Lab Report #4: Asset Pricing Fundamentals

Revised: March 20, 2013

Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Please include your Matlab code.

1. *State prices, pricing kernels, and risk-neutral probabilities.* Consider a 3-state distribution in which the state z takes on the values $\{-1, 0, 1\}$ with probabilities $\{\omega, 1-2\omega, \omega\}$. A random variable x is defined by $x(z) = \mu + \delta z$. Statisticians would say that x has a "categorical distribution."

   We'll build a representative-agent economy on this distribution by setting log consumption growth equal to x. Utility has the usual additive form,

   $$u(c_0) + \beta \sum_z p(z) u[c_1(z)],$$

   with $u(c) = c^{1-\alpha}/(1-\alpha)$ (power utility).

   (a) What are the mean and variance of x?

   (b) What are the traditional measures of skewness and excess kurtosis, γ_1 and γ_2? Under what conditions is γ_2 large?

   (c) What is the pricing kernel?

   (d) What are the state prices?

   (e) What are the risk-neutral probabilities? How do they differ from the true probabilities?

   (f) Which is more valuable, a claim to one unit of the good in state $z = -1$ or one unit in state $z = +1$? Why?

Solution: The idea here is to go through a concrete example and compute state prices, the pricing kernel, and risk-neutral probabilities, and think about what each one does.

(a) The easiest way to do this is with the cgf. The mgf is

$$h(s) \;=\; \omega e^{\mu-\delta} + (1-2\omega)e^{\mu} + \omega e^{\mu+\delta}$$

and the cgf is $k(s) = \log h(s)$. The first four cumulants are

$$\begin{aligned} \kappa_1 &= \mu \\ \kappa_2 &= 2\delta^2\omega \\ \kappa_3 &= 0 \\ \kappa_4 &= 2\delta^4\omega(1-6\omega). \end{aligned}$$

The first one is the mean, the second one is the variance.

(b) Skewness and excess kurtosis are

$$\begin{aligned}\gamma_1 &= \kappa_3/\kappa_2^{3/2} &= 0 \\ \gamma_2 &= \kappa_4/\kappa_2^2 &= 1/(2\omega) - 3.\end{aligned}$$

The first is clear, because the distribution is symmetric. Evidently γ_2 is large when ω is small. It's zero when $\omega = 1/6$

(c) The pricing kernel is

$$m[x(z)] \;=\; \beta e^{-\alpha x} \;=\; \beta e^{-\alpha(\mu+\delta z)}.$$

It's decreasing in x and z. In other words, m is higher in states where z is lower ("bad states").

(d) The state prices are

$$Q(z) \;=\; p(z)m(z) \;=\; p(z)\beta e^{-\alpha(\mu+\delta z)},$$

with $p(z)$ given in the previous question.

(e) The risk-neutral probabilities are

$$p^*(z) \;=\; p(z)m(z)/q^1 \;=\; \begin{cases} \omega\beta e^{-\alpha(\mu-\delta)}/q^1 & \text{for } z=-1 \\ (1-2\omega)\beta e^{-\alpha\mu}/q^1 & \text{for } z=0 \\ \omega\beta e^{-\alpha(\mu+\delta)}/q^1 & \text{for } z=1 \end{cases}$$

where

$$q^1 \;=\; \sum_z p(z)m(z) \;=\; \beta e^{-\alpha\mu}\left[\omega(e^{\alpha\delta}+e^{-\alpha\delta}) + (1-2\omega)\right].$$

If risk aversion α is zero, then $m(z) = \beta$ in all states. Otherwise, higher values of α raise the value of the bad state and lower it in the good state.

(f) The only difference is the term $e^{\pm\alpha\delta}$. As long as $\delta > 0$, then $e^{\alpha\delta} > e^{-\alpha\delta}$ and we have $Q(-1) > Q(1)$: state $z = -1$ is more valuable than state $z = 1$. That's the usual result, that bad states are more valuable than good states. If $\delta < 0$ the answer's the same, but $z = 1$ is the bad state.

2. *Excess kurtosis and the equity premium.* Continue with the same environment. We're going to vary ω and see how that affects the equity premium. The idea is to explore the role of excess kurtosis, which we've seen is controlled by ω. In all cases, set $\alpha = 10$ and $\beta = 0.99$.

   (a) We observe log consumption growth has a mean in US data of roughly 0.02 and a standard deviation of 0.035. Given a value of ω, what values of μ and δ reproduce these values?

   (b) Suppose $\omega = 1/6$. Why is this value a useful benchmark? What values of (μ, δ)

reproduce the mean and variance of log consumption growth?

(c) Suppose equity is a claim to the growth rate of consumption e^x. What is the equity premium with these values?

(d) Suppose $\omega = 1/20$. How do your choices of (μ, δ) adjust to keep the mean and variance of log consumption growth at their observed values? How does the equity premium change?

Solution: Here the plan is to explore the possible role of changes in the distribution, specifically changes in excess kurtosis.

(a) The idea here is to choose parameters to reproduce the mean and variance of log consumption growth in US data. The mean is μ, so we set $\mu = 0.02$. The variance is $2\delta^2\omega$. Given a value of $\omega > 0$, we set $2\delta^2\omega = 0.035^2$ or $\delta = 0.035/(2\omega)^{1/2}$.

(b) With $\omega = 1/6$, excess kurtosis is zero, the same value as the normal distribution. To match the variance, we then need $\delta = 0.0606$. μ remains 0.02.

(c) The equity premium is $E(r^e) - r^1 = 0.0143$ or 1.43%. See the Matlab code for the calculation.

(d) If we reduce ω to 1/20, excess kurtosis goes up to 7. The equity premium rises to 0.0160. The impact is modest, but it illustrates the potential role of adding more kurtosis to the distribution.

Matlab program:

```
%  hw4_s12
%  Matlab program for Lab Report #4, Spring 2012
%  NYU course ECON-UB 233, Macro foundations for asset pricing, Mar 2012.
%  Written by:  Dave Backus, March 2012

disp('Answers to Lab Report 4')

%%
disp(' ')
disp('-----------------------------------------------------------------')
disp('Question 1 (three-state example)')
format compact
clear all

syms s omega mu delta

disp(' ')
disp('Analytics for cumulants')
```

```
mgf = omega*exp(s*(mu-delta)) + (1-2*omega)*exp(s*mu) + omega*exp(s*(mu+delta));
cgf = log(mgf);

disp(' ')
disp('Cumulants')
kappa1 = subs(diff(cgf,s,1),s,0);    %  mean
kappa2 = subs(diff(cgf,s,2),s,0);    % variance
kappa3 = subs(diff(cgf,s,3),s,0);
kappa4 = subs(diff(cgf,s,4),s,0);

kappa1 = simplify(kappa1)
kappa2 = simplify(kappa2)
kappa3 = simplify(kappa3)
kappa4 = simplify(kappa4)

gamma1 = simplify(kappa3/kappa2^(3/2))
gamma2 = simplify(kappa4/kappa2^2)

% pricing kernel, state prices, and risk-neutral probs
syms beta alpha
p = [omega; 1-2*omega; omega];
z = [-1; 0; 1];
x = mu + delta*z

m = beta*exp(-alpha*x)
Q = p.*m
q1 = sum(Q)
pstar = Q/q1

% this is ugly, mu should drop out
pstar = simplify(pstar)


%%
disp(' ')
disp('-------------------------------------------------------------------')
disp('Question 2 (continued)')

% enter numbers
omeganum = 1/20
p = subs(p, omega, omeganum)
gamma2 = 1/(2*omeganum) - 3

munum = 0.02
deltanum = 0.035/sqrt(2*omeganum)
```

```
% asset pricing
alpha = 10;
beta = 0.99

x = munum + deltanum*z
m = beta*exp(-alpha*x)

q1 = sum(p.*m)
r1 = 1/q1

d = exp(x);
qe = sum(p.*m.*d)
re = d/qe
Ere = sum(p.*re)

eqprem = Ere - r1


% in logs
logr1 = log(r1)

qe = sum(p.*m.*d)
Elogre = sum(p.*log(d)) - log(qe);

eq_prem = Elogre - logr1

return
```