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Intermediate Macroeconomic Theory
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Problem Set 4 — Answers

Problem 1

1. **Derive the household's first order condition for consumption and saving.**

   To derive the household's first order condition for consumption/savings, we consider two adjacent time periods $t+i$ and $t+i+1$.

$$\max_{C_{t+i},C_{t+i+1},K_{t+i+1}} \quad \beta^i \log(C_{t+i}) + \beta^{i+1}\log(C_{t+i+1})$$

$$\text{s.t.} \qquad \begin{aligned} C_{t+i} &= W_{t+i}N + R_{t+i}K_{t+i} - K_{t+i+1} \\ C_{t+i+1} &= W_{t+i+1}N + R_{t+i+1}K_{t+i+1} - K_{t+i+2} \end{aligned}$$

   The household takes prices $W_{t+i}, W_{t+i+1}, R_{t+i}, R_{t+i+1}$ as given. At the start of period $t+i$ the capital stock K_{t+i} is given. The household also takes the future choice of K_{t+i+2} as given (and assumes this will be chosen optimally).

   As we did in the two-period model we can therefore substitute for C_{t+i} and C_{t+i+1} in the objective function. The problem then becomes:

$$\max_{K_{t+i+1}} \quad \beta^i \log(W_{t+i}N + R_{t+i}K_{t+i} - K_{t+i+1}) + \beta^{i+1}\log(W_{t+i+1}N + R_{t+i+1}K_{t+i+1} - K_{t+i+2})$$

   The first-order condition w.r.t. K_{t+i+1} is:

$$\beta^i \frac{1}{W_{t+i}N + R_{t+i}K_{t+i} - K_{t+i+1}}(-1) + \beta^{i+1}\frac{1}{W_{t+i+1}N + R_{t+i+1}K_{t+i+1} - K_{t+i+2}}R_{t+i+1} = 0$$

   Substituting back in for C_{t+i} and C_{t+i+1} implies:

$$C_{t+i+1} = R_{t+i+1}\beta C_{t+i}$$

   This is the first order condition we are after, and it holds for every value of i.

2. **Derive the firm's first order conditions for labor and capital demand.**

   The firm's problem is the same each period. We can therefore consider any time period $t+i$, but to simplify notation we'll just consider time period t. The problem is then given by:

$$\max_{K_t,N_t} \quad K_t^\alpha (A_t N_t)^{1-\alpha} - W_t N_t - (R_t - 1 + \delta)K_t$$

The first order conditions are:

$$\begin{aligned}
\text{FOC}(N_t): &\quad (1-\alpha)A_t K_t^{\alpha}(A_t N_t)^{-alpha} - W_t = 0 \\
\text{FOC}(K_t): &\quad \alpha K_t^{\alpha-1}(A_t N_t)^{1-\alpha} - (R_t - 1 + \delta) = 0
\end{aligned}$$

These can be rewritten as:

$$\begin{aligned}
\text{Labor demand}: &\quad (1-\alpha)\frac{Y_t}{N_t} = W_t \\
\text{Capital demand}: &\quad \alpha\frac{Y_t}{K_t} = R_t - 1 + \delta
\end{aligned}$$

3. **Present the set of equations that determine the balanced growth path equilibrium values of: $\frac{Y}{A}, \frac{C}{A}, \frac{I}{A}, \frac{K}{A}$. (Note we are not deflating these values by N since N is not growing.)**

   Since N is fixed in this part of the problem we don't need equations for labor supply/labor demand here. The equations we need are equations for technology, resource constraint[1], capital evolution, and the first order conditions for the household (consumption/savings) and the firm (capital demand). This gives us 5 equations.

$$\begin{aligned}
\text{Technology}: &\quad \frac{Y}{A} = \left(\frac{K}{A}\right)^{\alpha} N^{1-\alpha} \\
\text{Resource constraint}: &\quad \frac{Y}{A} = \frac{C}{A} + \frac{I}{A} \\
\text{Captial evolution}: &\quad (1+a)\frac{K}{A} = (1-\delta)\frac{K}{A} + \frac{I}{A} \\
&\Rightarrow\ (a+\delta)\frac{K}{A} = \frac{I}{A} \\
\text{Capital market equilibrium}: &\quad \\
\text{Consumption/savings}: &\quad (1+a)\frac{C}{A} = R\beta\frac{C}{A} \\
\text{Capital demand}: &\quad \alpha\frac{Y/A}{K/A} = R - 1 + \delta
\end{aligned}$$

   These 5 equations pin down the variables that we know are constant along a balanced growth path: $\frac{Y}{A}, \frac{C}{A}, \frac{I}{A}, \frac{K}{A}, R$. The questions only asks for equations for the first 4 of these. However, by using R in the equations, an extra variable is introduced, so we need one more equation. We could instead have noted that the consumption/savings equation tells us that $R = (1+a)\beta^{-1}$ and used that equation to substitute out for R.

4. **Now suppose the household choose labor supply endogenously. The households objective is now given by**:

$$\max_{N_t, C_t} \sum_{i=0}^{\infty} \beta^i [\log(C_{t+i}) - \nu N_{t+i}]$$

[1] In an earlier recitation I referred to this as "goods market clearing". This is the same thing.

Derive the household's first order condition for labor supply N_t. (Hint: the labor supply decision is a static period-by-period decision, i.e., you can solve for the optimal choice of N_t ignoring the future.) Then present the set of equations that determine the balanced growth path equilibrium values of $\frac{Y}{A}, \frac{C}{A}, \frac{I}{A}, \frac{K}{A}, N$.

As the hint points out, the choice of labor supply is a static period-by-period decision. We can therefore pick an arbitrary period and solve the problem

$$\begin{aligned} &\max_{C_{t+i}, N_{t+i}} \quad \beta^i \left(\log(C_{t+i}) - \nu N_{t+i}\right) \\ \text{s.t.} \quad &C_{t+i} = W_{t+i}N_{t+i} + R_{t+i}K_{t+i} - K_{t+i+1} \end{aligned}$$

When choosing N_{t+i} the household takes prices W_{t+i} and R_{t+i} as given. The capital stock K_{t+i} is given (since it was chosen last period), and the capital stock next period K_{t+i+1} is chosen optimally this period, so with respect to the choice of labor supply it is taken as given.

We can substitute for consumption in the objective function and solve:

$$\max_{N_{t+i}} \quad \beta^i \left(\log(W_{t+i}N_{t+i} + R_{t+i}K_{t+i} - K_{t+i+1}) - \nu N_{t+i}\right)$$

The first order condition w.r.t. N_{t+i} is then:

$$\beta^i \left(\frac{1}{C_{t+i}} W_{t+i} - \nu\right) = 0$$

where I have substituted back in C_{t+i} after taking the derivative. This implies the first order condition for labor supply:

$$\nu C_{t+i} = W_{t+i}$$

The set of equations that determine the balanced growth path with equlibrium values of $\frac{Y}{A}, \frac{C}{A}, \frac{I}{A}, \frac{K}{A}, N$ are then the same as those in part 3 with two additional equations for labor supply and labor demand. These additional equations are:

$$\begin{aligned} \text{Labor supply :} \qquad & \nu \frac{C}{A} = \frac{W}{A} \\ \text{Labor demand :} \qquad & (1-\alpha)\frac{Y}{A}\frac{1}{N} = \frac{W}{A} \end{aligned}$$

We are adding two extra equations that pin down the two extra variables N and $\frac{W}{A}$, but again we could combine them and eliminate $\frac{W}{A}$, since the question does not ask for this variable.

5. **Suppose there are two economies that are identical, except that in economy 1, productivity is a multiple of productivity in country 2, with $A_t^1 = \theta A_t^2$., with $\theta > 1$. Suppose the two economies are along balanced growth paths. Then at any time t, how do Y_t^1, C_t^1 and N_t^1 compare with Y_t^2, C_t^2 and N_t^2.? (Hint: you should be able to answer this questions mostly by inspecting your answer to the previous question.)**

   From the above question we have enough equations to solve for the relevant variables along the balanced growth path. But to answer this particular question we don't need to do that. Instead we follow the hint and inspect the equations that determine the balanced growth path values. The key thing to notice is that the equations that pin down $\frac{Y}{A}, \frac{C}{A}, \frac{I}{A}, \frac{K}{A}, N$ (and R, $\frac{W}{A}$), is that they are the same equations for both countries when the only difference between the coutries is that $A_t^1 = \theta A_t^2$. The two countries therefore have the same values of $\frac{Y}{A}, \frac{C}{A}, \frac{I}{A}, \frac{K}{A}, N$ (and R, $\frac{W}{A}$). Hence, we have that $Y_t^1 = \theta Y_t^2$, $C_t^1 = \theta C_t^2$ and $N_t^1 = N_t^2$.