ECON-UB 233 Dave Backus @ NYU

Lab Report #6: Options & Volatility

Revised: November 7, 2013

Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Please include your Matlab code.

I suspect some people will find the programming parts of this somewhat demanding. If so, do questions 1 and 2 and stop. And come see me if you have questions.

Solution: Brief answers follow, but see also the attached Matlab program; download the pdf, open, click on pushpin:

1. *Root-finding.* Consider the function $f(x) = x\cos(x)$. Our mission is to find solutions x^* between one and two for which $f(x) = 0$.

   (a) Plot f against a grid of points x between one and two. How many solutions do you see?

   (b) Write or adapt a Newton's method program to find a solution. What is it?

   (c) Change the function to $f(x) = x\cos(x) - c$. Modify your program to find solutions for $c = (1/2, 0, -1/2)$. What are they? Can you get your program to find them all at once?

   Comment: I find it useful to define f as a so-called "anonymous function" in Matlab:
   `f = @(x) x.*cos(x) - c`
   The first part (`f = @(x)`) defines f as a function of x. The rest gives us the function. The `.*` operator allows this to work for vectors, too. Doing it this way allows us to define the function once, then write `f(x)` whenever we want to use it. For examples, see the Matlab code posted on the course outline.

   Solution:

   (a) The function is monotonic in this interval and crosses the axis once.

   (b) The solution is 1.5708.

   (c) For the vector of values for c, the solutions are $(1.0980, 1.5708, 1.8452)$.

2. *Black-Scholes-Merton formula.* We'll examine the BSM formula in some numerical examples. In what follows, the current price of the underlying is 102, the option maturity is one year, and the one-year bond price is 0.95.

(a) If volatility $\sigma = 0.10$, what are the prices of call options at strike prices of 90, 100, and 110? Why do they decline with the strike price?

(b) What are the prices of put options with the same strikes?

(c) If volatility rises to $\sigma = 0.125$, what happens to the prices of calls?

(d) For strikes of 90, 100, and 110, graph the call price against volatility σ using a grid between (roughly) 0.01 and 0.50. (This gives you three lines, one for each strike.) How do call prices vary with volatility? Does the pattern vary with the strike price?

(e) For a strike of 110 and a call price of 3.50, what is the implied value of σ?

Solution:

(a) Call prices are 16.65, 8.38, and 2.99 at strikes of 90, 100, and 110, resp. They decline because the cash flows are lower at all outcomes s_{t+1}. [Draw a graph of $(s_{t+1} - k)^+$ v. s_{t+1} for two values of k.]

(b) For put options, the easiest route is to use put-call parity. The put prices are 0.15, 1.38, and 5.49.

(c) Calls rise to 16.92, 9.18, and 3.99. Why rise? Options prices are increasing in σ. We can show this by differentiating the function or by plotting it, as we do next.

(d) See the Matlab program. You see that call prices increase with volatility in all cases. (Puts, too, for that matter.) The relation is close to linear except for very small values of σ. (Think about the value of an option as σ approaches zero.)

(e) From the values computed for the figure, $\sigma = 0.113$ is about it.

3. *Volatilities for S&P 500 E-mini options.* For E-mini options, the prices are more conveniently expressed in terms of their implied volatilities. We'll compute them here for quotes reported on March 15, 2012:

	Call Price		Put Price	
Strike	Bid	Ask	Bid	Ask
1340	82.50	85.75	28.25	30.25
1350	75.25	78.25	30.75	32.75
1360	68.00	71.00	33.25	35.50
1370	61.25	64.00	36.25	38.75
1380	54.50	57.25	39.50	42.25
1390	48.25	50.75	43.25	45.75
1400	42.25	44.75	47.00	50.00
1410	36.75	39.25	51.25	54.50
1420	31.75	33.75	56.00	59.25
1430	27.00	29.00	61.25	64.50

The price of the underlying contract was 1395.75. The interest rate was essentially zero, so the appropriate bond price was one. The options expire June 15, so $\tau = 3/12 = 1/4$.

(a) Compute "mid" quotes as averages of bid and ask. Use put-call parity to compute call prices from mid puts. Plot call prices — bid, ask, and implied by puts — against the strike. How do they compare?

(b) Write a Newton's method program to compute implied volatilities for mid quotes of call options. Graph them against the strike. What shape does the resulting "smile" have? What does the shape suggest to you about the risk-neutral probabilities?

Comments:

- *If you're lazy like me, you can look up the derivative in Wikipedia. They refer to the derivative of the call price with respect to volatility as "vega." You can calculate the derivative with an anonymous function, too, just as we did in the Newton's method example posted on the course outline.*
- *I use this two-step definition of the BSM formula:*

```
d = @(sigma,k) (log(s./(q_tau.*k))+tau*sigma.^2/2)./(sqrt(tau)*sigma);
f = @(sigma,k) q*normcdf(d(sigma,k)) - ...
        q_tau.*k.*normcdf(d(sigma,k)-sqrt(tau)*sigma) - call_mid;
fp = @(d) s*sqrt(tau)*exp(-d.^2/2)/sqrt(2*pi);
```

  Here we've subtracted the call price from the formula to give us a function `f` *whose value is zero when we find the right volatility.*
- *We can do the same with the vega:*

```
fp = @(d) s*sqrt(tau)*exp(-d.^2/2)/sqrt(2*pi);
```

Solution:

(a) Run the Matlab program for the figure. Call prices computed from mid puts (asterisks) are within the bid-ask spread for call prices, so evidently people in this market understand the arbitrage possibilities from violations of put-call parity.

(b) This is a more involved calculation. The idea is to use a vectorized version of Newton's method to compute implied volatilities from call prices. One useful input is the derivative of the call price with respect to volatility:

$$\partial q^c / \partial \sigma \quad = \quad s \; p(d) \; \tau^{1/2},$$

where $p(d) = (2\pi)^{-1/2} \exp(-d^2/2)$ is the normal pdf and d is the usual component of the BSM formula. You can look this up in Wikipedia.

In the figure plotted by the Matlab program, we see that implied vols decline with strike. That means prices at low strikes are relatively more expensive than the BSM formula with constant σ would suggest. If you did this for a broader range of maturities, you would also see some convexity in the smile.