

Quiz #2

Fall 2014

Please write your name below. Then complete the exam in the space provided. There are THREE questions. You may refer to one page of notes: standard paper, both sides, any content you wish.

(Name and signature)

1. *Asset pricing fundamentals.* Consider an economy with the following properties:

State z	State Price $Q(z)$	Probability $p(z)$	Dividend $d(z)$
1	1/2	1/2	2
2	1/3	1/2	1

- (a) What is the price of a one-period riskfree bond? (10 points)
- (b) What is the price of a claim to the dividend? What are the returns? (10 points)
- (c) What is the expected excess return on a claim to the dividend? What is its Sharpe ratio? (10 points)
- (d) What is the maximum Sharpe ratio in this economy? (10 points)

Solution:

- (a) The price is $q^1 = Q(1) + Q(2) = 5/6$. (You could also find m and compute it as $E(m)$, but this is easier.)
- (b) The price of (let us call it) “equity” is $q^e = Q(1)d(1) + Q(2)d(2) = 4/3$. The returns are $r^e(1) = 3/2$ and $r^e(2) = 3/4$.
- (c) The excess returns are $r^e(1) - r^1 = 0.3$ and $r^e(2) - r^1 = -0.45$. The expected excess return is therefore -0.075 . The standard deviation is 0.375 , giving us a Sharpe ratio of -0.2 .
- (d) This is a call for the Hansen-Jagannathan bound. The maximum Sharpe ratio is

$$\frac{\text{Std}(m)}{E(m)} = \frac{1/6}{5/6} = 1/5.$$

The pricing kernel m comes from $m(z) = Q(z)/p(z)$, so that $m(1) = 1$ and $m(2) = 2/3$. This has the opposite sign as the equity premium but the same magnitude. The latter is a property of two-state distributions where, in a sense, everything is linearly related and correlations are $+1$ or -1 .

2. *Uniform option pricing.* We'll explore the logic of option pricing in a setting in which the underlying has a uniform distribution. More formally, let $x = s_{t+1}$ (not $x = \log s_{t+1}$!) have a uniform distribution over the interval $[\mu - \theta, \mu + \theta]$; that is, its probability density function is

$$p(x) = \begin{cases} (2\theta)^{-1} & \text{if } \mu - \theta \leq x \leq \mu + \theta \\ 0 & \text{otherwise.} \end{cases}$$

- (a) What is the mean of x ? The variance? (10 points)
- (b) What is the no-arbitrage condition here? (10 points)
- (c) What is the cash flow of a put option with strike price k ? What is its value? (20 points)
- (d) How does the price of a put option change when we increase θ ? Why? (10 points)

Solution:

- (a) You might guess the mean and variance. If not, they follow from

$$\begin{aligned} E(x) &= (2\theta)^{-1} \int_{\mu-\theta}^{\mu+\theta} x \, dx = (2\theta)^{-1} [(\mu + \theta)^2 - (\mu - \theta)^2]/2 = \mu \\ \text{Var}(x) &= (2\theta)^{-1} \int_{\mu-\theta}^{\mu+\theta} (x - \mu)^2 \, dx = (2\theta)^{-1} [\theta^3 - (-\theta)^3]/3 = \theta^2/3. \end{aligned}$$

So μ governs the mean and θ governs the variance.

- (b) The no-arb condition is

$$s_t = q^1 E^*(s_{t+1}) = q^1 \mu.$$

- (c) A put option is a claim to the cash flow $d = (k - x)^+$. If k is in the interval $[\mu - \theta, \mu + \theta]$ we have

$$\begin{aligned} q^p &= q^1 (2\theta)^{-1} \int_{\mu-\theta}^k (k - x) \, dx \\ &= q^1 k [k - (\mu - \theta)] / (2\theta) - q^1 [k^2 - (\mu - \theta)^2] / (4\theta). \end{aligned}$$

- (d) The idea is that option prices increase with risk. Here if we increase θ , the parameter that controls the variance, put prices seem to go up, but I haven't proven it.

3. *More fundamentals.* A well-known financial economist who spent years in the business world, commented on what he had learned as an academic: "I learned two basic lessons about financial mathematics that I've always found useful. One is that risk premiums come from covariances. The other is that asset prices come from risk-neutral probabilities."

- (a) Give an equation that illustrates his first lesson. (10 points)
- (b) Give an equation that illustrates his second lesson. (10 points)
- (c) Where did the covariance go in part (b)? (10 points)

Solution:

- (a) The first reflects $E(x) = -\text{Cov}(x, m)/E(m)$, where x is any excess return.
- (b) The second is reflected by $q = q^1 E^*(d) = q^1 \sum_z p^*(z) d(z)$, where q is the price of the dividend d and E^* is the expectation based on the risk-neutral probabilities.
- (c) The covariance is embedded in the risk-neutral probabilities. Asset prices, for example, can be expressed two ways:

$$\begin{aligned} q &= E(md) = q^1 E(d) + \text{Cov}(m, d) \\ q &= q^1 E^*(d). \end{aligned}$$

In the first, we have a covariance. In the second, that's built into E^* .