Option Pricing: Black-Scholes-Merton & Beyond

Revised: October 14, 2014

Options are one of the most common financial derivatives, and a nice application of asset pricing fundamentals. We derive option prices from risk-neutral probabilities and show how the prices of options depend on our choice of probability distribution. There are lots of moving parts, but that's the idea.

We have two specific goals. The first is to derive the Black-Scholes-Merton ("BSM") formula from a model in which the price of the *underlying* — the asset the option is based on — has a lognormal risk-neutral distribution. This is expressed most simply in terms of a flat *volatility smile*, something we'll define shortly. The second is to show how other distributions can generate other shapes for the smile, including shapes similar to those we observe for real-world options. The focus is on options on S&P 500 index futures, a popular and convenient contract. The same methods can be — and have been — applied to other option contracts, although the parameter values will be different.

A couple warnings about notation:

- Timing. We'll continue to use the $(t, t+1)$ timing, with today being t and next period (next year) $t+1$. In some cases we'll generalize this to $(t, t+\tau)$, where τ is the length of the time interval, so that we can value options of different maturities.
- Letters. We're going to repurpose some letters, including s and k, to stick closer to standard usage with options. Here s and k will have nothing to do with the cumulant generating function here, except in a couple instances I was unable to avoid. We use the letter d in two ways, as the dividend or cash flow and as a component of the BSM formula.

1 Review: risk-neutral asset pricing

Recall that one version of the no-arbitrage theorem connects an asset's price q_t to its cash flows d_{t+1} by

$$q_t \quad = \quad q_t^1 E^* \left(d_{t+1} \right), \tag{1}$$

where q_t^1 is the price of a one-period riskfree bond and E^* means the expectation using the risk-neutral distribution of d_{t+1}.

We have then the following steps to price an asset:

- Identify the cash flows d_{t+1}.
- Specify the risk-neutral distribution.
- Use equation (1) to price the asset.

This is the basic idea for any asset, whether an option or something else.

2 Option cash flows

An asset is defined by its cash flow or dividend, as we've called it. Here we buy an asset at t and realize its (generally uncertain) cash flow at $t + \tau$.

An option is a particular kind of cash flow. An option gives its owner the right to buy or sell another asset at a preset price within a given time period. For example, movie producers might have a 5-year option to produce a movie based on a book. Or a real estate developer might have a 6-month option to purchase a piece of property at a preset price. After that time, the option expires and the seller of the option can do what she wants with the asset. In financial markets we see options on lots of things: stocks, bonds, foreign currencies, pork belly futures, and so on. They're the classic "financial derivative" and show up all over the place. Options are useful to buyers because they give them flexibility. In return for this flexibility, the seller collects a fee — the price of the option.

A typical option is defined by these features:

- The underlying: the asset on which the option is based. In our case this is the S&P 500 futures contract.
- The strike price: the price at which you can buy or sell the underlying.
- The term or maturity: the period of time over which the option can be exercised.
- Call or put: whether it's an option to buy (a call) or sell (a put).
- Type of option: "European," "American," or something else ("Asian"? "Greek"?). Most traded options can be exercised any time up to maturity (American), but it's mathematically simpler to work with options that can be exercised only at maturity (European). That's what we'll do.

That's the basics.

With a small investment in notation, we can express an option's cash flows mathematically. Let us say we have a τ-period option. If the price of some underlying asset at time t is s_t then a τ-period call option at strike price k generates a cash flow at time $t + \tau$ of $d_{t+\tau} = (s_{t+\tau} - k)^+$, where $x^+ \equiv \max(x, 0)$. Why? Because you would only exercise the option if the price is above k. Otherwise you let it expire. Similarly, a τ-period put option at strike price k generates the cash flow $(k - s_{t+\tau})^+$.

If you graph the cash flow against the future price, you'll see that it's convex for both puts and calls. As a result, the value increases with the amount of risk in the underlying. The more risk there is, the more valuable they are. That's a loose statement, but one we could make tighter using logic from the chapter on risk. More simply: put and call options give you the upside without the downside.

3 Put-call parity

There's a useful connection between European put and call prices that tells us, in effect, that if you know the price of one you can easily compute the price of the other. With a call option, you buy the asset if its price is above k, and with a put you sell it if the price is below k. So buying a call and selling a put at the same strike leads you to own the stock in all cases. The price is k, paid at $t + \tau$, so we discount it using the price q_t^τ (the price at t of one dollar paid at $t + \tau$).

This gives us two ways to buy the stock at date t, directly and with options. Absence of arbitrage tells us both methods should cost the same:

$$\underbrace{q_t^c}_{\text{buy call}} - \underbrace{q_t^p}_{\text{sell put}} + \underbrace{q_t^\tau k}_{\text{present value of strike}} = \underbrace{s_t}_{\text{buy stock}} . \qquad (2)$$

This is known as *put-call parity*. A fine point: this is for an asset that doesn't pay a dividend between t and $t + \tau$. If it does, we need to work that cash flow into the equation.

4 The normal distribution function

Recall that if x is normal with mean zero and variance one ("standard normal"), its probability density function (pdf) is

$$p(x) = (2\pi)^{-1/2} \exp(-x^2/2) \qquad (3)$$

for any real number x. It's positive and integrates to one. It's also symmetric: $p(x) = p(-x)$.

We'll see shortly that the BSM formula involves the cumulative distribution function (cdf) N, defined by

$$\text{Prob}(x \leq x^*) = \int_{-\infty}^{x^*} p(x)dx = N(x^*).$$

There's no simple antiderivative of p, but it comes up enough that we give it the label N. It's also a common function in software packages. The function N corresponds to `normcdf` in Matlab and `NORMSDIST` in spreadsheet programs.

We can do the same with other normal random variables. The random variable $y = \mu + \sigma x$ is normal with mean μ and variance σ^2. Since $x = (y - \mu)/\sigma$ is standard normal, we can compute cumulative probabilities for y using

$$\text{Prob}(x \leq x^*) = \text{Prob}[x \leq (y^* - \mu)/\sigma] = N[(y^* - \mu)/\sigma] .$$

[We're using $\sigma > 0$ here, which we take for granted below.] Matlab does this calculation for us with the command `normcdf(ystar,mu,sigma)`.

One final property: since p is symmetric, $N(-x) = 1 - N(x)$. Why? Note that

$$\int_{-\infty}^{x^*} p(x)dx + \int_{x^*}^{\infty} p(x)dx = 1.$$

This implies

$$1 - N(x^*) \;=\; \int_{x^*}^{\infty} p(x)dx \;=\; \int_{-\infty}^{-x^*} p(-x)dx \;=\; N(-x^*)$$

by symmetry. It's easy to show in a picture: graph the pdf and its tail integrals.

5 The BSM formula and the volatility smile

We refer to this as a *formula* because that's what it is: a function that relates option prices to the strike price and a few other things. We'll derive it from a model shortly, but for now it's just a formula. We report it for an arbitrary option maturity τ, but in most of what we do we'll set $\tau = 1$. In the general case, q_t^τ is the price of a riskfree bond of maturity τ.

The BSM formula for the price q_t^c of a call option is

$$q_t^c \;=\; s_t N(d) - q_t^\tau k N(d - \tau^{1/2}\sigma), \tag{4}$$

where

$$d \;=\; \frac{\log(s_t/q_t^\tau k) + \tau\sigma^2/2}{\tau^{1/2}\sigma}.$$

With put-call parity, we can show that put prices are

$$q_t^p \;=\; q_t^\tau k N(-d + \tau^{1/2}\sigma) - s_t N(-d). \tag{5}$$

[You might try this. It uses the symmetry property of N.]

Let's review the ingredients:

- s_t is the current price of the underlying.
- τ is the maturity of the option.
- q_t^τ is the price at t of "one" at date $t + \tau$ (the bond price, in other words). This is commonly written in terms of the implied (continuously compounded) interest rate r: $q_t^\tau = \exp(-r\tau)$. We won't do that, because r means something else to us, but you might imagine a variant like $q_t^\tau = \exp(-y_t^\tau \tau)$ where y_t^τ is the continuously compounded yield on a bond with maturity τ. Not something we need, but it has one useful feature: the suggestion that we need not have a constant interest rate.
- k is the strike price.
- σ is a parameter we refer to as "volatility."

All but the last of these ingredients are observed. But if we know the others, and observe the option price, we back out σ. The output of this calculation is commonly referred to as *implied volatility* or simply volatility.

A plot of implied volatility against the strike price is commonly referred to as a *volatility smile*. The term smile refers to the shape, but for equity index options it's less a smile than a downward-sloping line, possible with some convexity to it. We'll treat the shape of the volatility smile as a feature of the data we'd like to explain.

A common variant is to graph implied volatility against some measure of "moneyness." Options are said to be "at the money" if $k = s_t$. We say a put is "in the money" if $k > s_t$ and "out of the money" if $k < s_t$. And the reverse for calls. So-called "moneyness" is often measured by $k - s_t$ or $(k - s_t)/s_t$, so that zero represents an at-the-money option.

6 Derivatives of the BSM formula

Most treatments go on to list derivatives of the BSM formula with respect to its inputs. You can look these up in Wikipedia. We'll use one: the derivative with respect to volatility σ. The so-called "vega" is

$$\partial q_t^p / \partial \sigma \quad = \quad s_t N'(d) \tau^{1/2},$$

where N' is the standard normal pdf, equation (3). We'll use this as an input to Newton's method when we compute implied volatilities.

7 The S&P 500 E-mini futures contract

We're interested in options on a broad-based equity index like the S&P 500, which we've seen is closed related to the state of the economy. (I have in mind here the scatterplot of equity returns against consumption growth.) As it happens, it's common to use options on the futures contract on the index, rather than the index itself. This is more information than we need, but if you're interested, this is the contract, as described in Wikipedia (lightly edited):

> The E-mini S&P futures contract, often called simply the "E-mini," is a stock market index futures contract traded on the Chicago Mercantile Exchange's Globex electronic trading platform. The notional value of one contract is US$50 times the value of the S&P 500 stock index. It was introduced by the CME on September 9th, 1997, after the value of the existing S&P contract became too large for many small traders. The E-mini quickly became the most popular equity index futures contract in the world.

Options on these contracts are among the most liquid we have.

8 Risk-neutral option pricing and the no-arbitrage condition

We reminded ourselves in Section 1 that if we know the risk-neutral probabilities, then we can price options. Commonly we guess a functional form for the probabilities and infer parameters from observed prices. When we do that, we need to make sure the probabilities satisfy equation (1) for the underlying. We refer to this as the *no-arbitrage condition*, because (1) reflects the no-arbitrage theorem.

The cash flow of the underlying is its price next period, so the no-arbitrage condition becomes

$$s_t \;=\; q_t^1 E^*(s_{t+1}). \tag{6}$$

[We could add a dividend, but this makes things more complicated and we end up in a similar place anyway.] Without this condition, the chosen probabilities violate the no-arbitrage theorem for this asset.

This is clearer if we look at an example. Suppose $x_{t+1} = \log s_{t+1} \sim \mathcal{N}(\kappa_1, \kappa_2)$. Then (6) implies

$$s_t \;=\; q_t^1 e^{\kappa_1 + \kappa_2/2}. \tag{7}$$

Typically we would choose κ_2 to match option prices (more on this shortly) and set $\kappa_1 = \log s_t - \log q_t^1 - \kappa_2/2$. We'll see this again, so don't panic if it hasn't sunk in yet. More generally, the no-arbitrage condition can be written

$$s_t \;=\; q_t^1 h(1),$$

where h is the moment generating function of $\log s_{t+1}$.

Once we have the risk-neutral distribution, we price options as we would any asset: we characterize the cash flows and apply the pricing relation (1). Prices of one-period put and call options are therefore

$$\begin{aligned} q_t^p &= q_t^1 E^*(k - s_{t+1})^+ \\ q_t^c &= q_t^1 E^*(s_{t+1} - k)^+. \end{aligned}$$

The expectation E^* is either a sum (if the distribution is discrete) or (more commonly) an integral (if the distribution is continuous). The problem now becomes a computational one.

There are two common shortcuts. One is to approximate a continuous distribution with a discrete grid. That's a brute-force approach, but works reasonably well. The other is to use distributions that we can solve by hand — distributions that lead to convenient formulas for option prices. Either would work. We'll do mostly the second because we can handle the challenge, and because it highlights the role of the distribution in the BSM model and alternatives to it.

9 The Black-Scholes-Merton model

The BSM formula follows directly from using a lognormal risk-neutral distribution for the underlying. Everything else is just calculus. We'll revert to a maturity of one, but consider alternatives in the next section.

We'll derive the put formula (5) using as an input a lognormal distribution of the underlying: $x_{t+1} = \log s_{t+1} \sim \mathcal{N}(\kappa_1, \kappa_2)$. This is, to be clear, the risk-neutral distribution. The no-arbitrage condition is therefore equation (7). We'll need it later.

The rest follows from evaluating moderately tedious integrals. There's no magic here, just first-year calculus. The price of a put option with strike k is

$$
\begin{aligned}
q_t^p &= q_t^1 E^*(k - s_{t+1})^+ \\
&= q_t^1 \int_{-\infty}^{\log k} (2\pi\kappa_2)^{-1/2} \exp[-(x_{t+1} - \kappa_1)^2/2\kappa_2]\,(k - e^{x_{t+1}})\,dx_{t+1} \\
&= q_t^1 k \int_{-\infty}^{\log k} (2\pi\kappa_2)^{-1/2} \exp[-(x_{t+1} - \kappa_1)^2/2\kappa_2] dx_{t+1} \quad \text{(term 1)} \\
&\quad - q_t^1 \int_{-\infty}^{\log k} (2\pi\kappa_2)^{-1/2} e^{x_{t+1}} \exp[-(x_{t+1} - \kappa_1)^2/2\kappa_2] dx_{t+1}. \quad \text{(term 2)}
\end{aligned}
$$

Term 1 is just the normal cdf evaluated at $x_{t+1} = \log s_{t+1} = \log k$:

$$
\text{term 1} \;=\; q_t^1 k N[(\log k - \kappa_1)/\kappa_2^{1/2}].
$$

This is the discounted value of the strike price times the risk-neutral probability the option is exercised.

Term 2 requires us to combine two exponential terms, something we've done a couple times before, starting with the normal moment generating function. (You might want to go back and remind yourself how that worked.) The key step is showing

$$
e^{x_{t+1}} \exp[-(x_{t+1} - \kappa_1)^2/2\kappa_2] \;=\; e^{\kappa_1 + \kappa_2/2} \exp\{-[x_{t+1} - (\kappa_1 + \kappa_2)]^2/2\kappa_2\}.
$$

You can verify this by expanding the exponents on both sides. That gives us

$$
\text{term 2} \;=\; q_t^1 e^{\kappa_1 + \kappa_2/2} N\{[\log k - (\kappa_1 + \kappa_2)]/\kappa_2^{1/2}\}.
$$

The put price is therefore

$$
q_t^p \;=\; q_t^1 k N(f) - q_t^1 e^{\kappa_1 + \kappa_2/2} N(f - \kappa_2^{1/2}), \tag{8}
$$

with

$$
f \;=\; (\log k - \kappa_1)/\kappa_2^{1/2}.
$$

This is a useful formula, and we'll come back to it later. But it's not the BSM formula — yet.

All that's left to derive the BSM formula (5) is to apply the no-arbitrage condition (7) to (8). That gives us

$$
q_t^p \;=\; q_t^1 k N(f) - s_t N(f - \kappa_2^{1/2}), \tag{9}
$$

with

$$
f \;=\; [\log(q_t^1 k/s_t) + \kappa_2/2]/\kappa_2^{1/2}.
$$

If we substitute $\kappa_2 = \sigma^2$ and $\tau = 1$ (more on that next), we get exactly the BSM put formula, equation (5).

From a practical perspective, we might ask how well the BSM model approximates observed option prices. The answer is that it misses the volatility smile. In the model, volatility σ is a parameter, the standard deviation of $\log s_{t+1}$, and it's the same for all strike prices. In the data, we typically find that the value of volatility that reproduces option prices — what we've called implied volatility — varies with the strike. With equity index options, implied volatility declines with the strike. This discrepancy between model and data lead us to search for a better model, one that can reproduce the volatility smile. That's the ultimate goal of this chapter, but we'll work up to it gradually.

10 Summing and dividing random variables

The goal here is to adapt the BSM and related formulas to time intervals different from a year. That has some practical importance, because most traded options have maturities significantly less than a year.

We'll attack this indirectly, starting with sums of independent random variables. As we've seen, the cumulant generating functions of these sums are sums of cgfs of the components. If $y = x_1 + x_2$, for example, then

$$k(s; y) \quad = \quad \log h(s; y) \quad = \quad \log h(s; x_1) + \log h(s; x_2) \quad = \quad k(s; x_1) + k(s; x_2).$$

If x_1 and x_2 have the same distribution, this becomes $k(s; y) = 2k(s; x)$. Similarly the cgf for $y = x_1 + x_2 + \cdots + x_n$, with n independent identically distributed components x_j, is $k(s; y) = nk(s; x)$.

We can always do this, but it doesn't always have a nice form. Two examples that do are the normal and Poisson. If $x_j \sim \mathcal{N}(\kappa_1, \kappa_2)$, then $k(s; x_j) = \kappa_1 s + \kappa_2 s^2/2$. The cgf of the sum of n such random variables is the same thing multiplied by n:

$$k(s; y) \quad = \quad n(\kappa_1 s + \kappa_2 s^2/2) \quad = \quad (n\kappa_1)s + (n\kappa_2)s^2/2.$$

That is, it's still normal, but the mean κ_1 and variance κ_2 are multiplied by n. Poisson random variables also scale nicely. Recall that if x is Poisson, it assumes the values $x = 0, 1, 2, \ldots$ with probabilities $p(x) = e^{-\omega}\omega^x/x!$ for some parameter $\omega > 0$ ("intensity"). Its cgf is (this is from the notes on random variables)

$$k(s; x) \quad = \quad \omega\,(e^s - 1)\,.$$

Now consider the sum y of n such random variables. Its cgf is

$$k(s; y) \quad = \quad n\omega\,(e^s - 1)\,.$$

That is, we just multiply the intensity parameter ω by n.

The obvious application of sums for us is time. If we have a random variable y that applies to (say) a month, and let the distribution be the same at all dates, then we can think of the random variable y for a year as the sum of 12 months x_j. If we're lucky, we can also go the other way. If we have the distribution for a year, we can divide it into months like this:

$$k(s; y) \;\; = \;\; \tau k(s; x),$$

where y is the random variable for a month, x is the random variable for a year, and $\tau = 1/12$ is the fraction of a year in one month. Random variables for which this is possible are said to be *divisible*.

It's not hard to see how this would work with normal and Poisson random variables. In the normal case, the mean κ_1 and variance κ_2 are proportional to τ. That's the one missing piece in our derivation of the BSM formula. We can now replace κ_2 with $\tau\sigma^2$ in equation (9) and get the BSM formula for any maturity. Generally we scale the interest rate, too, replacing $\log q_t^1$ with $\log q_t^\tau = \tau y_t^\tau$: we discount shorter intervals less than longer ones.

This doesn't work with all random variables. For example, the Bernoulli has cgf

$$k(s; y) \;\; = \;\; \log\left(1 - \omega + \omega e^s\right).$$

It's not divisible. That's one of the reasons the Poisson distribution is preferred in applications to finance: it scales more nicely to different time intervals.

11 The Merton model

Let's return to our problem: accounting for the shape of the volatility smile in equity index options. A lognormal risk-neutral distribution gives is a flat smile, so that doesn't work — evidently we need something else. But what? There's no shortage of possibilities, but we'll focus on one that's widely used in finance: a combination of a normal random variable and a Poisson mixture of normals. Robert Merton has the classic paper on the subject, so it's known as the Merton model.

Let us say that the log of the price of the underlying is the sum of two independent components:

$$\log s_{t+1} \;\; = \;\; x_1 + x_2.$$

The first component is normal:

$$x_1 \;\; \sim \;\; \mathcal{N}(\mu, \sigma^2).$$

If we stop there, we're back to BSM. The second component, sometimes called the "jump component," is a Poisson mixture of normals. If j is a Poisson random variable, then conditional on j

$$x_2 | j \;\; \sim \;\; \mathcal{N}(j\theta, j\delta^2).$$

The idea behind this is that during the relevant time interval, we may get 0, 1, 2, or more jumps, each of which is normal with mean θ and variance δ^2. If there are j jumps, then x_2 is a normal random variable with $j\theta$ and variance $j\delta^2$. When we compute the pdf for x_2, we take each of these normal pdfs and compute the sum weighted by the Poisson probabilities $e^{-\omega}\omega^j/j!$.

The beauty of this approach is that we have, essentially, a weighted average of BSM formulas. The density function is a weighted average of normal pdf's, one for each j. We know how to compute the put price for normals, namely equation (8) with the appropriate choices of κ_1 and κ_2. Then the price is the weighted average of the put prices for each j. This takes patience, but hang in there. For each j, we might write (8) as

$$\begin{aligned} q_t^p(j) &= q_t^1 k N(f_j) - q_t^1 e^{\kappa_{1j}+\kappa_{2j}/2} N(f_j - \kappa_{2j}^{1/2}) \\ f_j &= (\log k - \kappa_{1j})/\kappa_{2j}^{1/2}, \end{aligned}$$

where $\kappa_{1j} = \mu + j\theta$ and $\kappa_{2j} = \sigma^2 + j\delta^2$. The put price is then

$$q_t^p = \sum_{j=0}^{\infty} p(j) q_t^p(j) = \sum_{j=0}^{\infty} e^{-\omega}(\omega^j/j!) q_t^p(j).$$

In practice, you usually need only a few terms in j because $p(j)$ approaches zero rapidly.

There are two remaining issues. The first one is the no-arbitrage condition. In this setting, it takes the form

$$\log s_t - \log q_t^1 = (\mu + \sigma^2/2) + \omega\left(e^{\theta+\delta^2/2} - 1\right).$$

You can see the normal and Poisson inputs here. Given values of the other parameters, our operating procedure is to choose μ to satisfy this equation. The second issue is the time interval. If we change from a time interval of one to τ, we simply multiply (μ, σ^2, ω) by τ. That's the beauty of the normal and Poisson distributions, they adapt easily to different time intervals.

If we add the two components together, we have a distribution of $\log s_{t+1}$ that is normal for every j:

$$\log s_{t+1}|j \sim \mathcal{N}(\mu + j\theta, \sigma^2 + j\delta^2).$$

Therefore it can be computed using the BSM-like formula (8). The put price is the weighted average of these, with weights on each j of $p(j) = e^{-\omega}\omega^j/j!$.

This model has enough flexibility to reproduce many of the volatility smiles we see in actual markets. It's a common starting point for serious work on option pricing. [Talk about role of δ (generates curvature) and ω (generates slope).] A lot of this can be illustrated more simply with a Bernoulli mixture (one jump), but that model doesn't divide well into smaller time intervals.

Bottom line

The BSM formula is the result of a model in which the risk-neutral distribution of the underlying is lognormal. It's most useful as a formula that we use to compute implied volatilities. The resulting volatility smiles show clearly that the distribution isn't lognormal. That's a little weird, because we're using a formula that comes from a model that doesn't work very well, but that's what we do. That leaves us looking for alternative models, including the Merton model.

This is close to the state of the art in option pricing. The one thing most applications add is stochastic volatility: something like variation over time in σ in the model of Section 11. That generates movements up and down of volatility smiles, which we see all the time in these markets. The VIX, for example, is (roughly) the value of at-the-money volatility. If you're interested, let me know and I'll point you to the standard references. If you're really really interested, you might start by taking courses in stochastic processes and stochastic calculus.

Practice problems

1. *Put and calls.* Use put-call parity (2) to derive the BSM price of a call option (4) from the price of a put (5).

   Answer. This one's all yours.

2. *The BSM formula.* Consider the BSM call option formula, equation (4) with $s_t = 100$, $q_t^1 = 0.98$, and $\tau = 1$.

   (a) What is the price of a call option at strike $k = 95$ if $\sigma = 0.20$?

   (b) What is the price of a call option at strike $k = 105$? Why is it higher or lower than your answer to (a)?

   (c) What is the price of a call option at strike $k = 95$ if $\sigma = 0.25$? Why is it higher or lower than your previous answer to (a)?

   (d) Suppose you observe a call price of 12.00. Is implied volatility above or below 0.20? Why? What is implied volatility?

   Answer.

   (a) 11.625.

   (b) 6.713. The payoff is strictly less at all outcomes s_{t+1}.

   (c) 13.442. It's higher because options have convex payoffs: their value increases the more uncertainty there is.

   (d) We know the price is increasing in volatility. So if the price is higher, the volatility must be higher. If we play around with values, we find $\sigma = 0.2104$ does the trick.

3. *Digital options.* A digital or binary option either pays some fixed amount or not. Consider, for example, a digital option on the underlying s_{t+1}. A digital call with strike k pays 100 if $s_{t+1} > k$. A digital put with strike k pays 100 if $s_{t+1} \leq k$.

   (a) What is the analog of put-call parity for these options?

   (b) Suppose $\log s_{t+1}$ has a normal risk-neutral distribution with mean κ_1 and variance κ_2. What is the price of a put with strike k?

   (c) What is the no-arbitrage condition for this situation? Use it to simplify your answer to (b).

(d) *Extra credit.* How does the price of a put respond to an increase in volatility $\sigma = \kappa_2^{1/2}$? Why?

Answer.

(a) The idea behind put-call parity is that if we know the price of a put, we can use it to find the price of a call. And vice versa. Here we see that a put gives us 100 at date $t+1$ if $s_{t+1} \leq k$, and a call gives us 100 if $s_{t+1} > k$, so if we buy one of each we get 100 for sure. that gives us the parity relation

$$q_t^p + q_t^c \;=\; q_t^1 \cdot 100.$$

(b) The payoff from a put option is

$$d_{t+1} \;=\; \begin{cases} 100 & \text{if } s_{t+1} \leq k \\ 0 & \text{otherwise.} \end{cases}$$

The price is therefore

$$\begin{aligned} q_t^p &= q_t^1 E^*(d_{t+1}) \\ &= q_t^1 \text{Prob}(s_{t+1} \leq k) 100 \\ &= q_t^1 \text{Prob}(\log s_{t+1} \leq \log k) \cdot 100 \\ &= q_t^1 N[(\log k - \kappa_1)/\kappa_2^{1/2}] \cdot 100. \end{aligned}$$

The last line uses the lognormal (risk-neutral) distribution, but the others hold in general.

(c) This is the same as the one we used for BSM. The idea in general is that we must choose a risk-neutral distribution that's consistent with the current price of the asset. In this case we have

$$s_t \;=\; q_t^1 E^* s_{t+1} \;=\; q_t^1 e^{\kappa_1 + \kappa_2/2},$$

which should look familiar. In practice, we would use this condition to eliminate κ_1, as we did when we derived the BSM formula:

$$(\log k - \kappa_1)/\kappa_2^{1/2} \;=\; [\log(q_t^1 k/s_t) + \kappa_2/2]/\kappa_2^{1/2}.$$

There's a similar expression in BSM.

(d) This is terse, but the idea is that it depends on whether the option is in the money or not. Going beyond the formula, the payoff in this case isn't convex, so the impact of volatility is different.

4. *Option on an exponential underlying.* Suppose the risk-neutral distribution of the future value of the underlying $s_{t+1} = x$ has pdf

$$p(x) \;=\; \lambda e^{-\lambda x}$$

for $x \geq 0$ and $\lambda > 0$. In words: x has an exponential distribution. The mean of x is $1/\lambda$.

(a) What is the no-arbitrage condition for this asset?

(b) Consider a put option giving the owner the right to sell the asset for price k at $t+1$. What cash flow is generated by this option?

(c) What is the price of the option?

Answer.

(a) We have

$$s_t \;=\; q_t^1 E^*(s_{t+1}) \;=\; q_t^1 E^*(x) \;=\; q_t^1 (1/\lambda).$$

(b) The cash flow is (as usual) $(k - s_{t+1})^+$.

(c) The put price is

$$\begin{aligned} q_t^p &= q_t^1 E^*(k - s_{t+1})^+ \\ &= q_t^1 \int_0^k (k-x)\lambda e^{-\lambda x} dx \;=\; q_t^1 \big[k - (1 - e^{-\lambda k})/\lambda\big]. \end{aligned}$$

5. *Option on a mixture of exponentials.* Suppose the risk-neutral distribution of the future value of the underlying is a Bernoulli mixture of x_1 and x_2:

$$s_{t+1} \;=\; \begin{cases} x_1 & \text{with probability } 1-\omega \\ x_2 & \text{with probability } \omega \end{cases}$$

for some ω between zero and one. Each x_j is exponential with density

$$p(x_j) \;=\; \lambda_j e^{-\lambda_j x_j}$$

for $x_j \geq 0$ and $\lambda_j > 0$. Each x_j has a mean of $1/\lambda_j$.

(a) What is the no-arbitrage condition for this asset?

(b) Consider a put option giving the owner the right to sell the asset for price k at $t+1$. What cash flow is generated by this option?

(c) What is the price of a put option?

Answer.

(a) We have

$$\begin{aligned} s_t &= q_t^1 E^*(s_{t+1}) \\ &= q_t^1[(1-\omega)E^*(x_1) + \omega E^*(x_2)] \;=\; q_t^1[(1-\omega)/\lambda_1 + \omega/\lambda_2]. \end{aligned}$$

(b) The cash flow is (as usual) $(k - s_{t+1})^+$.

(c) The put price is

$$q_t^p \;=\; q_t^1 \Big\{(1-\omega)\big[k - (1 - e^{-\lambda_1 k})/\lambda_1\big] + \omega\big[k - (1 - e^{-\lambda_2 k})/\lambda_2\big]\Big\}.$$

The logic follows the previous problem.

6. *Divisibility.* Explain intuitively why the Bernoulli distribution isn't divisible.

Answer. A Bernoulli random variable takes on the values zero and one. If we repeat it n times, one way to think about it is that we have n chances of getting a one. The probabilities of various outcomes are binomial. If $n = 1$, for example, we can get zero ones, or one one. If $n = 2$, we can get zero ones, one one, or two ones. But what if $n = 1/2$? It's not possible to get half a one, it just doesn't work. Wikipedia has a more formal treat of this example and of the issue in general.

7. *Sums and mixtures (review).* Let us say that the log-price of the underlying has two components:

$$\log s_{t+1} \;=\; y_{t+1} \;=\; x_{1t+1} + x_{2t+1},$$

with (x_{1t+1}, x_{2t+1}) independent. The first component is normal: $x_{1t+1} \sim \mathcal{N}(\mu, \sigma^2)$. The second component, the "jump," is a mixture: with probability $1-\omega$, $x_{2t+1} = 0$, and with probability ω, $x_{2t+1} \sim \mathcal{N}(\theta, \delta^2)$.

With these inputs, the pdf for y is a weighted average of normals:

$$\begin{aligned} p(y) &= (1-\omega)\cdot(2\pi\sigma^2)^{-1/2}\exp[-(y-\mu)^2/2\sigma^2] \\ &\quad + \omega\cdot[2\pi(\sigma^2+\delta^2)]^{-1/2}\exp[-[y-(\mu+\theta)]^2/2(\sigma^2+\delta^2)]. \end{aligned} \tag{10}$$

If $\omega = 0$, the second component drops out. Otherwise, we have a weighted average of two normal densities.

(a) Show that the cumulant generating function for x_{1t+1} is

$$k(s; x_1) \;=\; \mu s + \sigma^2 s^2/2.$$

(b) Show that the cumulant generating function for x_{2t+1} is

$$k(s; x_2) \;=\; \log\left[(1-\omega) + \omega e^{\theta s + \delta^2 s^2/2}\right].$$

(c) What is the cgf for y_{t+1}? What are its mean, variance, skewness, and excess kurtosis? What parameters determine the sign of skewness?

Answer. This is a reminder that we can use mixtures of normals to generate nonnormal behavior.

(a) The usual normal cgf.

(b) This is a Bernoulli mixture with one twist: the "ω branch" has a normal mgf in it. If $\delta = 0$, it's just like the Bernoulli we looked at in Lab Report #1. Otherwise, we get some additional terms.

(c) The cgf of the sum is the sum of the cgfs:

$$\begin{aligned} k(s; y) &= k(s; x_1) + k(s; x_2) \\ &= (\mu s + \sigma^2 s^2/2) + \log\left[(1-\omega) + \omega e^{\theta s + \delta^2 s^2/2}\right]. \end{aligned}$$

We find it cumulants from derivatives, which are conveniently computed by Matlab. The mean and variance are

$$\begin{aligned} \kappa_1 &= \mu + \omega\theta \\ \kappa_2 &= \sigma^2 + \omega(1-\omega)\theta^2 + \omega\delta^2. \end{aligned}$$

Each has terms from both components. The third and fourth cumulants come only from x_2, because the normal component has zero cumulants beyond the first two. The third and fourth cumulants are

$$\begin{aligned} \kappa_3 &= \omega(1-\omega)\theta[3\delta^2 + (1-2\omega)\theta^2] \\ \kappa_4 &= \omega(1-\omega)\{\theta^4[1-6\omega(1-\omega)] + 3\delta^4 + (1-2\omega)6\delta^2\theta^2\} \end{aligned}$$

This is a bit of a mess, but for ω small, κ_3 depends on the sign of θ and $\kappa_4 > 0$, so the mixture introduces skewness and excess kurtosis, both of which are absent from normal random variables.

8. *Merton-like option pricing.* With the same setup as the previous problem, we can illustrate the value of mixtures in generating nonnormal distributions and, as a result, more flexible shapes of volatility smiles.

   (a) Risk-neutral asset pricing tells us, in general, that

   $$s_t \;=\; q_t^1 E^*(s_{t+1}) \;=\; q_t^1 E^*\left(e^{y_{t+1}}\right) \;=\; q_t^1 h(1; y) \;=\; q_t^1 e^{k(1;y)}.$$

   We refer to this as the no-arbitrage condition. What is the no-arbitrage condition for our example?

   We'll use this condition to set μ: given values for everything else, we'll choose μ to satisfy this condition.

   (b) Recall that if the risk-neutral distribution is $\log s_{t+1} = y_{t+1} \sim \mathcal{N}(\kappa_1, \kappa_2)$, then the put price at strike k is

   $$\begin{aligned} f(k; \kappa_1, \kappa_2) &= q_t^1 k N(d) - q_t^1 e^{\kappa_1 + \kappa_2/2} N(d - \kappa_2^{1/2}) \\ d &= (\log k - \kappa_1)/\kappa_2^{1/2}. \end{aligned}$$

   (Note: modest change in notation.) What is the call price?

   (c) Use (b) to show that the put price in the mixture model is a weighted average:

   $$q_t^p \;=\; (1-\omega) \cdot f(k; \mu, \sigma^2) + \omega \cdot f(k; \mu + \theta, \sigma^2 + \delta^2).$$

   (d) Consider these inputs: $\sigma = 0.04$, $\omega = 0.01$, $\theta = -0.3$, $\delta = 0.15$, $s_t = 100$, and $q_t^1 = 1$. What is μ? What are the prices of put options with strikes $k = (90, 94, 98, 102, 106, 110)$? (Use a finer grid if you have this automated.) What are the implied volatilities?

   (e) What happens to the volatility smile when you set
   - $\theta = 0$?
   - $\theta = +0.3$?
   - $\delta = 0.25$?

   Make sure you adjust μ in each case.

   Answer.

   (a) The condition is

   $$s_t \;=\; q_t^1 \exp(\mu + \sigma^2/2) \left[(1-\omega) + \omega e^{\theta + \delta^2/2}\right].$$

   It's not pretty, but given values of the other inputs we can use it to set μ.

   (b) This is a question of integrating over the distribution, as we did when we derived the BSM formula. We can then find the call price from put-call parity.

(c) Since the pdf is a sum, we can integrate each term separately (the integral of a sum is the sum of the integrals). Each integral gives us a BSM-like formula, as noted, but with different mean and variance. The put price is the weighted average of the two formulas, as stated.

(d) Now we can put all this to work. With these inputs, the arbitrage condition gives us $\mu = 4.6009$. The put prices are

Strike	Put Price
90	0.1606
94	0.2806
98	0.9285
102	2.8801
106	6.1523
110	10.0147

The implied volatilities give us a clearer picture; run the Matlab code below and see. We find that there's a distinct downward slope to the smile. That disappears if we set $\theta = 0$, since θ is the source of skewness in the model. Instead we get a U-shape. When we increase δ, the U is more pronounced. So roughly speaking, θ generates skewness and slope in the smile, and δ generates kurtosis and convexity in the smile.

Here's the code:

```
disp('Inputs')
tau = 1;
q1 = 1;
q_tau = q1;
s = 100.00
k = [85:2:115]';

sigma = 0.04;
omega = 0.01;
theta = -0.3;
delta = 0.15;

% apply arb condition
mu = log(s/q1) - sigma^2/2 - log((1-omega)+omega*exp(theta+delta^2/2))

% branch 1
d1 = (log(k)-mu)/sigma;
put1 = q1*k.*normcdf(d1) - q1*exp(mu+sigma^2/2)*normcdf(d1-sigma);

% branch 2
d2 = (log(k)-(mu+theta))/sqrt(sigma^2+delta^2);
put2 = q1*k.*normcdf(d2) - Q1*exp((mu+theta) ...
    + (sigma^2+delta^2)/2)*normcdf(d2-sqrt(sigma^2+delta^2));
```

```
puts = (1-omega)*put1 + omega*put2;
calls = puts + s - q1*k;

% BSM formula
% f = call price as function of sigma, two steps for clarity
% fp = the derivative (vega) for use in Newton routine
d = @(sigma,k) (log(s./(q_tau.*k))+tau*sigma.^2/2)./(sqrt(tau)*sigma);
f = @(sigma,k) s*normcdf(d(sigma,k)) - q_tau.*k.*normcdf(d(sigma,k) ...
        -sqrt(tau)*sigma) - calls;
fp = @(d) s*sqrt(tau)*exp(-d.^2/2)/sqrt(2*pi);

% convergence parameters
tol = 1.e-8;
maxit = 50;

% starting values
x_now = 0.12 + zeros(size(k));
f_now = f(x_now,k);

% compute implied vol
t0 = cputime;
for it = 1:maxit
    fp_now = fp(d(x_now,k));
    x_new = x_now - f_now./fp_now;
    f_new = f(x_new,k);
    diff_x = max(abs(x_new - x_now));
    diff_f = max(abs(f_new));

    if max(diff_x,diff_f) < tol, break, end

    x_before = x_now;
    x_now = x_new;
    f_before = f_now;
    f_now = f_new;
end

% plot smile
figure(1)
clf
plot(k, vol, 'b')
hold on
plot(k, vol, 'b+')
xlabel('Strike Price')
ylabel('Implied Volatility')
```

9. *Today's option prices.* Yahoo Finance has an Options Center, where you can look up prices of a wide range of equity options. Look up options on SPY, the ticker for an S&P

500 exchange traded fund. Choose a contract, note the prices, and compute the volatility smile. What does it look like?

Answer. This one's all yours, too.