Lab Report #7: Dynamics in Theory and Data

Revised: December 11, 2013

Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Please include your Matlab code.

1. *Linear models.* Consider the models

$$\begin{array}{lrcl} (a) & x_t & = & 0.9x_{t-1} + w_t \\ (b) & y_t & = & x_t + 2, \text{ with } x_t \text{ defined in (a)} \\ (c) & x_t & = & 0 \cdot w_t + 1 \cdot w_{t-1} \\ (d) & x_t & = & \varphi x_{t-1} + w_t + \theta w_{t-1} \\ (e) & x_t & = & 1.2x_{t-1} + 0.1x_{t-2} + w_t \end{array}$$

The disturbances w_t are independent standard normals.

For each model, answer the questions:

(i) Is the model Markov? For what definition of the state?

(ii) What is the conditional distribution of x_{t+1} given the state at date t?

(iii) Is the model stable?

(iv) If it's stable, what is the equilibrium distribution?

Solution: All of these are Markov. The state (z_t, say) is whatever you need to know at date $t-1$ to know the conditional distribution of x_t.

(a) This is an AR(1). (i) It's Markov with state x_{t-1}. (ii) Conditional distribution: normal with mean $0.9x_{t-1}$ and variance one. (iii) Yes, stable, because 0.9 is less than one in absolute value. (iv) The equilibrium distribution is normal with mean zero and variance $1/(1-0.9^2) = 5.2632$. The autocorrelation function is

$$\rho(k) \quad = \quad 0.9^k.$$

This includes $\rho(1) = 0.9$, $\rho(2) = 0.9^2 = 0.81$, and so on.

(b) Still an AR(1). (i) Doing the substitution $x_t = y_t - 2$ gives us

$$y_t \quad = \quad (1-0.9) \cdot 2 + 0.9y_{t-1} + w_t.$$

So it's Markov with state y_{t-1}. (ii) Conditional distribution: normal with mean $0.2 + 0.9y_{t-1}$ and variance one. (iii) Yes, stable, because 0.9 is less than one in absolute value. (iv) The equilibrium distribution is normal with mean two and variance $1/(1-0.9^2) = 5.2632$. All we've done here is shift the mean up by two. The autocorrelation function doesn't depend on the mean, so it's the same as before.

(c) This is an MA(1). (i) It's Markov with state w_{t-1}. (ii) Conditional distribution: normal with mean w_{t-1} and variance zero. (This is an unusual setup: since the coefficient of w_t is zero, we learn x_t one period ahead of time.) (iii) Yes, stable. For a moving average, all we need is that the coefficients are square summable. That's always true if there's a finite number of terms. (iv) The equilibrium distribution is normal with mean zero and variance one.

(d) This is an ARMA(1,1). (i) It's Markov with state (x_{t-1}, w_{t-1}). (ii) Conditional distribution: normal with mean $\varphi x_{t-1} + \theta w_{t-1}$ and variance one. (iii) It's stable if $|\varphi| < 1$. You can see this from the moving average representation, outlined in the notes:

$$x_t = w_t + (\varphi + \theta) w_{t-1} + (\varphi + \theta)\varphi w_{t-2} + (\varphi + \theta)\varphi^2 w_{t-3} + \cdots .$$

The first two moving average coefficients are arbitrary, then they decline at rate φ. (iv) The equilibrium distribution is normal with mean zero and variance equal to the sum of squared moving average coefficients:

$$\gamma(0) = 1 + (\varphi + \theta)^2/(1 - \varphi^2).$$

The autocovariances are

$$\gamma(k) = \varphi^{k-1}(\varphi + \theta)\left[1 + (\varphi + \theta)\varphi/(1 - \varphi^2)\right].$$

The autocorrelations are $\rho(k) = \gamma(k)/\gamma(0)$. They decline at rate φ after the first one.

(e) This is an AR(2). (i) It's Markov with state (x_{t-1}, x_{t-2}). (ii) The conditional distribution is normal with mean $\varphi x_{t-1} + \varphi x_{t-2}$ and variance one. (iii,iv) It's not stable. You can see this by substituting for a few periods and seeing how the impact of lagged x's works. So there's no equilibrium distribution, autocorrelation function, and so on.

2. *Combination models.* Consider the two-equation model

$$\begin{aligned} y_{t+1} &= x_t + \lambda w_{t+1} \\ x_t &= \varphi x_{t-1} + \sigma w_t, \end{aligned}$$

where w_t is our usual iid normal disturbance. The parameters are $\sigma > 0$, $|\varphi| < 1$, and λ (any sign).

(a) Express x as an infinite moving average; that is, an equation of the form

$$x_t = \theta_0 w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots .$$

What are the moving average coefficients θ_j?

(b) What is the acf of x?

(c) Now express y as an infinite moving average. What are its moving average coefficients?

(d) What is the acf of y? How does it depend on λ?

(e) What kind of ARMA model is y?

Solution:

(a) Repeated substitution gives us

$$x_t \;=\; \sigma(w_t + \varphi w_{t-1} + \varphi^2 w_{t-2} + \cdots).$$

The coefficients are evidently $\theta_j = \sigma\varphi^j$.

(b) The variance of x_t (the variance of its equilibrium distribution) is the sum of the squared MA coefficients:

$$\text{Var}(x_t) \;=\; \sigma^2/(1-\varphi^2).$$

The autocovariances are

$$\text{Cov}(x_t, x_{t-k}) \;=\; \sigma^2\varphi^k/(1-\varphi^2),$$

which makes the autocorrelation function

$$\rho(k) \;=\; \varphi^k$$

for integers $k \geq 0$.

(c) Here we have

$$y_{t+1} \;=\; \lambda w_{t+1} + \sigma(w_t + \varphi w_{t-1} + \varphi^2 w_{t-2} + \cdots).$$

(d,e) This is another ARMA(1,1), see the previous question.

3. *Dynamics of US interest rates.* We'll look at the autocorrelations of interest rates to get a sense of their dynamics. The first step is to download some data from the Fed. Go to

   http://www.federalreserve.gov/releases/h15/data.htm

   and download monthly data for Treasury constant maturities, specifically the 3-month and 10-year maturities, for the period 1985 to present. The second step is to download the function `acf.m` posted on the course website:

   http://pages.stern.nyu.edu/~dbackus/233/acf.m.

   It's my program, not part of Matlab. You need to save it in whatever Matlab thinks is the current working directory. The command `acf(x,n)` computes the first `n` terms of the sample acf of the vector of data `x`. To see how it works, look at

   http://pages.stern.nyu.edu/~dbackus/233/business_cycles.m.

Once you've done all this:

(a) Compute the mean, standard deviation, and autocorrelation function (acf) for the 3-month interest rate for lags k from 0 to 24 months.

(b) Graph the acf.

(c) Describe the acf for the AR(1):

$$x_t \;=\; \varphi x_{t-1} + \sigma w_t,$$

where $\{w_t\}$ is a sequence of standard normal random variables. How does the sample acf compare to the acf of a suitably chosen AR(1)?

(d) Compute the mean, standard deviation, autocorrelation function for the 10-year interest rate. How do they compare to those of the 3-month rate?

Solution:

(a,c,d) I didn't get a chance to update this. Last spring, the relevant statistics were

	Mean	Std Dev	Autocorr
3-month	4.00	2.52	0.989
10-year	5.79	2.17	0.978

They reflect some standard features of interest rates, including: (i) rates increase with maturity, on average; (ii) standard deviations decline with maturity; and (iii) all of them are very persistent. Which is the point of the exercise.

(b) With an AR(1), the autocorrelation has the form $\rho(k) = \varphi^k$: it declines geometrically. If we set $\varphi = \rho(1)$, we can compute estimated acf's corresponding to AR(1)s. They're plotted as the dashed lines above. You can see they're somewhat different from the AR(1). Whether this represents sampling variability of some more basic difference between the data and the AR(1) model isn't clear.