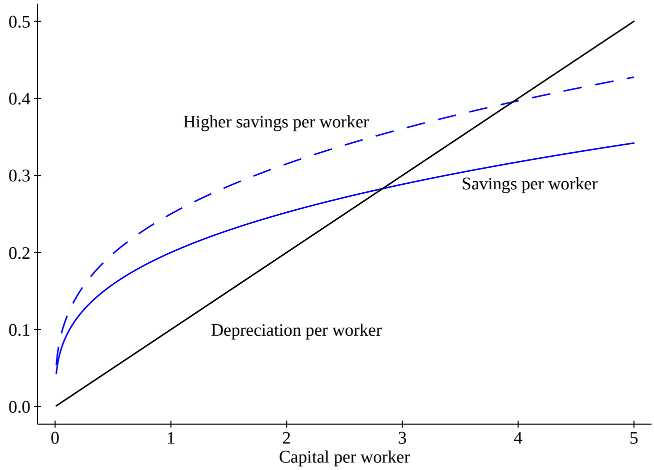
$$\Delta k_{t+1} \quad \equiv \quad k_{t+1} - k_t \quad = \quad sAk_t^{\alpha} - \delta k_t. \tag{4.5}$$

You'll note a resemblance to equation (4.4).

Figure 4.2 illustrates the model's dynamics. It's based on the same parameter values as our earlier example: $A = 1$, $s = 0.2$, $\delta = 0.1$, and $\alpha = 1/3$. The line marked "saving per worker" is the first expression on the right side of equation (4.5): sAk^{α}. The line marked "depreciation per worker" is the second expression on the right side of equation (4.5): δk. For small values of k, saving per worker is greater than depreciation per worker, so k increases. For large values of k, saving per worker is less than depreciation per worker, so k decreases. The two lines cross at the steady state, where the capital-labor ratio is constant. We can find the steady-state value of k from equation (4.5) by setting $\Delta k_{t+1} = 0$. This leads to

$$k_{ss} \;=\; \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)},$$

Figure 4.2: The impact of the saving rate in the Solow model.



a minor variant of our earlier expression for the steady-state capital stock.

We have shown that the capital-labor ratio eventually converges to its steady-state value. What about output per worker? The production function in per worker form is $Y/L = Ak^\alpha$, so steady-state output per worker depends on steady-state capital per worker:

$$(Y/L)_{ss} \;=\; Ak_{ss}^\alpha \;=\; A\left(\frac{sA}{\delta}\right)^{\alpha/(1-\alpha)} \;=\; A^{1/(1-\alpha)}\left(\frac{s}{\delta}\right)^{\alpha/(1-\alpha)}. \quad (4.6)$$

Similarly, the steady-state capital-output ratio is

$$(K/Y)_{ss} \;=\; \left(\frac{s}{\delta}\right).$$

The algebra isn't pretty, but it tells us how the steady state depends on the various parameters. The last equation tells us, for example, that countries with higher saving rates also have higher steady-state capital-output ratios — that is more saving leads to more capital. Equally important, the existence of a steady state tells us that if two countries have the same parameter values, they will converge to the same output per worker. We refer to this as the convergence property. In this model, any long-term differences between countries must come from differences in their parameters.

4.4 Impact of saving and investment

We can return to the question we began with: What is the impact of saving and investment rates on growth and income? The long-run impact of saving on growth is zero; the steady-state growth rate is zero, regardless of the saving rate. But there is an effect of saving on steady-state output per worker.

Consider our example. From equation (4.6), we see that steady-state output per worker is 1.4142. What if we increase the saving rate s from 20 percent to 25 percent? Then, steady-state output rises to 1.5811, an 11 percent increase. This isn't irrelevant, but it's a relatively modest increase for a substantial increase in saving. It clearly does not explain much of the enormous differences in GDP per capita that we see around the world.

We can see the same thing in Figure 4.2. The line marked "saving per worker" is based on a saving rate of $s = 0.20$, or 20 percent. If we raise the saving rate to 25 percent, the saving line shifts up, as shown by the dashed line marked "higher saving per worker." Why? Because sAk^α is higher at every value of k. With this new line, the steady-state value of capital per worker (where the saving line crosses the depreciation line) is higher, as shown.

4.5 Growth

If saving doesn't generate growth, what does? We add growth in the labor force and (critically) growth in total factor productivity with two goals in mind. The first goal is to account for the growth rate of output, showing how it depends on the growth rates of our two inputs. The second is to show that the economy approaches what we call a balanced growth path in which output and capital grow at the same rate. As before, the capital-output ratio approaches a constant, the features of which we can easily summarize. We do this with a striking example in mind: We know that China invests an astounding 40 percent of its GDP. Is this too much? A hint is that capital intensity (measured by the capital-output ratio) depends not only on the investment rate (which tells us how much new capital is added), but also on the growth rate (how fast the denominator is changing). A fast-growing economy needs a high investment rate simply to maintain a given capital-output ratio.

The new inputs into our analysis are growth in the labor force and productivity. Let us say, to be concrete, that labor and productivity grow at constant rates:

$$\begin{aligned} L_{t+1} &= (1+g_l)L_t \\ A_{t+1} &= (1+g_a)A_t. \end{aligned}$$

How fast do output and capital grow? Let's guess that output and capital grow at the same rate g_y, to be determined. (Why? Because we're good guessers.) From the production function, we then know that

$$\begin{aligned} (1+g_y) &= Y_{t+1}/Y_t \\ &= (A_{t+1}/A_t)(K_{t+1}/K_t)^\alpha (L_{t+1}/L_t)^{1-\alpha} \\ &= (1+g_a)(1+g_y)^\alpha (1+g_l)^{1-\alpha}. \end{aligned}$$

The growth rate, is therefore,

$$(1+g_y) = (1+g_a)^{1/(1-\alpha)}(1+g_l).$$

Just a formula, but it says that output growth is tied to the growth rates of productivity and labor. The saving rate does not affect this growth rate. Similarly, the growth rate in output per worker is

$$(1+g_y)/(1+g_l) = (1+g_a)^{1/(1-\alpha)},$$

which depends only on productivity growth. If α is positive, the growth rate of output per worker is higher than the growth rate of productivity, because the exponent $1/(1-\alpha)$ is greater than one. In words, the direct impact of productivity on output is magnified by the growth in the stock of capital; see the production function (4.1). This ties in with a remark we made earlier: That capital accumulation tends to reinforce the impact of productivity growth. Countries with high productivity also have a lot of capital.

What about capital — do countries with higher saving rates have more capital, relative to the size of their economies? Consider, again, a steady state in which capital and output grow at the same rate g_y. Then $K_{t+1} = (1+g_y)K_t$ and equation (4.3) becomes

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + sY_t \\ (1+g_y)(K_t/Y_t) &= (1-\delta)(K_t/Y_t) + s. \end{aligned}$$

Solving for K/Y gives us the steady-state capital-output ratio:

$$(K/Y)_{ss} = \left(\frac{s}{\delta + g_y}\right).$$

To return to our goal of understanding the sources of capital intensity, note the impact of growth on the steady-state capital-output ratio. For a given saving/investment rate s, countries with higher growth g_y will have relatively less capital per unit of output. Why? Because when output is growing quickly, you need to invest a lot to keep capital growing at the same rate.

Example. Here are some numbers based loosely on the US: $g_l = 0.005$ (0.5%), $g_a = 0.01$ (1%), $s = 0.15$, and $\delta = 0.06$. What is the growth rate of output? The steady-state capital-output ratio? The growth rate satisfies

$$1 + g_y \;=\; (1 + g_a)^{1/(1-\alpha)}(1 + g_l) \;=\; 1.015 \times 1.005 \;=\; 1.0201.$$

Here, we've used $\alpha = 1/3$, as usual. Using the same parameters as our earlier examples, the steady-state capital-output ratio is

$$1.872 \;=\; \frac{0.15}{0.06 + 0.020}.$$

Now consider numbers based on China. We keep $g_l = 0.005$ and $\delta = 0.06$, but change the others to $g_a = 0.04$ and $s = 0.40$. The growth rate is now $g_y = 0.0659$ and the capital-output ratio is 3.17. Note the moderate increase, despite the near tripling of the saving/investment rate. Is China investing too much? Perhaps not. Their capital-output ratio (by this calculation) is not much different from that of the US, so the 40% investment rate isn't delivering excessive capital intensity by this measure. They need to invest a lot simply to keep up with the growth of their economy.

Executive summary

1. Solow's model bases growth on saving and investment.
2. Saving affects steady-state GDP per worker, but not its growth rate. In this sense and others, saving is secondary to long-term economic performance.
3. Fast-growing countries must invest more to maintain the same capital-output ratio.

Review questions

1. The basics. Suppose $A = L = K = 1$, $\alpha = 1/3$, $\delta = 0.06$, and $s = 0.12$.
   (a) What is output Y?
   (b) What are saving S and investment I?
   (c) What is next period's capital stock?

Answer.

(a) $Y = AK^{1/3}L^{2/3} = 1$.

(b) $S = I = sY = 0.12$.

(c) Put t's on everything so far. Then $K_{t+1} = (1-\delta)K_t + I = 0.94 + 0.12 = 1.06$.

2. Example, continued. For the numerical example in the text:

(a) Suppose that the economy starts with the steady-state capital stock. What are the steady-state levels of output, investment, and consumption?

(b) If 25 percent of the capital stock is destroyed in a war, how long does it take the economy to eliminate half the fall in output?

Answer.

(a) The steady-state capital stock is (as we've seen) $K_{ss} = 282.8$. Using this value, the production function tells us that output is 141.4. Investment equals the depreciation of the capital stock, 28.3. We can find consumption in two ways. The first is through the expenditure identity: $Y = C + I$. We know Y and I, so C is 113.1. The second is through the flow identity. Saving is fraction s of output, so consumption is fraction $1-s$, $0.8 \times 141.4 = 113.1$.

(b) This requires a simulation. Let the capital stock fall to 212.1, 75 percent of its steady-state value. Then, output is 128.5, 90.9 percent of its steady-state value. We recover half the fall if output rises to 135.0. If we simulate the model, we see that it reaches 135.1 in 10 periods (years).

3. Government. We've ignored government so far. Suppose, instead, that the government purchases goods and services equal to a constant fraction of GDP (say, $G = dY$ for some fraction d) and collects taxes equal to the same fraction of output. Individuals have after-tax income of $(1-d)Y$ and save a fraction s of it. With these changes, how would the analysis of the basic Solow model change?

Answer. The critical ingredient here is the fraction of output allocated to investment. Investment here is $I = S = s(1-d)Y$. Effectively, d reduces the saving rate from s to $s(1-d)$ and takes resources away from investment. If the government invests, we'd have to include that, but we'd also have to decide how useful the investment was (does it count the same as other investment?).

If you're looking for more

This material is covered in many macroeconomics textbooks. Our favorites are

- Tyler Cowen and Alex Tabarrok, *Modern Principles: Macroeconomics*, ch 7.
- N. Gregory Mankiw, *Macroeconomics (6th edition)*, chs 7-8.

Any editions will do, but the chapter numbers may vary.

Goldman Sachs has used the Solow model (and some heroic assumptions about fundamentals) to forecast the importance of the BRICs (Brazil, Russia, India, and China) to the world economy in 50 years. See "Dreaming with BRICs." It's a good example of how assumptions about productivity, population growth, and education can be used to generate plausible scenarios for the sizes of economies in the distant future. (The equations on their page 18 should look familiar.) This doesn't make forecasting any less hazardous, but it tells you what the critical inputs are. The key one here, of course, is productivity growth.

Symbols used in this chapter

Table 4.2: Symbol table.

Symbol	Definition
Y	Output (real GDP)
A	Total factor productivity (TFP)
K	Stock of physical capital (plant and equipment)
L	Quantity of labor (number of people employed)
$F(K, L)$	Function of inputs K and L in production function
α	Exponent of K in Cobb-Douglas production function (= capital share of income)
S	Saving
I	Investment
C	Consumption
s	Saving rate as a percent of income Y
δ	Rate of depreciation of physical capital
ΔK	Change of K $(= K_{t+1} - K_t)$
K_{ss}	Steady-state capital stock
$(K/Y)_{ss}$	Steady-state capital-output ratio
k	Capital per worker, or capital-labor ratio $(= K/L)$
g_y	Discretely-compounded growth rate of Y
g_l	Discretely-compounded growth rate of L
g_a	Discretely-compounded growth rate of A
d	Government purchases as a share of output (G/Y)

Index