Lab Report #5: Returns & Risk Premiums

Revised: November 3, 2015

Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Please include your Matlab code.

Solution: Brief answers follow, but see also the attached Matlab program. Download the pdf, open, and click on the pushpin:

1. *The equity premium in a Bernoulli economy.* Consider a power utility agent with discount factor β and risk aversion parameter α facing Bernoulli consumption growth,

$$\log g \;=\; \begin{cases} a & \text{with probability } 1-\omega \quad \text{(state 1)} \\ a+b & \text{with probability } \omega \quad \text{(state 2).} \end{cases}$$

Here a and b are real numbers, ω is between zero and one, $\beta = 0.99$, and $\alpha = 5$. The pricing kernel is $m = \beta g^{-\alpha}$.

(a) Show that the mean and variance of $\log g$ are

$$E(\log g) \;=\; a + \omega b$$
$$\text{Var}(\log g) \;=\; (1-\omega)\omega b^2.$$

(b) If $\omega = 0.2$, what values of (a, b) give $\log g$ a mean of 0.0200 and a variance of 0.0350^2?

(c) With these parameter values, plus those for β and α, what is the pricing kernel in each state?

(d) What is the price of a riskless one-period bond? The return? How does it compare to what we see in US data?

(e) What is the price of a claim to $d = g$ ("equity")? What is its return in each state? Its expected return?

(f) What is the equity premium (that is, the expected excess return on equity)? How does it compare to what we see in US data?

(g) What is the maximum Sharpe ratio in this economy? How does it compare to the Sharpe ratio we see in US data?

Solution:

(a) I did this using the cgf; see the Matlab code.

(b) The expressions for the mean and variance give us

$$\begin{aligned} 0.0200 &= a + \omega b \\ 0.0350^2 &= (1-\omega)\omega b^2. \end{aligned}$$

If we take the positive square root, the second gives us $b = 0.0875$ and the first gives us $a = 0.0025$.

(c) m is 0.9777 in state 1, 0.6313 in state 2.

(d) With these values, we have $q^1 = 0.9084$, $r^1 = 1.1008$. In US data, the mean riskfree rate is 1.0198, so we're a bit high.

(e) The price is $q^e = 0.9223$, the expected return is $E(r^e) = 1.1069$.

(f) The equity premium $E(r^e - r^1) = 0.0061$. As we've come to expect, the equity premium is smaller than we see in the data (0.0571 was our estimate).

(g) The maximum Sharpe ratio comes from the Hansen-Jagannathan bound: $\text{Std}(m)/E(m) = 0.1526$. In the data, the Sharpe ratio for equity is 0.3049, so we're low, just as we are with the equity premium.

2. *Normal mixtures.* We can solve the lognormal representative agent asset pricing model by hand, but the cost is that we have eliminated the impact of skewness, excess kurtosis, and other high-order cumulants. We use a normal mixture here to explore their impact.

   More specifically, we use a discrete approximation to a normal mixture to produce departures from normality. (No, that's not an ozymoron.) The mixture uses two normal distributions as inputs. The first one has mean zero and variance one, the second has mean θ and variance δ. We denote their pdf's by p_1 and p_2, respectively. The pdf of our state variable z is then

$$p(z) = (1-\omega)p_1(z) + \omega p_2(z)$$

   for some mixing parameter ω between zero and one. Roughly speaking, θ governs skewness and δ governs excess kurtosis. We'll set $\omega = 0.2$, $\theta = -1$, and $\delta = 2$. We construct a discrete approximation to p with the Matlab commands

```
% grid
zmax = 6;
dz = 0.1;
z = [-zmax:dz:zmax]';
% mixture
omega = 0.2; theta = -1; delta = 2;
p1 = exp(-z.^2/2)*dz/sqrt(2*pi);
p2 = exp(-(z-theta).^2/(2*delta))*dz/sqrt(2*pi*delta);
p  = (1-omega)*p1 + omega*p2;
```

We build the consumption growth distribution from the random variable z. Log consumption growth is

$$\log g(z) \quad = \quad \mu + \sigma z.$$

The mean and variance are then

$$\begin{aligned} E(\log g) &= \mu + \omega\sigma\theta \\ \mathrm{Var}(\log g) &= \sigma^2[(1-\omega) + \omega\delta + \omega(1-\omega)\theta^2]. \end{aligned}$$

Finally, the pricing kernel is the representative agent's marginal rate of substitution,

$$m(z) \quad = \quad \beta \exp(-\alpha \log g) \quad = \quad \beta \exp[-\alpha(\mu + \sigma z)].$$

We'll set $\beta = 0.99$ and $\alpha = 10$.

(a) Plot p (the mixture) and p_1 (the standard normal). How do they differ?

(b) Choose values of μ and σ to reproduce the observed mean (0.0200) and standard deviation (0.0350) of $\log g$.

(c) Use the commands

```
loggbar  = sum(p.*logg)
stdlogg  = sqrt(sum(p.*(logg-loggbar).^2))
skewlogg = sum(p.*(logg-loggbar).^3)/stdlogg^3
xkurlogg = sum(p.*(logg-loggbar).^4)/stdlogg^4 - 3
```

to compute properties of the mixture distribution. What indications do they give us that the distribution is not normal? How do the skewness and excess kurtosis of $\log g$ compare to what we see in US data?

(d) What is the riskless one-period return r^1?

(e) Define equity as a claim to the dividend $d(z) = g(z)^3$. What is its price? Its expected return? The equity premium?

(f) How does your estimate of the equity premium change if you change θ to $+1$? That is: repeat (c,d,e) with $\theta = 1$ and compare your answers.

Solution:

(a) Run the Matlab code. The picture shows a fair amount of negative skewness in the mixture.

(b) I used $\mu = 0.0260$ and $\sigma = 0.0300$.

(c) The mean and standard deviation are what we put in, thank goodness. Skewness is -0.3588, which is what we saw in the figure. Excess kurtosis is 0.5528, another sign of non-normality. In the data, we see less skewness and more kurtosis.

(d) The riskfree rate is $r^1 = 1.1572$.

(e) The price of equity is $q^e = 0.8876$. Its expected return is 1.2027, giving us an equity premium of 0.0455. This is much better than we've seen before. the question is why. One possibility is the large risk aversion parameter. Another is the non-normal log consumption growth distribution. We check the latter below.

(f) If we change the sign of θ, that gives us positive skewness. What does that do to the equity premium? It falls to 0.0418, which is a modest effect. We could make the effect large by choosing a larger negative value of θ, but we already have more skewness than we see in the data.