Quiz #3

December 2013

Please write your name below. Then complete the exam in the space provided. There are THREE questions. You may refer to one page of notes: standard paper, both sides, any content you wish.

(Name and signature)

1. *Moving average dynamics.* Consider the stochastic process

$$x_t = \delta + w_t + \theta w_{t-1},$$

where $\{w_t\}$ is our usual collection of independent standard normal random variables.

(a) Is x Markov for some definition z_t of the state at date t? What is the distribution of x_{t+1} conditional on z_t? (5 points)

(b) What is the equilibrium distribution of x? (5 points)

(c) What are the autocovariance and autocorrelation functions? (10 points)

(d) What is the maximum first autocorrelation $\rho(1)$? For what value of θ does it occur? (10 points)

Solution:

(a) The natural definition of the state is $z_t = w_t$. Given z_t, x_{t+1} is normal with mean $\delta_\theta w_t$ and variance one. That's the one-period conditional distribution.

(b) The equilibrium distribution is normal with mean δ and variance $1 + \theta^2$.

(c) The autocovariance function is

$$\gamma(k) = \begin{cases} 1+\theta^2 & k = 0 \\ \theta & k = 1 \\ 0 & k \geq 2. \end{cases}$$

That is: an MA(1) has a one-period memory. The autocorrelation function is $\rho(k) = \gamma(k)/\gamma(0)$.

(d) The first autocorrelation is $\rho(1) = \theta/(1 + \theta^2$. This takes its highest value of one-half when $\theta = 1$.

2. *Inflation and government deficits.* Here's a variant of our forward-looking inflation model. We have, as usual, the quantity theory plus a velocity equation:

$$\begin{aligned} m_t + v_t &= p_t + y_t \\ v_t &= \alpha\left(E_t p_{t+1} - p_t\right). \end{aligned}$$

Then we set $y_t = 0$ (to keep things simple) and connect the money supply m_t to the government deficit d_t:

$$\begin{aligned} m_t &= \delta d_t \\ d_{t+1} &= \varphi d_t + \sigma w_{t+1}. \end{aligned}$$

(a) Express this model as a forward-looking difference equation in which the price level p_t is a function of its expected future value and the deficit. (15 points)

(b) How is the price level connected to the current deficit? (15 points)

Solution:

(a) Substitutions give us

$$\begin{aligned} p_t &= [\alpha/(1+\alpha)]E_t p_{t+1} + [\delta/(1+\alpha)]d_t \\ &= \lambda E_t p_{t+1} + \delta' d_t, \end{aligned}$$

where the second line is compact notation for the first.

(b) This has the usual forward-looking solution in which p_t is connected to expected future deficits. We could grind through this, but it's easier to use the method of undetermined coefficients. If we guess the solution has the form $p_t = a d_t$ for some coefficient a to be determined, then

$$\begin{aligned} p_{t+1} &= a d_{t+1} = a(\varphi d_t + \sigma w_{t+1}) \\ E_t(p_{t+1}) &= a\varphi d_t. \end{aligned}$$

If we substitute into the difference equation above and collect coefficients of d_t, we get

$$a = \frac{\delta'}{1-\lambda\varphi} = \frac{\delta}{1+\alpha(1-\varphi)}.$$

In words: the impact of the current deficit on the price is larger if (i) money is more strongly tied to deficits (large δ), (ii) deficits are more persistent (large φ), and (iii) velocity is less sensitive to expected inflation (small α). [Last one sounds wrong, mistake somewhere??]

3. *Expected returns on bonds.* Consider the bond pricing model

$$\begin{aligned} \log m_{t+1} &= -(\lambda_0 + \lambda_1 w_t)^2/2 - x_t + (\lambda_0 + \lambda_1 w_t) w_{t+1} \\ x_t &= \delta + \sigma(w_t + \theta w_{t-1}), \end{aligned}$$

where the w_t's are independent standard normal random variables.

(a) Suppose bond prices take the form

$$\log q_t^n \quad = \quad A_n + B_n w_t + C_n w_{t-1}.$$

How is $(A_{n+1}, B_{n+1}, C_{n+1})$ connected to (A_n, B_n, C_n)? What are the values of (A_n, B_n, C_n) for $n = 0, 1, 2$? (20 points)

(b) Express the expected log excess return on a two-period bond as a function of the coefficients (A_n, B_n, C_n) for $n = 1, 2$. Use your solution for the coefficients to describe how expected log excess returns vary with the interest rate innovation w_t. (20 points)

Solution:

(a) We use the usual formula, $q_t^{n+1} = E_t(m_{t+1} q_{t+1}^n)$. Then with substitutions,

$$\begin{aligned} \log(m_{t+1} q_{t+1}^n) &= A_n - \delta + (C_n - \sigma) w_t - \sigma\theta w_{t-1} - (\lambda_0 + \lambda_1 w_t)^2/2 \\ &\quad + [B_n + (\lambda_0 + \lambda_1 w_t)] w_{t+1}. \end{aligned}$$

The first line on the rhs gives us the conditional mean, the second line gives us the variance. Some intensive algebra gives us

$$\begin{aligned} A_{n+1} &= A_n - \delta + B_n^2/2 + B_n \lambda_0 \\ B_{n+1} &= C_n - \sigma + B_n \lambda_1 \\ C_{n+1} &= -\sigma\theta. \end{aligned}$$

That gives us

n	A_n	B_n	C_n
0	0	0	0
1	$-\delta$	$-\sigma$	$-\sigma\theta$
2	$-2\delta - \lambda_0\sigma + \sigma^2/2$	$-\sigma(1 + \theta + \lambda_1)$	$-\sigma\theta$

(b) The log excess return is

$$\begin{aligned} \log r_{t+1}^2 - \log r_{t+1}^1 &= \log q_{t+1}^1 - \log q_t^2 + \log q_t^1 \\ &= (2A_1 - A_2) + (C_1 + B_1 - B_2) w_t + (C_1 - C_2) w_{t-1} \\ &\quad + B_1 w_{t+1}. \end{aligned}$$

The expected excess return knocks out the last term.

If we substitute our solutions, we have

$$E_t\left(\log r_{t+1}^2 - \log r_{t+1}^1\right) \quad = \quad (\sigma\lambda_0 - \sigma^2/2) + (\sigma\lambda_1) w_t.$$

So the key parameter is λ_1: the sensitivity of the "price of risk" to w_t.