Lab Report #2: Sums, Mixtures, & Certainty Equivalents

Revised: February 13, 2013

Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Please include your Matlab code.

Solution: Most of the answers are computed in the Matlab program listed at the end.

1. *The sum of normals is normal.* The idea here is to use cumulant generating functions (cgfs) to show that the sum of independent normal random variables is also normal. It's helpful to break the problem into manageable pieces, like this:
   (a) Consider two independent random variables x_1 and x_2, not necessarily normal. Show that the cgf of the sum $y = x_1 + x_2$ is the sum of their cgfs:
   $$k(s; y) \;=\; k(s; x_1) + k(s; x_2).$$
   Hint: note the form of the pdf and apply the definition of the cgf.
   (b) Suppose $x_i \sim \mathcal{N}(\kappa_{i1}, \kappa_{i2})$. [This bit of notation means: x_i is normally distributed with mean κ_{i1} and variance κ_{i2}.] What is x_i's cgf? Hint: we did this in class.
   (c) Use (a) to find the cgf of $y = x_1 + x_2$, with (x_1, x_2) as described in (b) (namely, normal with given means and variances). How do you know that y is also normal? What are its mean and variance?
   (d) Extend this result to $y = ax_1 + bx_2$ for any real numbers (a, b).

Solution:

(a) If x_1 and x_2 are independent, their pdf factors: $p_{12}(x_1, x_2) = p_1(x_1)p_2(x_2)$. That means the mgf of x_1+x_2 is the product of their individual mgf's: $h_y(s) = h_1(s)h_2(s)$. We take the log to get the cgf's: $k_y(s) = \log h_y(s) = \log h_1(s) + \log h_2(s) = k_1(s) + k_2(s)$.

(b) $k(s; x_i) = s\kappa_{1i} + s^2\kappa_{2i}/2$. (If this isn't burned into your memory, please burn it in now.)

(c) Sum the cgf's:

$$\begin{aligned}
k(s; y) &= k(s, x_1) + k(s, x_2) \\
&= (s\kappa_{11} + s^2\kappa_{21}/2) + (s\kappa_{12} + s^2\kappa_{22}/2) \\
&= s(\kappa_{11} + \kappa_{12}) + s^2(\kappa_{21} + \kappa_{22})/2.
\end{aligned}$$

It's normal because it's mgf has the form of a normal random variable. In fact, we can pick the mean and variance right out of the formula.

(d) Still normal, but with a slight change in mean and variance:

$$k(s; y) = s(a\kappa_{11} + b\kappa_{12}) + s^2(a^2\kappa_{21} + b^2\kappa_{22})/2.$$

2. *Sums and mixtures.* More fun with cumulant generating functions (cgfs). The ingredients include two independent normal random variables: $x_1 \sim \mathcal{N}(0, \sigma^2)$ and $x_2 \sim \mathcal{N}(\theta, \delta^2)$. Suggestion: Have Matlab do the calculations.
   (a) *Sum.* Consider the random variable $y = x_1 + x_2$. Use the results of the previous question to derive the first four cumulants of y. What are the measures of skewness and excess kurtosis, γ_1 and γ_2?
   (b) *Mixture.* Now consider the mixture x_3:

$$x_3 = \begin{cases} 0 & \text{with probability } 1-\omega \\ x_2 & \text{with probability } \omega. \end{cases}$$

   What is its cgf? Hint: apply the definition.
   (c) Now consider the sum $y = x_1 + x_3$. What is its cgf? What are the measures of skewness and excess kurtosis, γ_1 and γ_2? What parameter configuration do you need to generate negative skewness?

Solution:

(a) Cumulants follow from differentiating. The cgf is

$$k_y(s) = k_1(s) + k_1(s) = s(0 + \theta) + s^2(\sigma^2 + \delta^2)/2.$$

The cumulants are $\kappa_1 = \theta$ (the mean) and $\kappa_2 = \sigma^2 + \delta^2$ (the variance). Since the cgf is quadratic (in s!), all the derivatives after the second one are zero. Therefore $\gamma_1 = \gamma_2 = 0$ as well.

(b) Mixtures are a little different. The mgf is

$$h(s) = (1 - \omega) + \omega e^{s\theta + s^2\delta^2/2}.$$

The cgf is the log of h.

(c) The cgf of a sum is the sum of the cgf's:

$$h_y(s) = s^2\sigma^2/2 + \log\left[(1 - \omega) + \omega e^{s\theta + s^2\delta^2/2}\right].$$

We find the cumulants by differentiating. The expressions are a mess, but include:

$$\begin{aligned} \kappa_1 &= \omega\theta \\ \kappa_2 &= \sigma^2 + \omega\delta^2 + \omega(1-\omega)\theta^2 \\ \kappa_3 &= \omega(1-\omega)\theta\left[3\delta^2 + \theta^2(1 - 2\omega)\right]. \end{aligned}$$

See the Matlab output for details. It looks like θ is the key. If $\omega < 1/2$, then the sign of κ_3 is the same as the sign of θ.

You might picture this: start with a normal pdf for x_1, now add a smaller pdf on top of it for x_2. If $\theta < 0$ this is to the left of the mean of the first one, which gives you negative skewness.

3. *Lognormal risks.* Our mission is to compute the *risk penalty* associated with lognormal risks and power utility. What risks matter? What role does the risk aversion parameter play?

   To be concrete, let $x = \log c \sim \mathcal{N}(\kappa_1, \kappa_2)$ and $u(c) = c^{1-\alpha}/(1-\alpha)$ for $\alpha > 0$. The risk penalty is

   $$rp \;=\; \log(\bar{c}/\mu) \;=\; \log\bar{c} - \log\mu,$$

   where $\bar{c} = E(c) = E(e^x)$ and μ is the certainty equivalent. We'll compute the terms one by one.

   (a) What is the cumulant generating function (cgf) of $x = \log c$?

   (b) Use the cgf to compute $\bar{c}$.

   (c) Use the cgf to compute the certainty equivalent μ.

   (d) What is the risk penalty? How does it depend on risk? Risk aversion?

   (e) Suppose $\kappa_2 = 0.02^2$. What is the risk penalty if $\alpha = 2$? $\alpha = 10$? $\alpha = 20$?

Solution: The idea is to explore risk and risk aversion in a lognormal setting, a setting we'll see over and over again. We're using the idea that if $x = \log c$, then $c = e^x$.

(a) The mgf of x is (get used to this, you'll see it a lot)

$$h(s) \;=\; E(e^{sx}) \;=\; \exp[s\kappa_1 + s^2\kappa_2/2].$$

The cgf is the log of this:

$$k(s) \;=\; \log h(s) \;=\; s\kappa_1 + s^2\kappa_2/2.$$

(b) We find the mean by setting $s = 1$: $\bar{c} = E(c) = E(e^x) = h(1) = \exp[k(1)] = e^{\kappa_1 + \kappa_2/2}$.

(c) We find expected utility by setting $s = 1-\alpha$: $E[u(c)] = E(c^{1-\alpha})/(1-\alpha) = E(e^{(1-\alpha)x})/(1-\alpha)$. The numerator is the mgf evaluated at $s = 1-\alpha$. So we have

$$\begin{aligned} E[u(c)] &= \exp[k(1-\alpha)/(1-\alpha)] \\ &= \exp[(1-\alpha)\kappa_1 + (1-\alpha)^2\kappa_2/2]/(1-\alpha) \\ &= \exp[(1-\alpha)\{\kappa_1 + (1-\alpha)\kappa_2/2\}]/(1-\alpha). \end{aligned}$$

What level of consumption gives us the same utility? Well, a constant level of consumption μ gives utility $E[u(\mu)] = u(\mu) = \mu^{1-\alpha}/(1-\alpha)$. Equating $u(\mu)$ and $E[u(c)]$ gives us

$$\mu = \exp[\kappa_1 + (1-\alpha)\kappa_2/2].$$

(d) The risk penalty is

$$\log(\bar{c}/\mu) = \log \bar{c} - \log \mu) = \alpha\kappa_2/2.$$

So it increases with risk κ_2 and risk aversion α.

Matlab program:

```
%  hw2_s13
%  Matlab program for Lab Report #2, Spring 2013
%  Written by:  Dave Backus, February 2013
format compact
clear all

%%
disp('Answers to Lab Report 2')

disp(' ')
disp('------------------------------------------------------------------')
clear
disp('Question 2')
syms s omega sigma theta delta

cgf1 = s^2*sigma^2/2;
mgf3 = 1-omega + omega*exp(s*theta+s^2*delta^2/2);
cgf3 = log(mgf3);

cgf = cgf1 + cgf3;

disp(' ')
disp('Cumulants')
kappa1 = subs(diff(cgf,s,1),s,0)    %  mean
simplify(kappa1)
kappa2 = subs(diff(cgf,s,2),s,0)    % variance
simplify(kappa2)
kappa3 = subs(diff(cgf,s,3),s,0)
simplify(kappa3)
kappa4 = subs(diff(cgf,s,4),s,0)
simplify(kappa4)

disp(' ')
gamma1 = kappa3/kappa2^(3/2)
simplify(gamma1)
gamma2 = kappa4/kappa2^2


%%
disp(' ')
disp('------------------------------------------------------------------')
disp('Question 3 (lognormal risks)')
syms s kappa_1 kappa_2 alpha
```

```
cgf_x = kappa_1*s + kappa_2*s^2/2
cgf = cgf_x

% cgf evaluated at s=1
log_cbar = subs(cgf,s,1)
% cgf evaluated at s=1-alpha, then divided by (1-alpha)
log_mu = subs(cgf,s,1-alpha)/(1-alpha)

rp = log_cbar - log_mu
rp = simplify(rp)

rp_alpha2  = subs(rp,[alpha,kappa2],[2,0.02^2])
rp_alpha10 = subs(rp,[alpha,kappa2],[10,0.02^2])
rp_alpha20 = subs(rp,[alpha,kappa2],[20,0.02^2])

return
```