Lab Report #1: Moments & Cumulants

Revised: August 19, 2014

Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Use Matlab where possible and attach your code to your answer.

1. *Moments of the standard normal.* This should be review, but will get you started with moments and generating functions.

   Suppose x is a standard normal random variable.

   (a) What is x's probability density function?

   (b) What is x's moment generating function (mgf)? (Don't derive it, just write it down.)

   (c) What is $E(e^x)$?

   (d) Differentiate the mgf to find the first two moments. How are they related to the mean and variance?

   (e) Find the third and fourth moments the same way. What are they?

2. *Sample moments of the standard normal.* It's often helpful to experiment with artificial test problems, just to remind ourselves how the code works. Here we compute sample moments of artificial data generated in Matlab and verify that calculations of various moments do what we think they do.

   This generates the data we'll use:

```
format compact          % single-spacing of output
nobs = 1000;            % number of observations
rng('default');         % sets "seed" so you can replicate the output
x = randn(nobs,1);      % generates a vector of standard normals
```

   These commands generate "pseudo-random" numbers from a standard normal distribution and put them in the vector `x`. (Standard normal means normal with mean equal to zero and variance equal to one.) As always, you can find out what Matlab commands do by typing `help command` at the prompt; for example, `help rng` or `help randn`.

   (a) Our first check is to see if the sample moments correspond, at least approximately, to our knowledge of normal random variables. Run the commands:

```
xbar = mean(x)
moments = mean([(x-xbar).^2 (x-xbar).^3 (x-xbar).^4])
```

   What do you get? How do your calculations compare to the analogous moments of the standard normal distribution?

   (b) Our second check is on the Matlab commands `std(x)`, `skewness(x)`, and `kurtosis(x)`. How do they compare to the calculations you did above?

3. *Cumulants of geometric random variables.* We say a random variable has a *geometric distribution* if it takes on the values $x = 0, 1, 2, \ldots$ with probabilities $p(x) = (1-\omega)\omega^x$ for some parameter ω satisfying $0 < \omega < 1/e$.

   Your mission: Compute the moments and cumulants of x. Use Matlab where possible.

   (a) Verify that this is a legitimate probability distribution.

   (b) Derive the moment generating function. (Hint: Apply the definition.) What is the cumulant generating function?

   (c) Use Matlab and the cgf to find the first four cumulants, labelled κ_1 through κ_4. What are the mean and variance?

   (d) Derive the standard measures of skewness and excess kurtosis:

$$
\begin{aligned}
\gamma_1 &= \kappa_3/(\kappa_2)^{3/2} \quad \text{(skewness)} \\
\gamma_2 &= \kappa_4/(\kappa_2)^2 \quad \text{(excess kurtosis)}
\end{aligned}
$$

4. *Sums of independent random variables.* Consider the sum of n random variables, say $y = x_1 + x_2 + \cdots + x_n$ with the x_i's "iid" (independent and identically distributed).

   (a) Suppose the expansion of the cgf for one of the x_i's is

$$
k(s;x) \quad = \quad \kappa_1(x)s + \kappa_2(x)s^2/2 + \kappa_3(x)s^3/3! + \kappa_4(x)s^4/4! + \cdots .
$$

   What is the analogous expansion for y? What does that tell you about the cumulants of y?

   (b) Compute our measures of skewness and excess kurtosis for y. How do they vary with n?