Lab Report #7: Dynamics in Theory and Data

Revised: May 1, 2013

Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Please include your Matlab code.

1. *Linear models.* Consider the models

$$\begin{array}{lrcl} (a) & x_t & = & 0.9x_{t-1} + w_t \\ (b) & y_t & = & x(t) + 2 \\ (c) & x_t & = & 0 \cdot w_t + 1 \cdot w_{t-1} \\ (d) & x_t & = & \varphi x_{t-1} + w_t + \theta w_{t-1} \\ (e) & x_t & = & 1.2x_{t-1} + 0.1x_{t-2} + w_t \end{array}$$

[The idea behind (b) is to take x_t from (a) and add 2.]

For each model, answer the questions:

(i) Is it Markov? For what definition of the state?

(ii) What is the conditional distribution of of x_{t+1} given the state at date t?

(iii) Is it stable?

(iv) If it's stable, what is the stationary distribution? What is the autocorrelation function?

Solution: All of these are Markov. The state (z_t, say) is whatever you need to know at date $t-1$ to know the conditional distribution of x_t.

(a) This is an AR(1). (i) It's Markov with state x_{t-1}. (ii) Conditional distribution: normal with mean $0.9x_{t-1}$ and variance one. (iii) Yes, stable, because 0.9 is less than one in absolute value. (iv) The stationary distribution is normal with mean zero and variance $1/(1-0.9^2) = 5.2632$. The autocorrelation function is

$$\rho(k) \quad = \quad 0.9^k.$$

This includes $\rho(1) = 0.9$, $\rho(2) = 0.9^2 = 0.81$, and so on.

(b) Still an AR(1). (i) Doing the substitution $x_t = y_t - 2$ gives us

$$y_t \quad = \quad (1-0.9) \cdot 2 + 0.9y_{t-1} + w_t.$$

So it's Markov with state y_{t-1}. (ii) Conditional distribution: normal with mean $0.2 + 0.9y_{t-1}$ and variance one. (iii) Yes, stable, because 0.9 is less than one in absolute value. (iv) The stationary distribution is normal with mean two and variance $1/(1-0.9^2) = 5.2632$. All we've done here is shift the mean up by two. The autocorrelation function doesn't depend on the mean, so it's the same as before.

(c) This is an MA(1). (i) It's Markov with state w_{t-1}. (ii) Conditional distribution: normal with mean w_{t-1} and variance zero. (This is an unusual setup: since the coefficient of w_t is zero, we learn x_t one period ahead of time.) (iii) Yes, stable. For a moving average, all we need is that the coefficients are square summable. That's always true if there's a finite number of terms. (iv) The stationary distribution is normal with mean zero and variance one.

(d) This is an ARMA(1,1). (i) It's Markov with state (x_{t-1}, w_{t-1}). (ii) Conditional distribution: normal with mean $\varphi x_{t-1} + \theta w_{t-1}$ and variance one. (iii) It's stable if $|\varphi| < 1$. You can see this from the moving average representation, outlined in the notes:

$$x_t \;=\; w_t + (\varphi+\theta)w_{t-1} + (\varphi+\theta)\varphi w_{t-2} + (\varphi+\theta)\varphi^2 w_{t-3} + \cdots .$$

The first two moving average coefficients are arbitrary, then they decline at rate φ. (iv) The stationary distribution is normal with mean zero and variance equal to the sum of squared moving average coefficients:

$$\gamma(0) \;=\; 1 + (\varphi+\theta)^2/(1-\varphi^2).$$

The autocovariances are

$$\gamma(k) \;=\; \varphi^{k-1}(\varphi+\theta)\left[1 + (\varphi+\theta)\varphi/(1-\varphi^2)\right].$$

The autocorrelations are $\rho(k) = \gamma(k)/\gamma(0)$. They decline at rate φ after the first one.

(e) This is an AR(2). (i) It's Markov with state (x_{t-1}, x_{t-2}). (ii) The conditional distribution is normal with mean $\varphi x_{t-1} + \varphi x_{t-2}$ and variance one. (iii,iv) It's not stable. You can see this by substituting for a few periods and seeing how the impact of lagged x's works. So there's no stationary distribution, autocorrelation function, and so on.

2. *Dynamics of interest rates 1.* We'll look at the autocorrelations of interest rates to get a sense of their dynamics. The first step is to download some data from the Fed. Go to

   http://www.federalreserve.gov/releases/h15/data.htm

   and download monthly data for Treasury constant maturities, specifically the 3-month and 10-year maturities, for the period 1985 to present. Read them into Matlab and:

   (a) Compute the mean, standard deviation, and autocorrelation function (acf) for the 1-month interest rate for lags k from 1 to 24 months. (You may recall that we used the program `acf.m` for the latter in class. It's our program, not part of Matlab, although Matlab's Econometrics Toolbox has a similar function. It works on time series objects, which is something I'd prefer to avoid for now. But by all means do whatever you wish.)

(b) Describe the acf for the AR(1):

$$x_t = (1-\varphi)\mu + \varphi x_{t-1} + \sigma w_t,$$

where $\{w_t\}$ is a sequence of standard normal random variables. How do the acf's compare for the data and a suitably estimated AR(1)?

(c) Compute the mean, standard deviation, autocorrelation function (acf) for the 10-year interest rate. How do they compare to the 1-month rate?

Solution:

(a,c) The relevant statistics are

	Mean	Std Dev	Autocorr
3-month	4.00	2.52	0.989
10-year	5.79	2.17	0.978

They reflect some standard features of interest rates, including: (i) rates increase with maturity, on average; (ii) standard deviations decline with maturity; and (iii) all of them are very persistent. Which is the point of the exercise.

We can see more of the autocorrelation functions in the Matlab figure (run the code to see it).

(b) With an AR(1), the autocorrelation has the form $\rho(k) = \varphi^k$: it declines geometrically. If we set $\varphi = \rho(1)$, we can compute estimated acf's corresponding to AR(1)s. They're plotted as the dashed lines above. You can see they're somewhat different from the AR(1). Whether this represents sampling variability of some more basic difference between the data and the AR(1) model isn't clear.

3. *Dynamics of interest rates 2.* Consider the barebones bond pricing model based on the pricing kernel

$$\begin{aligned}
\log m_{t+1} &= \delta + x_t + \lambda w_{t+1} \\
x_t &= \sigma(w_t + \theta w_{t-1}),
\end{aligned}$$

where w_t is our usual iid normal disturbance and $(\delta, \lambda, \sigma, \theta)$ are parameters.

(a) What are the dynamics of x_t?

(b) What is the moving average representation of $\log m_t$ in this model?

(c) What is the variance of x_t? Its first autocorrelation? How does the autocorrelation function compare to the one you computed for the 3-month treasury bill in the previous question?

(d) Suppose the continuously-compounded short rate is $y_t^1 = -\log q_t^1$. What is y_t^1 in this model?

Solution:

(a) It's an MA(1), which means it has a memory for one period only.

(b) The log pricing kernel is an MA(2):

$$\log m_t \;\; = \;\; \delta + \lambda w_t + \sigma(w_{t-1} + \theta w_{t-2}).$$

(c) The variance of x_t (the variance of its stationary distribution) is the sum of the squared MA coefficients:

$$\mathrm{Var}(x_t) \;\; = \;\; \sigma^2 + (\sigma\theta)^2.$$

The first autocovariance is

$$\mathrm{Cov}(x_t, x_{t-1}) \;\; = \;\; \sigma^2\theta,$$

so the first autocorrelation is the ratio, $\rho(1) = \theta/(1+\theta^2)$. All the autocovariances and autocorrelations after the first one are zero. That's very different from what we see in the data, where the autocorrelations declines more slowly.

(d) Note that the distribution of $\log m_{t+1}$, conditional on the state at date t, is normal with mean $\delta + x_t$ and variance λ^2. The one-period bond price is therefore

$$q_t^1 \;\; = \;\; E_t(m_{t+1}) \;\; = \;\; \exp(\delta + x_t + \lambda^2/2),$$

the usual "mean plus variance over two" formula. The one-period yield is

$$y_t^1 \;\; = \;\; -\log q_t^1 \;\; = \;\; -(\delta + x_t + \lambda^2/2),$$

which is, by design, an MA(1).

Matlab program:

```
%  hw7_s13.m
%  Matlab program for Lab Report #7, Spring 2013
%  NYU course ECON-UB 233, Macro foundations for asset pricing.
%  Written by:  Dave Backus
disp('Answers to Lab Report 7')
format compact
format short
clear all

%%
disp(' ')
disp('------------------------------------------------------------')
disp('Question 1 (e)')
disp('Are eigenvalues stable?')

% this is a linear algebra version of (e)
% it's a vector AR with autoregressive matrix A
% to be stable, we need all its eigenvalues to be less than one in abs val
phi1 = 1.2;
phi2 = 0.1
A = [phi1 phi2; 1 0]

eigs = eig(A)
abs(eigs)

%%
disp(' ')
disp('------------------------------------------------------------')
disp('Question 2 (properties of long and short rates)')

disp(' ')
disp('Data input from spreadsheet')
data = xlsread('fed_yield_data.xlsx');

y3m = data(:,1);
y10y = data(:,2);

disp(' ')
disp('Mean and Std Dev')
mean(data)
std(data)

disp(' ')
disp('Compute acfs')
```

```
nlag = 40;
lags = [0:nlag]';
acf_3m = acf(y3m,nlag);
acf_10y = acf(y10y,nlag);
acfs = [acf_3m(2) acf_10y(2)]

%  approximate with AR(1)s
acf_3m_ar1 = acf_3m(2).^lags;
acf_10y_ar1 = acf_10y(2).^lags;

figure(1)
clf
FontSize = 12;
FontName = 'Helvetica';  % or 'Times'
LineWidth = 1.5;

plot(lags,acf_3m,'b','LineWidth',LineWidth)
hold on
plot(lags,acf_3m_ar1,'b--','LineWidth',LineWidth)
plot(lags,acf_10y,'m','LineWidth',LineWidth)
plot(lags,acf_10y_ar1,'m--','LineWidth',LineWidth)
xlabel('Lag k in Months','FontSize',FontSize,'FontName',FontName)
ylabel('Autocorrelation Function','FontSize',FontSize,'FontName',FontName)
text(2,0.34,'blue is 3m, magenta is 10y','FontSize',FontSize,'FontName',FontName)
text(2,0.3,'solid is estimated acf, dashed is AR(1) extrapolation','FontSize',FontSize,'Fon
set(gca,'LineWidth',1,'FontSize',FontSize,'FontName',FontName)

print -dpdf hw7_q2.pdf


%%

return
```