Fundamentals of Asset Pricing
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Darrell Duffie notes that the 1970s were a "golden age" for asset pricing theory, but suggests that the period since has been "a mopping-up operation" (Duffie, *Dynamic Asset Pricing Theory*, preface). That takes some of the glamor out of the subject, but he's right, the basic theory has been worked out. The same is true of calculus, of course, and that doesn't make it any less useful. Our goal here is to summarize the concepts and results, including the remarkable *no-arbitrage theorem* and the mysterious *risk-neutral probabilities*.

One way to state the no-arbitrage theorem is that there is an m that makes $E(mr^j) = 1$ for every asset j. This is the foundation of almost all of modern asset pricing. In finance, it's common to find a statistical m that works reasonably well for the assets of interest. The source of this m is typically left unresolved. In macroeconomics, it's common to link m to the marginal rate of substitution of a representative agent. It's not a perfect theory, as we'll see, but it gives us some insight into asset returns, particularly the tendency for equity to pay higher returns, on average, than bonds. Why? Assets that pay off mostly in good states (states in which consumption is high) tend to have lower prices and (therefore) higher returns than those that pay off mostly in bad states.

1   Overview

The setting is our usual two-period event tree: there are two dates, 0 and 1, and at date 1 a state z occurs with probability $p(z)$. A particular asset j is a claim to a date-1 "dividend" or cash flow $d^j(z)$, a function of the state z (a random variable, in other words). If the date-0 price of this asset is q^j, the gross return between dates 0 and 1 is $r^j(z) = d^j(z)/q^j$.

[Draw the event tree, note where prices and dividends are paid.]

In this context, we can summarize modern asset pricing theory. The idea is to derive prices of assets from state prices — prices $Q(z)$ of Arrow securities. Why we can do this is a subtle issue that we'll address later, but for now note that if we know the dividends and state prices, the asset's price is the sum

$$q^j \;\; = \;\; \sum_z Q(z) d^j(z). \tag{1}$$

We'll refer to (1) and its successors as the *pricing relation*. Once we know the state prices $Q(z)$, we use the pricing relation to compute q^j. When we do this, we say we "price" the asset.

This is a good example of one of the standard tricks of finance: we decompose an asset into pieces that we value separately. Here the pieces are Arrow securities, which we might have mistakenly thought of as a purely theoretical concept. The theorem says we can always

decompose an asset into its component Arrow securities and value the security as the sum of the values of its parts. In this respect, the basic theory of asset pricing is very simple. What the theorem doesn't tell us is what the state prices are or how we might compute them.

We'll look at two other versions of the same equation — same theory, just different notation. One is based on a *pricing kernel* (or *stochastic discount factor*) m, defined implicitly by $Q(z) = p(z)m(z)$. [That is: $m(z) = Q(z)/p(z)$.] After substitution, equation (1) becomes

$$q^j \;=\; \sum_z p(z)m(z)d^j(z) \;=\; E(md^j). \tag{2}$$

Dividing by q^j gives us the familiar $E(mr^j) = 1$.

One of the assets we'll see repeatedly is the one-period "riskfree" bond. This asset is a claim to sure payoff of one: $d^1(z) = 1$ in all states z. Equation (1) tells us that its value is

$$q^1 \;=\; \sum_z Q(z).$$

In terms of the pricing kernel, its price is

$$q^1 \;=\; \sum_z p(z)m(z) \;=\; E(m).$$

However we compute the price, the return is connected to it by $r^1 = d^1/q^1 = 1/q^1$.

Another version of the pricing relation is based on so-called *risk-neutral probabilities* p^*. These objects,

$$p^*(z) \;=\; p(z)m(z)/q^1,$$

are positive if the m's are, and they sum to one, so they are legitimate probabilities. Substituting $p(z)m(z) = q^1 p^*(z)$ into (2) gives us

$$q^j \;=\; q^1 \sum_z p^*(z)d^j(z) \;=\; q^1 E^*(d^j), \tag{3}$$

where E^* means the expectation based on the p^*s. In (2), the pricing kernel performed two roles: discounting and risk adjustment. Here the same roles are divided between q^1 (discounting) and p^* (risk adjustment). If it were up to me, I'd call them *risk-adjusted probabilities*, but I haven't had much success selling that to others.

The idea in each of these cases is to go from state prices (or pricing kernel or risk-neutral probabilities) to prices of specific assets. We say we "price" these assets.

2   The no-arbitrage theorem

We start with the remarkable no-arbitrage theorem connecting arbitrage and state prices. Versions were developed by Steve Ross ("Return, risk, and arbitrage," 1977) and Michael Harrison and David Kreps ("Martingales and arbitrage," 1979) in the late 1970s. The focus on arbitrage wasn't completely new — Fischer Black and Myron Scholes used similar methods to value options — but it was a sharp break from the mean-variance approach to asset pricing that dominated finance at the time, and that still plays a central role in textbooks.

The central ingredient is arbitrage. An arbitrage here is defined as getting something for nothing: getting a positive payoff in one or more states for free. The condition of the no-arbitrage theorem is that the economy has no arbitrage possibilities: you can't get something for nothing. Think of an arbitrage as a portfolio position. "Get something" means (in this context) that the payoff is greater than or equal to zero in all states and positive in at least one. "For nothing" means that the cost of this position is zero or less. The condition of the theorem is that such possibilities don't exist. The rationale should be familiar if you've ever taken an economics course. If such an opportunity existed, then prices should adjust to eliminate them. In the general equilibrium model, they're ruled out by consumer maximization, but the theorem applies to other environments as well.

Formally, let q be a vector that contains all of the asset prices. The typical element is q^j. And let D be a matrix whose jth column is the dividend vector d^j for the jth asset, with each row corresponding to a specific state. [Draw a table to make sure you follow.] If J is the number of assets and Z the number of states, then q has dimension J and D is Z by J. The asset structure of an economy is thus summarized by D.

Now put the pieces together. A portfolio is a vector of quantities a, where a^j is the number of shares of asset j. Dividends on the portfolio as a whole are then

$$\sum_j d^j(z)a^j \;\; = \;\; Da.$$

The cost of the portfolio is

$$\sum_j a^j q^j \;\; = \;\; a^\top q.$$

We say there's an arbitrage opportunity if there's a portfolio a whose (i) payoffs are greater than or equal to zero in all states and positive in at least one and (ii) cost is non-positive: something for nothing, in other words. Conversely, we say (q, D) is *arbitrage free* if there are no such portfolios a.

Examples. Which of these price-dividend combinations are arbitrage free?

$$(a)\ q\ =\ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad D\ =\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(b)\ q\ =\ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad D\ =\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(c)\ q\ =\ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad D\ =\ \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

Do any of these assets have arbitrage possibilities? We'll come back to this, but (a) and (b) are arbitrage-free. But take a closer look at (c): if we sell two units of asset 1 and buy one unit of asset 2, the cost is zero and the payoff is $(0, 0, 1)$ in the three states. This is an arbitrage, so this example is not arbitrage free.

Back to the theorem. In its simplest form it's is based on state prices. We're looking for prices $Q(z)$ such that the price of every asset equals the sum of the values of its state-specific dividends — that is, equation (1). In matrix terms we might write

$$q\ =\ D^\top Q, \tag{4}$$

which is equation (1) rewritten in matrix form for all assets at once. Can we deduce state prices Q from the prices and dividends of less abstract assets?

The answer is yes:

Theorem. *There exist positive state prices $Q(z)$ consistent with (1) if and only if the economy is arbitrage free.*

Draw a box around this, it's the central result in the theory of asset pricing. What's remarkable about it is that it's so general. Arbitrage is ruled out in general equilibrium models because it's inconsistent with equilibrium, specifically the combination of agent maximization and market clearing. But here we get state prices with a lot less structure than that. The theorem tells us, under the no-arbitrage condition, that we can always find state prices that value assets correctly, but doesn't tell us much more. Note, too, that the theorem goes both ways: if we can reproduce the values of assets with positive state prices, then the model is arbitrage free.

We won't prove the theorem, but some examples should give you the idea.

Examples (continued). Which of these price-dividend combinations are arbitrage free?

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What are the state prices?

$$(a)\ q\ =\ \begin{bmatrix} 1 \\ 2 \end{bmatrix},\quad D\ =\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(b)\ q\ =\ \begin{bmatrix} 2 \\ 3 \end{bmatrix},\quad D\ =\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(c)\ q\ =\ \begin{bmatrix} 1 \\ 2 \end{bmatrix},\quad D\ =\ \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$(d)\ q\ =\ \begin{bmatrix} 3 \\ 6 \end{bmatrix},\quad D\ =\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

Let's go through them one at a time. In (a), the assets are Arrow securities so the state prices are the prices of the assets: $Q(1) = 1$ and $Q(2) = 2$. Both are positive so these assets are arbitrage free. In (b) the state prices are $Q(1) = Q(2) = 1$, which leads to the same conclusion. In (c) we noticed an arbitrage earlier. What happens to the state prices? The only way to reconcile the two asset prices with state prices is to set $Q(3) = 0$, thereby negating the payoff advantage of the second asset. But this violates the conclusion of the theorem, which says state prices are positive. In (d) we have fewer assets than states, but the same logic applies. There's no arbitrage. But here, because we have fewer equations than unknowns, the choice of Q isn't unique: There are lots of Q's that price the assets the same way. That's a classic feature of models with more states and assets — what we call *incomplete markets*. It's an illustration of the generality of the theory.

3 Pricing kernels and risk premiums

Here we shift from state prices to a pricing kernel and describe the theoretical foundation of risk premiums. We'll see that risk premiums are tied to the covariance of returns with the pricing kernel.

When we define the pricing kernel $m(z)$ from state prices by $Q(z) = p(z)m(z)$ [or $m(z) = Q(z)/p(z)$, if you prefer], we introduce probabilities into the pricing relation. That's helpful, because we often want to know about things like the mean return, which depends (obviously) on the probabilities of various outcomes. Consider the pricing of the one-period riskfree asset. As we've seen, its price is $q^1 = E(m)$ and its (gross) return is $r^1 = 1/E(m)$.

Now that we have probabilities, we can define *risk premiums*. We define the risk premium on asset j as its expected excess return over the one-period riskfree rate: $E(r^j - r^1)$. We'll see that it can be positive, negative, or zero, depending on the asset.

Consider the pricing of an arbitrary asset j with equation (2). The mean return now depends on the relation between the dividend and the pricing kernel. If m is constant, the price of the asset depends only on its expected dividend:

$$q^j\ =\ E(md^j)\ =\ mE(d^j)\ =\ q^1 E(d^j).$$

The return is therefore $r^j = d^j/q^j$ and the expected return $E(r^j) = E(d^j)/[q^1 E(d^j)] = r^1$. Since every asset has the same expected return, there are no risk premiums. Evidently variation in m is central to having nonzero risk premiums.

In general, the risk premium depends on the relation between the two random variables, m and d: whether $E(md)$ is larger or smaller than $E(m)E(d)$. If it's greater, the price is higher and the expected return lower than the riskfree rate. We can elaborate using

$$q^j \;=\; E(md^j) \;=\; E(m)E(d^j) + \mathrm{Cov}(m, d^j).$$

[Can you show this? Use the definition of the covariance and expand.] So if $\mathrm{Cov}(m, d) = 0$, we're back to the zero risk premium case. If the covariance is negative, the price is lower and the mean return is therefore higher.

We could demonstrate the same thing by looking directly at excess returns. Since $E(mr^j) = 1$ for all assets j, we can subtract asset 1 and express the pricing relation in terms of the excess return $x^j = r^j - r^1$:

$$E(mx^j) \;=\; 0.$$

Using the relation for the expectation of a product, we have

$$E(x^j) \;=\; -\mathrm{Cov}(m, x^j)/E(m).$$

Since m and $E(m)$ are positive, the expected excess return (the risk premium) has the opposite sign as the covariance with m.

The next question is where m comes from. In finance, it's common to think of m as an arbitrary random variable whose properties are chosen to reproduce observed asset prices. We'll see examples of this sort when we look at models to value options and bonds. In macroeconomics, it's common to think of m as connected to the state of the economy, whether it's growing rapidly or slowly. We give an example of that shortly.

4   Digression: The Capital Asset Pricing Model

The no-arbitrage theorem has been the foundation of asset pricing since the late 1970s. It's a general framework for thinking about asset prices that can, in principle, be applied to any financial asset: stocks, bonds, currencies, or derivatives of any of these broad categories. The Capital Asset Pricing Model or CAPM is a precursor from the 1960s that we still teach to undergrads and MBAs. It's used in the business world, I'm told, to value equity and to assess the riskiness of equity portfolios.

The model consists of the equation

$$E(r^j - r^1) \;=\; \beta_j E(r^m - r^1), \tag{5}$$

where r^m is the return on the "market" and $\beta_j = \mathrm{Cov}(r^j - r^1, r^m - r^1)/\mathrm{Var}(r^m - r^1)$. In practice, the market is generally a broad-based stock index. If you look carefully, you might

recognize β_j as a regression coefficient. The variance in the denominator simply scales the covariance: when $r^j = r^m$, $\beta^m = 1$ and the two sides of the equation are equal.

There's an analogy, which we can make precise, with the no-arbitrage theorem. The central issue is the covariance in β_j: the risk premium is proportional to the covariance with the market return. In the no-arbitrage approach, it's proportional to the covariance with the pricing kernel. They also have a similar macroeconomic flavor: assets that pay off most in good times have positive risk premiums. In the CAPM this stems from the connection with the market return, which tends to be high when the economy is growing rapidly and low when the economy is growing slowly or shrinking.

The challenge with applying asset pricing models, this one and others, is coming up with a practical version of the pricing kernel m. Here the issue is what to use for the market return r^m. They're the same issue, as we see next.

Further digression. We can derive the CAPM from the pricing kernel,

$$m(z) \quad = \quad a + b r^m(z).$$

The proof involves using $E(mr) = 1$ for $r = r^1, r^m$ to nail down the coefficients a and b. It's tedious. If you decide to take it on, keep in mind that $\mathrm{Var}(x) = E(x^2) - E(x)^2$ and $\mathrm{Cov}(x,y) = E(xy) - E(x)E(y)$.

5 Macrofoundations of the pricing kernel

What would make the covariance between the pricing kernel and dividends negative — or positive? At the level of generality of the theorem, we have little basis for an answer. That's where macroeconomic foundations come in, they supply an economic basis for saying which states have high prices and which states have low prices. The basic idea is that prices are lower when goods are abundant — namely, in booms. And prices are high when goods are scarce — in recessions. We'll see that this leads to positive risk premiums on assets who payoffs are larger in booms.

The simplest version of this is a representative agent model. In this case, m is the marginal rate of substitution of a representative agent. Since marginal utility is decreasing, m is lower in good states — states where consumption is high — than in bad states. An asset, like an equity index, that pays off most in good times will therefore have a lower price $[q^j = E(md^j) < E(m)E(d^j)]$ and a higher mean return $[E(r^j) = E(d^j)/q^j]$.

Example. Let's make this more concrete and consider a version of the one-agent exchange economy. We'll specify a distribution for consumption growth and use it to derive prices of assets. We illustrate the approach in a lognormal setting, where the math is unusually transparent. Suppose the state $z \sim \mathcal{N}(\kappa_1, \kappa_2)$. Let $c_1(z)/c_0 = e^z$, so that $z = \log(c_1/c_0) = \log c_1 - \log c_0$. Assume also that we have power utility: $u'(c) = c^{-\alpha}$ for some $\alpha > 0$. Now some questions, with answers:

- What is the pricing kernel? Its distribution? The pricing kernel is $m = \beta(c_1/c_0)^{-\alpha}$, so

$$\log m(z) \quad = \quad \log \beta - \alpha \log(c_1/c_0) \quad = \quad \log \beta - \alpha z \quad \sim \quad \mathcal{N}(\log \beta - \alpha \kappa_1, \alpha^2 \kappa_2).$$

- How is the pricing kernel related to consumption? Note the essential feature: when consumption $c_1(z)$ is high, the pricing kernel $m(z)$ is low. Good states have lower value. It's our old marginal utility result: consumption has less value in states where there's a lot of it. The strength of this effect depends on the risk aversion parameter α. If we set $\alpha = 0$ the effect goes away and the pricing kernel is constant.

- What are the price and return of a one-period (riskfree) bond? They solve

$$
\begin{aligned}
q^1 &= E(m) = \beta e^{-\alpha\kappa_1 + \alpha^2\kappa_2/2} \\
r^1 &= 1/q^1 = \beta^{-1} e^{\alpha\kappa_1 - \alpha^2\kappa_2/2}.
\end{aligned}
$$

  [This kind of calculation should be familiar by now.]

- What is the price of an asset ("equity") with dividend $d^e(z) = [c_1(z)/c_0]^\lambda$ for some arbitrary value of λ? This is a useful device, since it allows us to vary the sensitivity of the dividend to the endowment without killing off our convenient loglinear structure. As long as $\lambda > 0$, this is an asset that pays off more in good states than bad. The price of this "equity" is

$$
q^e = E(md) = \beta e^{(\lambda-\alpha)\kappa_1 + (\lambda-\alpha)^2\kappa_2/2}.
$$

  Note that when $\lambda = 0$ we get the one-period bond's dividend and price.

- What is the return on equity? The mean return? The return is $r^e(z) = d(z)/q^e$, which satisfies

$$
\log r^e(z) = \lambda z - \log q^e \sim \mathcal{N}(-\log\beta + \alpha\kappa_1 - (\lambda-\alpha)^2\kappa_2/2, \lambda^2\kappa_2).
$$

  (This takes patience, just stick with it. Or use Matlab for the substitutions.) The mean return is therefore

$$
E(r^e) = \beta^{-1} e^{\alpha\kappa_1 + [\lambda^2 - (\lambda-\alpha)^2]\kappa_2/2}.
$$

- Is there a risk premium on equity? Our definition of a risk premium is the expected excess return: $E(r^e - r^1)$. The β and κ_1 terms are the same, so we have (thankfully) no risk premium when $\kappa_2 = 0$. If we compare the κ_2 terms, we see that the risk premium is positive if

$$
0 < [\lambda^2 - (\lambda-\alpha)^2] + \alpha^2 = 2\alpha\lambda.
$$

  So the risk premium is positive if $\lambda > 0$. It's larger if risk aversion is larger (larger α) or the dividend is more sensitive to the endowment (larger λ).

This delivers on our hope for insight: assets that pay off more in good times ($\lambda > 0$) have positive risk premiums as a result. We also verify that the risk premium is zero if either risk aversion (α) or risk (κ_2) is zero.

6 Risk-neutral probabilities and the no-arbitrage condition

It's common in finance to talk about risk-neutral probabilities rather than pricing kernels. There's no difference in content, it's just a change in notation. As we noted earlier, risk-neutral probabilities p^* are defined implicitly by

$$q^1 p^*(z) = p(z)m(z) \tag{6}$$

and explicitly by $p^*(z) = p(z)m(z)/q^1$. Basically we're using $p(z)m(z)$ as probabilities (they're positive by construction) and dividing by $q^1 = E(m) = \sum_z p(z)m(z)$ (so that they sum to one). That makes them probabilities: positive numbers that sum to one. As before, the effects of risk come from m. If m is constant, then $p^*(z) = p(z)$. [Can you show this?]

So where does that leave us? We're still using the no-arbitrage theorem and the pricing relation. Here the latter becomes, as we've seen,

$$q^j = q^1 E^*(d^j),$$

which is (3) repeated. Dividing by q^j and substituting $r^1 = 1/q^1$ gives us

$$r^1 = E^*(r^j).$$

In words: under the risk-neutral probabilities, all assets have the same expected return. Their true expected returns aren't the same, of course, only their expected returns based on the artificial probabilities p^*. With these probabilities, pricing looks risk-neutral: only the expected return matters. The effects of risk, of course, are built into the risk neutral probabilities through m.

In practice, we might value assets by observing the bond price q^1 and making up a useful set of risk-neutral probabilities. However, the equations above tell us we can't do that arbitrarily: we need to do it in a way that's consistent with risk neutral pricing: the price equals the discounted risk-neutral expected payoff or, equivalently, the risk-neutral expected return is the riskfree rate r^1. We refer to this as the *no-arbitrage condition*.

Here's an example. Suppose $\log r^j$ has a normal risk-neutral distribution with mean κ_1 and variance κ_2. For the returns to be the same, we need

$$r^1 = E^*(r^j) = e^{\kappa_1 + \kappa_2/2}.$$

Sometimes we'll write the condition in logs: $\log r^1 = \kappa_1 + \kappa_2/2$. Typically we'll base κ_2 on data and let the mean κ_1 adjust to satisfy the equation. We could do the same with equation (3) using the substitution $r^j = d^j/q^j$.

7 Risk-neutral probabilities with power utility

We get a better sense of how risk-neutral probabilities work if we examine them in a familiar setting. Here we reconsider the macrofinance idea that payoffs in good states are worth less than payoffs in bad states. In macro-finance models, that happens because the pricing

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kernel is smaller in good states. Here we get the same thing by making the risk-neutral probabilities smaller.

Example (Bernoulli risks). Let $z = \log c_1 - \log c_0 = \log g$ (log consumption growth) take on two values,

$$z = \begin{cases} \gamma_1 & \text{with probability } 1 - \omega \\ \gamma_2 > \gamma_1 & \text{with probability } \omega. \end{cases}$$

The pricing kernel is $\beta g^{-\alpha} = \beta e^{-\alpha z}$. When we value assets with the pricing kernel, we put more weight on the bad state (state 1), because marginal utility is higher.

What happens to the risk-neutral probabilities? If we apply (6), the risk-neutral probability of state 1 is

$$p^*(z = \gamma_1) = \frac{(1 - \omega)\beta e^{-\alpha \gamma_1}}{(1 - \omega)\beta e^{-\alpha \gamma_1} + \omega \beta e^{-\alpha \gamma_2}} = \frac{(1 - \omega)}{(1 - \omega) + \omega e^{-\alpha(\gamma_2 - \gamma_1)}} > 1 - \omega.$$

The inequality follows because $\gamma_2 > \gamma_1$, which means the denominator is less than one. The risk neutral probabilities put more weight on the bad outcome, just as the pricing kernel does. We knew, of course, that the two approaches had to give us the same answer, but it's useful see it in action. It's also clear, here and in general, that the discount factor β drops out of the risk-neutral probabilities. It shows up, instead, in the bond price q^1 in the pricing relation (3).

Example (normal risks). Let $z = \log c_1 - \log c_0 \sim \mathcal{N}(\kappa_1, \kappa_2)$. The pdf is therefore

$$p(z) = (2\pi\kappa_2)^{-1/2} \exp[-(z - \kappa_1)^2/(2\kappa_2)]$$

and the pricing kernel is $m(z) = \beta \exp(-\alpha z)$. The one-period riskfree bond price is therefore $q^1 = E(m) = \beta \exp(-\alpha\kappa_1 + \alpha^2\kappa_2/2)$, as we saw earlier. That gives us the risk-neutral pdf

$$\begin{aligned} p^*(z) &= p(z)m(z)/q^1 \\ &= (2\pi\kappa_2)^{-1/2} \exp[-(z - \kappa_1)^2/(2\kappa_2)]\beta \exp(-\alpha z)/q^1 \\ &= (2\pi\kappa_2)^{-1/2} \exp[-(z - \kappa_1 + \alpha\kappa_2)^2/(2\kappa_2)]. \end{aligned}$$

That is, z's true distribution is $\mathcal{N}(\kappa_1, \kappa_2)$, but its risk-neutral distribution is $\mathcal{N}(\kappa_1 - \alpha\kappa_2, \kappa_2)$: we shift the distribution to the left (more pessimistic) to account for risk. How much depends on risk (κ_2) and risk aversion (α).

Note the effects of risk. In earlier sections, the effects of risk were captured by the pricing kernel m. Here the same effects are reflected in the difference between true and risk-neutral probabilities. In the lognormal case, the change is in the mean: risk-neutral probabilities build in risk aversion by reducing the mean. That's not true in general. In general, all of the cumulants of the distribution change. We'll see that shortly with numerical examples. In the next section, we derive an analytic expressions connecting their cumulants.

Let us say that we have an arbitrary continuous probability distribution over states z. We can approximate most such distributions with a grid over a finite number of states. If we make the grid fine enough, we can get as close to a continuous distribution as we like. Here's an example of such a grid in Matlab:

```
zmax = 4;
dz = 0.1;
z = [-zmax:dz:zmax]';
```

The next step is to define probabilities over the grid points. If we use a standard normal density, we can generate probabilities from

```
p = exp(-z.^2/2)*dz/sqrt(2*pi);
```

Here we'll do something a little different. Since we know how this works for the normal distribution (we shift the mean left, keep the variance the same), we'd like to consider alternatives. Gram-Charlier distributions are a useful class. The idea is (roughly) to approximate a distribution with nonzero skewness and excess kurtosis. If skewness and excess kurtosis are γ_1 and γ_2, then the density is

$$p(z) \quad = \quad p = (2\pi)^{-1/2} \exp(-z^2/2)[1 + \gamma_1(z^3 - 3z)/3! + \gamma_2(z^4 - 6z^2 + 3)/4!].$$

Our discrete approximation is therefore

```
p = exp(-z.^2/2).*(1 + gamma1*(z.^3-3*z)/6 + gamma2*(z.^4-6*z.^2+3)/24);
p = p*dz/sqrt(2*pi);
```

If $\gamma_1 = \gamma_2 = 0$ we get the standard normal. Otherwise, we get distributions with mean zero, variance one, and arbitrary amounts of skewness and excess kurtosis. None of this is essential to the idea, we just want a distribution with some degree of flexibility.

Given a distribution for the state, we then generate consumption growth, the pricing kernel, and risk-neutral probabilities, each of them as functions of the same state. Let us say that log consumption growth is

$$\log g(z) \quad = \quad \mu + \sigma z.$$

The pricing kernel is then

$$m(z) \quad = \quad \beta g(z)^{-\alpha}.$$

The price of a one-period riskfree bond is therefore

$$q^1 \quad = \quad E(m) \quad = \quad \sum_z p(z)m(z),$$

giving us the risk-neutral probabilities

$$p^*(z) \quad = \quad p(z)m(z)/q^1.$$

In Matlab, these steps are

```
logg = mug + sigmag*z;
g = exp(logg);
m = beta*g.^(-alpha);
q1 = p'*m
pstar = p.*m/q1;
```

So what do we get? This is easier to see than describe, but if you vary γ_1 and γ_2 you can generate a lot of different shapes for p. And you'll notice that p^* can have a much different shape. It really is true that the normal result (shift the mean) is special.

8   More fun with generating functions

We can also approach the difference between true and risk-neutral probabilities analytically — again, for the case of power utility. (The reason for power utility is that its form matches up nicely with the exponential in the moment generating function.) The idea comes from Ian Martin.

Suppose, as in our examples, that the state is $z = \log c_1 - \log c_0$. The cumulant generating function of z is

$$k(s) \quad = \quad \log E(e^{sz}).$$

In this notation, the log of the one-period bond price is

$$\log q^1 \quad = \quad \log E(\beta e^{-\alpha z}) \quad = \quad \log \beta + k(-\alpha).$$

The cgf of the risk-neutral distribution is

$$k^*(s) \quad = \quad \log[E(me^{sz})/q^1] \quad = \quad \log E\left(\beta e^{-\alpha z} e^{sz}\right) - \log q^1 \quad = \quad k(s - \alpha) - k(-\alpha). \quad (7)$$

This is a thing of beauty, a wonderfully compact summary of how true and risk-neutral distributions are connected.

Example. We'll redo the calculation of the previous section: the risk-neutral distribution of $z = \log c_1 - \log c_0$ when $z \sim \mathcal{N}(\kappa_1, \kappa_2)$. We know that $k(s) = s\kappa_1 + s^2\kappa_2/2$. Therefore

$$
\begin{aligned}
k^*(s) \quad &= \quad [(s - \alpha)\kappa_1 + (s - \alpha)^2 \kappa_2/2] - [-\alpha\kappa_1 + \alpha^2 \kappa_2/2] \\
&= \quad s(\kappa_1 - \alpha\kappa_2) + s^2 \kappa_2/2,
\end{aligned}
$$

the same answer we had before. Even better, you can have Matlab do the substitution.

For extra credit, show how cumulants are related. In the example, the only change is the mean. What happens in general?

Bottom line

The no-arbitrage theorem tells us that if arbitrage is ruled out, we can value assets with state prices. State prices, pricing kernels, and risk-neutral probabilities all represent the same idea.

More

The material is standard. The best textbook reference is Duffie, *Dynamic Asset Pricing Theory*, any edition. (Earlier editions are cheaper, and just as good for our purposes.) It's a PhD book, but well written and worth a look. See esp Chapters 1 and 2. Ross's classic paper is also a good read, esp Section 9.3. I scanned and posted it at this link.

Practice problems

1. *State prices.* Consider the following collections of asset prices q and dividends D:

$$(a) \ q \ = \ \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}, \ \ D \ = \ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(b) \ q \ = \ \begin{bmatrix} 1 \end{bmatrix}, \ \ D \ = \ \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

   Are they arbitrage free? What are the implied state prices?

   Answer.

   (a) If we solve for state prices, we get $Q(1) = Q(2) = 1/2$. The existence of positive state prices tells us it's arbitrage free.

   (b) There's only one asset, and it has a positive price, so it's arbitrage free. State prices aren't unique: any positive solution to $1 = Q(1) + 2Q(2)$ works. That means we need $0 < Q(2) < 1/2$ and $Q(1) = 1 - 2Q(2)$.

2. *Pricing kernel.* For the same problem, example (a), suppose the (true) probabilities are $p(1) = p(2) = 1/2$.

   (a) What is the pricing kernel?

   (b) What are the risk-neutral probabilities?

   Answer.

   (a) State prices are connected to the pricing kernel by $Q(z) = p(z)m(z)$. Here we have $m(1) = m(2) = 1$.

   (b) Risk-neutral probabilities are connected to the pricing kernel by $p^*(z) = p(z)m(z)/q^1$. Here $q^1 = E(m) = 1$ and $p^*(1) = p^*(2) = 1/2$.

3. *State prices and related objects.* Consider an economy with three states. State prices and probabilities are

State z	State Price $Q(z)$	Probability $p(z)$	Dividend $d(z)$
1	1/2	1/3	1
2	1/3	1/3	2
3	1/4	1/3	3

   (a) What is the pricing kernel in each state?

(b) What is the price of a one-period bond? What is its return?

(c) What are the risk-neutral probabilities? Why are they different from the true probabilities?

(d) Suppose equity is a claim to the dividend in the last column. What is its price? What is the return on equity in each state?

(e) What is the expected return on equity? The risk premium?

Answer.

State z	State Price $Q(z)$	Probability $p(z)$	Dividend $d(z)$	Pr Kernel $m(z)$	R-n probs $p^*(z)$	Return $r^e(z)$
1	1/2	1/3	1	3/2	0.4615	0.5217
2	1/3	1/3	2	1	0.3077	1.0435
3	1/4	1/3	3	3/4	0.2308	1.5652

(a) See the table.

(b) $q^1 = 1.0833$, $r^1 = 0.9231$.

(c) See table. They are a combination of the true probabilities p and the pricing kernel m, scaled to sum to one.

(d) The price is $q^e = 1.9167$. The returns are in the table.

(e) The expected return is $E(r^e) = 1.0435$ and the risk premium is $E(r^e - r^1) = 0.1204$. Why positive? You'll note that the asset's dividends are high when the pricing kernel is low, which makes the price low. That raises returns.

4. *Returns and risk premiums.* Consider the asset prices and dividends

$$\text{Asset 1:} \quad q^1 = 3/4, \ d^1(1) = 1, \ d^1(2) = 1$$
$$\text{Asset 2:} \quad q^2 = 1, \ d^2(1) = 1, \ d^2(2) = 2.$$

(a) What are the state prices? Is (q, D) arbitrage free?

(b) What are the returns on the two assets?

(c) If $p(1) = p(2) = 1/2$, what are the expected returns?

(d) What is the pricing kernel? Why does the second asset have a higher excess return?

(e) What are the risk-neutral probabilities? Why does the second asset have a higher excess return?

Answer.

(a) The state prices are $Q(1) = 1/2$ and $Q(2) = 1/4$. Since both are positive, the system is arbitrage free.

(b) Returns are $r^j(z) = d^j(z)/q^j$. That gives us returns

Asset	State 1	State 2
1	4/3	4/3
2	1	2

(c) Expected returns are $E(r^1) = 4/3$ and $E(r^2) = 3/2$. We would say that the second asset has a risk premium of $1/6$.

(d) The pricing kernel is connected to probabilities and state prices by $m(z) = Q(z)/p(z)$. That gives us $m(1) = 1$ and $m(2) = 1/2$. The second asset has a higher expected return because it pays off more in state 2, where m is lower.

(e) Risk-neutral probabilities are connected to the pricing kernel and probabilities by $p^*(z) = p(z)m(z)/q^1$. That gives us $p^*(1) = 2/3$ and $p^*(2) = 1/3$. The second asset has a higher expected return because it pays off more in state 2, where p^* is lower.

5. *Risk-neutral probabilities with exponential risk.* In our usual two-period setup, suppose the representative agent has power utility and $x(z) = \log c_1(z) - \log c_0$ has an exponential distribution:

$$p(x) = \lambda e^{-\lambda x},$$

for $x \geq 0$ and $\lambda > 0$. What is the pricing kernel? What is the risk-neutral distribution of x?

Answer. This is a little sloppy, but the following question nails down the details. The risk-neutral probability is the product of the true probability and the pricing kernel divided by q^1. The pricing kernel here is $m(x) = \beta e^{-\alpha x}$. Since q^1 is constant, the risk-neutral probabilities are

$$p^*(x) \propto p(x)m(x) = \lambda e^{-\lambda x} \beta e^{-\alpha x} = \text{constant} \times e^{-(\lambda+\alpha)x}.$$

The constant must be $\lambda + \alpha$ if the probabilities are to integrate to one. This tell us the risk-neutral distribution is also exponential, but with parameter $\lambda + \alpha$. The mean and standard deviation are both $1/(\lambda + \alpha)$, which decline with risk aversion α. So the risk-neutral distribution has a smaller mean and also a smaller standard deviation. The former is like the normal distribution, the latter is not.

6. *Exponential risk, cgf version.* Do the same problem using the cgf and equation (7).

Answer. You may show, or recall, that an exponential random variable with parameter $\lambda > 0$ has cgf

$$k(s) = -\log(1 - s/\lambda) = \log\left(\frac{\lambda}{\lambda - s}\right).$$

The cgf formula (7) gives us

$$k^*(s) = k(s - \alpha) - k(-\alpha) = \log\left(\frac{\lambda + \alpha}{\lambda + \alpha - s}\right),$$

which is the cgf for an exponential random variable with parameter $\lambda + \alpha$.

7. *Asset pricing with exponential risk.* We consider asset prices in the same setting: exponential risk, power utility, and so on. We assume $\lambda > 1$ throughout.

(a) What is the price of a bond paying a dividend of one in all states? What is its return?

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(b) What is the price of "equity," a claim to the aggregate growth rate e^x? What is its return? Its expected return?

(c) What is the risk premium on equity? How does it depend on α?

Answer.

(a) The price of the bond is

$$q^1 = \int_0^\infty m(x)p(x)dx = \int_0^\infty \beta\lambda e^{-(\lambda+\alpha)x}dx = -\left(\frac{\lambda\beta}{\lambda+\alpha}\right)e^{-(\lambda+\alpha)x}\bigg|_0^\infty$$
$$= \beta\lambda/(\lambda+\alpha).$$

Its return is $r^1 = 1/q^1 = (\lambda+\alpha)/(\beta\lambda)$.

(b) The price of "equity" is

$$q^e = \int_0^\infty \beta\lambda e^{(1-\lambda-\alpha)x}dx = \beta\lambda/(\lambda+\alpha-1).$$

The return is the dividend divided by the price: $r^e(x) = e^x/q^e$. The expected return is the expected dividend divided by the price. The expected dividend is

$$E(d^e) = E(e^x) = \int_0^\infty \lambda e^{(1-\lambda)x}dx = \lambda/(\lambda-1).$$

The expected return is $E(r^e) = [\lambda/(\lambda-1)][(\lambda+\alpha-1)/(\beta\lambda)]$.

(c) If $\alpha = 0$, both assets have expected return $1/\beta$. But if $\alpha > 0$, the return on equity is greater.

8. *Asset pricing fundamentals.* A well-known financial economist who spent years in the business world, commented on what he had learned as an academic: "I learned two basic lessons about financial mathematics that I've always found useful. One is that risk premiums come from covariances. The other is that asset prices come from risk-neutral probabilities."

(a) Write down an equation that illustrates his first lesson.

(b) Write down an equation that illustrates his second lesson.

(c) Where did the covariance go in part (b)?

Answer.

(a) The first reflects $E(x) = -\text{Cov}(x, m)/E(m)$, where x is any excess return.

(b) The second is reflected by $q = q^1 E^*(d) = q^1 \sum_z p^*(z)d(z)$, where q is the price of the dividend d and E^* is the expectation based on the risk-neutral probabilities.

(c) The covariance is embedded in the risk-neutral probabilities. Asset prices, for example, can be expressed two ways:

$$q = E(md) = q^1 E(d) + \text{Cov}(m, d)$$
$$q = q^1 E^*(d).$$

In the first, we have a covariance. In the second, that's built into E^*.

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Matlab code

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