Professor Mark Gertler
Intermediate Macroeconomic Theory
Spring 2011
Feb., 9

Lecture 4

Two-Period Competitive Equilibrium Model: Part 1

In the last lecture we developed a static compeitive equilibrium model of output and employment. Firms in the economy only produced consumption goods. Now we consider a two period model with investment goods as well as consumption. In the competiive equilibrium, the allocation of output between consumption and investment is determined, as well as the overall level of output and employment. In this lecture, we consider an economy with capital as the only productive input. We subsequently add labor. I stress that what I am presenting now are models of long run equilibrium behavior (or trend behavior) as opposed to business cycles.

1 Environment

Assume:

(i) Two periods: 1 and 2.

(ii) One representative household that: consumes, saves and receives dividend income (from ownership of firms.)

(iii) One representative firm that produces output using capital, invests in new capital, and pays dividends to households.

(iv) The household and the firm act competitively, i.e., each takes market prices as given

(v) Physical capital is the only productive input..

We next characterize preferences, technology and resource constraints:

Preferences

The household consumes in both periods. Let C_k^i be household i's consumption in period k. Then household preferences are given by

$$\frac{1}{1-\sigma}(C_1^i)^{1-\sigma} + \frac{1}{1-\sigma}\beta(C_2^i)^{1-\sigma} \quad : \quad \text{if } \sigma \neq 1 \tag{1}$$

$$\log C_1^i + \beta \log C_2^i \quad : \quad \text{if } \sigma = 1 \tag{2}$$

with $\sigma > 0$ and $0 < \beta < 1$. The parameter β is known as the household's subjective discount factor. It reflects how the household weighs consuming in the future relative to consuming today.

Technology

Firm j produces output in period 1 using capital input, K_1^j, and also produces output in period 2 using capital.input K_2^j Captial in period 2 depends upon the amount of period 1 investment, I. Let Y_k^j be output in period k and A_k total factor productivity. Then each period, production is given by

$$Y_1^j = A_1(K_1^j)^\alpha \tag{3}$$

$$Y_2^j = A_2(K_2^j)^\alpha \tag{4}$$

The link between capital and investment I^j is given by

$$K_2^j = (1-\delta)K_1^j + I^j \tag{5}$$

where δ is the rate of depreciation. Capital in period 2 depends on capital leftover from period 1.plus period 1 investment. Note that we take K_1^j as given in period 1.The cost of investing I^j units is

$$I^j + \frac{c}{2}(\frac{I^j}{K_1^j})^2 K_1^j$$

where I^j is the cost of buying new capital goods (where we have normalized the price of a unit of new capital and unitry) and where $\frac{c}{2}(\frac{I^j}{K_1^j})^2 K_1^j$ reflects costs to the firm of adjusting its capital stock.

Economy-Wide Resource Constraints

In period 1, output is divided between consumption and investment: there is no government and no external sector.

$$Y_1 = C_1 + I \tag{6}$$

In period 2, all output is consumed, as is the capital stock that is left over

$$(1-\delta)K_2 + Y_2 = C_2 \tag{7}$$

(We don't literally mean that households eat capital - rather including $(1-\delta)K_2$ in the resource constraint means that in the final period a unit of capital has the same value to the household as a unit of consumption goods. This is a simple way of assigning of value to capital in the final period.)

2 Household and Firm Behavior

2.0.1 The household decision problem

Let S^i be household i's saving, Π_k^i dividends in period $_k$, R the gross real interest rate (equal to one plus the net interest rate), all in units of consumption goods. Further, we normalize the price of consumption goods at unity. The representative household chooses C_1^i, C_2^i, and S^i,to solve

$$\max \frac{1}{1-\sigma}(C_1^i)^{1-\sigma} + \frac{1}{1-\sigma}\beta(C_2^i)^{1-\sigma} \tag{8}$$

subject to:

$$C_1^i = \Pi_1^i - S^i \tag{9}$$

$$C_2^i = \Pi_2^i + RS^i \tag{10}$$

The household takes as given R, Π_1^i and Π_2^i. Note that S^i may be positive or negative. Negative values of S^i imply borrowing. Implicit in our formulation, however, is the assumption of perfect capital markets; i.e., the households is able to borrow at the same rate R for which it is able to lend.

It is instructive to combine the two period budget constraints given by (9) and (10) into a single intertemporal budget constraint given by

$$C_1^i + \frac{C_2^i}{R} = \Pi_1^i + \frac{\Pi_2^i}{R}. \tag{11}$$

According to equation (11), the households lifetime consumption plan must satisfy the constraint that the present value of consumption is equal to the present value of income. The latter is given by the sum of labor income in the first period and the discounted stream of dividend income. Accordingly, consumption depends on lifetime income as opposed to income in the current period. This implication, however, depends on the assumption of perfect capital markets. What permitted collapsing the period budget constraints into the single intertemporal constraint for negative as well as positive values of S is that the value of R is independent of the sign of S; i.e., the individual can freely borrow or lend at the gross rate R.

To solve the household's decision problem, it is simplest to turn the constrained problem into an unconstrained one by plugging (9) and (10) into (**??**). This is possible as long as the two one-period budget constraints are always binding, so that there is no unused income. This latter condition is ensured by the assumptions we made on the utility function..

The representative household accordingly chooses S to solve

$$\max(\frac{1}{1-\sigma}(\Pi_1^i - S^i)^{1-\sigma} + \beta\frac{1}{1-\sigma}(\Pi_2^i + RS^i)^{1-\sigma},$$

given R, Π_1, and Π_2.

The first order necessary condition with respect to saving is given by:

$$\underbrace{-(C_1^i)^{-\sigma}}_{\text{MC of Savings}} + \underbrace{R\beta(C_2^i)^{-\sigma}}_{\text{MB of Savings}} = 0 \tag{12}$$

The opportunity cost of saving a unit of goods is the marginal utility of consumption. The marginal benefit is the gross real rate times the marginal utility of future consumption (from the standpoint of today.) Again, restrictions on the utility function ensure that the first order condition characterizes an optimum.

Rearranging equation (12) yields a relation between marginal utilities of consumption across time, known as a consumption euler equation:

$$(C_1^i)^{-\sigma} = R\beta(C_2^i)^{-\sigma} \tag{13}$$

Given concavity of the utility function, the consumption euler equation implies that individuals should try to smooth consumption over time. In the limiting case where $R\beta = 1$, it is optimal for individuals to consume the same amount each period.

Note that the consumption euler equation combined with the intertemporal budget constraint determines the optimal lifetime consumption plan, i.e., the optimal values of C_1^i and C_2^i. Note that this simple model captures the key aspects of the life-cycle permanent income hypothesis developed by Friedman and Modigliani. The key aspects are consumption smoothing and the dependence of consumption on lifetime resources.

To obtain an expression in terms of consumption goods, rearrange the consumption euler equation as follows:

$$1/R = \frac{\beta(C_2^i)^{-\sigma}}{(C_1^i)^{-\sigma}}. \tag{14}$$

The left side is the price in units of current consumption of a unit of future consumption (i.e., to buy a unit of second period consumption, the individual must save $1/R$ units of current consumption). The right side is the household's intertemporal marginal rate of substitution - the value the household places on a an additional unit of future consumption in units of current consumption.

Finally, to obtain a saving supply curve, invert equation (14) and combine with the period budget constraints to obtain:

$$R = \frac{(\Pi_1^i - S)^{-\sigma}}{\beta(\Pi_2^i + RS)^{-\sigma}} \tag{15}$$

We define the saving supply curve as combinations of R and S that satisfy equation (15). In analogy to the labor supply curve, there is both a substitution and an income effect of changes in R . A rise in R increases the opportunity cost of saving (the substitution effect). However, it also reduces the marginal value of second period consumption by reducing the marginal utility of second period consumption (the income effect). Given that the intertemporal marginal rate of substitution (the inverse of the right side of equation (15)) is decreasing in S, the substitution effect of a rise in R raises saving, while the income effect decreases it. We will assume the substitution effect dominates, as holds with standard preference specifications. The implication is that the saving supply curve will be upward sloping in (R, S) space.

An Explicit Solution

Our model is simple enough to be able to explicitly solve for consumption, as follows:

From rearranging the first order condition for consumption and saving.

$$C_2^i = (\beta R)^{\frac{1}{\sigma}} C_1^i \tag{16}$$

Note that the gross rate of consumption C_2/C_1 depends positively on the gross real rate of interest R. Intutiively, by making saving more attractive a rise in R tilts the houshold to raise consumption tomorrow relative to consumption today. The parameter σ is known as the intertemportal elasticity of substitution since it governs the response of consumption growth to interest rates. By combing this condition with the household's intertemporal budget constraint, we obtain

$$C_1^i(1 + \frac{1}{R}(\beta R)^{\frac{1}{\sigma}}) = \Pi_1^i + \frac{\Pi_2^i}{R}$$

which reduces to

$$C_1^i = \frac{1}{1+\beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1}} (\Pi_1^i + \frac{\Pi_2^i}{R}) \tag{17}$$

Let us define the households wealth $V_j^i \equiv \Pi_1^i + \frac{\Pi_2^i}{R}$ as the present value of its dividend income. Then we can write:

$$C_1^i = \frac{1}{1+\beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1}} V^i \tag{18}$$

Borrowing Constraints

Consumption depends on wealth as opposed to income. The timing of income receipts does not matter, only the present value. Note that this results depends on the assumption of perfect capital markets. If the household cannot borrow, then it must be the case that $C_1^i \leq \Pi_1$ This constraint is likely to be be binding if Π_1 is low relative to Π_2.

Infinite Horizon

One can show that if the household has an infinitie horizon:

$$C_1^i = (1 - \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1}) V^i$$

with

$$V^i = \sum_{k=0}^{\infty} (R^{-1})^k \Pi_k^i$$

If the period length is a year, then a reasonable value of β is 0.96. If σ is close to unity or C_1^i is close to C_2^i,then $(1 - \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1}) \approx 1 - \beta$. In this case, a dollar increase in household wealth leads to a four percent increase in consumption in a given year This is known as a wealth effect on consumption. The presence of this wealth effect is why there has been concern about the impact of the decline on the stock market on household consumption. Note that thus far we have abstracted from "human wealth", which is the discounted value of labor income.

Taxes

Finally, we can say something about the impact of tax cuts on consumption spending. Let T_1 and T_2 be taxes the individual pays in periods 1 and 2, respectively:

$$C_1^i = \Pi_1^i - S^i - T_1 \tag{19}$$

$$C_2^i = \Pi_2^i + RS^i - T_2 \tag{20}$$

Now consumption depends on the individual's wealth, net the discounted stream of taxes:

$$C_1^i = \frac{1}{1+\beta^{\frac{1}{\sigma}}R^{\frac{1}{\sigma}-1}}[(\Pi_1^i + \frac{\Pi_2^i}{R}) - (T_1 + \frac{T_2}{R})]$$

What matters is the present value of tax cuts, as opposed to the timing. Permanent tax cuts will have a bigger effect than temporary ones. On the other hand, if the household is credit constrained $C_1^i \leq \Pi_1 - T_1$ with the constraint binding, then a temporary tax cut can have a big effect on the household's consumption.

2.0.2 The Firm Decision Problem

The firm maximizes the discounted stream of profits returned to the household. Given that there is no uncertainty, the firm discounts future profits at the rate $1/R$. Accordingly, the firm chooses I, Π_1, and Π_2 to solve

$$\max \Pi_1^j + \frac{\Pi_2^j}{R} \tag{21}$$

subject to:

$$\Pi_1^j = A_1 K_{1s}^{j\alpha} - I - \frac{c}{2}(\frac{I^j}{K_1^j})^2 K_1^j + B^j \tag{22}$$

$$\Pi_2^j = A_2 K_2^{j\alpha} + (1-\delta)K_2^j - RB^j \tag{23}$$

where let B^j is the amount the firm borrows in period 1. taking as given R. R is the opportunity cost of funds that the firms faces in period 1. Further,

$$K_2^j = (1-\delta)K_1^j + I^j$$

.

Again, we can convert the problem into an unconstrained maximization problem by plugging the constraints (22) and (23) into (21). The firm then chooses I to solve

$$\max A_1 K_1^{j\alpha} - I^j - \frac{c}{2}(\frac{I^j}{K_1^j})^2 K_1^j + \frac{1}{R}\{A_2[(1-\delta)K_1^j + I_1^j]^\alpha + (1-\delta)[(1-\delta)K_1^j + I^j]\}$$

The first order necessary conditions for investment is given by:

$$-1 - c\frac{I^j}{K_1^j} + \frac{1}{R}[A_2 \alpha K_2^{j\alpha-1} + 1 - \delta] = 0 \tag{24}$$

Equivalently

$$\frac{I^j}{K_1^j} = \frac{1}{c}\{\frac{[A_2 \alpha K_2^{j\alpha-1} + 1 - \delta]}{R} - 1\} \tag{25}$$

$$\frac{I^j}{K_1^j} = \frac{1}{c}[Q - 1] \tag{26}$$

where $\frac{[A_1 \alpha K_2^{\alpha-1} + 1 - \delta]}{R}$ is the ratio of the marginal value of a unit of capital to the replacement cost and is known and Tobin's Q ratio. Investment depends positively on Tobin's Q. How sensitive investment is to Q depends on adjustmet costs as measured by c.

Note that we can obtain an investment demand curve by rerranging

$$R = \frac{[A_2 \alpha K_2^{j\alpha-1} + 1 - \delta]}{1 + c\frac{I^j}{K_1^j}}$$

Taking into account that K_2^j is increasing in I^j and $\alpha < 1$ yields an inverse relation between I^j and R..

Financial Policy

Note that the firms real investment decision does not depend on how much it borrowed in period 1. B^j drops out of the objective since the amount borrowed in period 1, B^j,is exactly offset by the present value of what the firm owes to pay of the debt in period 2,

$(1/R)RB^j = B^j$..Put differently, under perfect capital markets, it does not matter whether the firm finances investment by retained earnings or borrowing. This is because under perfect capital markets, the firm can borrow and lend freely at the market interest rate R. That is, the opportunity cost to the firm of using retained earnings to finance investment is the same as the cost of borrowing. The irrelevance of corporate financial policy to the firm's real investment decision is an example of the "Miller-Modligliani" theorem. Note that the Miller-Modigliani hypthesis applies only under perfect capital markets.

As we discuss below, if the capital market is imperfect - i.e., the firm faces a higher borrowing rate that lending rate, it's investment may be constrained by it's period 1 earnings.

Dividend Payouts

Under perfect capital markets, the present value of dividend payouts to households does not depend on whether the firm finances investment with retained earnings or borrowing. However, the timing of its dividend payouts does depend on its borrowing patterns. Let us suppose that B^j is given by:

$$B^j = \max\{0,\ I^j + \frac{c}{2}(\frac{I^j}{K_1^j})^2 K_1^j - A_1 K_1^{j\alpha}\} \tag{27}$$

What equation (27) states is that if revenues $A_1 K_1^{j\alpha}$ exceed investment costs $I^j + \frac{c}{2}(\frac{I^j}{K_1^j})^2 K_1^j$, the firm finances investment with retained earnings ($B^j = 0$). On the other hand, if investment costs exceeds revenues, the firm finances any excess by borrowing (i.e., $B^j = I^j + \frac{c}{2}(\frac{I^j}{K_1^j})^2 K_1^j - A_1 K_1^{j\alpha}$).

Under this borrowing policy, dividend payouts are given by

$$\Pi_1^j = \max\{A_1 K_{1s}^{j\alpha} - I - \frac{c}{2}(\frac{I^j}{K_1^j})^2 K_1^j, 0\}$$

$$\begin{aligned}
\Pi_2^j &= A_2 K_2^{j\alpha} \ + (1-\delta)K_2^j \text{ if } B^j = 0 \\
&= A_2 K_2^{j\alpha} + 1 - \delta)K_2^j - R[I^j + \frac{c}{2}(\frac{I^j}{K_1^j})^2 K_1^j - A_1 K_1^{j\alpha}] \text{ if } B^j > 0
\end{aligned}$$

Note that in either case

$$\Pi_1^j + \Pi_2^j/R = A_1 K_{1s}^{j\alpha} - I - \frac{c}{2}(\frac{I^j}{K_1^j})^2 K_1^j + (1/R)[A_2 K_2^{j\alpha} + 1 - \delta)K_2^j]$$

That is, the present value of dividend payouts is independent of firm financial policy. Note also that since households only care about the present value of dividend payments (with perfect capital markets), firm financial policy does not matter in general equilibrium

Imperfect Capital Markets

Suppose the rate at which the firm borrows, R^b, exceeds that rate at which it can lend, R. In this case it matters whether the firm finances investment with retained earnings or by borrowing. Suppose first that the firm needs to borrow in period 1, so that

$$B^j = I^j + \frac{c}{2}(\frac{I^j}{K_1^j})^2 K_1^j - A_1 K_1^{j\alpha}$$

In this instance the firms objective becomes

$$\begin{aligned} \Pi_1^j + \Pi_2^j/R &= \{A_1 K_{1s}^{j\alpha} - I - \frac{c}{2}(\frac{I^j}{K_1^j})^2 K_1^j + B^j\} \\ &+ \frac{1}{R}\{A_2 K_2^{j\alpha} + (1-\delta)K_2^j - R^b B^j\} \end{aligned}$$

Combining equations yields

$$\Pi_1^j + \Pi_2^j/R = \{A_1 K_{1s}^{j\alpha} - \frac{R^b}{R}(I^j + \frac{c}{2}(\frac{I^j}{K_1^j})^2 K_1^j) + \frac{1}{R}\{A_2 K_2^{j\alpha} + (1-\delta)K_2^j\}$$

In this instance the FONC are given by

$$R^b = (\frac{R^b}{R})R = \frac{[A_2 \alpha K_2^{j\alpha - 1} + 1 - \delta]}{1 + c\frac{I^j}{K_1^j}}$$

The cost of capital to the firm is R^b which is equal to the product of the gross credit spread $\frac{R^b}{R}$ and the lending rate R.

During a financial crisis, the gross credit spread $\frac{R^b}{R}$ goes up,which reduces investment.

3 Competitive Market Equilibrium

A competitive equilibrium for this economy (for the case of perfect capital markets) is an allocation (I, C_1, C_2, Y_1, Y_2, K_2) and a relative price vector (R) such that the household and the firm is each maximizing its respective objective, markets clear, and the economy resource constraints are satisfied.

Capital market:

$$\begin{aligned} 1 &= R\frac{\beta(C_2)^{-\sigma}}{(C_1)^{-\sigma}} && (28) \\ &= \frac{[A_2\alpha K_2^{\alpha-1}+1-\delta]}{1+c\frac{I^j}{K_1^j}}\frac{\beta(C_2)^{-\sigma}}{(C_1)^{-\sigma}} && (29) \end{aligned}$$

Resource Constraints:

$$Y_1 = C_1 + I + \frac{c}{2}(\frac{I}{K_1})^2K_1 \quad (30)$$

$$C_2 = Y_2 + (1-\delta)K_2 \quad (31)$$

Technology Constraints:

$$Y_1 = A_1(K_1)^\alpha \quad (32)$$

$$Y_2 = A_2(K_2)^\alpha \quad (33)$$

Evolution of Capital

$$K_2 = (1-\delta)K_1 + I$$

:

The equilibrium price vector is obtained from the respective market clearing condition, i.e.,

$$R = \frac{[A_2\alpha K_2^{\alpha-1}+1-\delta]}{1+c\frac{I^j}{K_1^j}} = \frac{(C_1)^{-\sigma}}{\beta(C_2)^{-\sigma}} \quad (34)$$

4 Adding Employment

4.1 Household Problem

$$\max \sum_{k=1}^{2} \beta^{k-1}[\frac{1}{1-\sigma}(C_k^i)^{1-\sigma} - \frac{1}{1+\varphi}(N_k^i)^{1+\varphi}]$$

s.t

$$C_1^i = W_1^i N_1^i + \Pi_1^i - S^i$$

$$C_2^i = W^i N_2^i + \Pi_2^i + RS^i$$

First Order Necessary Conditions:

(consumption/saving)

$$(C_1^i)^{-\sigma} = R\beta(C_2^i)^{-\sigma}$$

(labor supply)

$$W_k = \frac{(N_k^i)^{\varphi}}{(C_k^i)^{-\sigma}} : \text{ for } k = 1, 2$$

4.2 Firm Problem

$$\max \Pi_1^j + \frac{\Pi_2^j}{R}$$

$$\Pi_1 = Y_1 - I - \frac{c}{2}(\frac{I}{K_1})^2 K_1$$

$$\Pi_2 = Y_2 + (1-\delta)K_2$$

$$Y_k^j = A_k(K_k^j)^{\alpha}(N_k^j)^{1-\alpha} \text{ for } k = 1, 2$$

First Order Necessary Conditions:

(Investment)

$$\frac{[A_2 \alpha K_2^{j\alpha-1} + 1 - \delta]}{1 + c\frac{I^j}{K_1^j}} = R$$

(Labor)

$$A_k(1-\alpha)(\frac{N_k}{K_k})^{-\alpha} = W \text{ for } k = 1, 2$$

4.3 Competitive Equilibrium

A competitive equilibrium for this economy is an allocation (I, N_1, N_2, C_1, C_2, Y_1, Y_2, K_2) and a relative price vector (R) such that the household and the firm is each maximizing its respective objective, markets clear, and the economy resource constraints are satisfied.

Capital market:

$$\begin{aligned} 1 &= R\frac{\beta(C_2)^{-\sigma}}{(C_1)^{-\sigma}} && (35) \\ &= \frac{[A_2 \alpha K_2^{\alpha-1} + 1 - \delta]}{1 + c\frac{I^j}{K_1^j}} \frac{\beta(C_2)^{-\sigma}}{(C_1)^{-\sigma}} && (36) \end{aligned}$$

Labor Market

$$(1-\alpha)(\frac{N_k}{K_k})^{-\alpha} = \frac{(N_k)^{\varphi}}{(C_k)^{-\sigma}} : \text{ for } k = 1, 2$$

Resource Constraints:

$$Y_1 = C_1 + I + \frac{c}{2}(\frac{I}{K_1})^2 K_1 \quad (37)$$

$$C_2 = Y_2 + (1-\delta)K_2 \quad (38)$$

Technology Constraints:

$$Y_k = A_k(K_k)^{\alpha}(N_k)^{1-\alpha}; \text{ for k=1,2} \quad (39)$$

Evolution of Capital

$$K_2 = (1-\delta)K_1 + I$$

:

The equilibrium price vector is obtained from the respective market clearing condition, i.e.,

$$R = \frac{[A_2 \alpha K_2^{\alpha-1} + 1 - \delta]}{1 + c\frac{I}{K_1}} = \frac{(C_1)^{-\sigma}}{\beta^{-1}(C_2)^{-\sigma}} \tag{40}$$