# Lab Report #7: Dynamics in Theory and Data

Revised: December 4, 2015

*Due at the start of class. You may speak to others, but whatever you hand in should be your own work.*

1. *Is the light on?* Consider Hairer's example. A state variable $z_t$ equals one if the light is on at date $t$, zero if it's off. Between dates, the probability that the state stays the same is $(1+\varphi)/2$.
   (a) What is the probability that the state changes? How do you know?
   (b) How does the probability distribution over next period's state depend on this period's?
   (c) If the light is on at date $t$, what is the probability that it's on at $t+1$? At $t+2$? At $t+k$ for $k > 2$?
   (d) How does your uncertainty about the light's state change with the forecast horizon? Your ability to forecast?

**Solution:**

(a) Probabilities are positive numbers that sum to one. If the probability that the state stays the same is $(1+\varphi)/2$, then the probability that it changes is

$$1 - (1+\varphi)/2 \quad = \quad (1-\varphi)/2.$$

(b) If the light is on (that is, $z_1 = 1$), then the probability that it's on next period is $(1+\varphi)/2$. If the light is off (that is, $z_1 = 0$), then the probability that it's on next period is $(1-\varphi)/2$. So the probabilities over next period's states depend on this period's state.

(c) We did this in class. It's also in the notes. The answer is

$$p_k \quad = \quad \text{Prob}(z_{t+k} = 1 | z_t = 1) \quad = \quad (1 + \phi^k)/2.$$

(d) The further into the future we go, the more uncertainty there is. The question — and it's not a simple one — is how to measure that uncertainty. One possibility os the variance:

$$p_k \quad = \quad \text{Var}(z_{t+k} = 1 | z_t = 1) \quad = \quad p_k(1 - p_k).$$

(It's Bernoulli, so we apply the formula.) This has a peak at $p_k = 1/2$, which we approach gradually as we increase $k$.

2. *MA(1).* Consider the MA(1)

$$x_t = \theta_0 w_t + \theta_1 w_{t-1}$$

with iid standard normal innovations $w_t$ and coefficients $(\theta_0, \theta_1)$.

(a) What is the variance of $x$?

(b) What is the covariance of $x_t$ and $x_{t-1}$? How does it change if $\theta_1 = 0$? $\theta_0 = 0$?

(c) What is the autocorrelation function?

(d) We estimate the autocorrelation to be 0.4. What does that tell us about the coefficients $(\theta_0, \theta_1)$?

**Solution:**

(a) The variance is

$$\begin{aligned}\gamma(0) \;=\; \text{Var}(x_t) &= E\big[(\theta_0 w_t + \theta_1 w_{t-1})^2\big] \\ &= E\big[(\theta_0 w_t)^2 + (\theta_1 w_{t-1})^2\big] \;=\; \theta_0^2 + \theta_1^2.\end{aligned}$$

As usual, the cross terms cancel because the $w$'s are independent. This is the variance, not the conditional variance.

(b) The covariance is

$$\begin{aligned}\gamma(1) \;=\; \text{Cov}(x_t, x_{t-1}) &= E\big[(\theta_0 w_t + \theta_1 w_{t-1})(\theta_0 w_{t-1} + \theta_1 w_{t-2})\big] \\ &= E\big[\theta_1\theta_0 w_{t-1}^2\big] \;=\; \theta_0\theta_1.\end{aligned}$$

It's zero if either $\theta_0$ or $\theta_1$ is zero: we need both.

If we continued, we'd find that the "autocovariances" $\gamma(k) = \text{Cov}(x_t, x_{t-k})$ are zero for $k \geq 2$.

(c) The autocorrelations are the autocovariances divided by the variance: $\rho(k) = \gamma(k)/\gamma(0)$. Here we have

$$\rho(1) \;=\; \frac{\theta_0\theta_1}{\theta_0^2 + \theta_1^2} \;=\; \frac{\theta_1/\theta_0}{1 + (\theta_1/\theta_0)^2} \;=\; \frac{\lambda}{1 + \lambda^2},$$

where $\lambda = \theta_1/\theta_0$. Only the ratio matters. Think of $\theta_0$ as scaling the process variance up and down and $\theta_1$ controlling its dynamics.

The autocorrelation function (conventionally, the "acf") is then $\rho(0) = 1$, $\rho(1)$ as given, and $\rho(k) = 0$ for $k \geq 2$. As we've come to expect from moving averages, they have a finite memory, here one period.

(d) Suppose $\rho(1) = 0.4$. That gives us

$$\rho(1) \;=\; 0.4 \;=\; \frac{\lambda}{1 + \lambda^2},$$

which we can solve for $\lambda$. But it's a quadratic:

$$\rho\lambda^2 - \lambda + \rho \;=\; 0.$$

(Here I've replaced $\rho(1)$ with $\rho$ out of laziness.) It therefore has two roots:

$$\rho \;=\; \frac{1 \pm (1-4\rho^2)^{1/2}}{2\rho}$$

The solutions are $\lambda = 2$ and $\lambda = 1/2$. Both of them work, in the sense of giving us the same acf. We usually choose the smaller one, but the reasons for that are too complicated to go into.

3. *ARMA(1,1).* Consider the ARMA(1,1) model

$$x_t \;=\; \sum_{j=0}^{\infty} a_j w_{t-j}$$

with iid standard normal innovations $w_t$ and coefficients $a_0$, $a_1$, and $a_{j+1} = \varphi a_j$ for $j \geq 1$ and parameter $0 < \varphi < 1$.

(a) What is the variance of $x$?

(b) What is the covariance of $x_t$ and $x_{t-1}$?

(c) What is the autocovariance function? The autocorrelation function?

(d) What configuration of parameter values gives us negative autocorrelations?

(e) *Extra credit.* Show that the model can be expressed in traditional ARMA(1,1) form,

$$x_t \;=\; \varphi x_{t-1} + \sigma(w_t + \theta w_{t-1}).$$

**Solution:**

(a) This follows most naturally from the moving average formula. Because the $w$'s are independent, the cross terms are zero and we're left with the squares:

$$\begin{aligned}\mathrm{Var}(x) \;&=\; E\Big[\Big(\sum_{j=0}^{\infty} a_j w_{t-j}\Big)^2\Big] \;=\; \sum_{j=0}^{\infty} E\big(a_j w_{t-j}\big)^2 \\ &=\; \sum_{j=0}^{\infty} a_j^2 \;=\; a_0^2 + a_1^2/(1-\varphi^2).\end{aligned}$$

(b) The covariance is similar

$$\begin{aligned}\mathrm{Cov}(x_t, x_{t-1}) \;&=\; E\Big[\Big(\sum_{j=0}^{\infty} a_j w_{t-j}\Big)\Big(\sum_{j=0}^{\infty} a_j w_{t-j-1}\Big)\Big] \\ &=\; \sum_{j=0}^{\infty} a_j a_{j+1} \;=\; a_0 a_1 + a_1\varphi/(1-\varphi^2).\end{aligned}$$

(c) The autocovariance function is

$$\begin{aligned}
\gamma(0) &= a_0^2 + a_1^2/(1-\varphi^2) \\
\gamma(1) &= a_0 a_1 + a_1\varphi/(1-\varphi^2) \\
\gamma(k+1) &= \varphi\gamma(k) \text{ for } j \geq 1.
\end{aligned}$$

The autocorrelation function is $\rho(k) = \gamma(k)/\gamma(0)$. Like the moving average coefficients, the first two are arbitrary, then they decay at rate $\varphi$.

(d) Autocorrelations and autocovariances have the same sign. Looking at $\gamma(0) = \text{Cov}(x_t, x_{t-1})$, we see that it's sufficient to have $a_0 > 0$ and $a_1 < 0$.

(e) If we subtract $\varphi x_{t-1}$ from $x_t$ we get

$$\begin{aligned}
x_t - \varphi x_{t-1} &= \sum_{j=0}^{\infty} a_j w_{t-j} - \varphi \sum_{j=0}^{\infty} a_j w_{t-j-1} \\
&= a_0 w_t + (a_1 - \varphi a_0) w_{t-1} + (a_2 - \varphi a_1) w_{t-2} + \cdots \\
&= a_0 w_t + (a_1 - \varphi a_0) w_{t-1},
\end{aligned}$$

which is an ARMA(1,1).