ECON-UB 233 Dave Backus @ NYU

Quiz #3

December 2014

Please write your name below. Then complete the exam in the space provided. There are THREE questions. You may refer to one page of notes: standard paper, both sides, any content you wish.

____

(Name and signature)

1. *Short answers.* Provide short answers to the following:
   (a) Explain how to compute the continuously-compounded yield y_t^n from bond price(s). (10 points)
   (b) Explain how to compute the continuously-compounded forward rate f_t^n from bond price(s). (10 points)
   (c) If $\{x_t\}$ is a stochastic process, what property would make it a martingale? (10 points)

Solution:

(a) Use $y_t^n = -n^{-1} \log q_t^n$.

(b) Use $f_t^n = \log(q_t^n / q_t^{n+1})$.

(c) We would say x is a martingale if $E_t(x_{t+1}) = x_t$.

2. *Nonlinear dynamics.* Consider the stochastic process

$$x_{t+1} \quad = \quad x_t w_{t+1} + \theta w_t,$$

where $\{w_t\}$ is a sequence of independent normal random variables with means equal to zero and variances equal to one. Assume, as usual, that at date t we know the current and past values of x and w, but not the future values.

   (a) What is the distribution of x_{t+1} at date t — the one-period conditional distribution, in other words? (10 points)
   (b) Show that x is Markov for some definition z of the state. (10 points)
   (c) What is $E_t(x_{t+2})$? [*Hint: Use the law of iterated expectations.*] (10 points)
   (d) What is $\text{Var}_t(x_{t+2})$? [*Comment: This is difficult, skip if you're short of time.*] (10 points)

Solution:

(a) Conditional on information available at date t, x_{t+1} is normal with mean θw_t and variance x_t^2.

(b) The state is whatever we need to describe the conditional distribution of x_{t+1}. Here that's $z_t = (x_t, w_t)$.

(c) Using the law of iterated expectations, we find that the mean is $E_t[E_{t+1}(x_{t+2})] = E_t(\theta w_{t+1}) = 0$.

(d) The variance is moderately complicated:

$$\begin{aligned}
\mathrm{Var}_t(x_{t+2}) &= E_t[(x_{t+1}w_{t+2} + \theta w_{t+1})^2] \\
&= E_t[x_{t+1}^2 w_{t+2}^2 + 2\theta w_{t+1} x_{t+1} w_{t+2} + \theta^2 w_{t+1}^2] \\
&= E_t\left\{E_{t+1}[x_{t+1}^2 w_{t+2}^2 + 2\theta w_{t+1} x_{t+1} w_{t+2} + \theta^2 w_{t+1}^2]\right\} \\
&= E_t[x_{t+1}^2 + \theta^2 w_{t+1}^2] \\
&= x_t^2 + \theta^2 w_t^2 + \theta^2.
\end{aligned}$$

3. *Valuing dividend strips.* A dividend strip is a claim to a single dividend n periods in the future. We denote the price at date t of the dividend paid at $t+n$ by s_t^n. The term structure of strip prices – the sequence $s_t^1, s_t^2, \ldots$ — can be approached with methods similar to those we used with bonds.

We'll use the model

$$\begin{aligned}
\log m_{t+1} &= -\lambda^2/2 - z_t + \lambda w_{t+1} \\
z_t &= (1-\varphi)\delta + \varphi z_{t-1} + \sigma w_t \\
\log d_t &= \eta z_t.
\end{aligned}$$

Here $\log m_{t+1}$ is the pricing kernel, z_t is a state variable, w_t is one of a sequence of independent standard normal random variables, and d_t is the dividend. The parameter η controls the sensitivity of the dividend d_t to the state z_t.

(a) What is the short rate $f_t^0 = y_t^1$ in this model? (10 points)

(b) What is the price s_t^1 of next period's dividend d_{t+1}? (10 points)

(c) Value prices of future dividends recursively. If prices are loglinear functions of the state,

$$\log s_t^n = C_n + D_n z_t,$$

how would you compute the coefficients (C_n, D_n)? (20 points)

(d) Derive the excess log return on the strip of maturity one,

$$\log d_{t+1} - \log s_t^1 - f_t^0.$$

How does it vary with η? (10 points)

Solution:

(a) The short rate is

$$f_t^0 \quad = \quad -\log q_t^1 \quad = \quad -\log E_t(m_{t+1}) \quad = \quad z_t.$$

(b) The price the one-period strip is

$$\log s_t^1 \quad = \quad \log E_t(m_{t+1}d_{t+1}) \quad = \quad \log E_t\left[\exp(\log m_{t+1} + \log d_{t+1})\right].$$

Thus we need

$$\log m_{t+1} + \log d_{t+1} \quad = \quad -\lambda^2/2 + \eta(1-\varphi)\delta + (\eta\varphi - 1)z_t + (\eta\sigma + \lambda)w_{t+1}.$$

The usual "mean plus variance over two" gives us

$$\log s_t^1 \quad = \quad (\eta\sigma+\lambda)^2/2 - \lambda^2/2 + \eta(1-\varphi)\delta + (\eta\varphi - 1)z_t.$$

Thus we have $C_1 = (\eta\sigma+\lambda)^2/2 - \lambda^2/2 + \eta(1-\varphi)\delta$ and $D_1 = (\eta\varphi - 1)$.

(c) Strip prices of higher maturity follow from $s_t^{n+1} = E_t(m_{t+1}s_{t+1}^n)$. Given their loglinear form, we solve

$$\log m_{t+1} + \log s_{t+1}^n \quad = \quad -\lambda^2/2 + C_n + D_n(1-\varphi)\delta + (D_n\varphi - 1)z_t + (D_n\sigma + \lambda)w_{t+1}.$$

Then we have

$$\begin{aligned}
\log s_t^{n+1} &= \log E_t(m_{t+1}s_{t+1}^n) \\
&= (D_n\sigma+\lambda)^2/2 - \lambda^2/2 + C_n + D_n(1-\varphi)\delta + (D_n\varphi - 1)z_t \\
&= C_{n+1} + D_{n+1}z_t.
\end{aligned}$$

Lining up similar terms gives us recursions in the coefficients:

$$\begin{aligned}
C_{n+1} &= (D_n\sigma+\lambda)^2/2 - \lambda^2/2 + C_n + D_n(1-\varphi)\delta \\
D_{n+1} &= (D_n\varphi - 1).
\end{aligned}$$

We can start with (C_1, D_1) above, or note that a zero maturity strip gives us $\log s_t^0 = \log d_t = \eta z_t$, which gives us $C_0 = 0$ and $D_0 = \eta$.

(d) The log excess return is

$$\begin{aligned}
\log d_{t+1} - \log s_t^1 - f_t^0 &= \eta[(1-\varphi)\delta + \varphi z_t + \sigma w_{t+1} - (C_1 + D_1 z_t) - z_t \\
&= \lambda^2/2 - (\eta\sigma+\lambda)^2/2 + \eta\sigma w_{t+1}.
\end{aligned}$$

The (log) risk premium is the (conditional) mean, which we can simplify:

$$E_t(\log d_{t+1} - \log s_t^1 - f_t^0) \quad = \quad \lambda^2/2 - (\eta\sigma+\lambda)^2/2 \quad = \quad -(\eta\sigma)^2/2 - \lambda\eta\sigma.$$

Thus the risk premium depends on three parameters: λ, the sensitivity of the pricing kernel to risk in z; σ, the magnitude of this risk; and η, the sensitivity of the dividend to the same risk.