Quiz #2

(Fall 2013)

Please write your name below. Then complete the exam in the space provided. There are THREE questions. You may refer to one page of notes: standard paper, both sides, any content you wish.

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(Name and signature)

1. *State prices and related objects (30 points).* Consider a world with two dates ($t = 0$ and $t = 1$) and two states at date $t = 1$ ($z = 1$ and $z = 2$). The probabilities of the two states are $p(1) = 1/4$ and $p(2) = 3/4$. State prices are $Q(1) = 1/3$ and $Q(2) = 2/3$.
   (a) What is the price of a one-period bond in this world? (5 points)
   (b) What is the pricing kernel in each state? (5 points)
   (c) If this were a representative agent economy, which state would be the good one for the agent? Why? (5 points)
   (d) What are the risk-neutral probabilities? (5 points)
   (e) What is the maximum Sharpe ratio these state prices can generate? (10 points)

Solution:

(a) A bond consists of one unit of each Arrow security. Its price is $q^1 = Q(1) + Q(2) = 1$.

(b) The pricing kernel $m(z)$ is $m(z) = Q(z)/p(z)$, which gives us $m(1) = 4/3$ and $m(2) = 8/9$.

(c) In the good state, the pricin kernel is low: payoffs have low value when consumption is high. Here that's state 2.

(d) The risk-neutral probabilities are $p^*(z) = Q(z)/q^1$, which gives us back the state prices in this case: $p^*(1) = 1/3$ and $p^*(2) = 2/3$.

(e) The Hansen-Jagannathan bound gives us the maximum Sharpe ratio as the ratio of the standard deviation of the pricing kernel to its mean, the price of a (one-period) bond. Here the bond price is one, so all we need is the standard deviation. The usual calculations give us

$$\text{Var}(m) \quad = \quad E(m^2) - E(m)^2 \quad = \quad 4/9 + 16/27 - 1 \quad = \quad 0.03704.$$

The standard deviation is the square root, 0.1925, which is the largest Sharpe ratio possible in this world.

2. *Entropy with gamma risks (30 points).* Consider our usual representative agent economy with power utility and log consumption growth $\log g = x$. The log pricing kernel is therefore $\log m(x) = \log \beta - \alpha x$. Log consumption growth x has a gamma distribution with cgf $k(s;x) = -\theta \log(1 - s/\lambda)$ and density function $p(x) = x^{\theta-1} e^{-\lambda x}[\lambda^\theta/\Gamma(\theta)]$. The parameters (λ, θ) are positive.

   You may recall from earlier work that x has mean and variance

$$\begin{aligned} \kappa_1 &= \theta/\lambda \\ \kappa_2 &= \theta/\lambda^2 \end{aligned}$$

   and skewness and excess kurtosis

$$\begin{aligned} \gamma_1 &= 2/\theta^{1/2} \\ \gamma_2 &= 6/\theta. \end{aligned}$$

   (a) What is the cgf of the log pricing kernel? (10 points)

   (b) What is the entropy of the pricing kernel? (10 points)

   (c) How does entropy compare to a lognormal benchmark with the same mean and variance? (10 points)

Solution:

(a) Since it's a linear transformation of x, we have

$$k(s; \log m) \;=\; s \log \beta + k(-\alpha s; x) \;=\; s \log \beta - \theta \log(1 + \alpha s/\lambda).$$

[Recall: if $y = a + bx$, then $E(e^{sy}) = e^{sa} E(e^{sbx})$.]

(b) The term $\log E(m)$ is just the cgf of $\log m$ evaluated at $s = 1$, namely

$$\log E(m) \;=\; \log E(e^{\log m}) \;=\; \log \beta - \theta \log(1 + \alpha/\lambda).$$

The mean of $\log m$ is $\log \beta - \alpha\theta/\lambda$. The difference is entropy:

$$H(m) \;=\; \log E(m) - E(\log m) \;=\; -\theta \log(1 + \alpha/\lambda) + \alpha\theta/\lambda.$$

(c) In the lognormal case, entropy is the variance of $\log m$ over two, which here is $\alpha^2\theta/(2\lambda^2)$. The expressions aren't particularly friendly, but if you do a Taylor series expansion of the previous expression, you find

$$\begin{aligned} H(m) &= -\theta \left[(\alpha/\lambda) - (\alpha/\lambda)^2/2 + (\alpha/\lambda)^3/3! + \cdots \right] + \alpha\theta/\lambda \\ &= -\theta \left[-(\alpha/\lambda)^2/2 + (\alpha/\lambda)^3/3! + \cdots \right]. \end{aligned}$$

So the variance term corresponds to the lognormal case, but we have other terms after that.

3. *Option on mixture of exponentials (40 points).* Suppose the risk-neutral distribution of the future value of the underlying is a mixture of x_1 and x_2:

$$s_{t+1} \;=\; \begin{cases} x_1 & \text{with probability } 1-\omega \\ x_2 & \text{with probability } \omega \end{cases}$$

for some ω between zero and one. Each x_j is exponential with density

$$p(x_j) \;=\; \lambda_j \exp(-\lambda_j x_j)$$

for $x_j \geq 0$ and $\lambda_j > 0$. Each x_j has a mean of $1/\lambda_j$.

(a) What is the no-arbitrage condition for this asset? (10 points)

(b) Consider a put option giving the owner the right to sell the asset for price k at $t+1$. What cash flow is generated by this option? (10 points)

(c) Suppose $\omega = 0$. What is the option's value? (10 points)

(d) What is the option's value for some arbitrary value of ω? (10 points)

Solution:

(a) We have

$$\begin{aligned} s_t &= q_t^1 E^*(s_{t+1}) \\ &= q_t^1[(1-\omega)E^*(x_1) + \omega E^*(x_2)] \;=\; q_t^1[(1-\omega)/\lambda_1 + \omega/\lambda_2]. \end{aligned}$$

(b) The cash flow is (as usual) $(k - s_{t+1})^+$.

(c) If $\omega = 0$ we can skip the second term. The put price is

$$\begin{aligned} q_t^p &= q_t^1 E^*(k - s_{t+1})^+ \\ &= q_t^1 \int_0^k (k-x)\lambda_1 e^{-\lambda_1 x} dx \;=\; q_t^1[k - (1 - e^{-\lambda_1 k})/\lambda_1]. \end{aligned}$$

(d) Here we have, by the same logic,

$$q_t^p \;=\; q_t^1 \Big\{(1-\omega)[k - (1-e^{-\lambda_1 k})/\lambda_1] + \omega[k - (1-e^{-\lambda_2 k})/\lambda_2]\Big\}.$$