Quiz #2

November 2014

Please write your name below. Then complete the exam in the space provided. There are THREE questions. You may refer to one page of notes: standard paper, both sides, any content you wish.

(Name and signature)

1. *Asset pricing fundamentals.* Consider an economy with the following properties:

State z	State Price $Q(z)$	Probability $p(z)$	Dividend $d(z)$
1	1/2	1/2	2
2	1/3	1/2	1

(a) What is the price of a one-period riskfree bond? (10 points)

(b) What is the price of a claim to the dividend? What are the returns? (10 points)

(c) What is the expected excess return on a claim to the dividend? What is its Sharpe ratio? (10 points)

(d) What is the maximum Sharpe ratio in this economy? (10 points)

Solution:

(a) The price is $q^1 = Q(1) + Q(2) = 5/6$. (You could also find m and compute it as $E(m)$, but this is easier.)

(b) The price of (let us call it) "equity" is $q^e = Q(1)d(1) + Q(2)d(2) = 4/3$. The returns are $r^e(1) = 3/2$ and $r^e(2) = 3/4$.

(c) The excess returns are $r^e(1) - r^1 = 0.3$ and $r^e(2) - r^1 = -0.45$. The expected excess return is therefore -0.075. The standard deviation is 0.375, giving us a Sharpe ratio of -0.2.

(d) This is a call for the Hansen-Jagannathan bound. The maximum Sharpe ratio is

$$\frac{\text{Std}(m)}{E(m)} = \frac{1/6}{5/6} = 1/5.$$

The pricing kernel m comes from $m(z) = Q(z)/p(z)$, so that $m(1) = 1$ and $m(2) = 2/3$. This has the opposite sign as the equity premium but the same magnitude. The latter is a property of two-state distributions where, in a sense, everything is linearly related and correlations are $+1$ or -1.

2. *Uniform option pricing.* We'll explore the logic of option pricing in a setting in which the underlying has a uniform distribution. More formally, let $x = s_{t+1}$ (not $x = \log s_{t+1}$!) have a uniform distribution over the interval $[\mu-\theta, \mu+\theta]$; that is, its probability density function is

$$p(x) \;=\; \begin{cases} (2\theta)^{-1} & \text{if } \mu-\theta \le x \le \mu+\theta \\ 0 & \text{otherwise.} \end{cases}$$

   (a) What is the mean of x? The variance? (10 points)

   (b) What is the no-arbitrage condition here? (10 points)

   (c) What is the cash flow of a put option with strike price k? What is its value? (20 points)

   (d) How does the price of a put option change when we increase θ? Why? (10 points)

Solution:

(a) You might guess the mean and variance. If not, they follow from

$$\begin{aligned} E(x) &= (2\theta)^{-1}\int_{\mu-\theta}^{\mu+\theta} x\,dx \;=\; (2\theta)^{-1}[(\mu+\theta)^2-(\mu-\theta)^2]/2 \;=\; \mu \\ \mathrm{Var}(x) &= (2\theta)^{-1}\int_{\mu-\theta}^{\mu+\theta}(x-\mu)^2dx \;=\; (2\theta)^{-1}[\theta^3-(-\theta)^3]/3 \;=\; \theta^2/3. \end{aligned}$$

So μ governs the mean and θ governs the variance.

(b) The no-arb condition is

$$s_t \;=\; q^1E^*(s_{t+1}) \;=\; q^1\mu.$$

(c) A put option is a claim to the cash flow $d = (k-x)^+$. If k is in the interval $[\mu-\theta, \mu+\theta]$ we have

$$\begin{aligned} q^p &= q^1(2\theta)^{-1}\int_{\mu-\theta}^{k}(k-x)dx \\ &= q^1k\,[k-(\mu-\theta)]/(2\theta) - q^1[k^2-(\mu-\theta)^2]/(4\theta). \end{aligned}$$

(d) The idea is that option prices increase with risk. Here if we increase θ, the parameter that controls the variance, put prices seem to go up, but I haven't proven it.

3. *More fundamentals.* A well-known financial economist who spent years in the business world, commented on what he had learned as an academic: "I learned two basic lessons about financial mathematics that I've always found useful. One is that risk premiums come from covariances. The other is that asset prices come from risk-neutral probabilities."

(a) Give an equation that illustrates his first lesson. (10 points)

(b) Give an equation that illustrates his second lesson. (10 points)

(c) Where did the covariance go in part (b)? (10 points)

Solution:

(a) The first reflects $E(x) = -\text{Cov}(x, m)/E(m)$, where x is any excess return.

(b) The second is reflected by $q = q^1 E^*(d) = q^1 \sum_z p^*(z) d(z)$, where q is the price of the dividend d and E^* is the expectation based on the risk-neutral probabilities.

(c) The covariance is embedded in the risk-neutral probabilities. Asset prices, for example, can be expressed two ways:

$$
\begin{aligned}
q &= E(md) = q^1 E(d) + \text{Cov}(m, d) \\
q &= q^1 E^*(d).
\end{aligned}
$$

In the first, we have a covariance. In the second, that's built into E^*.