Lab Report #3: Consumption, Risk, & Portfolio Choice
Revised: October 6, 2015

Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Please include your Matlab code.

Solution: Answers follow, but see also the attached Matlab program for calculations related to Questions 3 and 4; download the pdf, open, click on pushpin:

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1. *Entropy.* Later in the course, we define the entropy of a positive random variable x by

$$H(x) = \log E(x) - E(\log x).$$

(We'll also explain why we call this entropy.)

   (a) Use Jensen's inequality to show that $H(x) > 0$ as long as x is random.

   (b) Suppose $x = e^y$ with $y \sim \mathcal{N}(\kappa_1, \kappa_2)$. What is the entropy of x?

Solution:

   (a) The log function is concave, so Jensen's inequality tells us that $\log E(x) > E(\log x)$ as long as x isn't constant.

   (b) If x is "lognormal" (it's log is normal), then

$$\begin{aligned}
E(x) &= e^{\kappa_1 + \kappa_2/2} \\
E(\log x) &= \kappa_1 \\
H(x) &= \log E(x) - E(\log x) = (\kappa_1 + \kappa_2/2) - \kappa_1 = \kappa_2/2.
\end{aligned}$$

   That is: the variance over two.

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2. *Constrained optimization.* Consider the problem: choose x and y to maximize

$$f(x, y) = \log(x - 1) + \log(y - 2)$$

   subject to $2x + y \leq 7$.

   (a) What is the Lagrangian associated with this problem?

   (b) What are the first-order conditions?

   (c) What values of x and y solve the problem? What is the Lagrange multiplier?

> **Solution:**
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> (a) The Lagrangian is
>
> $$\mathcal{L} = \log(x-1) + \log(y-2) + \lambda(7 - 2x - y).$$
>
> (b) The first-order conditions are
>
> $$1/(x-1) = 2\lambda$$
> $$1/(y-2) = \lambda.$$
>
> (c) The solution is $x = 7/4$, $y = 7/2$, and $\lambda = 2/3$.

3. *Assets, returns, and Arrow securities.* Consider a two-period event tree. At date 0, we purchase one unit of asset (or security, the words are interchangeable) j for price q^j. At date 1, we get dividend $d^j(z)$, which depends on the state z. Let us say, specifically, that there are two assets and two states, with dividends

Asset	State 1	State 2
1 ("bond")	2	2
2 ("equity")	2	3

The prices of the assets are $q^1 = 3/2$ (bond) and $q^e = 2$ (equity).

(a) Does it matter that the bond pays two in each state? How does it differ from a bond that pays one in each state? Or one hundred?

(b) What are the (gross) returns on these assets?

(c) An Arrow security pays one in a specific state, nothing in other states. Here we have two states, hence two Arrow securities. Their dividends are

Security	State 1	State 2
Arrow 1	1	0
Arrow 2	0	1

What quantities of the two assets (bond, equity) reproduce the dividend of the second Arrow security? What must the prices be of the two Arrow securities?

(d) Assets can be thought of as collections of Arrow securities. If we know the prices of Arrow securities, we can find the prices of other assets by adding up the values of their state-specific dividends. Use the prices of the Arrow securities computed above to find the prices of the two assets.

> **Solution:**
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> (a) It's a convention and has no impact on anything we care about. If we double the dividends and double the price, the returns are the same. In the business world, bonds quoted on a face value of 100, but this is a convention, too: Actual bonds generally have face values much larger than this.
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> (b) The (gross) returns on asset j are $r^j(z) = d^j(z)/q^j$ in state z. The returns depend on the state, which makes them random variables.
>
>   - The bond return is $r^1 = 2/(3/2) = 4/3$ in both states.
>   - The equity returns are $r^e(z) = d^e(z)/q^e$ or
>
> $$r^e(z) = \begin{cases} 2/2 = 1 & \text{in state } z = 1 \\ 3/2 & \text{in state } z = 2. \end{cases}$$
>
> (c) If we buy one unit of equity and sell one unit of the bond, we are left with a net dividend of zero in state 1 and one in state 2. So we've replicated the second Arrow security. The cost of this transaction is $q^e - q^1 = 1 - 3/4 = 1/4$, so that should be its price: $Q(2) = 1/4$. We can replicate the first Arrow security by purchasing $3/2$ units of the bond and selling one unit of equity. That gives us a net dividend of one in state 1 and zero in state 2, as needed. Its price must be $Q(1) = (3/2)q^1 - q^e = (3/2)(3/2) - 2 = 1/4$.
>
> (d) We can also do the reverse: combine Arrow securities to replicate the dividends of the two assets. The bond is two units of each Arrow security. Equity is two units of the first Arrow security and 3 of the second. Their prices must therefore be
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> $$\begin{aligned} q^1 &= 2Q(1) + 2Q(2) &= 3/2 \\ q^e &= 2Q(1) + 3Q(2) &= 2. \end{aligned}$$
>
>   That gives us $Q(1) = 1/2$ and $Q(2) = 1/4$. Lurking behind the scenes here is an arbitrage argument. Why do the assets and their replications sell for the same price? Because otherwise people would buy the cheaper one and sell the more expensive one, giving them a riskless profit. Markets are unlikely to let that happen: they should eliminate pure arbitrage opportunities like this.

4. *Portfolio choice.* An investor must decide how to allocate his saving between a riskfree bond and equity. We approximate the world with two states, each of which occurs with probability $1/2$. The returns by state are

Security	State 1	State 2
1 ("bond")	1.1	1.1
2 ("equity")	0.9	1.5

That is, one unit invested at date 0 yields the returns listed in the table at date 1.

The investor's problem is to choose current consumption c_0 and the fraction of saving a to invest in equity to solve

$$\max_{c_0, a} \quad u(c_0) + \beta \sum_z p(z) u[c_1(z)]$$

$$\text{s.t.} \quad c_1(z) = (y_0 - c_0)[(1-a)r^1 + ar^e(z)].$$

If $a > 1$, the agent has a levered position, borrowing to fund investments in equity greater than saving. As usual, $u(c) = c^{1-\alpha}/(1-\alpha)$. Where the problem calls for numbers, we'll use $\beta = 0.9$ and $\alpha = 2$.

(a) What is saving here?

(b) What are the upper and lower bounds on a consistent with positive consumption $c_1(z)$ in all states z? Lower bound?

(c) What are the first-order conditions for c_0 and a? *Comment: I recommend substituting the expression for $c_1(z)$ into the utility function.*

(d) Use Matlab to solve the first-order condition for a numerically. What value of a maximizes utility? Make sure it implies positive consumption in all states. *Comment: I did this by varying a manually until its first-order condition was satisfied. You could also compute the first-order condition for a grid of values for a, and choose the one that comes closest to satisfying the first-order condition.*

Solution:

(a) Saving is date-0 income minus date-0 consumption: $s = y_0 - c_0$.

(b) We need

$$(1-a)r^1 + ar^e(z) = r^1 + a[r^e(z) - r^1] > 0$$

for all states z. Given our numbers, that implies

$$\text{state 1}: \quad 1.1 - 0.2a > 0$$
$$\text{state 2}: \quad 1.1 + 0.2a > 0.$$

The first one gives us $a < 5.5$, the second $a > -5.5$.

(c) The first-order conditions are

$$c_0: \quad c_0^{-\alpha} = \beta \sum_z p(z) c_1(z)^{-\alpha}[(1-a)r^1 + ar^e(z)]$$

$$a: \quad 0 = \beta \sum_z p(z) \left\{ (y_0 - c_0)[(1-a)r^1 + ar^e(z)] \right\}^{-\alpha} [r^e(z) - r^1].$$

The term in curly brackets is $c_1(z)$. The second equation simplifies to

$$0 = \sum_z p(z) \left[(1-a)r^1 + ar^e(z) \right]^{-\alpha} [r^e(z) - r^1],$$

which depends on a but not c_0.

(d) This problem doesn't have a simple closed-form solution — that's a general feature of portfolio choice problems. But we can crack it with Matlab. I have used two methods. The first is this iterative procedure:

- Pick a value of a.
- Check the foc for a. If it equals zero, we're done. If not, pick another a and repeat.

I use a slightly different method in the attached code. I compute the foc for a grid of values for a and choose the value for which the foc is closest to zero. That gives me $a = 0.667$.