Lab Report #7: Dynamics in Theory and Data

Revised: December 2, 2014

Due at the start of class. You may speak to others, but whatever you hand in should be your own work.

1. *ARMA(1,1) models.* Consider the models

$$x_t = \sum_{j=0}^{\infty} a_j w_{t-j}$$

with iid standard normal innovations w_t and coefficients a_0, a_1, and $a_{j+1} = \varphi a_j$ for $j \geq 1$ and parameter $0 < \varphi < 1$.

(a) What is the variance of x?

(b) What is the covariance of x_t and x_{t-1}?

(c) What is the autocovariance function? The autocorrelation function?

(d) What configuration of parameter values gives us negative autocorrelations?

(e) *Extra credit.* Show that the model can be expressed in traditional ARMA(1,1) form,

$$x_t = \varphi x_{t-1} + \sigma(w_t + \theta w_{t-1}).$$

Solution:

(a) This follows most naturally from the moving average formula. Because the w's are independent, the cross terms are zero and we're left with the squares:

$$\begin{aligned} \mathrm{Var}(x) &= E\Big[\Big(\sum_{j=0}^{\infty} a_j w_{t-j}\Big)^2\Big] = \sum_{j=0}^{\infty} E\big(a_j w_{t-j}\big)^2 \\ &= \sum_{j=0}^{\infty} a_j^2 = a_0^2 + a_1^2/(1-\varphi^2). \end{aligned}$$

(b) The covariance is similar

$$\begin{aligned} \mathrm{Cov}(x_t, x_{t-1}) &= E\Big[\Big(\sum_{j=0}^{\infty} a_j w_{t-j}\Big)\Big(\sum_{j=0}^{\infty} a_j w_{t-j-1}\Big)\Big] \\ &= \sum_{j=0}^{\infty} a_j a_{j+1} = a_0 a_1 + a_1\varphi/(1-\varphi^2). \end{aligned}$$

(c) The autocovariance function is

$$\begin{aligned}\gamma(0) &= a_0^2 + a_1^2/(1-\varphi^2) \\ \gamma(1) &= a_0 a_1 + a_1\varphi/(1-\varphi^2) \\ \gamma(k+1) &= \varphi\gamma(k) \text{ for } j \geq 1.\end{aligned}$$

The autocorrelation function is $\rho(k) = \gamma(k)/\gamma(0)$. Like the moving average coefficients, the first two are arbitrary, then they decay at rate φ.

(d) Autocorrelations and autocovariances have the same sign. Looking at $\gamma(0) = \mathrm{Cov}(x_t, x_{t-1})$, we see that it's sufficient to have $a_0 > 0$ and $a_1 < 0$.

(e) If we subtract φx_{t-1} from x_t we get

$$\begin{aligned}x_t - \varphi x_{t-1} &= \sum_{j=0}^{\infty} a_j w_{t-j} - \varphi \sum_{j=0}^{\infty} a_j w_{t-j-1} \\ &= a_0 w_t + (a_1 - \varphi a_0) w_{t-1} + (a_2 - \varphi a_1) w_{t-2} + \cdots \\ &= a_0 w_t + (a_1 - \varphi a_0) w_{t-1},\end{aligned}$$

which is an ARMA(1,1).

2. *Forward-looking equity prices.* Suppose equity prices are given by the forward-looking difference equation

$$q_t = d_t + \delta E_t(q_{t+1})$$

where E_t means the expectation conditional on the state at date t.

(a) How is the price related to future dividends? (Ignore bubbles here.)

(b) Take the AR(1) state variable

$$x_t = \varphi x_{t-1} + \sigma w_t,$$

with the usual iid standard normal $w's$. What is the price of equity if $d_t = x_t$?

(c) What is the price of equity if $d_t = x_{t-1}$?

Solution:

(a) Repeated substitution gives us

$$q_t = d_t + \delta E_t(d_{t+1}) + \delta E_t(d_{t+2}) + \cdots = \sum_{j=0}^{\infty} \delta^j E_t(d_{t+j}).$$

In words: the price is the discounted value of expected future dividends.

(b) Here expected future dividends are

$$E_t(d_{t+j}) \quad = \quad E_t(x_{t+j}) \quad = \quad \varphi^j x_t,$$

which implies the equity price $q_t = x_t/(1-\delta\varphi)$.

(c) The one-period lag is a rough approximation to observed corporate dividend policy, which tends to lag performance. In that case, we have

$$q_t \quad = \quad x_{t-1} + \delta x_t + \delta\varphi x_t + \cdots \quad = \quad x_{t-1} + \delta x_t/(1-\delta\varphi).$$

Maturity n	Price q^n
1 year	0.9800
2 years	0.9600
3 years	0.9400
4 years	0.9200
5 years	0.9000

Table 1. Bond prices.

3. *Bond basics.* Consider the bond prices in Table 1.
   (a) What are the yields y^n?
   (b) What are the forward rates f^{n-1}?
   (c) How are the yields and forward rates related? Verify for y^3.

Solution:

Maturity n	Price q^n	Forward f^{n-1}	Yield y^n
0	1.0000		
1 year	0.9800	0.0202	0.0202
2 years	0.9600	0.0206	0.0204
3 years	0.9400	0.0211	0.0206
4 years	0.9200	0.0215	0.0208
5 years	0.9000	0.0220	0.0211

See the attached Matlab program; download the pdf, open, click on pushpin:

(a,b) Yields and forward rates are connected to bond prices by

$$\begin{aligned} y_t^n &= -n^{-1}\log q_t^n \\ f_t^n &= \log(q_t^n/q_t^{n+1}). \end{aligned}$$

See the table for the numbers.

(c) Yields are averages of forward rates:

$$y_t^n \;\; = \;\; n^{-1} \sum_{j=1}^{n} f_t^{j-1}.$$

Thus $y^3 = (0.0202 + 0.0206 + 0.0211)/3 = 0.0206$.