Lab Report #8: Bond Prices & Predictable Returns

Revised: December 10, 2015

Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Please include your Matlab code, if any.

Maturity n	Price q^n
1 year	0.9900
2 years	0.9800
3 years	0.9600
4 years	0.9300
5 years	0.8900

Table 1. Bond prices at various maturities. Here q^n is the price now of a claim to one dollar in n years.

1. *Equity prices and dividends.* Suppose the ex-dividend price of equity is

$$q_t \;=\; \delta E_t\left(d_{t+1} + q_{t+1}\right) \tag{1}$$

with discount factor $0 < \delta < 1$.

(a) Express the price as a function of expected future dividends.

(b) Suppose dividends follow

$$d_{t+1} \;=\; (1-\varphi)\mu + \varphi d_t + \sigma(w_{t+1} + \theta w_t),$$

where $\{w_t\}$ is a sequence of independent standard normal random variables. What definition of the state is enough to describe the conditional distribution of d_{t+1} at date t?

(c) How is the price q_t related to the state?

(d) Optional, extra credit. What are the variances of q and d? How do they relate to Shiller's observation that prices are more variable than dividends?

Solution:

(a) Repeated substitution gives us

$$q_t \;=\; \sum_{j=1}^{\infty} \delta^j E_t(d_{t+j}).$$

(b) The conditional distribution of d_{t+1} is normal with mean and variance

$$\begin{aligned} E_t(d_{t+1}) &= (1-\varphi)\mu + \varphi d_t + \sigma\theta w_t \\ \mathrm{Var}_t(d_{t+1}) &= \sigma^2. \end{aligned}$$

Evidently knowing $z_t = (d_t, w_t)$ is sufficient.

(c) We'll use the method of undetermined coefficients. If we guess $q_t = a + bd_t + cw_t$ for coefficients (a, b, c) to be determined, then the elements of (1) are

$$\begin{aligned} q_t &= a + bd_t + cw_t \\ E_t(q_{t+1}) &= a + b[(1-\varphi)\mu + \varphi d_t + \sigma\theta w_t] \\ E_t(d_{t+1}) &= (1-\varphi)\mu + \varphi d_t + \sigma\theta w_t. \end{aligned}$$

Substituting into (1) and lining up coefficients gives us

$$\begin{aligned} a &= \delta\big[a + (1+b)(1-\varphi)\mu\big] \\ b &= \delta\big(b\varphi + \varphi\big) \\ c &= \delta(b\sigma\theta + \sigma\theta). \end{aligned}$$

The second equation gives us $b = \delta\varphi/(1-\delta\varphi)$, and therefore $1+b = 1/(1-\delta\varphi)$. The first and third then give us

$$a = \frac{(1-\varphi)\mu\delta}{(1-\delta)(1-\delta\varphi)}, \qquad c = \frac{\sigma\theta\delta}{1-\delta\varphi}.$$

(d) d is ARMA(1,1), so its variance is

$$\mathrm{Var}(d) = \sigma^2 + \sigma^2(\varphi+\theta)^2/(1-\varphi^2).$$

This comes from the infinite moving average formula.

More coming...

2. *Bond basics.* Consider the bond prices in Table 1.

(a) What are the yields y^n?

(b) What are the forward rates f^{n-1}?

(c) How are the yields and forward rates related? Verify for y^3.

Solution: See the attached Matlab program; download the pdf, open, click on pushpin:

Maturity n	Price q^n	Forward f^{n-1}	Yield y^n
0	1.0000		
1 year	0.9900	0.0101	0.0101
2 years	0.9800	0.0102	0.0101
3 years	0.9600	0.0206	0.0136
4 years	0.9300	0.0317	0.0181
5 years	0.8900	0.0440	0.0233

(a,b) Yields and forward rates are connected to bond prices by

$$
\begin{aligned}
y_t^n &= -n^{-1} \log q_t^n \\
f_t^n &= \log(q_t^n / q_t^{n+1}).
\end{aligned}
$$

(c) Yields are averages of forward rates:

$$
y_t^n = n^{-1} \sum_{j=1}^{n} f_t^{j-1}.
$$

Thus $y^3 = (0.0101 + 0.0102 + 0.0206)/3 = 0.0136$.

3. *Bond prices with a moving average pricing kernel.* Consider the bond pricing model

$$
\log m_{t+1} = -\lambda^2/2 + \lambda w_{t+1} + \sigma w_t.
$$

(a) What kind of process is $\log m$?

(b) What is the short rate f_t^0?

(c) Suppose bond prices take the form

$$
\log q_t^n = A_n + B_n w_t.
$$

Use the pricing relation to derive recursions connecting (A_{n+1}, B_{n+1}) to (A_n, B_n). What are (A_n, B_n) for $n = 0, 1, 2, 3$?

(d) Express forward rates as functions of the state w_t. What are f_t^1 and f_t^2?

(e) What is $E(f^1 - f^0)$? What parameters govern its sign?

Solution:

(a) The log pricing kernel is MA(1).

(b) The price of a one-period bond is

$$
\log q_t^1 = \log E_t m_{t+1} = \lambda^2/2 + \sigma w_t - \lambda^2/2 = \sigma w_t.
$$

This is the usual "mean plus variance over two" formula. The short rate is therefore

$$y_t^1 \;=\; f_t^0 \;=\; -\log q_t^1 \;=\; -\sigma w_t.$$

(c) Bond prices follow from the pricing relation,

$$q_t^{n+1} \;=\; E_t(m_{t+1} q_{t+1}^n),$$

starting with $n = 0$ and $q_t^0 = 1$. With our loglinear guess, we have

$$\begin{aligned}
\log(m_{t+1} q_{t+1}^n) &= (-\lambda^2/2 + \lambda w_{t+1} + \sigma w_t) + (A_n + B_n w_{t+1}) \\
&= A_n - \lambda^2/2 + (\lambda + B_n) w_{t+1} + \sigma w_t.
\end{aligned}$$

That gives us the recursions

$$\begin{aligned}
A_{n+1} &= A_n - \lambda^2/2 + (\lambda + B_n)^2/2 \\
&= A_n + \lambda B_n + (B_n)^2/2 \\
B_{n+1} &= \sigma
\end{aligned}$$

for $n = 0, 1, 2, \ldots$. That gives us

n	A_n	B_n
0	0	0
1	0	σ
2	$\lambda\sigma + \sigma^2/2$	σ
3	$2(\lambda\sigma + \sigma^2/2)$	σ

(d) In general, forward rates are

$$f_t^n \;=\; (A_n - A_{n+1}) + (B_n - B_{n+1}) w_t.$$

That gives us

$$\begin{aligned}
f_t^0 &= -\sigma w_t \\
f_t^1 &= -(\lambda\sigma + \sigma^2/2) \\
f_t^2 &= -(\lambda\sigma + \sigma^2/2).
\end{aligned}$$

Evidently f^1 and f^2 are constant: the one-period memory of the MA(1) isn't enough to generate variation in long rates.

(e) The means are the same with $w_t = w_{t-1} = 0$, their mean. Therefore

$$E(f^1 - f^0) \;=\; -(\lambda\sigma + \sigma^2/2).$$

A necessary condition for this to be positive is that λ and σ have opposite signs.