ECON-UB 233 Dave Backus @ NYU

Lab Report #3: Consumption, Risk, & Portfolio Choice

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Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Please include your Matlab code.

Solution: Brief answers follow, but see also the attached Matlab program for calculations related to Questions 3 and 4; download the pdf, open, click on pushpin:
If this doesn't work for you, let me know and I'll explain how it works.

1. *Static risk and return.* Consider an agent with utility

$$U \quad = \quad E[u(c)],$$

where $u(c) = c^{1-\alpha}/(1-\alpha)$ for some risk aversion parameter $\alpha > 0$. She invests one and consumes the gross return r.

(a) What is her expected utility if she invests everything in a riskfree asset whose (gross) return is $r = e^x$ for some constant $x > 0$? (Her consumption is therefore e^x in every state.) What is the certainty equivalent of this outcome?

(b) What is her expected utility if she invests in an asset whose return is lognormal: $\log r \sim \mathcal{N}(\kappa_1, \kappa_2)$? What is her certainty equivalent?

(c) For what values of κ_1 and κ_2 is the risky asset preferred? How does your answer depend on α?

Solution:

(a) Expected utility is $E[u(e^x)] = u(e^x)$. The certainty equivalent of a sure thing is the sure thing, which here is e^x. More formally, μ solves

$$U(\mu, \mu, \ldots, \mu) \quad = \quad u(\mu) \quad = \quad u(e^x),$$

so the certainty equivalent is e^x.

(b) We're using properties of lognormal random variables here. We know $E(r^{1-\alpha}) = \exp[(1-\alpha)\kappa_1 + (1-\alpha)^2\kappa_2/2]$. The certainty equivalent is therefore $\mu^{1-\alpha} = E(r^{1-\alpha})$ or $\mu = E(r^{1-\alpha})^{1/(1-\alpha)} = \exp[\kappa_1 + (1-\alpha)\kappa_2/2]$.

(c) Evidently we need $\kappa_1 + (1-\alpha)\kappa_2/2 > x$. So large κ_1 helps. If $\alpha > 1$, small κ_2 helps, too, otherwise the reverse. [There's a small technical issue here: the reason we need $\alpha > 1$, rather than just $\alpha > 0$, is that increasing κ_2 here increases the mean of r. That's where the one comes from. When $\alpha > 1$ this effect is dominated by the impact on risk.]

2. *Constrained optimization.* Consider the problem: choose x and y to maximize

$$f(x, y) \quad = \quad 2x^{1/2} + 4y^{1/2}$$

subject to $x + y \leq 9$.

(a) What is the Lagrangian associated with this problem?

(b) What are the first-order conditions?

(c) What values of x and y solve the problem? What is the Lagrange multiplier?

Solution:

(a) The Lagrangian is

$$\mathcal{L} \quad = \quad 2x^{1/2} + 4y^{1/2} + \lambda(9 - x - y).$$

(b) The first-order conditions are

$$\begin{aligned} 1/x^{1/2} &= \lambda \\ 2/y^{1/2} &= \lambda. \end{aligned}$$

(c) The solution includes $y = 4x$, $x = 9/5$, $y = 36/5$, and $\lambda = 5^{1/2}/3$.

3. *Securities and returns.* Consider our usual two-period event tree. At date 0, we purchase one unit of security or asset j (an arbitrary label) for price q^j. At date 1, we get dividend $d^j(z)$, which depends on the state z. Let us say, specifically, that there are two securities and two states, with dividends

Security	State 1	State 2
1 (“bond”)	1	1
2 (“equity”)	1	2

The prices of the securities are $q^1 = 3/4$ (bond) and $q^e = 1$ (equity).

(a) Why does it make sense for the bond to pay one in each state?

(b) What are the (gross) returns on the assets?

(c) An Arrow security pays one in a specific state, nothing in other states. Here we have two states, hence two Arrow securities, whose dividends are

Security	State 1	State 2
Arrow 1	1	0
Arrow 2	0	1

What quantities of bonds and equity reproduce the dividend of the second Arrow security?

(d) Securities can be thought of as collections of Arrow securities. If we know the prices of Arrow securities, we can find the prices of other securities by adding up the values of their dividends. Here we do the reverse: use the prices of bonds and equity to find the prices $Q(z)$ of Arrow securities. What are $Q(1)$ and $Q(2)$ here?

Solution:

(a) A riskfree bond is riskfree because its dividend is the same in all states. Making the dividend one is simply a convention.

(b) The bond return is $r^1 = 1/(3/4) = 4/3$ in all states. The equity return is $r^e(z) = d^e(z)/q^e$ or

$$r^e(z) \quad = \quad \begin{cases} 1/1 = 1 & \text{for } z = 1 \\ 2/1 = 2 & \text{for } z = 2 \end{cases}$$

(c) If we buy one unit of equity and sell one unit of the bond, we are left with a net dividend of zero in state 1 and one in state 2. So we've replicated the second Arrow security. The costs of this transaction is $q^e - q^1 = 1 - 3/4 = 1/4$, so that should be its price: $Q(2) = 1/4$.

(d) We can also do the reverse: combine Arrow securities to replicate the dividends of the two assets. If the assets and their replications sell for the same price, we have

$$\begin{aligned} 3/4 &= Q(1) + Q(2) \\ 1 &= Q(1) + 2Q(2). \end{aligned}$$

That gives us $Q(1) = 1/2$ and $Q(2) = 1/4$. Lurking behind the scenes here is an arbitrage argument. Why do the assets and their replications sell for the same price? Because otherwise people would buy the cheaper one and sell the more expensive one, giving them a riskless profit. Markets are unlikely to let that happen: they should eliminate pure arbitrage opportunities like this.

4. *Portfolio choice.* An investor must decide how to allocate his saving between a riskfree bond and equity. We approximate the world with three states, each of which occurs with probability 1/3. The returns by state are

Security	State 1	State 2	State 3
1 ("bond")	1.1	1.1	1.1
2 ("equity")	0.6	1.2	1.6

The investor's problem is to choose current consumption c_0 and the fraction of saving

a to invest in equity to solve

$$\begin{aligned} \max_{c_0, a} \quad & u(c_0) + \beta \sum_z p(z) u[c_1(z)] \\ s.t. \quad & c_1(z) \;=\; (y_0 - c_0)[(1-a)r^1 + a r^e(z)]. \end{aligned}$$

If $a > 1$, the agent has a levered position, borrowing to fund investments in equity greater than saving. As usual, $u(c) = c^{1-\alpha}/(1-\alpha)$. Where the problem calls for numbers, we'll use $\beta = 0.9$ and $\alpha = 2$.

(a) What is the upper bound on a consistent with positive consumption $c_1(z)$ in all states z? Lower bound?

(b) What are the first-order conditions for c_0 and a? *Comment:* I recommend you substituting the expression for $c_1(z)$ into the utility function.

(c) Use Matlab to solve the first-order condition for a numerically. What value of a maximizes utility? *Comment:* I did this by varying a manually until its first-order condition was satisfied. You could also compute the first-order condition for a grid of values for a, and choose the one that comes closest to satisfying the first-order condition.

Solution:

(a) We need $(1-a)r^1 + a r^e(z) > 0$. Only the extremes matter here, so in the bad state (state 1) we need $a < 2.2$ and in the good state (state 2) we need $a > -2.2$. If you find a solution to the first-order conditions outside this range, it's not really a solution.

(b) The first-order conditions are

$$\begin{aligned} c_0 : \quad c_0^{-\alpha} \;&=\; \beta \sum_z p(z) c_1(z)^{-\alpha} [(1-a)r^1 + a r^e(z)] \\ a : \quad 0 \;&=\; \beta \sum_z p(z) \left\{ (y_0 - c_0)[(1-a)r^1 + a r^e(z)] \right\}^{-\alpha} [r^e(z) - r^1]. \end{aligned}$$

The term in curly brackets is $c_1(z)$. The second equation simplifies to

$$0 \;=\; \sum_z p(z) [(1-a)r^1 + a r^e(z)]^{-\alpha} [r^e(z) - r^1],$$

which depends on a but not c_0.

(c) This problem doens't have a simple closed-form solution, but we can crack it easily with Matlab. We can automate this, and will later in the course, but for now consider this iterative procedure:

- Pick a value of a.
- Check the foc. If it equals zero, we're done. If not, pick another a.

The solution is $a = 0.107$.