Practice Problem: Perpetual Options

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Consider an American call option with no expiration date. The problem underlies the extensive literature on so-called *real options*, in which a firm considers, for example, the decision to extract oil from land it owns. They can do it any time, what time is best?

A general statement of the problem might be expressed this way. We have a Markov process for some state variable x_t. The "underlying" (the asset on which the option is written) pays dividend $d(x_t)$ at date t and satisfies the (ex-dividend) valuation relation

$$V(x_t) \quad = \quad E_t \left\{ m(x_t, x_{t+1}) \left[d(x_{t+1}) + V(x_{t+1}) \right] \right\}. \qquad (1)$$

The process for x_t gives us one for $V(x_t)$.

Now consider a perpetual American call option with strike k. Its value solves

$$J(x_t) \quad = \quad \max \left\{ E_t m(x_t, x_{t+1}) J(x_{t+1}), V(x_t) - k \right\}. \qquad (2)$$

This is a Bellman equation for the value function J. The first element on the right is what we get if we wait — the current value of the same option next period. The second is what we get if we exercise — the value now of the underlying minus the strike. Many such problems have a threshold property: we exercise the option when $V_t \geq V^*$ for some threshold V^*.

There's a nice example that we can solve by guess and verify. It has what we might call the Black-Scholes-Merton lognormal structure. Its ingredients include: (i) We skip the state x_t and deal directly with V_t. (ii) Risk-neutral pricing with $m = e^{-r}$ and lognormal risk-neutral distribution of V: $\log V_{t+1} - \log V_t \sim \mathcal{N}(\kappa_1, \kappa_2)$. (iii) Constant dividend rate δ: $d_{t+1} = \delta V_{t+1}$.

Your mission is to work through this example and find the value function J and threshold V^*.

(a) What is the Bellman equation for this example?

(b) Apply the valuation equation (1) to our assumptions about the distribution of V_{t+1}. What restriction on $(r, \delta, \kappa_1, \kappa_2)$ is implied?

(c) Guess that the value function has the form $J(V) = AV^a$ for parameters (A, a) to be determined. With this function, what is the value of the "wait" branch of the option?

(d) At the threshold V^*, we are indifferent between waiting and exercising. Use this to establish two conditions on (A, a, V^*).

(e) We need one more condition. Use the envelope condition, evaluated at $V_t = V^*$, to produce it.

(f) Now use all three conditions to find the unknowns (A, a, V^*).

(g) Verify that J is increasing and convex. Why does it make sense for the option value to be convex?

(h) Choose numbers for the inputs $(r, \delta, \kappa_1, \kappa_2, k)$ and graph the value function.