ECON-UB 233 Dave Backus @ NYU

Quiz #1

Revised: October 8, 2013

Please write your name below, then complete the exam in the space provided. You may refer to one page of notes: standard paper, both sides, any content you wish. **There are FOUR questions.**

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(Name and signature)

1. *Moments (30 points).* Consider a random variable x that takes on the values

$$x = \begin{cases} -\delta & \text{with probability } 1-\omega \\ +\delta & \text{with probability } \omega \end{cases}$$

with $\delta > 0$.

(a) Do the probabilities constitute a legitimate probability distribution? (5 points)

(b) What is the moment generating function (mgf) for x? (10 points)

(c) Use the mgf to derive the mean and variance of x. (15 points)

Solution:

(a) The probabilities must be nonnegative and sum to one. They sum to one by construction. They're nonnegative if $0 \leq \omega \leq 1$.

(b) The mgf is

$$h(s) = E(e^{sx}) = (1-\omega)e^{-\delta s} + \omega e^{\delta s}.$$

(c) The first two derivatives of the mgf are

$$\begin{aligned} h'(s) &= -\delta(1-\omega)e^{-\delta s} + \delta\omega e^{\delta s} \\ h''(s) &= \delta^2(1-\omega)e^{-\delta s} + \delta^2\omega e^{\delta s}. \end{aligned}$$

The mean is $k'(0) = \delta(2\omega - 1)$. The variance is $k''(0) - k'(0)^2 = 4\delta^2\omega(1-\omega)$.

2. *Geometric risk (25 points).* Consider a power utility agent facing "geometric risk." Utility is $u(c) = c^{1-\alpha}/(1-\alpha)$ with risk aversion parameter $\alpha > 0$. Consumption is $c = e^x$ for $x = 0, 1, 2, \ldots$ with probabilities $p(x) = (1-\omega)\omega^x$ for some parameter $0 < \omega < 1$. We say that x has a *geometric distribution*.

(a) Show that $p(x)$ is a legitimate probability distribution. (5 points)

(b) What is the mgf of x? (5 points)

(c) Suppose $\omega e < 1$. What is the mean of c? (5 points)

(d) What is expected utility? (5 points)

(e) What is the certainty equivalent? (5 points)

Solution:

(a) Probabilities must be nonnegative and sum to one. Here they're positive, and the sum is

$$\sum_x p(x) \;=\; \sum_{x=0}^{\infty}(1-\omega)\omega^x \;=\; (1-\omega)/(1-\omega) \;=\; 1.$$

(b) The mgf is

$$h(s) \;=\; E(e^{sx}) \;=\; \sum_{x=0}^{\infty}(1-\omega)\omega^x e^{sx} \;=\; (1-\omega)/(1-\omega e^s).$$

(c) The mean of c is

$$E(c) \;=\; E(e^x) \;=\; h(1) \;=\; (1-\omega)/(1-\omega e).$$

(d) Expected utility is

$$E(c^{1-\alpha})/(1-\alpha) \;=\; h(1-\alpha)/(1-\alpha) \;=\; (1-\omega)/[(1-\omega e^{1-\alpha})(1-\alpha)].$$

(e) The certainty equivalent μ is the solution to

$$\mu^{1-\alpha}/(1-\alpha) \;=\; (1-\omega)/[(1-\omega e^{1-\alpha})(1-\alpha)],$$

namely

$$\mu \;=\; \left[\frac{1-\omega}{1-\omega e^{1-\alpha}}\right]^{1/(1-\alpha)}.$$

3. *General equilibrium (20 points).* Consider a two-period economy with dates $t=0$ and $t=1$. At $t=1$, a state z occurs, an element of a finite set $\mathcal{Z}$. There is one good in each date and state. A single agent's preferences over these goods are described by the utility function

$$U \;=\; u(c_0) + \beta \sum_z p(z) u[c_1(z)].$$

There is also a production opportunity: if we use x units of the date-0 good as an input, we produce $f(x)$ units of the date-1 good in all states z.

(a) What ingredients do you need to turn this into a complete model economy? (10 points)

(b) What is the Pareto problem associated with this economy? (10 points)

Solution:

(a) The standard list:

- List of commodities: 1 at $t = 0$, as many as states at $t = 1$.
- List of agents: one.
- Preferences and endowments: preferences above; need to specify endowments.
- Technology: $f(x)$.
- Resource constraints: If the endowments are y, we have

$$
\begin{aligned}
c_0 + x &\leq y_0 \\
c_1(z) &\leq y_1(z) + f(x) \text{ in each state } z.
\end{aligned}
$$

(b) We want to maximize the utility of the agent subject to the resource constraints. You could simply write down the problem: utility function and resource constraints. Or you could report the Lagrangian,

$$
\mathcal{L} = u(c_0) + \beta \sum_z p(z) u[c_1(z)] + q_0(y_0 - c_0 - x) + \sum_z q_1(z)[y_1(z) + f(x) - c_1(z)],
$$

which contains the same information.

4. *Short answers (25 points).*

(a) How would you compute excess kurtosis in a sample of data? (10 points)

(b) Consider the asset prices and dividends

$$
\begin{aligned}
&\text{Asset 1:} \quad q^1 = 3/4, \ \ d^1(1) = 1, \ \ d^1(2) = 1 \\
&\text{Asset 2:} \quad q^2 = 1, \ \ d^2(1) = 1, \ \ d^2(2) = 2.
\end{aligned}
$$

What are the returns on the two assets? What are the implied prices of Arrow securities? (15 points)

Solution:

(a) First, we need to compute the sample mean and variance:

$$
\begin{aligned}
\bar{x} &= T^{-1} \sum_t x_t \\
s^2 &= T^{-1} \sum_t (x_t - \bar{x})^2.
\end{aligned}
$$

Excess kurtosis is then

$$\gamma_2 = \frac{T^{-1}\sum_t (x_t - \bar{x})^4}{s^4} - 3.$$

(b) The returns on the first asset are $r^1(1) = r^1(2) = 4/3$. The returns on the second are $r^2(1) = 1$ and $r^2(2) = 2$. The state prices are $Q(1) = 1/2$ and $Q(2) = 1/4$.