ECON-UB 233 Dave Backus @ NYU

Lab Report #4: Asset Pricing Fundamentals

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Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Please include your Matlab code.

Solution: Brief answers follow, but see also the attached Matlab program. Download this document as a pdf, open it, and click on the pushpin:

1. *State prices and related objects.* Consider an economy with three states. State prices and probabilities are

State z	State Price $Q(z)$	Probability $p(z)$	Dividend $d(z)$
1	1/3	1/2	1
2	1/3	1/4	2
3	1/3	1/4	3

   (a) What is the pricing kernel in each state?

   (b) What is the price of a one-period bond? What is its return?

   (c) What are the risk-neutral probabilities? Why are they different from the true probabilities?

   (d) Suppose equity is a claim to the dividend in the last column. What is its price? What is the return on equity in each state?

   (e) What is the expected return on equity? The risk premium?

Solution:

State z	State Price $Q(z)$	Probability $p(z)$	Dividend $d(z)$	Pr Kernel $m(z)$	R-n probs $p^*(z)$	Return $r^e(z)$
1	1/3	1/2	1	2/3	1/3	1/2
2	1/3	1/4	2	4/3	1/3	1
3	1/3	1/4	3	4/3	1/3	3/2

(a) See table. In each state, $m(z) = Q(z)/p(z)$.

(b) $q^1 = Q(1) + Q(2) + Q(3) = E(m) = 1$. Its return is $r^1 = 1/q^1 = 1$.

(c) See table. In each case, $p^*(z) = p(z)m(z)/q^1$.

(d) The price is $q^e = \sum_z Q(z)d(z) = \sum_z p(z)m(z)d(z) = 2$. The returns are in the table.

(e) The expected return is $E(r^e) = \sum_z p(z)r^e(z) = 0.875$. The risk premium is $E(r^e - r^1) = -0.125$. Why negative? Because the dividend is highest in the states when m is also high.

2. *Pricing kernels and risk-neutral probabilities with geometric risk.* Consider a representative agent economy with a power utility agent. Utility is

$$u(c_0) + \beta \sum_z p(z) u[c_1(z)]$$

with $u(c) = c^{1-\alpha}/(1-\alpha)$ and risk aversion parameter $\alpha > 0$. Log consumption growth $z = \log g = \log c_1 - \log c_0$ is geometric: z takes on the values $0, 1, 2, \ldots$ with probabilities $p(z) = (1-\omega)\omega^z$ and "intensity" parameter $0 < \omega < 1$.

(a) What is the pricing kernel $m(z)$ in each state z?

(b) What are the state prices $Q(z)$?

(c) What are the risk-neutral probabilities $p^*(z)$? What is the risk-neutral distribution?

(d) How do the risk-neutral probabilities $p^*(z)$ differ from the true probabilities $p(z)$? Why?

(e) Set $\omega = 2/3$ and $\alpha = 1$ and plot $p(z)$ and $p^*(z)$ for z between zero and 10. How do they differ? Why?

Matlab mini-tutorial on bar charts. Suppose we have vectors `z`, `p`, and `pstar`. The order of inputs in Matlab plot commands is x variable first (horizontal axis), then the y variable (vertical axis): `plot(x,y)`, `bar(x,y)`, etc. We can plot probabilities against `z` with the commands

```
bar(z, p)           % just p
bar(z, [p pstar])   % p and pstar together
```

The second differs only in having two y's.

Solution:

(a) The pricing kernel is $m(z) = \beta[c_1(z)/c_0]^{-\alpha} = \beta e^{-\alpha z}$.

(b) State prices are

$$Q(z) \quad = \quad p(z)m(z) \quad = \quad (1-\omega)\omega^z \beta e^{-\alpha z} \quad = \quad (1-\omega)\beta(\omega e^{-\alpha})^z.$$

(c) Risk-neutral probabilities are $p^*(z) = Q(z)/q^1 = p(z)m(z)/q^1$. Here we have

$$q^1 \;=\; \sum_{z=0}^{\infty} Q(z) \;=\; (1-\omega)\beta \sum_{z=0}^{\infty} (\omega e^{-\alpha})^z \;=\; (1-\omega)\beta/(1-e^{-\alpha}\omega).$$

Risk-neutral probabilities are therefore

$$p^*(z) \;=\; Q(z)/q^1 \;=\; (1-e^{-\alpha}\omega)(\omega e^{-\alpha})^z.$$

This is geometric with parameter $\omega^* = \omega e^{-\alpha} < \omega$.

We could also attack this using the cumulant generating function. See the notes.

(d) The true probabilities start at $1-\omega$. The risk-neutral probabilities start at $1-\omega^*$, which is larger. In this sense we're putting more weight on the bad outcomes.

(e) You can see how this looks in the figure generated by the Matlab code.

3. *Option pricing.* We're going to value an option and persaude ourselves that option valuation is just an application of the no-arbitrage theorem. We'll examine the structure of option prices in greater depth in a couple weeks.

A *call option* gives the owner the right to purchase an asset — which we refer to as the *underlying* — one period from now at a price k — the so-called *strike price*. As with other assets, we set the option price now.

The question is what that price is. One input is the current price of the underlying, which we label s_0. We set $s_0 = 100$ here. Another input is the risk-neutral distribution of future prices of the underlying, which we label $s(z)$. The owner of a call option with strike price k will exercise the option and purchase the stock only if s is greater than (or equal to?) k. That gives rise to the option cash flow

$$d(z) \;=\; \max\{0, s(z)-k\}.$$

Given this cash flow, we value the option as we would any other asset. We'll use specifically the risk-neutral valuation equation

$$q^c \;=\; q^1 \sum_z p^*(z) d(z) \;=\; q^1 \sum_z p^*(z) \max\{0, s(z)-k\}, \tag{1}$$

where q^c is the price of the call option, q^1 is the price of a one-period riskfree bond, and $p^*(z)$ is the risk-neutral probability of state z.

The final input is the risk-neutral probabilities. We'll work with a discrete approximation to a standard normal distribution for z and connect the future price to it by $\log s(z) = \mu + \sigma z$. A discrete approximation is easier to work with than the real thing (sums are easy, but numerical integration is neither pretty nor efficient). In Matlab terms, we set up a grid of points for z and assign probabilities to them from the standard normal pdf:

```
zmax = 4;
dz = 0.1;
z = [-zmax:dz:zmax]';
pstar = exp(-z.^2/2)*dz/sqrt(2*pi);
```

We can make this approximation as close to the original as we want by shrinking `dz`.

(a) What did we just do there with the discrete grid?

(b) One check on the approximation is the sum of the probabilities. Do they sum to one?

(c) Set up a related grid of values for $s(z)$: that is, for each point z we compute the related point $s(z)$ using the connection between them. When you do this, use $q^1 = 0.95$, $\sigma = 0.1$, and

$$\mu \;=\; \log(100/q^1) - \sigma^2/2.$$

More on this later. What value of μ do you get?

(d) Compute the cash flows $d(z)$ for an option with strike price $k = 110$. Graph the cash flow $d(z)$ against the future price $s(z)$ of the underlying.

You may find these Matlab commands helpful:

```
d_positive = s >= k
d = d_positive.*(s-k);
```

The first line generates a vector that equals one if $s \geq k$ and zero otherwise.

(e) Use the risk-neutral pricing equation (1) to compute the option's value.

(f) *Optional, extra credit.* Compute the option price with $\sigma = 0.2$, making sure to update your value of μ. How does it compare to your earlier calculation? Can you guess why?

Solution:

(a) We approximated a continuous random variable with a discrete one. As long as the pdf of the former is smooth, this works pretty well.

(b) To four digits: 1.0000.

(c) $\mu = 4.6515$.

(d) The cash flows are $d(z) = \max\{0, s(z) - k\}$. The Matlab program produces the graph.

(e) The value of the option is

$$q \;=\; q^1 \sum_z p^*(z) d(z).$$

In this case we have $q = 2.1471$.

(f) If the future price is $s(z)$, then the current price follows from the usual pricing formula. This gives us a price of 99.9981. If we were to do this exactly, we'd get 100 — the same 100 that shows up in the formula in part (c).

The next part is a little obscure, but we will see it again when we look at options more closely. The idea is to choose the risk-neutral distribution so that it's consistent with the (known) price of the underlying asset. If the price is 100, the formula in part (c) sets μ to reproduce this value.