Two-Period Macroeconomic Models
Revised: October 5, 2015

Our next step is to look at equilibrium prices and quantities in the same two-period setting we used earlier. The idea is to combine the consumption, saving, and portfolio decisions of agents we studied earlier with the production and investment decisions of firms and find the prices and quantities that clear markets. We simplify each of these things as much as we can in the interest of clarity. The result is a relatively simple environment in which we can talk, for the first time, about the effects of risk on asset prices and returns.

1 Keep it simple

Einstein is reported to have said: "Make things as simple as possible, but not simpler." It's not clear exactly what that means, or even whether Einstein said it, but I'd like to think he'd agree we should keep things simple. Economics, in any case, is a blend of realism and simplicity, with most of the weight on simplicity. Some people find the apparent lack of realism of economic models puzzling, but it's often necessary to gain sharp insights. The idea is to focus on the things that matter, and simplify or ignore the things that don't. It's an art, of course, to be able to tell the difference. The artist Georgia O'Keeffe said something similar: "Nothing is less real than realism. Details are confusing. It is only by selection, by elimination, by emphasis that we get to the real meaning of things."

Here's another version of the same sentiment, an exchange between economist Robert Mundell and banker Alex McLeod. The paper in question helped Mundell get the Nobel prize, but McLeod thought it was unrealistic. As a banker, he knew how things worked in the real world. I've taken some liberty with the quotations, but the exchange went something like this:

McLeod:

> Mundell's article [includes] a number of incongruous assumptions. One is that complications associated with speculation in the forward market do not exist. It can only bring discredit on the economics profession to leave unchallenged his attempt to draw from the model policy conclusions that are applicable in the real world.

Mundell:

> Theory is the poetry of science. It is simplification, abstraction, the exaggeration of truth. Through simplification, theory creates a caricature of reality. The caricature itself is not the real world — it mocks it. Yet mind true things by their mockeries!

Dr McLeod does not like my caricature; he calls my assumptions unrealistic. I certainly hope he is right. I hope my assumptions are unrealistic. I left out a million variables that made my caricature of reality unrealistic. At the same time, it enabled me to find fruitful empirical generalizations.

[From: *Canadian Journal of Economics and Political Science*, vol 30, 1964, pp 413-414, 419-421.]

The theory of Mundell's paper is a long way from anything we'll do in this class, but you have to love a guy who has the nerve to say "theory is the poetry of science." The substantive point is that our goal isn't realism, it's insight and understanding. The art of economics lies in making unrealistic simplifying assumptions that deliver clear results that we believe generalize to more complex environments. The art lies in making that distinction — in deciding which assumptions are crucial and which simply make the analysis clearer.

2   General equilibrium models

In the 1950s, economists formalized ideas that date back to Adam Smith and before. Lots of people were involved, but the central players were Kenneth Arrow, Gerard Debreu, and Lionel McKenzie. The first idea is to consider equilibrium (think: supply equals demand) for all markets at once — what economists refer to as general equilibrium. That's missing in the usual supply and demand diagram, one of the reasons we don't stop there. The second is to address the welfare properties of general equilibrium. Is the equilibrium allocation of resources a good one in some sense? Does Smith's invisible hand work? We'll do a quick summary of the principle results.

Ingredients. Here are the ingredients of a general equilibrium model — what we might term the physical environment, which includes the objects involved but says nothing about how they interact. The list would include:

- List of commodities. What's produced? What's consumed?

- List of agents. Who are the people in the model? How many are there?

- Preferences and endowments. Each agent has a utility function that describes her preferences over commodities. Also a vector of endowments: quantities of the various commodities that she starts with.

- Technologies. Production functions that transform combinations of some commodities (inputs) into other commodities (outputs).

- Resource constraints. Uses (consumption) of each commodity are limited by its sources (endowment net of other uses plus production).

Most theoretical economies consist of these ingredients in one form or other. If there are no technologies, we refer to it as an *exchange economy*; since production is ruled out, all agents can do is exchange their endowments.

Equilibrium. A *competitive equilibrium* for an environment like this consists of

- Agents maximize. Agents choose consumption quantities that maximize utility given prices and budget constraints.

- Firms maximize. Firms choose inputs and outputs to maximize profits given prices and technologies.

- Markets clear. Sources ("supply") equal uses ("demand") for every commodity.

That sounds simple, but it was a significant achievement to formalize it and prove (under suitable conditions on preferences and technologies) that an equilibrium exists. Note that it's a *competitive* equilibrium: agents and firms take prices as given, which my industrial organization colleagues find amusing. (Equilibrium when agents or firms can influence prices is a much more demanding problem, not one we can leave to macroeconomists, I'm told.)

Welfare. Now the invisible hand: Is there a sense in which a competitive equilibrium delivers a good allocation of resources? It depends what you mean by "good." Suppose we have two agents and consider two allocations, one in which agent 1 gets a lot and one in which agent 2 gets a lot, with the other getting what's left in each case. Which one do we think is better? We generally resist making comparisons between allocations like this, because it forces us to decide which agent we like best. Like parents, we care about all our agents equally. We settle for a weaker definition of good: we say an allocation is *Pareto optimal* (ie, good) if we can't make one agent better off without making at least one other agent worse. This leaves some allocations noncomparable, but seems like a useful condition to apply to an economy. If we can make someone better off without hurting anyone, surely the allocation isn't all that good.

With that background, we have two classical results:

- First welfare theorem. A competitive equilibrium produces a Pareto optimal allocation of resources.

- Second welfare theorem. A Pareto optimal allocation of resources can be reproduced as a competitive equilibrium for some allocation of endowments.

The first is the modern version of Adam Smith's invisible hand. The second is (among other things) a useful shortcut for computing a competitive equilibrium: we compute a Pareto optimum instead.

All of this might seem like excessive formalism. On the contrary, I find it extremely helpful to keep track of the ingredients and how they fit together.

3   Finding an equilibrium as a Pareto problem

We'll generally attack equilibrium indirectly by finding a Pareto optimum. By the second welfare theorem, this corresponds to a competitive equilibrium for some allocation of endowments. In much of macroeconomics, finding the right allocation of endowments is easy because we only have one agent, who therefore owns everything.

Example (exchange economy). Let's see how this works in an example, a really simple one. The ingredients are:

- List of commodities. Two: apples and bananas.

- List of agents. One.

- Preferences and endowments. The utility function is

$$U(c_a, c_b) = \beta \log c_a + (1 - \beta) \log c_b,$$

 where (c_a, c_b) is consumption. The endowment is (y_a, y_b).

- Technologies. None, this is an exchange economy.

- Resource constraints. They are

$$c_a \leq y_a$$
$$c_b \leq y_b.$$

That's the environment: one agent, two goods, no production. You can't get much simpler than that, but it serves to illustrate our approach.

The Pareto problem for this economy consists of maximizing utility subject to the resource constraints:

$$\max_{c_a, c_b} \quad \beta \log c_a + (1 - \beta) \log c_b$$
$$\text{s.t.} \quad c_a \leq y_a$$
$$c_b \leq y_b.$$

The solution is obvious (consume the endowment), but we'll solve it by Lagrangean methods using multipliers q_a and q_b on the two constraints. If you're more comfortable use λ_a and λ_b, you're welcome to do that. We use q's because we know something about the answer: the multipliers turn out to be the market prices in the competitive equilibrium associated with this Pareto optimum.

The Lagrangean for this problem is

$$\mathcal{L} = \beta \log c_a + (1 - \beta) \log c_b + q_a(y_a - c_a) + q_b(y_b - c_b).$$

The first-order conditions are

$$\beta/c_a - q_a = 0$$
$$(1 - \beta)/c_b - q_b = 0.$$

The constraints tell us to consume the endowments, which nails down the Lagrange multipliers: $q_a = \beta/y_a$ and $q_b = (1 - \beta)/y_b$.

The nice thing about this approach is that we get the prices from the Lagrange multipliers. (You might recall that the multipliers give us the cost, in utility terms, of the constraints.)

It's not hard to show that these are, in fact, the competitive equilibrium prices. The first condition for a competitive equilibrium is that agents maximize utility given their budget constraints. Here the budget constraint is

$$q_a c_a + q_b c_b \;\le\; q_a y_a + q_b y_b \;=\; Y.$$

(The letter Y is here out of laziness: it's easier than writing the stuff it's equal to.) We saw earlier that this leads to the demand functions $c_a = \beta Y/q_a$ and $c_b = (1-\beta)Y/q_b$. Does demand equal supply when we substitute the prices we derived from the Pareto problem? With those prices, $Y = q_a y_a + q_b y_b = \beta + (1-\beta) = 1$. Demand for apples is therefore

$$c_a \;=\; \beta Y/q_a \;=\; \beta/(\beta/y_a) \;=\; y_a,$$

so demand equals supply for apples. The same is true for bananas, too.

The bottom line: When we solve a Pareto problem, we not only get the equilibrium quantities, we get the equilibrium prices, too.

[Draw picture: indifference curves, endowment point, slope of tangent line.]

Example (production economy). We can easily modify this economy to incorporate production. Suppose we have no endowment of bananas ($y_b = 0$), but can transform apples into bananas with the production function $f(k) = k^\theta$. If we use k to denote use of apples to produce bananas, the Pareto problem becomes

$$\max_{c_a,k,c_b} \quad \beta \log c_a + (1-\beta)\log c_b$$
$$\text{s.t.} \quad c_a + k \;\le\; y_a$$
$$c_b \;\le\; k^\theta,$$

with parameters $0 < \beta, \theta < 1$. The Lagrangean is

$$\mathcal{L} \;=\; \beta \log c_a + (1-\beta)\log c_b + q_a(y_a - c_a - k) + q_b(k^\theta - c_b).$$

The first-order conditions are

$$\beta/c_a - q_a \;=\; 0$$
$$-q_a + q_b \theta k^{\theta-1} \;=\; 0$$
$$(1-\beta)/c_b - q_b \;=\; 0.$$

They imply

$$(y_a - k)\beta/(1-\beta) \;=\; k/\theta,$$

which we can solve for k. Consumption quantities (c_a, c_b) then follow from the constraints.

[Update picture, add production possibilities frontier.]

4   Two-period deterministic economies

In macroeconomics, a useful starting point is an economy with a single agent and some form of dynamics. We keep the dynamics simple here with two periods, dates 0 and 1. Then we develop a series of examples of increasing complexity to illustrate how the approach works and what it implies for asset pricing. The insight is to treat consumption now and consumption next period as different commodities, just as we treated apples and bananas as different commodities above.

If we follow the formalism described above, we have the following ingredients:

- List of commodities. Two: the consumption good at dates 0 and 1.

- List of agents. One: a "representative" agent.

- Preferences and endowments. The utility function is

$$U(c_0, c_1) \;=\; u(c_0) + \beta u(c_1).$$

  Endowments are (y_0, y_1).

- Technologies. The production function f transforms input of the date-0 good (label this k) into output of the date-1 good.

- Resource constraints. For the two commodities, we have

$$c_0 + k \;\leq\; y_0$$
$$c_1 \;\leq\; y_1 + f(k).$$

  The idea is that date-0 investment k produces date-1 output $f(k)$.

That's the physical environment.

We could compute a competitive equilibrium directly, but it's easier to compute a Pareto optimum and use the second welfare theorem to tell us it's also an equilibrium. With only one agent, a Pareto optimum is the solution to:

$$\max_{c_0, k, c_1} \quad u(c_0) + \beta u(c_1)$$
$$\text{s.t.} \quad c_0 + k \;\leq\; y_0$$
$$c_1 \;\leq\; y_1 + f(k).$$

We could simply substitute the constraints into the objective function and maximize, but using Lagrange multipliers gives us the prices, too. The Lagrangean is

$$\mathcal{L} \;=\; u(c_0) + \beta u(c_1) + q_0(y_0 - c_0 - k) + q_1(y_1 + f(k) - c_1).$$

The first-order conditions are

$$c_0: \quad u'(c_0) - q_0 \;=\; 0$$
$$k: \quad -q_0 + q_1 f'(k) \;=\; 0$$
$$c_1: \quad \beta u'(c_1) - q_1 \;=\; 0.$$

They imply that the marginal rate of substitution (mrs) and marginal rate of transformation (mrt) equal the price ratio:

$$\frac{\beta u'(c_1)}{u'(c_0)} = \frac{q_1}{q_0} = \frac{1}{f'(k)}.$$

If we define $Q = q_1/q_0$ and the (gross) interest rate r by $Q = 1/r$, then we also have $[\beta u'(c_1)/u'(c_0)]r = 1$ and $f'(k) = r$. This differs from the consumption-saving problem in (possibly) having curvature in f, so that $f'(k) = r$ isn't constant.

[Draw indifference curves and ppf again.]

Where does the interest rate r come from? It depends on preferences (u and β) and technology (f). A high interest rate is associated with productive capital (high f') and high date-1 consumption [hence low $u'(c_1)$].

5 Two-period stochastic economies: exchange version

Now let's add uncertainty to the model, so that we can give a general equilibrium interpretation to asset prices. We deal with uncertainty as we did with consumption and portfolio choice. We make decisions at date 0. At date 1, any of a number of states z can occur. [Draw our usual event tree.] If there are Z states (a finite number), the ingredients of an exchange economy with a single representative agent are

- List of commodities. $Z + 1$: the consumption good at date 0 and in each state z at date 1.

- List of agents. One: a "representative" agent.

- Preferences and endowments. Preferences are governed by the utility function

$$U(c_0, c_1) = u(c_0) + \beta \sum_z p(z)u[c_1(z)].$$

  Endowments are $\{y_0, y_1(z)\}$.

- Technologies. None.

- Resource constraints. For the two commodities, we have

$$c_0 \leq y_0$$
$$c_1(z) \leq y_1(z), \text{ for each } z.$$

This has what we call *complete markets*: we have one market for every state z.

The Pareto problem has a similar form: maximize utility subject to resource constraints. The main difference from the previous section is that we have lots of resource constraints, one for each state. The problem is

$$\max_{c_0, c_1(z)} \quad u(c_0) + \beta \sum_z p(z)u[c_1(z)]$$
$$\text{s.t.} \quad c_0 \leq y_0$$
$$c_1(z) \leq y_1(z) \text{ for each } z.$$

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The Lagrangean is

$$\mathcal{L} = u(c_0) + \beta \sum_z p(z)u[c_1(z)] + q_0(y_0 - c_0) + \sum_z q_1(z)[y_1(z) - c_1(z)].$$

As before, we represent the Lagrange multipliers by q's: q_0 for the date-0 constraint and $q_1(z)$ for the date-1 constraint in state z. The first-order conditions are

$$c_0 : \quad u'(c_0) - q_0 = 0$$
$$c_1(z) : \quad \beta p(z)u'[c_1(z)] - q_1(z) = 0.$$

(We have one of the second equation for each state z.) The solution, of course, is to consume the endowment in all states, but this gives us the prices too.

[Draw indifference curves between states and endowment point.]

If we take the ratio of the second equation to the first, we have

$$\frac{\beta p(z)u'[c_1(z)]}{u'(c_0)} = \frac{\beta p(z)u'[y_1(z)]}{u'(y_0)} = \frac{q_1(z)}{q_0} = Q(z). \tag{1}$$

The first equality substitutes the equilibrium consumption quantities: the endowments. The subsequent equalities connect them to prices. The ratio $Q(z) = q_1(z)/q_0$ is the price at date 0 of one unit in state z at date 1: what we have called the *state price*, the price of an Arrow security. Note that the state price is connected to the agent's marginal rate of substitution, just as it was in the portfolio choice problem. More formally, we can write $Q(z) = \beta p(z)u'[c_1(z)]/u'(c_0) = p(z)m(z)$, with m defined by $m(z) = \beta u'[c_1(z)]/u'(c_0)$. We will typically refer to $m(z)$ as the mrs in state z, rather than $p(z)m(z)$, but that's a story for another time.

How do prices differ across states? Consider the role of probability. The state price formula (1) tells us that states with higher probability have higher prices, other things equal. More interesting is the role of the date-1 endowment. The same equations tell us that states with higher output have lower prices and higher returns. Why? Because $u'[c_1(z)] = u'[y_1(z)]$ is a decreasing function. If we increase $y_1(z)$, then we decrease marginal utility $u'[y_1(z)]$, the marginal rate of substitution $m(z) = \beta u'[y_1(z)]/u'(y_0)$, and the state price $Q(z) = p(z)m(z)$. In words: a payoff of one unit is more valuable when output is scarce than when output is plentiful. Securities that pay off in good states (high endowment) are therefore cheaper than those that pay off in bad states (low endowment). We'll see later that this provides a basis for positive risk premiums on such assets. Securities that pay off mostly in good states have lower prices and higher average returns than those that pay off mostly in bad states.

6 Asset pricing in the exchange economy: an example

We can be more specific about asset prices and returns if we put more structure on the economy. We apply two kinds of structure: we give the agent power utility and make the date-1 endowment lognormal (that is, the log of the endowment is normal). This is our first look at a collection of equations we'll see over and over again.

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Here's how asset pricing works in settings like this. An asset consists, in general, of a state-dependent dividend or payoff $d(z)$. If its price is q, its return is $r(z) = d(z)/q$. We typically value assets in this order: specify the dividend, compute its price, then calculate its return. Given a marginal rate of substitution $m(z)$ and probabilities $p(z)$, pricing follows from

$$1 \quad = \quad E(mr) \quad = \quad \sum_z p(z)m(z)[d(z)/q] \quad = \quad E[m(d/q)]].$$

Since q doesn't depend on z, we can take it outside the expectation. If we multiply both sides by q, we have

$$q \quad = \quad E(md).$$

Hence the term asset "pricing."

Since we'll be using normal and lognormal random variables, it's helpful to remind ourselves of some of their properties. Let us say that $x = \log y$ is normal; in short-hand notation, $x \sim \mathcal{N}(\kappa_1, \kappa_2)$. Then we say $y = \exp(x)$ is lognormal (its log is normal). Result 1: The mgf of x is $h(s) = E(e^{sx}) = e^{\kappa_1 s + \kappa_2 s^2/2}$. The cgf, of course, is the log of this. Result 2: If $x = \log y$, the mean of $y = e^x$ is $E(y) = E(e^x) = e^{\kappa_1 + \kappa_2/2}$. This also follows from the mgf of x: evaluate h at $s = 1$. We say y is lognormal (its log is normal). Result 3: Linear functions of x are also normal. Specifically: $y = a + bx \sim \mathcal{N}(a + b\kappa_1, b^2\kappa_2)$. (If you have any question about this, write down the definition of the mgf of y.)

Now the example. Let $u(c) = c^{1-\alpha}/(1-\alpha)$ with $\alpha > 0$ (power utility) and $\log y_1(z) \sim \mathcal{N}(\kappa_1, \kappa_2)$ (lognormal endowment). For convenience, we set $y_0 = 1$ so that it drops out of calculations. With power utility, the mrs is

$$m(z) \quad = \quad \beta u'[y_1(z)]/u'(y_0) \quad = \quad \beta[y_1(z)/y_0]^{-\alpha},$$

a function of the growth rate of the endowment. Since $y_0 = 1$, $\log m(z)$ is

$$\log m(z) \quad = \quad \log \beta - \alpha \log y_1(z) \quad \sim \quad \mathcal{N}(\log \beta - \alpha\kappa_1, \alpha^2\kappa_2).$$

The second step follows from Result 3 for lognormals.

Now consider two specific assets: a "bond" that pays one in every state and a share of "equity" that pays a dividend equal to the endowment in each state. If the bond has price q^1 then its return is $r^1 = 1/q^1$. The price satisfies

$$q^1 \quad = \quad E(m \cdot 1) \quad = \quad \exp(\log \beta - \alpha\kappa_1 + \alpha^2\kappa_2/2) \quad = \quad \beta \exp(-\alpha\kappa_1 + \alpha^2\kappa_2/2),$$

which gives us the return

$$r^1 \quad = \quad 1/q^1 \quad = \quad \beta^{-1} \exp(\alpha\kappa_1 - \alpha^2\kappa_2/2).$$

Higher mean growth κ_1 is associated with a lower price and therefore a higher interest rate. The greater is risk aversion the greater is this effect. The same is true in the deterministic case, which corresponds to $\kappa_2 = 0$. An increase in the variance κ_2 raises the price and lowers the interest rate.

Now consider equity, a claim to the dividend $d^e(z) = y_1(z)$. Its return is $r^e(z) = d^e(z)/q^e = y_1(z)/q^e$. Its price is therefore $q^e = E(md^e) = E(my_1)$. Taking logs and collecting terms, we have

$$\log m(z) + \log d^e(z) = \log\beta + (1-\alpha)\log y_1(z) \sim \mathcal{N}(\log\beta + (1-\alpha)\kappa_1, (1-\alpha)^2\kappa_2).$$

Result 1 for lognormals gives us the price:

$$
\begin{aligned}
q^e &= E\left(md^e\right) \\
&= \exp[\log\beta + (1-\alpha)\kappa_1 + (1-\alpha)^2\kappa_2/2] \\
&= \beta\exp[(1-\alpha)\kappa_1 + (1-\alpha)^2\kappa_2/2].
\end{aligned}
$$

The return is

$$r^e(z) = y_1(z)/q^e = y_1(z)\beta^{-1}\exp[-(1-\alpha)\kappa_1 - (1-\alpha)^2\kappa_2/2].$$

The expected return is (using Result 1 again)

$$
\begin{aligned}
E(r^e) &= \exp(\kappa_1 + \kappa_2/2)\beta^{-1}\exp[-(1-\alpha)\kappa_1 - (1-\alpha)^2\kappa_2/2] \\
&= \beta^{-1}\exp\{\alpha\kappa_1 + [1 - (1-\alpha)^2]\kappa_2/2\}.
\end{aligned}
$$

(Hang in there, we're making progress.)

Now the question: Does equity have a higher or lower expected return than the bond? We refer to the difference as a risk premium. We suggested that an asset that pays off mostly in good states, as equity does, would demand a higher return as a result. We would say, then, that equity has a positive risk premium. Let's see if that's true here. In general we have $E(r^e) > r^1$ if

$$
\begin{aligned}
\exp\{[1-(1-\alpha)^2]\kappa_2/2\} &> \exp(-\alpha^2\kappa_2/2) \\
(-\alpha^2 + 2\alpha)\kappa_2/2 &> -\alpha^2\kappa_2/2.
\end{aligned}
$$

So it's true since α and κ_2 are positive. When either is zero, the expected difference in returns is also zero.

In loglinear models like this, including some of the most common ones in finance, it's simpler to compute the expected difference in returns in logs; namely, $E\log r^e - \log r^1$. Here we get

$$E\log r^e - \log r^1 = (2\alpha - 1)\kappa_2/2,$$

which is positive if $\alpha > 1/2$.

7 Two-period stochastic economies 2: production version

We'll use the exchange economy repeatedly, not because it's realistic but because it's simple. If we expand the economy to include production, we get the same relation between asset returns and consumption. Why consumption and not output? Because it's consumption that shows up in the portfolio choice problem. Production is useful, but for another reason: it gives us a source of dividends paid by firms.

Let's consider production then. We allow investment at date 0 to increase output and consumption at date 1. The standard version is the same as above, except:

- Technologies. An input of k units of the date-0 good produces $a(z)f(k)$ units of the date-1 good. The output of this investment is, therefore, uncertain.

- Resource constraints. They become

$$
\begin{aligned}
c_0 &\leq y_0 - k \\
c_1(z) &\leq y_1(z) + a(z)f(k), \text{ for each } z.
\end{aligned}
$$

The idea is that investment is risky: the marginal product of an additional unit of capital is $a(z)f'(k)$, which depends on the date-1 state z.

With these changes, the Pareto problem becomes

$$
\begin{aligned}
\max_{c_0, k, c_1(z)} \quad & u(c_0) + \beta \sum_z p(z)u[c_1(z)] \\
\text{s.t.} \quad c_0 + k &\leq y_0 \\
c_1(z) &\leq y_1(z) + a(z)f(k).
\end{aligned}
$$

The Lagrangean is

$$
\mathcal{L} = u(c_0) + \beta \sum_z p(z)u[c_1(z)] + q_0(y_0 - c_0 - k) + \sum_z q_1(z)[y_1(z) + a(z)f(k) - c_1(z)].
$$

(As before, we've used q's for the Lagrange multipliers, because we know that they'll end up being the equilibrium prices.) The first-order conditions are

$$
\begin{aligned}
c_0 : \quad & u'(c_0) - q_0 = 0 \\
c_1(z) : \quad & \beta p(z)u'[c_1(z)] - q_1(z) = 0 \\
k : \quad & -q_0 + \sum_z q_1(z)a(z)f'(k) = 0.
\end{aligned}
$$

That gives us, in every state z,

$$
\frac{\beta p(z)u'[c_1(z)]}{u'(c_0)} = \frac{q_1(z)}{q_0} = Q(z) = \frac{1}{a(z)f'(k)}.
$$

The first part is the same as the exchange economy: the marginal rate of substitution $\beta u'[c_1(z)]/u'(c_0)$ equals the state price $Q(z)$. The second part concerns production: the marginal rate of transformation $1/[a(z)f'(k)]$ also equals the state price.

Example. We can solve this by hand for a special case. Let $u(c) = \log c$ and $f(k) = k^\theta$, with $0 \leq \theta \leq 1$, and $a(z) \sim \mathcal{N}(\kappa_1, \kappa_2)$. Equity is a claim to $d(z) = a(z)f'(k)k$: Each unit of capital is paid its marginal product $a(z)f'(k)$. I'll leave it to you to derive its expected excess return.

What's useful about this model is that we can say where aggregate dividends come from: they're payments to capital. At the margin, an increase in k at date 0 generates $a(z)f'(k)$ units of output at date 1 in state z. Since z is uncertain, so is the dividend.

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Bottom line

General equilibrium models are useful devices for thinking about where prices and quantities come from. From a macro-finance perspective, even simple models give us some insights. Among them are:

- Asset prices and returns reflect the marginal rates of substitution of the people who buy them. Our old favorite, $E(mr) = 1$, is a good example, with m the mrs of a representative agent.

- Assets that pay off mostly in good states have less value and higher returns, on average, than either a riskfree asset that has the same payoff in all states or an asset that pays off mostly in bad states. Why? Because marginal utility is lower in good states than bad.

Practice problems

1. *A three-good economy.* Consider a static economy with three goods, labelled 1, 2, and 3. A representative agent has utility over consumption of the three goods of

$$U(c_1, c_2, c_3) \quad = \quad \log c_1 + \beta \log c_2 + \gamma \log c_3.$$

   The same agent's endowments are (y_1, y_2, y_3).

   (a) What are the "ingredients" for this economy?

   (b) What is the Lagrangian problem associated with a Pareto optimum?

   (c) What are its first-order conditions?

   (d) What are the optimal values of consumption? What are the prices that support this as a competitive equilibrium?

   Answer.

   (a) The ingredients are
   - List of commodities: 1, 2, and 3.
   - List of agents: one.
   - Preferences and endowments: see above.
   - Technology: none.
   - Resource constraints: $c_i \leq y_i$ for each good i.

   (b) The Lagrangian is

$$\mathcal{L} \quad = \quad \log c_1 + \beta \log c_2 + \gamma \log c_3 + \sum_i q_i(y_i - c_i).$$

   The q_i's are Lagrange multipliers on the resource constraints.

   (c) The first-order conditions are

$$\begin{aligned} 0 &= 1/c_1 - q_1 \\ 0 &= \beta/c_2 - q_2 \\ 0 &= \gamma/c_3 - q_3. \end{aligned}$$

(d) Obviously we consume the whole endowment: $c_i = y_i$ for $i = 1, 2, 3$. Why? Because more is better, and that's the most we can consume. The first-order conditions then give us the multipliers: $q_1 = 1/y_1$, $q_2 = \beta/y_2$, and $q_3 = \gamma/y_3$.

2. *A two-good economy with power utility.* Consider a static economy with two goods, labelled 1 and 2. A representative agent has utility

$$U(c_1, c_2) \ = \ c_1^{1-\alpha}/(1-\alpha) + \beta c_2^{1-\alpha}/(1-\alpha).$$

The endowments are (y_1, y_2).

(a) What are the "ingredients" for this economy?

(b) What is the Lagrangian problem associated with a Pareto optimum?

(c) What are its first-order conditions?

(d) What are the optimal values of consumption? What are the prices that support this as a competitive equilibrium?

Answer.

(a) The ingredients are
- List of commodities: 1 and 2.
- List of agents: one.
- Preferences and endowments: see above.
- Technology: none.
- Resource constraints: $c_i \le y_i$ for each good i.

(b) The Lagrangian is

$$\mathcal{L} \ = \ c_1^{1-\alpha}/(1-\alpha) + \beta c_2^{1-\alpha}/(1-\alpha) + \sum_i q_i(y_i - c_i).$$

(c) The first-order conditions are

$$0 \ = \ c_1^{-\alpha} - q_1$$
$$0 \ = \ \beta c_2^{-\alpha} - q_2.$$

(d) We consume the whole endowment: $c_i = y_i$ for $i = 1, 2$. The prices are $q_1 = y_1^{-\alpha}$ and $q_2 = \beta y_2^{-\alpha}$.

3. *Consumption and leisure.* Most macroeconomic models include a role for work. Our agent (let's stick with one) has one unit of time. She supplies n units to firms, who use it to produce output. That leaves her with $1 - n$ units to consume a leisure, which includes here everything other than paid work: sleeping, cleaning, watching television, biking, and so on. Let us say, to be concrete, that the agent has utility function

$$U(c, 1-n) \ = \ \log c + \theta \log(1 - n),$$

and that each unit of labor hired by a firm generates a units of output. The overall resource constraint is therefore

$$c \ \le \ an.$$

Your mission is to find a Pareto optimum for this economy.

(a) What are the "ingredients" for this economy?

(b) What is the Lagrangian problem associated with a Pareto optimum?

(c) What are its first-order conditions?

(d) What are the optimal values of c and n?

Answer.

(a) The ingredients are

- List of commodities: the consumption good, labor.
- List of agents: one.
- Preferences: see utility function above. Endowment: one unit of labor.
- Technology: $y = an$, where y is output of the consumption good.
- Resource constraints: for leisure we have $1 - n$ of leisure plus n of work is no greater than one; for consumption, we have the constraint listed above.

(b) The Lagrangian is

$$\mathcal{L} = \log c + \theta \log(1 - n) + \lambda(an - c)$$

(c) The first-order conditions are

$$0 = 1/c - \lambda$$
$$0 = -\theta/(1 - n) + \lambda a.$$

(d) The first-order conditions plus the constraint give us $n = 1/(1 + \theta)$ and $c = an = a/(1 + \theta)$.

It wasn't asked, but this environment gives us prices, too. This is a little terse, but we read them from the first-order conditions: the price of consumption is λ, the price of labor is λa.

4. *Pareto problem with production.* Consider a theoretical economy with these ingredients:

- One agent.
- Two goods, apples and bananas.
- Utility function: $\log c_a + \log c_b$.
- Endowment: 6 apples, 4 bananas.
- Technology: apples can be converted to bananas one for one.

(a) What are the resource constraints for this economy?

(b) What is the Pareto problem?

(c) What is the solution to the Pareto problem? What are the optimal quantities? What are the implied prices?

Answer.

(a) If we let a be the number of apples we convert to bananas, the resource constraints for apples and bananas are

$$
\begin{aligned}
c_a + a &\leq y_a = 6 \\
c_b &\leq y_b + a = 4 + a.
\end{aligned}
$$

We might also say $a \geq 0$ if we're worried about the reverse technology: converting bananas to apples. We'll ignore that from here on.

(b) The Pareto problem is: maximize utility subject to the resource constraints. The Lagrangian is

$$
\mathcal{L} = \log c_a + \log c_b + \lambda_a(y_a - c_a - a) + \lambda_b(y_b + a - c_b).
$$

The first-order conditions (the derivatives of $\mathcal{L}$) include

$$
\begin{aligned}
c_a : &\quad 1/c_a - \lambda_a = 0 \\
c_b : &\quad 1/c_b - \lambda_b = 0 \\
a : &\quad -\lambda_a + \lambda_b = 0.
\end{aligned}
$$

That gives us $\lambda_a = \lambda_b$ and (therefore) $c_a = c_b$. From the resource constraints we have $c_a = a_b = 5$ and $a = 1$. The relative price of bananas comes from the Lagrange multipliers: $q_b/q_a = \lambda_b/\lambda_a = 1$.

5. *Asset pricing with production and dividends.* Consider a two-period economy with a linear technology. What is equilibrium consumption growth? What are the state prices? What is a claim to one unit of capital worth?

To address these questions, we use a variant of our two-period economy, with dates 0 and 1 and states z at date 1 that occur with probability $p(z)$. The representative agent has utility function

$$
u(c_0) + \beta \sum_z p(z)u[c_1(z)],
$$

with $u(c) = c^{1-\alpha}/(1 - \alpha)$ for $\alpha > 0$ (power utility). She is endowed with y_0 units of the date-0 good, nothing at date 1. The technology is linear: k units of the date-0 good invested in capital generate zk units of the good in state z at date 1. The resource constraints are therefore

$$
\begin{aligned}
c_0 + k &= y_0 \\
c_1(z) &= zk,
\end{aligned}
$$

with one of the latter for each state z. Think of the productivity factor z as $a(z)$ with $a(z) = z$.

(a) What are the classical "ingredients" of this economy?

(b) What is the associated Pareto problem? What are its first-order conditions?

(c) Suppose z is lognormal: that is, $\log z \sim \mathcal{N}(\kappa_1, \kappa_2)$. Use the properties of lognormal random variables to show that $E(z^a) = \exp(a\kappa_1 + a^2\kappa_2/2)$ for any real number a.

(d) Use this result to find the optimal values of c_0 and k. Given these values, what is saving?

(e) What is $c_1(z)$? What are the state prices?

(f) What is the value of one unit of capital, that is, a claim to z units of output in each state z?

Answer.

(a) Commodities: the good at date 0 plus the good in each state at date 1.
Agents: one
Preferences, endowment, and technologies: given above.
Resource constraints: ditto.

(b) Pareto problem based on the Lagrangean:

$$\mathcal{L} = \log c_0 + \beta \sum_z p(z) \log c_1(z) + q_0(y_0 - c_0 - k) + \sum_z q_1(z)[zk - c_1(z)].$$

We choose c_0, k, and $c_1(z)$ (one for each z) to maximize this. The first-order conditions are

$$
\begin{aligned}
c_0 : && c_0^{-\alpha} &= q_0 \\
k : && q_0 &= \sum_z q_1(z)z \\
c_1(z) : && \beta p(z) c_1(z)^{-\alpha} &= q_1(z)
\end{aligned}
$$

(c) Define $x = \log z$. Its mgf is $e^{sx} = E(e^{s\log z}) = E(z^s) = e^{s\kappa_1 + s^2\kappa_2/2}$. Setting $s = a$ gives us the answer.

(d) This is moderately demanding, but here's how it works. With apologies for mixing sums and integrals, we use the first and third first-order conditions to substitute for q_0 and $q_1(z)$ in the second:

$$
\begin{aligned}
1 &= \sum_z \frac{q_1(z)}{q_0} z = \sum_z \frac{\beta p(z)c_1(z)^{-\alpha}}{c_0^{-\alpha}} z = \sum_z \beta p(z) \frac{(zk)^{-\alpha}}{(y_0 - k)^{-\alpha}} z \\
&= \beta(y_0 - k)^\alpha k^{-\alpha} \sum_z p(z) z^{1-\alpha} = \beta(y_0 - k)^\alpha k^{-\alpha} E(z^{1-\alpha}).
\end{aligned}
$$

To simplify, denote $E(z^{1-\alpha}) = Z$. Then consumption and capital are

$$
k = \frac{(\beta Z)^{1/\alpha}}{1 + (\beta Z)^{1/\alpha}} y_0, \quad c_0 = \frac{1}{1 + (\beta Z)^{1/\alpha}} y_0
$$

Using the lognormal result, we have

$$
Z = E(z^{1-\alpha}) = e^{(1-\alpha)\kappa_1 + (1-\alpha)^2\kappa_2/2}
$$

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(e) Evidently

$$c_1(z) \quad = \quad zk \quad = \quad zy_0 \frac{(\beta Z)^{1/\alpha}}{1 + (\beta Z)^{1/\alpha}}.$$

The big part at the end is a constant, so it's ugly but innocuous. The state prices are (more tedious substitution)

$$q(z) \quad = \quad p(z)\beta[c_1(z)/c_0]^{-\alpha} \quad = \quad p(z)z^{-\alpha}/Z.$$

As usual, the state price is the product of a pricing kernel [here $m(z) = z^{-\alpha}/Z$] and a probability [the normal density for $\log z$, which we haven't bothered to write out].

(f) This is claim to z next period, with value at date 0 of

$$q^e \quad = \quad E(z^{1-\alpha})/Z \quad = \quad 1.$$

Hmmmm... Why does this make sense? Because one unit of capital is one unit of the good at date 0, whose price is one since we're valuing assets in units of the date-0 good.

Matlab code

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