Lab Report #1: Moments & Cumulants

Revised: September 17, 2015

Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Use Matlab where possible and attach your code to your answer.

Solution: Brief answers follow, but see also the attached Matlab program. Download this document as a pdf, open it, and click on the pushpin:

1. *Moments of normal random variables.* This should be review, but will get you started with moments and generating functions.

   Suppose x is a normal random variable with mean $\mu = 0$ and variance σ^2.

   (a) What is x's standard deviation?

   (b) What is x's probability density function?

   (c) What is x's moment generating function (mgf)? (Don't derive it, just write it down.)

   (d) What is $E(e^x)$?

   (e) Let $y = a + bx$. What is $E(e^{sy})$? How does it tell you that y is normal?

Solution:

(a) The standard deviation is the (positive) square root of the variance, namely σ if $\sigma > 0$ (or $|\sigma|$ if you want to be extra precise).

(b) The pdf is

$$p(x) \quad = \quad (2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2).$$

(c) The mgf is $h(s) = \exp(s^2\sigma^2/2)$.

(d) $E(e^x)$ is just the mgf evaluated at $s = 1$: $h(1) = e^{\sigma^2/2}$.

(e) The mfg of y is

$$\begin{aligned} h_y(s) \quad &= \quad E(e^{sy}) \quad = \quad E(e^{s(a+bx)}) \quad = \quad e^{sa}E(e^{sbx}) \quad = \quad e^{sa}h_x(bs) \\ &= \quad e^{sa+(bs)^2\sigma^2/2} \quad = \quad e^{sa+s^2(b\sigma)^2/2}. \end{aligned}$$

This has the form of a normal random variable with mean a (the coefficient of s) and variance $(b\sigma)^2$ (the coefficient of $s^2/2$).

2. *Sample moments of normal random variables.* It's often helpful to experiment with artificial test problems, just to remind ourselves how the code works. Here we compute sample moments of artificial data generated in Matlab and verify that calculations of various moments do what we think they do.

   This generates the data we'll use:

```
format compact              % single-spacing of output
nobs = 1000;                % number of observations
rng('default');             % sets "seed" so you can replicate the output
x = -1 + 2*randn(nobs,1);   % vector of normals with mean -1 and sd 2
```

   These commands generate "pseudo-random" numbers from a normal distribution with mean -1 and standard deviation 2 and puts them in the vector `x`. As always, you can find out more about Matlab commands by typing `help command` at the prompt; for example, `help rng` or `help randn`.

   (a) Our first check is to see if the sample moments correspond, at least approximately, to our knowledge of normal random variables. Run the commands:

```
xbar = mean(x)
moments = mean([(x-xbar).^2 (x-xbar).^3 (x-xbar).^4])
```

   What do you get? How do your calculations compare to the analogous true (or population) moments?

   (b) Our second check is on the Matlab commands `std(x)`, `skewness(x)`, and `kurtosis(x)`. Can you reproduce them with the sample moments computed in (a)?

Solution:

(a) We get and `moments = [-1.0653 3.9877 0.7566 52.1768]`. The first one is the sample mean, which is close to the theoretical mean of -1. The second is the sample variance, which is close to the theoretical variance of $2^2 = 4$. We'll touch on the others in (b). [If you did this in Python, you got different numbers, but you should find that you don't get exactly the theoretical values.)

(b) The `std` command gives us a sample standard deviation of 1.9979. The square root of our sample variance, however, is 1.9969. There's not much difference, but where does the difference come from? If we type `help std`, we see "normalizes by $N - 1$ rather than the number of observations N."

We exactly reproduce the `skewness` and `kurtosis` commands with

```
skw_x = moments(3)/moments(2)^(3/2)
krt_x = moments(4)/moments(2)^2
```

which assures us they do what we want them to do. We get values of 0.0950 (skewness) and 3.2811 (kurtosis). The theoretical values are zero and three, so we're close — not as close as with the mean and standard deviation because the higher-order powers are more sensitive to variation in a small number of observations. We'll typically subtract three from kurtosis to give us excess kurtosis, which is theoretically zero in the normal case.

3. *Sums of independent random variables.* Consider the sum of n random variables, say $y = x_1 + x_2 + \cdots + x_n$ with the x's "iid" (independent and identically distributed) Poisson random variables. That is, each x_i takes on values $x = 0, 1, 2, \ldots$ with probability $e^{-\omega}\omega^x/x!$. The parameter ω ("intensity") is positive.
   (a) What is the cumulant generating function (cgf) of x? (Don't derive it, just write it down.)
   (b) Use the cgf to derive the first four cumulants of x.
   (c) Now consider y. What is its cgf? How do we know that y is Poisson? What is its intensity parameter?
   (d) What are y's first four cumulants? How do our measures of skewness and excess kurtosis vary with n?

Solution:

(a) The cgf is $k(s; x) = \omega(e^s - 1)$.

(b) Differentiating k four times times us $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \omega$.

(c) The cgf of a sum of independent random variables is the sum of the individual cgfs. Here we have

$$k(s; y) \quad = \quad nk(s; x) \quad = \quad n\omega(e^s - 1).$$

We see from its form that y is Poisson with intensity parameter $n\omega$.

(d) The cumulants of y are evidently $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = n\omega$. Skewness and excess kurtosis are

$$\begin{aligned} \gamma_1 &= \kappa_3/(\kappa_2)^{3/2} \quad = \quad n\omega/(n\omega)^{3/2} \quad = \quad (n\omega)^{-1/2} \\ \gamma_2 &= \kappa_4)/(\kappa_2)^2 \quad = \quad n\omega/(n\omega)^2 \quad = \quad (n\omega)^{-1}. \end{aligned}$$

Both decline with n, so if n gets big enough they'll be arbitrarily close to zero. This is a close relative of the *central limit theorem*, in which the mean of iid random variables gets closer and closer to normal as we increase the number of components n. Here we see skewness and excess kurtosis, which are zero in the normal case, go to zero, so in that sense the sum is getting more normal.

4. *Squared normal random variable.* Suppose x is standard normal (normal with mean zero and variance one). What are the mgf and cgf of $y = x^2$? What are y's mean and variance?

Hints. (a) Apply the definition of the mgf. (b) Then combine the exponential terms — similar to what we did to derive the mgf of a normal random variable.

Solution: The starting point is the integral of the normal pdf,

$$\int_{\infty}^{\infty} (2\pi\sigma^2)^{-1/2} \exp[-(x-\mu)^2/2\sigma^2] dx \;\; = \;\; 1.$$

This is true for any values of μ and σ.

Here we're dealing with a standard normal random variable x, so $\mu = 0$ and $\sigma = 1$. The mgf of $y = x^2$ is

$$\begin{aligned} h(s) &= E(e^{sy}) \;\; = \;\; E(e^{sx^2}) \\ &= \int_{\infty}^{\infty} (2\pi)^{-1/2} \exp[-x^2/2] e^{sx^2} dx \\ &= \int_{\infty}^{\infty} (2\pi)^{-1/2} \exp[-(1-2s)x^2/2] dx. \end{aligned}$$

(This works as long as $1 - 2s > 0$ or $s < 1/2$. Since we're interested in a neighborhood around $s = 0$, that's fine.) This looks like a normal integral with $\sigma^2 = 1/(1 - 2s)$, but we need to fix up the constant term:

$$\begin{aligned} h(s) &= \int_{\infty}^{\infty} (2\pi)^{-1/2} \exp(-x^2/2\sigma^2) dx \\ &= (\sigma^2)^{1/2} \int_{\infty}^{\infty} (2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2) dx \\ &= (\sigma^2)^{1/2} \;\; = \;\; (1-2s)^{-1/2}. \end{aligned}$$

The cgf is its log: $k(s) = \log h(s) = -\log(1 - 2s)/2$. We differentiate twice to find that the mean is one and the variance is two.

The random variable y has a *chi-squared distribution* with one degree of freedom. You can verify its properties at Wikipedia.