Stephen A. Ross*
Yale University

A Simple Approach to the Valuation of Risky Streams

In an asset market where there are no unexploited arbitrage opportunities, there will exist a linear valuation operator that can unambiguously price return streams with perfect market substitutes or bound values for streams bounded by market combinations. This is possible, without further assumptions, only if the project returns can be duplicated (or bounded) by a deterministic intertemporal program of purchasing a portfolio of marketed assets. These results are proven and used to simplify and unify a number of topics in financial economics, including project valuation, Modigliani-Miller theory, forward pricing, the closed-end mutual fund paradox, and efficient market theories.

The optimal investment decision or the capital budgeting problem in a competitive environment are equivalent to the problem of valuing a stream of returns. If the returns from a project can be valued, then the criterion for acceptance is simply whether value exceeds cost. More generally, if returns are net of costs then the criterion for acceptance is whether the value of the net stream is positive.[1]

There are a number of methodologies currently available for determining the value of a stream of returns. If the returns are certain, then there is widespread agreement that value is determined as present discounted value. If returns are uncertain, matters become more complex. The classical method of simply substituting expected returns—or, in its refined version, certainty equivalents—and discounting at some risk-adjusted rate has passed from fashion, at least in academic circles if not in business practice.

The current view is that the valuation problem calls for the use of dynamic programming techniques coupled with capital asset pricing models. At each stage a capital asset pricing model is

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1. This is true under capital budgeting constraints as well; value then includes the shadow prices of rationed resources, but then the statement is simply true as a tautology at this level of generality.

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employed to value end-of-period returns, and the resulting valuation is carried into the previous period (see, e.g., Bogue and Roll 1974). A more modern form of this argument utilizes the Black-Scholes techniques to value streams following diffusion processes—ostensibly eliminating the need for a capital asset pricing theory. For a large number of problems, though, there is a much simpler and apparently unexplored approach. It may well be that individual agents go through the complicated calculus described above. But, if that is the case, there may be no need for investors to repeat the same analysis; the answer may already be recorded in the marketplace. As we shall see, some quite complicated return streams can be valued by simply using existing market valuations and the assumption that financial markets are in equilibrium.

The basic tool for accomplishing this is a theory, introduced in Ross (1976*a*, 1976*b*), that is built on the premise that all arbitrage opportunities will be exhausted in markets with open access. The intent of this paper is to show how this arbitrage theory can be used to value streams generated by risky assets in a simple and straightforward fashion that makes use only of information available in the market. Section I illustrates this approach with some simple examples. Many of these results have been derived elsewhere by more complicated means. Section II develops the general theory of the approach and, putting more technical portions in an Appendix, proves some theorems that describe what can be done and what are the limitations of the simple approach. Section III uses the results of Section II to explore some more complicated applications, and Section IV concludes the paper by considering some possible generalizations.

I. Arbitrage and Investment Criteria—Some Examples

A word on notation is necessary before we begin. Several types of symbols will be used to denote returns. For simple flows a variable such as x_t or a_t will denote the dollar flow at time t. If the variable can take on discrete values then we will adopt the (Lebesgue) notation, dx_t. For example, a bond that pays a coupon of c at unit time intervals could be represented as dx_t where x_t is the simple cumulative dollar flow at time t, with

$$x_t = \begin{cases} x_0 + nc \\ \text{for } t \epsilon (n,\ n+1), \text{ where } n = 0, 1, 2, \ldots, \end{cases}$$

$$dx_t = \begin{cases} 0 \text{ if } t \neq \text{integer} \\ c \text{ if } t = \text{integer}. \end{cases}$$

More generally, dx_t can represent an arbitrary random increment, perhaps a diffusion or a quantum jump—the results of this paper will be valid under quite general interpretations of these return streams. A bracket around terms such as $<x>$ or $<dx>$ will be used to denote the whole return stream over some understood time interval. When the implicit time interval over which the stream is being evaluated is unclear, a notation such as $<x_t>_\tau^T$ will be used to describe a flow on (τ, T).

Throughout the paper we will assume that there is a level term structure with an interest rate of r and a complete market in bonds of the required maturities. All of the results will generalize in a trivial fashion to the case of a variable term structure, and the reader is free to do so by simply interpreting symbols such as e^{-rt} as the price of a unit pure discount bond of maturity t.

Now, let us consider some simple propositions concerning the value of return streams in competitive markets. I believe that most of these are well known (at least at some level of consciousness), but the failure to understand either their universality or their applicability is endemic.

Example 1

Suppose that we consider the simplest possible return stream, a stream that pays a fixed payment at a prespecified future date,

$$dx_t = \begin{cases} 0 \text{ for } t \neq T \\ b \text{ for } t = T. \end{cases}$$

Clearly, the current value of this stream is given by $P \equiv P(<dx>) = b\ e^{-rt}$.

If the return stream $<dx>$ were from a marketed asset, P would obviously be its market value. This result is so simple and so ingrained in our thinking that it might be worth pausing a moment to see in what sense it is true. In the first place, b units of a unit discount bond of maturity T would be a perfect substitute for the stream $<dx_t>$ and must, therefore, sell for the same price. This is simply a condition to prevent arbitrage. If the stream were valued at $P > b\ e^{-rT}$, then by borrowing for a term of T and investing in the stream, competitive agents would believe that they could realize pure arbitrage profits, that is, positive returns with no net investment. Similarly, if $P < b\ e^{-rT}$, shorting the project and purchasing the T period bond would produce the same sort of profits. Even if short selling were precluded for some reason, the two assets would still sell at the same price in equilibrium if they both had positive net supplies; otherwise the demand for the higher-priced asset would be zero. Notice that it is crucial for these arguments that $<dx>$ be the return stream on a capital asset. If $<dx>$ were the dollar return on an asset that also yielded a service flow, for example, a consumer durable like a house or a car or even money, then arbitrage would assure that the price of the asset P would exceed $b\ e^{-rT}$ by the market value of the service flow.

Suppose, though, that $<dx>$ is not a marketed asset but rather is the return stream offered by a prospective investment project. If C denotes the cost of undertaking this project then, ignoring capital budgeting constraints, and following the lead of Irving Fisher, the criterion for acceptance will be whether $P = b\ e^{-rT} > C$. Why? The familiar argument is that borrowing the cost, C, and undertaking the project will yield a return b at time T which will exceed the debt repayment of $C\ e^{rT}$. Equivalently, though, the returns of the project can be duplicated by purchasing b units of the bond for $b\ e^{-rT}$. Financial claims on the project can thus be sold for this amount yielding a current gain of $b\ e^{-rT} - C$. A project should be undertaken if combining it with existing market opportunities expands the opportunity set defined by any budget constraint. Since taking the above project can generate wealth by borrowing against its returns net of cost, it clearly satisfies this condition. This suggests the following general proposition connecting arbitrage, valuation, and acceptance criteria that is applicable to any project whether sure or uncertain. Let P be the value of a combination of marketed assets whose returns would duplicate or be dominated by those of the project.

Acceptance Criterion Proposition (ACP)

Let the (strong) criterion for undertaking an investment project in a competitive market be that it expands the opportunity set of returns available to the investor. Under this criterion a project will be accepted if $P - C > 0$. Equivalently, the project will be undertaken if at a price of C it appears to offer arbitrage opportunities. Conversely, if a combination of marketed assets can be found whose returns duplicate or dominate those of the project and for which $P - C < 0$, then the project should be rejected. (Of course, with either acceptance or rejection, if the streams dominate, then the criterion will hold at $P = C$.)

PROOF: If $P - C > 0$, then in a competitive market the investor can finance the project by issuing claims to its returns that will be valued at (least at) P by the market. The excess of P over C will then be an increment to the investor's current wealth and will therefore expand the investor's opportunity set. Equivalently, if the project were a marketed asset with a price of C, then its purchase would appear to offer arbitrage opportunities if and only if $P > C$. The converse is similar. Q.E.D.

The above proposition is quite general, and the example of a certain project was only used to motivate it. But the key to its application lies in the ability to combine existing assets so as to duplicate the return stream of the proposed project, and with uncertain streams this can be quite difficult (or impossible). The examples below will illustrate two cases where it is possible to duplicate the return stream—independent of any assumptions on the stochastic processes governing return streams.

Example 2

Consider a project,

$$dc_t = \begin{cases} 0 \text{ if } t \neq T \\ bX_T \text{ at } t = T, \end{cases}$$

where X_T is the (possibly random) time T value of a marketed capital asset with no payouts on it. The value of this project as a marketed asset is $P = bX_0$, and, therefore, by the ACP the project should be undertaken if $bX_0 > C$, the cost.

To verify that the project value must be bX_0, we use the fact that the value of the original asset, X_0, is simply the market value of a claim to receiving X_T at time T. The project is thus equivalent to holding b units of the capital asset. Now let us consider a somewhat more difficult example.

Example 3

Let X_t be the (possibly random) value of a marketed asset with payouts of aX_t for a constant a. Let the project to be evaluated have flows of

$$c_t = \begin{cases} bX_t \text{ for } t \in (0, T), \\ 0 \text{ otherwise.} \end{cases}$$

It takes a bit of work to figure out what the value of this project must be. Consider the policy of purchasing $(b/a)(1 - e^{-aT})$ units of the marketed asset at a cost of X_0 and, irrespective of returns, selling off the asset at the rate of $b\, e^{a(t - T)}$ units. At any time $t \in (0, T)$, the number of units sold will be

$$\int_0^t b\, e^{a(\tau - T)}\, d\tau = \frac{b\, e^{-aT}}{a} (e^{at} - 1),$$

and the number remaining will be $(b/a)(1 - e^{-aT}) - (b/a)e^{-aT}(e^{at} - 1) = (b/a)[1 - e^{a(t - T)}]$. The payout flow from this stock at time t will be $a \cdot (b/a)[1 - e^{a(t - T)}]X_t = b\,[1 - e^{a(t - T)}]X_t$, and the proceeds from the current sales will be $X_t\, b\, e^{a(t - T)}$. Combining the payout flow and the proceeds from sales yields bX_t, the flow of the original project. By time T all of the asset will have been sold off, hence the remaining flow will be zero.

By purchasing $(b/a)(1 - e^{-aT})$ units of the marketed asset, then, we have been able to mimic the flow of returns on the investment project. It follows from the ACP that, for investment purposes, the current value of this project will be given by $P = (b/a)(1 - e^{-aT})X_0$.

As example 3 illustrates, though, we are reaching the limits of our ability to price streams by this technique, if only because we are making increasing demands on our cleverness at constructing such arguments on an ad hoc basis for every new investment project. Furthermore, it should also be apparent that not all projects can be mimicked and, therefore, priced. The next section develops a general theory to price investment projects and lays out the limitations of this approach.

II. A General Theory of Arbitrage Valuation

The purpose of this section is to develop the use of arbitrage principles in valuation problems in a sufficiently general setting to be useful in pricing. The successful pricing of a return stream requires that there be some (perhaps time and value dependent) combination of marketed assets whose returns equal those of the given stream in all contingencies. Finding such a combination is not always straightforward—as example 3 illustrated—but fortunately there is a basic principle that serves as a guide to when such a combination can be formed. The principle was developed in a discrete time and state space context in Ross (1976*a*, 1976*b*), and it has been applied to the theory of option pricing in Ross (1976*c*), Cox and Ross (1976*a*, 1976*b*), Garman (1975, 1977), and Rubinstein (1976).

We will state the principle formally below, but essentially it comes to this: If there are no arbitrage opportunities in a market, then there must exist a positive linear operator, L, that can be used to value all marketed assets. A detailed proof and analysis of this argument is presented in an Appendix, and in this section we will content ourselves with displaying the basic properties of this operator.

Linearity

The operator L is linear; that is, given any two return streams $<dx>$ and $<dy>$ and any two constants a and b, if the values of the two streams are given by $L \cdot <dx>$ and $L \cdot <dy>$, respectively, then the value of the combined stream $<a \cdot dx + b \cdot dy>$ is given by $L \cdot <a \cdot dx + b \cdot dy> = a\, L \cdot <dx> + b\, L \cdot <dy>$.

The linearity property is the basic property for any valuation rule in a competitive market; nonlinearities in pricing capital assets are really equivalent to the perception of monopoly power in the financial markets.

Positivity

Let $<dx>$ be any return stream that is nonzero and nonnegative. It follows that $L \cdot <dx> \geqslant 0$, with strict inequality if dx_t can be replicated by a combination of marketed assets.

Positivity simply says that a return stream on a nontrivial limited liability asset will have a positive value. The final property is less fundamental than the first two since it depends on the existence of the given riskless asset.

Riskless Valuation

If $<1_T>$ denotes a return stream yielding one dollar for certain at time T and nothing at any other date, then $L \cdot <1_T> = e^{-rT}$.

It is important to stress that the existence of a valuation operator with the above properties is not simply assumed ad hoc; rather it is a consequence of the absence of arbitrage possibilities in the capital markets. For example, in a finite Arrow-Debreu market in which there are prices for pure contingent claims, the valuation operator, L, will be the vector of state-space prices. This follows since the value of any asset will be the sum of the current values of returns in different states of nature which, in turn, are given by multiplying state returns by the respective contingent claim prices. More generally, even if the market is not complete in the Arrow-Debreu sense, there will still exist a linear valuation operator. In this case, though, the particular operator will not be uniquely defined from the values of existing assets, and applying different admissible operators to a return stream that is not spanned by existing assets will lead to different valuations. Conversely, if all admissible operators lead to the same value, then the return stream must be spanned by existing assets and the resulting value is the proper one for the investment criterion.

We will refer to the existence of a valuation operator as the basic valuation theorem and for easy reference it is stated below, leaving a formal treatment to the Appendix.

Basic Valuation Principle

If there are no arbitrage opportunities in the capital markets, then there exists a (not generally unique) valuation operator, L, that satisfies the linearity, positivity, and riskless valuation properties.

It should be stressed again that the return stream under consideration must be duplicable by combining existing assets for all admissible valuation operators to yield the same valuation. Of course, if for all L the project price is above its cost, then from the ACP it should be accepted, and if it is less it should be rejected, otherwise the criterion is ambiguous. Given this limitation, though, the valuation operator, L, can be used to derive a number of well-known results in finance. For example, from linearity and riskless valuation we can derive the result that the value of a certain stream is the sum of its discounted returns. The point of the basic valuation principle is that this method of valuing the certain stream is the only one that precludes arbitrage possibilities. We can also apply the operator in more complex situations, but rather than considering more specific examples we will prove a general theorem and some corollaries that describe a large class of return streams that can be unambiguously valued by the basic valuation principle.[2] A further result will describe the sense in which this is the largest class which can be so valued. In the situations which follow we will refer to the underlying marketed assets on which valuation is based as primary assets and the assets or projects that can be valued will be called derivative assets (see Ross [1976*c*]).

Asset Valuation Theorem

Let X_t be the value of a capital asset with a payout stream $da(t, \cdot)$. Let P be the value of a payout stream

$$dc = f\,da - X\,df, \tag{1}$$

where $f = f(t)$ is a sure function, and where

$$P_T = f(T)\,X_T. \tag{2}$$

It follows that

$$P_t = f(t)\,X_t. \tag{3}$$

PROOF: From the basic valuation theorem there is a linear price operator, L, that can be used to value assets. Hence,

$$\begin{aligned}
P_t &= L\cdot <f\,da - X\,df> + L\cdot f(T)\,X_T \\
&= L\cdot <[\int_t^T df + f(t)]\,da - X\,df> + f(T)\,L\cdot X_T \\
&= f(t)\,L\cdot <da> + \int_t^T (L\cdot <da>_\tau^T - L\cdot X_\tau)\,df_\tau + f(T)\,L\cdot X_T \\
&= f(t)\,L\cdot <da> + \int_t^T (-L\cdot X_T)\,df_\tau + f(T)\,L\cdot X_T \\
&= f(t)\,L\cdot <da> + f(t)\,L\cdot X_T \\
&= f(t)\,L\cdot (<da> + X_T) \\
&= f(t)\,X_t.^3
\end{aligned}$$

Q.E.D.

2. As Garman (1975, 1977) has pointed out, by using techniques similar to those developed in Cox and Ross (1976*b*), the basic valuation theorem can also be applied to price options written on assets that follow diffusions. This is an alternative to the direct arbitrage derivations.

3. The propriety of interchanging the linear operator and the integral is valid under general circumstances; see, e.g., Dunford and Schwartz (1957), p. 113.

Corollary 1: a general asset valuation theorem. The statement and proof of this theorem are identical to those of the asset valuation theorem, with X interpreted as a vector and, similarly, all flows given vector interpretations.

Corollary 2. The value, P, of a payout stream of

$$dc = f\,da - x\,df + dh(t), \tag{4}$$

and with

$$P_T = f(T)X_T + b(T), \tag{5}$$

is given by

$$P_t = f(t)X_t + g(t), \tag{6}$$

where

$$dg = rg - dh \tag{7}$$

and

$$g(T) = e^{-rt}\,b(T). \tag{8}$$

PROOF: The first term of the flow, dependent on X, can be valued by the asset valuation theorem. The second term can also be valued in this fashion by considering it as a flow from an option written on a unit bond. Alternatively, a direct proof follows from present value computation. Defining $q(\tau) = e^{-rt}\,g(\tau)$, (7) yields $dh = -e^{rt}\,dq$. Hence, the present value of the sure flow and the balloon payment, $b(T)$, is given by

$$\begin{aligned}\int_t^T e^{-r(\tau-t)}\,dh + e^{-rT}\,b(T) &= -\int_t^T e^{rt}\,dq + e^{-rT}\,b(T)\\ &= g(t) - g(T) + e^{-rT}\,b(T)\\ &= g(t),\end{aligned}$$

from (8). Q.E.D.

The asset valuation theorem is really an extension of the technique applied in example 3. If $f(t)$ denotes the quantity being held of an asset with payouts of da, then $f\,da$ is the return flow from this holding. In addition, since df denotes the rate of acquisition of the asset, if the current value of the asset is X, then the net return flow will be

$$dc = f\,da - X\,df. \tag{1}$$

It follows that by purchasing $f(t)\,X_t$ at time t the asset flow dc can be duplicated with an appropriate policy of acquisition.

The next theorem verifies that this technique is the only one that can be used with no specific information on the actual return-generating process and the movement of the value of the underlying market asset. In other words, the next theorem verifies that the conditions of the asset valuation theorem are the most general that can be found without further assumptions on technology or preferences or anticipations. First, we need the following definition: a *null function* is an asset stream whose current value is zero for any truncation time.

Knowing that two streams have the same value for all truncation times, then, implies that the streams differ by at most a null function. There is, of course, no reason for a null function to be the zero function, although the zero function is null. For example, if valuation is by the expectation, then any even-money lottery is a null function. In the theorem below we will only require that the relevant streams differ by a null function.

Equivalence Theorem

The conditions of the asset valuation theorem are both necessary and sufficient for valuing a derivative asset as a function only of the observables, time, and the value of the primary asset.

PROOF: Sufficiency follows directly from the asset valuation theorem. To prove necessity we have to show that the conditions of the asset valuation theorem are the only ones for which the value of the derivative asset is process independent. Letting X_t denote the value of the primary marketed asset, by assumption $P = P(X_t, t)$ and is unaffected by the stochastic process generating X_t values.[4] From the Basic Valuation Theorem, for any $T, X_t = L \cdot (\langle da \rangle + X_T)$, and

$$P(X_t, t) = L \cdot [\langle dc \rangle + P(X_T, T)], \tag{9}$$

where $\langle da \rangle$ and $\langle dc \rangle$ are the flows from the primary and the derivative assets, respectively. Letting $\langle da \rangle = 0$, we have $X_t = L \cdot X_T$, hence from the independence of valuation and process,

$$L \cdot P(X_T, T) = F(L \cdot X_T). \tag{10}$$

Partition the states of nature at time T into two subsets, θ_1 and θ_2, and let $X_\theta = X_T(\theta)$ be constant on each subset. Restricted to these subsets, L takes the form of a positive two-vector (l_1, l_2) and from (10), $l_1P(X_1, T) + l_2P(X_2, T) = F(l_1X_1 + l_2X_2)$. Setting $X_2 = 0$ implies that $l_1P(X_1, T) + l_2P(0, T) = F(l_1X_1)$, hence

$$P(X_1, T) + \left(\frac{l_2}{l_1}\right) P(X_2, T) = P\left(X_1 + \frac{l_2}{l_1} X_2, T\right) + \frac{l_2}{l_1} P(0, T).$$

4. This ignores the possibility of valuing a derivative asset whose returns depend on other observable features of the primary asset, e.g., some function of the past payout flow.

Setting $X_1 = 0$ yields

$$\left(\frac{l_2}{l_1}\right) P(X_2, T) = P\left(\frac{l_2}{l_1} X_2, T\right) + \frac{l_2}{l_1} P(0, T),$$

hence $P(u, T) + P(v, T) = P(u + v, T)$, where $u \equiv X_1$ and $v \equiv (l_2/l_1)$ X_2. Since this holds for all (u, v), P is linear and for all t $P(X, t) = f(t)$ X. From (9)

$$\begin{aligned} L \cdot \langle dc \rangle &= P(X_t, t) - L \cdot P(X_T, T) \\ &= P[L \cdot (\langle da \rangle + X_T), t] - L \cdot P(X_T, T) \\ &= f(t)\, L \cdot (\langle da \rangle + X_T) - f(T)\, L \cdot X_T \\ &= L \cdot \{f(t) \langle da \rangle + [f(t) - f(T)]\, X_T\} \\ &= L \cdot \langle f\, da - x\, df \rangle. \end{aligned}$$

It follows that the flow on the derivative asset can only differ from (1) by a null function. Q.E.D.

There are several different ways to prove the equivalence theorem. Those more comfortable with the continuous time diffusion pricing theory may find the following analysis more familiar—although less general.

Suppose that x_t follows a diffusion process with instantaneous mean μ and variance σ^2. From the extension of the Black-Scholes pricing theory to assets with payouts, in Merton (1974) and Cox and Ross (1976*b*), we know that P must satisfy the partial differential equation

$$\frac{1}{2} \sigma^2 P_{xx} + (rx - a) P_x - rP + c = -P_t, \tag{11}$$

to prevent arbitrage possibilities. For ease of reference to earlier work we are also assuming the existence of a sure rate of interest, r. Notice that by arbitrage μ does not enter into the pricing relation.

For the solution to (11) to be process independent for nontrivial boundary conditions, that is, for P to depend only on X, r, and t and be independent of the form of the process, the solution must be independent of σ^2. In particular, then, the solution will be independent of σ^2 if and only if

$$P_{xx} = 0. \tag{12}$$

From (12) $P = f(t)\, x + g(t)$, which implies that

$$[rx - a(x,t)]\, f(t) - r\, f(t) x - r\, g(t) + c(x,t) = -\dot{f}(t) x - \dot{g}(t). \tag{13}$$

From (13), $c(x,t) = f(t) a - x\dot{f} + r\, g(t) - \dot{g}(t)$, which is the flow found in (4).

The equivalence theorem is important because the resulting valuation depends only on observables. This distinguishes it from the

Black-Scholes result, for example, in which the unknown process variance must be estimated to price derivative assets. Nevertheless, despite the generality of the underlying assumptions, the next section will use the basic valuation principle and the asset valuation theorem to evaluate some surprisingly complex financial instruments and investment projects.

III. Applications of the Arbitrage Valuation Principles

The next five subsections will illustrate some important applications of the basic valuation principle and the asset valuation theorem. We will choose whichever approach seems more straightforward for the problem at hand, but the reader is encouraged to reinterpret the analysis using the alternative.

Valuation of Investment Projects

A classic problem in finance is that of valuing investment projects. The usual approach has been to invoke an equilibrium model and apply dynamic programming to recursively value the return stream from the investment.

Recently, several authors have applied a continuous time version of these arguments—the option analysis of diffusions—to this problem (see Black and Scholes [1973], Merton [1974], or Cox and Ross [1976*a*]). Brennan (1973) used this approach, and Myers and Turnball (1975) and Turnball (1978) have extended the analysis. Brennan priced an asset whose returns depended on the movement of a market index and Myers and Turnball did the same. The latter authors have worked both in discrete and continuous time and have devoted considerable effort to modeling the expectational mechanism underlying the index. The intent of this section is to simplify much of the work in this area by deriving the results from the asset valuation theorem which, in turn, exploits only the absence of arbitrage possibilities. This is not to say that the previous analyses of this problem have been incorrect; rather, our point is to show that some of the earlier results are much more robust than had been suspected and may be obtained in a more straightforward manner.

Now, for simplicity, consider an investment whose return stream takes the form

$$a_t + b_t I_t, \tag{14}$$

where I_t is the value of a traded capital asset index. This is similar to a β model, with returns having sensitivity b_t with respect to the index. By assumption, dividends are reinvested and the index has no payouts. (It is sufficient to simply reinvest all payouts in the index.) At time T the investment will return a stock amount $g_T + h_T I_T$.

We can value this project by applying the asset valuation theorem. Define

$$\dot{f}_t = -b_t, \tag{15}$$

and $f(T) = h_T$. Now, from (6), the value of the project

$$P(I, t) = f(t)I + \int_t^T e^{-r(\tau-t)}\, a_\tau d\tau + e^{-r(T-t)}\, g_T, \tag{16}$$

and solving (15) yields

$$f(t) = \int_t^T b_\tau d\tau + h_T.$$

More generally, if the return stream on the investment satisfies $c(X, t) = \Sigma_i\, b_t^i\, X_T^i + a_t$, and $P(X, T) = \Sigma_i\, h_T^i\, X_T^i + g_T$, then from the general asset valuation theorem,

$$P(X, t) = \sum_i \left(\int_t^T b_\tau^i d\tau + h_T^i\right) X^i + \int_t^T e^{-r(\tau-t)}\, a_\tau d\tau + e^{-r(T-t)} g_T, \tag{17}$$

where X is a vector of indexes that are capital assets with no payouts. In addition, (17) can be further extended to a version with proportional payouts.

Let us stress again the power of these results. To begin with, expectational mechanisms to describe how we anticipate the index, X_t^i, will behave are irrelevant. Furthermore, there is no need for X_t to follow a diffusion process to apply the usual option analysis or, for that matter, for the stream c to follow any specific process. All that is necessary is the assumption that current asset values do not permit arbitrage. In particular, then, we do not even have to assume that the current index price is in equilibrium to correctly appraise the project. (Of course, out of equilibrium we are still—heroically—assuming that the project can be financed for its computed value.) Notice, too, that a project whose returns dominate (14) will be accepted if its cost does not exceed (17), and if (14) dominates the project, it will be rejected if (17) does not exceed its cost.

Modigliani-Miller Theory

The Basic Valuation Principle can be used to prove a very general form of the Modigliani-Miller arguments on the irrelevancy of financial structure. Consider a firm with a specified investment policy that yields a total return stream of $<dx>$ that is known by the market.[5] The firm has a vector of financial claims to these total returns with payouts to the claims of $<df^1, \ldots, df^n>$. At any moment of time we must have $\sum_i$

5. For an alternative theoretical development based on the view that the financial structure can communicate information about the firm, see Ross (1977, 1978).

$df_t^i = dx_t$, to conserve the return flow. (If dividends, for example, exceed the return flow, then some claim must have an inflow to maintain the investment policy, since $<dx>$ is given exogenously.)

Applying the basic valuation principle, the values of the claims will be given by $L \cdot <df^i>$. From linearity the total sum of these values, that is, the market value of the firm will be given by

$$\sum_i L \cdot <df^i> = L \sum_i <df^i>$$

$$= L \cdot <dx>,$$

independent of the nature of the individual financial claims if L is independent of them. This is a restatement of the Modigliani-Miller irrelevance theorem.

Inherent in the above argument is the assumption that altering the financial packages issued by the firm would not change the valuation operator. A sufficient condition for this to be true is that there is an Arrow-Debreu market. More plausible weaker conditions are that there are restrictions on preferences, for example, to those with von Neumann–Morgenstern utility representations, and restrictions on the distributions of asset returns that fix L independently of the financial claims. Alternatively, we could simply constrain the class of allowable financial claims to those that are already spanned by market assets; if all claims can be issued and there are no further demand or supply restrictions, then there must be an Arrow-Debreu market to support the Modigliani-Miller conclusions (see Grossman and Stiglitz [1976]).

As a final point, there is an important sense in which the analysis could be reversed and a strong version of the Modigliani-Miller propositions could be shown to be equivalent to the basic valuation principle. One way to see this is by using the fact that if the Modigliani-Miller theory is applied to all return streams, then it rules out arbitrage possibilities. From the results in the Appendix, the absence of arbitrage implies the basic valuation principle.

The Price of a Forward Contract

A forward contract is an agreement made at a time t to purchase a unit amount of a commodity at a date $T > t$, for a specified contract price C. Clearly, the value or price of such a contract will vary with both the contract price and the maturity date, and we can define a variable X_t, called the futures price, as that contract price which sets the value of the forward contract at zero. Over time, then, the value of any given forward contract can be written as a function of this futures price. Recently, Black (1976) applied the diffusion option pricing theory to this valuation problem and derived the following very simple formula that the price of such a contract must satisfy,

$$P(X_t, t, C, T) = (X_t - C)\, e^{-r(T-t)}. \tag{18}$$

Some care is required, however, in interpreting this analysis. First, implicit in the application of modern diffusion theory to option pricing is the assumption that the underlying commodity is a pure capital asset. For commodities which yield service flows, though, this will not be the case, and the application of the Black-Scholes analysis to such commodities may lead to incorrect answers. For example, suppose wheat cannot be stored (i.e., storage costs are infinite) and the current harvest simply changes with the weather. It would be wrong to develop an arbitrage argument on such a commodity by ignoring the service (consumption) rentals. About the only legitimate use of the diffusion formalism in these situations is as an analytic tool to derive the local expected returns and covariance with exogenous variables. To then assume that a riskless hedge can be formed from the commodity and the contracts written on it, and that such a hedge must earn the riskless rate, would be incorrect.[6]

Although the underlying commodity may not be a capital asset, financial instruments written on the commodity can be pure capital assets. In the particular case of a forward contract, there is no need to assume that the commodity spot price follows a diffusion or any specified stochastic process: the forward prices can be related by arbitrage considerations alone. At time T, the value of a forward contract will be given by $X_T - C$, where the futures price at time T must, of course, equal the prevailing spot price. Now, by definition the futures price at time t, X_t, is given by solving the equation $P(X_t, t, X_t, T) = 0$, that is, it is the contract price such that the forward contract has no value. Applying the basic valuation principle,

$$\begin{aligned} P(X_t, t, C, T) &= L\cdot(X_T - C) \\ &= L\cdot X_T - L\cdot C \\ &= L\cdot X_T - C\, e^{-r(T-t)} \\ &= 0, \end{aligned}$$

if $X_t\, e^{-r(T-t)} = L\cdot X_T$. Simply discounting the futures price thus gives the value of a claim to the future spot price. It follows that the value of any forward contract with a contract price of C is given by

$$\begin{aligned} P(X_t, t, C, T) &= L\cdot(X_T - C) \\ &= L\cdot X_T - C\, e^{-r(T-t)} \\ &= (X_t - C)\, e^{-r(T-t)}. \end{aligned} \tag{18}$$

This result thus is quite general and depends only upon the assumption that no arbitrage opportunities are present.[7]

6. Recently Garman (1977) has made the same point; see also Constantinides (1978). Of course, I do not mean to imply that Black (1976) is guilty of such reasoning.

7. A new wrinkle is introduced, though, when the interest rate is stochastic; see Cox, Ingersoll, and Ross (1976).

Notice that we did not assume that the underlying random variable, the commodity spot price, was the price of an asset. Essentially, the formula values one forward contract in terms of another, and forward contracts are pure capital assets even when the commodity they are written on is not. Applying the basic valuation principle as above, we can show that if P and P^* are two forward contracts with contract prices of C and C^*, respectively, then

$$P - P^* = (C^* - C)\, e^{-r(T-t)}. \tag{19}$$

If we set C^* so that $P^* = 0$, then C^* is the futures price and (19) reduces to (18).

Closed-End Mutual Funds—an Exercise in Terminal Valuation

A variety of paradoxes can arise in infinite horizon valuation problems. Consider a closed-end fund that holds marketed assets, has no termination date, and makes no payouts. Like a savings account that is legally constrained to simply accumulate without any withdrawals, its value must be zero. Of course, if the fund could be purchased and dissolved, then the value would be the current asset value. To the extent that the possibility exists of altering the no-withdrawal or no-payout rules, this will give a value to the fund, and to the extent to which this possibility is not a certainty, there will be a discount.[8]

However, even though there is no fixed termination date, if the payout flow is sufficiently great, the value of the fund will be the value of its current asset holdings. A very general proof of this can be constructed by using the basic valuation principle.

Let A_t denote the value of the fund's assets. Suppose that the fund makes payouts, C_t, that are in excess of some proportional payout flow, that is, for some $\delta > 0$,

$$C_t > \delta A_t. \tag{20}$$

Let M_t denote the market value of the fund, and let S_t denote the value of the initial assets, A_0, at time t if the fund had made no payouts. From the basic valuation principle,

$$A_0 = S_0 = L \cdot S_t. \tag{21}$$

Since the same payout stream given by the fund can always be realized by holding the assets themselves and liquidating the holding at the rate δ_t, the fund cannot sell at a premium, that is,

$$M_0 \leq A_0 = S_0. \tag{22}$$

The proportional outflow, δ_t, is given by

$$C_t = \delta_t A_t, \tag{23}$$

8. None of this explains how these funds could get started in the first place.

that is, δ_t is the rate at which assets are realized to make payouts. Hence, the value of assets in the fund, A_t, must be given by

$$A_t = (e^{-\int_0^t \delta_\tau d_\tau})\, S_t, \tag{24}$$

where the exponential term in (24) represents the number of unliquidated units.

Applying the basic valuation principle, then, yields for any T

$$\begin{aligned} M_0 &= L\cdot(<C_t>_0^T + M_T) \\ &= \mathrm{L}\cdot <C_t>_0^T + L\cdot M_T \\ &= \mathrm{L}\cdot <\delta_t A_t>_0^T + L\cdot M_T \\ &= \mathrm{L}\cdot <\delta_t e^{-\int_0^t \delta_\tau d_\tau} S_t>_0^T + L\cdot M_T \\ &= \int_0^T \delta_t\, e^{-\int_0^t \delta_\tau d_\tau}(L\cdot S_t)\, dt + L\cdot M_T \\ &= (1 - e^{-\int_0^T \delta_\tau d_\tau})S_0 + L\cdot M_T. \end{aligned} \tag{25}$$

It follows from (20), (22), and (25) that $M_0 = A_0 = S_0$, and, furthermore, the residual value of the fund, M_T, must go to zero, $L\cdot M_T \to 0$, as $T \to \infty$.

Hence, the paradox of a closed-end fund selling at a discount will, at least in theory, be avoided when the payouts from the fund are sufficiently large. Conversely, if a fund makes sufficiently small payouts (and if the rules can be altered only at a cost), then it is easy to see from the same analysis that the fund will sell at a discount from asset value.

Efficient Market Theories

Can the operator analysis provide us with any testable propositions on asset price movements? Not on its own, but it can permit us to draw inferences about the relative pricing of assets. The situation is similar to that in option pricing where options are priced given stock prices, but stock values are not determined relative to other assets.

Consider, for example, the valuation of a claim with no payouts whose value at time T is given by $\tilde{P}_T$. The current value at time o of such a claim is given by

$$\begin{aligned} P_o &= L_o\,(\tilde{X}_T) \\ &\equiv L\,(\tilde{X}_T \mid I_o), \end{aligned}$$

where I_o denotes the time zero information set that conditions the linear valuation operator. Similarly, the current price at any intermediate date $t \in (o, T)$, will be given by $\tilde{P}_t = L(\tilde{X}_T \mid I_t)$. Since the claim at date o on $\tilde{P}_t$ at time t is equivalent to a claim on $\tilde{P}_T$ at T, to prevent arbitrage we must have that

$$\begin{aligned} P_o &= L\,(\tilde{X}_T \mid I_o) \\ &= L\,(\tilde{P}_t \mid I_o) \\ &= L[L\,(\tilde{X}_T \mid I_t) \mid I_0]. \end{aligned}$$

This is of course, a version of Bayes's identity (or the Chapman-Kolmogorov relations). If L were simply the discounted expected value operator, it would specialize to the statement that prices must follow a martingale, which is Samuelson's (1972) result that "properly" anticipated prices must fluctuate randomly. More generally, it follows that defining $\tilde{\epsilon}_t \equiv \tilde{P}_t - e^{rt} P_o$, we have

$$\begin{aligned} L\,(\tilde{\epsilon}_t \mid I_o) &= L\,(\tilde{P}_t \mid I_o) - L\,(e^{rt} P_o \mid I_o) \\ &= L\,(\tilde{P}_t \mid I_o - P_o) \\ &= 0. \end{aligned}$$

This result generalizes the martingale theorem in a significant fashion. In the martingale literature on efficient markets it is required that $E\,(\tilde{\epsilon}_t \mid I_o) = o$, while we have shown that $\tilde{\epsilon}_t$ has no value. Similarly, whereas in the efficient markets theory, $E\,(\tilde{P}_t \mid I_o) = e^{rt} P_o \geqslant P_o$, we require that $L\,(\tilde{P}_t \mid I_o) = e^{rt} P_o \geqslant P_o$. This verifies the following theorem.

Efficient market theorem. In a competitive capital market where values are determined relative to a specific information set, I_o, the deviation of a future price, P_t, from the amortized current price, $e^{rt} P_o$, must be a null stream, that is, a stream of returns with no economic value.

PROOF: See above.

This is not by itself easily testable without further restrictions on the valuation operator. (Of course, in principle it is testable and would be rejected, say, by the discovery of an asset for which $\tilde{\epsilon}_t$ was bounded above zero.) An example is where $L(\cdot)$ is determined by a capital asset pricing model or by an arbitrage model. In these instances pricing is by covariance with certain factors and the argument is similar to that given above. Specifying such a model, then, is solely for testing and is not part of the conceptual framework. This frees the results of efficient market theory from a particular pricing framework; in particular, risk neutrality is irrelevant to our version of informational efficiency. For example, the traditional one-period Sharpe-Lintner valuation relation, $P_{t-1} = [E(\tilde{P}_t \mid I_{t-1}) - \text{cov}(\tilde{P}_t + \tilde{D}_t, \tilde{M}_t \mid I_{t-1})]/(1 + r)$, is a linear valuation with $L(\cdot \mid I_{t-1}) \equiv \frac{1}{1+r}[E(\cdot \mid I_{t-1}) - R \text{ cov}(\cdot, \tilde{M}_t \mid I_{t-1})]$.

Another somewhat less familiar route to testing is to directly restrict the information set, I_t. As a simple example, suppose $\tilde{I}_t$ is unchanging on $(0, T)$ or suppose that $L(\cdot \mid \tilde{I}_t)$ is known, that is, deterministic. It follows that

$$\begin{aligned} P_t &= L(\tilde{X}_T \mid \tilde{I}_t) \\ &= L(\tilde{X}_T \mid \tilde{I}_t = I_o), \end{aligned}$$

which is nonstochastic. Since the price at $t \in (0, T)$ is sure, we must have $P_t = e^{rt} P_o$. More interestingly, specific restrictions on the essential rank of I_t when it is parametrized as a vector space will provide theorems analogous to those derived from the arbitrage pricing theory.

IV. Conclusion and Generalizations

This paper has developed two tools for use in pricing financial assets, the basic valuation theorem and a derivative result, the asset valuation theorem. These were successfully applied to show, by arbitrage considerations alone, how a number of familiar problems involving the valuation of risky return streams could be successfully attacked. The strength of the theoretical underpinnings and the minimal data requirements of these approaches makes them particularly attractive.

This work, however, is only a first step, and much remains to be done. One possible generalization would be to approximate pricing as in the arbitrage theory of capital asset pricing (Ross [1976*a*]). Additionally, the emphasis we have taken has been on discovering principles that could be used with no assumptions on the underlying stochastic processes in the economy. The introduction of such assumptions would enrich the theory. Garman (1977) and Cox and Ross (1976*b*), for example, have shown that if diffusions are used then the operator L will take the form of the Ito parabolic differential operator, and this integration of the operator and the continuous time approaches should lead to new results. Nevertheless, it is surprising how much of what is central to modern finance is based solely on the arbitrage principles embodied in the basic valuation theorem.

Appendix

The principal mathematical tool used to prove the existence of a valuation operator is the following version of the Hahn-Banach separation theorem.

Separation Theorem:

Let A and B be disjoint, nonempty convex sets in a topological space, X. If A is open, then there exists $x^* \in X^*$ (the dual space composed of the continuous linear functionals on X) and $\gamma \in R$ such that $(\forall\ a \in A,\ b \in B)$

$$x^*b \leqslant \gamma < x^*a \cdot$$

PROOF: See Rudin (1973).

We will use the separation theorem to prove the basic valuation principle in a somewhat more general setting than is necessary for this paper, but the additional generality might prove useful in other applications. A rather high level of abstraction is unavoidable, though, due to the use of continuous time stochastic processes in financial pricing models. The spaces which describe such processes are sophisticated, and the abstract form of the separation theorem given above will be used.

The space X will denote returns (per unit asset holding) and will be a space of functions on the domain $R^{\dagger} \times \Omega$, where Ω is a universal set and the functions are measurable with respect to a given σ-algebra on Ω. For example, we could associate with $t \in R^{\dagger}$ a σ-field, P_t, such that for $t' > t$, $P_{t'} \supset P_t$ and $X(t, \cdot)$ is measurable, for each t, with respect to P_t. This would be the typical case where X is an unanticipating random function. The space X is a vector space, and we will endow it with a strong enough topology to insure that the positive orthant, $(x \in X | x > 0)$, is an open set, where $x \geqslant 0$ if $x \geqq 0$ on all non-null sets with strict inequality on some non-null set, and $x > 0$ if the inequality holds on all non-null sets.

The set of marketed assets, M, is a subset of the returns space, X. Let $\psi = \psi(M)$ denote the resulting linear space generated by M. Associated with M is a function, p, that assigns prices to elements of M. This function has an obvious extension to a correspondence on ψ. If α is a bounded additive set function on M, $p(\alpha) \equiv \int_M p(y)\, d\alpha$ and $(\forall x \epsilon \psi)$ if $x(\alpha) \equiv \int_M x d\alpha$, then the extension is given by $p(x) = [p(\alpha) \mid (\exists \alpha) x = x(\alpha)]$. We can now prove the following valuation theorem.

Basic Valuation Theorem:

Suppose that there are no arbitrage possibilities, that is, for all $x \in \psi$ with $x \geqslant 0$, $p(x) > 0$. In addition, $\exists x^o \in \psi$ with $x^o > 0$. It follows that $p(x)$ is linear on ψ and may be extended to $p^* \in X^*$ such that for all $x > 0$, and for all $x \in \psi$ with $x \geqslant 0$, $p^*x > 0$. In particular, then, the value of any pure contingent claim in ψ must be positive.

PROOF: Let A be the interior of the positive orthant of X and let $B = [x \mid (\exists \alpha) x = x(\alpha)$ and $p(\alpha) \leqslant 0]$. Both A and B are convex (cones) and nonempty, and the absence of arbitrage implies that they are disjoint. Since A is open we can apply the separation theorem to obtain $p^* \in X^*$ such that

$$p^*b \leqslant \gamma < p^*a, \tag{A1}$$

for all $a \in A$, $b \in B$. Since $0 \in B$ and the origin is in the closure of A, $\gamma = 0$. Hence for all $x \in \psi$, $p(\alpha) \leqslant 0$ implies that $p^*x \leqslant 0$. Normalize p^* such that for some $y = y(\beta)$ (e.g., $-x^0$), $p^*y = p(\beta) < 0$. By the definition of $p(\cdot)$, we must have $(\forall\ e, f \in R)$ $e\ p^*x + f\ p^*y \leqslant 0$ if $e\ p(\alpha) + f\ p(\beta) \leqslant 0$.

It follows that for all $x \in \psi, p^*x = p(\alpha)$. In particular, then, $p(\alpha)$ generates a function on $\psi, p(x) \equiv p(\alpha)$ where $x = x(\alpha)$. From (A1), for all $x > 0, p^*x > 0$. In addition, since p^* coincides with $p(\cdot)$ on ψ, $(\forall\, x \in \psi)\, x \geqslant 0$ implies that $p^*x = p(x) > 0$. Q.E.D.

Notice that p^* is not a strictly positive operator on all of X; to prove such a result would require further assumptions on the structure of ψ and on the underlying topology.

The need to assume the existence of some $x^0 \in \psi$ with $x^0 > 0$ is made clear in the following example.[9] (Of course, if current wealth can serve as a numeraire, the problem does not arise.)

Example (See fig. A1):

Suppose that $M = (a_1, a_2)$, where $a_1 = \binom{1}{-1}$, $a_2 = \binom{2}{-2}$, and $p(a_1) = 1, p(a_2) = 3$.

The feasible space of returns, ψ, is the line through a_1 and a_2 and does not offer positive returns in both states. The returns from holding two units of a_1 and shorting a_2 are zero in both states of nature and $p(2a_1 - a_2) = -1 < 0$, but since there is no nonzero limited liability asset ($x^0 \geqslant 0$, the money generated by this operation can only be used to attain different points on ψ and not to achieve arbitrarily high $x > 0$.

The p^* operator will generally change over time. To examine its intertemporal properties we can define the suboperator, p^*_t, that will apply at time $t > 0$

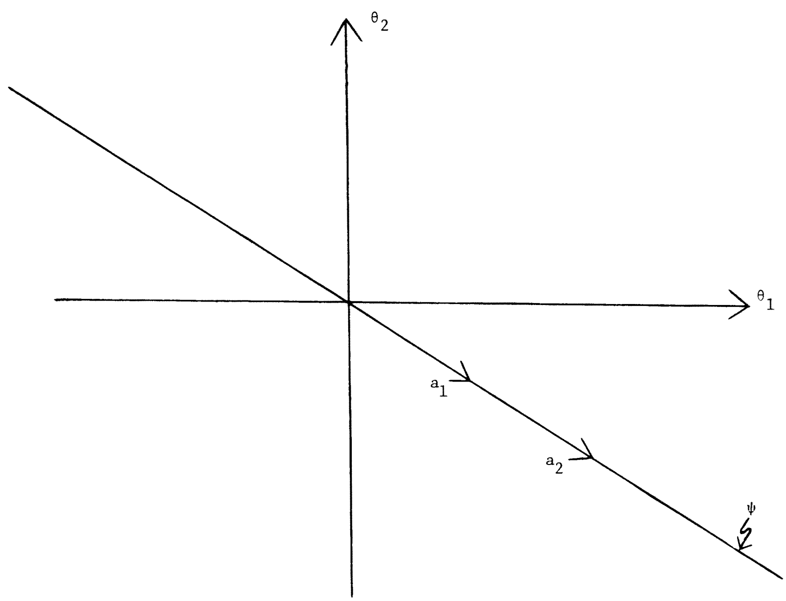

FIG. A1

9. The strict inequality of x^0 in the proof of the basic valuation theorem can be weakened to $x^0 \geqslant 0$ in spaces with more structure (e.g., finite dimensional), but I have not been able to obtain a stronger result at the level of generality assumed in the theorem. It is possible to directly extend $p(a)$ to a $p^* \in X^*$ that agrees with $p(\cdot)$ on ψ, but it appears that to show positivity of p^* requires (for example) an enlarged ψ space.

to return streams on (t, ∞). As currently viewed, p^*_t, is random, and if X is a stream that is 0 on $(0,t)$ then $p^* \cdot X = p^* (p^*_t \cdot X)$. With additional structure on X, for example, a restriction to diffusions, the valuation operator can be shown to have a differential interpretation (e.g., as a parabolic operator, see Garman [1977]).

It should also be noted that there sometimes appears in the literature a confusion between the Basic Valuation Theorem and what has come to be known as the "singe-price law of markets" (see Rubinstein [1976]). The latter asserts that if two portfolios yield the same return streams then they must have the same price. This is, of course, a consequence of the basic valuation theorem, but it is not equivalent to it. The simple two-asset, two-state example with returns (1, 2) and (3, 4), and equal prices provides many counterexamples.

The basic valuation theorem is the heart of the pricing theory in the paper, and it might be useful to see it developed in the more familiar context of discrete state spaces. Consider a two-period model and suppose that Ω is a discrete space with $M \equiv (x^1, \ldots, x^n)$ as the set of marketed assets where x^j is an m-vector whose components are the returns of asset j in the different states of nature. Now, $\psi = [x | x = \sum_j x^j \alpha_j$ for some $(\alpha_j)]$, and we will assume that $\exists x^o \geq 0$ with $x^o \in \psi$. If $p = (p_j)$ is the price vector of the assets, then the no-arbitrage condition asserts that if $\sum_j x^j \alpha_j \geq 0$, then $\sum_j p_j \alpha_j > 0$.

If $M \equiv (x_{ij})$, then we can display this condition in matrix notation as $M \alpha \geq 0 => p \alpha > 0$. The pricing operator, $p(x)$, on ψ has the form $p(x) = \sum_j p_j \alpha_j$, where $x = \sum_j x^j \alpha_j$.

Now the valuation theorem asserts that $p(x)$ can be extended to a positive operator p^* on all of X. Since X is an m-dimensional Euclidian space, p^* is a row vector, $p^* = (p^*_1, \ldots, p^*_m)$, and since p^* agrees with p on ψ, for each j, $p_j = p^* x^j = \Sigma_i p^*_i x_{ij}$. In addition, for all $x > 0$, $p^* x = \sum_i p^*_i x_i > 0$, which implies that $p^* \geq 0$. Furthermore, since p^* agrees with p on ψ, if a unit vector $e_i \in \psi$, then $e_i = M\alpha$ for some α and $p^* e_i = p^*_i = p \alpha > 0$, and it follows that the implicit prices for all states in which contingent claims can be formed are strictly positive.

In this framework the operator p^* may be interpreted as a vector of positive state space prices. The existence of such a price system, thus, is equivalent to the absence of arbitrage possibilities. Of course, there is no assertion that p^* is unique. To the contrary, if n is the number of marketed basic assets and $m > n$, then the space of possible valuation operators will be of dimension $m - n$. In the more complex spaces of the basic valuation theorem, the space of possible operators will generally be of the same size—in the sense of category or measure—as the space of returns itself.

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