Lab Report #8: Bond Prices & Predictable Returns

Revised: December 12, 2013

Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Please include your Matlab code.

1. *Equity prices and dividends.* Suppose the ex-dividend price of equity is

$$q_t \;=\; \delta E_t\left(d_{t+1} + q_{t+1}\right) \tag{1}$$

with discount factor $0 < \delta < 1$.

(a) Express the price as a function of expected future dividends.

(b) Suppose dividends follow

$$d_{t+1} \;=\; (1-\varphi)\mu + \varphi d_t + \sigma(w_{t+1} + \theta w_t),$$

where $\{w_t\}$ is a sequence of independent standard normal random variables. What definition of the state is enough to describe the conditional distribution of d_{t+1} at date t?

(c) How is the price q_t related to the state?

(d) Optional, extra credit. What are the variances of q and d? How do they relate to Shiller's observation that prices are more variable than dividends?

Solution:

(a) Repeated substitution gives us

$$q_t \;=\; \sum_{j=1}^{\infty} \delta^j E(d_{t+j}).$$

(b) The date-t state for an ARMA(1,1) can be expressed by $z_t = (d_t, w_t)$. The conditional distribution of d_{t+1} is normal with mean and variance

$$\begin{aligned} E_t(d_{t+1}) &= (1-\varphi)\mu + \varphi d_t + \sigma\theta w_t \\ \mathrm{Var}_t(d_{t+1}) &= \sigma^2. \end{aligned}$$

(c) We'll use the method of undetermined coefficients. If we guess $q_t = a + b d_t + c w_t$ for coefficients (a, b, c) to be determined, then the elements of (1) are

$$\begin{aligned} q_t &= a + b d_t + c w_t \\ E_t(q_{t+1}) &= a + b[(1-\varphi)\mu + \varphi d_t + \sigma\theta w_t] \\ E_t(d_{t+1}) &= (1-\varphi)\mu + \varphi d_t + \sigma\theta w_t. \end{aligned}$$

Substituting into (1) and lining up coefficients gives us

$$\begin{aligned} a &= \delta[a + (1+b)(1-\varphi)\mu] \\ b &= \delta(b\varphi + \varphi) \\ c &= \delta(b\sigma\theta + \sigma\theta). \end{aligned}$$

The second equation gives us $b = \delta\varphi/(1-\delta\varphi)$, and therefore $1+b = 1/(1-\delta\varphi)$. The first and third then give us

$$a = \frac{(1-\varphi)\mu\delta}{(1-\delta)(1-\delta\varphi)}, \quad c = \frac{\sigma\theta\delta}{1-\delta\varphi}.$$

2. *Bond basics.* Consider the following bond prices at some date t:

Maturity n	Price q^n
0	1.0000
1 year	0.9704
2 years	0.9324
3 years	0.8914
4 years	0.8479
5 years	0.8065

(a) What are the yields y^n?

(b) What are the forward rates f^{n-1}?

(c) How are the yields and forward rates related? Verify for y^3.

Solution:

Maturity n	Price q^n	Yield y^n	Forward f^{n-1}
0	1.0000		
1 year	0.9704	0.0300	0.0300
2 years	0.9324	0.0350	0.0400
3 years	0.8914	0.0383	0.0450
4 years	0.8479	0.0413	0.0500
5 years	0.8065	0.0430	0.0500

See the attached Matlab program; download the pdf, open, click on pushpin:

(a) See above.

(b) See above.

(c) Yields are averages of forward rates:

$$y_t^n \quad = \quad n^{-1} \sum_{j=1}^{n} f_t^{j-1}.$$

Thus $y^3 = (0.0300 + 0.0400 + 0.0450)/3 = 0.0383$.

3. *Moving average bond pricing.* Consider the bond pricing model

$$\begin{aligned} \log m_{t+1} &= -\lambda^2/2 - x_t + \lambda w_{t+1} \\ x_t &= \delta + \sigma(w_t + \theta w_{t-1}). \end{aligned}$$

(a) What is the short rate f_t^0?

(b) Suppose bond prices take the form

$$\log q_t^n \quad = \quad A_n + B_n w_t + C_n w_{t-1}.$$

Use the pricing relation to derive recursions connecting $(A_{n+1}, B_{n+1}, C_{n+1})$ to (A_n, B_n, C_n). What are (A_n, B_n, C_n) for $n = 0, 1, 2, 3$?

(c) Express forward rates as functions of the state (w_t, w_{t-1}). What are f_t^1 and f_t^2?

(d) What is $E(f^1 - f^0)$? What parameters govern its sign?

Solution:

(a) The short rate is

$$f_t^0 \quad = \quad -\log E_t m_{t+1} \quad = \quad \lambda^2/2 + x_t - \lambda^2/2 \quad = \quad x_t.$$

The second equality is the usual "mean plus variance over two" with the sign flipped (as indicated by the first equality). In other words: the usual setup. In what follows, we'll kill off x_t by substituting.

(b) Bond prices follow from the pricing relation,

$$q_t^{n+1} \quad = \quad E_t(m_{t+1} q_{t+1}^n),$$

starting with $n = 0$ and $q_t^0 = 1$. The state in this case is (w_t, w_{t-1}), a simple example of a two-dimensional model, hence the extra term in the form of the bond price. We need

$$\log(m_{t+1} q_{t+1}^n) \quad = \quad A_n - (\lambda^2/2 + \delta) + (\lambda + B_n) w_{t+1} + (C_n - \sigma) w_t - \sigma\theta w_{t-1}.$$

The (conditional) mean and variance are

$$\begin{aligned} E_t[\log(m_{t+1} q_{t+1}^n)] &= A_n - (\lambda^2/2 + \delta) + (C_n - \sigma) w_t - \sigma\theta w_{t-1} \\ \mathrm{Var}_t[\log(m_{t+1} q_{t+1}^n)] &= (\lambda + B_n)^2. \end{aligned}$$

Using "mean plus variance over two" and lining up terms gives us

$$
\begin{aligned}
A_{n+1} &= A_n - (\lambda^2/2 + \delta) + (\lambda + B_n)^2/2 \\
&= A_n - \delta + \lambda B_n + (B_n)^2/2 \\
B_{n+1} &= C_n - \sigma \\
C_{n+1} &= -\sigma\theta
\end{aligned}
$$

for $n = 0, 1, 2, \ldots$. That gives us

n	A_n	B_n	C_n
0	0	0	0
1	$-\delta$	$-\sigma$	$-\sigma\theta$
2	$-2\delta - \lambda\sigma + \sigma^2/2$	$-\sigma(1+\theta)$	$-\sigma\theta$
3	X	$-\sigma(1+\theta)$	$-\sigma\theta$

with $X = -3\delta - \lambda(2+\theta) + [1 + (1+\theta)^2]\sigma^2/2$.

(c) In general, forward rates are

$$
f_t^n = (A_n - A_{n+1}) + (B_n - B_{n+1})w_t + (C_n - C_{n+1})w_{t-1}.
$$

That gives us

$$
\begin{aligned}
f_t^0 &= \delta + \sigma w_t + \sigma\theta w_{t-1} \\
f_t^1 &= \delta + \lambda\sigma - \sigma^2/2 + \sigma\theta w_t \\
f_t^2 &= \delta - (1+\theta)^2\sigma^2/2 + \lambda\sigma(1+\theta).
\end{aligned}
$$

(d) The means are the same with $w_t = w_{t-1} = 0$, their mean. Therefore

$$
E(f^1 - f^0) = \lambda\sigma - \sigma^2/2.
$$

Therefore we need $\lambda\sigma > \sigma^2/2$, so a necessary condition is that λ and σ have the same sign.