Quiz #1

Spring 2013

Please write your name below, then complete the exam in the space provided. There are FOUR questions. You may refer to one page of notes: standard paper, both sides, any content you wish.

____

(Name and signature)

1. *Moments, cumulants, and generating functions (20 points).* Consider an arbitrary random variable x.

   (a) Define the moment generating function of x. How is the cumulant generating function related to it? (5 points)

   (b) How is the cumulant generating function of $y = \alpha + \beta x$ related to the cgf of x? (5 points)

   (c) What is the second central moment μ_2 of x? How is it connected to the moment generating function? (5 points)

   (d) What is the third cumulant κ_3 of x? How is it connected to the cumulant generating function? (5 points)

Solution:

(a) The mgf h is defined by: $h(s) = E(e^{sx})$. The cgf is its log: $k(s) = \log h(s)$.

(b) The mgf of y is $h_y(s) = E(e^{sy}) = E(e^{s(\alpha+\beta x)}) = e^{s\alpha} h_x(\beta s)$. The cgf is its log: $k_y(s) = s\alpha + k_x(\beta s)$.

(c) The second central moment is the variance. If the mean is $\bar{x}$, it's $\text{Var}(x) = E(x-\bar{x})^2 = E(x^2) - \bar{x}^2$. $E(x^2)$ is the second raw or noncentral moments, and $\bar{x}$ is the first. They are the second and first derivatives, resp, of the mgf, evaluated at $s = 0$.

(d) The third cumulant is the third derivative of the cgf, also evaluated at $s = 0$. It's also the third central moment.

2. *Risk and return (30 points).* Consider an agent with utility

$$U \;=\; E[u(c)]$$

where $u(c) = c^{1-\alpha}/(1-\alpha)$ for some $\alpha > 0$. She invests one and consumes the gross return r.

(a) What is her expected utility if she invests everything in a riskfree asset whose (gross) return is 1.1? (Her consumption is therefore 1.1 in every state.) What is the certainty equivalent of this outcome? (10 points)

(b) What is her expected utility if she invests in an asset whose return is lognormal: $\log r \sim \mathcal{N}(\kappa_1, \kappa_2)$? What is her certainty equivalent? (10 points)

(c) For what values of κ_1 and κ_2 is the risky asset preferred? (10 points)

Solution:

(a) The certainty equivalent of a sure thing $\bar{c}$ is $\bar{c}$. More formally, if consumption is $\bar{c}$ in all states, then the certainty equivalent μ solves

$$U(\bar{c}, \bar{c}, \ldots, \bar{c}) \quad = \quad U(\mu, \mu, \ldots, \mu),$$

so $\mu = \bar{c}$. So the certainty equivalent here is 1.1.

(b) We're using properties of lognormal random variables here. We know $E(r^{1-\alpha}) = \exp[(1-\alpha)\kappa_1 + (1-\alpha)^2\kappa_2/2]$. The certainty equivalent is $\mu = E(r^{1-\alpha})^{1/(1-\alpha)} = \exp[\kappa_1 + (1-\alpha)\kappa_2/2]$.

(c) Evidently we need $\kappa_1 + (1-\alpha)\kappa_2/2 > \log 1.1$. So large κ_1 helps. If $\alpha > 1$, small κ_2 helps, too, otherwise the reverse.

3. *Securities and returns (30 points).* Consider an economy with two assets and two equally likely states. The assets have dividends

Asset	State 1	State 2
1 ("bond")	1	1
2 ("equity")	2	5

The prices of the two assets are $q^1 = 0.7$ and $q^e = 2$.

(a) What is the mean return on Asset 1? Asset 2? The risk premium on Asset 2? (15 points)

(b) How can you decompose each asset into Arrow securities? What are the implied prices of Arrow securities? (15 points)

Solution:

(a) The bond has a sure return $r^1 = 1/0.7 = 1.43$. Equity has returns

$$r^e(z) \quad = \quad \begin{cases} 2/2 = 1 & \text{for } z = 1 \\ 5/2 = 2.5 & \text{for } z = 2 \end{cases}$$

The states have probability one-half each ("equally likely"), so its expected return is 1.75. Its risk premium is $1.75 - 1.43 = 0.32$.

(b) The bond pays one in each state, so it consists of the two Arrow securities. Equity pays 2 in state 1 and 5 in state 2, so it consists of 2 units of the state-1 Arrow securities and 5 units of the state-2 Arrow security. If the prices of Arrow securities are denoted by $Q(z)$, then we have

$$\begin{aligned} 0.7 &= Q(1) + Q(2) \\ 2 &= 2Q(1) + 5Q(2). \end{aligned}$$

That gives us $Q(1) = 0.5$ and $Q(2) = 0.2$.

4. *Saving and investment (20 points).* Consider the Pareto problem of choosing (c_0, k) to maximize

$$U = u(c_0) + \beta \sum_z p(z) u[c_1(z)],$$

subject to the resource constraints

$$\begin{aligned} c_0 + k &\leq y_0 \\ c_1(z) &\leq z f(k). \end{aligned}$$

There is one of the second constraint for each state z. Here k is capital — plant and equipment — produced at date 0 and used to produce output $zf(k)$ at date 1. The amount of output is random and depends on the state z.

(a) What is the associated Lagrangian? (10 points)

(b) What are the first-order conditions for c_0 and k? (10 points)

Solution:

(a) If we use q_0 and $q_1(z)$ as multipliers on the constraints, the Lagrangian is

$$\mathcal{L} = u(c_0) + \beta \sum_z p(z) u[c_1(z)] + q_0(y_0 - c_0 - k) + \sum_z q_1(z)[zf(k) - c_1(z)].$$

(b) The first-order conditions are

$$\begin{aligned} c_0 :& \quad u'(c_0) - q_0 = 0 \\ k :& \quad -q_0 + \sum_z q_1(z) z f'(k). \end{aligned}$$