# Quiz #3

(Revised: December 5, 2013)

*Please write your name below. Then complete the exam in the space provided. There are TWO questions. You may refer to one page of notes: standard paper, both sides, any content you wish.*

(Name and signature)

1. (moving averages) (50 points) Consider the MA(2),

$$x_t = \delta + w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2},$$

with $\{w_t\} \sim \text{NID}(0, 1)$ (the $w$'s are independent normals with mean zero and variance one). Our mission is to explore its properties.

(a) What is the mean of $x$? The variance? (10 points)

(b) What are the conditional means, $E_t(x_{t+1})$, $E_t(x_{t+2})$, and $E_t(x_{t+3})$? (10 points)

(c) What are the conditional variances, $\text{Var}_t(x_{t+1})$, $\text{Var}_t(x_{t+2})$, and $\text{Var}_t(x_{t+3})$? (10 points)

(d) What is the autocovariance function,

$$\gamma(k) = \text{Cov}(x_t, x_{t-k}),$$

for $k = 0, 1, 2, 3$? (10 points)

(e) What is the autocorrelation function? Under what conditions are $\rho(1)$ and $\rho(2)$ positive? (10 points)

**Solution:**

(a) The mean is $\delta$,

$$E(x_t) = E(\delta + w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2}) = \delta.$$

The variance is

$$\text{Var}(x_t) = E(x_t - \delta)^2 = 1 + \theta_1^2 + \theta_2^2.$$

(b) The conditional means are

$$\begin{aligned}
E_t(x_{t+1}) &= E_t(\delta + w_{t+1} + \theta_1 w_t + \theta_2 w_{t-1}) &= \delta + \theta_1 w_t + \theta_2 w_{t-1} \\
E_t(x_{t+2}) &= E_t(\delta + w_{t+2} + \theta_1 w_{t+1} + \theta_2 w_t) &= \delta + \theta_2 w_t \\
E_t(x_{t+3}) &= E_t(\delta + w_{t+3} + \theta_1 w_{t+2} + \theta_2 w_{t+1}) &= \delta.
\end{aligned}$$

You can see that as we increase the forecast horizon, the conditional mean approaches the mean.

(c) The conditional variances are

$$\begin{aligned}
\mathrm{Var}_t(x_{t+1}) &= E_t[(w_{t+1})^2] &= 1 \\
\mathrm{Var}_t(x_{t+2}) &= E_t[(w_{t+2} + \theta_1 w_{t+1})^2] &= 1 + \theta_1^2 \\
\mathrm{Var}_t(x_{t+3}) &= E_t[(w_{t+3} + \theta_1 w_{t+2} + \theta_2 w_{t+1})^2] &= 1 + \theta_1^2 + \theta_2^2.
\end{aligned}$$

You see here that as we increase the forecast horizon, the conditional variance approaches the variance.

(d) The autocovariance function is

$$\mathrm{Cov}(x_t, x_{t-k}) = \begin{cases} 1 + \theta_1^2 + \theta_2^2 & k = 0 \\ \theta_1 + \theta_1\theta_2 & k = 1 \\ \theta_2 & k = 2 \\ 0 & k \geq 3. \end{cases}$$

(e) Autocorrelations are scaled autocovariances: $\rho(k) = \gamma(k)/\gamma(0)$. $\rho(2)$ is positive if $\theta_2$ is. $\rho(1)$ is positive if $\theta_1(1+\theta_2)$ is. Both are therefore positive if $\theta_1$ and $\theta_2$ are positive.

2. (moving average bond pricing) (50 points) Consider the bond pricing model

$$\begin{aligned}
\log m_{t+1} &= -\lambda^2/2 - x_t + \lambda w_{t+1} \\
x_t &= \delta + \sigma(w_t + \theta w_{t-1}).
\end{aligned}$$

(a) What is the short rate $f_t^0$? (10 points)

(b) Suppose bond prices take the form

$$\log q_t^n = A_n + B_n w_t + C_n w_{t-1}.$$

Use the pricing relation to derive recursions connecting $(A_{n+1}, B_{n+1}, C_{n+1})$ to $(A_n, B_n, C_n)$. What are $(A_n, B_n, C_n)$ for $n = 0, 1, 2, 3$? (20 points)

(c) Express forward rates as functions of the state $(w_t, w_{t-1})$. What are $f_t^1$ and $f_t^2$? (10 points)

(d) What is $E(f^1 - f^0)$? What parameters govern its sign? (10 points)