Lab Report #3: Consumption, Risk, and Portfolio Choice

Revised: February 19, 2013

Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Please include your Matlab code.

1. *Exponential risks.* Consider risk when when $x = -\log c$ is exponential, a tractable nonnormal distribution. The minus sign gives us negative skewness, which power utility agents dislike. The goal is to see how this shows up in the risk penalty.

   The connection between c and x implies $c = e^{\log c} = e^{-x}$. The pdf is

   $$p(x) \quad = \quad \lambda e^{-\lambda x}$$

   for $x \geq 0$ and $\lambda > 0$. The cgf is

   $$k(s) \quad = \quad -\log\left(1 - s/\lambda\right).$$

   In terms of x, the mean of c is $\bar{c} = E(c) = E(e^{-x})$.

   (a) Use the cgf to compute the variance, skewness, and excess kurtosis of x. How do they compare to those of the normal distribution?

   (b) Use the cgf to compute $\bar{c}$.

   (c) Choose the value of λ that sets the variance equal to 0.02^2, the number we used earlier.

   (d) For extra credit: how does this differ from the lognormal example we did last week? Why?

Solution: Same approach as the previous problem.

(a) The first four cumulants are $\kappa_1 = 1/\lambda$, $\kappa_2 = 1/\lambda^2$, $\kappa_3 = 2/\lambda^3$, and $\kappa_4 = 6/\lambda^4$. Skewness and excess kurtosis are therefore $\gamma_1 = 2$ and $\gamma_2 = 6$.

(b) If $k(s)$ is the cgf of x, then $\log \bar{c} = k(-1)$ (k evaluated at $s = -1$): $k(-1) = -\log(1 + 1/\lambda)$.

(c) We solve $0.02^2 = 1/\lambda^2$ or $\lambda = 1/0.02 = 50$.

(d) The risk penalties are

	$\alpha = 2$	$\alpha = 10$	$\alpha = 20$
lognormal	0.0004	0.0020	0.0040
exponential	0.0004	0.0022	0.0054

The point is that departures from normality matter, although they're not especially large here. The numbers suggest that the impact increases with risk aversion α. This may seem mysterious now, but we could elaborate on this using the power series expansion of the cgf. That would show us that the jth cumulant of $\log g$ is multiplied by $(-\alpha)^j/j!$, so as we increase α some of these terms can increase dramatically. Amir Yaron (Wharton prof) refers to this as a "bazooka," since it allows you to take a little nonnormality and generate enormous risk premiums. More on this kind of thing later in the course.

2. *Consumption, saving, and the interest rate.* Here's a variant of the consumption-saving problem we did in class. Let the agent have the problem

$$\begin{aligned} \max_{c_0,c_1} & \quad u(c_0)+\beta u(c_1) \\ s.t. & \quad c_0+qc_1 \;=\; y_0+qy_1 \end{aligned}$$

with $u(c)=c^{1-\alpha}/(1-\alpha)$. The idea of the budget constraint is that the present value of consumption equals the present value of income.

(a) How is q related to the interest rate?

(b) What is saving in this setting? How is saving connected to second-period consumption c_1?

(c) What are the first-order conditions for the problem?

(d) What are the optimal choices of c_0 and c_1 given the price q?

Solution: We did all of this in class but the last part: solve explicitly for (c_0,c_1). If λ is the Lagrange multiplier on the constraint, the first-order conditions are

$$\begin{aligned} c_0: &\quad u'(c_0) \;=\; \lambda \\ c_1: &\quad \beta u'(c_1) \;=\; \lambda q. \end{aligned}$$

With power utility, we have $u'(c)=c^{-\alpha}$.

We have three equations (the two foc's plus the budget constraint) in the three unknowns (c_0,c_1,λ). The goal here is to find c_0 and c_1 as functions of q. Here's a line of attack that works. Solve the foc's for consumption:

$$\begin{aligned} c_0 &= (1/\lambda)^{1/\alpha} \\ c_1 &= (1/\lambda)^{1/\alpha}(q/\beta)^{1/\alpha}. \end{aligned}$$

Then we have

$$c_0+qc_1 \;=\; (1/\lambda)^{1/\alpha}\left[1+(q/\beta)^{1/\alpha}\right].$$

From the budget constraint, this is $y_0+qy_1=Y$ (say), so we have

$$(1/\lambda)^{1/\alpha} \;=\; Y/[1+(q/\beta)^{1/\alpha}].$$

That gives us

$$
\begin{aligned}
c_0 &= Y/[1+(q/\beta)^{1/\alpha}] &= Y\beta^{1/\alpha}/(\beta^{1/\alpha}+q^{1/\alpha}) \\
c_1 &= Yq^{1/\alpha}/(\beta^{1/\alpha}+q^{1/\alpha}). &
\end{aligned}
$$

You'll note that both are proportional to Y. These preferences have the convenient property that they scale nicely: if we double income, we double both consumptions. The technical term is "homothetic."

3. *Two-state portfolio choice.* Portfolio choice problems are notoriously unfriendly, but we can get a sense of their properties with an example. The agent's consumption-saving-portfolio-choice problem is

$$
\begin{aligned}
\max_{c_0,a} \quad & u(c_0)+\beta\sum_z p(z)u[c_1(z)] \\
s.t. \quad & c_1(z) \;=\; (y_0-c_0)[(1-a)r^1+ar^e(z)].
\end{aligned}
$$

The idea is that we have two assets, a riskfree asset with gross return r^1 ("1" for one-period bond) and a risky asset with gross return r^e ("e" for equity). We invest a fraction a of our saving in equity and the complementary fraction $1-a$ in the riskfree bond. If $a>1$, the agent has a levered position. Note that this version of the problem is slightly different from the one we used in class.

We'll deal with a special case: two states, $z=1$ and $z=2$; equally likely, $p(1)=p(2)=1/2$; and power utility, $u(c)=c^{1-\alpha}/(1-\alpha)$. Parameter values include $\beta=1/1.1$, $r^1=1.1$, $r^e(1)=1.0$, $r^e(2)=1.4$, $y_0=1$.

(a) What are the mean and variance of the return on equity?

(b) What are the implied prices of Arrow securities? Hint: Recall that Arrow securities pay off in one state only. Our assets are combinations of Arrow securities, so it's a question of unbundling them.

(c) What are the first-order conditions for c_0 and a? Show that with power utility, the latter can be determined without knowing c_0.

(d) Solve these conditions numerically for $\alpha=5$. What values do you get for a, c_0, $c_1(1)$, and $c_1(2)$? Comment: I did this by varying a until its first-order condition was satisfied. You could also compute the first-order condition for a grid of values for a, and choose the one that works best.

(e) How does your answer change if you use $\alpha=2$? Does the difference make sense to you?

Solution:

(a) The mean is 1.2 (the average of 1.0 and 1.4) and the variance is $0.2^2=0.04$.

(b) Let $Q(z)$ be the price of the Arrow security that pays one in state z. A one-period bond pays one in each state, so its price is the sum:

$$q^1\ (= 1/r^1) \quad = \quad Q(1) + Q(2).$$

Equity is a little more complicated, because we have to agree on units. Let's choose units to that the price of a share is one unit of date-0 consumption. Then it pays off 1.0 units of date-1 consumption in state $z = 1$ and 1.4 units in state $z = 2$:

$$q^e\ (= 1) \quad = \quad 1.0Q(1) + 1.4Q(2).$$

Now it's a matter of algebra to find $q^A(1) = 0.6818$ and $q^A(2) = 0.2273$.

(c) The first-order conditions are

$$\begin{aligned} c_0: \qquad u'(c_0) &= \beta \sum_z p(z) u'[c_1(z)][(1-a)r^1 + ar^e(z)] \\ a: \qquad 0 &= \beta \sum_z p(z) u'[c_1(z)][r^e(z) - r^1]. \end{aligned}$$

Written out more completely, the latter becomes

$$\begin{aligned} 0 &= \beta \sum_z p(z)(y_0 - c_0)^{1-\alpha}[(1-a)r^1 + ar^e(z)]^{-\alpha}[r^e(z) - r^1] \\ &= \sum_z p(z)[(1-a)r^1 + ar^e(z)]^{-\alpha}[r^e(z) - r^1]. \end{aligned}$$

which depends on a but not c_0.

(d) I did this numerically. We'll talk about numerical methods for solving equations shortly ("root-finding"), but the simplest way is to compute the right side of the previous equation for a grid of values for a. We take the value that produces the first-order condition closest to zero. A grid of 0.01 gives us $a = 1.70$ when $\alpha = 2$.

Now that we know a, we can find c_0 from the first first-order condition:

$$c_0^{-\alpha} \quad = \quad \beta(y_0 - c_0)^{-\alpha} \sum_z p(z)[(1-a)r^1 + ar^e(z)]^{1-\alpha}.$$

We compute the sum on the rhs first — denote it by S (for sum). Then we solve for c_0:

$$(y_0 - c_0)/c_0 \quad = \quad (\beta S)^{-1/\alpha} \;\Rightarrow\; c_0 \quad = \quad y_0/[1 + (\beta S)^{-1/\alpha}].$$

Second-period consumption follows from the budget constraint: $c_1(z) = (y_0 - c_0)[(1-a)r^1 + ar^e(z)]$.

The numbers are listed below:

	a	S	c_0	$c_1(1)$	$c_1(2)$
$\alpha = 2$	1.702	0.8482	0.4676	0.4951	0.8576
$\alpha = 5$	0.637	0.6135	0.4708	0.5484	0.6832

(e) When $\alpha = 5$, we get $a = 0.64$ by the same method. The punchline: when we increase risk aversion, we hold less of the risky asset and less risky second-period consumption as a result.

Matlab program:

```
%  hw3_s12
%  Matlab program for Lab Report #3, Spring 2012
%  Written by:  Dave Backus, February 2012
format compact
clear all

%disp(' ')
disp('Answers to Lab Report 3')

%%
disp(' ')
disp('------------------------------------------------------------------')
clear
disp('Question 1')
syms s lambda alpha

cgf_x = -log(1-s./lambda)
cgf = subs(cgf_x,s,-s)

disp(' ')
disp('Cumulants')
kappa1 = subs(diff(cgf,s,1),s,0)    %  mean
kappa2 = subs(diff(cgf,s,2),s,0)    % variance
kappa3 = subs(diff(cgf,s,3),s,0)
kappa4 = subs(diff(cgf,s,4),s,0)

disp(' ')
gamma1 = kappa3/kappa2^(3/2)
simplify(gamma1)
gamma2 = kappa4/kappa2^2

% cgf evaluated at s=1
log_cbar = subs(cgf,s,1)
% cgf evaluated at s=1-alpha, then divided by (1-alpha)
log_mu = subs(cgf,s,1-alpha)/(1-alpha)

rp = log_cbar - log_mu
rp = simplify(rp)

rp_alpha2  = subs(rp,[alpha,lambda],[2,1/0.02])
rp_alpha10 = subs(rp,[alpha,lambda],[10,1/0.02])
rp_alpha20 = subs(rp,[alpha,lambda],[20,1/0.02])
```

```
%%
disp(' ')
disp('------------------------------------------------------------------')
disp('Question 3')
disp('Portfolio choice:  2-state case')
disp(' ')
% set parameters
beta = 1/1.1;
r1 = 1.1;
re_1 = 1.0;
re_2 = 1.4;
y0 = 1;

% Arrow securities
disp('Prices and returns of Arrow securities')
Q = inv([1 1; 1 1.4])*[1/r1; 1]
rA = 1./Q

% approximate portfolio weight a: find value that sets foc closest to zero
agrid = [0.1:0.001:2];

disp(' ')
disp('Portfolio weight a')
alpha = 2;
foc2 = 0.5*((1-agrid)*r1+agrid*re_1).^(-alpha).*(re_1-r1) + ...
        0.5*((1-agrid)*r1+agrid*re_2).^(-alpha).*(re_2-r1);
[focmin,i] = min(abs(foc2));
a2 = agrid(i)

alpha = 5;
foc5 = 0.5*((1-agrid)*r1+agrid*re_1).^(-alpha).*(re_1-r1) + ...
        0.5*((1-agrid)*r1+agrid*re_2).^(-alpha).*(re_2-r1);
axis = 0*agrid;
[focmin,i] = min(abs(foc5));
a5 = agrid(i)

% Digression on Merton's formula
% See:  http://en.wikipedia.org/wiki/Merton's_portfolio_problem
disp('Compare Merton share')
% next few lines aren't needed
E_rx = 0.5*(re_1+re_2) - r1
var_re = 0.2^2
E_logre = 0.5*(log(re_1)+log(re_2))
E_logrx = 0.5*(log(re_1)+log(re_2)) - log(r1)
var_logre = 0.5*(log(re_1)-E_logre)^2 + 0.5*(log(re_2)-E_logre)^2
E_rx_approx = E_logrx + var_logre/2
```

```
amerton5 = (0.1+var_re/2)/(alpha*var_re)
amerton2 = (0.1+var_re/2)/(2*var_re)
ratio2 = amerton2/a2
ratio5 = amerton5/a5
% end of digression

plot(agrid,foc2,'b')
hold on
plot(agrid,foc5,'m')
plot(agrid,axis,'k')
ylabel('First-order condition')
xlabel('Portfolio weight a')
title('Blue is \alpha=2, Magenta is \alpha=5')

disp(' ')
disp('Consumption quantities')
alpha = 5; a = a5;          % change as needed
S = 0.5*((1-a)*r1+a*re_1)^(1-alpha) + 0.5*((1-a)*r1+a*re_2)^(1-alpha)
SBA = (S*beta)^(-1/alpha)

c0 = y0/(1+SBA)
saving = y0 - c0

c1_1 = saving*((1-a)*r1+a*re_1)
c1_2 = saving*((1-a)*r1+a*re_2)

return
```