

**Quiz #1**

Revised: October 3, 2014

Please write your name below, then complete the exam in the space provided. You may refer to one page of notes: standard paper, both sides, any content you wish. **There are FOUR questions.**

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(Name and signature)

1. *Linear transformations.* Consider the random variable  $x$ , which is normal with mean  $\mu$  and standard deviation  $\delta$ .
  - (a) What are the variance, skewness, and excess kurtosis of  $x$ ? (10 points)
  - (b) What are  $x$ 's moment generating function (mgf) and cumulant generating function (cgf)? What is the expectation of  $e^x$ ? (10 points)
  - (c) What is the cgf of  $y = a + bx$ ? (10 points)
  - (d) What is  $y$ 's distribution? What are its mean, variance, skewness, and excess kurtosis? (10 points)

**Solution:**

- (a) The variance is the standard deviation squared,  $\delta^2$ . Since  $x$  is normal, its skewness and excess kurtosis are zero.
- (b) The mgf is  $h(s) = E(e^{sx}) = e^{\mu s + \delta^2 s^2/2}$ . The cgf is the log of this:  $k(s) = \log h(s) = \mu s + \delta^2 s^2/2$ . The expectation of  $e^x$  is  $h(1) = e^{\mu + \delta^2/2}$ .
- (c) If we run through the definitions, we see  $k_y(s) = as + k_x(bs) = (a + b\mu)s + (b\delta)^2 s^2/2$ .
- (d) From the form of the cgf (quadratic in  $s$ ), we see that  $y$  is normal. Its mean is  $a + b\mu$  (the linear term) and its variance is  $(b\delta)^2$  (the quadratic term). And since it's normal, skewness and excess kurtosis are still zero.

2. *Option payoffs.* An option on the random variable  $y$  has a payoff of  $f(y) = \max\{0, y - k\}$  for some strike price  $k$ .

Consider random variables  $y$  constructed from these independent inputs:

- $x_1$  takes on the values  $\{4, 6\}$  with probability one-half each.
- $x_2$  takes on the values  $\{-2, +2\}$  with probability one-half each.

We'll explore the expected payoff of options with strike price  $k = 5$  and two versions of  $y$ :  $y = x_1$  and  $y = x_1 + x_2$ .

- (a) What is the expected value of  $y = x_1$ ? (5 points)
- (b) What is the expected option payoff — that is,  $E(f(y))$  — in this case? How does it compare to  $f[E(y)]$ ? What mathematical result does this illustrate? (10 points)
- (c) What values does  $y = x_1 + x_2$  take on? With what probabilities? What is its expected value? (10 points)
- (d) What is the expected option payoff in this case? (5 points)
- (e) How does the option payoff differ when  $y = x_1$  and  $y = x_1 + x_2$ ? What mathematical result does this illustrate? (10 points)

**Solution:**

- (a)  $E(y) = (1/2)(4) + (1/2)(6) = 5$ .
- (b)  $E[f(y)] = (1/2)(0) + (1/2)(1) = 1/2$ .  $f[E(y)] = 0$ . The former is greater because  $f$  is a convex function: Jensen's inequality in action.
- (c)  $y$  takes on the values  $\{2, 4, 6, 8\}$  with probability one-fourth each. The expected value is  $E(y) = (1/4)(2 + 4 + 6 + 8) = 5$ . That is: the mean hasn't changed, we've just spread out the distribution a bit. (Draw the two distributions if you like.)
- (d) The expected option payoff is  $E[f(y)] = (1/2)(0) + (1/4)(1) + (1/4)(3) = 1$ .
- (e) This is greater than the option payoff with  $y = x_1$  because we've spread out the distribution. We don't have a name for this, but an increase in the variance holding constant the mean increases the expected value of a convex function.

3. *State prices.* Suppose three assets have dividends

Asset	State 1	State 2	State 3
Asset A	1	1	1
Asset B	1	2	1
Asset C	1	1	3

If the prices of assets are  $q^A = 11/12$  and  $q^B = q^C = 13/12$ , what are the state prices? (15 points)

**Solution:** The asset payoffs are connected to state prices  $Q(z)$  by

$$\begin{aligned} q^A &= Q(1) + Q(2) + Q(3) \\ q^B &= Q(1) + 2Q(2) + Q(3) \\ q^C &= Q(1) + Q(2) + 3Q(3), \end{aligned}$$

which we can solve for the  $Q$ 's. The solution is  $Q(1) = 8/12$ ,  $Q(2) = 2/12$ , and  $Q(3) = 1/12$ .

4. *Pareto problem.* Consider a theoretical economy with these ingredients:

- One agent.
  - Two goods, apples and bananas.
  - Utility function:  $\log c_a + \log c_b$ .
  - Endowment: 6 apples, 4 bananas.
  - Technology: apples can be converted to bananas one for one.
- (a) What are the resource constraints for this economy? (5 points)
- (b) What is the Pareto problem? (5 points)
- (c) What is the solution to the Pareto problem? What are the optimal quantities? What are the implied prices? (15 points)

**Solution:**

- (a) If we let  $a$  be the number of apples we convert to bananas, the resource constraints for apples and bananas are

$$\begin{aligned} c_a + a &\leq y_a = 6 \\ c_b &\leq y_b + a = 4 + a. \end{aligned}$$

We might also say  $a \geq 0$  if we're worried about the reverse technology: converting bananas to apples. We'll ignore that from here on.

- (b) The Pareto problem is: maximize utility subject to the resource constraints. The Lagrangian is

$$\mathcal{L} = \log c_a + \log c_b + \lambda_a(y_a - c_a - a) + \lambda_b(y_b + a - c_b).$$

The first-order conditions (the derivatives of  $\mathcal{L}$ ) include

$$\begin{aligned} c_a : \quad 1/c_a - \lambda_a &= 0 \\ c_b : \quad 1/c_b - \lambda_b &= 0 \\ a : \quad -\lambda_a + \lambda_b &= 0. \end{aligned}$$

That gives us  $\lambda_a = \lambda_b$  and (therefore)  $c_a = c_b$ . From the resource constraints we have  $c_a = a_b = 5$  and  $a = 1$ . The relative price of bananas comes from the Lagrange multipliers:  $q_b/q_a = \lambda_b/\lambda_a = 1$ .