Quiz #1

Fall 2015

Please write your name below, then complete the exam in the space provided. You may refer to one page of notes: standard paper, both sides, any content you wish. **There are FOUR questions.**

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(Name and signature)

1. *State prices.* Consider our usual two-period economy with two states, 1 and 2, and two assets, A and B. The assets have prices at date $t = 0$ of $q^A = 1$ and $q^B = 5/4$ and dividends at $t = 1$ of

Asset	State 1	State 2
Asset A	1	2
Asset B	2	1

   (a) What are the Arrow securities in this setting? (5 points)

   (b) What are their prices (ie, the state prices)? (10 points)

   (c) What is the price of an asset that has a dividend of one in each state? What is its return? (10 points)

Solution:

(a) Arrow securities pay one in one state, zero in the others. Here we have two Arrow securities, one for each state.

(b) State prices $Q(z)$ satisfy

$$\begin{aligned} q^A &= 1Q(1) + 2Q(2) \\ q^B &= 2Q(1) + 1Q(2). \end{aligned}$$

That gives us $Q(1) = 1/2$ and $Q(2) = 1/4$.

(c) This asset must have price $q = Q(1) + Q(2) = 3/4$. Its return is $r = d/q = 1/q = 4/3$ in each state.

2. *Moments of exponentials.* Consider a random variable x with moment generating function $h_x(s) = E(e^{sx})$. Our goal is to use what we know about x to describe $y = e^x$.

(a) What is the mean of y? (10 points)

(b) What is the variance of y? *Hint: Note that* $y^2 = (e^x)^2 = e^{2x}$. (15 points)

Solution:

(a) We apply the definition:

$$E(y) \;=\; E(e^x) \;=\; h_x(1).$$

(b) The direct route is

$$\mathrm{Var}(y) \;=\; E(y^2) - E(y)^2 \;=\; h_x(2) - h_x(1)^2.$$

3. *Aversion to Poisson risk.* Consider the risk preference of a power utility agent facing Poisson risk. Specifically:

- The agent has utility function $u(c) = c^{1-\alpha}/(1-\alpha)$ with $\alpha \geq 0$.
- Log consumption is Poisson with intensity parameter $\omega > 0$. That is: $x = \log c$ takes on the values $x = 0, 1, 2, \ldots$ with probabilities $p(x) = e^{-\omega}\omega^x/x!$.

(a) What is the agent's certainty equivalent if she has constant consumption $\bar{c}$? (5 points)

(b) What is $\bar{c} = E(c)$ with Poisson risk? That is, with $x = \log c$ Poisson. (10 points)

(c) What expected utility with Poisson risk? (10 points)

(d) What is the certainty equivalent with Poisson risk? How does it compare to mean consumption when $\alpha = 0$? Why? (10 points)

Solution:

(a) The certainty equivalent μ is the constant consumption that delivers the same utility. If consumption is constant, the certainty equivalent is the constant: $\mu = \bar{c}$.

(b) Mean consumption is $E(c) = E(e^x) = e^{\omega(e-1)}$. You might recognize this as related to the mgf of x: $h_x(s) = E(e^x) = e^{\omega(e^s-1)}$, so $E(c) = h_x(1)$. [It's not necessary to derive the Poisson mgf, you can just write it down if you know it.]

(c) Expected utility is

$$E[u(c)] \;=\; E(e^{(1-\alpha)x})/(1-\alpha) \;=\; e^{\omega(e^{1-\alpha}-1)}/(1-\alpha) \;=\; h_x(1-\alpha)/(1-\alpha).$$

(d) The certainty equivalent μ is the solution to $u(\mu) = E[u(c)]$, or

$$\mu^{1-\alpha}/(1-\alpha) \;=\; e^{\omega(e^{1-\alpha}-1)}/(1-\alpha).$$

When $\alpha = 0$, μ is precisely mean consumption. Why? Because with $\alpha = 0$ utility is linear and the agent is risk-neutral. Risk doesn't matter.

4. *Saving and risk.* A two-period agent has utility function

$$u(c_0) + \beta E[u(c_1)]$$

with $u(c) = c^{1-\alpha}/(1-\alpha)$ and $\alpha \geq 0$. The agent has incomes in the two periods of $y_0 > 0$ and $y_1 = 0$. Any saving is invested in an asset with risky (gross) return r, with $\log r \sim \mathcal{N}(\kappa_1, \kappa_2)$.

(a) What is the expected return $E(r)$? (5 points)

(b) What is the budget constraint? That is: given a choice of c_0, what is c_1? (10 points)

(c) What is the optimal choice of c_0? (The condition is enough, you don't need to solve for c_0.) (15 points)

Solution:

(a) The usual lognormal result. Suppose $x = \log r$ and x has mgf $h_x(s) = e^{sx}$. In the normal case, $h_x(s) = e^{s\kappa_1 + s^2\kappa_2/2}$. Then $E(r) = h_x(1) = e^{\kappa_1 + \kappa_2/2}$. This is more general than we need, but makes the notation simpler later on.

(b) The agent starts with y_0. If he spends c_0, that leaves saving of $s = y_0 - c_0$. Since $y_1 = 0$, $c_1 = r(y_0 - c_0)$.

(c) The direct approach is to substitute for c_1 in the utility function and compute the expectation:

$$u(c_0) + \beta E[u(c_1)] = c_0^{1-\alpha}/(1-\alpha) + \beta(y_0 - c_0)^{1-\alpha} h_x(1-\alpha)/(1-\alpha).$$

If we differentiate with respect to c_0 and set the result equal to zero, we have

$$\begin{aligned} c_0^{-\alpha} &= \beta(y_0 - c_0)^{-\alpha} h_x(1-\alpha) \\ \Leftrightarrow \quad y_0 - c_0 &= c_0 \left[\beta h_x(1-\alpha)\right]^{1/\alpha}, \end{aligned}$$

which we can easily solve for c_0.