Lab Report #6: Sums & Mixtures

(Started: March 26, 2012; Revised: October 31, 2013)

Due at the start of class. You may speak to others, but whatever you hand in should be your own work.

We'll look at a Bernoulli mixture and show that it can generate signficant departures from the lognormal model and the Black-Scholes-Merton formula. The good news is that it's simpler than the Poisson mixture. The bad news is that it doesn't scale easily to different time intervals, so we'll stick with a time interval of one year.

1. (sums and mixtures) Let us say that the log-price of the underlying has two components:

$$\log s_{t+1} \;=\; y_{t+1} \;=\; x_{1t+1} + x_{2t+1},$$

   with (x_{1t+1}, x_{2t+1}) independent. The first component is normal: $x_{1t+1} \sim \mathcal{N}(\mu, \sigma^2)$. The second component, the "jump," is a mixture: with probability $1-\omega$, $x_{2t+1} = 0$, and with probability ω, $x_{2t+1} \sim \mathcal{N}(\theta, \delta^2)$.

   With these inputs, the pdf for y is a weighted average of normals:

$$\begin{aligned} p(y) \;=\;& (1-\omega)\cdot(2\pi\sigma^2)^{-1/2}\exp[-(y-\mu)^2/2\sigma^2] \\ & + \omega\cdot[2\pi(\sigma^2+\delta^2)]^{-1/2}\exp[-[y-(\mu+\theta)]^2/2(\sigma^2+\delta^2)]. \end{aligned} \tag{1}$$

   If $\omega = 0$, the second component drops out. Otherwise, we have a weighted average of two normal densities.

   (a) Show that the cumulant generating function for x_{1t+1} is

$$k(s; x_1) \;=\; \mu s + \sigma^2 s^2/2.$$

   (b) Show that the cumulant generating function for x_{2t+1} is

$$k(s; x_2) \;=\; \log\left[(1-\omega) + \omega e^{\theta s + \delta^2 s^2/2}\right].$$

   (c) What is the cgf for y_{t+1}? What are its mean, variance, skewness, and excess kurtosis? What parameters determine the sign of skewness?

   (d) Suppose we started with (1). How do we know it integrates to one? What is its cgf?

2. (Merton-like option pricing) With the same setup, we can illustrate the value of mixtures in generating nonnormal distributions and their impact on option prices and volatility smiles.

(a) Risk-neutral asset pricing tells us, in general, that

$$s_t \;=\; q_t^1 E^*(s_{t+1}) \;=\; q_t^1 E^*\left(e^{y_{t+1}}\right) \;=\; q_t^1 e^{k(1;y)}.$$

We refer to this as the no-arbitrage condition. What is the no-arbitrage condition for our example?

We'll use this condition to set μ: given values for everything else, we'll choose μ to satisfy this condition.

(b) Recall that if the risk-neutral distribution is $\log s_{t+1} = y_{t+1} \sim \mathcal{N}(\kappa_1, \kappa_2)$, then the put price at strike k is

$$\begin{aligned} f(k; \kappa_1, \kappa_2) &= q_t^1 k N(d) - q_t^1 e^{\kappa_1 + \kappa_2/2} N(d - \kappa_2^{1/2}) \\ d &= (\log k - \kappa_1)/\kappa_2^{1/2}. \end{aligned}$$

(Note: this isn't quite the usual d.) What is the call price?

(c) Use (b) to show that the put price in the mixture model is a weighted average:

$$q_t^p \;=\; (1-\omega) \cdot f(k; \mu, \sigma^2) + \omega \cdot f(k; \mu + \theta, \sigma^2 + \delta^2).$$

(d) Consider these inputs: $\sigma = 0.04$, $\omega = 0.01$, $\theta = -0.3$, $\delta = 0.15$, $s_t = 100$, and $q_t^1 = 1$. What is μ? What are the prices of put options with strikes $k = (80, 90, 100, 110, 120)$? (Use a finer grid if you have this automated.) What are the implied volatilities?

(e) What happens to the volatility smile when you set

- $\theta = 0$?
- $\theta = +0.3$?
- $\delta = 0.25$?

Make sure you adjust μ in each case.

Matlab program: