ECON-UB 233 Dave Backus @ NYU

Quiz #1

Fall 2014

Please write your name below, then complete the exam in the space provided. You may refer to one page of notes: standard paper, both sides, any content you wish. **There are FOUR questions.**

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(Name and signature)

1. *Linear transformations.* Consider the random variable x, which is normal with mean μ and standard deviation δ.
   (a) What are the variance, skewness, and excess kurtosis of x? (10 points)
   (b) What are x's moment generating function (mgf) and cumulant generating function (cgf)? What is the expectation of e^x? (10 points)
   (c) What is the cgf of $y = a + bx$? (10 points)
   (d) What is y's distribution? What are its mean, variance, skewness, and excess kurtosis? (10 points)

Solution:

(a) The variance is the standard deviation squared, δ^2. Since x is normal, its skewness and excess kurtosis are zero.

(b) The mgf is $h(s) = E(e^{sx}) = e^{\mu s + \delta^2 s^2/2}$. The cgf is the log of this: $k(s) = \log h(s) = \mu s + \delta^2 s^2/2$. The expectation of e^x is $h(1) = e^{\mu + \delta^2/2}$.

(c) If we run through the definitions, we see $k_y(s) = as + k_x(bs) = (a + b\mu)s + (b\delta)^2 s^2/2$.

(d) From the form of the cgf (quadratic in s), we see that y is normal. Its mean is $a + b\mu$ (the linear term) and its variance is $(b\delta)^2$ (the quadratic term). And since it's normal, skewness and excess kurtosis are still sero.

2. *Option payoffs.* An option on the random variable y has a payoff of $f(y) = \max\{0, y-k\}$ for some strike price k.

   Consider random variables y constructed from these independent inputs:

   - x_1 takes on the values $\{4, 6\}$ with probability one-half each.
   - x_2 takes on the values $\{-2, +2\}$ with probability one-half each.

We'll explore the expected payoff of options with strike price $k = 5$ and two versions of y: $y = x_1$ and $y = x_1 + x_2$.

(a) What is the expected value of $y = x_1$? (5 points)

(b) What is the expected option payoff — that is, $E(f(y))$ — in this case? How does it compare to $f[E(y)]$? What mathematical result does this illustrate? (10 points)

(c) What values does $y = x_1 + x_2$ take on? With what probabilities? What is its expected value? (10 points)

(d) What is the expected option payoff in this case? (5 points)

(e) How does the option payoff differ when $y = x_1$ and $y = x_1 + x_2$? What mathematical result does this illustrate? (10 points)

Solution:

(a) $E(y) = (1/2)(4) + (1/2)(6) = 5$.

(b) $E[f(y)] = (1/2)(0) + (1/2)(1) = 1/2$. $f[E(y)] = 0$. The former is greater because f is a convex function: Jensen's inequality in action.

(c) y takes on the values $\{2, 4, 6, 8\}$ with probability one-fourth each. The expected value is $E(y) = (1/4)(2 + 4 + 6 + 8) = 5$. That is: the mean hasn't changed, we've just spread out the distribution a bit. (Draw the two distributions if you like.)

(d) The expected option payoff is $E[f(y)] = (1/2)(0) + (1/4)(1) + (1/4)(3) = 1$.

(e) This is greater than the option payoff with $y = x_1$ because we've spread out the distribution. We don't have a name for this, but an increase in the variance holding constant the mean increases the expected value of a convex function.

3. *State prices.* Suppose three assets have dividends

Asset	State 1	State 2	State 3
Asset A	1	1	1
Asset B	1	2	1
Asset C	1	1	3

If the prices of assets are $q^A = 11/12$ and $q^B = q^C = 13/12$, what are the state prices? (15 points)

Solution: The asset payoffs are connected to state prices $Q(z)$ by

$$\begin{aligned} q^A &= Q(1) + Q(2) + Q(3) \\ q^B &= Q(1) + 2Q(2) + Q(3) \\ q^C &= Q(1) + Q(2) + 3Q(3), \end{aligned}$$

which we can solve for the Q's. The solution is $Q(1) = 8/12$, $Q(2) = 2/12$, and $Q(3) = 1/12$.

4. *Pareto problem with production.* Consider a theoretical economy with these ingredients:
   - One agent.
   - Two goods, apples and bananas.
   - Utility function: $\log c_a + \log c_b$.
   - Endowment: 6 apples, 4 bananas.
   - Technology: apples can be converted to bananas one for one.

   (a) What are the resource constraints for this economy? (5 points)

   (b) What is the Pareto problem? (5 points)

   (c) What is the solution to the Pareto problem? What are the optimal quantities? What are the implied prices? (15 points)

Solution:

(a) If we let a be the number of apples we convert to bananas, the resource constraints for apples and bananas are

$$\begin{aligned} c_a + a &\leq y_a = 6 \\ c_b &\leq y_b + a = 4 + a. \end{aligned}$$

We might also say $a \geq 0$ if we're worried about the reverse technology: converting bananas to apples. We'll ignore that from here on.

(b) The Pareto problem is: maximize utility subject to the resource constraints. The Lagrangian is

$$\mathcal{L} = \log c_a + \log c_b + \lambda_a(y_a - c_a - a) + \lambda_b(y_b + a - c_b).$$

The first-order conditions (the derivatives of $\mathcal{L}$) include

$$\begin{aligned} c_a: &\quad 1/c_a - \lambda_a = 0 \\ c_b: &\quad 1/c_b - \lambda_b = 0 \\ a: &\quad -\lambda_a + \lambda_b = 0. \end{aligned}$$

That gives us $\lambda_a = \lambda_b$ and (therefore) $c_a = c_b$. From the resource constraints we have $c_a = a_b = 5$ and $a = 1$. The relative price of bananas comes from the Lagrange multipliers: $q_b/q_a = \lambda_b/\lambda_a = 1$.