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Intermediate Macroeconomic Theory
Spring 2009
Feb., 1

Lecture 6

Economic Growth

1 Basic Determinants of Growth

1.1 Growth Accounting

Let output be determined by the following constant returns to scale production function, where Y_t is output, K_t is capital and A_t is labor-augmenting technological change.

$$Y_t = (K_t)^{\alpha}(A_t N_t)^{1-\alpha} \tag{1}$$

Taking logs of each side:

$$\log Y_t = (1-\alpha)\log A_t + \alpha \log K_t + (1-\alpha)\log N_t \tag{2}$$

and then taking first differences yields

$$\log Y_t - \log Y_{t-1} = (1-\alpha)(\log A_t - \log A_{t-1}) + \alpha(\log K_t - \log K_{t-1}) + (1-\alpha)(\log N_t - \log N_{t-1}) \tag{3}$$

It the time period is not too long we can approximate the log difference of a variable by the percentage change:

$$\frac{\Delta Y}{Y} = (1-\alpha)\frac{\Delta A}{A} + \alpha\frac{\Delta K}{K} + (1-\alpha)\frac{\Delta N}{N}$$

Note that even though A_t is labor-augmenting technological change, we still refer to it as total factor productivity because a percentage change yields a percentage change in output that is independent of factor inputs.

The growth rate of output thus depends on three factors: the growths rate of productivity, capital and labor.

It is useful to distinguish between total factor productivity and average labor productivity, $\frac{Y}{N}$.

$$\frac{\Delta Y}{Y} - \frac{\Delta N}{N} = (1-\alpha)\frac{\Delta A}{A} + \alpha(\frac{\Delta K}{K} - \frac{\Delta N}{N})$$

or equivalently:

$$\frac{\Delta(Y/N)}{Y/N} = (1-\alpha)\frac{\Delta A}{A} + \alpha\frac{\Delta(K/N)}{K/N}$$

Thus far we do not have a complete model because we have said nothing about how the three determinant of growth evolve. Before completely the model, we first analyze the balanced growth path, which can be thought of us defining the economies low frequency trend.

We first assume that

$$\frac{\Delta A}{A} = a$$

$$\frac{\Delta N}{N} = n$$

Productivity grows at the rate a and the labor force grows at the rate n. It follows that the growth rate of the effective labor force, AN is

$$\frac{\Delta AN}{AN} = a + n$$

1.2 Steady State and the Golden Rule

In steady state all quantity variables grow at the rate AN. We refer to this kind of steady state as a balanced growth path. In particular it is convenient to define the normalize variables $\frac{Y}{AN}$, $\frac{C}{AN}$, $\frac{I}{AN}$, $\frac{K}{AN}$ and $\frac{G}{AN}$, all of which have the property that they are constant along a balanced growth path. Along a balance growth path the following relations must hold

$$Y = C + I + G$$

$$K^{'} = I + (1-\delta)K$$

$$Y = K^{\alpha}(AN)^{1-\alpha}$$

which we can rewrite as

$$\frac{Y}{AN} = \frac{C}{AN} + \frac{I}{AN} + \frac{G}{AN}$$

We assume the government adjusts $\frac{G}{AN}$ to keep it constant. Similarly, the evolution of capital may be expressed as:

$$\frac{K^{'}}{A'N'}\frac{A'N'}{AN} = \frac{I}{AN} + (1-\delta)\frac{K}{AN}$$

$$\frac{Y}{AN} = (\frac{K}{AN})^{\alpha} \tag{4}$$

Along a balanced growth path:

$$\frac{K^{'}}{A'N'} = \frac{K}{AN}$$

which implies

$$\frac{I}{AN} = (\frac{\Delta A}{A} + \frac{\Delta N}{N} + \delta)\frac{K}{AN} \tag{5}$$

$$\begin{aligned} \frac{C}{AN} &= \frac{Y}{AN} - \frac{I}{AN} - \frac{G}{AN} \qquad (6) \\ &= (\frac{K}{AN})^{\alpha} - (\frac{\Delta A}{A} + \frac{\Delta N}{N} + \delta)\frac{K}{AN} - \frac{G}{AN} \\ &= (\frac{K}{AN})^{\alpha} - (a + n + \delta)\frac{K}{AN} - \frac{G}{AN} \end{aligned}$$

Note that we have four unknowns - $\frac{Y}{AN}, \frac{K}{AN}, \frac{I}{AN}$, $\frac{C}{AN}$ - and only three independent equations - (4), (5), and (6). Since we have not said anything about what determines consumption/saving behavior, the model is not comptete. Put differently, given $\frac{K}{AN}$,have enough to determine $\frac{C}{AN}$, $\frac{Y}{AN}$, and $\frac{I}{AN}$. To determine the former along with the latter, we need to say something about consumption/saving.

In the meantime, we can ask what value of $\frac{K}{AN}$ maximizes $\frac{C}{AN}$. Maximizing with respect to $\frac{K}{AN}$ yields the "golden rule"

$$[(\alpha\frac{K}{AN})^{\alpha-1} - \delta] = a + n$$

Consumption per capita maximized when $\frac{K}{N}$ adjusts to the point where the net marginal return to capital equals the population growth. beyond this point the economy has to save too much to maintain the per capita capital stock. below it, the return to capital is sufficiently high to have additional saving increase per capita consumption on net. note that at the golden rule the net interest rate equals the population growth rate.

1.3 A competitive equilibrium model of growth

We now examine how growth is determined in a decentralized competitive equilibrium.

Suppose there is a representative family with the following objective. Note that the horizon is infinite but the future is discounted by the preference parameter β^i. We assume that labor supply is exogenous but that within the family the supply of labor grows at the rate n.

$$\max \sum_{i=0}^{\infty} \beta^i \log(C_{t+i})$$

The budget constraint is given by

$$\begin{aligned} C_t &= W_t N_t + (R_t - 1)K_t - (K_{t+1} - K_t) \\ &= W_t N_t + R_t K_t - K_{t+1} \end{aligned}$$

First Order Necessary Conditions:

$$C_t^{-1} = R_{t+1}\beta C_{t+1}^{-1}$$

Assuming perfect capital markets, we can collapse the sequence of single period budget constraints into a single intertemporal budget constraint:

$$C_t + \sum_{i=1}^{\infty} C_{t+i}\Pi_{j=1}^{i}(\frac{1}{R_{t+j}}) = R_t K_t + W_t N_t + \sum_{i=1}^{\infty} W_{t+i} N_t (1+n)^i \Pi_{j=1}^{i}(\frac{1}{R_{t+j}})$$

$$C_{t+i} = \Pi_{j=1}^{i} R_{t+i}\beta^i C_t$$

$$C_t \sum_{i=0}^{\infty} \beta^i = R_t K_t + W_t N_t + \sum_{i=1}^{\infty} W_{t+i} N_t (1+n)^i \Pi_{j=1}^{i}(\frac{1}{R_{t+j}})$$

$$C_t = (1-\beta)[R_t K_t + W_t N_t + \sum_{i=1}^{\infty} W_{t+i} N_t (1+n)^i \Pi_{j=1}^{i}(\frac{1}{R_{t+j}})]$$

Firms:

$$Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha}$$

$$Y_t + (1-\delta)K_t - W_t N_t - R_t K_t$$

F.O.N.C:

$$\max K_t^{\alpha}(A_t N_t)^{1-\alpha} + (1-\delta)K_t - W_t N_t - R_t K_t$$

:

$$\alpha(\frac{K_t}{A_t N_t})^{\alpha-1} + (1-\delta) = R_t$$

$$(1-\alpha)A_t(\frac{K_t}{A_t N_t})^{\alpha} = W_t$$

Resource Constraints

$$\frac{Y_t}{A_t N_t} = \frac{C_t}{A_t N_t} + \frac{I_t}{A_t N_t} + \frac{G_t}{A_t N_t}$$

$$
\begin{aligned}
\frac{K_{t+1}}{A_{t+1}N_{t+1}}\frac{A_{t+1}N_{t+1}}{A_tN_t} &= \frac{I_t}{A_tN_t} + (1-\delta)\frac{K_t}{A_tN_t} \\
\frac{K_{t+1}}{A_{t+1}N_{t+1}}(1+a+n) &= \frac{I_t}{A_tN_t} + (1-\delta)\frac{K_t}{A_tN_t}
\end{aligned}
$$

Steady state:

$$
\begin{aligned}
\frac{C_{t+1}}{C_t} &= R\beta \\
\frac{C_{t+1}/A_{t+1}N_{t+1}}{C_t/A_tN_t}\frac{A_{t+1}N_{t+1}}{A_tN_t} &= R\beta \\
1+a+n &= R\beta \\
R &= (1+a+n)\beta^{-1}
\end{aligned}
$$

$$
\alpha(\frac{K}{AN})^{\alpha-1} + (1-\delta) = R = (1+a+n)\beta^{-1}
$$

$$
\frac{Y}{AN} = (\frac{K}{AN})^{\alpha}
$$

$$
\frac{C}{AN} = (\frac{K}{AN})^{\alpha} - (a+n+\delta)\frac{K}{AN} - \frac{G}{AN}
$$

$$
A_t(\frac{K}{AN})^{\alpha} = W_t
$$

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