

Solow's Model of Economic Growth

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We see large differences in saving and investment rates across countries, with (for example) the US investing 20% of GDP, China 30%, and India 12% in recent years (ratios of real investment to real GDP from the Penn World Tables). How important are these differences to the income and growth rates of countries? The answer: not very important. Why? Because diminishing returns to capital means (in practice) that additional capital generates smaller and smaller additions to output. That's somewhat disappointing, but is nevertheless useful because it tells us to look elsewhere if we want to account for the enormous differences we see across countries. The insight comes from Robert Solow, the 1987 recipient of the Nobel Prize in economics.

The model

Solow's model has four relatively simple components. The first is our friend the production function:

$$Y_t = A_t F(K_t, L_t) = A_t K_t^\alpha L_t^{1-\alpha}. \quad (1)$$

Changes in output therefore come from changes in (total factor) productivity, capital, and/or labor. Recall that one of the properties of this production function is diminishing returns to capital: each additional unit of capital leads to a smaller addition to output. This is the critical ingredient in what follows. The second component is a link between investment and saving. You'll recall that the flow identity, $S_t = I_t + (G_t - T_t) + NX_t$, linked household or personal saving to investment, the government deficit, and net exports. (In an earlier note we called S personal saving and labelled it S_p , but we'll skip the subscript here.) Solow ruled out the last two (we can put them back in later if we like), giving us

$$S_t = I_t.$$

Lurking behind the scenes here is the expenditure identity, $Y_t = C_t + I_t$ in this case. The third component is a description of saving behavior: people save a constant fraction s of their income,

$$S_t = sY_t,$$

where the saving rate s is a (constant) number between zero and one. This is a little simplistic — you might expect saving to depend on the rate of return and/or expectations of future income — but simplicity has a lot to be said for it. For our purposes, s is really the investment rate (the ratio of investment to GDP), but since

saving and investment are the same here, we can call it the saving rate. Finally, the capital stock depreciates at a constant rate δ , so that

$$K_{t+1} = (1 - \delta)K_t + I_t,$$

where the depreciation rate δ is a number between zero and one.

The model consists of these four equations. This seems kind of simple for a Nobel Prize, but they really are good equations. Now let's see where they lead.

Capital dynamics

Let's think about how the model behaves if the labor input L and productivity A are constant. Analysis of the model in this case consists of describing how the capital stock evolves through time. Other variables follow from their relations to the capital stock: we can find output from the production function, saving (= investment) from output, and consumption from the expenditure identity ($Y_t = C_t + I_t$). All of this is easily done in a spreadsheet.

The key step is to describe how the capital stock changes from one period to the next. With a little work, we see that the capital stock behaves like this:

$$\begin{aligned} K_{t+1} &= (1 - \delta)K_t + I_t \\ &= (1 - \delta)K_t + S_t \\ &= (1 - \delta)K_t + sY_t \\ &= (1 - \delta)K_t + sAK_t^\alpha L^{1-\alpha}. \end{aligned} \tag{2}$$

Note that each step follows from one of the components of the model. [Suggestion: label them yourself to make sure you follow the logic.] If we have values for the parameters (A, α, s, δ) , we can program this up and see how K moves through time.

Example. A numerical example — a “simulation” — will show you how this works. Let $L = 100$, $A = 1$, $s = 0.2$, $\delta = 0.1$, and $\alpha = 1/3$. (We'll use the same parameters throughout.) If the initial capital stock is 250, we can compute future values of the capital stock by applying equation (2) repeatedly. We then compute output from the capital stock using the production function, equation (1).

The results for this case are summarized in Table 1. [Suggestion: try to reproduce a couple periods of the table to make sure you understand how it works. If you get stuck, read the last two pages again. The trick is to set up formulas that tie each period to the previous one.] You can see in the table that capital and output both increase over time. Will they increase forever? The answer is no, but it takes a little work to show. (Alternatively, you could extend the simulation and see what happens.) This is an important conclusion, because it tells us saving and capital formation can't

Date t	Capital Stock K	Output Y
0	250.0	135.7
1	252.1	136.1
2	254.2	136.5
3	256.0	136.8
4	257.8	137.1
5	259.4	137.4
6	261.0	137.7
7	262.4	137.9
8	263.8	138.2
9	265.0	138.4
10	266.2	138.6

Table 1: Simulation of the Solow model.

be the reason (in this model, anyway) that some countries grow faster than others — indefinitely. More on this soon.

The dynamics of the capital stock reflect a balance of two factors: (i) saving tends to increase the capital stock by financing new investment and (ii) depreciation tends to reduce it. We see why if we subtract K_t from both sides of equation (2):

$$\Delta K_{t+1} = K_{t+1} - K_t = sAK_t^\alpha L^{1-\alpha} - \delta K_t. \quad (3)$$

You can see that the change is zero (the capital stock doesn't change) when $\Delta K_{t+1} = 0$ or

$$K_{ss} = (sA/\delta)^{1/(1-\alpha)} L, \quad (4)$$

where K_{ss} is the “steady state” capital stock. This is a little complicated, but remember, it's just a formula. In our example, $K_{ss} = 282.8$, so we have a ways to go before the model reaches its steady state.

Let's return to dynamics. What happens if we are above or below K_{ss} ? You can get a sense of the dynamics from Figure 1. The top line is output, which is related to the capital stock through the production function. The next line is saving, a constant fraction of output and the first expression on the right side of equation (3): $sAK_t^\alpha L^{1-\alpha}$. The third line is depreciation, a constant fraction δ of the capital stock and the second object on the right side of equation (3): δK_t . Diminishing returns to capital gives the saving line its curvature. It leads to higher saving than depreciation at low values of the capital stock, so the capital stock is increasing. Similarly, saving is lower than depreciation at high values of the capital stock, so the capital stock falls. The crossing point is K_{ss} , where saving is just enough to make up for depreciation, leaving the capital stock unchanged.

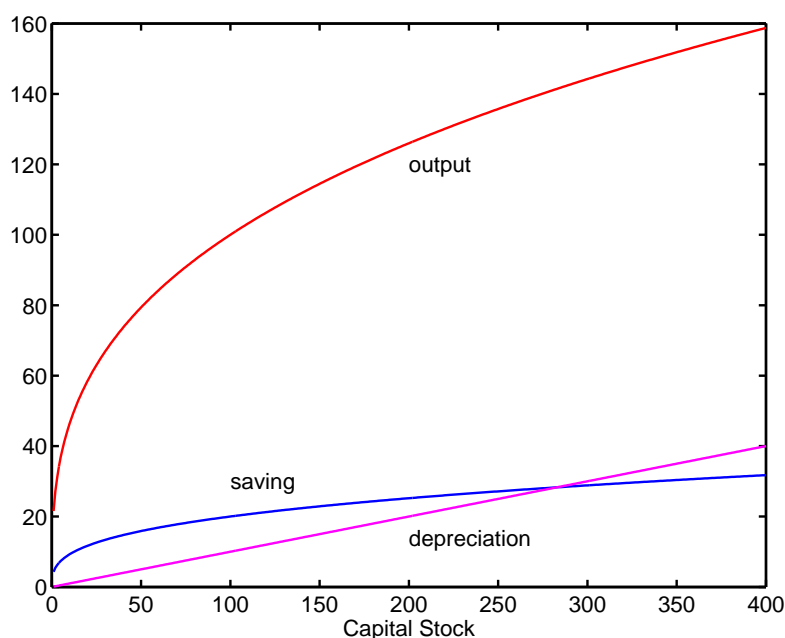


Figure 1: The Solow model.

Impact of saving and investment on income

So what have we seen? We've seen that if productivity A and labor input L are constant, the saving rate has no impact on the long-term growth of the economy. At least in this model, the capital stock (and therefore output) will eventually converge to its steady state value and the economy will not grow. Differences in saving rates are therefore not an explanation of the differences we see in growth rates across countries. That's not to say saving is irrelevant, but it can't be the primary factor in country performance.

So if saving has no impact on long-term growth, what does it do? It affects the level of income. Countries with higher saving rates have higher steady state capital stocks and therefore higher steady state income. We can see that in Figure 2. The solid lines at the bottom are the saving and depreciation lines from the previous figure. They cross at the steady state capital stock. The top line (the dashed one) shows what happens to saving if we increase the saving rate from 0.2 to 0.25. Saving is higher at every value of the capital stock. As a result, the steady state capital stock (where the dashed line crosses depreciation) is higher. And since capital is higher, output will be higher, too.

We can see the same thing in the mathematics. Equation (4) shows how the steady state capital stock varies with the saving rate. Evidently higher saving leads to a larger stock of capital. If we plugged in numbers, we'd see that steady state output rises by 11%. That's not irrelevant, but it's a relatively modest increase for a significant

increase in saving.

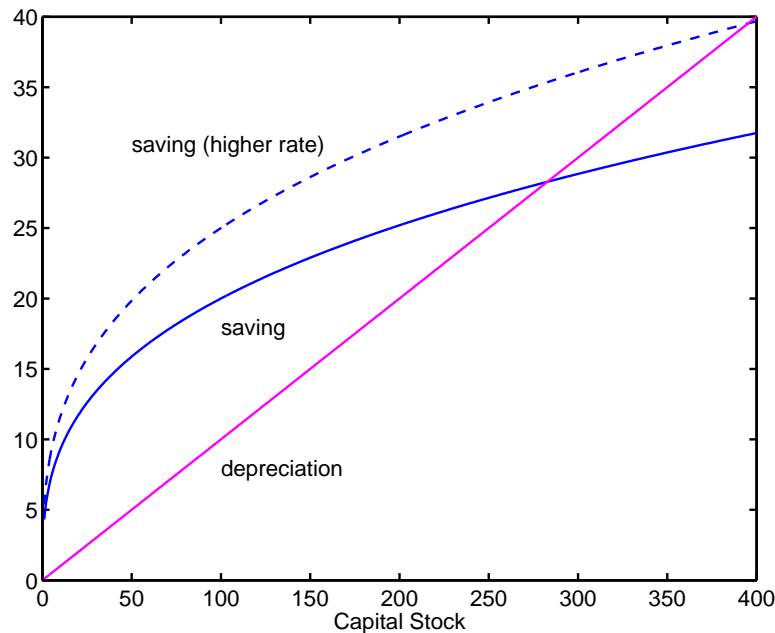


Figure 2: The impact of higher saving in the Solow model.

Growth

If saving doesn't generate growth, what does? We add growth in the labor force and (critically) growth in total factor productivity with two goals in mind. The first goal is to account for the growth rate of output, showing how it depends on the growth rates of our two inputs. The second is to show that the economy approaches what we call a balanced growth path in which output and capital grow at the same rate. As before, the capital-output ratio approaches a constant, whose features we can easily summarize. We do this with a striking example in mind: we know that China invests an astounding 40% of its GDP. Is this too much? A hint of the answer is that capital intensity (measured by the capital-output ratio) depends not only on the investment rate (which tells us how much new capital is added) but on the growth rate (how fast the denominator is changing). A fast-growing economy needs a high investment rate simply to maintain a given capital-output ratio.

We start with some ugly algebra (sorry), then return to the issue of China. The new inputs into our analysis are growth in the labor force and productivity. Let us say, to be concrete, that labor and productivity grow at constant rates:

$$\begin{aligned} L_{t+1} &= (1+n)L_t \\ A_{t+1} &= (1+a)A_t. \end{aligned}$$

How fast do output and capital grow? Let's guess that output and capital grow at the same rate g , to be determined. (Why? Because we're good guessers.) From the production function, we then know

$$\begin{aligned}(1 + g) &= Y_{t+1}/Y_t \\ &= (A_{t+1}/A_t)(K_{t+1}/K_t)^\alpha(L_{t+1}/L_t)^{1-\alpha} \\ &= (1 + a)(1 + g)^\alpha(1 + n)^{1-\alpha}.\end{aligned}$$

The growth rate is therefore

$$(1 + g) = (1 + a)^{1/(1-\alpha)}(1 + n).$$

Just a formula, but it says that output growth is tied to the growth rates of productivity and labor. Note, once again, that the saving rate does not affect this growth rate. Similarly, the growth rate in output per worker is

$$(1 + g)/(1 + n) = (1 + a)^{1/(1-\alpha)},$$

which depends only on productivity growth. If a is positive, the growth rate of output per worker is higher than the growth rate of productivity, because the exponent $1/(1 - \alpha)$ is greater than one. In words: the direct impact of productivity on output is magnified by the growth in the stock of capital; see the production function (1). This ties in with a remark we made earlier note: that capital accumulation tends to reinforce the impact of productivity growth.

What about capital: Do countries with high saving rates have more capital, relative to the size of their economies? The answer is yes, but it's also true that fast-growing countries need to invest a lot simply to maintain a constant ratio of capital to output. China has a large investment rate, but it needs it to maintain a capital-output ratio comparable to the US. If you buy that, you can safely skip the rest of the section.

Consider, again, a "steady state" in which capital and output grow at the same rate g . Then equation (2) becomes

$$\begin{aligned}K_{t+1} &= (1 - \delta)K_t + sY_t \\ (1 + g)K_t/Y_t &= (1 - \delta)K_t/Y_t + s.\end{aligned}$$

Solving for K/Y gives us the steady state capital-output ratio:

$$(K/Y)_{ss} = \left(\frac{s}{\delta + g} \right). \quad (5)$$

You may note that this line of argument doesn't actually require the Solow model: all we need is the equation connecting capital today to capital tomorrow. If we divide equation (2) by Y_t and substitute $Y_t = Y_{t+1}/(1 + g)$, we get

$$(1 + g)K_{t+1}/Y_{t+1} = (1 - \delta)K_t/Y_t + s.$$

The steady state capital-output ratio (the ratio that makes K_{t+1}/Y_{t+1} and K_t/Y_t equal) is exactly what we just derived.

To return to our goal of understanding the sources of capital intensity, note the impact of growth on the steady state capital-output ratio. For a given saving/investment rate s , countries with higher growth will have relatively less capital per unit of output. Why? Because output is growing quickly, you need to invest a lot to keep capital in line with output.

Example. Here are some numbers based loosely on the US: $n = 0.005$ (0.5%), $a = 0.01$ (1%), $s = 0.15$, and $\delta = 0.06$. What is the growth rate of the output? The steady state capital-output ratio? The growth rate satisfies

$$1 + g = (1 + a)^{1/(1-\alpha)}(1 + n) = 1.015 \times 1.005 = 1.0201.$$

Here we've used $\alpha = 1/3$, as usual. Using the same parameters as our earlier examples, the steady state capital-output ratio is

$$1.872 = \frac{0.15}{0.06 + 0.020}.$$

Now consider numbers based on China. We keep $n = 0.005$ and $\delta = 0.06$, but change the others to $a = 0.06$ and $s = 0.30$. The growth rate is now $g = 0.0968$ and the capital-output ratio falls to 1.013. Note the modest increase, despite the large increase in the saving/investment rate. Is China saving too much? Maybe not: their capital-output ratio (by this calculation) is not much different from that in the US. They need to invest a lot simply to keep up with the growth of their economy.

Executive summary

1. Solow's model bases growth on saving and investment.
2. Saving has a modest impact on steady state capital and output, and none on their growth rate.
3. Fast-growing countries must save and invest more than slower-growing countries to have maintain the same capital-output ratio.

Review questions

1. For the numerical example in the text:
 - (a) Suppose the economy starts with the steady state capital stock. What are the steady state levels of output, investment, and consumption?
 - (b) If 25% of the capital stock is destroyed in a war, how much does output fall? How long does it take the economy to reverse half the fall in output?

Answer.

- (a) The steady state capital stock is (as we've seen) $K_{ss} = 282.8$. Using this value, the production function tells us that output is 141.4. Investment equals the depreciation of the capital stock, 28.3. We can find consumption two ways. The first is through the expenditure identity: $Y = C + I$. We know Y and I , so C is 113.1. The second is through the flow identity. Saving is fraction s of output, so consumption is fraction $1 - s$, $0.8 \times 141.4 = 113.1$.
 - (b) This requires a simulation. Let the capital stock fall to 212.1, 75% of its steady state value. Then output falls to 128.5, 90.9% of its steady state value. We recover half the fall if output rises to 135.0. If we simulate the model, we see that it reaches 135.1 in 10 periods (years).
2. We've ignored government so far. Suppose, instead, that the government purchases goods and services equal to a constant fraction of GDP (say, $G = gY$ for some fraction g) and collects taxes equal to the same fraction of output. Individuals have after-tax income of $(1 - g)Y$ and save a fraction s of it. With these changes, how would the analysis of the basic Solow model change?

Answer. The critical ingredient here is the fraction of output allocated to investment. Investment here is $I = S = s(1 - g)Y$. Effectively, g reduces the saving rate from s to $s(1 - g)$ and takes resources away from investment. If the government invests, we'd have to include that, but we'd also have to decide how useful the investment was (does it count the same as other investment?).

3. (optional) Consider the model with population growth, and suppose $\delta = 0.1$, $s = 0.2$, $A = 1$, and $n = 0.01$. How much does steady state output per worker fall if n rises to 0.02?

Warning: This is moderately difficult. You need to work out the steady state on your own, not apply a formula in the notes.

Answer. With $n = 0.01$, steady state output per worker is 1.348. When $n = 0.02$, it falls to 1.291, a drop of about 4%.

If you're looking for more

The Solow model is covered in most macroeconomics textbooks. It's sometimes used as a forecasting tool. It doesn't make forecasting any less hazardous than usual, but it tells us what the critical inputs are. The critical one, of course, is productivity growth. Goldman Sachs did this a few years back to estimate the importance of the BRICs (Brazil, Russia, India, and China) in the world economy. See their [BRICs page](#).