# Forward-Looking Models

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Lots of things in economics and finance are forward-looking: decisions made now depend on things we expect to happen in the future. We'll take what we might call a classical approach to this issue, illustrating in one-dimensional linear models how expectations of the future affect the present. This builds on work done in the 1970s by lots of people, Tom Sargent and Neil Wallace among them. In one respect, it's a step backwards for us: we'll focus on the conditional mean and ignore risk. No need to panic, we'll bring back risk shortly.

Our leading example here is inflation. It's an introduction of sorts to bond yields. Since bonds are claims to money in the future, the inflation rate is a central component.

## 1 Hyperinflation and the quantity theory

*Hyperinflation* is the term we use for high rates of inflation: over one hundred percent a year. There have been many examples of inflation rates above a thousand percent. See, for example, the Wikipedia entry.

So where do hyperinflations come from? As economists, we're incredibly pleased with ourselves that we figured this one out. Tom Sargent put it this way in an interview:

> The way to start a hyperinflation is run sustained government deficits and then have the monetary authority print money to pay for it. That always works. How do you stop a hyperinflation? You stop doing it. This isn't high economic theory.

Here's a link to the whole interview. The comments about hyperinflation come at 23:40.

The "print money" step of this process is often interpreted with the *quantity theory of money*:

$$MV \;=\; PY.$$

Here $M$ is the quantity of money (currency, for example) in circulation, $V$ is "velocity" (more shortly), $P$ is the price level, and $Y$ is GDP. This is a description of how money is used to make transactions. The dollar value of transactions is $PY$, the product of the price and the quantity. Velocity $V$ is a measure of how quickly a unit of money is reused in another transaction. In a hyperinflation, rapid growth in money is associated with rapid growth in prices — inflation, in other words.

This theory works reasonably well in the sense that we always see high money growth in periods of high inflation. We also see government deficits, as Sargent suggested. In these respects the theory works pretty well. But it misses on the timing at both the start and the end. Often hyperinflations break out before money growth picks up, and they end before money growth stops. The question is why.

## 2 Digression: the law of iterated expectations

Consider a stochastic process $x$ that evolves somehow over time. What is the expectation of $x_{t+k}$ conditional on whatever we know at date $t$? If $I_t$ is the information available at date $t$, we might write

$$E_t(x_{t+k}) \;=\; E\left(x_{t+k}|I_t\right).$$

The left side here is shorthand notation for the right side. In a Markovian environment with state $z_t$, we can replace $I_t$ with $z_t$.

The *law of iterated expectations* says that we can compute this one period at a time. If $k = 2$, we have

$$E_t(x_{t+2}) \;=\; E_t\big[E_{t+1}(x_{t+2})\big].$$

And so on for longer time horizons $k$. [Write this out to make sure you're following.]

*Example.* Let's see how this works for an AR(1):

$$x_{t+1} \;=\; \varphi x_t + \sigma w_{t+1},$$

where $\{w_t\}$ is (as usual) a sequence of iid standard normal random variables. Here

$$E_{t+1}(x_{t+2}) \;=\; \varphi x_{t+1},$$

so

$$E_t(x_{t+2}) \;=\; E_t\big[E_{t+1}(x_{t+2})\big] \;=\; E_t(\varphi x_{t+1}) \;=\; \varphi^2 x_t.$$

You get the idea. We do this one period at a time, but end up with the same answer we had before.

## 3 Forward-looking models

Now let's think about forward-looking models. There are lots of situations in economics in which current decisions are based on what people expect to happen in the future: consumption (what is future income?), investment (what is future demand for my product?), asset valuation (what are future cash flows?), and so on.

A canonical version of a *forward-looking difference equation* looks like this:

$$y_t \;=\; \lambda E_t(y_{t+1}) + x_t. \tag{1}$$

This idea here is that $y_t$ depends on the variable $x_t$, and also on the expectation of itself at $t + 1$. That leads to the question: What drives $y_t$? Is it $x_t$ or the expectation $E_t y_{t+1}$?

*Example.* Here's an example that shows up in finance classes: equity as a claim to future dividends. Suppose the price of equity is

$$q_t = \delta(d_t + E_t q_{t+1}).$$

Here $\delta$ is the discount factor we apply to future cash flows, something you might write as $1/(1+i)$ in a finance or accounting class. So we see that today's stock price depends on dividends, but also on what we expect the stock price to be next period. So we wonder: Is the price is high today because we expect it to be high tomorrow? How do we get out of this circular reasoning?

Let's go back to our canonical example. In the deterministic version, with no uncertainty, equation (1) becomes

$$y_t \;=\; \lambda y_{t+1} + x_t.$$

If we substitute, we get

$$\begin{aligned} y_t &= x_t + \lambda x_{t+1} + \lambda^2 y_{t+2} \\ &= x_t + \lambda x_{t+1} + \cdots \lambda^k x_{t+k} + \lambda^{k+1} y_{t+k+1}. \end{aligned}$$

If $|\lambda| < 1$ and $y$ doesn't explode somehow (we're glossing over some technical details here), then we have

$$y_t \;=\; \sum_{j=0}^{\infty} \lambda^j x_{t+j}.$$

That is: $y_t$ depends on the current and future values of $x_t$. That's the sense in which this is a forward-looking model.

In the stochastic version, the same logic (repeated substitution) plus the law of iterated expectations gives us

$$y_t \;=\; \sum_{j=0}^{\infty} \lambda^j E_t(x_{t+j}). \tag{2}$$

Thus we have connected the price to expectations of future dividends, discounted back to the present. We might say that the variable $y$ responds to changes in expected fundamentals $x$.

*Example.* With some structure on $x$, we can be more specific. Suppose, for example, that $x$ is AR(1): $x_{t+1} = \varphi x_t + \sigma w_{t+1}$. Then $E_t(x_{t+j}) = \varphi^j x_t$ and

$$y_t \;=\; \sum_{j=0}^{\infty} \lambda^j \varphi^j x_t \;=\; x_t/(1-\lambda\varphi).$$

This works as long as $|\lambda\varphi| < 1$.

There's a more direct route that you might have run across in a differential equations course: the *method of undetermined coefficients*. Continuing with the same example, we guess a linear solution: $y_t = a x_t$ for some parameter $a$ to be determined. Then (1) becomes

$$a x_t \;=\; \lambda a(\varphi x_t) + x_t.$$

Since this holds for all $x_t$, we must have $a = \lambda\varphi a + 1$ or $a = 1/(1-\lambda\varphi)$. This works, but since it shortcuts the infinite sum it doesn't remind us that we need $|\lambda\varphi| < 1$.

## 4 Hyperinflation revisited

So what does this have to do with hyperinflation? There's a more sophisticated version of the quantity theory in which velocity varies with the expected inflation rate. In logs, the quantity theory is

$$m_t + v_t \;=\; p_t + y_t.$$

Now add velocity:

$$v_t \;=\; \alpha(E_t p_{t+1} - p_t)$$

for some $\alpha > 0$. Here we see that as expected inflation rises, velocity rises, too. The idea is that money loses value through inflation. At faster rates of inflation people work harder to get rid of money, which raises velocity.

Putting these two pieces together, we have

$$m_t + \alpha(E_t p_{t+1} - p_t) \;=\; p_t + y_t$$

or

$$p_t \;=\; [\alpha/(1+\alpha)]E_t(p_{t+1}) + (1+\alpha)^{-1}(m_t - y_t).$$

This has the same form as (1), so we can apply the solution (2). As a result, the price level $p$ depends not only on the current money supply, it also depends on expectations of the future money supply. Anything that changes expectations of future $m$ will change current $p$. This line of reasoning offers hope, and perhaps more, of fixing up the timing problems noted earlier.

## 5 Bubbles

The 2013 Nobel Prize in economics went to three people known for their contributions to financial economics: Gene Fama, Lars Hanse, and Robert Shiller. We ran across some of Hansen's work when we derived the Hansen-Jagannathan bound. Fama and Shiller are known for having very different views of We've noted that Shiller often sees bubbles in asset markets, while Fama says he doesn't know what a bubble is. Oddly enough, I'd say both make useful points. The term bubble is often used to describe a market in which prices seem to have lost connection to "fundamentals." We might think that equity prices, for example, should be connected to dividends. Shiller argues that the loose connection between the two suggests we need another model, perhaps one in which investors are "irrational," or even a "bubble." Fama argues instead that the term bubble is typically used to mean "we don't know," which is hardly a theory of anything.

Whether any particular situation is a bubble is hard to say, but we do have mathematical theories in which the behavior of asset prices is disconnected from fundamentals. The most popular version gives us solutions to (1) other than (2). The idea? Solving a difference equation like (1) is analogous to solving a differential equation, where we have a constant of integration to think about.

Here's how that works. We add a term $c\lambda^{-t}$ to the solution, turning (2) into

$$y_t = \sum_{j=0}^{\infty} \lambda^j E_t(x_{t+j}) + c\lambda^{-t}$$

for some constant $c$. Does this satisfy (1)? We have

$$\begin{aligned} y_{t+1} &= \sum_{j=0}^{\infty} \lambda^j E_{t+1}(x_{t+j+1}) + c\lambda^{-(t+1)} \\ \lambda E_t(y_{t+1}) &= \sum_{j=0}^{\infty} \lambda^{j+1} E_t(x_{t+j+1}) + c\lambda^{-t} = \sum_{j=1}^{\infty} \lambda^j E_t(x_{t+j}) + c\lambda^{-t}. \end{aligned}$$

Subtracting this from $y_t$ gives us zero, so our new solution satisfies (1) for any constant $c$.

How does this new solution behave? The term $c\lambda^{-t}$ tends to grow, since $|\lambda| < 1$. The traditional approach is to set $c = 0$, since any other solution blows up. Since we don't see asset prices growing forever, or turning negative, that seems persuasive.

But here's a variant that avoids this problem. Consider the term $c_t\lambda^{-t}$. The same logic as before tells us that this satisfies (1) as long as

$$E_t(c_{t+1}\lambda^{-t}) = c_t\lambda^{-t}.$$

This is true if

$$E_t(c_{t+1}) = c_t.$$

This condition defines $c_t$ as a *martingale*: a stochastic process whose conditional mean is its current value.

Here's an example. Suppose

$$c_{t+1} = \begin{cases} \lambda^{-1} c_t & \text{with probability } \lambda \\ 0 & \text{with probability } 1-\lambda. \end{cases}$$

It should be clear that this is a martingale. This is like the previous example, in which prices rise at rate $1/\lambda$. But eventually the bubble pops, as the term goes to zero. [Graph this term over time.]

[This section is adapted from Tom Sargent's course notes, pages 27-29 and 55-58.]

## Bottom line

Solutions to forward-looking models connect current decisions to the expected future values of fundamentals. Bubbles add an extra term to these solutions.

## Practice problems

1. *Stock prices.* Suppose stock prices are ex-dividend, so that

$$q_t = \delta E_t(d_{t+1} + q_{t+1}).$$

How is the current stock price $q_t$ related to expected future dividends?

Answer. The same logic we used earlier gives us

$$q_t = \sum_{j=1}^{\infty} \delta^j E_t(d_{t+j}).$$

That is: we start the sum at $j = 1$ rather than $j = 0$.

2. *ARMA(1,1) fundamentals.* Suppose in our canonical model (1) that $x_t$ is ARMA(1,1):

$$x_{t+1} = \varphi x_t + w_{t+1} + \theta w_t,$$

with $|\varphi| < 1$ and $\{w_t\}$ the usual sequence of independent standard normals. What is the solution? That is: how is the endogenous variable $y_t$ connected to $x_t$ and $w_t$?

Answer. The simplest approach is the method of undetermined coefficients. The state here is $z_t = (x_t, w_t)$. We guess a solution of the form $y_t = ax_t + bw_t$. Since $E_t(x_{t+1}) = \varphi x_t + \theta w_t$, we have

$$E_t(y_{t+1}) = aE_t(x_{t+1}) = a(\varphi x_t + \theta w_t).$$

Equation (1) then implies

$$ax_t + bw_t = \lambda a(\varphi x_t + \theta w_t) + x_t.$$

Equating coefficients of $x_t$ and $w_t$, respectively, gives us $a = 1/(1 - \lambda\varphi)$ and $b = \lambda a\theta = \lambda\theta/(1 - \lambda\varphi)$.

3. *Inflation and the Taylor rule.* We can get a sense of the impact of monetary policy on inflation with the equations

$$\begin{aligned} i_t &= r + E_t(\pi_{t+1}) \\ i_t &= r + \tau\pi_t + s_t. \end{aligned}$$

The first equation is the Fisher equation, which says that the nominal interest rate $i_t$ equals the (constant) real interest rate $r$ plus expected inflation. The second equation is Taylor's rule, which tells the central bank to set the nominal interest rate equal to the real rate plus an adjustment for current inflation. By assumption $\tau > 1$, so that an increase in inflation leads to a greater increase in the nominal interest rate. The "shock" $s_t$ to monetary policy is AR(1): $s_{t+1} = \varphi s_t + \sigma w_{t+1}$ with $0 < \varphi < 1$ and independent standard normal innovations $w_t$.

The idea behind this policy rule is to respond aggressively to inflation; the larger is $\tau$ the more aggressive the response.

(a) Express these two equations in the form of a forward-looking difference equation in the inflation rate $\pi_t$.

(b) In what sense does future monetary policy affect the current inflation rate?

(c) Solve the model. How does inflation depend on the shock $s_t$?

(d) In what sense might we say that larger values of $\tau$ are more successful in eliminating inflation?

Answer.

(a) If we set the first equation equal to the second, we get

$$E_t(\pi_{t+1}) \quad = \quad \tau\pi_t + s_t.$$

This isn't quite in the form of (1), but it's close.

(b) Repeated substitution gives us

$$\pi_t \quad = \quad \sum_{j=0} \tau^{-j} E_t\big(\tau^{-1} s_{t+j}\big),$$

so that future shocks to monetary policy affect the current inflation rate.

(c) We guess the solution has the form: $\pi_t = a s_t$ for some coefficient $a$. The equation becomes

$$a\varphi s_t \quad = \quad \tau a s_t + s_t,$$

which implies $a = -1/(\tau - \varphi) < 0$.

(d) As we increase $\tau$, we reduce the impact $a$ of the shock on the inflation rate. More precisely, the variance of the inflation rate is

$$\text{Var}(\pi_t) \quad = \quad (\tau - \varphi)^{-2}\text{Var}(s_t) \quad = \quad \frac{1}{(\tau - \varphi)^2(1 - \varphi^2)},$$

which is decreasing in $\tau$.