Dynamic Asset Pricing Theory

THIRD EDITION

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I
Discrete-Time Models

This first part of the book takes place in a discrete-time setting with a discrete set of states. This should ease the development of intuition for the models to be found in Part II. The three pillars of the theory, *arbitrage*, *optimality*, and *equilibrium*, are developed repeatedly in different settings. Chapter 1 is the basic single-period model. Chapter 2 extends the results of Chapter 1 to many periods. Chapter 3 specializes Chapter 2 to a Markov setting and illustrates dynamic programming as an alternate solution technique. The Ho-and-Lee and Black-Derman-Toy term-structure models are included as exercises. Chapter 4 is an infinite-horizon counterpart to Chapter 3 that has become known as the *Lucas model*.

The focus of the theory is the notion of state prices, which specify the price of any security as the state-price weighted sum or expectation of the security's state-contingent dividends. In a finite-dimensional setting, there exist state prices if and only if there is no arbitrage. The same fact is true in infinite-dimensional settings under mild technical regularity conditions. Given an agent's optimal portfolio choice, a state-price vector is given by that agent's utility gradient. In an equilibrium with Pareto optimality, a state-price vector is likewise given by a representative agent's utility gradient at the economy's aggregate consumption process.

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Introduction to State Pricing

THIS CHAPTER INTRODUCES the basic ideas in a finite-state one-period setting. In many basic senses, each subsequent chapter merely repeats this one from a new perspective. The objective is a characterization of security prices in terms of "state prices," one for each state of the world. The price of a given security is simply the state-price weighted sum of its payoffs in the different states. One can treat a state price as the "shadow price," or Lagrange multiplier, for wealth contingent on a given state of the world. We obtain a characterization of state prices, first based on the absence of arbitrage, then based on the first-order conditions for optimal portfolio choice of a given agent, and finally from the first-order conditions for Pareto optimality in an equilibrium with complete markets. State prices are connected with the "beta" model for excess expected returns, a special case of which is the Capital Asset Pricing Model (CAPM). Many readers will find this chapter to be a review of standard results. In most cases, here and throughout, technical conditions are imposed that give up much generality so as to simplify the exposition.

A. Arbitrage and State Prices

Uncertainty is represented here by a finite set $\{1, \ldots, S\}$ of states, one of which will be revealed as true. The N securities are given by an $N \times S$ matrix D, with D_{ij} denoting the number of units of account paid by security i in state j. The security prices are given by some q in $\mathbb{R}^N$. A *portfolio* $\theta \in \mathbb{R}^N$ has *market value* $q \cdot \theta$ and *payoff* $D^\top \theta$ in $\mathbb{R}^S$. An *arbitrage* is a portfolio θ in $\mathbb{R}^N$ with $q \cdot \theta \leq 0$ and $D^\top \theta > 0$, or $q \cdot \theta < 0$ and $D^\top \theta \geq 0$. An arbitrage is therefore, in effect, a portfolio offering "something for nothing." Not surprisingly, it will later be shown that an arbitrage is naturally ruled out, and this gives a characterization of security prices as follows. A

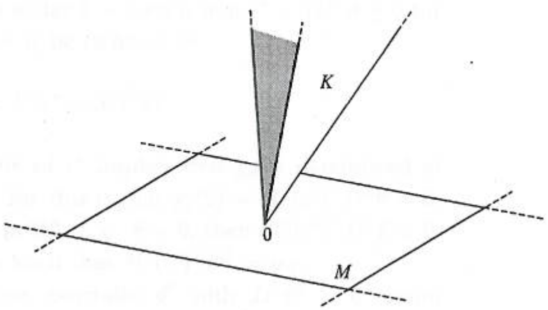

Figure 1.1. *Separating a Cone from a Linear Subspace*

state-price vector is a vector ψ in $\mathbb{R}^S_{++}$ with $q = D\psi$. We can think of ψ_j as the marginal cost of obtaining an additional unit of account in state j.

Theorem. *There is no arbitrage if and only if there is a state-price vector.*

Proof: The proof is an application of the Separating Hyperplane Theorem. Let $L = \mathbb{R} \times \mathbb{R}^S$ and $M = \{(-q \cdot \theta, D^\top \theta) : \theta \in \mathbb{R}^N\}$, a linear subspace of L. Let $K = \mathbb{R}_+ \times \mathbb{R}^S_+$, which is a cone (meaning that if x is in K, then λx is in K for each strictly positive scalar λ). Both K and M are closed and convex subsets of L. There is no arbitrage if and only if K and M intersect precisely at 0, as pictured in Figure 1.1.

Suppose $K \cap M = \{0\}$. The Separating Hyperplane Theorem (in a version for closed cones that is found in Appendix B) implies the existence of a nonzero linear functional $F : L \to \mathbb{R}$ such that $F(z) < F(x)$ for all z in M and nonzero x in K. Since M is a linear space, this implies that $F(z) = 0$ for all z in M and that $F(x) > 0$ for all nonzero x in K. The latter fact implies that there is some $\alpha > 0$ in $\mathbb{R}$ and $\psi \gg 0$ in $\mathbb{R}^S$ such that $F(v, c) = \alpha v + \psi \cdot c$, for any $(v, c) \in L$. This in turn implies that $-\alpha q \cdot \theta + \psi \cdot (D^\top \theta) = 0$ for all θ in $\mathbb{R}^N$. The vector ψ/α is therefore a state-price vector.

Conversely, if a state-price vector ψ exists, then for any θ, we have $q \cdot \theta = \psi^\top D^\top \theta$. Thus, when $D^\top \theta \geq 0$, we have $q \cdot \theta \geq 0$, and when $D^\top \theta > 0$, we have $q \cdot \theta > 0$, so there is no arbitrage. ▮

B. Risk-Neutral Probabilities

We can view any p in $\mathbb{R}^S_+$ with $p_1 + \cdots + p_S = 1$ as a vector of probabilities of the corresponding states. Given a state-price vector ψ for the dividend-price pair (D, q), let $\psi_0 = \psi_1 + \cdots + \psi_S$ and, for any state j, let $\hat{\psi}_j = \psi_j / \psi_0$.

We now have a vector $(\hat{\psi}_1, \ldots, \hat{\psi}_S)$ of probabilities and can write, for an arbitrary security i,

$$\frac{q_i}{\psi_0} = \hat{E}(D_i) \equiv \sum_{j=1}^{S} \hat{\psi}_j D_{ij},$$

viewing the normalized price of the security as its expected payoff under specially chosen "risk-neutral" probabilities. If there exists a portfolio $\bar{\theta}$ with $D^\top \bar{\theta} = (1, 1, \ldots, 1)$, then $\psi_0 = \bar{\theta} \cdot q$ is the discount on riskless borrowing and, for any security i, $q_i = \psi_0 \hat{E}(D_i)$, showing any security's price to be its discounted expected payoff in this sense of artificially constructed probabilities.

C. Optimality and Asset Pricing

Suppose the dividend-price pair (D, q) is given. An *agent* is defined by a strictly increasing *utility function* $U : \mathbb{R}^S_+ \to \mathbb{R}$ and an *endowment* e in $\mathbb{R}^S_+$. This leaves the *budget-feasible set*

$$X(q, e) = \{e + D^\top \theta \in \mathbb{R}^S_+ : \theta \in \mathbb{R}^N, q \cdot \theta \leq 0\},$$

and the problem

$$\sup_{c \in X(q,e)} U(c). \tag{1}$$

We will suppose for this section that there is some portfolio θ^0 with payoff $D^\top \theta^0 > 0$. Because U is strictly increasing, the wealth constraint $q \cdot \theta \leq 0$ is then binding at an optimum. That is, if $c^* = e + D^\top \theta^*$ solves (1), then $q \cdot \theta^* = 0$.

Proposition. *If there is a solution to* (1), *then there is no arbitrage. If U is continuous and there is no arbitrage, then there is a solution to* (1).

Proof is left as an exercise.

Theorem. *Suppose that c^* is a strictly positive solution to* (1), *that U is continuously differentiable at c^*, and that the vector $\partial U(c^*)$ of partial derivatives of U at c^* is strictly positive. Then there is some scalar $\lambda > 0$ such that $\lambda \partial U(c^*)$ is a state-price vector.*

Proof: The first-order condition for optimality is that for any θ with $q \cdot \theta = 0$, the marginal utility for buying the portfolio θ is zero. This is expressed more precisely in the following way. The strict positivity of c^* implies that

D. Efficiency and Complete Markets

Suppose there are m agents, defined as in Section C by strictly increasing utility functions $U_1, \ldots, U_m$ and by endowments $e^1, \ldots, e^m$. An *equilibrium* for the *economy* $[(U_i, e^i), D]$ is a collection $(\theta^1, \ldots, \theta^m, q)$ such that, given the security-price vector q, for each agent i, θ^i solves $\sup_\theta U_i(e^i + D^\top\theta)$ subject to $q \cdot \theta \leq 0$, and such that $\sum_{i=1}^m \theta^i = 0$. The existence of equilibrium is treated in the exercises and in sources cited in the Notes.

With $\mathrm{span}(D) \equiv \{D^\top\theta : \theta \in \mathbb{R}^N\}$ denoting the set of possible portfolio payoffs, markets are *complete* if $\mathrm{span}(D) = \mathbb{R}^S$, and are otherwise *incomplete.*

Let $e = e^1 + \cdots + e^m$ denote the aggregate endowment. A consumption *allocation* $(c^1, \ldots, c^m)$ in $(\mathbb{R}^S_+)^m$ is *feasible* if $c^1 + \cdots + c^m \leq e$. A feasible allocation $(c^1, \ldots, c^m)$ is *Pareto optimal* if there is no feasible allocation $(\hat{c}^1, \ldots, \hat{c}^m)$ with $U_i(\hat{c}^i) \geq U_i(c^i)$ for all i and with $U_i(\hat{c}^i) > U_i(c^i)$ for some i. Complete markets and the Pareto optimality of equilibrium allocations are almost equivalent properties of any economy.

Proposition. *Suppose markets are complete and $(\theta^1, \ldots, \theta^m, q)$ is an equilibrium. Then the associated equilibrium allocation is Pareto optimal.*

This is sometimes known as *The First Welfare Theorem.* The proof, requiring only the strict monotonicity of utilities, is left as an exercise. We have established the sufficiency of complete markets for Pareto optimality. The necessity of complete markets for the Pareto optimality of equilibrium allocations does not always follow. For example, if the initial endowment allocation $(e^1, \ldots, e^m)$ happens by chance to be Pareto optimal, then any equilibrium allocation is also Pareto optimal, regardless of the span of securities. It would be unusual, however, for the initial endowment to be Pareto optimal. Although beyond the scope of this book, it can be shown that with incomplete markets and under natural assumptions on utility, for almost every endowment, the equilibrium allocation is not Pareto optimal.

E. Optimality and Representative Agents

Aside from its allocational implications, Pareto optimality is also a convenient property for the purpose of security pricing. In order to see this, consider, for each vector $\lambda \in \mathbb{R}^m_+$ of "agent weights," the utility function $U_\lambda : \mathbb{R}^S_+ \to \mathbb{R}$ defined by

$$U_\lambda(x) = \sup_{(c^1,\ldots,c^m)} \sum_{i=1}^m \lambda_i U_i(c^i) \qquad \text{subject to } c^1 + \cdots + c^m \leq x. \qquad (6)$$

Lemma. *Suppose that, for all i, U_i is concave. An allocation $(c^1, \ldots, c^m)$ that is feasible is Pareto optimal if and only if there is some nonzero $\lambda \in \mathbb{R}^m_+$ such that $(c^1, \ldots, c^m)$ solves (6) at $x = e = c^1 + \cdots + c^m$.*

Proof: Suppose that $(c^1, \ldots, c^m)$ is Pareto optimal. For any allocation x, let $U(x) = (U_1(x^1), \ldots, U_m(x^m))$. Next, let

$$\mathcal{U} = \{U(x) - U(c) - z : x \in \mathcal{A},\ z \in \mathbb{R}^m_+\} \subset \mathbb{R}^m,$$

where $\mathcal{A}$ is the set of feasible allocations. Let $J = \{y \in \mathbb{R}^m_+ : y \neq 0\}$. Since $\mathcal{U}$ is convex (by the concavity of utility functions) and $J \cap \mathcal{U}$ is empty (by Pareto optimality), the Separating Hyperplane Theorem (Appendix B) implies that there is a nonzero vector λ in $\mathbb{R}^m$ such that $\lambda \cdot y \leq \lambda \cdot z$ for each y in $\mathcal{U}$ and each z in J. Since $0 \in \mathcal{U}$, we know that $\lambda \geq 0$, proving the first part of the result. The second part is easy to show as an exercise. ∎

Proposition. *Suppose that for all i, U_i is concave. Suppose that markets are complete and that $(\theta^1, \ldots, \theta^m, q)$ is an equilibrium. Then there exists some nonzero $\lambda \in \mathbb{R}^m_+$ such that $(0, q)$ is a (no-trade) equilibrium for the single-agent economy $[(U_\lambda, e), D]$ defined by (6). Moreover, the equilibrium consumption allocation $(c^1, \ldots, c^m)$ solves the allocation problem (6) at the aggregate endowment. That is, $U_\lambda(e) = \sum_i \lambda_i U_i(c^i)$.*

Proof: Since there is an equilibrium, there is no arbitrage, and therefore there is a state-price vector ψ. Since markets are complete, this implies that the problem of any agent i can be reduced to

$$\sup_{c \in \mathbb{R}^S_+} U_i(c) \quad \text{subject to } \psi \cdot c \leq \psi \cdot e^i.$$

We can assume that e^i is not zero, for otherwise $c^i = 0$ and agent i can be eliminated from the problem without loss of generality. By the Saddle Point Theorem of Appendix B, there is a Lagrange multiplier $\alpha_i \geq 0$ such that c^i solves the problem

$$\sup_{c \in \mathbb{R}^S_+} U_i(c) - \alpha_i(\psi \cdot c - \psi \cdot e^i).$$

(The Slater condition is satisfied since e^i is not zero and $\psi \gg 0$.) Since U_i is strictly increasing, $\alpha_i > 0$. Let $\lambda_i = 1/\alpha_i$. For any feasible allocation $(x^1, \ldots, x^m)$, we have

$$\sum_{i=1}^m \lambda_i U_i(c^i) = \sum_{i=1}^m [\lambda_i U_i(c^i) - \lambda_i \alpha_i(\psi \cdot c^i - \psi \cdot e^i)]$$

$$\begin{aligned}
&\geq \sum_{i=1}^{m} \lambda_i [U_i(x^i) - \alpha_i(\psi \cdot x^i - \psi \cdot e^i)] \\
&= \sum_{i=1}^{m} \lambda_i U_i(x^i) - \psi \cdot \sum_{i=1}^{m} (x^i - e^i) \\
&\geq \sum_{i=1}^{m} \lambda_i U_i(x^i).
\end{aligned}$$

This shows that $(c^1, \ldots, c^m)$ solves the allocation problem (6). We must also show that no trade is optimal for the single agent with utility function U_λ and endowment e. If not, there is some x in $\mathbb{R}^S_+$ such that $U_\lambda(x) > U_\lambda(e)$ and $\psi \cdot x \leq \psi \cdot e$. By the definition of U_λ, this would imply the existence of an allocation $(x^1, \ldots, x^m)$, not necessarily feasible, such that $\sum_i \lambda_i U_i(x^i) > \sum_i \lambda_i U_i(c^i)$ and

$$\sum_i \lambda_i \alpha_i \psi \cdot x^i = \psi \cdot x \leq \psi \cdot e = \sum_i \lambda_i \alpha_i \psi \cdot c^i.$$

Putting these two inequalities together, we have

$$\sum_{i=1}^{m} \lambda_i [U_i(x^i) - \alpha_i \psi \cdot (x^i - e^i)] > \sum_{i=1}^{m} \lambda_i [U_i(c^i) - \alpha_i \psi \cdot (c^i - e^i)],$$

which contradicts the fact that, for each agent i, (c^i, α_i) is a saddle point for that agent's problem. ∎

Corollary 1. *If, moreover, $e \gg 0$ and U_λ is continuously differentiable at e, then λ can be chosen so that $\partial U_\lambda(e)$ is a state-price vector, meaning*

$$q = D\partial U_\lambda(e). \tag{7}$$

The differentiability of U_λ at e is implied by the differentiability, for some agent i, of U_i at c^i. (See Exercise 10(C).)

Corollary 2. *Suppose there is a fixed vector p of state probabilities such that, for all i, $U_i(c) = E[u_i(c)] \equiv \sum_{j=1}^{S} p_j u_i(c_j)$, for some $u_i(\cdot)$. Then $U_\lambda(c) = E[u_\lambda(c)]$, where, for each y in $\mathbb{R}_+$,*

$$u_\lambda(y) = \max_{x \in \mathbb{R}^m_+} \sum_{i=1}^{m} \lambda_i u_i(x_i) \quad \text{subject to } x_1 + \cdots + x_m \leq y.$$

In this case, (7) is equivalent to $q = E[Du'_\lambda(e)]$.

Extensions of this *representative-agent* asset pricing formula will crop up frequently in later chapters.

F. State-Price Beta Models

We fix a vector $p \gg 0$ in $\mathbb{R}^S$ of probabilities for this section, and for any x in $\mathbb{R}^S$ we write $E(x) = p_1 x_1 + \cdots + p_S x_S$. For any x and π in $\mathbb{R}^S$, we take $x\pi$ to be the vector $(x_1\pi_1, \ldots, x_S\pi_S)$. The following version of the *Riesz Representation Theorem* can be shown as an exercise.

Lemma. *Suppose $F : \mathbb{R}^S \to \mathbb{R}$ is linear. Then there is a unique π in $\mathbb{R}^S$ such that, for all x in $\mathbb{R}^S$, we have $F(x) = E(\pi x)$. Moreover, F is strictly increasing if and only if $\pi \gg 0$.*

Corollary. *A dividend-price pair (D, q) admits no arbitrage if and only if there is some $\pi \gg 0$ in $\mathbb{R}^S$ such that $q = E(D\pi)$.*

Proof: Given a state-price vector ψ, let $\pi_s = \psi_s / p_s$. Conversely, if π has the assumed property, then $\psi_s = p_s \pi_s$ defines a state-price vector ψ. ∎

Given (D, q), we refer to any vector π given by this result as a *state-price deflator.* (The terms *state-price density* and *state-price kernel* are often used synonymously with state-price deflator.) For example, the representative-agent pricing model of Corollary 2 of Section E shows that we can take $\pi_s = u'_\lambda(e_s)$.

For any x and y in $\mathbb{R}^S$, the *covariance* $\text{cov}(x, y) \equiv E(xy) - E(x)E(y)$ is a measure of covariation between x and y that is useful in asset pricing applications. For any such x and y with $\text{var}(y) \equiv \text{cov}(y, y) \neq 0$, we can always represent x in the form $x = \alpha + \beta y + \epsilon$, where $\beta = \text{cov}(y, x)/\text{var}(y)$, where $\text{cov}(y, \epsilon) = E(\epsilon) = 0$, and where α is a scalar. This *linear regression* of x on y is uniquely defined. The coefficient β is called the associated *regression coefficient.*

Suppose (D, q) admits no arbitrage. For any portfolio θ with $q \cdot \theta \neq 0$, the *return* on θ is the vector R^θ in $\mathbb{R}^S$ defined by $R^\theta_s = (D^\top \theta)_s / \, q \cdot \theta$. Fixing a state-price deflator π, for any such portfolio θ, we have $E(\pi R^\theta) = 1$. Suppose there is a *riskless portfolio,* meaning some portfolio θ with constant return R^0. We then call R^0 the *riskless return.* A bit of algebra shows that for any portfolio θ with a return, we have

$$E(R^\theta) - R^0 = -\frac{\text{cov}(R^\theta, \pi)}{E(\pi)}.$$

Thus, covariation with π has a negative effect on expected return, as one might expect from the interpretation of state prices as shadow prices for wealth.

The *correlation* between any x and y in $\mathbb{R}^S$ is zero if either has zero variance, and is otherwise defined by

$$\operatorname{corr}(x, y) = \frac{\operatorname{cov}(x, y)}{\sqrt{\operatorname{var}(x)\operatorname{var}(y)}}.$$

There is always a portfolio θ^* solving the problem

$$\sup_{\theta} \operatorname{corr}(D^\top\theta, \pi). \tag{8}$$

If there is such a portfolio θ^* with a return R^* having nonzero variance, then it can be shown as an exercise that, for any return R^θ,

$$E(R^\theta) - R^0 = \beta_\theta[E(R^*) - R^0], \tag{9}$$

where

$$\beta_\theta = \frac{\operatorname{cov}(R^*, R^\theta)}{\operatorname{var}(R^*)}.$$

If markets are complete, then R^* is of course perfectly correlated with the state-price deflator.

Formula (9) is a *state-price beta model*, showing excess expected returns on portfolios to be proportional to the excess return on a portfolio having maximal correlation with a state-price deflator, where the constant of proportionality is the associated regression coefficient. The formula can be extended to the case in which there is no riskless return. Another exercise carries this idea, under additional assumptions, to the *Capital Asset Pricing Model*, or *CAPM*.

Exercises

1.1 The dividend-price pair (D, q) of Section A is defined to be *weakly arbitrage-free* if $q \cdot \theta \geq 0$ whenever $D^\top\theta \geq 0$. Show that (D, q) is weakly arbitrage-free if and only if there exist ("weak" state prices) $\psi \in \mathbb{R}^S_+$ such that $q = D\psi$. This fact is known as *Farkas's Lemma.*

1.2 Prove the assertion in Section A that (D, q) is arbitrage-free if and only if there exists some $\psi \in \mathbb{R}^S_{++}$ such that $q = D\psi$. Instead of following the proof given in Section A, use the following result, sometimes known as the *Theorem of the Alternative.*

Stiemke's Lemma. *Suppose A is an $m \times n$ matrix. Then one and only one of the following is true:*

(a) *There exists x in $\mathbb{R}^n_{++}$ with $Ax = 0$.*
(b) *There exists y in $\mathbb{R}^m$ with $y^\top A > 0$.*

1.3 Show, for $U(c) \equiv E[u(c)]$ as defined by (3), that (2) is equivalent to (4).

1.4 Prove the existence of an equilibrium as defined in Section D under these assumptions: There exists some portfolio θ with payoff $D^\top\theta > 0$ and, for all i, $e^i \gg 0$ and U_i is continuous, strictly concave, and strictly increasing. This is a demanding exercise, and calls for the following general result.

Kakutani's Fixed Point Theorem. *Suppose Z is a nonempty convex compact subset of $\mathbb{R}^n$, and for each x in Z, $\varphi(x)$ is a nonempty convex compact subset of Z. Suppose also that $\{(x, y) \in Z \times Z : x \in \varphi(y)\}$ is closed. Then there exists x^* in Z such that $x^* \in \varphi(x^*)$.*

1.5 Prove Proposition D. Hint: The maintained assumption of strict monotonicity of $U_i(\cdot)$ should be used.

1.6 Suppose that the endowment allocation $(e^1, \ldots, e^m)$ is Pareto optimal.

(A) Show, as claimed in Section D, that any equilibrium allocation is Pareto optimal.

(B) Suppose that there is some portfolio θ with $D^\top\theta > 0$ and, for all i, that U_i is concave and $e^i \gg 0$. Show that $(e^1, \ldots, e^m)$ is itself an equilibrium allocation.

1.7 Prove Proposition C. Hint: A continuous real-valued function on a compact set has a maximum.

1.8 Prove Corollary 1 of Proposition E.

1.9 Prove Corollary 2 of Proposition E.

1.10 Suppose, in addition to the assumptions of Proposition E, that

(a) $e = e^1 + \cdots + e^m$ is in $\mathbb{R}^S_{++}$;
(b) for all i, U_i is concave and twice continuously differentiable in $\mathbb{R}^S_{++}$;
(c) for all i, c^i is in $\mathbb{R}^S_{++}$ and the Hessian matrix $\partial^2 U(c^i)$, which is negative semi-definite by concavity, is in fact negative definite.

Property (c) can be replaced with the assumption of *regular preferences*, as defined in a source cited in the Notes.

(A) Show that the assumption that U_λ is continuously differentiable at e is justified and, moreover, that for each i there is a scalar $\gamma_i > 0$ such that $\partial U_\lambda(e) = \gamma_i \partial U_i(c^i)$. (This co-linearity is known as "equal marginal rates of substitution," a property of any Pareto optimal allocation.) Hint: Use the following:

Implicit Function Theorem. *Suppose for given m and n that $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is C^k (k times continuously differentiable) for some $k \geq 1$. Suppose also that the $n \times n$ matrix $\partial_y f(\bar{x}, \bar{y})$ of partial derivatives of f with respect to its second argument is nonsingular at some $(\bar{x}, \bar{y})$. If $f(\bar{x}, \bar{y}) = 0$, then there exist scalars $\epsilon > 0$ and $\delta > 0$ and a C^k function $Z : \mathbb{R}^m \to \mathbb{R}^n$ such that if $\|x - a\| < \epsilon$, then $f[x, Z(x)] = 0$ and $\|Z(x) - b\| < \delta$.*

(B) Show that the negative-definite part of condition (c) is satisfied if $e \gg 0$ and, for all i, U_i is an expected utility function of the form $U_i(c) = E[u_i(c)]$, where u_i is strictly concave with an unbounded derivative on $(0, \infty)$.

(C) Obtain the result of part (A) without assuming the existence of second derivatives of the utilities. (You would therefore not exploit the Hessian matrix or Implicit Function Theorem.) As the first (and main) step, show the following. Given a concave function $f : \mathbb{R}^S_+ \to \mathbb{R}$, the *superdifferential* of f at some x in $\mathbb{R}^S_+$ is

$$\partial f(x) = \{z \in \mathbb{R}^S : f(y) \le f(x) + z \cdot (y - x), \quad y \in \mathbb{R}^S_+\}.$$

For any feasible allocation $(c^1, \ldots, c^m)$ and $\lambda \in \mathbb{R}^m_+$ satisfying $U_\lambda(e) = \sum_i \lambda_i U_i(c^i)$,

$$\partial U_\lambda(e) = \bigcap_{i=1}^m \lambda_i \partial U_i(c_i).$$

1.11 (Binomial Option Pricing). As an application of the results in Section A, consider the following two-state ($S = 2$) option-pricing problem. There are $N = 3$ securities:

(a) a stock, with initial price $q_1 > 0$ and dividend $D_{11} = Gq_1$ in state 1 and dividend $D_{12} = Bq_1$ in state 2, where $G > B > 0$ are the "good" and "bad" gross returns, respectively;
(b) a riskless bond, with initial price $q_2 > 0$ and dividend $D_{21} = D_{22} = Rq_2$ in both states (that is, R is the riskless return and R^{-1} is the discount);
(c) a *call option* on the stock, with initial price $q_3 = C$ and dividend $D_{3j} = (D_{1j} - K)^+ \equiv \max(D_{1j} - K, 0)$ for both states $j = 1$ and $j = 2$, where $K \ge 0$ is the *exercise price* of the option. (The call option gives its holder the right, but not the obligation, to pay K for the stock, with dividend, after the state is revealed.)

(A) Show necessary and sufficient conditions on G, B, and R for the absence of arbitrage involving only the stock and bond.

(B) Assuming no arbitrage for the three securities, calculate the call-option price C explicitly in terms of q_1, G, R, B, and K. Find the state-price probabilities $\hat{\psi}_1$ and $\hat{\psi}_2$ referred to in Section B in terms of G, B, and R, and show that $C = R^{-1}\hat{E}(D_3)$, where $\hat{E}$ denotes expectation with respect to $(\hat{\psi}_1, \hat{\psi}_2)$.

1.12 (CAPM). In the setting of Section D, suppose $(c^1, \ldots, c^m)$ is a strictly positive equilibrium consumption allocation. For any agent i, suppose utility is of the expected-utility form $U_i(c) = E[u_i(c)]$. For any agent i, suppose there are fixed positive constants $\bar{c}$ and b_i such that, for any state j, we have $c^i_j < \bar{c}$ and $u_i(x) = x - b_i x^2$ for all $x \le \bar{c}$.

(A) In the context of Corollary 2 of Section E, show that $u'_\lambda(e) = k - Ke$ for some positive constants k and K. From this, derive the CAPM

$$q = AE(D) - B\ \mathrm{cov}(D, e), \tag{10}$$

for positive constants A and B, where $\mathrm{cov}(D, e) \in \mathbb{R}^N$ is the vector of covariances between the security dividends and the aggregate endowment.

Suppose for a given portfolio θ that each of the following is well defined:

- the return $R^\theta \equiv D^\top\theta / q \cdot \theta$;
- the return R^M on a portfolio M with payoff $D^\top M = e$;
- the return R^0 on a portfolio θ^0 with $\mathrm{cov}(D^\top\theta^0, e) = 0$;
- $\beta_\theta = \mathrm{cov}(R^\theta, R^M)/\mathrm{var}(R^M)$.

The return R^M is sometimes called the *market return*. The return R^0 is called the *zero-beta return* and is the return on a riskless bond if one exists. Prove the "beta" form of the CAPM

$$E(R^\theta - R^0) = \beta_\theta E(R^M - R^0). \tag{11}$$

(B) Part (A) relies on the completeness of markets. Without any such assumption, but assuming that the equilibrium allocation $(c^1, \ldots, c^m)$ is strictly positive, show that the same beta form (11) applies, provided we extend the definition of the market return R^M to be the return on any portfolio solving

$$\sup_{\theta \in \mathbb{R}^N} \mathrm{corr}(R^\theta, e). \tag{12}$$

For complete markets, $\mathrm{corr}(R^M, e) = 1$, so the result of part (A) is a special case.

(C) The CAPM applies essentially as stated without the quadratic expected-utility assumption provided that each agent i is *strictly variance-averse*, in that $U_i(x) > U_i(y)$ whenever $E(x) = E(y)$ and $\mathrm{var}(x) < \mathrm{var}(y)$. Formalize this statement by providing a reasonable set of supporting technical conditions.

We remark that a common alternative formulation of the CAPM allows security portfolios in initial endowments $\hat{\theta}^1, \ldots, \hat{\theta}^m$ with $\sum_{i=1}^m \hat{\theta}^i_j = 1$ for all j. In this case, with the total endowment e redefined by $e = \sum_{i=1}^m (e^i + D^\top\hat{\theta}^i)$, the same CAPM (11) applies. If $e^i = 0$ for all i, then even in incomplete markets, $\mathrm{corr}(R^M, e) = 1$, since (12) is solved by $\theta = (1, 1, \ldots, 1)$. The Notes provide references.

1.13 An *Arrow-Debreu equilibrium* for $[(U_i, e^i), D]$ is a nonzero vector ψ in $\mathbb{R}^S_+$ and a feasible consumption allocation $(c^1, \ldots, c^m)$ such that for each i, c^i solves $\sup_c U_i(c)$ subject to $\psi \cdot c^i \le \psi \cdot e^i$. Suppose that markets are complete, in that $\mathrm{span}(D) = \mathbb{R}^S$. Show that $(c^1, \ldots, c^m)$ is an Arrow-Debreu consumption allocation if and only if it is an equilibrium consumption allocation in the sense of Section D.

1.14 Suppose (D, q) admits no arbitrage. Show that there is a unique state-price vector if and only if markets are complete.

1.15 (Aggregation). For the "representative-agent" problem (6), suppose for all i that $U_i(c) = E[u(c)]$, where $u(c) = c^\gamma/\gamma$ for some nonzero scalar $\gamma < 1$.

(A) Show, for any nonzero agent weight vector $\lambda \in \mathbb{R}^m_+$, that $U_\lambda(c) = E[kc^\gamma/\gamma]$ for some scalar $k > 0$ and that (6) is solved by $c^i = k_i x$ for some scalar $k_i \ge 0$ that is nonzero if and only if λ_i is nonzero.

(B) With this special utility assumption, show that there exists an equilibrium with a Pareto efficient allocation, without the assumption that markets are complete, but with the assumption that $e^i \in \text{span}(D)$ for all i. Calculate the associated equilibrium allocation.

1.16 (State-Price Beta Model). This exercise is to prove and extend the state-price beta model (9) of Section F.

(A) Show problem (8) is solved by any portfolio θ such that $\pi = D^\top \theta + \epsilon$, where $\text{cov}(\epsilon, D^j) = 0$ for any security j, where $D^j \in \mathbb{R}^S$ is the payoff of security j.

(B) Given a solution θ to (8) such that R^θ is well defined with nonzero variance, prove (9).

(C) Reformulate (9) for the case in which there is no riskless return by redefining R^0 to be the expected return on any portfolio θ such that R^θ is well defined and $\text{cov}(R^\theta, \pi) = 0$, assuming such a portfolio exists.

1.17 Prove the Riesz representation lemma of Section F. The following hint is perhaps unnecessary in this simple setting but allows the result to be extended to a broad variety of spaces called *Hilbert spaces*. Given a vector space L, a function $(\cdot \mid \cdot) : L \times L \to \mathbb{R}$ is called an *inner product* for L if, for any x, y, and z in L and any scalar α, we have the five properties:

(a) $(x \mid y) = (y \mid x)$
(b) $(x + y \mid z) = (x \mid z) + (y \mid z)$
(c) $(\alpha x \mid y) = \alpha(x \mid y)$
(d) $(x \mid x) \geq 0$
(e) $(x \mid x) = 0$ if and only if $x = 0$.

Suppose a finite-dimensional vector space L has an inner product $(\cdot \mid \cdot)$. (This defines a special case of a Hilbert space.) Two vectors x and y are defined to be *orthogonal* if $(x \mid y) = 0$. For any linear subspace H of L and any x in L, it can be shown that there is a unique y in H such that $(x - y \mid z) = 0$ for all z in H. This vector y is the orthogonal projection in L of x onto H, and solves the problem $\min_{h \in H} \|x - h\|$. Let $L = \mathbb{R}^S$. For any x and y in L, let $(x \mid y) = E(xy)$. We must show that given a linear functional F, there is a unique π with $F(x) = (\pi \mid x)$ for all x. Let $J = \{x : F(x) = 0\}$. If $J = L$, then F is the zero functional, and the unique representation is $\pi = 0$. If not, there is some z such that $F(z) = 1$ and $(z \mid x) = 0$ for all x in J. Show this using the idea of orthogonal projection. Then show that $\pi = z/(z \mid z)$ represents F, using the fact that for any x, we have $x - F(x)z \in J$.

1.18 Suppose there are $m = 2$ consumers, A and B, with identical utilities for consumption c_1 and c_2 in states 1 and 2 given by $U(c_1, c_2) = 0.2\sqrt{c_1} + 0.5 \log c_2$. There is a total endowment of $e_1 = 25$ units of consumption in state 1.

(A) Suppose that markets are complete and that, in a given equilibrium, consumer A's consumption is 9 units in state 1 and 10 units in state 2. What is the total endowment e_2 in state 2?

(B) Continuing under the assumptions of part (A), suppose there are two securities. The first is a riskless bond paying 10 units of consumption in each state. The second is a risky asset paying 5 units of consumption in state 1 and 10 units in state 2. In equilibrium, what is the ratio of the price of the bond to that of the risky asset?

1.19 There are two states of the world, labeled 1 and 2, two agents, and two securities, both paying units of the consumption numeraire good. The risky security pays a total of 1 unit in state 1 and pays 3 units in state 2. The riskless security pays 1 unit in each state. Each agent is initially endowed with half of the total supply of the risky security. There are no other endowments. (The riskless security is in zero net supply.) The two agents assign equal probabilities to the two states. One of the agents is risk-neutral, with utility function $E(c)$ for state-contingent consumption c, and can consume negatively or positively in both states. The other, risk-averse, agent has utility $E(\sqrt{c})$ for nonnegative state-contingent consumption. Solve for the equilibrium allocation of the two securities in a competitive equilibrium.

1.20 Consider a setting with two assets A and B, only, both paying off the same random variable X, whose value is nonnegative in every state and nonzero with strictly positive probability. Asset A has price p, while asset B has price q. An arbitrage is then a portfolio $(\alpha, \beta) \in \mathbb{R}^2$ of the two assets whose total payoff $\alpha X + \beta X$ is nonnegative and whose initial price $\alpha p + \beta q$ is strictly negative, or whose total payoff is nonzero with strictly positive probability and always nonnegative, and whose initial price is negative or zero.

(A) Assuming no restrictions on portfolios, and no transactions costs or frictions, state the set of arbitrage-free prices (p, q). (State precisely the appropriate subset of $\mathbb{R}^2$.)

(B) Assuming no short sales ($\alpha \geq 0$ and $\beta \geq 0$), state the set of arbitrage-free prices (p, q).

(C) Now suppose that A and B can be short sold, but that asset A can be short sold only by paying an extra fee of $\phi > 0$ per unit sold short. There are no other fees of any kind. Provide the obvious new definition of "no arbitrage" in precise mathematical terms, and state the set of arbitrage-free prices.

Notes

The basic approach of this chapter follows Arrow (1953), taking a general equilibrium perspective originating with Walras (1877). Black (1995) offers a perspective on the general equilibrium approach and a critique of other approaches.

(A) The state-pricing implications of no arbitrage found in Section A originate with Ross (1978).

(B) The idea of "risk-neutral probabilities" apparently originates with Arrow (1970), a revision of Arrow (1953), and appears as well in Drèze (1971).

(C) This material is standard.

(D) Proposition D is the First Welfare Theorem of Arrow (1951) and Debreu (1954). The generic inoptimality of incomplete-markets equilibrium allocations can be gleaned from sources cited by Geanakoplos (1990). Indeed, Geanakoplos and Polemarchakis (1986) show that even a reasonable notion of constrained optimality generically fails in certain incomplete-markets settings. See, however, Kajii (1994) and references cited in the Notes of Chapter 2 for mitigating results. Mas-Colell (1987) and Werner (1991) also treat constrained optimality.

(E) The "representative-agent" approach goes back, at least, to Negishi (1960). The existence of a representative agent is no more than an illustrative simplification in this setting, and should not be confused with the more demanding notion of aggregation of Gorman (1953) found in Exercise 15. In Chapter 10, the existence of a representative agent with smooth utility, based on Exercise 1.11, is important for technical reasons.

(F) The "beta model" for pricing goes back, in the case of mean-variance preferences, to the capital asset pricing model, or CAPM, of Sharpe (1964) and Lintner (1965). The version without a riskless asset is due to Black (1972). Allingham (1991), Berk (1992), Nielsen (1990a), and Nielsen (1990b) address the existence of equilibrium in the CAPM. Characterization of the mean-variance model and two-fund separation is provided by Bottazzi, Hens, and Löffler (1994), Nielsen (1993b), and Nielsen (1993a). Löffler (1996) provides sufficient conditions for variance aversion in terms of mean-variance preferences.

Additional Topics: Ross (1976) introduced the *arbitrage pricing theory*, a multifactor model of asset returns that, in terms of expected returns, can be thought of as an extension of the CAPM. In this regard, see also Bray (1994a), Bray (1994b), and Gilles and LeRoy (1991). Balasko and Cass (1986) and Balasko, Cass, and Siconolfi (1990) treat equilibrium with constrained participation in security trading. See also Hara (1994).

Debreu (1972) provides a notion of regular preferences that substitutes for the existence of a negative-definite Hessian matrix of each agent's utility function at the equilibrium allocation. For more on regular preferences and the differential approach to general equilibrium, see Mas-Colell (1985) and Balasko (1989). Kreps (1988) reviews the theory of choice and utility representations of preferences. For Farkas's and Stiemke's Lemmas, and other forms of the Theorem of the Alternative, see Gale (1960).

Arrow and Debreu (1954) and, in a slightly different model, McKenzie (1954) are responsible for a proof of the existence of complete-markets equilibria. Debreu (1982) surveys the existence problem. Standard introductory treatments of general equilibrium theory are given by Debreu (1959) and Hildenbrand and Kirman (1989). In this setting, with incomplete markets, Polemarchakis and Siconolfi (1993) address the failure of existence unless one has a portfolio θ with payoff $D^\top \theta > 0$. Geanakoplos (1990) surveys other literature on the existence of equilibria in incomplete markets, some of which takes the alternative of defining security payoffs in nominal units of account, while allowing consumption

of multiple commodities. Most of the literature allows for an initial period of consumption before the realization of the uncertain state. For a survey, see Magill and Shafer (1991). Additional results on incomplete-markets equilibrium include those of Araujo and Monteiro (1989), Berk (1997), Boyle and Wang (1999), and Weil (1992).

For related results in multiperiod settings, references are cited in the Notes of Chapter 2.

The superdifferentiability result of Exercise 10(C) is due to Skiadas (1995).

Hellwig (1996), Mas-Colell and Monteiro (1996), and Monteiro (1996) have recently shown existence of equilibrium with a continuum of states. Geanakoplos and Polemarchakis (1986) and Chae (1988) show existence in a model closely related to that studied in this chapter. Grodal and Vind (1988) and Yamazaki (1991) show existence with alternative formulations. With multiple commodities or multiple periods, existence is not guaranteed under any natural conditions, as shown by Hart (1975), who gives a counterexample. For these more delicate cases, the literature on generic existence is cited in the Notes of Chapter 2.

The binomial option-pricing formula of Exercise 1.11 is from an early edition of Sharpe (1985), and is extended in Chapter 2 to a multiperiod setting. The hint given for the demonstration of the Riesz representation exercise is condensed from the proof given by Luenberger (1969) of the *Riesz-Frechet Theorem*: For any Hilbert space H with inner product $(\cdot \mid \cdot)$, any continuous linear functional $F : H \to \mathbb{R}$ has a unique π in H such that $F(x) = (\pi \mid x)$, $x \in H$. The Fixed Point Theorem of Exercise 1.4 is from Kakutani (1941).

On the role of default and collateralization, see Geanakoplos and Zame (1999) and Sabarwal (1999). Gottardi and Kajii (1999) study the role and existence of sunspot equilibria. Pietra (1992) treats indeterminacy. Lobo, Fazel, Boyd (1999) address portfolio choice with fixed transactions costs.

2

The Basic Multiperiod Model

THIS CHAPTER EXTENDS the results of Chapter 1 on arbitrage, optimality, and equilibrium to a multiperiod setting. A connection is drawn between state prices and martingales for the purpose of representing security prices. The exercises include the consumption-based capital asset pricing model and the multiperiod "binomial" option pricing model.

A. Uncertainty

As in Chapter 1, there is some finite set, say Ω, of states. In order to handle multiperiod issues, however, we will treat uncertainty a bit more formally as a *probability space* $(\Omega, \mathcal{F}, P)$, with $\mathcal{F}$ denoting the *tribe* of subsets of Ω that are *events* (and can therefore be assigned a probability), and with P a *probability measure* assigning to any event B in $\mathcal{F}$ its probability $P(B)$. Those not familiar with the definition of a probability space can consult Appendix A. The terms "σ-algebra" and "σ-field," among others, are often used in place of the word "tribe."

There are $T+1$ dates: $0, 1, \ldots, T$. At each of these, a tribe $\mathcal{F}_t \subset \mathcal{F}$ denotes the set of events corresponding to the information available at time t. In effect, an event B in $\mathcal{F}_t$ is known at time t to be true or false. (A definition of tribes in terms of "partitions" of Ω is given in Exercise 2.11.) We adopt the usual convention that $\mathcal{F}_t \subset \mathcal{F}_s$ whenever $t \leq s$, meaning that events are never "forgotten." For simplicity, we also take it that every event in $\mathcal{F}_0$ has probability 0 or 1, meaning roughly that there is no information at time $t = 0$. Taken altogether, the *filtration* $\mathbb{F} = \{\mathcal{F}_0, \ldots, \mathcal{F}_T\}$ represents how information is revealed through time. For any random variable Y, we let $E_t(Y) = E(Y \mid \mathcal{F}_t)$ denote the conditional expectation of Y given $\mathcal{F}_t$. (Appendix A provides definitions of random variables and of conditional expectation.) An *adapted process* is a sequence $X = \{X_0, \ldots, X_T\}$ such that for each t, X_t is a random variable with respect to $(\Omega, \mathcal{F}_t)$. Informally, this means that X_t is observable at time t. An adapted process X is a *martingale* if, for any times t and $s > t$, we have $E_t(X_s) = X_t$. As we shall see, martingales are useful in the characterization of security prices. In order to simplify things, for any two random variables Y and Z, we always write "$Y = Z$" if the probability that $Y \neq Z$ is zero.

B. Security Markets

A security is a claim to an adapted *dividend process*, say δ, with δ_t denoting the dividend paid by the security at time t. Each security has an adapted *security-price process* S, so that S_t is the price of the security, *ex dividend*, at time t. That is, at each time t, the security pays its dividend δ_t and is then available for trade at the price S_t. This convention implies that δ_0 plays no role in determining ex-dividend prices. The *cum-dividend* security price at time t is $S_t + \delta_t$.

Suppose there are N securities defined by the $\mathbb{R}^N$-valued adapted dividend process $\delta = (\delta^{(1)}, \ldots, \delta^{(N)})$. These securities have some adapted price process $S = (S^{(1)}, \ldots, S^{(N)})$. A *trading strategy* is an adapted process θ in $\mathbb{R}^N$. Here, $\theta_t = (\theta_t^{(1)}, \ldots, \theta_t^{(N)})$ represents the portfolio held after trading at time t. The dividend process δ^θ *generated* by a trading strategy θ is defined by

$$\delta_t^\theta = \theta_{t-1} \cdot (S_t + \delta_t) - \theta_t \cdot S_t, \tag{1}$$

with "θ_{-1}" taken to be zero by convention.

C. Arbitrage, State Prices, and Martingales

Given a dividend-price pair (δ, S) for N securities, a trading strategy θ is an *arbitrage* if $\delta^\theta > 0$. (The reader should become convinced that this is the same notion of arbitrage defined in Chapter 1.) Let Θ denote the space of trading strategies. For any θ and φ in Θ and scalars a and b, we have $a\delta^\theta + b\delta^\varphi = \delta^{a\theta + b\varphi}$. Thus the *marketed subspace* $M = \{\delta^\theta : \theta \in \Theta\}$ of dividend processes generated by trading strategies is a linear subspace of the space L of adapted processes.

Proposition. *There is no arbitrage if and only if there is a strictly increasing linear function $F : L \to \mathbb{R}$ such that $F(\delta^\theta) = 0$ for any trading strategy θ.*

Proof: The proof is almost identical to that of Theorem 1A. Let $L_+ = \{c \in L : c \geq 0\}$. There is no arbitrage if and only if the cone L_+ and the marketed subspace M intersect precisely at zero. Suppose there is no arbitrage. The Separating Hyperplane Theorem, in a form given in Appendix B for cones, implies the existence of a nonzero linear functional F such that $F(x) < F(y)$ for each x in M and each nonzero y in L_+. Since M is a linear subspace, this implies that $F(x) = 0$ for each x in M, and thus that $F(y) > 0$ for each nonzero y in L_+. This implies that F is strictly increasing. The converse is immediate. ∎

The following result gives a convenient *Riesz representation* of a linear function on the space of adapted processes. Proof is left as an exercise, extending the single-period Riesz representation lemma of Section 1F.

Lemma. *For each linear function $F : L \to \mathbb{R}$, there is a unique π in L, called the Riesz representation of F, such that*

$$F(x) = E\left(\sum_{t=0}^{T} \pi_t x_t\right), \qquad x \in L.$$

If F is strictly increasing, then π is strictly positive.

For convenience, we call any strictly positive adapted process a *deflator.* A deflator π is a *state-price deflator* if, for all t,

$$S_t = \frac{1}{\pi_t} E_t\left(\sum_{j=t+1}^{T} \pi_j \delta_j\right). \tag{2}$$

A state-price deflator is variously known in the literature as a *state-price density*, a *pricing kernel*, and a *marginal-rate-of-substitution process.*

For $t = T$, the right-hand side of (2) is zero, so $S_T = 0$ whenever there is a state-price deflator. The notion here of a state-price deflator is a natural extension of that of Chapter 1. It can be shown as an exercise that a deflator π is a state-price deflator if and only if, for any trading strategy θ,

$$\theta_t \cdot S_t = \frac{1}{\pi_t} E_t\left(\sum_{j=t+1}^{T} \pi_j \delta_j^\theta\right), \qquad t < T, \tag{3}$$

meaning roughly that the market value of a trading strategy is, at any time, the state-price discounted expected future dividends generated by the strategy. The *cum-dividend* value process V^θ of a trading strategy θ is defined by $V_t^\theta = \theta_{t-1} \cdot (S_t + \delta_t)$. If π is a state-price deflator, we have

$$V_t^\theta = \frac{1}{\pi_t} E_t\left(\sum_{j=t}^{T} \pi_j \delta_j^\theta\right).$$

The *gain process* G for (δ, S) is defined by $G_t = S_t + \sum_{j=1}^{t} \delta_j$, the price plus accumulated dividend. Given a deflator γ, the *deflated gain process* G^γ is defined by $G_t^\gamma = \gamma_t S_t + \sum_{j=1}^{t} \gamma_j \delta_j$. We can think of deflation as a change of numeraire.

Theorem. *The dividend-price pair (δ, S) admits no arbitrage if and only if there is a state-price deflator. A deflator π is a state-price deflator if and only if $S_T = 0$ and the state-price-deflated gain process G^π is a martingale.*

Proof: It can be shown as an easy exercise that a deflator π is a state-price deflator if and only if $S_T = 0$ and the state-price-deflated gain process G^π is a martingale.

Suppose there is no arbitrage. Then $S_T = 0$, for otherwise the strategy θ is an arbitrage when defined by $\theta_t = 0$, $t < T$, $\theta_T = -S_T$. The previous proposition implies that there is some strictly increasing linear function $F : L \to \mathbb{R}$ such that $F(\delta^\theta) = 0$ for any strategy θ. By the previous lemma, there is some deflator π such that $F(x) = E(\sum_{t=0}^{T} x_t \pi_t)$ for all x in L. This implies that $E(\sum_{t=0}^{T} \delta_t^\theta \pi_t) = 0$ for any strategy θ.

We must prove (2), or equivalently, that G^π is a martingale. From Appendix A, an adapted process X is a martingale if and only if $E(X_\tau) = X_0$ for any stopping time $\tau \le T$. Consider, for an arbitrary security n and an arbitrary stopping time $\tau \le T$, the trading strategy θ defined by $\theta^{(k)} = 0$ for $k \ne n$ and $\theta_t^{(n)} = 1$, $t < \tau$, with $\theta_t^{(n)} = 0$, $t \ge \tau$. Since $E(\sum_{t=0}^{T} \pi_t \delta_t^\theta) = 0$, we have

$$E\left(- S_0^{(n)} \pi_0 + \sum_{t=1}^{\tau} \pi_t \delta_t^{(n)} + \pi_\tau S_\tau^{(n)} \right) = 0,$$

implying that the deflated gain process $G^{n\pi}$ of security n satisfies $G_0^{n,\pi} = E(G_\tau^{n,\pi})$. Since τ is arbitrary, $G^{n,\pi}$ is a martingale, and since n is arbitrary, G^π is a martingale.

This shows that absence of arbitrage implies the existence of a state-price deflator. The converse is easy. ∎

D. Individual Agent Optimality

We introduce an agent, defined by a strictly increasing utility function U on the set L_+ of nonnegative adapted "consumption" processes, and by an *endowment process* e in L_+. Given a dividend-price process (δ, S), a trading strategy θ leaves the agent with the total consumption process $e + \delta^\theta$. Thus the agent has the *budget-feasible consumption set*

$$X = \{e + \delta^\theta \in L_+ : \theta \in \Theta\},$$

and the problem

$$\sup_{c \in X} U(c). \tag{4}$$

The existence of a solution to (4) implies the absence of arbitrage. Conversely, it can be shown as an exercise that if U is continuous, then the absence of arbitrage implies that there exists a solution to (4). For purposes of checking continuity or the closedness of sets in L, we will say that c_n converges to c if $E[\sum_{t=0}^{T} |c_n(t) - c(t)|] \to 0$. Then U is continuous if $U(c_n) \to U(c)$ whenever $c_n \to c$.

Suppose that (4) has a strictly positive solution c^* and that U is continuously differentiable at c^*. We can use the first-order conditions for optimality (which can be reviewed in Appendix B) to characterize security prices in terms of the derivatives of the utility function U at c^*. Specifically, for any c in L, the derivative of U at c^* in the direction c is the derivative $g'(0)$, where $g(\alpha) = U(c^* + \alpha c)$ for any scalar α sufficiently small in absolute value. That is, $g'(0)$ is the marginal rate of improvement of utility as one moves in the direction c away from c^*. This derivative is denoted $\nabla U(c^*; c)$. Because U is continuously differentiable at c^*, the function $c \mapsto \nabla U(c^*; c)$, on L into $\mathbb{R}$, is linear. Since δ^θ is a budget-feasible direction of change for any trading strategy θ, the first-order conditions for optimality of c^* imply that

$$\nabla U(c^*; \delta^\theta) = 0, \qquad \theta \in \Theta.$$

We now have a characterization of a state-price deflator.

Proposition. *Suppose that (4) has a strictly positive solution c^* and that U has a strictly positive continuous derivative at c^*. Then there is no arbitrage and a state-price deflator is given by the Riesz representation π of $\nabla U(c^*)$:*

$$\nabla U(c^*; x) = E\left(\sum_{t=0}^{T} \pi_t x_t \right), \qquad x \in L.$$

Despite our standing assumption that U is strictly increasing, $\nabla U(c^*; \cdot)$ need not in general be strictly increasing, but is so if U is concave.

As an example, suppose U has the *additive* form

$$U(c) = E\left[\sum_{t=0}^{T} u_t(c_t) \right], \qquad c \in L_+, \tag{5}$$

for some $u_t : \mathbb{R}_+ \to \mathbb{R}$, $t \geq 0$. It is an exercise to show that if $\nabla U(c)$ exists, then

$$\nabla U(c; x) = E\left[\sum_{t=0}^{T} u_t'(c_t)x_t\right]. \tag{6}$$

If, for all t, u_t is concave with an unbounded derivative and e is strictly positive, then any solution c^* to (4) is strictly positive.

Corollary. *Suppose U is defined by (5). Under the conditions of the proposition, for any times t and $\tau \geq t$,*

$$S_t = \frac{1}{u_t'(c_t^*)} E_t\left[S_\tau u_\tau'(c_\tau^*) + \sum_{j=t+1}^{\tau} \delta_j u_j'(c_j^*)\right].$$

For the case $\tau = t + 1$, this result is often called the *stochastic Euler equation.* Extending this classical result for additive utility, the exercises include other utility examples such as *habit-formation* utility and *recursive* utility. As in Chapter 1, we now turn to the multi-agent case.

E. Equilibrium and Pareto Optimality

Suppose there are m agents. Agent i is defined as above by a strictly increasing utility function $U_i : L_+ \to \mathbb{R}$ and an endowment process $e^{(i)}$ in L_+. Given a dividend process δ for N securities, an *equilibrium* is a collection $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$, where S is a security-price process and, for each i, $\theta^{(i)}$ is a trading strategy solving

$$\sup_{\theta \in \Theta} U_i(c) \text{ subject to } c = e^{(i)} + \delta^\theta \in L_+, \tag{7}$$

with $\sum_{i=1}^m \theta^{(i)} = 0$.

We define markets to be *complete* if, for each process x in L, there is some trading strategy θ with $\delta_t^\theta = x_t$, $t \geq 1$. Complete markets thus means that any consumption process x can be obtained by investing some amount at time 0 in a trading strategy that generates the dividend x_t in each future period t. With the same definition of Pareto optimality, Proposition 1D carries over to this multiperiod setting. Any equilibrium $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$ has an associated feasible consumption allocation $(c^{(1)}, \ldots, c^{(m)})$ defined by letting $c^{(i)} - e^{(i)}$ be the dividend process generated by $\theta^{(i)}$.

Proposition. *Suppose $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$ is an equilibrium and markets are complete. Then the associated consumption allocation is Pareto optimal.*

The completeness of markets depends on the security-price process S itself. Indeed, the dependence of the marketed subspace on S makes the existence of an equilibrium a nontrivial issue. We ignore existence here and refer to the Notes for some relevant sources.

F. Equilibrium Asset Pricing

Again following the ideas in Chapter 1, we define for each λ in $\mathbb{R}_+^m$ the utility function $U_\lambda : L_+ \to \mathbb{R}$ by

$$U_\lambda(x) = \sup_{(c^{(1)}, \ldots, c^{(m)})} \sum_{i=1}^{m} \lambda_i U_i(c^i) \qquad \text{subject to } c^{(1)} + \cdots + c^{(m)} \leq x. \tag{8}$$

Proposition. *Suppose for all i that U_i is concave and strictly increasing. Suppose that $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$ is an equilibrium and that markets are complete. Then there exists some nonzero $\lambda \in \mathbb{R}_+^m$ such that $(0, S)$ is a (no-trade) equilibrium for the one-agent economy $[(U_\lambda, e), \delta]$, where $e = e^{(1)} + \cdots + e^{(m)}$. With this λ and with $x = e = e^{(1)} + \cdots + e^{(m)}$, problem (8) is solved by the equilibrium consumption allocation.*

Proof is assigned as an exercise. The result is essentially the same as Proposition 1E. A method of proof, as well as the intuition for this proposition, is that with complete markets, a state-price deflator π represents Lagrange multipliers for consumption in the various periods and states for all of the agents simultaneously, as well as for the *representative agent* (U_λ, e).

Corollary 1. *If, moreover, U_λ is differentiable at e, then λ can be chosen so that for any times t and $\tau \geq t$, there is a state-price deflator π equal to the Riesz representation of $\nabla U_\lambda(e)$.*

Differentiability of U_λ at e can be shown by the arguments used in Exercise 1.10.

Corollary 2. *Suppose for each i that U_i is of the additive form*

$$U_i(c) = E\left[\sum_{t=0}^{T} u_{it}(c_t)\right].$$

Then U_λ is also additive, with

$$U_\lambda(c) = E\left[\sum_{t=0}^{T} u_{\lambda t}(c_t)\right],$$

where

$$u_{\lambda t}(y) = \sup_{x \in \mathbb{R}^m_+} \sum_{i=1}^{m} \lambda_i u_{it}(x_i) \qquad \textit{subject to } x_1 + \cdots + x_m \leq y.$$

In this case, the differentiability of U_λ at e implies that for any times t and $\tau \geq t$,

$$S_t = \frac{1}{u'_{\lambda t}(e_t)} E_t\left[u'_{\lambda\tau}(e_\tau) S_\tau + \sum_{j=t+1}^{\tau} u'_{\lambda j}(e_j)\delta_j \right]. \tag{9}$$

G. Arbitrage and Martingale Measures

This section shows the equivalence between the absence of arbitrage and the existence of a probability measure Q with the property, roughly speaking, that the price of a security is the sum of Q-expected discounted dividends.

There is *short-term riskless borrowing* if, for each given time $t < T$, there is a security trading strategy θ with $\delta^\theta_{t+1} = 1$ and with $\delta^\theta_s = 0$ for $s < t$ and $s > t+1$. The associated *discount* is $d_t = \theta_t \cdot S_t$. If there is no arbitrage, the discount d_t is uniquely defined and strictly positive, and we may define the associated *short rate* r_t by $1 + r_t = 1/d_t$. This means that at any time $t < T$, one may invest one unit of account in order to receive $1 + r_t$ units of account at time $t+1$. We refer to $\{r_0, r_1, \ldots, r_{T-1}\}$ as the associated "short-rate process," even though r_T is not defined.

We suppose throughout this section that there is short-term riskless borrowing at some uniquely defined short-rate process r. We can define, for any times t and $\tau \leq T$,

$$R_{t,\tau} = (1 + r_t)(1 + r_{t+1}) \cdots (1 + r_{\tau-1}),$$

the payback at time τ of one unit of account borrowed risklessly at time t and "rolled over" in short-term borrowing repeatedly until date τ.

It would be a simple situation, both computationally and conceptually, if any security's price were merely the expected discounted dividends of the security. Of course, this is unlikely to be the case in a market with risk-averse investors. We can nevertheless come close to this sort of characterization of security prices by adjusting the original probability measure P. For this, we define a new probability measure Q to be *equivalent* to P if Q and P assign zero probabilities to the same events. An equivalent probability measure Q is an *equivalent martingale measure* if

$$S_t = E^Q_t\left(\sum_{j=t+1}^{T} \frac{\delta_j}{R_{t,j}} \right), \qquad t < T,$$

where E^Q denotes expectation under Q, and likewise $E^Q_t(x) = E^Q(x|\mathcal{F}_t)$ for any random variable x. An equivalent martingale measure is often called a *risk-neutral measure.*

It is easy to show that Q is an equivalent martingale measure if and only if, for any trading strategy θ,

$$\theta_t \cdot S_t = E^Q_t\left(\sum_{j=t+1}^{T} \frac{\delta^\theta_j}{R_{t,j}} \right), \qquad t < T. \tag{10}$$

If interest rates are deterministic, (10) is merely the total discounted expected dividends, after substituting Q for the original measure P. We will show that the absence of arbitrage is equivalent to the existence of an equivalent martingale measure.

The deflator γ defined by $\gamma_t = R^{-1}_{0,t}$ defines the *discounted gain process* G^γ. The word "martingale" in the term "equivalent martingale measure" comes from the following equivalence.

Lemma. *A probability measure Q equivalent to P is an equivalent martingale measure for (δ, S) if and only if $S_T = 0$ and the discounted gain process G^γ is a martingale with respect to Q.*

We already know from Theorem C that the absence of arbitrage is equivalent to the existence of a state-price deflator π. As explained in Appendix A, a probability measure Q equivalent to P can be defined in terms of a Radon-Nikodym derivative, a strictly positive random variable $\frac{dQ}{dP}$ with $E(\frac{dQ}{dP}) = 1$, via the definition of expectation with respect to Q given by $E^Q(Z) = E(\frac{dQ}{dP}Z)$, for any random variable Z. We will choose a particular equivalent probability measure Q by the Radon-Nikodym derivative $\frac{dQ}{dP} = \xi_T$, where

$$\xi_T = \frac{\pi_T R_{0,T}}{\pi_0}.$$

(Indeed, one can check that ξ_T is strictly positive and of expectation 1.) The *density process* ξ for Q is defined by $\xi_t = E_t(\xi_T)$. Relation (A.2) of Appendix A implies that for any times t and $j > t$, and any $\mathcal{F}_j$-measurable random variable Z_j,

$$E^Q_t(Z_j) = \frac{1}{\xi_t} E_t(\xi_j Z_j). \tag{11}$$

Fixing some time $t < T$, consider a trading strategy θ that invests one unit of account at time t and repeatedly rolls the value over in short-term riskless borrowing until time T, with final value $R_{t,T}$. That is, $\theta_t \cdot S_t = 1$ and $\delta^\theta_T = R_{t,T}$. Relation (3) then implies that

$$\pi_t = E_t(\pi_T R_{t,T}) = \frac{E_t(\pi_T R_{0,T})}{R_{0,t}} = \frac{E_t(\xi_T \pi_0)}{R_{0,t}} = \frac{\xi_t \pi_0}{R_{0,t}}. \qquad (12)$$

From (11), (12), and the definition of a state-price deflator, (10) is satisfied, so Q is indeed an equivalent martingale measure. We have shown the following result.

Theorem. *There is no arbitrage if and only if there exists an equivalent martingale measure. Moreover, π is a state-price deflator if and only if an equivalent martingale measure Q has the density process ξ defined by $\xi_t = R_{0,t}\pi_t/\pi_0$.*

Proposition. *Suppose that $\mathcal{F}_T = \mathcal{F}$ and there is no arbitrage. Then markets are complete if and only if there is a unique equivalent martingale measure.*

Proof: Suppose that markets are complete and let Q_1 and Q_2 be two equivalent martingale measures. We must show that $Q_1 = Q_2$. Let A be any event. Since markets are complete, there is a trading strategy θ with dividend process δ^θ such that $\delta^\theta_T = R_{0,T}1_A$ and $\delta^\theta_t = 0$, $0 < t < T$. By (10), we have $\theta_0 \cdot S_0 = Q_1(A) = Q_2(A)$. Since A is arbitrary, $Q_1 = Q_2$.

Exercise 2.18 outlines a proof of the converse part of the result. ∎

This martingale approach simplifies many asset pricing problems that might otherwise appear to be quite complex. This approach also applies much more generally than indicated here. For example, the assumption of short-term borrowing is merely a convenience. More generally, as elaborated in Chapter 6, one can typically obtain an equivalent martingale measure after normalizing prices and dividends by the price of some particular security (or trading strategy).

H. Valuation of Redundant Securities

Suppose that the given dividend-price pair (δ, S) is arbitrage-free, with an associated state-price deflator π. Now consider the introduction of a new security with dividend process $\hat{\delta}$ and price process $\hat{S}$. We say that $\hat{\delta}$ is *redundant* given (δ, S) if there exists a trading strategy θ, with respect to only the original security dividend-price process (δ, S), that *replicates* $\hat{\delta}$, in the sense that $\delta^\theta_t = \hat{\delta}_t$, $t \geq 1$. In this case, the absence of arbitrage for the "augmented" dividend-price process $[(\delta, \hat{\delta}), (S, \hat{S})]$ implies that $\hat{S}_t = Y_t$, where

$$Y_t = \frac{1}{\pi_t} E_t\left(\sum_{j=t+1}^{T} \pi_j \hat{\delta}_j\right), \qquad t < T.$$

If this were not the case, there would be an arbitrage, as follows. For example, suppose that for some stopping time τ, we have $\hat{S}_\tau > Y_\tau$, and that $\tau \leq T$ with strictly positive probability. We can then define the strategy:

(a) Sell the redundant security $\hat{\delta}$ at time τ for $\hat{S}_\tau$, and hold this position until T.

(b) Invest $\theta_\tau \cdot S_\tau$ at time τ in the replicating strategy θ, and follow this strategy until T.

Since the dividends generated by this combined strategy (a)–(b) after τ are zero, the only dividend is at τ for the amount $\hat{S}_\tau - Y_\tau > 0$, which means that this is an arbitrage. Likewise, if $\hat{S}_\tau < Y_\tau$ for some nontrivial stopping time τ, the opposite strategy is an arbitrage. We have shown the following.

Proposition. *Suppose (δ, S) is arbitrage-free with state-price deflator π. Let $\hat{\delta}$ be a redundant dividend process with price process $\hat{S}$. Then the augmented dividend-price pair $[(\delta, \hat{\delta}), (S, \hat{S})]$ is arbitrage-free if and only if it has π as a state-price deflator.*

In applications, it is often assumed that (δ, S) generates complete markets, in which case any additional security is redundant. Exercise 2.1 gives a classical example in which the redundant security is an option on one of the original securities.

I. American Exercise Policies and Valuation

We now extend our pricing framework to include a family of securities, called "American," for which there is discretion regarding the timing of cash flows.

Given an adapted process X, each finite-valued stopping time τ generates a dividend process $\delta^{X,\tau}$ defined by $\delta^{X,\tau}_t = 0$, $t \neq \tau$, and $\delta^{X,\tau}_\tau = X_\tau$. In this context, a finite-valued stopping time is an *exercise policy*, determining the time at which to accept payment. Any exercise policy τ is constrained by $\tau \leq \bar{\tau}$, for some *expiration time* $\bar{\tau} \leq T$. (In what follows, we might take $\bar{\tau}$ to be a stopping time, which is useful for the case of certain *knockout options*, as shown for example in Exercise 2.1.) We say that $(X, \bar{\tau})$ defines an *American security*. The exercise policy is selected by the holder of the security. Once exercised, the security has no remaining cash flows. A standard example is an American put option on a security with price process p. The American put gives the holder of the option the right, but not the obligation, to sell the underlying security for a fixed exercise price at any time before a given expiration time $\bar{\tau}$. If the option has an exercise price K and expiration time $\bar{\tau} < T$, then $X_t = (K - p_t)^+$, $t \leq \bar{\tau}$, and $X_t = 0$, $t > \bar{\tau}$.

We will suppose that, in addition to an American security $(X, \bar{\tau})$, there are securities with an arbitrage-free dividend-price process (δ, S) that generates complete markets. The assumption of complete markets will dramatically simplify our analysis since it implies, for any exercise policy τ, that the dividend process $\delta^{X,\tau}$ is redundant given (δ, S). For notational convenience, we assume that $0 < \bar{\tau} < T$.

Let π be a state-price deflator associated with (δ, S). From Proposition H, given any exercise policy τ, the American security's dividend process $\delta^{X,\tau}$ has an associated cum-dividend price process, say V^τ, which, in the absence of arbitrage, satisfies

$$V_t^\tau = \frac{1}{\pi_t} E_t(\pi_\tau X_\tau), \qquad t \leq \tau.$$

This value does not depend on which state-price deflator is chosen because, with complete markets, state-price deflators are all equal up to a positive rescaling, as one can see from the theorem and proposition of Section G.

We consider the optimal stopping problem

$$V_0^* \equiv \max_{\tau \in \mathcal{T}(0)} V_0^\tau, \tag{13}$$

where, for any time $t \leq \bar{\tau}$, we let $\mathcal{T}(t)$ denote the set of stopping times bounded below by t and above by $\bar{\tau}$. A solution to (13) is called a *rational exercise policy* for the American security X, in the sense that it maximizes the initial arbitrage-free value of the resulting claim.

We claim that in the absence of arbitrage, the actual initial price V_0 for the American security must be the "rational value" V_0^*. In order to see this, suppose first that $V_0^* > V_0$. Then one could buy the American security, adopt for it a rational exercise policy τ, and also undertake a trading strategy replicating $-\delta^{X,\tau}$. Since $V_0^* = E(\pi_\tau X_\tau)/\pi_0$, this replication involves an initial payoff of V_0^*, and the net effect is a total initial dividend of $V_0^* - V_0 > 0$ and zero dividends after time 0, which defines an arbitrage.

Thus the absence of arbitrage easily leads to the conclusion that $V_0 \geq V_0^*$. It remains to show that the absence of arbitrage also implies the opposite inequality $V_0 \leq V_0^*$.

Suppose that $V_0 > V_0^*$. One could sell the American security at time 0 for V_0. We will show that for an initial investment of V_0^*, one can "super-replicate" the payoff at exercise demanded by the holder of the American security, *regardless of the exercise policy used.* Specifically, a *super-replicating trading strategy* for $(X, \bar{\tau}, \delta, S)$ is a trading strategy θ involving only the securities with dividend-price process (δ, S) that has the properties:

(a) $\delta_t^\theta = 0$ for $0 < t < \bar{\tau}$, and
(b) $V_t^\theta \geq X_t$ for all $t \leq \bar{\tau}$,

where, we recall, V_t^θ is the cum-dividend value of θ at time t. Regardless of the exercise policy τ used by the holder of the security, the payment of X_τ demanded at time τ is dominated by the market value V_t^θ of a super-replicating strategy θ. (In effect, one modifies θ by liquidating the portfolio θ_τ at time τ, so that the actual trading strategy φ associated with the arbitrage is defined by $\varphi_t = \theta_t$ for $t < \tau$ and $\varphi_t = 0$ for $t \geq \tau$.) Now, suppose θ is super-replicating, with $V_0^\theta = V_0^*$. If, indeed, $V_0 > V_0^*$, then the strategy of selling the American security and adopting a super-replicating strategy, liquidating at exercise, effectively defines an arbitrage.

This notion of arbitrage for American securities, an extension of the notion of arbitrage used earlier in the chapter, is reasonable because a super-replicating strategy does not depend on the exercise policy adopted by the holder (or sequence of holders over time) of the American security. It would be unreasonable to call a strategy involving a short position in the American security an "arbitrage" if, in carrying it out, one requires knowledge of the exercise policy for the American security that will be adopted by other agents that hold the security over time, who may after all act "irrationally."

Proposition. *Given $(X, \bar{\tau}, \delta, S)$, suppose (δ, S) is arbitrage-free and generates complete markets. Then there is a super-replicating trading strategy θ for $(X, \bar{\tau}, \delta, S)$ with the initial value $V_0^\theta = V_0^*$.*

In order to construct a super-replicating strategy, we will make a short excursion into the theory of optimal stopping. For any process Y in L, the *Snell envelope* W of Y is defined by

$$W_t = \max_{\tau \in \mathcal{T}(t)} E_t(Y_\tau), \qquad 0 \leq t \leq \bar{\tau}.$$

It can be shown as an exercise that for any $t < \bar{\tau}$, $W_t = \max[Y_t, E_t(W_{t+1})]$. Thus $W_t \geq E_t(W_{t+1})$, implying that W is a supermartingale. As explained in Appendix A, this implies that we can decompose W in the form $W = Z - A$, for some martingale Z and some increasing adapted process A with $A_0 = 0$. This decomposition is illustrated in Figure 2.1 for the case in which Y is a deterministic process, which implies that W, Z, and A are also deterministic.

In order to prove Proposition I, we define Y by $Y_t = X_t \pi_t$, and let W, Z, and A be defined as above. By the definition of complete markets, there is a trading strategy θ with the property that

- $\delta_t^\theta = 0$ for $0 < t < \bar{\tau}$;
- $\delta_{\bar{\tau}}^\theta = Z_{\bar{\tau}}/\pi_{\bar{\tau}}$;
- $\delta_t^\theta = 0$ for $t > \bar{\tau}$.

Property (a) of a super-replicating strategy is satisfied by this strategy θ. From the fact that Z is a martingale and the definition of a state-price deflator, the cum-dividend value V^θ of the trading strategy θ satisfies

$$\pi_t V_t^\theta = E_t(\pi_{\bar{\tau}} \delta_{\bar{\tau}}^\theta) = E_t(Z_{\bar{\tau}}) = Z_t, \qquad t \leq \bar{\tau}. \tag{14}$$

From (14) and the fact that $A_0 = 0$, we know that $V_0^\theta = V_0^*$ because $Z_0 = W_0 = \pi_0 V_0^*$. Since $Z_t - A_t = W_t \geq Y_t$ for all t, from (14) we also know that

$$V_t^\theta = \frac{Z_t}{\pi_t} \geq \frac{1}{\pi_t}(Y_t + A_t) = X_t + \frac{A_t}{\pi_t} \geq X_t, \qquad t \leq \bar{\tau},$$

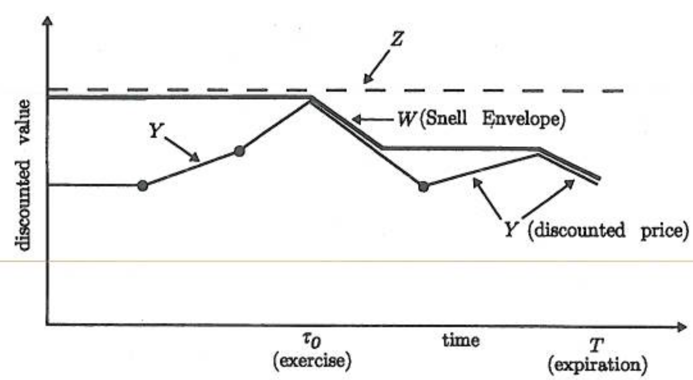

Figure 2.1. *Snell Envelope and Optimal Stopping Rule: Deterministic Case*

the last inequality following from the fact that $A_t \geq 0$ for all t. Thus the dominance property (b) is also satisfied, and θ is indeed a super-replicating strategy with $V_0^\theta = V_0^*$. This proves the proposition and implies that unless there is an arbitrage, the initial price V_0 of the American security is equal to the market value V_0^* associated with a rational exercise policy.

The Snell envelope W is also the key to finding a rational exercise policy. As for the deterministic case illustrated in Figure 2.1, a rational exercise policy is given by $\tau^0 = \min\{t : W_t = Y_t\}$. We now show the optimality of τ^0. First, we know that if τ is a rational exercise policy, then $W_\tau = Y_\tau$. (This can be seen from the fact that $W_\tau \geq Y_\tau$, and if $W_\tau > Y_\tau$, then τ cannot be rational.) From this fact, any rational exercise policy τ has the property that $\tau \geq \tau^0$. For any such τ, we have

$$E_{\tau^0}[Y(\tau)] \leq W(\tau^0) = Y(\tau^0),$$

and the law of iterated expectations implies that $E[Y(\tau)] \leq E[Y(\tau^0)]$, so τ^0 is rational.

We have shown the following.

Theorem. *Given $(X, \bar{\tau}, \delta, S)$, suppose (δ, S) generates complete markets. Suppose there is a state-price deflator π for (δ, S), and let W be the Snell envelope of $X\pi$ up to the expiration time $\bar{\tau}$. Then a rational exercise policy for $(X, \bar{\tau}, \delta, S)$ is given by $\tau^0 = \min\{t : W_t = \pi_t X_t\}$. The unique initial cum-dividend arbitrage-free price of the American security is*

$$V_0^* = \frac{1}{\pi_0} E[X(\tau^0)\pi(\tau^0)].$$

J. Is Early Exercise Optimal?

With the equivalent martingale measure Q defined in Section G, we can also write the optimal stopping problem (13) in the form

$$V_0^* = \max_{\tau \in \mathcal{T}(0)} E^Q\left(\frac{X_\tau}{R_{0,\tau}}\right). \tag{15}$$

This representation of the rational exercise problem is sometimes convenient. For example, let us consider the case of an American call option on a security with price process p. We have $X_t = (p_t - K)^+$ for some exercise price K. Suppose the underlying security has no dividends before or at the expiration time $\bar{\tau}$. We suppose positive interest rates, meaning that $R_{t,s} \geq 1$ for all t and $s \geq t$. With these assumptions, we will show that it is never optimal to exercise the call option before its expiration date $\bar{\tau}$.

This property is sometimes called "no early exercise," or "better alive than dead."

We define the "discounted price process" p^* by $p_t^* = p_t/R_{0,t}$. The fact that the underlying security pays dividends only after the expiration time $\bar{\tau}$ implies, by Lemma G, that p^* is a Q-martingale at least up to the expiration time $\bar{\tau}$. That is, for $t \leq s \leq \bar{\tau}$, we have $E_t^Q(p_s^*) = p_t^*$.

Jensen's Inequality can be used to show the following fact about convex functions of martingales, which we will use to obtain conditions for the no-early-exercise result.

Lemma. *Suppose $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is convex with respect to its first argument, Y is a martingale, $\tau(1)$ and $\tau(2)$ are two stopping times with $\tau(2) \geq \tau(1)$, and Z is an adapted process. Then $f(Y_{\tau(1)}, Z_{\tau(1)}) \leq E_{\tau(1)}[f(Y_{\tau(2)}, Z_{\tau(1)})]$. Moreover, the law of iterated expectations implies that $E[f(Y_{\tau(1)}, Z_{\tau(1)})] \leq E[f(Y_{\tau(2)}, Z_{\tau(1)})]$.*

With the benefit of this lemma and positive interest rates, we have, for any stopping time $\tau \leq \bar{\tau}$,

$$
\begin{aligned}
E^Q\left[\frac{1}{R_{0,\tau}}(p_\tau - K)^+\right] &= E^Q\left[\left(p_\tau^* - \frac{K}{R_{0,\tau}}\right)^+\right] \\
&\leq E^Q\left[\left(p_{\bar{\tau}}^* - \frac{K}{R_{0,\tau}}\right)^+\right] \\
&\leq E^Q\left[\left(p_{\bar{\tau}}^* - \frac{K}{R_{0,\bar{\tau}}}\right)^+\right] \\
&= E^Q\left[\frac{1}{R_{0,\bar{\tau}}}(p_{\bar{\tau}} - K)^+\right].
\end{aligned}
$$

It follows that $\bar{\tau}$ is a rational exercise policy. In typical cases, $\bar{\tau}$ is the unique rational exercise policy.

If the underlying security pays dividends before expiration, then early exercise of the American call is, in certain cases, optimal. From the fact that the put payoff is increasing in the strike price (as opposed to decreasing for the call option), the second inequality above is reversed for the case of a put option, and one can guess that early exercise of the American put is sometimes optimal.

Exercise 2.1 gives a simple example of American security valuation in a complete-markets setting. Chapter 3 presents the idea in a Markovian setting, which offers computational advantages in solving for the rational exercise policy and market value of American securities. In Chapter 3 we also consider the case of American securities that offer dividends before expiration.

The real difficulties with analyzing American securities begin with incomplete markets. In that case, the choice of exercise policy may play a role in determining the marketed subspace, and therefore a role in pricing securities. If the state-price deflator depends on the exercise policy, it could even turn out that the notion of a rational exercise policy is not well defined.

Exercises

2.1 Suppose in the setting of Section B that S is the price process of a security with zero dividends before T. We assume that

$$S_{t+1} = S_t H_{t+1}; \qquad t \geq 0; \qquad S_0 > 0,$$

where H is an adapted process such that for all $t \geq 1$, H_t has only two possible outcomes $U > 0$ and $D > 0$, each with positive conditional probability given $\mathcal{F}_{t-1}$. Suppose β is the price process of a security, also with no dividends before T, such that

$$\beta_{t+1} = \beta_t R; \qquad t \geq 0; \qquad \beta_0 > 0,$$

where $R > 1$ is a constant. We can think of β as the price process of a riskless bond. Consider a third security, a *European call option* on S with *expiration* at some fixed date $\tau < T$ and exercise price $K \geq 0$. This means that the price process C^τ of the third security has expiration value

$$C_\tau^\tau = (S_\tau - K)^+ \equiv \max(S_\tau - K, 0),$$

with $C_t^\tau = 0$, $t > \tau$. That is, the option gives its holder the right, but not the obligation, to purchase the stock at time τ at price K.

The absence of arbitrage implies that $U = R = D$, or that $U > R > D$. We will assume the latter (nondegeneracy of returns) for the remainder of the exercise.

(A) Assuming no arbitrage, show that for $0 \leq t < \tau$,

$$C_t^\tau = \frac{1}{R^{\tau-t}} \sum_{i=0}^{\tau-t} b(i; \tau - t, p)(U^i D^{\tau-t-i} S_t - K)^+, \tag{16}$$

where $p = (R - D)/(U - D)$ and where

$$b(i; n, p) = \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} \tag{17}$$

is the probability of i successes, each with probability p, out of n Bernoulli (independent binomial) trials. One can thus view (16) as the discounted expected exercise value of the option, with expectation under some probability measure constructed from the stock and bond returns. In order to model this viewpoint, let $\hat{S}$ be the process defined by

$$\hat{S}_{t+1} = \hat{S}_t \hat{H}_{t+1}; \qquad t \geq 0; \qquad \hat{S}_0 = S_0, \tag{18}$$

where $\{\hat{H}_0, \hat{H}_1, \dots\}$ are Bernoulli trials with outcomes U and D of probability p and $1-p$, respectively. Then (18) implies that

$$C_0^\tau = E\left[\frac{(\hat{S}_\tau - K)^+}{R^\tau}\right]. \tag{19}$$

(B) We take it that $\mathbb{F}$ is the filtration generated by the return process H, meaning that for all $t \geq 1$, $\mathcal{F}_t$ is the tribe generated by $\{H_1, \dots, H_t\}$. We extend the definition of the option described in part (A) by allowing the expiry date τ to be a stopping time. Show that (19) is still implied by the absence of arbitrage.

(C) Show that markets are complete.

(D) The *moneyness* of a call option with expiration at a deterministic time $\tau < T$ and with strike price K is $(S_0R^\tau - K)/K$. If the moneyness is positive, the option is said to be *in the money*. If the moneyness is negative, the option is said to be *out of the money*. If the moneyness is zero, the option is said to be *at the money*. For a put option with same strike K and expiration time τ, the moneyness is $(K - S_0R^\tau)/K$ (minus the call moneyness), and the same terms apply. Let $S_0 = 100$, $U = 1.028$, $R = 1.001$, and $D = 0.978$. Plot (or tabulate) the price of European call and American call and put options, with expiration at $\tau = 100$, against moneyness.

(E) For the parameters in Part (D), suppose that up and down returns for the stock are equally likely in each period, and that one time period represents 0.01 years. Compute the annualized continuously compounding interest rate r, and the mean and standard deviation of the annual return $\rho = (\log S_{100} - \log S_0)/100$. Plot the likelihood of ρ as a frequency diagram, that is, showing the probability of each outcome of ρ above that outcome.

(F) A *barrier option* is one that can be exercised or not depending on whether or not the underlying price has crossed a given level before expiration. For example, a *down-and-out call*, at barrier $\underline{S}$, exercise time τ (possibly a stopping time), and exercise price $K \geq \underline{S}$, is a security that pays $1_A(S_\tau - K)^+$ at τ, where A is the event $\{\omega : \min\{S_0(\omega), S_1(\omega), \dots, S_{\tau(\omega)}(\omega)\} > \underline{S}\}$. That is, A is the event that the minimum price achieved through τ is larger than the barrier $\underline{S}$. Show that an American down-and-out call, exercisable at any time $\tau \leq \bar{\tau}$, is rationally exercised (if at all) only at $\bar{\tau}$.

(G) For the parameters in Part (D), and $\bar{\tau} = 100$, price a down-and-out call with barrier $\underline{S} = 80$, for strikes K from 80 to 120. Plot (or, if you cannot, tabulate) the prices for integer K in this range.

(H) For the parameters in Part (D), price European and American up-and-out put options, with knockout barrier $\overline{S} = 120$. Again, obtain the prices for strikes ranging from 80 to 120. Plot (or, if you cannot, tabulate) the prices for integer K

in this range. For the special case of $K = 100$, plot the optimal exercise region for the American up-and-out put. That is, for each $t \leq 100$, show the set of outcomes for S_t at which it is optimal to exercise at time t.

2.2 Suppose in the context of problem (4) that (δ, S) admits no arbitrage and that U is continuous. Show the existence of a solution. Hint: A continuous function on a compact set has a maximum. In this setting, a set is compact if closed and bounded.

2.3 Suppose in the context of problem (4) that $e \gg 0$ and that U has the additive form (5), where for each t, u_t is concave with an unbounded derivative. Show that any solution c^* is strictly positive. Show that the same conclusion follows if the assumption that $e \gg 0$ is replaced with the assumption that markets are complete and that e is not zero.

2.4 Prove Lemma C. Hint: For any x and y in L, let

$$(x \mid y) = E\left(\sum_{t=0}^{T} x_t y_t\right).$$

Then follow the hint given for Exercise 1.17, remembering that we write "$x = y$" whenever $x_t = y_t$ for all t almost surely.

2.5 For U of the additive form (5), show that the gradient $\nabla U(c)$, if it exists, is represented as in (6).

2.6 (Consumption-based CAPM) Suppose $(c^{(1)}, \dots, c^{(m)})$ is a strictly positive equilibrium consumption allocation and that for all i, U_i is of the additive form: $U_i(c) = E[\sum_{t=0}^{T} u_{it}(c_t)]$. Assume there is a constant $\bar{c}$ larger than c_t^i for all i and t such that, for all i and t, $u_{it}(x) = A_{it}x - B_{it}x^2$, $x \leq \bar{c}$, for some positive constants A_{it} and B_{it}. That is, utility is quadratic in the relevant range.

(A) In the context of Corollary 2 of Section F, show that for each t, there are some constants k_t and K_t such that $u'_{\lambda t}(e_t) = k_t + K_t e_t$. Suppose for a given trading strategy θ and time t that the following are well defined:

- $R_t^\theta = \theta_{t-1} \cdot (S_t + \delta_t)/\theta_{t-1} \cdot S_{t-1}$, the return on θ at time t;
- R_t^M, the return at time t on a strategy φ maximizing $\mathrm{corr}_{t-1}(R_t^\varphi, e_t)$, where $\mathrm{corr}_t(\cdot)$ denotes $\mathcal{F}_t$-conditional correlation;
- $\beta_{t-1}^\theta = \mathrm{cov}_{t-1}(R_t^\theta, R_t^M)/\mathrm{var}_{t-1}(R_t^M)$, the conditional beta of the trading strategy θ with respect to the market return, where $\mathrm{cov}_t(\cdot)$ denotes $\mathcal{F}_t$-conditional covariance and $\mathrm{var}_t(\cdot)$ denotes $\mathcal{F}_t$-conditional variance;
- R_t^0, the return at time t on a strategy η with $\mathrm{corr}_{t-1}(R_t^\eta, e_t) = 0$.

Derive the following beta-form of the *consumption-based CAPM*:

$$E_{t-1}(R_t^\theta - R_t^0) = \beta_{t-1}^\theta E_{t-1}(R_t^M - R_t^0). \tag{20}$$

(B) Prove the same beta-form (20) of the CAPM holds in equilibrium even without assuming complete markets.

(C) Extend the state-price beta model of Section 1F to this setting, as follows, without using the assumptions of the CAPM. Let π be a state-price deflator. For each t, suppose R_t^* is the return on a trading strategy solving

$$\sup_{\theta} \operatorname{corr}_{t-1}(R_t^\theta, \pi_t).$$

Assume that $\operatorname{var}_{t-1}(R_t^*)$ is nonzero almost surely. Show that for any return R_t^θ,

$$E_{t-1}(R_t^\theta - R_t^0) = \beta_{t-1}^\theta E_{t-1}(R_t^* - R_t^0),$$

where $\beta_{t-1}^\theta = \operatorname{cov}_{t-1}(R_t^\theta, R_t^*)/\operatorname{var}_{t-1}(R_t^*)$ and $\operatorname{corr}_{t-1}(R_t^0, \pi_t) = 0$.

2.7 Prove Proposition E.

2.8 In the context of Section D, suppose that U is the *habit-formation* utility function defined by $U(c) = E[\sum_{t=0}^T u(c_t, h_t)]$, where $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable on the interior of its domain and, for any t, the "habit" level of consumption is defined by $h_t = \sum_{j=0}^t \alpha_j c_{t-j}$ for some $\alpha \in \mathbb{R}^T$. For example, we could take $\alpha_j = \gamma^j$ for $\gamma \in (0, 1)$, which gives geometrically declining weights on past consumption. Calculate the Riesz representation of the gradient of U at a strictly positive consumption process c.

2.9 Consider a utility function U defined by $U(c) = V_0$, where the *utility process* V is defined recursively, backward from T in time, by $V_T = J(c_T, h(0))$ and, for $t < T$, by $V_t = J(c_t, E_t[h(V_{t+1})])$, where $J : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is increasing and continuously differentiable on the interior of its domain and $h : \mathbb{R} \to \mathbb{R}$ is increasing and continuously differentiable. This is a special case of what is known as *recursive utility*, and also a special case of what is known as *Kreps-Porteus* utility. Note that the utility function can depend nontrivially on the filtration $\mathbb{F}$, which is not true for additive or habit-formation utility functions. This utility model is reconsidered in an exercise in Chapter 3.

(A) Compute the Riesz representation π of the gradient of U at a strictly positive consumption process c.

(B) Suppose that h and J are concave and increasing functions. Show that U is concave and increasing.

2.10 In the setting of Section E, an Arrow-Debreu equilibrium is a feasible consumption allocation $(c^{(1)}, \ldots, c^{(m)})$ and a nonzero linear function $\Psi : L \to \mathbb{R}$ such that for all i, c^i solves $\max_{c \in L_+} U_i(c)$ subject to $\Psi(c^i) \le \Psi(e^i)$. Suppose that $(c^{(1)}, \ldots, c^{(m)})$ and Ψ form an Arrow-Debreu equilibrium and that π is the Riesz representation of Ψ. Let S be defined by $S_T = 0$ and by taking π to be a state-price deflator. Suppose, given (δ, S), that markets are complete. Show the existence of trading strategies $\theta^{(1)}, \ldots, \theta^{(m)}$ such that $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$ is an equilibrium with the same consumption allocation $(c^{(1)}, \ldots, c^{(m)})$.

2.11 Given a finite set Ω of states, a *partition* of Ω is a collection of disjoint nonempty subsets of Ω whose union is Ω. For example, a partition of $\{1, 2, 3\}$ is given by $\{\{1\}, \{2, 3\}\}$. The tribe on a finite set Ω generated by a given partition p of Ω, denoted $\sigma(p)$, is the smallest tribe $\mathcal{F}$ on Ω such that $p \subset \mathcal{F}$. Conversely, for any tribe $\mathcal{F}$ on Ω, the partition $\mathcal{P}(\mathcal{F})$ generated by $\mathcal{F}$ is the smallest partition p of Ω such that $\mathcal{F} = \sigma(p)$. For instance, the tribe $\{\varnothing, \Omega, \{1\}, \{2, 3\}\}$ is generated by the partition in the above example. Since partitions and tribes on a given finite set Ω are in one-to-one correspondence, we could have developed the results of Chapter 2 in terms of an increasing sequence $p_0, p_1, \ldots, p_T$ of partitions of Ω rather than a filtration of tribes, $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_T$. (In infinite-state models, however, it is more convenient to use tribes than partitions.)

We suppose for simplicity that $P(B) > 0$ for any nonempty event B. Given a subset B of Ω and a partition p of Ω, let $n(B, p)$ denote the minimum number of elements of p whose union contains B. In a sense, this is the number of distinct nonempty events that might occur if B is to occur. For $t < T$, let

$$n_t = \max_{B \in p_t} n(B, p_{t+1}).$$

Finally, the *spanning number* of the filtration $\mathbb{F}$ generated by $p_0, \ldots, p_T$ is $n(\mathbb{F}) \equiv \max_{t<T} n_t$. In a sense, $n(\mathbb{F})$ is the maximum number of distinct events that could be revealed between two periods.

Show that complete markets requires at least $n(\mathbb{F})$ securities, and that given the filtration $\mathbb{F}$, there exists a set of $n(\mathbb{F})$ dividend processes and associated arbitrage-free security-price processes such that markets are complete. This issue is further investigated in sources indicated in the Notes.

2.12 Given securities with a dividend-price pair (δ, S), extend Theorem G to show, in the presence of riskless borrowing at a strictly positive discount at each date, the equivalence of these statements:

(a) There exists a state-price deflator.
(b) There exists a deflator π such that (3) holds for any trading strategy θ.
(c) $S_T = 0$ and there exists a deflator π such that the deflated gain process G^π is a martingale.
(d) There is no arbitrage.
(e) There is an equivalent martingale measure.

2.13 Show, from (11) and (12), that (10) is indeed satisfied, confirming that Q is an equivalent martingale measure.

2.14 Show, as claimed in Section I, that if τ^* is a rational exercise policy for the American security X and if V^* is the cum-dividend price process for the American security with this rational exercise policy, then $V_\tau^* \ge X_\tau$ for any stopping time $\tau \le \tau^*$.

2.15 (Aggregation Revisited). Suppose, in the context of the supremum (8), that $x \gg 0$ and, for all i, $U_i(c) = E[\sum_{t=0}^T u_t(c_t)]$, where, for all t, $u_t(x) = k_t x^{\gamma(t)}/\gamma(t)$, where k_t and $\gamma(t) < 1$, $\gamma(t) \neq 0$, are constants (depending on t).

(A) Show that U_λ is of the same utility form as U_i.

(B) Suppose that $\gamma(t)$ is a constant independent of t. Replace the assumption of complete markets in Proposition F with the assumption that $e \gg 0$ and that, for all i, there is a security whose dividend process is e^i. (This can easily be weakened.) Demonstrate the existence of an equilibrium with the same properties described in Proposition F, including a consumption allocation that is Pareto optimal.

2.16 (Put-Call Parity). In the general setting explained in Section B, suppose there exist the following securities:

(a) a "stock," with price process X;
(b) a European call option on the stock with strike price K and expiration τ;
(c) a European put option on the stock with strike price K and expiration τ;
(d) a τ-period zero-coupon riskless bond.

Let X_0, C_0, P_0, and B_0 denote the initial respective prices of the securities. Suppose there is no arbitrage, and that the stock pays no dividends before time τ. Solve for C_0 explicitly in terms of X_0, P_0, and B_0.

2.17 (Futures-Forward Price Equivalence). This exercise defines (in ideal terms) a *forward contract* and a *futures contract*, and gives simple conditions under which the *futures price* and the *forward price* coincide. We adopt the setting of Section B, in the absence of arbitrage. Fixed throughout are a *delivery date* τ and a settlement amount W_τ (an $\mathcal{F}_\tau$-measurable random variable).

Informally speaking, the associated forward contract made at time t is a commitment to pay an amount F_t (the forward price), which is agreed upon at time t and paid at time τ, in return for the amount W_τ at time τ. Formally speaking, the forward contract made at time t for delivery of W_τ at time τ for a forward price of F_t is a security whose price at time t is zero and whose dividend process δ is defined by $\delta_t = 0$, $t \neq \tau$, and $\delta_\tau = W_\tau - F_t$.

(A) Suppose that Q is an equivalent martingale measure and that there is riskless short-term borrowing at any date t at a discount d_t that is deterministic. Show that $\{F_0, F_1, \ldots, F_\tau\}$ is a Q-martingale, in that $F_t = E_t^Q(F_\tau)$ for all $t \leq \tau$.

A futures contract differs from a forward contract in several practical ways that depend on institutional details. One of the details that is particularly important for pricing purposes is *resettlement*. For theoretical modeling purposes, we can describe resettlement as follows: A futures-price process $\Phi = \{\Phi_0, \ldots, \Phi_\tau\}$ for delivery of W_τ at time τ is taken as given. At any time t, an investor can adopt a position of θ futures contracts by agreeing to accept the resettlement payment $\theta(\Phi_{t+1} - \Phi_t)$ at time $t+1$, $\theta(\Phi_{t+2} - \Phi_{t+1})$ at time $t+2$, and so on, until the position is changed (or eliminated). This process of paying or collecting any changes in the futures price, period by period, is called *marking to market*, and serves in practice to reduce the likelihood or magnitude of potential defaults. Formally, all of this means simply that the dividend process δ of the futures contract is defined by $\delta_t = \Phi_t - \Phi_{t-1}$, $1 \leq t \leq \tau$.

For our purposes, it is natural to assume that the delivery value Φ_τ is contractually equated with W_τ. (In a more detailed model, we could equate Φ_τ and W_τ by the absence of *delivery arbitrage*.)

(B) Suppose Q is an equivalent martingale measure and show that for all $t \leq \tau$, $\Phi_t = E_t^Q(W_\tau)$. It follows from parts (A) and (B) that with deterministic interest rates and the absence of arbitrage, futures and forward prices coincide.

(C) We now suppose that W_τ is the market value S_τ of a security with dividend process δ. Suppose that δ and the discount process $d = \{d_1, \ldots, d_T\}$ on riskless borrowing are both deterministic. Calculate the futures and forward prices, Φ_t and F_t, explicitly in terms of S_t, d, and δ.

2.18 Provide details fleshing out the following outline of a proof of the converse part of Proposition G.

Let $J = \{(x_1, \ldots, x_T) : x \in L\}$ and $H = \{(\delta_1^\theta, \ldots, \delta_T^\theta) : \theta \in \Theta\}$. Markets are complete if and only if $J = H$. By Theorem G, there is a unique equivalent martingale measure if and only if there is a unique state-price deflator π such that $\pi_0 = 1$. Suppose $H \neq J$. Since H is a linear subspace of J, there is some nonzero y in J "orthogonal" to H, in the sense that $E(\sum_{t=1}^T y_t h_t) = 0$ for all h in H. Let $\hat{\pi} \in L$ be defined by $\hat{\pi}_0 = 1$ and $\hat{\pi}_t = \pi_t + \alpha y_t$, $t \geq 1$, where $\alpha > 0$ is a scalar small enough that $\hat{\pi} \gg 0$. Then $\hat{\pi}$ is a distinct state-price deflator with $\hat{\pi}_0 = 1$. This shows that if there is a unique state-price deflator π with $\pi_0 = 1$, then markets must be complete. Hint: Let

$$(y \mid h) \equiv E\left(\sum_{t=1}^T y_t h_t\right), \qquad h \in H,$$

define an inner product $(\cdot \mid \cdot)$ for H in the sense of Exercise 1.17.

2.19 It is asserted in Section I that if W is the Snell envelope of Y, then $W_t = \max[Y_t, E_t(W_{t+1})]$. Prove this natural property.

2.20 Prove Lemma J.

2.21 Consider the "tree" of prices for securities A and B shown in Figure 2.2. At each node in the tree, a pair (p_A, p_B) of prices is shown, the first of which is the price of A at that node, the second of which is the price of B.

(A) Construct a probability space $(\Omega, \mathcal{F}, P)$, a filtration of tribes, $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$, and a vector security price process V, that formally encode the information in the figure. Please be explicit. Take the security price process to be *cum dividend*, so that V_2 is both the price and the dividend payoff vector of the securities at time 2. There are no dividend payments in periods 0 and 1.

(B) Suppose there is no arbitrage. Find the price at time 0 of an American put option on asset B, with an exercise price of 95 and expiring at time 2. (Remember, this is an option to sell B for 95 at any of times 0, 1, or 2.)

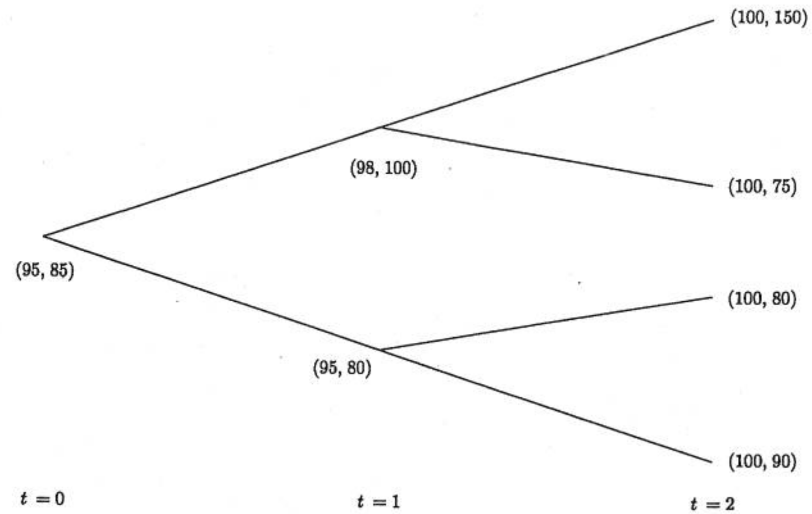

Figure 2.2. *An Event Tree With Prices*

(C) Suppose the price at time 0 in the market of this put option is in fact 10 percent lower than the arbitrage-free price you arrived at in part B. Show explicitly how to create a riskless profit of 1 million dollars at time 0, with no cash flow after time 0.

(D) Suppose the price in the market of this put option is in fact 10 percent higher than the arbitrage-free price you arrived at in part B. Show explicitly how to create a riskless profit of 1 million dollars at time 0, with nonnegative cash flow after time 0. Hint: If you decide to sell the option, you should not assume that the person to whom you sold it will exercise it in any particular fashion.

2.22 Let $T = 1$, and suppose there are three equally likely states of world, ω_1, ω_2, and ω_3, one of which is revealed as true at time 1. A particular agent has utility function U and has equilibrium consumption choices $c_0 = 25$ and

$$c_1(\omega_1) = 9, \qquad c_1(\omega_2) = 16, \qquad c_1(\omega_3) = 4.$$

In each case below, compute the price of a security that pays 3 in state ω_1, 6 in state ω_2, and 5 in state ω_3. Show your work.

(A) Expected, but not time-additive, utility $U(c_0, c_1) = E[u(c_0, c_1)]$, with $u(x, y) = \sqrt{xy}$.

(B) Nonexpected utility $U(c_0, c_1) = \sqrt{c_0\, c_1(\omega_1)\, c_1(\omega_2)\, c_1(\omega_3)}$.

2.23 For concreteness, the length of one period is one year. There are two basic types of investments. The first is riskless borrowing or lending. The equilibrium one-year short rate is 25 percent (simple interest per year), each year. (So one can invest 1 at time zero and collect 1.25 at the end of the first year, or invest 1 at the end of the first year and collect 1.25 at the end of the second year.) At the end of each year, a fair coin is flipped. A risky security has zero initial market value. Its market value goes up by one unit at the end of each year if the outcome of the coin flip for that year is heads. Its market value goes down by one unit at the end of each year if the outcome of the coin flip for that year is tails. For example, the price of the risky security at the end of the second year is -2, 0, or $+2$, with respective probabilities 0.25, 0.50, and 0.25.

There is also a European option to purchase the risky security above at the end of the second year (only) at an exercise price of 1 unit of account.

(A) Suppose there is no arbitrage. State the initial market price q of the option. (Show your reasoning.)

(B) Now suppose the option is actually selling for $q/2$. Construct a trading strategy that generates a net initial positive cash flow of 1000 units of account and no subsequent cash flows. (State a precise recipe for the quantities of each security to buy or sell, at each time, in each contingency.)

2.24 Prove Corollary F.

2.25 (Numeraire Invariance). Consider a dividend-price pair $(\delta, S) \in L^N \times L^N$, and a deflator γ. Let $\hat{S} = S\gamma$ and $\hat{\delta} = \delta\gamma$ denote the deflated price and dividend processes. Let θ be any given trading strategy. Show that the dividend process $\hat{\delta}^\theta$ generated by θ given $(\hat{\delta}, \hat{S})$ and the dividend process δ^θ generated by θ under (δ, S) are related by $\hat{\delta}^\theta = \gamma\delta^\theta$. Show that θ is an arbitrage with respect to (δ, S) if and only if θ is an arbitrage with respect to $(\hat{\delta}, \hat{S})$. If π is a state-price deflator for (δ, S), compute a state-price deflator $\hat{\pi}$ for $(\hat{\delta}, \hat{S})$ in terms of π and γ.

Notes

Radner (1967, 1972) originated the sort of dynamic equilibrium model treated in this chapter. The monograph by Magill and Quinzii (1996b) is a comprehensive survey of the theory of general equilibrium in incomplete markets.

(A–B) The model of uncertainty and information is standard. The model of uncertainty is equivalent to that originated in the general equilibrium model of Debreu (1953), which appears in Chapter 7 of Debreu (1959). For more details in a finance setting, see Dothan (1990).

(C–D) The connection between arbitrage and martingales given in Sections C and G is from the more general model of Harrison and Kreps (1979). Girotto and Ortu (1996) present general results, in this finite-dimensional setting, on the equivalence between no arbitrage and the existence of an equivalent martingale measure. The spirit of the results on optimality and state prices is also from Harrison and Kreps (1979). Girotto and Ortu (1994, 1997a, 1997b) fully explore this equivalence in finite-dimensional multiperiod economies.

(E) The spirit of this section is from Kreps (1982) and Duffie and Huang (1985).

(F) The representative-agent state-pricing model for this setting was shown by Constantinides (1982). An extension of this notion to incomplete markets, where one cannot generally exploit Pareto optimality, is given by Cuoco and He (1992a).

(G–H) These sections are based on the ideas of Harrison and Kreps (1979).

(I) The modeling of American security valuation given here is similar to the continuous-time treatments of Bensoussan (1984) and Karatzas (1988), who do not formally connect the valuation of American securities with the absence of arbitrage, but rather deal with the similar notion of "fair price." Merton (1973b) was the first to attack American option valuation systematically using arbitrage-based methods and to point out the inoptimality of early exercise of certain American options in a Black-Scholes style setting. American option valuation is reconsidered in Chapters 3 and 8, whose literature notes cite many additional references.

(J) These results were developed in a continuous-time setting by Merton (1973b).

Additional Topics: The habit-formation utility model was developed by Dunn and Singleton (1986) and in continuous time by Ryder and Heal (1973). An application of habit formation to state-pricing in this setting appears in Chapman (1998b). The recursive-utility model, in various forms, is due to Selden (1978), Kreps and Porteus (1978), and Epstein and Zin (1989), and is surveyed by Epstein (1992). Koopmans (1960) presented an early precursor. The recursive-utility model allows for preference for earlier or later resolution of uncertainty (which have no impact on additive utility). This is relevant, for example, in the context of the remarks by Ross (1989), as shown by Skiadas (1998), and Duffie, Schroder, and Skiadas (1997). See Grant, Kajii, and Polak (2000) for more on preference for resolution of information. For a more general form of recursive utility than that appearing in Exercise 2.9, the von Neumann-Morgenstern function h can be replaced with a function of the conditional distribution of next-period utility. Examples are the local-expected-utility model of Machina (1982) and the *betweenness certainty equivalent* model of Chew (1983, 1989), Dekel (1989), and Gul and Lantto (1990). The equilibrium state-price associated with recursive utility is computed in a Markovian version of this setting by Kan (1995). For further justification and properties of recursive utility, see Chew and Epstein (1991), Skiadas (1998), and Skiadas (1997). For further implications for asset pricing, see Epstein (1988), Epstein (1992), Epstein and Zin (1999), and Giovannini and Weil (1989). Kan (1993) explored the utility gradient representation of recursive utility in this setting.

The basic approach to existence given in Exercise 2.11 is suggested by Kreps (1982), and is shown to work for "generic" dividends and endowments, under technical regularity conditions, in McManus (1984), Repullo (1986), and Magill and Shafer (1990), provided the number of securities is at least as large as the spanning number of the filtration $\mathbb{F}$ (as suggested in Exercise 2.11). This literature is reviewed in depth by Geanakoplos (1990). See Duffie and Huang (1985) for the definition of spanning number in more general settings and for a continuous-time version of a similar result. Duffie and Shafer (1985, 1986b) show generic existence of equilibrium in incomplete markets; Hart (1975) gives a counterexample. Bottazzi (1995) has a somewhat more advanced version of this result in its single-period multiple-commodity version. See also Won (1996a, 1996b). Related existence topics are studied by Bottazzi and Hens (1996), Hens (1991), and Zhou (1997b). Dispersed expectations, in a temporary-equilibrium variant of the model, is shown to lead to existence by Henrotte (1994) and by Honda (1992). Alternative proofs of existence of equilibrium are given in the two-period version of the model by Geanakoplos and Shafer (1990), Hirsch, Magill, and Mas-Colell (1990), and Husseini, Lasry, and Magill (1990); and in a T-period version by Florenzano and Gourdel (1994). If one defines security dividends in nominal terms, rather than in units of consumption, then equilibria always exist under standard technical conditions on preferences and endowments, as shown by Cass (1984), Werner (1985), Duffie (1987), and Gottardi and Hens (1996), although equilibrium may be indeterminate, as shown by Cass (1989) and Geanakoplos and Mas-Colell (1989). On this point, see also Kydland and Prescott (1991), Mas-Colell (1991), and Cass (1991). Likewise, one obtains existence in a one-period version of the model provided securities have payoffs in a single commodity (the framework of most of this book), as shown by Chae (1988) and Geanakoplos and Polemarchakis (1986). Surveys of general-equilibrium models in an incomplete-markets setting are given by Cass (1991), Duffie (1992), Geanakoplos (1990), Magill and Quinzii (1996b), and Magill and Shafer (1991). In the presence of price-dependent options, existence can be more problematic, as shown by Polemarchakis and Ku (1990), but variants of the formulation will suffice for existence in many cases, as shown by Huang and Wu (1994) and Krasa and Werner (1991). Detemple and Selden (1991) examine the implications of options for asset pricing in a general equilibrium model with incomplete markets. Bajeux-Besnainou and Rochet (1996) explore the dynamic spanning implications of options. The importance of the timing of information in this setting is described by Berk and Uhlig (1993). Hindy and Huang (1993b) show the implications of linear collateral constraints on security valuation. Hara (1993) treats the role of "redundant" securities in the presence of transactions costs.

Hahn (1994, 1999) raises some philosophical issues regarding the possibility of complete markets and efficiency. The Pareto inefficiency of incomplete-markets equilibrium consumption allocations, and notions of constrained efficiency, are discussed by Hart (1975), Kreps (1979) (and references therein), Citanna, Kajii, and Villanacci (1994), Citanna and Villanacci (1993), and Pan (1993, 1995).

The optimality of individual portfolio and consumption choices in incomplete markets in this setting is given a dual interpretation by He and Pagès (1993). (Girotto and Ortu (1994) offer related remarks.) Methods for computation of equilibrium with incomplete markets are developed by Brown, DeMarzo, and Eaves (1996a), Brown, DeMarzo, and Eaves (1996b), and DeMarzo and Eaves (1996). See also the notes of Chapter 12.

Kraus and Litzenberger (1975) and Stapleton and Subrahmanyam (1978) present parametric examples of equilibrium. Hansen and Richard (1987) explore the state-price beta model in a much more general multiperiod setting.

Ross (1987) and Prisman (1985) show the impact of taxes and transactions costs on the state-pricing model. Hara (1993) discusses the role of redundant securities in the presence of transactions costs. The consumption-based CAPM of Exercise 2.6 is found, in a different form, in Rubinstein (1976). The aggregation result of Exercise 2.15 is based on Rubinstein (1974a). Rubinstein (1974b) has a detailed treatment of asset pricing results in the setting of this chapter. Rubinstein (1987) is a useful expository treatment of derivative asset pricing in this setting.

Cox, Ross, and Rubinstein (1979) developed the multiperiod binomial option pricing model analyzed in Exercise 2.1, and further analyzed in terms of convergence to the Black-Scholes formula in Chapter 12.

The role of production is considered by Duffie and Shafer (1986a) and Naik (1994). The Modigliani-Miller Theorems are reconsidered in this setting by DeMarzo (1988), Duffie and Shafer (1986a), and Gottardi (1995).

3 The Dynamic Programming Approach

THIS CHAPTER PRESENTS portfolio choice and asset pricing in the framework of dynamic programming, a technique for solving dynamic optimization problems with a recursive structure. The asset pricing implications go little beyond those of the previous chapter, but there are computational advantages. After introducing the idea of dynamic programming in a deterministic setting, we review the basics of a finite-state Markov chain. The Bellman equation is shown to characterize optimality in a Markov setting. The first-order condition for the Bellman equation, often called the "stochastic Euler equation," is then shown to characterize equilibrium security prices. This is done with additive utility in the main body of the chapter, and extended to more general recursive forms of utility in the exercises. The last sections of the chapter show the computation of arbitrage-free derivative security values in a Markov setting, including an application of Bellman's equation for optimal stopping to the valuation of American securities such as the American put option. An exercise presents algorithms for the numerical solution of term-structure derivative securities in a simple "binomial" setting.

A. The Bellman Approach

To get the basic idea, we start in the T-period setting of the previous chapter, with no securities except those permitting short-term riskless borrowing at any time t at the discount $d_t > 0$. The endowment process of a given agent is e. Given a consumption process c, it is convenient to define the agent's *wealth process* W^c by $W_0^c = 0$ and

$$W_t^c = \frac{W_{t-1}^c + e_{t-1} - c_{t-1}}{d_{t-1}}, \qquad t \geq 1. \tag{1}$$