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Intermediate Macroeconomic Theory
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# Midterm — Answers

## Problem 1

1. **Derive the first order condition for the household consumption/saving relation.**

   To derive the household's first order condition for consumption/savings, we substitute the budget constraints for $C_1^i$ and $C_2^i$ into the objective function, and solve the unconstrained problem:

$$\max_{S^i} \quad \log\left(\Pi_1^i - S^i\right) + \beta \log\left(\Pi_2^i + RS^i\right)$$

   The household takes the income it receives from the firm $\Pi_1^i$ and $\Pi_2^i$, and the interest rate $R$ as given. The first order condition for the choice of $S^i$ is then:

$$\text{FOC}(S^i): \quad \frac{1}{\Pi_1^i - S^i}(-1) + \beta \frac{1}{\Pi_2^i + RS^i} R = 0$$

   Substituting back in using the budget constraints and rearranging gives:

$$\frac{1}{C_1^i} = \frac{\beta R}{C_2^i}$$
$$\Rightarrow \; C_2^i = \beta R C_1^i \tag{1}$$

2. **Derive the first order condition for the firm investment decision.**

   To solve the firm's problem we substitute in for the profits $\Pi_1^j$ and $\Pi_2^j$ using the budget constraints. We also substitute for the amount of output produced $Y_1^j$ and $Y_2^j$ using the technology constraints. Then we have:

$$\max_{I^j} \quad A_1 K_1^j - I^j - \frac{c}{2}\left(\frac{I^j}{K_1^j}\right)^2 K_1^j + \frac{A_2 K_2^j + (1-\delta)K_2^j}{R}$$
$$\text{s.t.} \quad K_2^j = (1-\delta)K_1^j + I^j$$

   Substituting for $K_2^j$ using the capital evolution equation gives us an unconstrained problem:

$$\max_{I^j} \quad A_1 K_1^j - I^j - \frac{c}{2}\left(\frac{I^j}{K_1^j}\right)^2 K_1^j + \frac{A_2\left((1-\delta)K_1^j + I^j\right) + (1-\delta)\left((1-\delta)K_1^j + I^j\right)}{R}$$

   The first order condition for the choice of $I^j$ is then:

$$\text{FOC}(I^j): \quad -1 - c\frac{I^j}{K_1^j} + \frac{A_2 + 1 - \delta}{R} = 0$$

Rearranging this gives

$$R = \frac{A_2 + 1 - \delta}{1 + c\frac{I^j}{K_1^j}} \tag{2}$$

3. **Present the set of equations that describe the equilibrium.**

   These equations are as follows:

$$\begin{aligned}
\text{Savings supply}: &\quad & C_2 &= \beta R C_1 \\
\text{Investment demand}: &\quad & R &= \frac{A_2 + 1 - \delta}{1 + c\frac{I}{K_1}} \\
\text{Resource constraints}: &\quad & Y_1 &= C_1 + I + \frac{c}{2}\left(\frac{I}{K_1}\right)^2 K_1 + XM \\
& & Y_2 + (1-\delta)K_2 &= C_2 - R \cdot XM \\
\text{Technology constraints}: &\quad & Y_1 &= A_1 K_1 \\
& & Y_2 &= A_2 K_2 \\
\text{Evolution of Capital}: &\quad & K_2 &= (1-\delta)K_1 + I \\
\text{Interest rate}: &\quad & R &= \Omega R^w
\end{aligned}$$

   The equilibrium is defined by these 8 equations that pin down the allocation $\{Y_1, Y_2, I, C_1, C_2, K_2, XM\}$ and the interest rate $R$. So we have 8 equations and 8 unknowns. The world interest rate $R^w$ is taken as given and it pins down the interest rate faced by both the household and the firm in the country under consideration.

   Note that we could write the equations differently. For example, we could have substituted out for $R$ in the equations for savings supply and investment demand, and dropped the equation for the interest rate. Then we would have 7 equations for 7 unknowns, and the key is to have enough equations to solve for the variables used in the equations presented.

4. **Now suppose that international investors perceive that lending to the country carries increased risk, leading to an increase in $\Omega$. Suppose further that the economy is in in initial equilibrium with $XM < 0$. Describe the impact on investment $I$, output in period 2, $Y_2$, and net foreign exports $XM$. Explain.**

   We're given that $XM < 0$, so we know that $\Omega > 1$. The firm faces the interest rate $R = \Omega R^w$, so from the investment demand equation we have

$$\Omega R^w = \frac{A_2 + 1 - \delta}{1 + c\frac{I}{K_1}}$$

   We can solve for $I$ to obtain:

$$I = \frac{K_1}{c}\left(\frac{A_2 + 1 - \delta}{\Omega R^w} - 1\right)$$

We see from this equation that an increase in $\Omega$ leads to a decrease in $I$. If we consider a firm that is borrowing from abroad to finance investment, the intuition is that when the borrowing costs increase the firm responds by reducing investment.

The capital evolution equation says that $K_2 = (1-\delta)K_1 + I$, and the technology constraint in period 2 says that $Y_2 = A_2K_2$. When the firm reduces investment, there is less capital in period 2 ($K_2 \downarrow$), and hence less output. Thus output in period 2, $Y_2$, decreases.

Finally, we want to use the resource constraint in period 1 to determine the effect on $XM$, but to do that we need to know what happens to $C_1$. By combining the two budget constraints for the household, we have

$$C_1 + \frac{C_2}{R} = \Pi_1 + \frac{\Pi_2}{R}$$

Substituting out for $C_2$ using the savings supply equation, and for $R$ using the interest rate equation, we have:

$$C_1 = \frac{1}{1+\beta}\left(\Pi_1 + \frac{\Pi_2}{\Omega R^w}\right)$$

Since $\Pi_2 = Y_2 + (1-\delta)K_2$ we know that the increase in $\Omega$ decreases $\Pi_2$. Financing investment is more expensive, so investment is reduced and there is less output in period 2. This negative income effect decreases $C_1$.

In addition, the increase in $\Omega$ makes $C_1$ more expensive relative to $C_2$. To see this, we can rewrite the savings supply equation as:

$$\frac{C_2}{C_1} = \beta\Omega R^w$$

Hence, the substitution effect also serves to decrease $C_1$.

The resource constraint in period 1 is

$$Y_1 = C_1 + I + \frac{c}{2}\left(\frac{I}{K_1}\right)^2 K_1 + XM$$

Since $Y_1$ is fixed by initial capital and productivity, $I$ decreases, and $C_1$ decreases, we must have that $XM$ increases. The intuition is that an increase in $\Omega$ makes it more expensive to borrow from abroad, so this borrowing is reduced. $XM < 0$ so an increase means that $XM$ moves closer to zero (we cannot say exactly how much it moves since we have not quantified the parameters of the model and the size of the increase in $\Omega$).

## Problem 2

1. **Derive an expression for the firm's optimal choice of investment.**

   To derive the required expression we substitute for $Y_1$ and $Y_2$ in the firm's objective function which then becomes:

$$\max_I \quad A_1 K_1^\alpha - P_K I - \frac{c}{2}\left(\frac{I}{K_1}\right)^2 K_1 + \frac{A_2 K_2^\alpha + (1-\delta)K_2}{R}$$
$$\text{s.t.} \quad K_2 = (1-\delta)K_1 + I$$

   We can also substitute out for $K_2$ using the capital evolution equation:

$$\max_I \quad A_1 K_1^\alpha - P_K I - \frac{c}{2}\left(\frac{I}{K_1}\right)^2 K_1 + \frac{A_2((1-\delta)K_1 + I)^\alpha + (1-\delta)((1-\delta)K_1 + I)}{R}$$

   The first order condition with respect to $I$ is:

$$-P_K - c\frac{I}{K_1} + \frac{A_2\alpha((1-\delta)K_1 + I)^{\alpha-1} + 1 - \delta}{R} = 0$$

   Substituting back in for $K_2$ and rearranging gives the expression that we are after which must be satisfied by the firm's optimal choice of investment:

$$R = \frac{\alpha A_2 K_2^{\alpha-1} + 1 - \delta}{P_K + c\frac{I}{K_1}}$$

2. **Show that one can relate investment to the difference between the market value of another unit of capital and its $P_K$.**

   An additonal unit of capital in period 2 allows the firm to produce more in period 2. The additional production is given by the marginal product of capital which is $\alpha A_2 K_2^{\alpha-1}$. In addition, capital depreciates after production and can be consumed at the end of the period 2, so the additional unit of capital leads to an increase of $(1-\delta)$. The total marginal value of an additonal unit of capital in period 2 is therefore $\alpha A_2 K_2^{\alpha-1} + 1 - \delta$. It follows that the market value of this additional unit discounted to period 1 is given by:

$$\frac{\alpha A_2 K_2^{\alpha-1} + 1 - \delta}{R}$$

   Hence, we obtain the expression we are after by rearranging the first order condition from part 1:

$$I = \frac{K_1}{c}\left(\frac{\alpha A_2 K_2^{\alpha-1} + 1 - \delta}{R} - P_K\right) \tag{3}$$

3. **What is the impact of a decline in the price of new capital on investment? Explain.**

   A decline in $P_K$ increases investment. We see this from equation (3). A decrease in $P_K$ increases the difference on the right hand side of the equation. For the equation to hold investment must increase, but increasing investment has two effects on the terms in the equation: 1) $I$ on the left hand side increases, and 2) $K_2$ on the right hand side increases. As more investment leads to more capital in period 2 the marginal value of additional units of capital goes down. Hence, if $P_K$ decreases by 1 unit the increase in investment is less than $\frac{K_1}{c}$ as long as $\alpha < 1$.

   The intuition is pretty straightforward: if the price of capital goods declines it becomes cheaper to invest and firms respond by investing more. We see that a fall in the price of capital goods can be a different reason for increased investment (earlier we have seen that investment can increase when it becomes cheaper to obtain funds in financial markets, for example when the cost of borrowing changes).

## Problem 3

1. **Derive the household's first order condition for consumption and saving.**

   To derive the household's first order condition we consider the two adjacent time periods $t$ and $t+1$, and solve

$$\max_{C_t, C_{t+1}, K_{t+1}} \quad \log C_t + \beta \log C_{t+1}$$

   with budget constraints:

$$\begin{aligned} C_t &= W_t N_t - T_t + R_t K_t - K_{t+1} \\ C_{t+1} &= W_{t+1} N_{t+1} - T_{t+1} + R_{t+1} K_{t+1} - K_{t+2} \end{aligned}$$

   The first-order condition with respect to $K_{t+1}$ is:

$$\frac{1}{C_t}(-1) + \frac{\beta}{C_{t+1}} R_{t+1} = 0$$

   This implies the condition $C_{t+1} = \beta R_{t+1} C_t$, and this relation holds for any two adjacent time periods. Hence, for any $i > 0$ we have $C_{t+1+i} = \beta R_{t+1+i} C_{t+i}$.

2. **Derive the firm's first order condition for labor and capital demand.**

   The firm's problem is a static one. We substitute in for $Y_t$ and solve:

$$\max_{N_t, K_t} \quad K_t^{\alpha} (A N_t)^{1-\alpha} - W_t N_t - (R_t - 1 + \delta) K_t$$

   The first order conditions are:

$$\begin{aligned} \text{FOC}(N_t): &\quad (1-\alpha) K_t^{\alpha} A^{1-\alpha} N_t^{-\alpha} - W_t = 0 \\ \text{FOC}(K_t): &\quad \alpha K_t^{\alpha-1} (A N_t)^{1-\alpha} - (R_t - 1 + \delta) = 0 \end{aligned}$$

   By rearranging these two equations we get

$$\begin{aligned} \text{Labor demand}: &\quad (1-\alpha) \frac{Y_t}{N_t} = W_t \\ \text{Capital demand}: &\quad \alpha \frac{Y_t}{K_t} = (R_t - 1 + \delta) \end{aligned}$$

3. **Present the set of equations that determine the balanced growth path equilibrium values of: $\frac{Y}{AN}, \frac{C}{AN}, \frac{I}{AN}, \frac{K}{AN}$.**

   To obtain these equations, we first note that

$$\frac{A_{t+1}}{A_t} \frac{N_{t+1}}{N_t} = (1+a)(1+n) \approx 1 + a + n$$

   This approximatio is valid when $a$ and $n$ are small, since then the product $a \cdot n \approx 0$.

Using this in the household's first order condition for savings supply, we have:

$$\frac{C_{t+1}}{A_{t+1}N_{t+1}}(1+a+n) = \beta R \frac{C_t}{A_t N_t}$$

It follows that along a balanced growth path when $\frac{C}{AN}$ is constant, we have

$$1+a+n = \beta R$$

and hence $R$ is a constant as well.

We also use the approximation in the capital evolution equation which becomes

$$\frac{K_{t+1}}{A_{t+1}N_{t+1}}(1+a+n) = (1-\delta)\frac{K_t}{A_t N_t} + \frac{I_t}{A_t N_t}$$

Along the balanced growth path we therefore have

$$\frac{I}{AN} = (a+n+\delta)\frac{K}{AN}$$

The complete set of equations that determine the balanced growth path equilibrium are:

$$\begin{aligned}
\text{Technology :} &\quad \frac{Y}{AN} = \left(\frac{K}{AN}\right)^{\alpha} \\
\text{Capital evolution :} &\quad \frac{I}{AN} = (a+n+\delta)\frac{K}{AN} \\
\text{Resource constraint :} &\quad \frac{Y}{AN} = \frac{C}{AN} + \frac{I}{AN} + g \\
\text{Savings supply :} &\quad 1+a+n = \beta R \\
\text{Capital demand :} &\quad \alpha\left(\frac{K}{AN}\right)^{\alpha-1} = R - 1 + \delta
\end{aligned}$$

4. **Now suppose the government increases spending along the balanced growth from $g$ to $\bar{g} > g$. What is the impact on the economy's balanced growth path? (i.e. on $\frac{Y}{AN}$, $\frac{C}{AN}$, $\frac{I}{AN}$, and $\frac{K}{AN}$). Explain intuitively.**

   An increase in $g$ to $\bar{g} > g$ has no effect on the interest rate along the balanced growth path which is pinned down by $1+a+n = \beta R$. Hence, the factor prices faced by the firm do not change, and we see from the capital demand equation that $\frac{K}{AN}$ does not change. From the technology equation and the capital evolution equation we see that $\frac{Y}{AN}$ and $\frac{I}{AN}$ are therefore unchanged as well. Since there is no change in conditions for the firm, the firm's decisions on investment and production are unchanged.

   The resource constraint shows that $\frac{C}{AN}$ decreases when government spending increases. From the household's budget constraint we see that the increase in the lump sum tax needed to finance an increase in government spending lowers the household's wealth. Intuitively, since government spending is financed by taxing the household only, a change in spending along the balanced growth path will reduce consumption, but since the prices (wage and interest rate) faced by the firm are unchanged it will not affect the firm's demand for capital (or labor).

5. **Would your answer to 4. be affected if the government could finance its expenditures by borrowing as well as current taxes? Explain.**

   If the government can finance expenditures by borrowing as well as taxes, an increase in spending does not necessarily lead to an increase in taxes immediately if the increase in spending is financed by borrowing. However, taxes will have to rise eventually to finance the repayment of that borrowing. Since there are perfect capital markets in this model, an alternative form of financing government spending will not matter. The household's consumption choices are determined by the present value of taxes, and alternative ways for the government to finance its expenditure will not change this present value.

   [To see this more formally (which I don't think is strictly necessary when answering this question), we use the budget constraints from the household's problem. Remember that there is one budget constraint for each time period $t$:

$$C_t = W_t N_t - T_t + R_t K_t - K_{t+1}$$
$$C_{t+1} = W_{t+1} N_{t+1} - T_{t+1} + R_{t+1} K_{t+1} - K_{t+2}$$

   Substituting for $K_{t+1}$ in the first budget constraint using the second we have

$$C_t + \frac{C_{t+1}}{R_{t+1}} = R_t K_t + W_t N_t + \frac{W_{t+1} N_{t+1}}{R_{t+1}} - \left(T_t + \frac{T_{t+1}}{R_{t+1}}\right) - \frac{K_{t+2}}{R_{t+1}}$$

   Next, by using the household's first order condition $C_{t+1} = \beta R_{t+1} C_t$ we can substitute out for $C_{t+1}$:

$$C_t(1+\beta) = R_t K_t + W_t N_t + \frac{W_{t+1} N_{t+1}}{R_{t+1}} - \left(T_t + \frac{T_{t+1}}{R_{t+1}}\right) - \frac{K_{t+2}}{R_{t+1}}$$

   We can continue in this way using the budget constraint at time $t+2$ to substitute for $K_{t+2}$, and then use the first order condition $C_{t+2} = \beta R_{t+2} C_{t+1} = \beta^2 R_{t+2} R_{t+1} C_t$ to substitute for $C_{t+2}$ etc. (see lecture notes 6 page $5-6$), but to illustrate the point we can just consider a government that lowers taxes at $t$ by financing expenditure via borrowing, and then raises taxes at $t+1$ to repay the borrowed amount. If taxes are lowered by 1 at $t$ the government has to repay $R_{t+1}$ at time $t+1$. This change has no effect on the term

$$T_t + \frac{T_{t+1}}{R_{t+1}}$$

   and hence no effect on consumption $C_t$.]