ECON-UB 233 Dave Backus @ NYU

Quiz #2

November 2015

Please write your name below. Then complete the exam in the space provided. There are THREE questions. You may refer to one page of notes: standard paper, both sides, any content you wish.

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(Name and signature)

1. *Representative agent asset pricing.* Consider asset pricing in a two-period economy with a representative agent. The agent has utility function

$$u(c_0) + \beta E u(c_1)$$

with $u(c) = c^{1-\alpha}/(1-\alpha)$. Consumption growth is lognormal: $\log g(z) = \log[c_1(z)/c_0] \sim \mathcal{N}(\kappa_1, \kappa_2)$.

(a) What is the pricing kernel? (5 points)

(b) What is the price of a one-period bond? (5 points)

(c) Consider a claim to the dividend $d(z) = g(z)^\lambda$. What is the value of such a claim? The return? (10 points)

(d) Consider log excess returns

$$\log r(z) - \log r^1.$$

For what value of λ does our asset have the highest expected excess return? How is your answer connected to the entropy of the pricing kernel? (10 points)

Solution: This question uses the lognormal distribution extensively. Specifically, if $\log x \sim \mathcal{N}(\kappa_1, \kappa_2)$ then $E(y) = e^{\kappa_1 + \kappa_2/2}$.

(a) The pricing kernel is $m = \beta g^{-\alpha}$.

(b) The price is $q^1 = E(m) = \beta e^{-\alpha\kappa_1 + \alpha^2\kappa_2/2}$.

(c) More lognormal calculations: $q^e = E(md)$ where

$$\log(md) \quad = \quad \log\beta + (\lambda - \alpha)\log g.$$

Therefore $q^e = \beta e^{(\lambda-\alpha)\kappa_1 + (\lambda-\alpha)^2\kappa_2/2}$. The return is $d/q^e = g^\lambda/q^e$.

(d) The log expected excess return is (after lots of cancelling)

$$E(\log r - \log r^1) \;=\; -\lambda(\lambda - 2\alpha)\kappa_2.$$

The maximum value is at $\lambda = \alpha$, which gives us an expected excess return of $\alpha^2\kappa_2/2$. This is, in fact, the maximum possible log excess return for this pricing kernel, given to us by the entropy bound:

$$E(\log r - \log r^1) \;\leq\; H(m) \;=\; \log E(m) - E(\log m) \;=\; \alpha^2\kappa_2.$$

2. *Two-state option pricing.* Consider a two-period economy in which assets are purchased at date t and pay off at date $t+1$. Payoffs depend on the state z, which takes on the values 1 and 2 with probability one-half each. The inputs include the future value $s(z)$ of a specific asset, which we'll call the underlying:

State z	State Price $Q(z)$	Probability $p(z)$	Future Value $s(z)$
1	2/3	1/2	10
2	1/3	1/2	20

(a) What is the pricing kernel in each state? (5 points)

(b) What is the price of a one-period riskfree bond? (5 points)

(c) What is the price of a claim to the future value of the underlying? (5 points)

(d) What is the cash flow of an option to buy the underlying at date $t+1$ for price $k = 15$? Of an option to sell the underlying at the same price? (10 points)

(e) What are the prices of the two options? (10 points)

(f) Verify that put-call parity works in this setting. (10 points)

(g) In what sense might you say that option prices can be used to recover state prices? (5 points)

Solution:

State z	State Price $Q(z)$	Probability $p(z)$	Future Value $s(z)$	Pr Kernel $m(z)$	Risk-N Prob $p^*(z)$
1	2/3	1/2	10	4/3	2/3
2	1/3	1/2	20	2/3	1/3

(a) The pricing kernel m is the solution to $Q(z) = p(z)m(z)$ in each state z; in other words, $m(z) = Q(z)/p(z)$. See the table.

(b) The price is $q^1 = Q(1) + Q(2) = 1$. (You could also find m and compute it as $E(m)$, but this is easier.)

(c) The price is $s_t = Q(1) \cdot 10 + Q(2) \cdot 20 = 40/3$.

(d) The cash flow from the option to buy is $(s(z) - k)^+$, which gives us 5 in state 2 and nothing in state 1. The cash flow from the option to sell is $(k - s(z))^+$ which gives us 5 in state 1 and nothing in state 2.

(e) The prices are

$$\begin{aligned} q_t^c &= Q(2) \cdot (20 - 15) = 5/3 \\ q_t^p &= Q(1) \cdot (15 - 10) = 10/3. \end{aligned}$$

(f) The parity relation is

$$\begin{aligned} q_t^c - q_t^p + q^1 k &= s_t \\ 5/3 - 0/3 + 15 &= 40/3, \end{aligned}$$

which works with our numbers.

(g) Here we see that the options pay off in one state only. So we might imagine recovering them from the option prices. More generally, we might imagine doing the same thing if we had a fine enough grid of options to work with.

3. *Risk-neutral probabilities.* Using the same data as the previous question:
   (a) What are the risk-neutral probabilities? (5 points)
   (b) What is the return on the risk-free bond? (5 points)
   (c) What are the returns in each state of a claim to the future value of the underlying? (10 points)
   (d) What is the expected return on the underlying if we use the risk-neutral probabilities? Explain. (10 points)

Solution:

(a) Risk-neutral probabilities p^* satisfy $Q(z) = q^1 p^*(z)$. Here $q^1 = 1$, so $p^*(z) = Q(z)$ for each state z.

(b) Return is $r^1 = 1/q^1 = 1$.

(c) Returns are

$$\begin{aligned} r(1) &= 10/(40/3) = 3/4 \\ r(2) &= 20/(40/3) = 3/2. \end{aligned}$$

(d) The expected return using risk-neutral probabilities is 1. This is the same as the return on the riskfree bond. That's a feature of expected returns based on risk-neutral probabilities: they're all the same.