

Quiz #1

Spring 2013

Please write your name below, then complete the exam in the space provided. There are FOUR questions. You may refer to one page of notes: standard paper, both sides, any content you wish.

(Name and signature)

1. *Moments, cumulants, and generating functions (20 points).* Consider an arbitrary random variable x .
 - (a) Define the moment generating function of x . How is the cumulant generating function related to it? (5 points)
 - (b) How is the cumulant generating function of $y = \alpha + \beta x$ related to the cgf of x ? (5 points)
 - (c) What is the second central moment μ_2 of x ? How is it connected to the moment generating function? (5 points)
 - (d) What is the third cumulant κ_3 of x ? How is it connected to the cumulant generating function? (5 points)

Solution:

- (a) The mgf h is defined by: $h(s) = E(e^{sx})$. The cgf is its log: $k(s) = \log h(s)$.
- (b) The mgf of y is $h_y(s) = E(e^{sy}) = E(e^{s(\alpha + \beta x)}) = e^{s\alpha} h_x(\beta s)$. The cgf is its log: $k_y(s) = s\alpha + k_x(\beta s)$.
- (c) The second central moment is the variance. If the mean is \bar{x} , it's $\text{Var}(x) = E(x - \bar{x})^2 = E(x^2) - \bar{x}^2$. $E(x^2)$ is the second raw or noncentral moments, and \bar{x} is the first. They are the second and first derivatives, resp, of the mgf, evaluated at $s = 0$.
- (d) The third cumulant is the third derivative of the cgf, also evaluated at $s = 0$. It's also the third central moment.

2. *Risk and return (30 points).* Consider an agent with utility

$$U = E[u(c)]$$

where $u(c) = c^{1-\alpha}/(1-\alpha)$ for some $\alpha > 0$. She invests one and consumes the gross return r .

- (a) What is her expected utility if she invests everything in a riskfree asset whose (gross) return is 1.1? (Her consumption is therefore 1.1 in every state.) What is the certainty equivalent of this outcome? (10 points)
- (b) What is her expected utility if she invests in an asset whose return is lognormal: $\log r \sim \mathcal{N}(\kappa_1, \kappa_2)$? What is her certainty equivalent? (10 points)
- (c) For what values of κ_1 and κ_2 is the risky asset preferred? (10 points)

Solution:

- (a) The certainty equivalent of a sure thing \bar{c} is \bar{c} . More formally, if consumption is \bar{c} in all states, then the certainty equivalent μ solves

$$U(\bar{c}, \bar{c}, \dots, \bar{c}) = U(\mu, \mu, \dots, \mu),$$

so $\mu = \bar{c}$. So the certainty equivalent here is 1.1.

- (b) We're using properties of lognormal random variables here. We know $E(r^{1-\alpha}) = \exp[(1-\alpha)\kappa_1 + (1-\alpha)^2\kappa_2/2]$. The certainty equivalent is $\mu = E(r^{1-\alpha})^{1/(1-\alpha)} = \exp[\kappa_1 + (1-\alpha)\kappa_2/2]$.
- (c) Evidently we need $\kappa_1 + (1-\alpha)\kappa_2/2 > \log 1.1$. So large κ_1 helps. If $\alpha > 1$, small κ_2 helps, too, otherwise the reverse.

3. *Securities and returns (30 points)*. Consider an economy with two assets and two equally likely states. The assets have dividends

Asset	State 1	State 2
1 ("bond")	1	1
2 ("equity")	2	5

The prices of the two assets are $q^1 = 0.7$ and $q^e = 2$.

- (a) What is the mean return on Asset 1? Asset 2? The risk premium on Asset 2? (15 points)
- (b) How can you decompose each asset into Arrow securities? What are the implied prices of Arrow securities? (15 points)

Solution:

- (a) The bond has a sure return $r^1 = 1/0.7 = 1.43$. Equity has returns

$$r^e(z) = \begin{cases} 2/2 = 1 & \text{for } z = 1 \\ 5/2 = 2.5 & \text{for } z = 2 \end{cases}$$

The states have probability one-half each ("equally likely"), so its expected return is 1.75. Its risk premium is $1.75 - 1.43 = 0.32$.

- (b) The bond pays one in each state, so it consists of the two Arrow securities. Equity pays 2 in state 1 and 5 in state 2, so it consists of 2 units of the state-1 Arrow securities and 5 units of the state-2 Arrow security. If the prices of Arrow securities are denoted by $Q(z)$, then we have

$$\begin{aligned} 0.7 &= Q(1) + Q(2) \\ 2 &= 2Q(1) + 5Q(2). \end{aligned}$$

That gives us $Q(1) = 0.5$ and $Q(2) = 0.2$.

4. *Saving and investment (20 points)*. Consider the Pareto problem of choosing (c_0, k) to maximize

$$U = u(c_0) + \beta \sum_z p(z) u[c_1(z)],$$

subject to the resource constraints

$$\begin{aligned} c_0 + k &\leq y_0 \\ c_1(z) &\leq zf(k). \end{aligned}$$

There is one of the second constraint for each state z . Here k is capital — plant and equipment — produced at date 0 and used to produce output $zf(k)$ at date 1. The amount of output is random and depends on the state z .

- (a) What is the associated Lagrangian? (10 points)
 (b) What are the first-order conditions for c_0 and k ? (10 points)

Solution:

- (a) If we use q_0 and $q_1(z)$ as multipliers on the constraints, the Lagrangian is

$$\mathcal{L} = u(c_0) + \beta \sum_z p(z) u[c_1(z)] + q_0(y_0 - c_0 - k) + \sum_z q_1(z)[zf(k) - c_1(z)].$$

- (b) The first-order conditions are

$$\begin{aligned} c_0 : \quad & u'(c_0) - q_0 = 0 \\ k : \quad & -q_0 + \sum_z q_1(z)zf'(k) = 0. \end{aligned}$$