

Quiz #1

Fall 2013

Please write your name below, then complete the exam in the space provided. You may refer to one page of notes: standard paper, both sides, any content you wish. **There are FOUR questions.**

(Name and signature)

1. *Moments (30 points).* Consider a random variable x that takes on the values

$$x = \begin{cases} -\delta & \text{with probability } 1 - \omega \\ +\delta & \text{with probability } \omega \end{cases}$$

with $\delta > 0$.

- (a) Do the probabilities constitute a legitimate probability distribution? (5 points)
- (b) What is the moment generating function (mgf) for x ? (10 points)
- (c) Use the mgf to derive the mean and variance of x . (15 points)

Solution:

- (a) The probabilities must be nonnegative and sum to one. They sum to one by construction. They're nonnegative if $0 \leq \omega \leq 1$.

- (b) The mgf is

$$h(s) = E(e^{sx}) = (1 - \omega)e^{-\delta s} + \omega e^{\delta s}.$$

- (c) The first two derivatives of the mgf are

$$h'(s) = -\delta(1 - \omega)e^{-\delta s} + \delta\omega e^{\delta s}$$

$$h''(s) = \delta^2(1 - \omega)e^{-\delta s} + \delta^2\omega e^{\delta s}.$$

The mean is $k'(0) = \delta(2\omega - 1)$. The variance is $k''(0) - k'(0)^2 = 4\delta^2\omega(1 - \omega)$.

2. *Geometric risk (25 points).* Consider a power utility agent facing “geometric risk.” Utility is $u(c) = c^{1-\alpha}/(1 - \alpha)$ with risk aversion parameter $\alpha > 0$. Consumption is $c = e^x$ for $x = 0, 1, 2, \dots$ with probabilities $p(x) = (1 - \omega)\omega^x$ for some parameter $0 < \omega < 1$. We say that x has a *geometric distribution*.

- (a) Show that $p(x)$ is a legitimate probability distribution. (5 points)
- (b) What is the mgf of x ? (5 points)
- (c) Suppose $\omega e < 1$. What is the mean of c ? (5 points)
- (d) What is expected utility? (5 points)
- (e) What is the certainty equivalent? (5 points)

Solution:

- (a) Probabilities must be nonnegative and sum to one. Here they're positive, and the sum is

$$\sum_x p(x) = \sum_{x=0}^{\infty} (1-\omega)\omega^x = (1-\omega)/(1-\omega) = 1.$$

- (b) The mgf is

$$h(s) = E(e^{sx}) = \sum_{x=0}^{\infty} (1-\omega)\omega^x e^{sx} = (1-\omega)/(1-\omega e^s).$$

- (c) The mean of c is

$$E(c) = E(e^x) = h(1) = (1-\omega)/(1-\omega e).$$

- (d) Expected utility is

$$E(c^{1-\alpha})/(1-\alpha) = h(1-\alpha)/(1-\alpha) = (1-\omega)/[(1-\omega e^{1-\alpha})(1-\alpha)].$$

- (e) The certainty equivalent μ is the solution to

$$\mu^{1-\alpha}/(1-\alpha) = (1-\omega)/[(1-\omega e^{1-\alpha})(1-\alpha)],$$

namely

$$\mu = \left[\frac{1-\omega}{1-\omega e^{1-\alpha}} \right]^{1/(1-\alpha)}.$$

3. *General equilibrium (20 points).* Consider a two-period economy with dates $t = 0$ and $t = 1$. At $t = 1$, a state z occurs, an element of a finite set \mathcal{Z} . There is one good in each date and state. A single agent's preferences over these goods are described by the utility function

$$U = u(c_0) + \beta \sum_z p(z) u[c_1(z)].$$

There is also a production opportunity: if we use x units of the date-0 good as an input, we produce $f(x)$ units of the date-1 good in all states z .

- (a) What ingredients do you need to turn this into a complete model economy? (10 points)
- (b) What is the Pareto problem associated with this economy? (10 points)

Solution:

- (a) The standard list:

- List of commodities: 1 at $t = 0$, as many as states at $t = 1$.
- List of agents: one.
- Preferences and endowments: preferences above; need to specify endowments.
- Technology: $f(x)$.
- Resource constraints: If the endowments are y , we have

$$\begin{aligned} c_0 + x &\leq y_0 \\ c_1(z) &\leq y_1(z) + f(x) \text{ in each state } z. \end{aligned}$$

- (b) We want to maximize the utility of the agent subject to the resource constraints. You could simply write down the problem: utility function and resource constraints. Or you could report the Lagrangian,

$$\mathcal{L} = u(c_0) + \beta \sum_z p(z) u[c_1(z)] + q_0(y_0 - c_0 - x) + \sum_z q_1(z)[y_1(z) + f(x) - c_1(z)],$$

which contains the same information.

4. *Short answers (25 points).*

- (a) How would you compute excess kurtosis in a sample of data? (10 points)
- (b) Consider the asset prices and dividends

$$\begin{aligned} \text{Asset 1: } & q^1 = 3/4, \quad d^1(1) = 1, \quad d^1(2) = 1 \\ \text{Asset 2: } & q^2 = 1, \quad d^2(1) = 1, \quad d^2(2) = 2. \end{aligned}$$

What are the returns on the two assets? What are the implied prices of Arrow securities? (15 points)

Solution:

- (a) First, we need to compute the sample mean and variance:

$$\begin{aligned} \bar{x} &= T^{-1} \sum_t x_t \\ s^2 &= T^{-1} \sum_t (x_t - \bar{x})^2. \end{aligned}$$

Excess kurtosis is then

$$\gamma_2 = \frac{T^{-1} \sum_t (x_t - \bar{x})^4}{s^4} - 3.$$

- (b) The returns on the first asset are $r^1(1) = r^1(2) = 4/3$. The returns on the second are $r^2(1) = 1$ and $r^2(2) = 2$. The state prices are $Q(1) = 1/2$ and $Q(2) = 1/4$.