Syllabus

Wednesday, March 10, 2010
10:40 AM

Math and advanced macro and finance

1. Math tools

   Laplace transform

   z-transform

   state space representation of linear difference equation

   stochastic linear difference equation

   Forecasting a geometric sum of future random variables

   "Solving stable roots backwards and unstable roots forward"

   or

   Effects of foresight and transient decay

   Kuhn-Tucker and the min-max theorem

   Markov chains - joint distribution over histories that they induce

2. "A little knowledge of geometric series goes a long way"

   . A linear asset pricing formula

- "dynamic programming solution" of asset pricing formula
- "bubbles"
- Friedman's permanent income model with rational expectations.
- Borrowing limits.
- Friedman meets Muth
- consumption growth as forecaster of income
- Theory of interest rates - term structure

3. Complete markets
   - The event tree & the joint distribution
   - Arrow-Debreu complete markets
   - Generalization of Lucas model — asset pricing in an endowment economy.

<u>To be completed!</u>

Useful math:

$t \geq 0$, $t \in T = \mathbb{R}^+$

$f(t) : T \to R$

examples: (1) $f(t) = e^{+\delta t}$, $\delta > 0$

(2) $f(t) = t e^{\delta t}$, $\delta > 0$

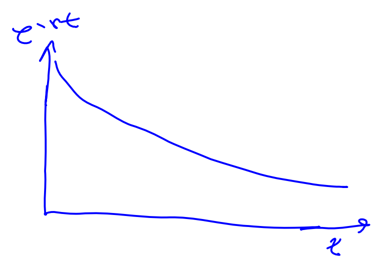
(3) $f(t) = t^n e^{\delta t}$, $\delta > 0$

Want to compute present value

$$F(r) = \int_0^\infty e^{-rt} f(t)\, dt$$

where $r > 0$.

$e^{-rt} \rightarrow$



$r > 0$ is the "instantaneous interest rate"

$F(r)$ is the (one sided) Laplace transform of $f(t)$.

$F(r) \longleftrightarrow F(t)$

Remark: There are tables of Laplace transforms. — see Wikepedia.

Example: $f(t) = B e^{+\delta t}$, $0 < \delta < r$.

$$F(r) = B\int_0^\infty e^{-rt} e^{\delta t}\, dt$$

$$= B \int_0^\infty e^{-(r-\delta)t}\, dt$$

$$= -\frac{B}{(r-\delta)} e^{-(r-\delta)t} \Big|_0^\infty$$

$$\boxed{F(r) = \frac{B}{r-\delta}}$$

This is called the "Gordon formula".

Special case: $\delta = 0 \implies$

$$F(r) = \frac{B}{r}$$ — Ricardo formula for price of a constant income stream.

Richard Feynman secret weapon: differentiate under integral sign.

$$F(r) = \int_0^\infty e^{-rt} f(t)\, dt$$

$$F'(r) = \int_0^\infty -t\, e^{-rt} f(t)\, dt$$

$$\implies -F'(r) \longleftrightarrow t f(t)$$

Application:

$$g(t) = B\, t\, e^{\delta t}, \quad t \geq 0$$

$$t f(t)$$

$$F(r) = \frac{B}{r-\delta}$$

$$F'(r) = \frac{-B}{(r-\delta)^2}$$

$$\Rightarrow \quad \beta t e^{-\delta t} \leftrightarrow \frac{\beta}{(r-\delta)^2} = \beta \int_0^\infty e^{-rt} t e^{\delta t} dt$$

Exercise: show that

$$F''(r) \longleftrightarrow t^2 f(t)$$

Digression: another trick with Laplace transforms

$$G(r) = \int_0^\infty e^{-rt} f'(t)\,dt = r F(r) - f(0^-), \text{ where again}$$

$$F(r) \doteq \int_0^\infty e^{-rt} f(t)\,dt$$

$$\int u \cdot dv = uv - \int v\,du$$

$$dv = f'(t)\,dt \qquad v = f(t)$$

$$u = e^{-rt} \qquad du = -r e^{-rt} dt$$

$$\int e^{-rt} f'(t)\,dt = e^{-rt} f(t) \Big|_0^\infty + r \int e^{-rt} f(t)\,dt$$

$$f'(t) \leftrightarrow -f(0^-) + r F(r)$$

e.g. $f(t) = e^{+\delta t}$, $f'(t) = \delta e^{+\delta t}$

Then $F(r) = \frac{1}{r-\delta}$

$$G(r) = \frac{r}{r-\delta} - 1 = \frac{r}{r-\delta} - \frac{r-\delta}{r-\delta} = \frac{\delta}{r-\delta} \quad \checkmark$$

Discrete time formulas:

$$x_t, \quad t \geq 0, \quad t = 0, 1, 2, \ldots$$

e.g. $x_t = \lambda^t$,

To compute:

$$\sum_{t=0}^{\infty} \beta^t x_t \qquad \text{where} \qquad \beta = \frac{1}{1+r}$$

$t=0$

r = one-period interest rate

$$= \sum_{t=0}^{\infty} \beta^t \lambda^t = \sum_{t=0}^{\infty} (\beta\lambda)^t = \frac{1}{1-\beta\lambda}$$

provided $|\beta\lambda| < 1$.

Some matrix algebra

vector $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ column vector

$x' = (x_1 \; x_2 \; \cdots \; x_n)$ row vector

matrix $A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ A_{21} & \cdots & A_{2m} \\ \vdots & & \\ A_{n1} & \cdots & A_{nm} \end{pmatrix}$

$[A_{ij}]$, $i = 1, \ldots, n$; $j = 1, \ldots, m$

matrix multiplication:

$$\underset{n \times p}{C} \quad \underset{n \times m}{A} \cdot \underset{m \times p}{B}$$

$$C_{ik} = \sum_j A_{ij} B_{jk}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$c_{11} = a_{11} b_{11} + a_{12} \cdot b_{12}$$

Inverse of a matrix: A^{-1}

$$A^{-1} A = I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \\ & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

identity matrix.

Matrix equation:

$$\underset{n\times n}{A} \underset{n\times 1}{x} = \underset{n\times 1}{b}$$

$$A^{-1} A x = A^{-1} b$$

$$x = A^{-1} b$$

Application of matrix algebra to model dynamics.

First order linear difference equation:

$$x_{t+1} = A x_t \quad , A \text{ is } n \times n$$

$$y_t = G x_t \quad , G \text{ is } 1 \times n$$

x_0 given

Example:

$$y_{t+1} = \rho_1 y_t + \rho_2 y_{t-1}$$

let $\quad x_t = \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix}$

Then

$$\underbrace{\begin{pmatrix} y_{t+1} \\ y_t \end{pmatrix}}_{= x_{t+1}} = \underbrace{\begin{pmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{pmatrix}}_{= A} \underbrace{\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix}}_{= x_t}$$

$$y_t = \underbrace{[1 \quad 0]}_{= G} \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix}$$

Want:

$$p_t = \sum_{j=0}^{\infty} \beta^j y_{t+j} \qquad \beta = \frac{1}{1+r}$$

asset price

discount factor

dividends

note: $\quad x_{t+1} = A x_t$

$$x_{t+2} = A^2 x_t$$

$$\vdots$$

$$x_{t+j} = A^j x_t$$

$$\Rightarrow \quad p_t = \sum_{j=0}^{\infty} \beta^j G A^j x_t$$

$$= G \sum_{j=0}^{\infty} (\beta A)^j x_t$$

remark: $\displaystyle\sum_{j=0}^{\infty} \beta^j A^j = (I - \beta A)^{-1}$

this converges if the maximum eigenvalue of A is less than $\frac{1}{\beta}$ in modulus.

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Eigenvalue of A: a scalar (possibly complex) such that

$A x = \lambda x$ for some $x \neq 0$. such an x is called an eigenvector.

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evidently

$$(*) \qquad p_t = G (I - \beta A)^{-1} x_t$$

Other way to solve for p_t - "guess and verify" or "undetermined coefficients" method

recursive argument

$$p_t = \sum_{j=0}^{\infty} \beta^j y_{t+j}$$

$$= y_t + \beta \sum_{j=0}^{\infty} \beta^j y_{t+j+1}$$

$$p_t = y_t + \beta p_{t+1}$$

Guess $p_t = H x_t$ $\qquad x_t$ is state

$$p_t = y_t + \beta H x_{t+1} \quad , \quad x_{t+1} = A x_t$$
$$y_t = G x_t$$

$$H x_t = G x_t + \beta H A x_t$$

$$H = G + \beta H A$$

$$H (I - \beta A) = G$$

$$H(I-\beta A) = G$$

$$(*) \qquad H = G(I-\beta A)^{-1} .$$

thus $p_t = G(I-\beta A)^{-1} x_t$

Example:

$$y_{t+1} = .9\, y_t + .04\, y_{t-3}$$

Compute p_t for $\beta = .95$.

Probability theory:

$i = 1, \ldots, I$ there are I possible "states of the world"

$\pi_i = \text{Prob}\{\text{state of world is } i\}$

$\pi_i \geq 0 , \sum_{i=1}^{I} \pi_i = 1$

A random variable - a vector

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_I \end{pmatrix}$$

where y_i is value in state of the world i.

mathematical expectation (mean)

$$E\, y = \sum_{i=1}^{I} \pi_i y_i = \pi \cdot y$$

inner product

Here are some Laplace transform tables from Wikipedia:

Laplace transform - Wikipedia, the free encyclopedia - Mozilla Firefox

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http://en.wikipedia.org/wiki/Laplace_transform

	Time domain	's' domain	Comment
Linearity	$af(t)+bg(t)$	$aF(s)+bG(s)$	Can be proved using basic rules of integration.
Frequency differentiation	$tf(t)$	$-F'(s)$	F' is the first derivative of F.
Frequency differentiation	$t^n f(t)$	$(-1)^n F^{(n)}(s)$	More general form, (n)th derivative of F(s).
Differentiation	$f'(t)$	$sF(s)-f(0)$	f is assumed to be a differentiable function, and its derivative is assumed to be of exponential type. This can then be obtained by integration by parts
Second Differentiation	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	f is assumed twice differentiable and the second derivative to be of exponential type. Follows by applying the Differentiation property to $f'(t)$.
General Differentiation	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	f is assumed to be n-times differentiable, with n^{th} derivative of exponential type. Follow by mathematical induction.
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	
Integration	$\int_0^t f(\tau)\,d\tau=(u*f)(t)$	$\frac{1}{s}F(s)$	$u(t)$ is the Heaviside step function. Note $(u*f)(t)$ is the convolution of $u(t)$ and $f(t)$.
Scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right)$	

Done

Lecture 1 - ... OneNote N... Laplace tran... 10:43 AM

Laplace transform - Wikipedia, the free encyclopedia - Mozilla Firefox

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http://en.wikipedia.org/wiki/Laplace_transform

ID	Function	Time domain $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace s-domain $F(s) = \mathcal{L}\{f(t)\}$	Region of convergence
1	ideal delay	$\delta(t-\tau)$	$e^{-\tau s}$	
1a	unit impulse	$\delta(t)$	1	all s
2	delayed *n*th power with frequency shift	$\frac{(t-\tau)^n}{n!}e^{-\alpha(t-\tau)} \cdot u(t-\tau)$	$\frac{e^{-\tau s}}{(s+\alpha)^{n+1}}$	$\mathrm{Re}\{s\} > -\alpha$
2a	*n*th power (for integer *n*)	$\frac{t^n}{n!} \cdot u(t)$	$\frac{1}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$ $(n > -1)$
2a.1	*q*th power (for complex *q*)	$\frac{t^q}{\Gamma(q+1)} \cdot u(t)$	$\frac{1}{s^{q+1}}$	$\mathrm{Re}\{s\} > 0$ $(\mathrm{Re}\{q\} > -1)$
2a.2	unit step	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
2b	delayed unit step	$u(t-\tau)$	$\frac{e^{-\tau s}}{s}$	$\mathrm{Re}\{s\} > 0$
2c	ramp	$t \cdot u(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$
2d	*n*th power with frequency shift	$\frac{t^n}{n!}e^{-\alpha t} \cdot u(t)$	$\frac{1}{(s+\alpha)^{n+1}}$	$\mathrm{Re}\{s\} > -\alpha$
2d.1	exponential decay	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$	$\mathrm{Re}\{s\} > -\alpha$
3	exponential approach	$(1-e^{-\alpha t}) \cdot u(t)$	$\frac{\alpha}{s(s+\alpha)}$	$\mathrm{Re}\{s\} > 0$
4	sine	$\sin(\omega t) \cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$	$\mathrm{Re}\{s\} > 0$
5	cosine	$\cos(\omega t) \cdot u(t)$	$\frac{s}{[illegible]}$	$\mathrm{Re}\{s\} > 0$

Done

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http://en.wikipedia.org/wiki/Laplace_transform

5	cosine	$\cos(\omega t)\cdot u(t)$	$\frac{s}{s^2+\omega^2}$	$\operatorname{Re}\{s\}>0$
6	hyperbolic sine	$\sinh(\alpha t)\cdot u(t)$	$\frac{\alpha}{s^2-\alpha^2}$	$\operatorname{Re}\{s\}>\lvert\alpha\rvert$
7	hyperbolic cosine	$\cosh(\alpha t)\cdot u(t)$	$\frac{s}{s^2-\alpha^2}$	$\operatorname{Re}\{s\}>\lvert\alpha\rvert$
8	Exponentially-decaying sine wave	$e^{\alpha t}\sin(\omega t)\cdot u(t)$	$\frac{\omega}{(s-\alpha)^2+\omega^2}$	$\operatorname{Re}\{s\}>\alpha$
9	Exponentially-decaying cosine wave	$e^{\alpha t}\cos(\omega t)\cdot u(t)$	$\frac{s-\alpha}{(s-\alpha)^2+\omega^2}$	$\operatorname{Re}\{s\}>\alpha$
10	*n*th root	$\sqrt[n]{t}\cdot u(t)$	$s^{-(n+1)/n}\cdot\Gamma\left(1+\frac{1}{n}\right)$	$\operatorname{Re}\{s\}>0$
11	natural logarithm	$\ln\left(\frac{t}{t_0}\right)\cdot u(t)$	$-\frac{t_0}{s}\,[\,\ln(t_0 s)+\gamma\,]$	$\operatorname{Re}\{s\}>0$
12	Bessel function of the first kind, of order *n*	$J_n(\omega t)\cdot u(t)$	$\frac{\omega^n\left(s+\sqrt{s^2+\omega^2}\right)^{-n}}{\sqrt{s^2+\omega^2}}$	$\operatorname{Re}\{s\}>0$ $(n>-1)$
13	Modified Bessel function of the first kind, of order *n*	$I_n(\omega t)\cdot u(t)$	$\frac{\omega^n\left(s+\sqrt{s^2-\omega^2}\right)^{-n}}{\sqrt{s^2-\omega^2}}$	$\operatorname{Re}\{s\}>\lvert\omega\rvert$
14	Bessel function of the second kind, of order 0	$Y_0(\alpha t)\cdot u(t)$	$-\frac{2\sinh^{-1}(s/\alpha)}{\pi\sqrt{s^2+\alpha^2}}$	$\operatorname{Re}\{s\}>0$
15	Modified Bessel function of the second kind, of order 0	$K_0(\alpha t)\cdot u(t)$		
16	Error function	$\operatorname{erf}(t)\cdot u(t)$	$\frac{e^{s^2/4}\left(1-\operatorname{erf}(s/2)\right)}{s}$	$\operatorname{Re}\{s\}>0$

Laplace transform - Wikipedia, the free encyclopedia - Mozilla Firefox

http://en.wikipedia.org/wiki/Laplace_transform

Formal definition

[edit]

The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, defined by:

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^\infty e^{-st} f(t)\,dt.$$

The parameter s is a complex number:

$s = \sigma + i\omega$, with real numbers σ and ω.

The meaning of the integral depends on types of functions of interest. For functions that decay at infinity or are of exponential type, it can be understood as a (proper) Lebesgue integral. However, for many applications it is necessary to regard it as a conditionally convergent improper integral at ∞. Still more generally, the integral can be understood in a weak sense, and this is dealt with below.

One can define the Laplace transform of a finite Borel measure μ by the Lebesgue integral[8]

$$(\mathcal{L}\mu)(s) = \int_{[0,\infty)} e^{-st} d\mu(t).$$

An important special case is where μ is a probability measure or, even more specifically, the Dirac delta function. In operational calculus, the Laplace transform of a measure is often treated as though the measure came from a distribution function f. In that case, to avoid potential confusion, one often writes

$$(\mathcal{L}f)(s) = \int_{0^-}^\infty e^{-st} f(t)\,dt$$

where the lower limit of 0^- is short notation to mean

$$\lim_{\varepsilon \to 0+} \int_{-\varepsilon}^\infty.$$

This limit emphasizes that any point mass located at 0 is entirely captured by the Laplace transform. Although with the Lebesgue integral, it is not necessary to take such a limit, it does appear more naturally in connection with the Laplace–Stieltjes transform.

Probability theory

[edit]

In pure and applied probability, the Laplace transform is defined by means of an expectation value. If X is a random variable with probability

Done

Kuhn Tucker

Wednesday, January 20, 2010
1:01 PM

Notes on Kuhn-Tucker —

(taken from Wikepedia, but changed "min" to "max")

Problem: $\max f(x)$ s.t. $g(x) \geq 0$

$h(x) = 0$

x a scalar

Form:

$$L = f(x) + \mu g(x) + \lambda h(x)$$

First order necessary conditions:

$$f'(x) + \mu g'(x) + \lambda h'(x) = 0$$

$$g(x) \geq 0 \qquad , \mu \geq 0$$

$$\mu g(x) = 0 \qquad ; \quad h(x) = 0$$

e.g. $g(x) = x \iff x \geq 0$ non negativity constraint on x.

$$f'(x) + \lambda h'(x) + \mu = 0 \quad , \quad \mu \cdot x = 0 \quad \Rightarrow$$

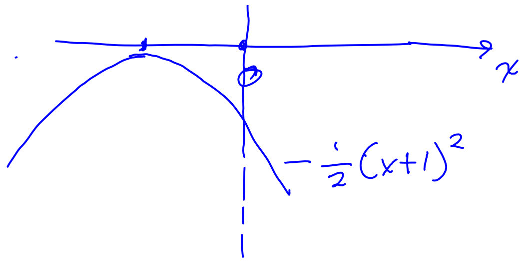
$\Rightarrow$

$$f'(x) + \lambda h'(x) \leq 0 \quad , = \text{if } x > 0$$

because $\mu x = 0$ and $\mu \geq 0$.

Example:

Example:



$f'(x) = -(x+1) = 0$

$\Rightarrow x + 1 = 0$

$x = -1$

$$\max \ -\frac{1}{2}(x+1)^2 \quad \text{s.t. } x \geq 0$$

$$L = -\frac{1}{2}(x+1)^2 + \mu x$$

FONC: $x: \ -(x+1) + \mu = 0$

$$-(1+x) + \mu = 0 \qquad (*)$$

or

$$-(x+1) \leq 0 \ , \ = 0 \text{ if } x > 0$$

$$\mu \cdot x = 0 \ ,$$

Soln is $x = 0$ and

$$-(1+x) + \mu = 0$$

$$-1 + \mu = 0 \Rightarrow \mu = 1 .$$

$$\mu x = 0 .$$

So for this problem, we can write the Kuhn-Tucker conditions in short hand as

$$-(x+1) \leq 0 \ , \ = 0 \text{ if } x > 0$$

$$\Rightarrow x = 0 .$$

Z-transform

Sunday, January 24, 2010
11:33 AM

Z-transform

Unilateral Z-transform

$\{x_n\}_{n=0}^{\infty}$ a one-sided sequence

$\{x_n\} \leftrightarrow X(z)$, $z \in \mathbb{C}$ space of complex variables

$$X(z) = \sum_{n=0}^{\infty} x_n z^n$$, z^n is a "generating function"

differentiate w.r.t. z =>

$$X'(z) = \sum_{n=0}^{\infty} n x_n z^{n-1} = z^{-1} \sum_{n=0}^{\infty} n x_n z^n$$

or

$$z X'(z) = \sum_{n=0}^{\infty} n x_n z^n$$

so

$$z X'(z) \leftrightarrow \{n x_n\}$$

Example:

let $x_n = \lambda^n$, $\lambda \in \mathbb{R}$

$$X(z) = \sum_{n=0}^{\infty} \lambda^n z^n = \frac{1}{1-\lambda z}$$

$$X'(z) = \frac{\lambda}{(1-\lambda z)^2}$$

$$z X'(z) = \frac{\lambda z}{(1-\lambda z)^2}$$

$$\therefore \{n \lambda^n\} \leftrightarrow \frac{\lambda z}{(1-\lambda z)^2}$$

Application 1: Computes present value of $\{n\lambda^n\}_{n=0}^{\infty}$:

$$v(\beta) = \sum_{n=0}^{\infty} n \lambda^n \beta^n$$

$$= \frac{\lambda \beta}{(1-\lambda\beta)^2} .$$

Application 2: let $\{p_j\}_{j=0}^{\infty}$ with $p_j \geq 0$, $\sum_{j=0}^{\infty} p_j = 1$

be a **probability density** for a nonnegative random variable N taking values $j = 0, 1, 2, 3, \ldots$. Here

$$p_j = \text{Prob}(N = j) .$$

The **probability generating function** is

$$P(z) = \sum_{j=0}^{\infty} p_j z^j .$$

here $\text{Prob}\{N = j\}$ = coefficient on j^{th} power of z.

The **mean** of N is defined as

$$\mu = \sum_{j=0}^{\infty} j\, p_j .$$

Exercise: Use the probability generating function to compute μ.

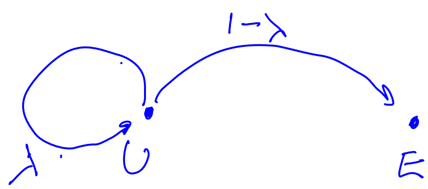
Note.

$$z\,p'(z) = \sum_{j=0}^{\infty} j\,p_j\,z^j$$

evaluate at $z=1$ to get

$$\boxed{p'(1) = \sum_{j=0}^{\infty} j \cdot p_j \; = \mu}$$

example: waiting time distribution.
Each period, an unemployed person finds a job with probability $0<(1-\lambda)<1$. With probability λ, each period the previously unemployed person remains unemployed. So the picture is



Let N be the random variable "length of time it takes to find a job", with $N=1$ meaning that it takes 1 period.

Let $p_j =$ probability $(N=j)$

Evidently,

$p_1 = (1-\lambda)$ succeed

$p_2 = \lambda(1-\lambda)$ fail, succeed

$p_3 = \lambda^2(1-\lambda)$ fail, fail, succeed

$p_4 = \lambda^3(1-\lambda)$ $\underbrace{\text{fail, fail, fail}}_{\lambda^3}$, succeed

$\vdots$

Thus $$P(z) = (1-\lambda)\sum_{j=1}^{\infty} \lambda^{j-1} z^j$$ ← note the sum starts at $j=1$.

$j=1$ starts at $j=1$.

$p_0 = 0$

The mean waiting time is $z P'(z)$:

But $$P(z) = \frac{(1-\lambda) z}{(1-\lambda z)}$$

$$P'(z) = \frac{(1-\lambda)(1-\lambda z) + \lambda (1-\lambda) z}{(1-\lambda z)^2}$$

$$= \frac{(1-\lambda)\cdot 1}{(1-\lambda z)^2}$$

$$z P'(z) = \frac{(1-\lambda)}{(1-\lambda z)^2}$$

$$\boxed{P'(1) = \frac{1}{(1-\lambda)}}$$ = mean waiting time for a geometric distribution

Bubbles

Monday, February 01, 2010

10:16 AM

More asset pricing.

$$p_t = \sum_{j=0}^{\infty} \beta^j y_{t+j} \quad , \quad \beta = \frac{1}{1+r}$$

For example, $r = .05 \Rightarrow \beta = \frac{1}{1.05} \approx .95$

$$p_t = y_t + \beta y_{t+1} + \beta^2 y_{t+2} + \cdots$$

$$p_{t+1} = \qquad y_{t+1} + \beta y_{t+2} + \beta^2 y_{t+3} + \cdots$$

$$\beta p_{t+1} = \qquad \beta y_{t+1} + \beta^2 y_{t+2} + \cdots$$

$$\therefore \quad p_t - \beta p_{t+1} = y_t$$

or

$$(*) \qquad p_t = y_t + \beta p_{t+1}$$

$(*)$ is a difference equation.

Take a model of y_t:

$$X_{t+1} = A x_t$$

$$y_t = G x_t$$

we have shown

$$(+) \qquad p_t = G(I - \beta A)^{-1} x_t$$

is a solution of the difference equation (*)

Question: is (†) the unique solⁿ of (*)?

Answer: no. To show this, consider the special case

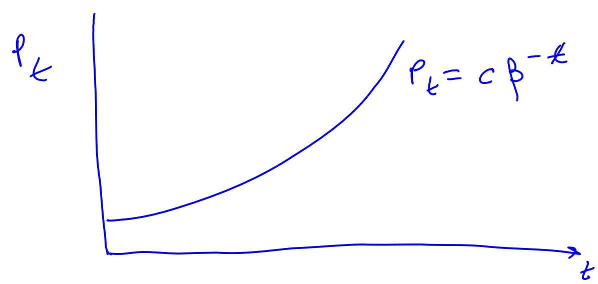
(**) $p_t = \beta p_{t+1}$

here $y_t = 0$. Clearly $p_t = 0$ is a solⁿ of (**). But so is

$$p_t = c\left(\frac{1}{\beta}\right)^t = c\beta^{-t}$$

for any constant c.

This is a pure bubble, because $\beta < 1$.



The general solⁿ of (*) is

$$p_t = G(I - \beta A)^{-1} x_t + c\beta^{-t}$$

for any c. The solⁿ is not unique.

$c\beta^{-t}$ is the bubble.

"prices rise because the are expected to rise"

Homework 1

Tuesday, January 26, 2010
12:05 PM

Homework number 1.

1. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of real numbers. Let $\beta \in (0,1)$ be a constant discount factor.

The <u>present value</u> of $\{x_n\}_{n=0}^{\infty}$ is

$$PV(\beta) = \sum_{n=0}^{\infty} \beta^n x_n \quad .$$

Please compute the present values of the following sequences as functions of β:

a. $x_n = A\lambda^n$, $n \geq 0$, $|\lambda| < \frac{1}{\beta}$

b. $x_n = A n \lambda^n$, $n \geq 0$, $|\lambda| < \frac{1}{\beta}$

c. $x_n = A_0 n \lambda^n + A_1 n^2 \delta^n$, $n \geq 0$, $|\lambda| < \frac{1}{\beta}$, $|\delta| < \frac{1}{\beta}$

d. $x_n = A n^3 \lambda^n$, $n \geq 0$, $|\lambda| < \frac{1}{\beta}$

2. A stock price P_t obeys the following formula:

$$(*) \qquad p_t = y_t + \beta\, p_{t+1}$$

price today — dividend today — discount factor — price tomorrow

The dividend obeys

$$\underset{n\times 1}{x_{t+1}} = \underset{n\times n}{A}\ \underset{n\times 1}{x_t}$$

, where maximum eigenvalue is less than $\frac{1}{\beta}$ in modulus

$$\underset{1\times 1}{y_t} = \underset{1\times n}{G}\ \underset{n\times 1}{x_t}$$

A friend of yours guesses that (*) implies that the following equation for p_t must be true that

$$p_t = H x_t + c\,\lambda^t$$

for a matrix $H_{1\times n}$ and scalars c and λ.

a. Get as far as you can in finding formulas for H, λ, and c. (Hint: use the guess and verify method.)

b. Is H unique? Is λ unique? Is c unique?

3. "Finding the state is an art"

A dividend process obeys the law of motion

$$y_{t+1} = \lambda_0 + \lambda_1 y_t + \lambda_2 y_{t-1}$$

The price of a stock obeys

$$p_t = \sum_{j=0}^{\infty} \beta^j \, y_{t+j} \,, \qquad \beta \in (0,1)$$

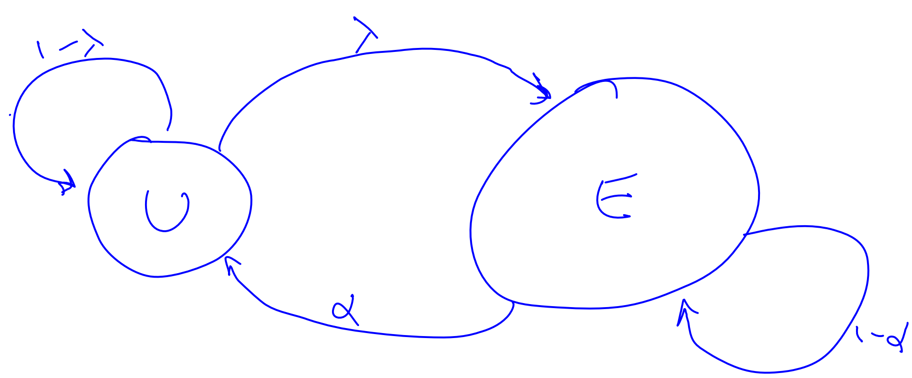
Please give another formula for p_t of the form

$$p_t = a_0 + a_1 y_t + a_2 y_{t-1}$$

Please tell how to compute a_0, a_1, and a_2.

4. There are two states U (for unemployment) and E (for employment). With probability $\lambda \in (0,1)$ a person unemployed today becomes employed tomorrow. With probability $\alpha \in (0,1)$ a person employed today becomes unemployed tomorrow.

So the situation is



a. Let $N \geq 1$ be the number of periods elapsed until a currently unemployed person becomes employed.

Let $p_n = \text{Prob}\{N = n\}$

Please compute p_n. Please compute

$$E\, N = \sum_{n=1}^{\infty} n\, p_n$$

b. Let $M \geq 1$ be the number of periods elapsed until a currently employed person loses his or her job. Please compute $q_n = \text{Prob}\{M = n\}$.

Please compute $E\, M = \sum_{n=0}^{\infty} q_n n$.

c. Please compute the fraction of time an infinitely lived person can expect to be unemployed, and the fraction that he or she can expect to be employed.

Permanent income model

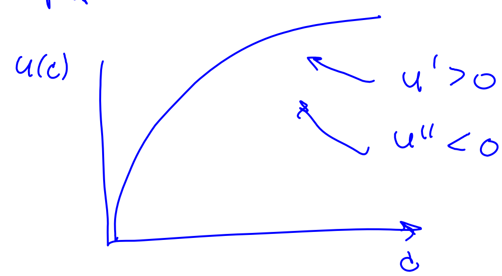
Sunday, January 24, 2010
11:08 AM

Permanent income model. (Friedman)

$$\max_{\{c_{t+j}\}_{j=0}^{\infty}} \sum_{j=0}^{\infty} \beta^j u(c_{t+j}), \quad \beta \in (0,1)$$

$$\text{s.t.} \quad \sum_{j=0}^{\infty} \beta^j (\underbrace{y_{t+j}}_{\text{labor income}} - \underbrace{c_{t+j}}_{\text{consumption}}) + \underbrace{A_t}_{\text{assets}} = 0$$

$$u'(c) > 0, \quad u''(c) < 0$$



Formulate as a Lagrangian:

$$L = \sum_{j=0}^{\infty} \beta^j u(c_t) + \lambda \left[\sum_{j=0}^{\infty} \beta^j (y_{t+j} - c_{t+j}) + A_t \right]$$

Use Feynman trick ~ "differentiate under integral"

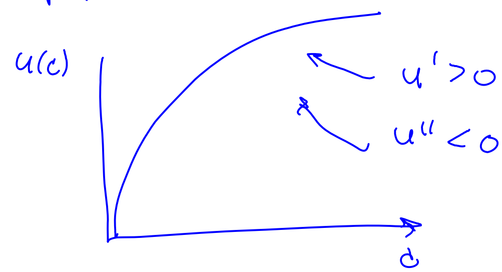
FONC:

$$c_{t+j}: \quad \beta^j u'(c_{t+j}) - \lambda \beta^j = 0, \quad j = 0, \ldots, \infty$$

s.t. $\sum_{j=0}^{\infty} \beta^j (y_{t+j} - c_{t+j}) + A_t = 0$

labor income   consumption   assets

$u'(c) > 0, \quad u''(c) < 0$



Formulate as a Lagrangian:

$$L = \sum_{j=0}^{\infty} \beta^j u(c_t)$$

$$+ \lambda \left[\sum_{j=0}^{\infty} \beta^j (y_{t+j} - c_{t+j}) + A_t \right]$$

Use Feynman trick ~ "differentiate under integral"

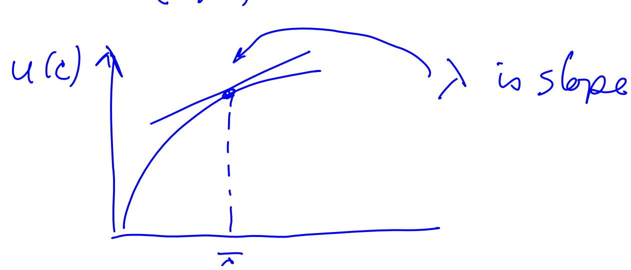
FONC:

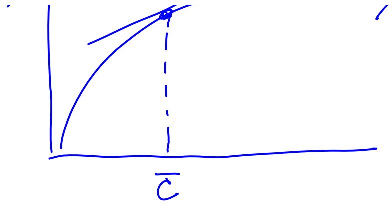
c_{t+j}: $\quad \beta^j u'(c_{t+j}) - \lambda \beta^j = 0, \quad j = 0, \dots, \infty$

$$\Rightarrow \quad u'(c_{t+j}) = \lambda \quad \forall j$$

a constant

$$\Rightarrow \quad c_{t+j} = u'^{-1}(\lambda) \quad \text{— a constant.}$$





Now find λ:

FONC w.r.t. λ (minimize wrt λ)

$$\sum_{j=0}^{\infty} \beta^j \left(y_{t+j} - c_{t+j} \right) + A_t = 0$$

or

$$\sum_{j=0}^{\infty} \beta^j y_{t+j} + A_t = \sum_{j=0}^{\infty} \beta^j c_{t+j}$$

But $c_{t+j} = \bar{c}$ (from above) $\Rightarrow$

$$\sum_{j=0}^{\infty} \beta^j y_{t+j} + A_t = \bar{c} \sum_{j=0}^{\infty} \beta^j$$

$$= \frac{\bar{c}}{(1-\beta)}$$

or

$$\bar{c} = (1-\beta) \left[\sum_{j=0}^{\infty} \beta^j y_{t+j} + A_t \right]$$

marginal propensity to consume out of wealth.

"human wealth"

financial wealth

Note: $\beta = \frac{1}{1+r}$ net interest rate

$$\Rightarrow (1-\beta) = \frac{r}{1+r}$$

so $\bar{c} = \frac{r}{(1+r)} \left(\sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} + A_t \right)$

Remark: manipulate adjacent budget constraints:

t $$\sum_{j=0}^{\infty} \beta^j y_{t+j} + A_t = \sum_{j=0}^{\infty} \beta^j c_{t+j}$$

$t+1$ $$\sum_{j=0}^{\infty} \beta^j y_{t+j+1} + A_{t+1} = \sum_{j=0}^{\infty} \beta^j c_{t+j+1}$$

multiply second equation by β & subtract second from first

$\Rightarrow$

$$y_t + A_t - \beta A_{t+1} = c_t$$

or

$$A_{t+1} = \beta^{-1} (A_t + y_t - c_t)$$

or

$$A_{t+1} = (1+r) (A_t + \underbrace{y_t - c_t}_{\text{savings}})$$

Permanent Income model, 2

Sunday, January 24, 2010
7:59 PM

When asked "what is the most powerful force in nature?"
Albert Einstein answered "compound interest!"

Finite horizon permanent income or life-cycle model.

A household has exogenous labor income process $\{y_t\}_{t=0}^{T}$ and chooses $\{c_t\}_{t=0}^{T}$ to maximize

$$\sum_{t=0}^{T} \beta^t u(c_t) \qquad , u' > 0, u'' < 0$$

$$\beta \in (0,1)$$

s.t.

$$A_{t+1} = R(A_t + y_t - c_t), \quad t = 0, 1, \ldots, T$$

$$R > 1$$

R is gross rate of return on assets

$R \sim (1+\rho)$
net rate of return

A_0 is given

Form Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t \Big\{ u(c_t)$$

$$+ \lambda_t \left[R(A_t + y_t - c_t) - A_{t+1} \right] \Big\}$$

one for each $t = 0, \ldots, T$

Note: we impose $A_{T+1} \geq 0$

FONC:

c_t : $\beta^t u'(c_t) - \beta^t \lambda_t R = 0$

A_t : $\lambda_t R - \lambda_{t-1} \beta^{-1} = 0$, $t = 1, \ldots, T$

A_{T+1} : $-\beta^T \lambda_T \leq 0$, $= 0$ if $A_{T+1} > 0$

or $\beta^T \lambda_T A_{T+1} = 0$

$\Rightarrow A_{T+1} = 0$ if

FONC $\Rightarrow$ for c_t $u'(c_t) = \beta R \, u'(c_{t+1})$, $t = 0, \ldots, T-1$

budget constraint $A_t = R^{-1} A_{t+1} + c_t - y_t$

$\Rightarrow A_t = \sum_{j=0}^{T-t} R^{-j} (c_{t+j} - y_{t+j})$

$A_{T+1} = 0$

$A_T = c_T - y_T$

$A_{T-1} = R^{-1} (c_T - y_T) + (c_{T-1} - y_{T-1})$

$\vdots$

Special case : $T = 1$

$\sum_{t=0}^{1} \beta^t u(c_t) = u(c_0) + \beta u(c_1) = U$

$$dU = u'(c_0)\,dc_0 + \beta u'(c_1)\,dc_1 = 0$$

$$\Rightarrow \frac{dc_1}{dc_2} = -\frac{u'(c_0)}{\beta u'(c_1)}$$

$$\max_{c_0, c_1, A_2} \; u(c_0) + \beta u(c_1)$$

s.t. $A_1 = R(A_0 + y_0 - c_0)$

$A_2 = R(A_1 + y_1 - c_1)$

FONC:

$$u'(c_0) = \beta R u'(c_1)$$

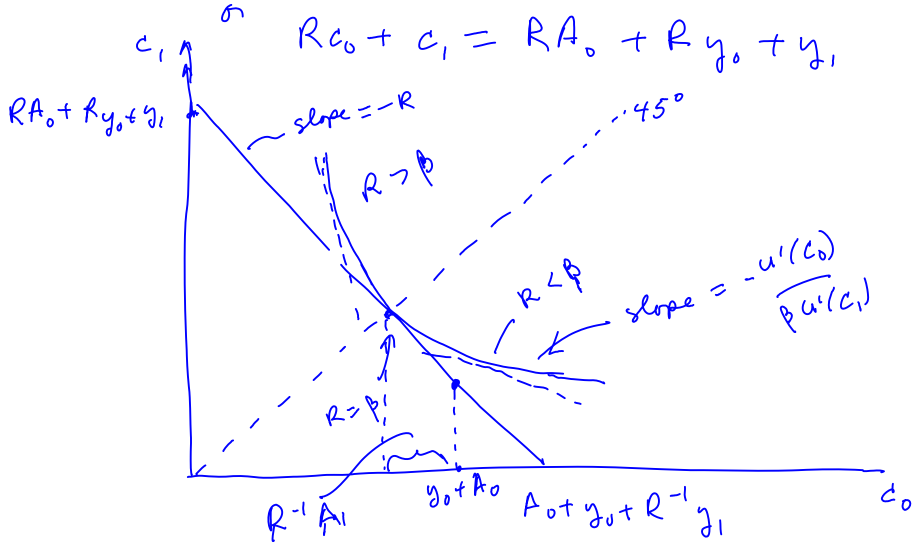
$$A_0 = R^{-1}(c_1 - y_1) + (c_0 - y_0)$$

or

$$c_0 + R^{-1} c_1 = A_0 + y_0 + R^{-1} y_1$$

or

$$R c_0 + c_1 = R A_0 + R y_0 + y_1$$



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Suppose $u(c_t) = \ln c_t \Rightarrow u'(c_t) = 1/c_t$

then $u'(c_t) = \beta R\, u'(c_{t+1})$

$$\Rightarrow \frac{1}{c_t} = \beta R \frac{1}{c_{t+1}}$$

or

$$c_{t+1} = \beta R\, c_t, \qquad t = 0, \ldots, T-1$$

Examples:

set $T = +\infty$, $y_t = \delta^t y_0$, $c_t = (\beta R)^t c_0$

$A_0 = 0$. Then

$$0 = \sum_{j=0}^{\infty} R^{-j}\left((\beta R)^j c_0 - \delta^j y_0 \right)$$

or

$$c_0 \sum_{j=0}^{\infty} \beta^j = \sum_{j=0}^{\infty} (\delta R^{-1})^j \, y_0$$

or

$$\frac{1}{1-\beta} c_0 = \frac{1}{1-\delta/R} y_0$$

Assume $\left|\left(\delta/R\right)\right| < 1$

Example 2: $T < +\infty$.

$c_t = (\beta R)^t c_0$, $y_t = \delta^t y_0$, $A_0 = 0$

$$0 = \sum_{j=0}^{T} R^{-j}\left((\beta R)^j c_0 - \delta^j y_0 \right)$$

$\left|\delta/R\right| < 1$

$$\frac{(1-\beta^{T+1})}{1-\beta^T} c_0 = \left[\frac{(1-(\delta/R)^{T+1})}{(1-(\delta/R))} \right] y_0$$

Example 3: $T = 70$

$$c_t = (\beta R)^t c_0 \; ; \quad y_t = \begin{cases} \delta^t y_0 & , t = 0, \dots, 50 \\ 0 & , t = 51, \dots, 90 \end{cases}$$

"0" stands for age 20

"70" stands for age 90 , $A_0 = 0$, $A_{T+1} = 0$

$\sum^{70} R^{-t}(\beta R)^t \ldots \quad \sum^{50} \delta^t R^{-t} \ldots$

$$\sum_{t=0} R^{?} (\beta R)^{?} c_0 = \sum_{t=0} \delta^{?} R^{?} y_0$$

$$\Rightarrow \sum_{t=0}^{70} \beta^t c_0 = \sum_{t=0}^{50} \left(\delta/R\right)^t y_0$$

$$c_0 \frac{1-\beta^{71}}{1-\beta} = \frac{1-\left(\delta/R\right)^{51}}{1-\left(\delta/R\right)} y_0$$

$$c_0 = \frac{(1-\beta)}{(1-\beta^{71})} \frac{\left(1-\left(\delta/R\right)^{51}\right)}{\left(1-\delta/R\right)}$$

this OK even if $\left|\frac{\delta}{R}\right| > 1$

e.g. $(\beta R) = 1 \Rightarrow c_t = c_0 \quad \forall\, t = 0, 1, \ldots, 70$

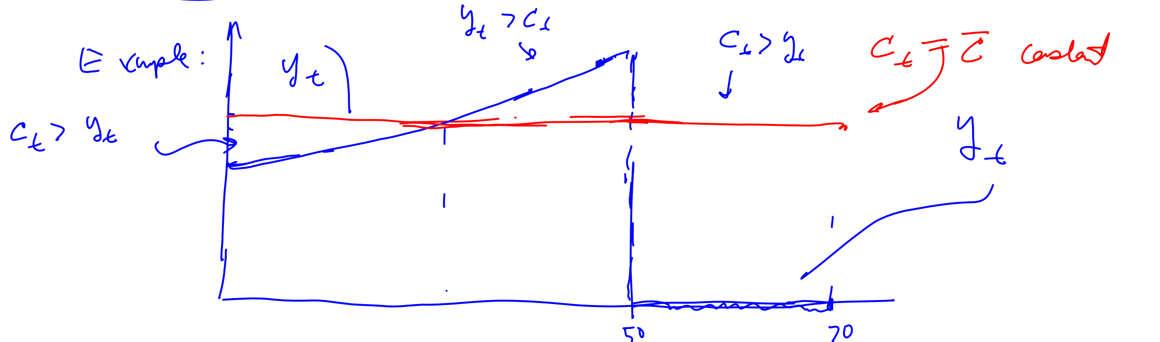
Interesting case: $\beta R = 1$, $R > 1$, $\frac{\delta}{R} < 1$, $\delta > 1$.

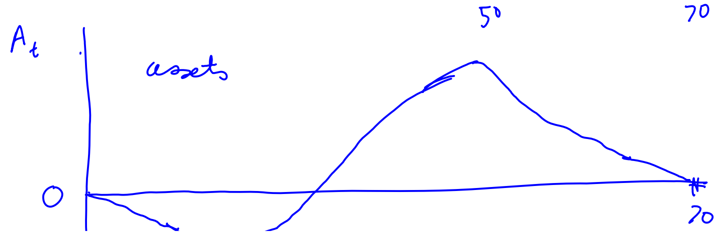
Note:

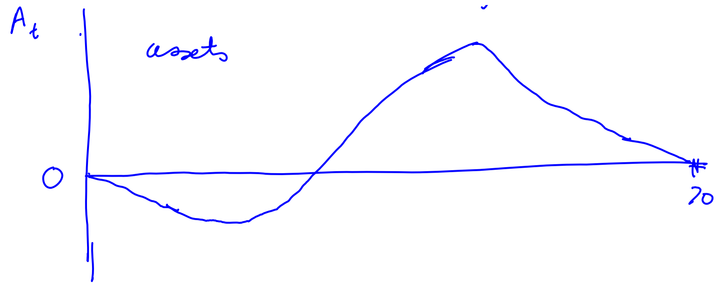
$$\begin{array}{r l}
 & 1 + b + \cdots + b^T \\
1-b \,\big) & 1 - b^{T+1} \\
 & \underline{1-b} \\
 & b \\
 & \underline{b - b^2} \\
 & b^2 \\
 & \ddots \\
 & b^T \\
 & \underline{b^T - b^{T+1}} \\
 & b^{T+1}
\end{array}$$

OK for $|b| > 1$

Example:







More on permanent income model.

Consider infinite horizon problem with

no-borrowing allowed. Problem becomes

$$L = \sum_{t=0}^{\infty} \beta^t \{ u(c_t) + \lambda_t [R(A_t + y_t - c_t) - A_{t+1}] \}$$

$\beta < 1$
$R > 1$

max s.t. $\underline{c_t \geq 0}$, $\underline{A_t \geq 0}$ no borrowing

FONC:

c_t: $\beta^t u'(c_t) - \beta^t \lambda_t R = 0$

$t = 0, 1, 2, \ldots$

A_t: $\beta^t \lambda_t R - \beta^{t-1} \lambda_{t-1} \leq 0$, $=$ if $A_{t+1} > 0$.

$t = 1, 2, \ldots,$

$\Rightarrow$ $\lambda_t = u'(c_t)/R$,

and $u'(c_{t-1}) \geq \beta R \, u'(c_t)$

$=$ if $c_{t-1} < A_{t-1} + y_{t-1}$

so that $A_t > 0$

$\Bigg\}$ $t = 1, 2, \ldots$

Digression: Explanation of first-order conditions. (Kuhn Tucker)

Constraints: multiplier

$$R(A_t + y_t - c_t) - A_{t+1} \leq 0 \; ; \qquad \beta^t \lambda_t \;, \lambda_t \geq 0$$

$$c_t \geq 0 \; ; \qquad \beta^t \alpha_t \;, \alpha_t \geq 0$$

$$A_{t+1} \geq 0 \; ; \qquad \beta^t \nu_{t+1} \;, \nu_{t+1} \geq 0$$

$$t \geq 0 \qquad \boxed{\begin{array}{l} \alpha_t c_t = 0 \\ \nu_{t+1} A_{t+1} = 0 \end{array}}$$

Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t \Big\{ u(c_t) + \lambda_t \left[R(A_t + y_t - c_t) - A_{t+1} \right] + \nu_{t+1} A_{t+1} + \alpha_t c_t \Big\}$$

FONC:

$$(*) \quad c_t: \; \beta^t u'(c_t) + \beta^t \alpha_t - \beta^t \lambda_t R = 0 \;, \; t \geq 0$$

$$(**) \quad A_t: \; \beta^t \lambda_t R - \beta^{t-1} \lambda_{t-1} + \beta^{t-1} \nu_t = 0 \;, \; t \geq 1$$

$$(*) \Rightarrow \lambda_t R = u'(c_t) + \alpha_t$$

where $\alpha_t \geq 0$ and $\alpha_t c_t = 0$

$$\alpha_t \geq 0 \Rightarrow \lambda_t R \geq u'(c_t)$$

<u>but</u>

Inada condition $u'(0) = +\infty \Rightarrow c_t > 0$

$$\Rightarrow \alpha_t = 0 \Rightarrow$$

$$\lambda_t R = u'(c_t) \Rightarrow \lambda_t = \frac{u'(c_t)}{R}$$

$(**)$ and $\lambda_t = \dfrac{u'(c_t)}{R} \Rightarrow$

$$R\beta u'(c_t) + R N_t = u'(c_{t-1}) \quad , t \geq 1$$

where $N_t \geq 0$ and $N_t A_t = 0$

Thus $R\beta u'(c_t) \leq u'(c_{t-1}) \quad , t \geq 1$

$= \text{ if } A_t > 0$, (because then $N_t = 0$)

End of digression on how to apply the Kuhn Tucker conditions.

Thus, along an optimal path either

$$* \begin{cases} u'(c_{t-1}) = \beta R\, u'(c_t) \; ; \text{ or} \\ u'(c_{t-1}) > \beta R\, u'(c_t) \text{ and } c_{t-1} = A_{t-1} + y_{t-1} \end{cases}$$

Examples: Assume $\beta \cdot R = 1$, $u(c) = \ln c$.

then (*) becomes:

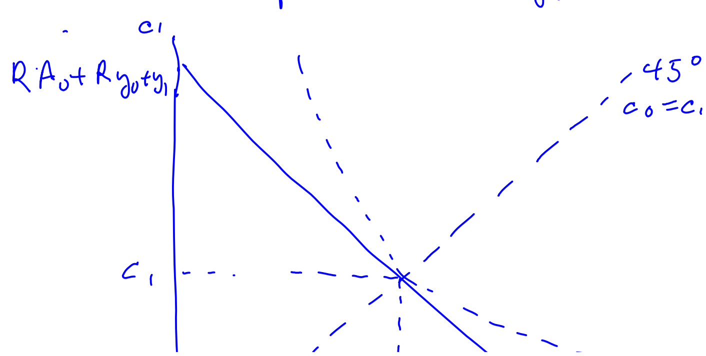
$$\frac{1}{c_{t-1}} = \frac{1}{c_t} \Rightarrow c_t = c_{t-1} \quad \text{or}$$

$$\frac{1}{c_{t-1}} > \frac{1}{c_t} \Rightarrow c_t > c_{t-1} \text{ and } c_{t-1} = A_{t-1} + y_{t-1}$$

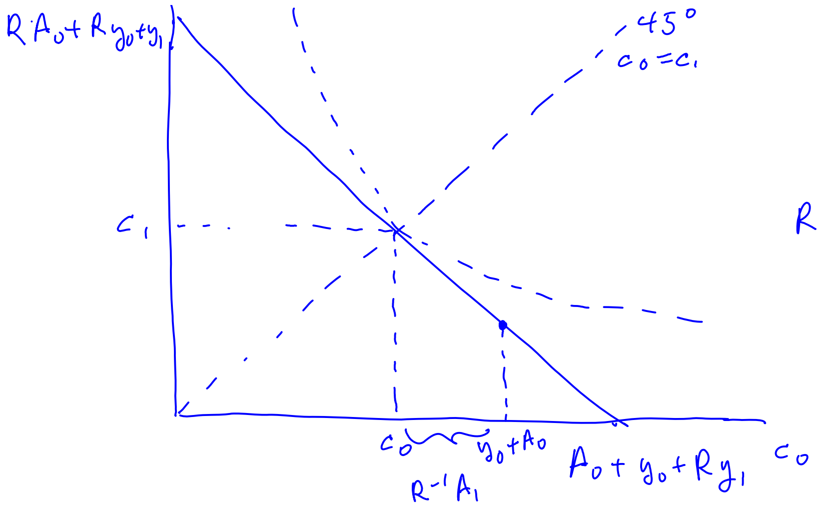
Example: $T=1$, no borrowing, $A_1 \geq 0$.

$u'(c_0) \geq \beta R\, u'(c_1)$

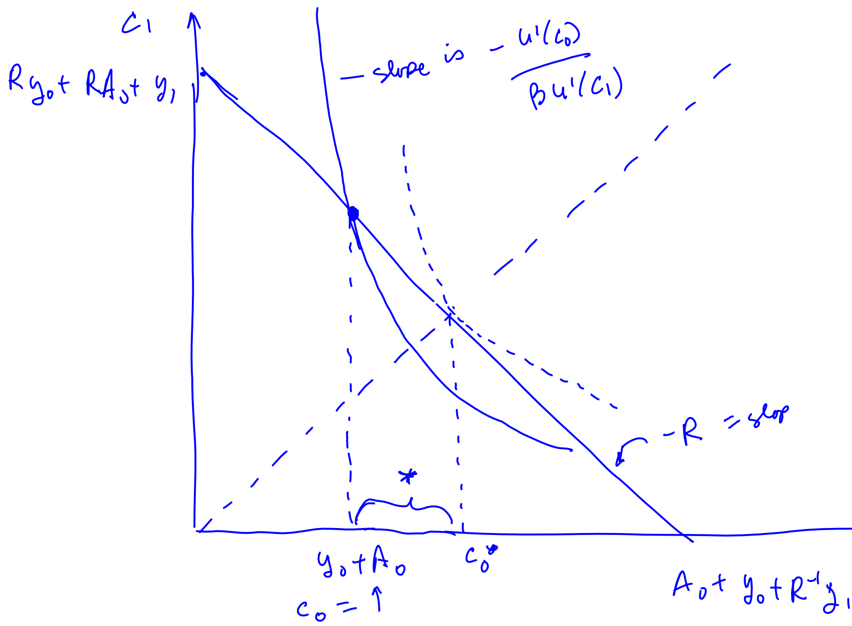
$= \text{ if } c_0 < A_0 + y_0 \text{ and } A_1 > 0$



$R = \beta$



$R = \beta$



$-\frac{u'(c_0)}{\beta u'(c_1)} < -R$ or

$u'(c_0) > \beta R u'(c_1)$

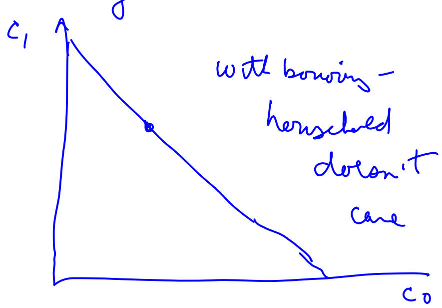
person would like to borrow

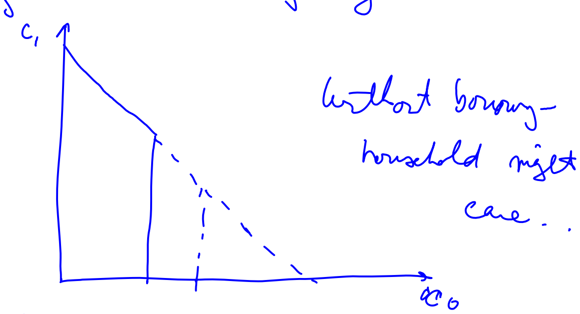
$R^{-1}A_p =$

$y_0 + A_0 - c_0^*$

but cannot.

Do "Ricardian" experiment - move endowment along budget line under borrowing & no-borrowing regimes.





Return to $T = +\infty$.

Example 1:

Example 1:

$$y_{t+1} = \delta y_t \Rightarrow y_t = \delta^t y_0 \qquad \delta > 1,$$

$$A_0 = 0. \qquad |\beta\delta| < 1$$

Then solution is

$$c_t = y_t \quad \forall\, t,$$

$$A_t = 0 \quad \forall\, t.$$

the person is always borrowing constrained.

Example 2:

$$y_t = \delta^t y_0 \quad , \quad 0 < \delta < 1$$

Then $c_t = \bar{c}$ $\forall\, t$ where $\bar{c}$ solves

$$\frac{1}{1-\beta}\,\bar{c} = y_0 \sum_{t=0}^{\infty} \delta^t \beta^t = \frac{y_0}{1-\beta\delta}$$

$$\bar{c} = (1-\beta)\,\frac{y_0}{1-\beta\delta} \qquad \forall\, t \geq 0$$

$$A_{t+1} = \beta^{-1}(A_t + y_t - \bar{c})$$

Note: $c_0 = \dfrac{(1-\beta)}{(1-\beta\delta)}\, y_0 < \dfrac{1-\beta}{1-\beta}\, y_0 = y_0$

because $0 < \delta < 1$.

$\therefore$ for $t = 0$, the individual saves

$$A_1 = R(y_0 - c_0) \qquad \text{because } A_0 = 0$$

In the limit as $t \to \infty$ $y_t \to 0$

but $c_t = \bar{c}$, $\therefore$

$$A_{t+1} = R(A_t + 0 - \bar{c})$$

and $A_{t+1} = A_t$ — seek a steady state $\bar{A}$

we get

$$R\,\bar{c} = (R-1)\,\bar{A}$$

$$\bar{c} = \frac{R-1}{R} A \qquad , \frac{R-1}{R} = \frac{r}{1+r}$$

$$\bar{c} = \frac{r}{1+r} A$$ live off of savings eventually

Example 3: (Remember Einstein's answer)

$$y_t = \begin{cases} y_0 & \text{for } t = 0, \dots, T_w . \\ 0 & \text{for } t > T_w \end{cases}$$

$A_t > 0$ constraint never binds.

$c_t = \bar{c} \quad \forall t.$ (from our FONC's (*) above)

$$\bar{c} = (1-\beta) \sum_{t=0}^{T} \beta^t y_0$$

$$= \cancel{(1-\beta)} \frac{(1-\beta^{T_w+1})}{\cancel{(1-\beta)}} y_0$$

$$\bar{c} = (1-\beta^{T_w+1})\, y_0$$

$$A_{t+1} = R\left[A_t + y_0 - \bar{c}\right] \quad \text{for } t = 0, \dots, T_w$$

$$A_{t+1} = R\left[A_t - \bar{c}\right] \quad \text{for } t > T_w$$

or

$$\bar{c} = \left(\frac{R-1}{R}\right) \bar{A} \qquad \text{for } t > T_w$$

Saving while early in life:

$$y_0 - \bar{c} = \beta^{T_w + 1} \cdot y_0$$

Stochastic asset prices

Wednesday, February 03, 2010
1:35 PM

Asset pricing of random dividend stream under risk-neutrality.

Assume

$$\underset{n\times 1}{x_{t+1}} = \underset{n\times n}{A}\,\underset{n\times 1}{x_t} + \underset{n\times m}{C}\,\underset{m\times 1}{\tilde{\varepsilon}_{t+1}}$$

$$\underset{1\times 1}{y_t} = \underset{1\times n}{G}\,\underset{n\times 1}{x_t}$$

where $\{\varepsilon_{t+1}\}$ is an independent & identically distributed sequence of random variables that are Gaussian – i.e., multivariate normal with mean $0_{m\times 1}$ and covariance matrix $I_{m\times m}$

$$f(\varepsilon_{t+1}) \propto \exp\left(-\tfrac{1}{2}\varepsilon_{t+1}' I \varepsilon_{t+1}\right)$$

Gaussian density

Notice that

$$E_t x_{t+1} = E_t\left(A x_t + C\varepsilon_{t+1}\right)$$
$$= A x_t + C E_t \varepsilon_{t+1}$$
$$= A x_t + C\cdot 0$$

Because $E_t \varepsilon_{t+1} \equiv \int f(\varepsilon_{t+1})\, d\varepsilon_{t+1} = 0$

We will write $E_t z_{t+1} \equiv E\, z_{t+1} \mid x_t$

$\uparrow$ conditional on

Knowing x_t

In summary, $E_t x_{t+1} = A x_t$

Problem: to solve

$$* \quad p_t = \beta E_t p_{t+1} + y_t \qquad \beta = \frac{1}{1+r}, \; r > 0$$

digression: $(*) \Rightarrow p_t = \sum_{j=0}^{\infty} \beta^j E_t y_{t+j}$ + possible bubble (see below)

where $E_t y_{t+j}$ = best forecast of y_{t+j} given information at time t.

end of digression

method: guess and verify

Guess: $p_t = \overset{1\times n}{H} \overset{n\times 1}{x_t}$, H to be determined

verify: - substitute guess into $(*)$ and rearrange -

$$H x_t = \beta E_t (H x_{t+1}) + y_t$$

$$= \beta H E_t (A x_t + C \varepsilon_{t+1}) + y_t$$

$$= \beta H A x_t + \beta H C E \varepsilon_{t+1} + y_t$$

$$H x_t = \beta H A x_t + G x_t \qquad \forall x_t$$

$$\Rightarrow \quad H = \beta H A + G$$

$$H (I - \beta A)^{-1} = G$$

$$H = G (I - \beta A)^{-1} x_t$$

$$\therefore \quad p_t = G (I - \beta A)^{-1} x_t$$

Remark: this is the same formula we had earlier without the random term ε_{t+1}.

Stochastic bubbles:

return to (*) and consider the special case

$$y_t \equiv 0 \quad \forall t$$

want to solve

$$p_t = \beta E_t \, p_{t+1} \quad , \quad \beta = \frac{1}{1+r}$$

to find a solution - use guess and verify

Guess:

$$p_t = c_t \beta^{-t} \qquad , \; c_t \sim 1\times 1 - \text{scalar}$$

where c_t is any martingale.

A martingale $\{c_t\}$ is a random process that satisfies

$$E_t \, c_{t+1} = c_t \qquad -$$

best forecast of future value is today's value. (random walk)

verify:

$$p_t = \beta E_t \, p_{t+1}$$

$$c_t \beta^{-t} = \beta \, E_t \, \beta^{-t+1} c_{t+1}$$

$$= \beta^{-t} E_t c_{t+1}$$

$$= \beta^{-t} c_t \quad . \quad \therefore \text{ we have}$$

verified that $p_t = c_t \beta^{-t}$

satisfies $p_t = \beta E_t p_{t+1}$.

Example of a martingale:

$$c_{t+1} = \begin{cases} \lambda^{-1} c_t & \text{with probability } \lambda \in (0,1) \\ 0 & \text{with probability } (1-\lambda) \end{cases}$$

$$E_t c_{t+1} = \lambda \left[\lambda^{-1} c_t \right] + (1-\lambda) 0$$

$$= c_t \,. \qquad \therefore \{c_t\} \text{ is a martingale}$$

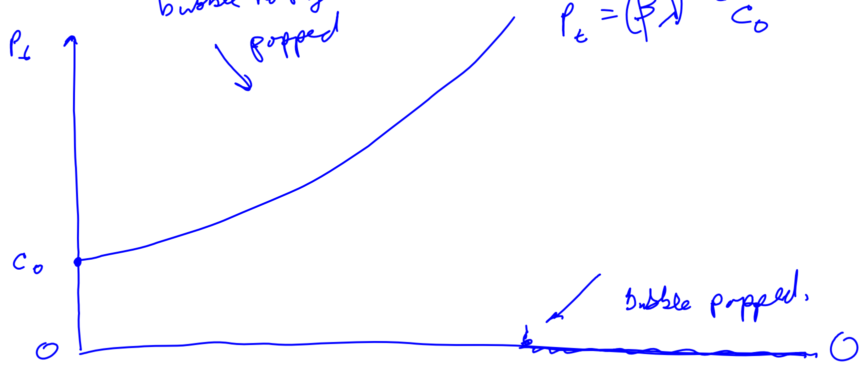
Sample path of

$$c_t = \begin{cases} \lambda^{-t} c_0 & \text{- if bubble has not yet popped} \\ 0 & \text{- after bubble has popped} \end{cases}$$

$$p_t = c_t \beta^{-t}$$

$$p_t = (\beta\lambda)^{-t} c_0$$

bubble not yet popped



Time Series Models

Wednesday, February 03, 2010
11:08 AM

Consider model

$$\underset{n\times 1}{x_{t+1}} = \underset{n\times n}{A}\, x_t + \underset{n\times m}{C}\, \underset{m\times 1}{\varepsilon_{t+1}}$$

$$\underset{1\times 1}{y_t} = \underset{1\times n}{G}\, \underset{n\times 1}{x_t}$$

$$\varepsilon_{t+1} \sim \mathcal{N}(0, I) \qquad \text{i.i.d.}$$

$$E\,\varepsilon_{t+1} = 0, \quad E\,\varepsilon_{t+1}\varepsilon_{t+1}' = I_{m\times m}$$

$$x_{t+2} = A x_{t+1} + C\varepsilon_{t+2}$$

$$= A(Ax_t + C\varepsilon_{t+1}) + C\varepsilon_{t+2}$$

$$= A^2 x_t + AC\varepsilon_{t+1} + C\varepsilon_{t+2}$$

$$\vdots$$

$$x_{t+j} = \underbrace{A^j x_t}_{\text{known at } t} + \underbrace{A^{j-1}C\varepsilon_{t+1} + A^{j-2}C\varepsilon_{t+2} + \cdots + C\varepsilon_{t+j}}_{\text{unknown at } t}$$

$$E_t x_{t+j} = A^j x_t$$

forecasting formula

transpose

$$\Sigma_j \equiv E(x_{t+j} - E_t x_{t+j})(x_{t+j} - E_t x_{t+j})' = \text{variance-covariance matrix of } j\text{-step ahead prediction errors}$$

prediction error

$$\Sigma_j = E\left(A^{j-1} C \varepsilon_{t+1} + A^{j-2} C \varepsilon_{t+2} + \cdots + C \varepsilon_{t+1} \right)$$
$$\left(A^{j-1} C \varepsilon_{t+1} + A^{j-2} C \varepsilon_{t+2} + \cdots + C \varepsilon_{t+1} \right)'$$
$$= A^{j-1} C \, \overbrace{E \varepsilon_{t+1} \varepsilon_{t+1}'}^{=I} \, C' (A^{j-1})' + \cdots + C \, \overbrace{E \varepsilon_{t+1} \varepsilon_{t+1}'}^{=I} \, C'$$

$$\Sigma_j = A^{j-1} C C' (A^{j-1})' + A^{j-2} C C' A^{j-2\,\prime} + \cdots + C C'$$

As $j \to \infty$ Σ_j might diverge (if eigenvalues of A ~~exceed~~ equal or exceed 1 in absolute value

or converge (if eigenvalues of A are all less than 1 in absolute value)

Impulse response function:

response of x_{t+j} to shock vector ε_{t+1} is

$$A^{j-1} C$$

response of y_{t+j} to shock vector ε_{t+1} is

$$G A^{j-1} C$$

Exercise: Compute present value of moving average coefficients: $A^j C$

$$(H \quad \sum_{j=0}^{\infty} \beta^{j-1} A^{j-1} C = (I - \beta A)^{-1} C$$

What question does (+) answer?

Well, note that

$$\sum_{j=0}^{\infty} \beta^j A^j C = C + \beta A C + \beta^2 A^2 C + \cdots$$

response of x_{t+1} to ε_{t+1} response of βx_{t+2} to ε_{t+1} response of $\beta^2 x_{t+3}$ to ε_{t+1}

= response of $x_{t+1} + \beta x_{t+2} + \beta^2 x_{t+3} + \cdots$

to ε_{t+1}

i.e. "response of present value of x to ε_{t+1}"

- - - - - -

Law of iterated expectations:

$$E_t\, x_{t+2} = E_t \left(E_{t+1}\, x_{t+2} \right)$$

$$A^2 x_t = E_t \left(A x_{t+1} \right)$$

$$= E_t\, A \left(A x_t + C \varepsilon_{t+1} \right)$$

$$= A^2 x_t$$

Competitive equilibrium

Wednesday, February 03, 2010
5:52 PM

Introduction to general equilibrium

Commodities: c, ℓ, h, k

household: preferences over c, ℓ

technology: production function

endowments: of ℓ & k

Household

$u(c, \ell)$, $0 \le \ell \le 1$

cons'n, leisure

$h = 1 - \ell$, $\ell = 1 - h$

hours of work

$$U = u(c, \ell) = u(c, 1-h)$$

$$dU = u_c\, dc + u_\ell\, d\ell = 0$$ indifference curve

$$\frac{dc}{d\ell} = -\frac{u_\ell}{u_c} = MRS$$

Examples:

(1) $u(c, 1-h) = \log c + \alpha(1-h)$ Prescott (2002)

(2) $u(c, 1-h) = \log c - B h$ Hansen-Rogerson

$\downarrow$

$$dU = \frac{1}{c} dc - B\, dh = 0$$

$$\Rightarrow \quad \frac{dc}{dh} = c B \quad , \Rightarrow \frac{dc}{d\ell} = -cB$$

HH problem in a market economy:

$$\max_{c,h} \; u(c, 1-h)$$

s.t. $c \leq w(1-\tau_n) h + Z$

w = real wage,

τ_n = marginal labor tax rate

Z = other income.

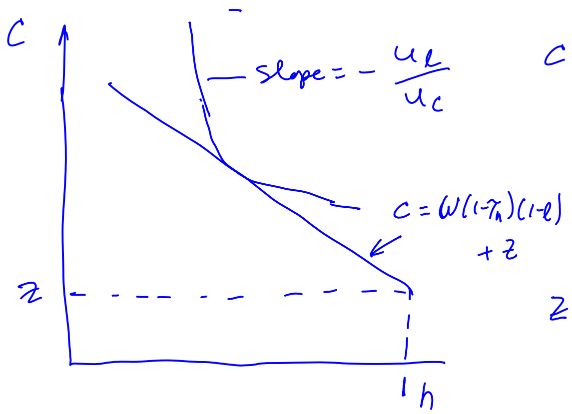
$$L = u(c, 1-h) + \lambda \left[w(1-\tau_n) h + Z - c \right]$$

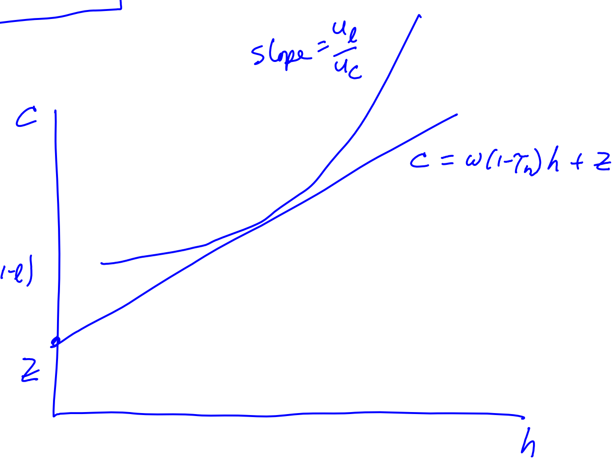
FONC:

c: $u_c - \lambda \leq 0$, $= 0$ if $c > 0$

h: $-u_\ell + \lambda w (1-\tau_n) \leq 0$, $= 0$ if $h > 0$

at equality: $\boxed{\frac{u_\ell}{u_c} = w(1-\tau_h)}$





Production Function:

$$Y = F(k, h)$$

assume constant returns to scale.

$$\lambda F(k,h) = F(\lambda k, \lambda h)$$

differentiate both sides w.r.t. h

$$\lambda F_h(k,h) = \lambda F_h(\lambda k, \lambda h)$$

or

(*) $$F_h(k,h) = F_h(\lambda k, \lambda h)$$

this is "homogeneous of degree zero"

marginal products only depend on ratio (h/k): set $\lambda = \frac{1}{k}$ in (*) to get

$$F_h(k,h) = F_h\left(1, \frac{h}{k}\right)$$

Example:

$$F(k,h) = A k^{\alpha} h^{1-\alpha}, \quad \alpha \in (0,1)$$

$$\underbrace{F_k > 0, \; F_h > 0}_{\text{positive marginal products}}, \; \underbrace{F_{kk} < 0, \; F_{hh} < 0}_{\text{diminishing returns}}$$

Feasibility:

$$c + G \leq F(k,h)$$

G: govt purchases

G exogenous (given)

Planning problem: – Command economy

$$\max_{c,h} \; u(c, 1-h)$$

$$\text{s.t.} \quad c + G \leq F(k,h)$$

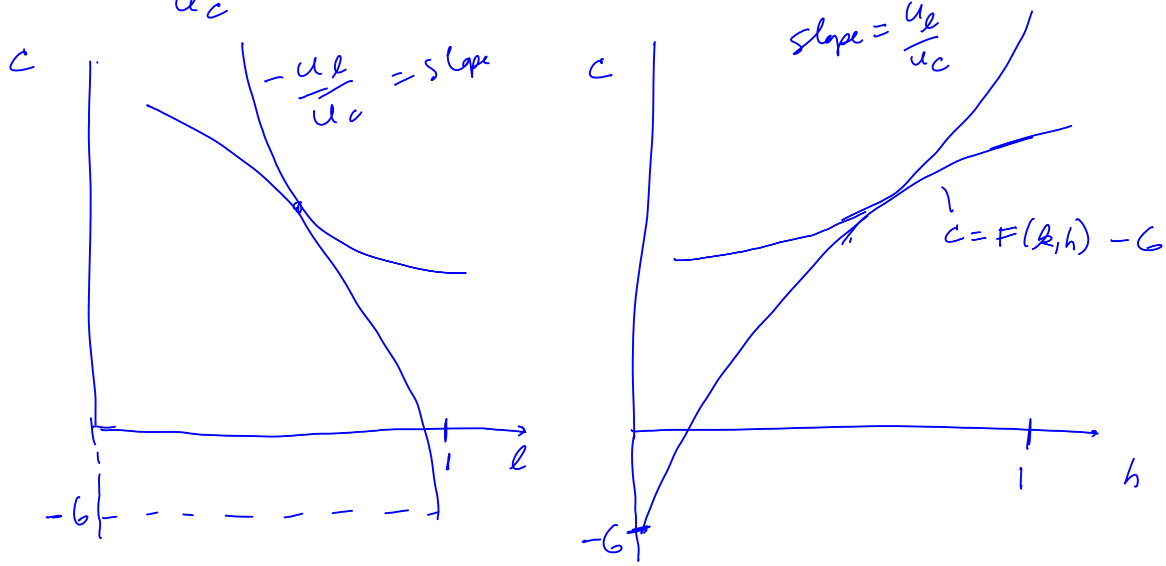
$$L = u(c, 1-h) + \lambda\left[F(k,h) - c - G\right]$$

FONC:

$$c: \quad u_c - \lambda = 0$$

$$h: \quad -u_\ell + \lambda F_h = 0$$

$$\Rightarrow \quad \frac{u_\ell}{u_c} = F_h$$



Firm problem:

household owns capital & labor & rents them to the firm. Firm is <u>competitive</u> – operate as a price taker.

$$\max_{k,h} \ \tilde{p}Y - \tilde{r}k - \tilde{\omega}h$$

$\tilde{p}$ – price in \$

$\tilde{\omega}$ – wage in \$/

$\tilde{r}$ – rental of capital in \$

$$Y = F(k,h)$$

$$\max_{k,h} \ \tilde{p}F(k,h) - \tilde{r}k - \tilde{\omega}h$$

<u>FONC:</u>

$$k: \quad \tilde{p}F_k - \tilde{r} = 0$$

$$\Rightarrow \quad \frac{\tilde{r}}{\tilde{p}} = F_k$$

$n: \quad \tilde{p}\, F_h - \tilde{\omega} = 0$

$\Rightarrow \quad F_h = \tilde{\omega} / \tilde{p}$

call $\dfrac{\tilde{\omega}}{\tilde{p}} = \omega$ real wage - $\dfrac{\text{goods}}{\text{unit of labor}}$

$\dfrac{\tilde{r}}{\tilde{p}} = r$ real rental rate of capital

$\sim \dfrac{\text{goods}}{\text{units of capital}}$

$$(\text{++}) \quad \begin{cases} F_k(k, h) = r \\ F_h(k, h) = \omega \end{cases}$$

Note: F_k, F_h are homogeneous of degree zero —

given ω: $F_h(k, h)$ only determines h/k

given r: $F_k(k, h)$ only determines h/k

The size of the firm — the levels of k & h cannot be determined by these equations.

We'll have to use the eqs (++) carefully.

Digression: factor price frontier. Example

$Y = A k^{\alpha} h^{1-\alpha}$

$F_k = \alpha A k^{\alpha-1} h^{1-\alpha} = \alpha A \left(\dfrac{h}{k}\right)^{1-\alpha}$

$$F_h = (1-\alpha) A k^{\alpha} h^{-\alpha} = (1-\alpha) A \left(\frac{h}{k}\right)^{-\alpha}$$

$$\Rightarrow \quad r = \alpha A \left(\frac{h}{k}\right)^{1-\alpha}$$

$$\omega = (1-\alpha) A \left(\frac{h}{k}\right)^{-\alpha}$$

$$\frac{\omega}{(1-\alpha)A} = (z)^{-\alpha} \qquad\qquad z \equiv \left(\frac{h}{k}\right)$$

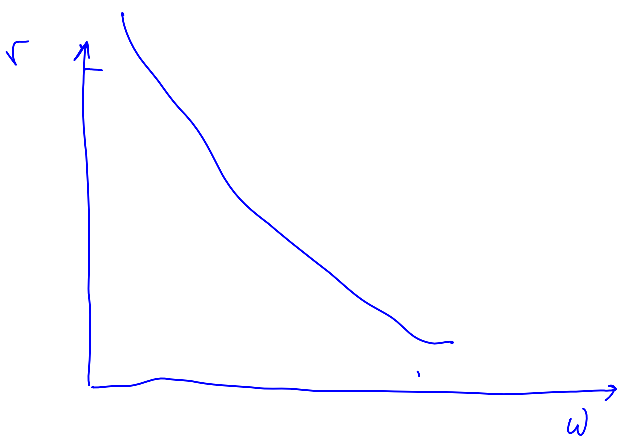
$$\left(\frac{\omega}{(1-\alpha)A}\right)^{-\frac{1}{\alpha}} = z$$

$$r = \alpha A \left(\left(\frac{\omega}{(1-\alpha)A}\right)^{-\frac{1}{\alpha}}\right)^{1-\alpha}$$

factor price frontier

$$= \alpha A \left(\frac{\omega}{(1-\alpha)A}\right)^{1-1/\alpha}$$

$$\frac{1-\alpha}{-\alpha} = \frac{\alpha-1}{\alpha} = 1-\frac{1}{\alpha}$$



draw in matlab or mathematica

end of digression

Competitive equilibrium (with $G=0$).

A feasible allocation is a k, h, c that satisfy

$$c \leq F(k, h)$$

with k given.

A price system is a pair (w, r).

A c.e. is a feasible allocation and a price system such that

(1) Given (w, r), (c, h) solves the household problem.

(2) Given (w, r), (h, k) solve the firm's problem.

Remark: It is redundant to add "markets clear".

Example: Compute a c.e.

$$u(c, 1-h) = \ln c - B h$$

$$F(k, h) = A k^{\alpha} h^{1-\alpha}$$

Method - solve planning problem, then reverse engineer the required prices, then verify c.e. condition hold.

Solve planning problem:

$$F_h = \frac{u_\ell}{u_c}$$

$$(1-\alpha) A \left(\frac{h}{k}\right)^{-\alpha} = c B$$

$$c = A k^\alpha h^{1-\alpha}$$

$$\Rightarrow (1-\alpha) \cancel{A} \cancel{k^\alpha} h^{-\alpha} = B \cancel{A} \cancel{k^\alpha} h^{1-\alpha}$$

$$B h = (1-\alpha)$$

(1) $$\boxed{h = \frac{(1-\alpha)}{B}}$$

(2) $$c = A k^\alpha \left(\frac{(1-\alpha)}{B}\right)^{1-\alpha}$$

(3) $$w = (1-\alpha) A k^\alpha \left(\frac{1-\alpha}{B}\right)^{-\alpha}$$

(4) $$r = \alpha A k^{\alpha-1} \left(\frac{1-\alpha}{B}\right)^{1-\alpha}$$

The FONC's for the firm's problem & the hh's problem are satisfied.

Interest rates

Tuesday, February 02, 2010
10:11 AM

Theory of interest rates.

A consumer has preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad u'>0, u''<0$$

$$0<\beta<1$$

There is no uncertainty.

At time 0, the consumer can buy or sell a claim to one unit of consumption at date $t \geq 1$ for a price q_t^0.

Here 0 denotes the date of the "trade" (now) and $t>0$ denotes the delivery date of the one unit of consumption. q_t^0 is the time 0 price of a zero coupon bond maturing at t.

Remark: We can convert q_0^0 to a t-period interest rate or yield to maturity using the formula

$$q_t^0 = \frac{1}{(1+\rho_{0,t})^t} \approx \exp(-t\,\rho_{0,t}) .$$

A graph of $\rho_{0,t}$ against t is called the "term structure of interest rates" at $t=0$.

Assume that the consumer owns an endowment stream $\{y_t\}_{t=0}^{\infty}$. The consumer can <u>sell</u> this endowment stream for the present value

$$W_0 = \sum_{t=0}^{\infty} q_t^0 \, y_t$$

" sum of price × quantity "

We think of the consumer as a <u>price-taker</u> — he faces $\{q_t^0\}_{t=0}^{\infty}$ as an exogenous sequence.

We consider the following problem of a utility maximizing household who owns endowment $\{y_t\}_{t=0}^{\infty}$ and faces price system $\{q_t^0\}_{t=0}^{\infty}$ as a price taker.

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.} \quad \underbrace{\sum_{t=0}^{\infty} q_t^0 \, c_t}_{\text{present value of consumption}} \leq \underbrace{\sum_{t=0}^{\infty} q_t^0 \, y_t}_{\text{present value of endowment}}$$

To solve: form Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \theta \left[\sum_{t=0}^{\infty} q_t^0 (y_t - c_t) \right]$$

FONC:

$$c_t: \quad \beta^t u'(c_t) = \theta q_t^0, \quad t = 0, 1, \ldots$$

$$\theta: \quad \sum_{t=0}^{\infty} q_t^0 (y_t - c_t) \geq 0, \quad = 0 \text{ if } \theta > 0$$

Because $u' > 0$ (no satiation) budget constraint will be satisfied with equality and $\theta > 0$.

Thus, FONC for consumption implies that

$$(1) \qquad \beta^t u'(c_t) = \theta q_t^0, \quad \theta > 0.$$

Competitive equilibrium: Consider an economy consisting of I consumers with preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i), \quad u' > 0, \quad u'' < 0$$

and endowments

$$\{ y_t^i \}_{t=0}^{\infty}, \quad i = 1, \ldots, I$$

Definitions :

A <u>feasible allocation</u> is a set of sequences $\{c_t^i\}_{t=0}^{\infty}$ that satisfies

$$\sum_{i=1}^{I} c_t^i \leq \sum_{i=1}^{I} y_t^i \qquad , t=0,1,\ldots$$

A <u>price system</u> is a sequence $\{q_t^0\}_{t=0}^{\infty}$

A <u>competitive equilibrium</u> is a feasible allocation and price system such that $\{c_t^i\}_{t=0}^{\infty}$ solves household i's problem, taking $\{q_t^0\}_{t=0}^{\infty}$ as given, for $i=1,\ldots,I$.

<u>Example</u> 1. Consider an economy with I consumers each having <u>identical</u> endowment sequences

$$y_t^i = y_t^1 \qquad i=2,\ldots,I$$
$$t=0,1,\ldots$$

Let's construct a competitive equilibrium using the "guess and verify" method.

Guess: no-trade — $c_t^i = y_t^i \quad , i=1,\ldots,I$
$t=0,1,\ldots$

Guess: $q_t^0 = \beta^t u'(y_t^i)$

$\theta = 1$

Verify: $\beta^t u'(c_t^i) = q_t^0$ ✓

$$\sum_{t=0}^{\infty} q_t^0 (y_t^i - c_t^i) = 0$$ ✓

∴ we have a no-trade competitive equilibrium.

Example 2: (Lucas's famous asset pricing model)

set $I = 1$ in above example. It still works. — this is a "representative consumer" construction.

Example 3: $I = 1$, $y_t^1 = y_0$

then $\theta q_t^0 = \beta^t u'(y_0)$

constant

$c_t = y_0$

any $\theta > 0$ will work — choice of a numeraire. Only relative prices matter. Here $q_t^0 \propto \beta^t$

This is our "$R\beta = 1$" specification.

Remark: remember formula

$$q^0_t = \frac{1}{(1+\rho_{0,t})^t} \approx \exp(-t\,\rho_{0,t})$$

for this to hold we must set $q^0_0 = 1$. We can do this by choosing $q^0_t = \beta^t \dfrac{u'(c_t)}{u'(c_0)}$.

Example 4: $I = 1$

$$u(c_t) = \ln c_t \quad , \quad \Rightarrow u'(c_t) = \frac{1}{c_t}$$

$$y_t = y_0\,\delta^t$$

Competitive equilibrium: no-trade

$$\forall\; q^0_t = \beta^t \frac{1}{c_t} = \frac{\beta^t \delta^{-t}}{c_0}$$

thus $q^0_t \propto \left(\beta/\delta\right)^t$

we require $\left|\beta/\delta\right| < 1$

for wealth $\sum_{t=0}^{\infty} q^0_t\, y_t$ to be finite.

Example 5: $I = 2$,

$$y_t^1 = \{1, 0, 1, 0, \dots\}$$
$$y_t^2 = \{0, 1, 0, 1, \dots\}$$

Note $y_t^1 + y_t^2 = 1$

guess: (i) $c_t^1 = c^1$, $c_t^2 = c^2$, $t = 0, 1, \dots$

(total consumption smooths), $c^1 + c^2 = 1$

(ii) $q_t^0 = \beta^t$

verify:

$$\theta^1 q_t^0 = \beta^t u'(c^1) \quad \checkmark$$

$$\theta^1 q_t^0 = \beta^t u'(c^2) \quad \checkmark$$

because $\theta^1, \theta^2, c^1, c^2$ are constants

to find: c^1 & c^2

use budget constraint:

$$\sum_{t=0}^{\infty} \beta^t c^1 = (1 + \beta^2 + \beta^4 + \dots)$$

$$\frac{c_1}{(1-\beta)} = \frac{1}{(1-\beta^2)}$$

$$c_1 = \frac{(1-\beta)}{(1-\beta)(1+\beta)} = \frac{1}{1+\beta}$$

$$\sum_{t=0}^{\infty} \beta^t c^2 = (0 + \beta + 0 + \beta^2 + \dots)$$

as superscript, t a

$$\frac{c^2}{(1-\beta)} = \frac{\beta}{(1-\beta^2)}$$

not a "square"

$$\overline{(1-\beta)} = \overline{(1-\beta^2)}$$

$$c^2 = \frac{\beta(1-\beta)}{(1-\beta)(1+\beta)} = \frac{\beta}{1+\beta}$$

$$c^1 + c^2 = \frac{1}{1+\beta} + \frac{\beta}{1+\beta} = 1 \quad \checkmark$$

total consumption equals endowment.

Here there is a lot of "borrowing & lending"

Example 6:

$$I = 2$$

$$y_t^1 = \begin{cases} 1 & \text{for } t = 0, \dots, 10 \\ 0 & \text{for } t \geq 11 \end{cases}$$

$$y_t^2 = \begin{cases} 0 & \text{for } t = 0, \dots, 10 \\ 1 & \text{for } t \geq 11 \end{cases}$$

Guess $c_t^1 = c^1$, $c_t^2 = c^2$ $\forall\ t \geq 0$

$$q_t^0 \propto \beta^t .$$

as in example 5, the guess verifies the household's FONC (same argument)

to determine c^1, c^2, use budget constraints

$$\sum_{t=0}^{\infty} \beta^t c^1 = \frac{1-\beta^{11}}{1-\beta}$$

$$\frac{c^1}{1-\beta} = \frac{1-\beta^{11}}{1-\beta} \implies c^1 = 1-\beta^{11}$$

$$\sum_{t=0}^{\infty} \beta^t c^1 = \frac{1-\beta^{11}}{1-\beta}$$

$$\frac{c^1}{1-\beta} = \frac{1-\beta^{11}}{1-\beta} \Rightarrow c^1 = 1-\beta^{11}$$

$$\sum_{t=0}^{\infty} \beta^t c^2 = \frac{\beta^{11}}{1-\beta} = \beta^{11} + \beta^{12} + \cdots$$

$$\Rightarrow c^2 = \beta^{11}$$

$$c^1 + c^2 = 1 - \beta^{11} + \beta^{11} = 1 \quad . \quad \checkmark$$

which consumer consumes more? It depends on β.

Permanent income, 3

Wednesday, February 03, 2010
3:30 PM

Permanent income model -

"pure consumption loans model"

Example 1: There are two consumers with identical preferences $\sum_{\tau=0}^{\infty} \beta^t u(c_t^i)$, $i=1,2$

$$u' > 0 , u'' < 0 , \beta \in (0,1)$$

Each has intertemporal budget constraint

$$\sum_{t=0}^{\infty} \beta^t c_t^i = \sum_{t=0}^{\infty} \beta^t y_t^i$$

where y_t^i is income of person i.

Assume that: person 1:

$$y_t^1 = \{0, 1, 0, 1, \dots \}$$
$$= \begin{cases} 0 & t \text{ even} \\ 1 & t \text{ odd} \end{cases} \quad t \geq 0$$

Person 2:

$$y_t^2 = \{1, 0, 1, 0, \dots \}$$
$$= \begin{cases} 1 & , t \text{ even} \\ 0 & , t \text{ odd} \end{cases} \quad t \geq 0$$

The total endowment for the two people is

$$y_t^1 + y_t^2 = \bar{y}_t = 1 \quad \forall \ t \geq 0$$

Apply permanent income result:

$$c^i = (1-\beta) \sum_{t=0}^{\infty} \beta^t y_t^i \quad , \quad c_t^i = c^i \ \forall t .$$

for each person $i = 1, 2$

Person 1:

$$c^1 = (1-\beta)\left[0 + \beta \cdot 1 + \beta^2 \cdot 0 + \beta^3 1 + \cdots \right]$$

$$= (1-\beta)\left[\ \beta (1 + \beta^2 + \beta^4 + \cdots)\right]$$

$$= (1-\beta) \frac{\beta}{(1-\beta^2)} = \beta \frac{\cancel{(1-\beta)}}{\cancel{(1-\beta)}(1+\beta)}$$

$$* \quad c^1 = \frac{\beta}{1+\beta} = c_t^1 \quad \text{for all } t \geq 0$$

For person 2:

$$c^2 = (1-\beta)\left[1 + \beta \cdot 0 + \beta^2 \cdot 1 + \beta^3 0 + \beta^4 \cdot 1 + \cdots \right]$$

$$= (1-\beta)(1 + \beta^2 + \beta^4 + \cdots)$$

$$= (1-\beta) \cdot \frac{1}{(1-\beta^2)} = \cancel{(1-\beta)} \frac{1}{\cancel{(1-\beta)}(1+\beta)}$$

$$* \quad c^2 = \frac{1}{1+\beta}$$

So $c_1 = \frac{\beta}{1+\beta}$, $c_2 = \frac{1}{1+\beta}$ $(c_2 > c_1)$

and $c_1 + c_2 = \frac{\beta}{1+\beta} + \frac{1}{1+\beta} = 1$

Compute asset trades that support this.

$$A^i_{t+1} = R(A^i_t + y^i_t - c^i_t), \quad i = 1, 2$$

$$R = \beta^{-1}$$

Initial $A^i_0 = 0$ for both people.

Person 1:

$$A^1_1 = \beta^{-1}\left(0 + 0 - \frac{\beta}{1+\beta}\right)$$

$A^1_1 = -\frac{1}{1+\beta}$ = borrowing of $-\frac{\beta}{1+\beta}$, payback $\frac{1}{1+\beta}$ at time 1

$$A^1_2 = \beta^{-1}\left(-\frac{1}{1+\beta} + 1 - \frac{\beta}{1+\beta}\right) \quad \text{time } 1$$

$$= \beta^{-1}\left(\frac{1+\beta}{1+\beta} - \frac{(1+\beta}{1+\beta}\right) = 0$$

$A^1_2 = 0$

and so on

debt oscillates between $-\frac{\beta}{1+\beta}$ and 0.

Person 2:

$$A^2_1 = \beta^{-1}\left(0 + 1 - \frac{1}{1+\beta}\right) \quad t = 0$$

$$= \beta^{-1}\left(\frac{1+\beta}{1+\beta} - \frac{1}{1+\beta} \right)$$

$$A_1^2 = \frac{1}{1+\beta}$$ lends $\frac{\beta}{1+\beta}$ at $t=0$ & is repaid $\frac{1}{1+\beta}$ at $t=1$

$$A_2^2 = \beta^{-1}\left(\frac{1}{1+\beta} + 0 \frac{1}{1+\beta} \right) = 0$$

and so on.

Thus person 2's assets oscillate between $\frac{1}{1+\beta}$ and 0.

Consumers 1 and 2 borrow & lend with eachother

Example 2:

Two consumers. $\sum_{t=0}^{\infty} \beta^t u(c_t^i)$, $0 < \beta < 1$

$u' > 0$, $u'' < 0$

Consumer 1:

$$y_t^1 = \begin{cases} y_1 & t = 0, \ldots, T \\ 0 & t = T+1, T+2, \ldots, \infty \end{cases}$$

Consumer 2:

$$y_t^2 = \begin{cases} 0 & , t = 0, \ldots, T \\ y_2 & t = T+1, T+2, \ldots, \infty \end{cases}$$

assume that $y_2 = \left(\frac{1}{\beta^{T+1}} - 1 \right) y_1$

assume that $y_2 = \left(\frac{1}{\beta^{T+1}} - 1\right) y_1$

Apply permanent income consumption

$$c_t^i = c^i = (1-\beta) \sum_{t=0}^{\infty} \beta^t y_t^i$$

person 1:

$$c^1 = (1-\beta)\, y_1 \sum_{t=0}^{T} \beta^t = \cancel{(1-\beta)} \frac{(1-\beta^{T+1})}{\cancel{1-\beta}} \cdot y_1$$

$$= (1-\beta^{T+1})\, y_1$$

Person 2:

$$c^2 = (1-\beta)\, y_2 \sum_{t=T+1}^{\infty} \beta^t$$

$$= \cancel{(1-\beta)} \frac{\beta^{T+1}}{\cancel{(1-\beta)}} y_2$$

$$c^2 = \beta^{T+1} y_2$$

But $y_2 = \left(\frac{1}{\beta^{T+1}} - 1\right) y_1$

so

$$c^2 = \beta^{T+1} \left(\frac{1}{\beta^{T+1}} - 1\right) y_1$$

$$= (1-\beta^{T+1})\, y_1 = c^1$$

Compute asset level of person 1: $A_0^1 = A_0^2 = 0$

$$A_1' = \beta^{-1}(A_0' + y_0' - c')$$

$$= \beta^{-1}(0 + y_1 - (1-\beta^{T+1})y_1)$$

$$= \beta^{-1}(\beta^{T+1})y_1$$

$$A_1' = \beta^{T} y_1$$

$$A_2' = \beta^{-1}(\beta^{T} y_1 + y_1 - (1-\beta^{T+1})y_1)$$

$$= \beta^{-1}(\beta^{T} + \beta^{T+1})y_1$$

$$A_2' = (\beta^{T-1} + \beta^{T})y_1$$

$$\vdots$$

$$A_{T+1}' = \left[(1+\beta+\beta^2+\dots+\beta^T)\right] y_1 = \frac{1-\beta^{T+1}}{1-\beta} y_1$$

$$A_{T+2}' = \beta^{-1}\left[\frac{1-\beta^{T+1}}{1-\beta} y_1 - (1-\beta^{T+1})y_1\right]$$

$$= \beta^{-1}\left[\frac{1-\beta^{T+1}}{1-\beta} - \frac{(1-\beta)(1-\beta^{T+1})}{(1-\beta)}\right] y_1$$

$$= \beta^{-1}\left[\frac{1-\beta^{T+1} - (1-\beta^{T+1}) + \beta(1-\beta^{T+1})}{1-\beta}\right] y_1$$

$$= \beta^{-1}\left[\frac{\beta(1-\beta^{T+1})}{1-\beta}\right] y_1$$

$$= \frac{1-\beta^{T+1}}{1-\beta} y_1 = A_{T+1}' \Rightarrow$$ roll over his assets & just lives off interest earnings.

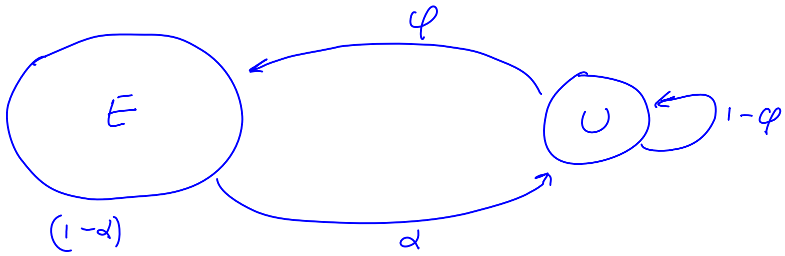
Person 2 does the opposite — he builds up debt —

Person 2 does the opposite – he builds up <u>debt</u> –
then rolls it over.

Lake model

Wednesday, February 03, 2010
8:54 PM

Lake model of gross labor mkt flows



Markov Chain: transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \varphi & 1-\lambda \end{pmatrix}$$

state 1 $\longleftrightarrow$ E
state 2 $\longleftrightarrow$ U

To compute invariant distribution (i.e. "long run" distribution

π = prob of E

$1-\pi$ = prob of U

method: $\begin{pmatrix} \pi \\ 1-\pi \end{pmatrix}$ is a <u>left</u> eigenvector of P affiliated with a unit eigenvalue

thus it solves

$$\begin{bmatrix} \pi & 1-\pi \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ \varphi & 1-\varphi \end{bmatrix} = \begin{bmatrix} \pi & 1-\pi \end{bmatrix}$$

$\Rightarrow$

$$\pi(1-\alpha) + \varphi(1-\pi) = \pi$$

$$-\alpha\pi + (1-\pi)\varphi = 0$$

$$-\varphi - \pi(\alpha+\varphi) = 0$$

$$\pi = \frac{\varphi}{\alpha+\varphi} = \text{probability of being employed}$$

E

$$1-\pi = \frac{\alpha}{\alpha+\varphi} = \text{probability of being unemployed}$$

U

$$1-\pi = \frac{\frac{1}{\varphi}}{\frac{1}{\varphi}+\frac{1}{\alpha}}$$

divide both numerator & denominator by $\alpha \cdot \varphi$

Some numbers for U.S. - 2007
(Source: President's Economic Report, 2009).

average duration of unemployment = 16.8 weeks
median " " = 8.5 weeks

Civilian unemployment rate: 4.7 %

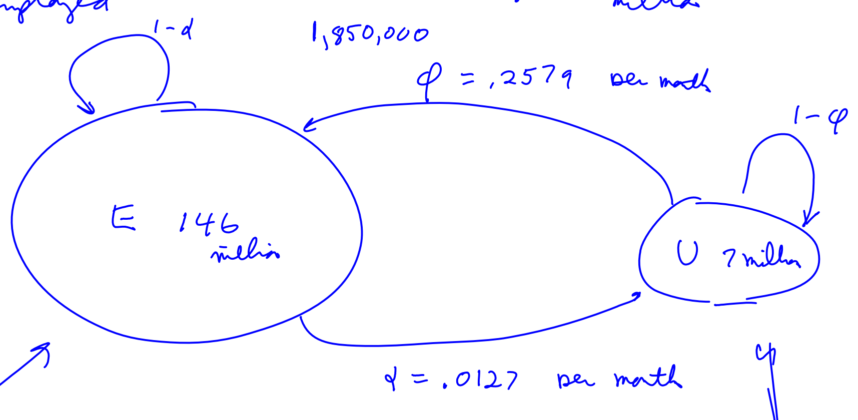
α employment / population 63.%

labor force / population 66%

Civilian population (non institutional) 231 million

civilian labor force 153 million

unemployed 7 million



average duration
78.62 months
= 6.55 years

average duration
3.87 months

≈ 1,850,000 jobs created

1,850,000 jobs destroyed.

$$\frac{U}{\text{labor force}} = \frac{\frac{1}{\varphi}}{\frac{1}{\alpha}+\frac{1}{\varphi}} = \frac{\alpha}{\alpha+\varphi}$$

$\overline{\text{labor force}}$ $\frac{1}{\varphi} + \frac{1}{\alpha}$ $\alpha + \varphi$

↓ rate inflow, 1 − outflow

$$u_{t+1} = \alpha(1 - u_t) + (1-\varphi)u_t \qquad u = \frac{U}{\text{labor force}} = \text{unemploy. rate}$$

this eqn describes unemployment dynamics

Note: to estimate α we took the 2007 data

on $u = .047$ and $\varphi = .2579$ per month

and then solved $u = \alpha(1-u) + (1-\varphi)u$

for α. $\Rightarrow$

$$\alpha = \varphi \frac{u}{1-u} = .0127 \text{ per month.}$$

Then we computed the gross flows from, eq.

gross job creation $= U \cdot \varphi$

Homework 2

Wednesday, February 10, 2010
6:13 AM

1. An economy consists of two consumers, $i=1,2$, each of whom orders consumption streams according to the intertemporal utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i) \quad \text{when } 0<\beta<1 \text{ and}$$
$$u'>0 \text{ and } u''<0$$

each person has intertemporal budget constraint

[Note: we are assuming that "$R=\beta^{-1}$".]

$$\sum_{t=0}^{\infty} \beta^t c_t^i = \sum_{t=0}^{\infty} \beta^t y_t^i .$$

Consumer 1 has endowment stream

$$\begin{array}{cccccccccccc} t= & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots & \\ y_t^1 = \{ & 1, & 1, & 0, & 1, & 1, & 0, & 1, & 1, & 0, & \ldots & \} \end{array}$$

while consumer 2 has stream

$$y_t^2 = \{0,0,1,0,0,1,0,0,1,\ldots$$

a. Find the optimal consumption paths $\{c_t^i\}_{t=0}^{\infty}$ for $i=1,2$.

b. Compute $c_t^1 + c_t^2$ for $t=0,1,2,\ldots$

c. For $t=0,1,2,\ldots$, please compute

A^1_{t+1} where

$$A^1_{t+1} = \beta^{-1}\left[A^1_t + y^1_t - c^1_t\right], \quad A^1_0 = 0$$

d. For $t = 0, 1, 2, \ldots$, please compute

A^2_{t+1} where

$$A^2_{t+1} = \beta^{-1}\left[A^2_t + y^2_t - c^2_t\right].$$

---

2. Consider an economy consisting of two people, each of whom orders consumption streams according to

$$\sum_{t=0}^{\infty} \beta^t u(c^i_t), \quad i = 1, 2, \quad u' > 0, u'' < 0$$

$$0 < \beta < 1$$

Person 1 has income stream

$$y^1_t = \delta^t, \quad t \geq 0, \quad \delta \in (0, 1).$$

Person 2 has income stream

$$y^2_t = 1 - \delta^t, \quad t \geq 0, \quad \delta \in (0, 1).$$

Each person has the intertemporal budget constraint

$$\sum_{t=0}^{\infty} \beta^t \left[c^i_t - y^i_t\right] = 0$$

a. Find the optimal consumption path

$\{c_t^i\}_{t=0}^{\infty}$ for $i=1,2$.

b. Compute $c_t^1 + c_t^2$ for $t=0,1,2,\ldots$

c. For $t=0,1,2,\ldots$, please compute

A_{t+1}^1 where

$$A_{t+1}^1 = \beta^{-1}\left[A_t^1 + y_t^1 - c_t^1\right], \quad A_0^1 = 0.$$

d. For $t=0,1,2,\ldots$, please compute

A_{t+1}^2 where

$$A_{t+1}^2 = \beta^{-1}\left[A_t^2 + y_t^2 - c_t^2\right].$$

e. Given β with $0<\beta<1$, please find a value of δ that makes consumers 1 and 2 consume identical amounts for $t=0,1,2,\ldots$.

e. Assume the same two consumers with the same income processes mentioned above. But now assume that each consumer has the <u>sequence</u> of budget constraints

$$A_{t+1}^i = \beta^{-1}\left[A_t^i + y_t^i - c_t^i\right],$$

where $A_0^i = 0$ and

(*) $A_{t+1}^{i} \geq 0$ for $t = 0, 1, \ldots$

Here (*) is a no-borrowing constraint.

Please find the optimal consumption path $\{c_t^i\}_{t=0}^{\infty}$ for $i = 1, 2$.

---

3. A government wants to minimize the following measure of tax distortions:

$$\sum_{t=0}^{\infty} \beta^t D(T_t) \quad , \quad 0 < \beta < 1$$

where $D(T_t)$ is a measure of the costs of distortions at date t, T_t are total tax revenues, and $D' > 0$, $D'' > 0$.

The government has an exogenous stream of expenditures $\{G_t\}_{t=0}^{\infty}$ and faces the sequence of government budget constraints

$$B_{t+1} = R\left[B_t + G_t - T_t\right] \quad , \quad B_0 = 0$$

where B_{t+1} is government debt issued at t and due at $t+1$. The government can borrow or lend. $R = 1 + r > 1$ is the gross rate of return on government debt. Assume that $R = \beta^{-1}$.

debt. Assume that $R = \beta^{-1}$.

Assume that $\lim_{s \to \infty} \beta^s D'(T_s) B_s \doteq 0$

a condition to rule out "Ponzi schemes"

a. Consider the government expenditure process

$$G_t = \begin{cases} 1 & t \text{ even} \\ 0 & t \text{ odd} \end{cases}$$

Find the optimal setting for taxes $\{T_t\}_{t=0}^{\infty}$ for $t = 0, 1, 2, \ldots$

b. Consider the expenditure process

$$G_t = \begin{cases} 1 & t \neq 10j, \; j = 0, 1, 2, \ldots; t = 0, 1, 2 \\ 10 & t = 10j, \; j = 0, 1, 2, \ldots \end{cases}$$

here a "war" happens in periods 0, 10, 20, ...
and "peace" occurs the rest of the time.

Find the optimal level of $\{T_t\}_{t=0}^{\infty}$.

c. Does the analysis in this problem remind you of any other problem or model you have seen? If so, explain.

---

4. The unemployment rate at month t, u_t,

satisfies the difference equation

$$(*) \quad u_{t+1} = \alpha(1-u_t) + (1-\varphi)u_t$$

where α = rate of inflow into unemployment

φ = rate of exit from unemployment.

Definition: A steady state unemployment rate $\bar{u}$ satisfies (*) with $\bar{u} = u_{t+1} = u_t$.

a. Compute a formula for the steady state unemployment rate $\bar{u}$ as a function of α and φ.

b. In 2007 in the U.S., $\alpha = .0127$ and $\varphi = .2579$. Compute the implied steady state unemployment rate.

c. Suppose that in a recession year 2xxx, $\varphi = .15$ and $\alpha = .0175$. These are monthly rates.
Compute the steady state unemployment rate in year 2xxx.

d. Extra credit. Compute the average duration of a job in 2xxx. Compute the average duration of unemployment in 2xxx.

Trees

Wednesday, February 10, 2010
6:37 AM

Consider a two-state Markov chain

with state space $S = \{\bar{s}_1, \bar{s}_2\}$

and time-invariant transition matrix

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

where $P_{ij} = \text{Prob}\{s_{t+1} = \bar{s}_j \mid s_t = \bar{s}_i\}$;

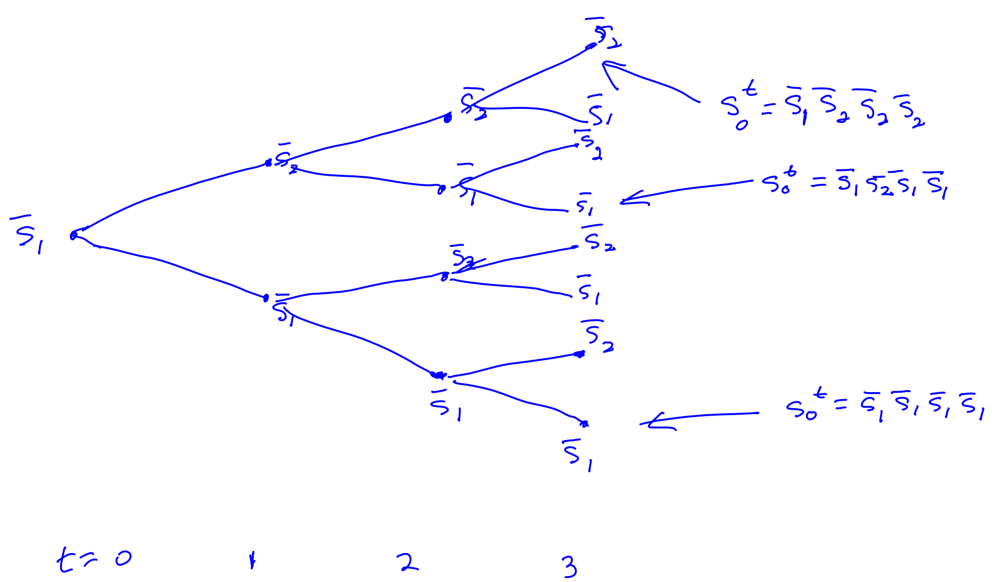
and $\pi_0 = \begin{pmatrix} \pi_{0,1} \\ \pi_{0,2} \end{pmatrix} = \begin{bmatrix} \text{Prob}(s_0 = \bar{s}_1) \\ \text{Prob}(s_0 = \bar{s}_2) \end{bmatrix}$

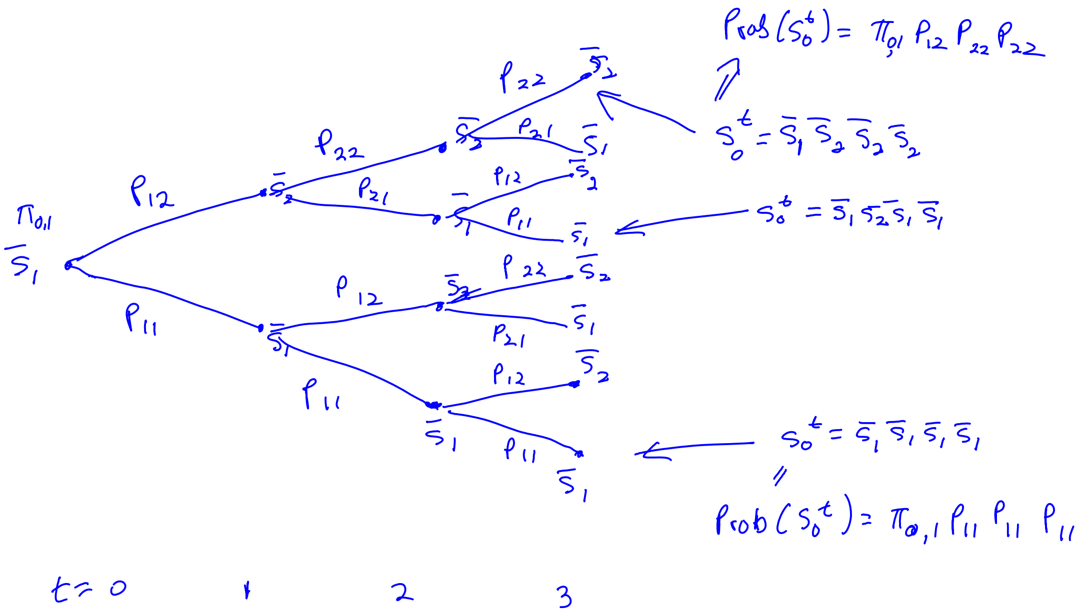
being the initial probability vector over states.

Here $\pi_{0,i} \geq 0$ and $\sum \pi_{0,i} = 1$ (i.e. the $\pi_{0,i}$'s are probabilities)

$P_{ij} \geq 0$ and $\sum_j P_{ij} = 1$ (i.e., the P_{ij}'s are probabilities)

For example. ($\bar{s}_1 = 0$, $\bar{s}_2 = 1$).





In this way, we build up probabilities over histories s_0^t (or s^t for shorthand)

$$\text{Prob}(s^t = \bar{s}^t) \equiv \pi_t(s^t)$$

$$\pi_t((\bar{s}_i, \bar{s}_j, \bar{s}_k, \bar{s}_\ell \ldots)) = \pi_{0,i} P_{ij} P_{jk} P_{k\ell} \cdots$$

Evidently $\sum_{s^t} \pi_t(s^t) = 1$

$\pi_t(s^t) \geq 0$

Probability over a history of length t.

$(s_0, s_1, s_2, \ldots, s_t)$

$\uparrow$ values of state at $t = 0, 1, 2, \ldots$

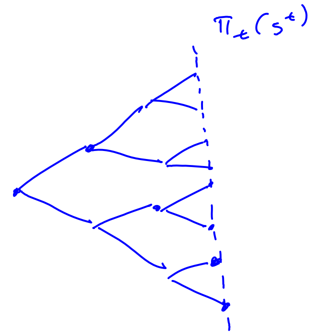
So for each $t \geq 0$, we have a collection of histories $s^t \in S^{t+1} = S \times S \times \cdots S$

t+1 times
Cartesian product

for each $s^t \in S^{t+1}$, we have probabilities

$\pi_t(s^t) \geq 0$

$\sum_{s^t} \pi^t(s^t) = 1$



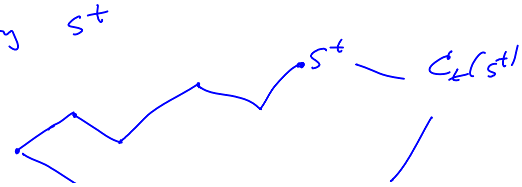
sum across all these probabilities for all histories s^t at time t.

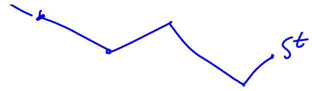
Modeling risk. At time $t=0$, the consumer observes s_0 but does not know subsequent realizations $s_1, s_2, \ldots$. He knows that any history s^t with $\pi_t(s^t) > 0$ for $t > 0$ might happen. His discounted utility along one history s^t is

$$\sum_{t=0}^{\infty} \beta^t u(c_t(s^t)) \quad , 0 < \beta < 1$$

$u' > 0, u'' < 0$

where $c_t(s^t)$ is consumption at date t at history s^t





$c_t(s^t)$

thus we allow consumption at t to depend on the particular history realized at date t.

The consumer's expected utility at time 0 is

$$(*) \quad \sum_t \sum_{s^t} \beta^t \, u(c_t(s^t)) \, \pi_t(s^t)$$

Here we are averaging discounted utilities $\beta^t u(c_t(s^t))$ across histories s^t at date t, using the probabilities $\pi_t(s^t)$ as weights.

$(*)$ is called "expected utility".

---

history-date contingent claim on consumption

$c_t(s^t)$ time - t

history s^t

consumer values these for every $t \geq 0$, for every s^t

Note: a huge number of commodities

many t's, and for each t, many s^t's.

Complete markets: at time 0 the consumer can buy or sell $c_t(s^t)$ at price

$q_t^0(s^t)$ = price of one unit of consumption at

t at history s^t

The consumer faces $q_t^0(s^t)$ as a price-taker.

Consumer problem:

$$\max_{\{c_t(s^t)\}} \sum_t \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t)$$

$$\text{s.t.} \quad \underbrace{\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, c_t(s^t)}_{\text{price} \cdot \text{quantity}} \leq \underbrace{W_0}_{\text{time 0 wealth}}$$

One budget constraint

Formulate as a Lagrangian

$$L = \sum_t \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t)$$

$$+ \underbrace{\theta}_{\text{Lagrange multiplier}} \left[W_0 - \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, c_t(s^t) \right]$$

FONC:

$$c_t(s^t): (*) \quad \beta^t u'(c_t(s^t)) \pi_t(s^t) = \theta\, q_t^0(s^t)$$

to be

given as price taker

determined from budget constraint

$$(**) \quad \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, c_t(s^t) = W_0$$

(*) $\forall\, (t, s^t)$ and (**) determine

$c_t(s^t)\ \forall\, (t, s^t)$ and θ

"marginal utility of wealth"

Examples: (1) suppose that $q_t^0(s^t) = \beta^t \pi_t(s^t)$

(why? - just a guess - it might happen in a very special case)

then (*) $\Rightarrow$

$$\cancel{\beta^t} u'(c_t(s^t))\, \pi_{\cancel{t}}\cancel{(s^t)} = \theta\, \cancel{\beta^t}\, \pi_{\cancel{t}}\cancel{(s^t)}$$

or $u'(c_t(s^t)) = \theta$ a constant

$\Rightarrow c_t(s^t) = \bar{c}$ a constant $\forall t, \forall s^t$

and (**) $\Rightarrow$

$$\bar{c} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) = W_0$$

$$\bar{c} \sum_{t=0}^{\infty} \beta^t \underbrace{\sum_{s^t} \pi_t(s^t)}_{=1} = W_0$$

$$\bar{c} \sum_{t=0}^{\infty} \beta^t = W_0$$

$$\bar{c} \frac{1}{1-\beta} = W_0$$

$$\boxed{\bar{c} = (1-\beta) W_0}$$

Complete consumption smoothing across $\underline{\text{time}}$ & across histories s^t.

<u>Remark</u>: Source of wealth: Consumer owns an endowment $y_t(s^t)$ of time t, history s^t consumption good.

His <u>wealth</u> W_0 is

$$W_0 = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, y_t(s^t)$$

example: $S = \{0, 1\}$ — only depends on s_t, not history s^t

$$y_t(\cdot s^t) = y_t(s_t) = \begin{cases} \varepsilon & \text{if } s_t = 0 \\ 1-\varepsilon & \text{if } s_t = 1 \end{cases}$$

Thus, a sample endowment path is

s_t	0	0	1	0	1	1
y_t	ε,	ε,	$1-\varepsilon$,	ε,	$1-\varepsilon$,	$1-\varepsilon$, ...

Example 2: (Lucas, 1978) - a pure endowment representative agent economy.

$$c_t(s^t) = y_t(s^t) \leftarrow \text{exogenous}$$

representative consumer has to consume endowment in equilibrium. Our job: to find the equilibrium price vector $q_t^0(s^t)$ that reconciles a competitive, price-taking consumer to consuming only his endowment each period.

$\Rightarrow$ Substitute $c_t(s^t) = y_t(s^t)$ into (*) and "stare":

$$(+) \qquad \beta^t u'(y_t(s^t))\, \pi_t(s^t) = \theta\, q_t(s^t)$$

if prices satisfy (+), the consumer's FONC are satisfied: what about budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, c_t(s^t) \le \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, y_t(s^t)$$

automatic for $c_t(s^t) = y_t(s^t)$

Note: (1) any $\theta > 0$ works - we are just scaling prices.

(2) multiplying all prices $q_t^0(s^t)$ by a scalar > 0 just rescales θ.

"Choice of numeraire" - only relative prices matter.

Example 3: Two consumers with preferences ordered by , person i

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t(s^t) \qquad , i=1,2$$

$u' > 0, u'' < 0$

$\beta \in (0,1)$

Note: no i on u or $\pi_t(s^t)$ or β

$S = \{0, 1\}$

Consumer i's endowment streams

$y_t^1(s_t) = s_t$

$y_t^2(s_t) = 1 - s_t$

Note: $y_t^1(s_t) + y_t^2(s_t) = 1$ no aggregate risk

Try to find a competitive equilibrium price system $q_t^0(s^t)$ and allocation $\{c_t^1(s^t), c_t^2(s^t)\}_{t, s^t}$

Definition: A feasible allocation is an allocation $\{c_t^1(s^t), c_t^2(s^t)\}_{t, s^t}$ that satisfies

$$c_t^1(s^t) + c_t^2(s^t) \le y_t^1(s^t) + y_t^2(s^t)$$
$$\forall t, \forall s^t$$

Definition

A c.e. is a price system and feasible allocation such that given the price system, the allocation solves each household's optimum problem (i.e., it satisfies (*) and (**) for each household) (i.e., it satisfies (*) and (**) for each household).

How to <u>find</u> a competitive equilibrium. In general, a difficult problem: you must simultaneously solve all the conditions for household's optimality and feasibility. We'll study this later. For now we'll use guess & verify - <u>guess</u> an equilibrium price vector.

Example: make a lucky guess

$$q_t^0(s^t) = \beta^t \pi_t(s^t)$$

At this guess, FONC for i

$$\cancel{\beta^t} u'(c_t^i(s^t)) \cancel{\pi_t(s^t)} = \theta_i \cancel{\beta^t} \cancel{\pi_t(s^t)}$$

$$\Rightarrow \quad u'(c_t^i(s^t)) = \theta_i \quad \text{for } i = 1, 2$$

$$\Rightarrow \quad c_t^i(s^t) = \bar{c}^i \quad \text{for } i = 1, 2$$

Constant consumption $\forall$ t and s^t for each consumer i.

Feasibility: $c^1 + c^2 = 1$

To determine: division of 1 between c^1 & c^2

Use budget constraints:

for i:

$$\bar{c}^i \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) = \sum_{t=0} \sum_{s^t} \beta^t \cdot y_t^i(s^t)\, \pi_t(s^t)$$

or

$$\bar{c}^i \frac{1}{1-\beta} = W_0^i$$

Note: $$W_0 + W_1 = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t)\, 1 \quad y_t^1 + y_t^2 = 1$$

$$= \sum_{t=0}^{\infty} \beta^t \underbrace{\sum_{s^t} \pi_t(s^t)}_{1} = \frac{1}{1-\beta}.$$

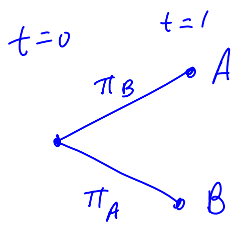
this is as far as we can get – to determine W_0^1 & W_0^2 separately, we need to know the $\pi_t(s^t)$'s numerical values.

Trees, 2

Saturday, February 20, 2010
5:24 PM

Very simple example of a tree.

$t = \{0, 1\}$ two periods



$\pi_A + \pi_B = 1$

$\pi_A \geq 0, \pi_B \geq 0$

Preferences:

$$u(c_0) + \beta u(c_1(A))\pi_A + \beta u(c_1(B))\pi_B$$

$$u' > 0, u'' < 0$$

expected utility. randomness realized at date 1.

endowment of consumer:

at $t=0$: y_0

at $t=1$: $y_1(A)$ in state A

$y_2(B)$ in state B

Commodities:

c_0 – consumption at date 0

$c_1(A)$ – consumption in state A at date 1

$c_1(A)$ – consumption in state A at date 1

$c_2(A)$ – consumption in state B at date 1

Prices:

q_0 = price of one unit of consumption at $t=0$

$q_1(A)$ = price of one unit of consumption at $t=1$ in state A

$q_1(B)$ = price of one unit of consumption at $t=1$ in state B

e.g. state A – rain

state B – no rain

or

state A – Yankees win

state B – Yankees lose

or ...

Consumer's problem:

Choose $\{c_0, c_1(A), c_1(B)\}$ to maximize

$$u(c_0) + \beta u(c_1(A))\pi_A + \beta u(c_1(B))\pi_B$$

s.t. $q_0 c_0 + q_1(A) c_1(A) + q_1(B) c_1(B)$

$$\leq q_0 y_0 + q_1(A) y_1(A) + q_1(B) y_1(B) \equiv W_0$$

Form Lagrangian:

$$L = u(c_0) + \beta u(c_1(A)) \pi_A + \beta u(c_1(B)) \pi_B$$

$$+ \underset{\substack{\uparrow \\ \text{Lagrange multiplier}}}{\theta} \Big[W_0 - \underbrace{q_0 c_0 - q_1(A) c_1(A) - q_2(B) c_2(B)}_{\text{– present value of consumption}} \Big]$$

FONC:

c_0 : $u'(c_0) - \theta q_0 = 0$ (interior solution because $u' > 0$)

$c_1(A)$: $\beta u'(c_1(A)) \pi_A - \theta q_1(A) = 0$

$c_2(B)$: $\beta u'(c_1(B)) \pi_B - \theta q_1(B) = 0$

θ: $W - q_0 c_0 - q_1(A) c_1(A) - q_1(B) c_1(B) = 0$

↑ budget constraint at equality because $u' > 0$

First FONC $\Rightarrow$

$$\theta = u'(c_0)/q_0$$

Substitute into 2nd & 3rd FONC's

$$\beta u'(c_1(A))\pi_A = \frac{q_1(A)}{q_0} u'(c_0)$$

$$\beta u'(c_1(B))\pi_B = \frac{q_1(B)}{q_0} u'(c_0)$$

Asset pricing

Monday, February 22, 2010
9:31 AM

Asset pricing with redundant assets or

"arbitrage free pricing"

Suppose there are complete markets

$s_t \in S \quad \forall t$

$s^t \in S^{t+1} \quad \forall t \quad , s^t = (s_t, s_{t-1}, \dots, s_0)$

$\pi_t(s^t) = \text{prob}(s^t)$

Preferences

$$\sum_t \beta^t \sum_{s^t} u(c_t(s^t))\, \pi_t(s^t)$$

Budget constraint

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, c_t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, y_t^0(s^t)$$

Suppose we add one (redundant) asset

it promises to pay one unit of consumption

for sure at a particular time T, e.g. $T=10$

One unit of the asset has price v_0 units of

the time 0 numeraire good. — the consumer can

buy or sell unlimited amounts of the

asset at time 0. The

consumer's budget constraint now becomes

$$\sum_t \sum_{s^t} q_t^0(s^t)\, c_t(s^t) + v_0 \cdot B$$

$$\leq \sum_t \sum_{s^t} q_t^0(s^t)\, y_t^0(s^t)$$

$$+ B \cdot \sum_{s^T} q_T^0(s^T)$$

where B = amount of the bond that he buys at $t=0$

After buying the bond, his entitlement to time T consumption is augmented by B for all histories s^T at T.

$\Rightarrow$ budget constraint is

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\left(c_t(s^t) - y_t^0(s^t)\right)$$

$$\leq B\left(\sum_{s^T} q_T^0(s^T) - v_0\right)$$

Stare:

If $\sum_{s^T} q_T^0(s^T) - v_0 > 0$

set B as large as possible "$B = +\infty$"

Buy the bond for v_0 and unpackage & sell

If $\sum_{s^T} q_T^0(s^T) - v_0 < 0$

set $B < 0 \implies B = -\infty$

sell the bond, then buy & repackage

" $y_t(s^t)$ " commodities

$\therefore$ If $\sum_{s^T} q_T^0(s^T) - v_0 \neq 0$

$\exists$ arbitrage profits: Perfectly sure, risk free profits. A "money machine". impossible in equilibrium. $\therefore$

(c) $v_0 = \sum_{s^T} q_T^0(s^t)$

* is the price of a risk free bond paying off for sure at T

Permanent income (stochastic)

Wednesday, February 03, 2010
8:59 AM

Permanent income model with random income:

$$\underset{n\times 1}{x_{t+1}} = \underset{n\times n}{A}\,\underset{n\times 1}{x_t} + \underset{n\times m}{C}\,\underset{m\times 1}{\varepsilon_{t+1}}$$

$$y_t = G x_t \qquad , \; y_t = \text{"labor income"}$$

$$\varepsilon_{t+1} \sim \mathcal{N}(0, \underset{m\times m}{I})$$

Budget constraint at $t \geq 0$:

$$(1) \quad F_{t+1} = \beta^{-1}\left[F_t + y_t - c_t\right]$$

β^{-1}: gross interest rate; F_t: financial assets; y_t: income from labor; c_t: cons^n

"Consumption function":

$$(2) \quad c_t = (1-\beta)\left[F_t + E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}\right]$$

derive this by replacing $\sum_{j=0}^{\infty} \beta^j y_{t+j}$ by its best forecast $E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}$.

Remark: This is valid when $u'(c_t)$ is linear in c_t. otherwise it is an approximation that ignores effects of risk aversion.

(1) & (2) $\Rightarrow$

$$F_{t+1} = \beta^{-1}\left[F_t + y_t - (1-\beta)\left(F_t + E_t\sum_{j=0}^{\infty}\beta^j y_{t+j}\right)\right]$$

$$= \beta^{-1}\left[(\cancel{1} - \cancel{1}-\beta))F_t + y_t - (1-\beta)E_t\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]$$

$$(3) \quad F_{t+1} = F_t + \beta^{-1}\left[y_t - (1-\beta)E_t\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]$$

decision rule: add difference between y_t and permanent income to financial assets.

---

(2) $\Rightarrow$

$$c_{t+1} = (1-\beta)\left[F_{t+1} + E_{t+1}\sum_{j=0}^{\infty}\beta^j y_{t+j+1}\right]$$

$$c_t = (1-\beta)\left[F_t + E_t\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]$$

$\Rightarrow$

$$c_{t+1} - c_t = (1-\beta)(F_{t+1} - F_t) + (1-\beta)\left[E_{t+1}\sum_{j=0}^{\infty}\beta^j y_{t+j+1} - E_t\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]$$

$$= (1-\beta)\cdot\underbrace{\beta^{-1}\left[y_t - (1-\beta)E_t\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]}_{F_{t+1}-F_t}$$

(*)

$$+ (1-\beta)\left[E_{t+1}\sum_{j=0}^{\infty}\beta^j y_{t+j+1} - E_t\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]$$

$$c_{t+1} - c_t = (1-\beta)\sum_{j=0}^{\infty}\beta^j\left(E_{t+1}y_{t+j+1} - E_t y_{t+j+1}\right)$$

Proof:

$$c_{t+1} - c_t = (1-\beta)\beta^{-1}\left[y_t - (1-\beta) E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$$

$$+ (1-\beta)\left[E_{t+1} \sum_{j=0}^{\infty} \beta^j y_{t+j+1} - E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$$

$$= (1-\beta) E_{t+1} \sum_{j=0}^{\infty} \beta^j y_{t+j+1} + (\beta^{-1} - 1) y_t$$

$$\underbrace{- (1-\beta)\left[(1-\beta)\beta^{-1} + 1 \right]}_{\substack{- (1-\beta)\left[\beta^{-1} - \cancel{1} + \cancel{1} \right] \\ - \beta^{-1}(1-\beta)}} E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}$$

$\Rightarrow$

$$c_{t+1} - c_t = (1-\beta) E_{t+1} \sum_{j=0}^{\infty} \beta^j y_{t+j+1} + (\beta^{-1} - 1) y_t$$

$$\underbrace{- \beta^{-1}(1-\beta) E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}}$$

$$\shortparallel$$

$$\underbrace{- (1-\beta) E_t \sum_{j=0}^{\infty} \beta^{j-1} y_{t+j}}$$

$$\underbrace{- (1-\beta) E_t \sum_{j=0}^{\infty} \beta^j y_{t+j+1} - (1-\beta)\beta^{-1} y_t}$$

$\Rightarrow$

$$c_{t+1} - c_t = (1-\beta) \sum_{j=0}^{\infty} \beta^j \left(E_{t+1} y_{t+j+1} - E_t y_{t+j+1} \right)$$

qed.

Now,

$$E_{t+1} \sum_{j=0}^{\infty} \beta^j y_{t+j+1} = G(I - \beta A)^{-1} x_{t+1}$$

$$E_t \sum_{j=0}^{\infty} \beta^j y_{t+j+1} = E_t y_{t+1} + \beta E_t y_{t+2} + \cdots$$

$$= G(I - \beta A)^{-1} A x_t$$

$\therefore$

$$\sum_{j=0}^{\infty} \beta^j \left(E_{t+1} y_{t+j+1} - E_t y_{t+j+1} \right)$$

$$= G(I - \beta A)^{-1} x_{t+1} - G(I - \beta A) A x_t$$

$$= G(I - \beta A)^{-1} \left[A x_t + C \varepsilon_{t+1} - A x_t \right]$$

$$= G(I - \beta A)^{-1} C \varepsilon_{t+1}$$

Therefore:

$$c_{t+1} - c_t = (1 - \beta) \underbrace{G(I - \beta A)^{-1} C}_{\text{present value of impulse response coefficients}} \varepsilon_{t+1}$$

$$c_{t+1} - c_t = \bar{H} \varepsilon_{t+1}, \quad \bar{H} = (1 - \beta) G(I - \beta A)^{-1} C$$

Example: Friedman-Muth

$$y_t = z_t + u_t \leftarrow \text{transitory}$$

z_t: persistent or "permanent" income

$$z_{t+1} = z_t + \sigma_1 \varepsilon_{1t+1}$$

$$u_{t+1} = \sigma_2 \varepsilon_{2t+1}, \qquad \varepsilon_{t+1} = \begin{pmatrix} \varepsilon_{1t+1} \\ \varepsilon_{2t+1} \end{pmatrix} \sim N(0, I_{2\times 2})$$

$$\begin{pmatrix} z_{t+1} \\ u_{t+1} \end{pmatrix} = \overset{A}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \overset{x_t}{\begin{pmatrix} z_t \\ u_t \end{pmatrix}} + \overset{C}{\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}} \begin{pmatrix} \varepsilon_{1t+1} \\ \varepsilon_{2t+1} \end{pmatrix}$$

$$y_t = \overset{G}{\begin{bmatrix} 1 & 1 \end{bmatrix}} \overset{x_t}{\begin{bmatrix} z_t \\ u_t \end{bmatrix}}$$

$$(I - \beta A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\beta & 0 \\ 0 & 1 \end{pmatrix}$$

$$(1 - \beta A)^{-1} = \begin{pmatrix} \frac{1}{1-\beta} & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore$

$$G(I - \beta A)^{-1} = (1 \quad 1) \begin{pmatrix} \frac{1}{1-\beta} & 0 \\ 0 & 1 \end{pmatrix} = \left(\frac{1}{1-\beta} \quad 1 \right) \begin{pmatrix} z \\ u \end{pmatrix}$$

$\therefore$

$$c_t = (1-\beta) \left[F_t + \frac{1}{1-\beta} z_t + u_t \right]$$

or

$$c_t = (1-\beta) F_t + z_t + (1-\beta) u_t$$

marginal propensity to consume out of permanent income = 1

marginal propensity to consume out of transitory income = $= (1-\beta)$

$$G(I-\beta A)^{-1} C \, \varepsilon_{t+1}$$

$$= \begin{bmatrix} \frac{1}{1-\beta} & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{pmatrix} \varepsilon_{1\,t+1} \\ \varepsilon_{2\,t+1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{1-\beta}\sigma_1 & \sigma_2 \end{pmatrix} \begin{pmatrix} \varepsilon_{1t+1} \\ \varepsilon_{2t+1} \end{pmatrix}$$

$$= \frac{1}{1-\beta}\sigma_1 \varepsilon_{1t+1} + \sigma_2 \varepsilon_{2t+1}$$

$$\therefore \quad c_{t+1} - c_t = \sigma_1 \varepsilon_{1t+1} + (1-\beta)\sigma_2 \varepsilon_{2t+1}$$

What about savings in financial assets F_t?

Earlier we showed

$$F_{t+1} - F_t = \beta^{-1}\left[y_t - (1-\beta) E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$$

but using our forecasting formulas here $\Rightarrow$

$$F_{t+1} - F_t = \beta^{-1} \left[G x_t - (1-\beta) G (I - \beta A)^{-1} x_t \right]$$

$$* \quad = \beta^{-1} G \left[I - (1-\beta)(I - \beta A)^{-1} \right] x_t$$

Apply (*) to the Friedman-Muth example:

$$(1-\beta) G (I - \beta A)^{-1} x_t = \begin{bmatrix} 1 & (1-\beta) \end{bmatrix} \begin{pmatrix} z_t \\ u_t \end{pmatrix}$$

$$G I x_t = \begin{pmatrix} z_t \\ u_t \end{pmatrix}$$

$$\Rightarrow G \left(I - (1-\beta)(I - \beta A)^{-1} \right] x_t$$

$$= \left(\begin{bmatrix} 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1-\beta \end{bmatrix} \right) \begin{pmatrix} z_t \\ u_t \end{pmatrix}$$

$$= \begin{bmatrix} 0 & \beta \end{bmatrix} \begin{pmatrix} z_t \\ u_t \end{pmatrix}$$

multiply by $\beta^{-1} \Rightarrow$

$\Rightarrow$

$$F_{t+1} - F_t = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{pmatrix} z_t \\ u_t \end{pmatrix}$$

$$F_{t+1} - F_t = u_t$$

Thus: the consumer spends <u>all</u> of z_t, saves nothing.

He spends $(1-\beta)$ of u_t and saves β of u_t.

Incomplete vs complete

Tuesday, February 23, 2010
12:29 PM

Complete markets setting

$$\max_{\{c_t(s^t)\}_{t,s^t}} \sum_t \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t)$$

$$\text{s.t.} \sum_t \sum_{s^t} q_t^0(s^t)\left(c_t(s^t) - y_t(s^t)\right) \le 0$$

FONC: $\beta^t u'(c_t(s^t)) \cdot \pi_t(s^t) = \theta\, q_t^0(s^t)$

divide for two histories at t, $t+1$

$\Rightarrow$

$$(*) \quad \frac{\beta u'(c_{t+1}(s^{t+1}))\, \pi_{t+1}(s^{t+1})}{u'(c_t(s^t))\, \pi_t(s^t)} = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)}$$

$$\frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \equiv \pi_{t+1}(s^{t+1} \mid s^t)$$

~ conditional probability

very easy to compute in our case because we generated $\pi_t(s^t)$ from Markov chain

$$\frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} \equiv q_{t+1}^t(s^{t+1} \mid s^t)$$

~ one step ahead "pricing kernel"

Thus we can write $(*)$ as

Thus we can write (*) as

$$(+) \qquad \beta \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \pi_{t+1}(s^{t+1}|s^t) = q^t_{t+1}(s^{t+1}|s^t)$$

Remark:

$$\sum_{s^{t+1}|s^t} q^t_{t+1}(s^{t+1}|s^t) = \text{price at node } s^t$$

of a risk-free or <u>sure</u> claim on consumption at time $t+1$

thus

$$\sum_{s^{t+1}|s^t} q^t_{t+1}(s^{t+1}|s^t) \equiv \left(R_t(s^t)\right)^{-1}$$

= reciprocal of gross one period interest rate at node s^t

then $(+) \Rightarrow$

$$(\#) \qquad \beta R_t \sum_{s^{t+1}} \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \pi_t(s^{t+1}|s^t) = 1$$

Note: $(+) \Rightarrow (\#)$ but <u>not</u> $(\#) \Rightarrow (+)$

(#) is weaker - holds "on average" while (+) is "state by state"

<u>Incomplete</u> markets setting:

Consumer has preferences

$$\sum_t \beta^t \sum_{s^t} u(c_t(s^t))\,\pi_t(s^t)$$

but now is subject to sequence of budget constraints:

$$A_{t+1}(s^t) = R_t(s^t)\left[y_t(s^t) - c_t(s^t) + A_t(s^{t-1})\right]$$

$\uparrow$ not s^{t+1} $\qquad\qquad$ $\uparrow$ s^{t-1} not s^t

Form Lagran gian:

$$L = \sum_t \sum_{s^t} \beta^t u(c_t(s^t))\,\pi_t(s^t)$$

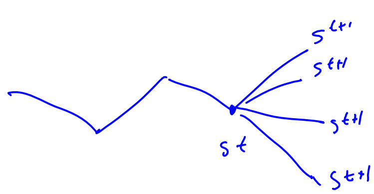
$$+ \sum_t \sum_{s^t} \lambda_t(s^t)\left[A_t(s^{t-1}) + y_t(s^t) - c_t(s^t) - R_t^{-1}(s^t)A_{t+1}(s^t)\right]$$

FONC:

(1) $\quad c_t(s^t)$: $\quad \beta^t u'(c_t(s^t))\,\pi_t(s^t) = \lambda_t(s^t)$

(2) $\quad A_{t+1}(s^t)$: $-\lambda_t(s^t)R_t^{-1}(s^t) + \sum_{s^{t+1}|s^t} \lambda_{t+1}(s^{t+1})$

$\uparrow$ Sum over all s^{t+1} stemming from s^t



Substitute (2) into (1) $\quad \Rightarrow$

$$R_t^{-1}(s^t)\,\beta^t u'(c_t(s^t))\,\pi_t(s^t) = \beta^{t+1} \sum_{s^{t+1}|s^t} u'(c_{t+1}(s^{t+1}))\,\pi_{t+1}(s^{t+1})$$

$\Rightarrow$

$$1 = \beta R_t(s^t) \sum_{s^{t+1}|s^t} \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \underbrace{\pi_{t+1}(s^{t+1}|s^t)}_{\left(\frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)}\right)}$$

or

$$1 = \beta E_t R \frac{u'(c_{t+1})}{u'(c_t)}$$

This is a version of (H) under complete markets.
But (H) does not hold.

Punch lines:

(1) under complete markets

intertemporal marginal rates of substitution

$$\beta \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \pi_t(s^{t+1}|s^t)$$

are equated for all consumers able to trade at relative prices $q^t_{t+1}(s^{t+1}|s^t)$

(2) Under incomplete markets with only risk-free security with gross return $R_t(s^t)$,

only "average" intertemporal rate of substitution

$$\beta \sum_{s^{t+1}|s^t} \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \pi_{t+1}(s^{t+1}|s^t)$$

are equated.

Search theory

Thursday, March 04, 2010
10:08 AM

- First some facts about non-negative random variables.

$p \sim$ a nonnegative r.v. with p.d.f. (probability density function) $f(p)$ and c.d.f. (cumulative density function)

$F(p) = \int_0^s f(s)\,ds$.

Assume $F(0) = 0$, $F(B) = 1$.

B is the upper bound on the random variable.

Mean or expectation of p $= \int_0^B p f(p)\,dp \equiv E(p)$

To get another useful formula for $E(p)$ remember integration by parts:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

apply integration by parts to

$\int_0^B p\, f(p)\,dp$ by taking

$u = p$, $dv = f(p)\,dp$ $\Rightarrow$

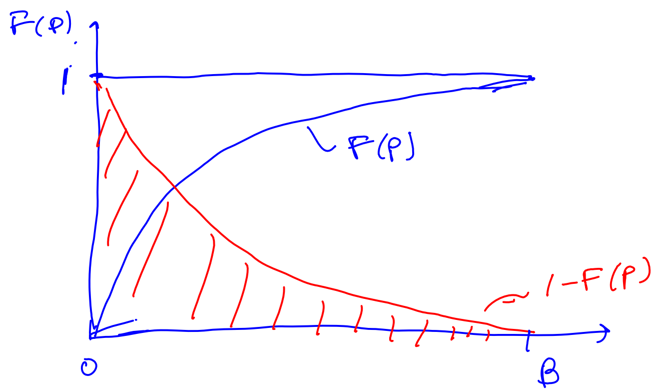
$du = dp$, $v = F(p)$ $\Rightarrow$

$$\int_0^B \rho f(\rho)\,d\rho = \rho F(\rho)\Big|_0^B - \int_0^B F(\rho)\,d\rho$$

$$= B - \int_0^B F(\rho)\,d\rho$$

$$(*) \qquad = \int_0^B [1 - F(\rho)]\,d\rho = E(\rho)$$

(*) is alternative formula for the mean of a nonnegative r.v.



$E\rho = \int_0^1 (1 - F(\rho))\,d\rho$

Consider two independent random variables ρ_1 and ρ_2, both drawn from F. Event

$$\{\rho_1 < \rho\} \cap \{\rho_2 < \rho\}$$

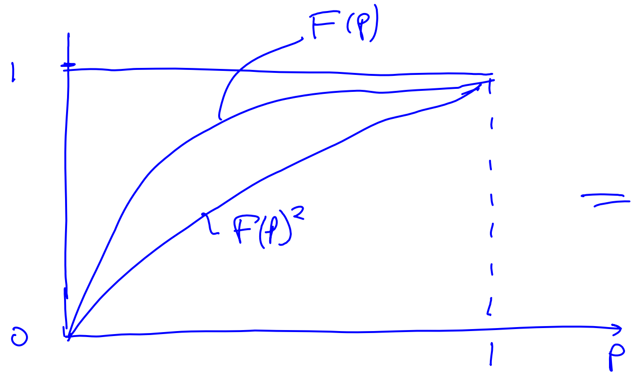
"intersection" or "and"

which means $\max(\rho_1, \rho_2) < \rho$

The probability that $\max(\rho_1, \rho_2) < \rho$ is evidently

$$F(\rho) \cdot F(\rho) = F(\rho)^2$$

Then $E \max(p_1, p_2) = \int_0^B [1 - F(p)^2]\,dp$

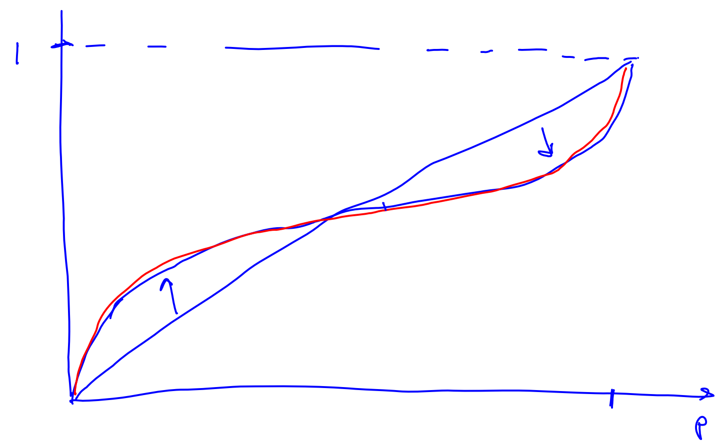


$\Rightarrow E \max(p_1, p_2) > E p_1$

Mean preserving spread:

Keep $\int_0^B F(p)\,dp$ constant –

but more probability to both tails:



both F's have same $\int_0^B F(p)\,dp$

red one has more spread

McCall search model.

Infinitely lived worker wants to maximize

$$E \sum_{t=0}^{\infty} \beta^t y_t \qquad \beta \in (0,1)$$

expectation

$\{ w$ if employed

$$y_t = \begin{cases} c & \text{if unemployed.} \end{cases}$$

Worker starts out unemployed. Each period, a previously unemployed worker draws one, and only one, offer to work forever at wage w. The wage is drawn from a c.d.f. F, where $F(0) = 0$, $F(B) = 1$ for $B > 0$, $F'(w) = f(w)$

c.d.f. density

There is no recall of past offers. Each period, the unemployed worker can either accept the current wage drawn from F, or reject it, collect $c > 0$ this period, and wait to draw another offer next period. Again, if he <u>accepts</u> the offer, he works forever at the wage w; he neither quit nor be fired.

<u>Problem:</u> Find the worker's optimal strategy.

<u>Solution:</u> <u>Let</u> Q = optimal value of the problem for a worker who is about to draw a wage offer; i.e.,

$$Q = E_{-1} \sum_{t=0}^{\infty} \beta^t y_t$$

↑ expectation *before* wage is drawn.

Let $v(w)$ be the optimal value of the problem for a previously unemployed worker who has just drawn offer w and is about to decide what to do.

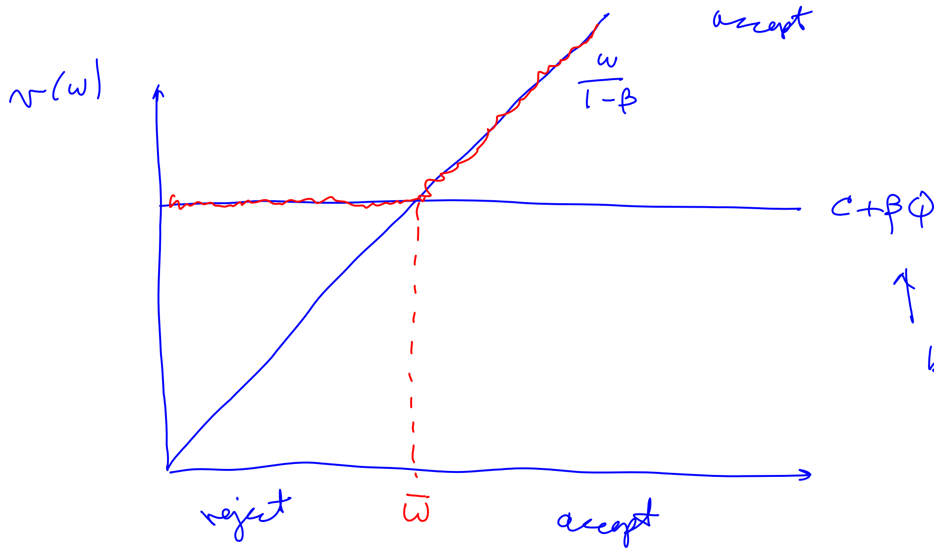
Evidently,

$$Q = \int_0^B v(w) f(w)\, dw$$

Key observation:

$$(*) \quad v(w) = \max_{\{accept,\ reject\}} \left\{ \frac{w}{1-\beta},\ c + \beta Q \right\}$$

↑ accept ↑ reject



↑ independent of w because $Q = \int v(w) f(w) dw$ — w has been integrated out

(*) is called a Bellman equation. (Dynamic programming)

How can we solve (*)?

Method 1: Iterate on

dummy for integration

$$v_{j+1}(w) = \max\left\{ \frac{w}{1-\beta},\ c + \beta \int_0^B v_j(\tilde{w}) f(\tilde{w})\, d\tilde{w} \right\}$$

next period's wage

start from $v_0(w) = 0$

This is guaranteed to converge.

<u>Method 2:</u>

$$v(w) = \max\left\{ \frac{w}{1-\beta},\ c + \beta \int_0^B v(\tilde{w}) f(\tilde{w})\, d\tilde{w} \right\}$$

evidently

$$\frac{\bar{w}}{1-\beta} = c + \beta \int_0^B v(\tilde{w}) f(\tilde{w})\, d\tilde{w}$$

$$= c + \beta \int_0^{\bar{w}} \frac{\bar{w}}{1-\beta} f(\tilde{w})\, d\tilde{w} + \beta \int_{\bar{w}}^B \frac{\tilde{w}}{1-\beta} f(\tilde{w})\, d\tilde{w}$$

$$\frac{\bar{w}}{1-\beta}\left[\int_0^{\bar{w}} f(\tilde{w})\, d\tilde{w} + \int_{\bar{w}}^B f(\tilde{w})\, d\tilde{w}\right] = c + \beta \int_0^{\bar{w}} \frac{\bar{w}}{1-\beta} f(\tilde{w})\, d\tilde{w}$$

$$+ \beta \int_{\bar{w}}^B \frac{\tilde{w}}{1-\beta} f(\tilde{w})\, d\tilde{w}$$

$$\bar{w} \int_0^{\bar{w}} f(\tilde{w})\, d\tilde{w} - c = \frac{1}{1-\beta} \int_{\bar{w}}^B (\beta\tilde{w} - \bar{w}) f(\tilde{w})\, d\tilde{w}$$

add $\bar{w} \int_{\bar{w}}^B f(\tilde{w})\, d\tilde{w}$ to both sides to get

$$(\bar{w} - c) = \frac{\beta}{1-\beta} \int_{\bar{w}}^B (w' - \bar{w}) f(w')\, dw'$$

up

Cost of searching one more period

benefit of searching one more period (switch $\tilde{w}$ to w')

Benefit of search:

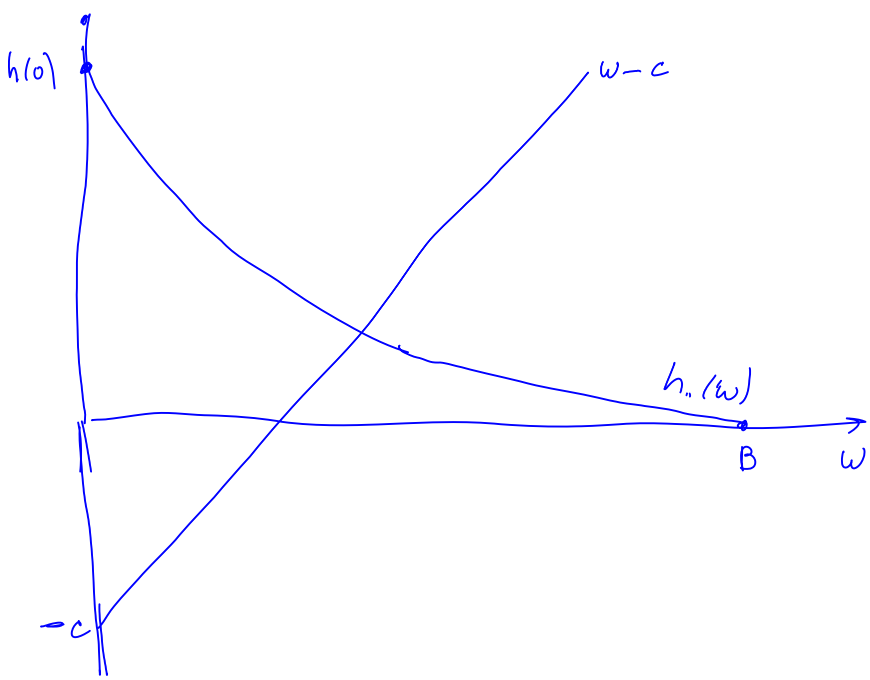
$$h(w) = \frac{\beta}{1-\beta} \int_{w}^{B} (w' - w) f(w') \, dw'$$

$$h(0) = \frac{\beta}{1-\beta} E(w) > 0$$

$$h(B) = 0$$

$$h'(w) = \frac{-\beta}{1-\beta} [1 - F(w)] < 0$$

$$h''(w) = \frac{\beta}{1-\beta} F'(w) > 0$$



Firing: probability of $\alpha > 0$ of being fired at the beginning of each period, after the first period on the job.

Let $\hat{v}(w)$ be the optimal value for a previously unemployed worker who has offer w in hand and is about to choose what to do.

If reject offer, get

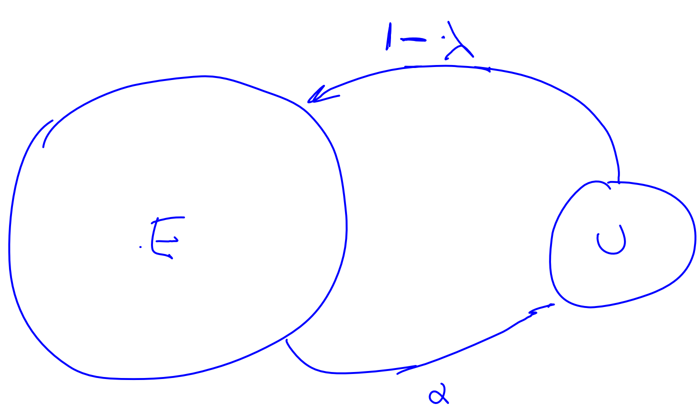
$$\hat{v}(w) = c + \beta \int \hat{v}(w') f(w')\, dw'$$

If accept offer, get

unemployment compensation tomorrow

$$\hat{v}(w) = \underbrace{w}_{\text{today}} + \underbrace{\beta(1-\alpha)\hat{v}(w)}_{\text{not fired tomorrow}} + \underbrace{\beta\alpha\,[c}_{\text{fired tomorrow}} + \underbrace{\beta \int \hat{v}(w') f(w')\, dw']}_{\text{day after tomorrow}}$$

$$\tilde{v}(w) = \max_{\text{accept, reject}} \left\{ w + \beta(1-\alpha)\hat{v}(w) + \beta\alpha\,[c + \beta E \hat{v}],\ c + \beta E \hat{v} \right\}$$



$\lambda = \int_0^{\bar{w}} f(w')\, dw'$

$\|$

prob stay unemployed

$1 - \lambda = \int_{\bar{w}}^{B} f(w')\, dw'$

= prob leave

Unemployment

Kareken-Wallace

Wednesday, March 10, 2010
1:57 PM

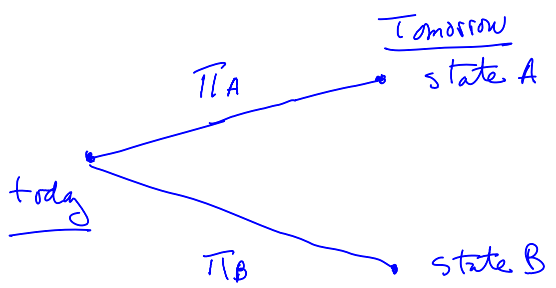
Two quite different models of "financial crises" and what government can do to contribute to them or stop them.

- Moral hazard (Kareken-Wallace)
- Deposit insurance and bank runs (Diamond-Dybvig)

Kareken-Wallace moral hazard analysis, 1977

message: deposit insurance without regulation creates incentives for banks to take too much risk.

A very simple two period ("today" and "tomorrow") analysis that assumes two states tomorrow and that takes as exogenous prices for state contingent claims.



today = time 0

tomorrow = time 1

π_A = prob of state A

π_B = prob of state B

Exogenous prices at which a bank can trade:

p_A = price of one unit of consumption tomorrow if state A occurs, measured in time 0 good per good at time 1 in state A

p_B = price of one unit of consumption tomorrow if state B occurs, measured in time 0 good per good at time 1 in state B

Risk-free interest rate: $(1+r)$

$$p_A + p_B = \frac{1}{1+r}$$ = Price of one unit of consumption tomorrow for sure in units of consumption at $t=0$

Model of a bank.

Today $(t=0)$

receives deposits D

choose a portfolio:

claims paying off at $t=1$ in state A: Q_A

claims pay off in state B at $t=1$: Q_B

Budget constraint: $p_A Q_A + p_B Q_B = D$

Assets	Liabilities

Balance sheet at $t=0$

Balance sheet at t = 0

$p_A Q_A$	D
$p_B Q_B$	

$t=1$: • the state $S \in \{A, B\}$ is realized; say state S.

- collect Q_S
- pay off depositors $(1+r)D$
- keep any profits $Q_S - (1+r)D$
- if $Q_S - (1+r)D < 0$, pay depositors Q_S.

Bank profits:

$$\text{Profit} = \max\{Q_S - (1+r)D,\ 0\}$$

Assume

(1) There is no deposit insurance

(2) Depositors will put D into the bank at $t=0$ only if

in state A $\quad Q_A \geq (1+r)D$

in state B $\quad Q_B \geq (1+r)D$

Guess and verify: the only portfolio that works is

$$Q_A = Q_B = \bar{Q}$$

Note: this is a risk-free portfolio — same Q for states A & B

$$\rho_A \bar{Q} + \rho_B \bar{Q} = D$$

$$(\rho_A + \rho_B)\bar{Q} = D$$

$$\bar{Q} = \frac{D}{\rho_A + \rho_B}, \quad \text{but } \frac{1}{\rho_A + \rho_B} = 1+r$$

So,

in state A: $Q_A = \bar{Q} = (1+r)D$

in state B: $Q_B = \bar{Q} = (1+r)D$

expected profit of bank:

$$\pi_A \max\{\underbrace{Q_A - (1+r)D, 0}_{0}\} + \pi_B\{\underbrace{Q_B - (1+r)D, 0}_{0}\}$$

$$= 0.$$

<u>Remark</u>: Any other portfolio would involve the bank taking on risk and would lead to

$Q_s - (1+r)D < 0$ in some state,

causing all depositors to desert the bank.

Conclusion: without deposit insurance,

banks only attract depositors if they have risk-free portfolios. Doing so gives the bank zero profits.

Remark: Free entry into the business of banking can rationalize the zero profits condition that banks offer deposit rates $(1+r) = (p_A + p_B)^{-1}$.

Deposit insurance:

The government offers deposit insurance that guarantees a return of $(1+r)$ to depositors. If the bank's assets can't cover $(1+r)D$ at $t=1$, the government makes up the difference.

Consider a bank portfolio with

$$Q_A > \bar{Q} \quad \text{and} \quad Q_B < \bar{Q}$$

$$\text{where} \quad \bar{Q} = \frac{D}{p_A + p_B}$$

$\Rightarrow$ when $s = B$ (state B occurs), then the bank defaults at $t=1$, activating the deposit insurer to pay the depositors the residual.

- With deposit insurance, the bank can attract depositors regardless of its portfolio, even if it is risky.

if it is risky.

- Key fact when there is deposit insurance, it can happen that

$$D > p_A Q_A + p_B Q_B$$

$$D - \underbrace{(p_A Q_A + p_B Q_B)}_{\text{value of bank portfolio}} = \text{value of deposit insurance to the bank}$$

when $Q_A > \bar{Q} = \dfrac{D}{p_A + p_B}$,

the deposit insurer is taking on risk - and is implicitly issuing claims to the depositors in the event that $s = B$.

- ———

expected profits of the bank with $Q_A > \bar{Q}$, $Q_B < \bar{Q}$:

$$= \pi_A \cdot \max\{\underbrace{Q_A - (1+r)D, 0}_{> 0}\} + \pi_B \cdot \max\underbrace{\{Q_B - (1+r)D, 0\}}_{= 0}$$

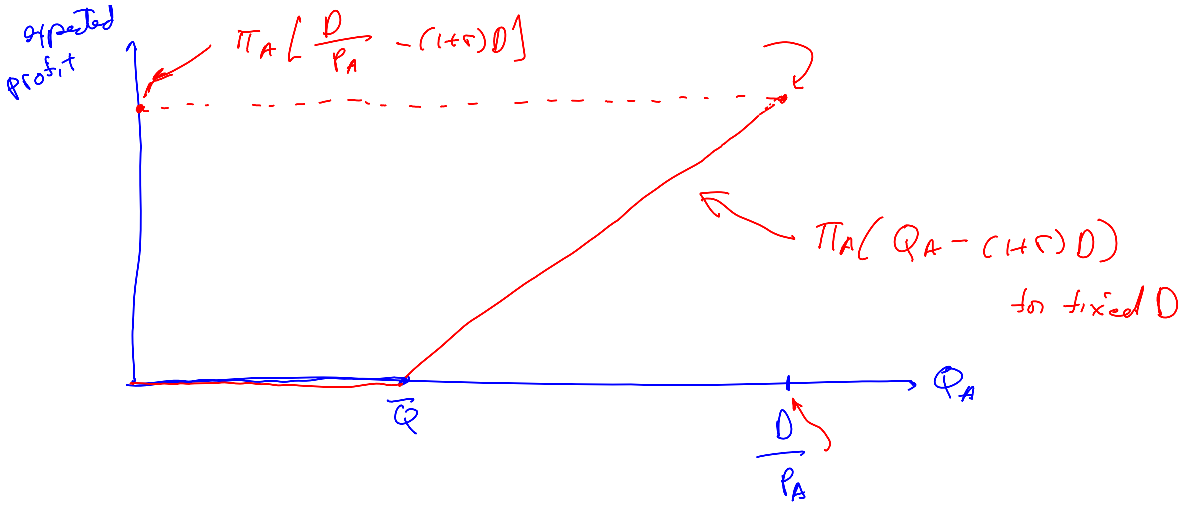
$$= \pi_A \max\{Q_A - (1+r)D\}$$

Highest profits occur where

$$Q_A = \frac{D}{p_A}, \quad Q_B = 0$$

(very risky portfolio!)

expected π ↑ , $\pi_A\left[\frac{D}{p_A} - (1+r)D\right]$



Highest profit value for Q_A

$$Q_A = \frac{D}{p_A}, \quad Q_B = 0 \qquad \text{when} \quad Q_A > \bar{Q}$$

Profits when $Q_A > \bar{Q}$

Expected profits =

$$\pi_A\left[\frac{D}{p_A} - (1+r)D\right]$$

$$= \pi_A D\left[\frac{1}{p_A} - \frac{1}{p_A + p_B}\right] > 0$$

Conclusion: set D arbitrarily large, and take on maximum risk. A force for banks to grow large and to expose the insurance agency to risks of big losses.

When $Q_A = \frac{D}{p_A}$ and $Q_B = 0$, the expected value of the losses of the insurance agency are

$$\pi_B \underbrace{(r+1)D}_{\text{payment in state B because } Q_B = 0.}$$

Now consider a bank portfolio with $Q_A < \bar{Q}$ and
$Q_B > \bar{Q}$

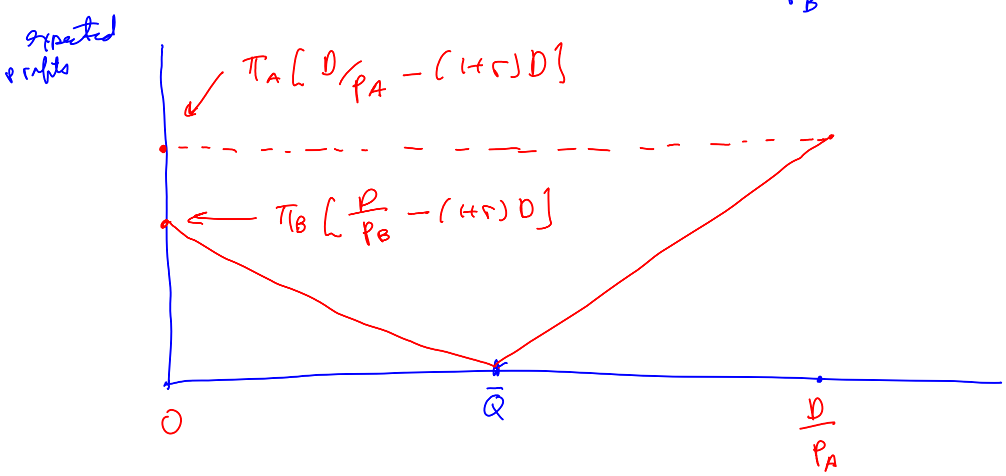
expected profit of bank =

$\pi_A \cdot \max\{Q_A - (1+r)D, 0\} + \pi_B \max\{Q_B - (1+r)D, 0\}$

$= 0 + \pi_B [Q_B - (1+r) D]$

$= \pi_B \cdot \left[\frac{D - p_A Q_A}{p_B} - (1+r) D \right]$

highest profit : $Q_A = 0$, $Q_B = \frac{D}{p_B}$



Whether bank sets $Q_A = 0$ or $Q = \frac{D}{p_A}$ for a given D

depends on $\frac{\pi_B}{p_B}$ relative to $\frac{\pi_A}{p_A}$.

- bank wants to set D arbitrarily large when it has deposit insurance.

Conclusion: Banks maximize expected profits by taking on much risk. (and becoming very large)

Remarks of Kareken & Wallace:

(1) Without deposit insurance and without regulation, bankruptcy does not occur because banks take on no risk. Banks take on no risk because if they did, depositors would desert them. Force for making banks be cautious - depositors watching banks' portfolios

(2) If bank liabities are insured, as under FDIC at a premium that is independent of portfolio risk (we set the premium to 0 in the above model) and if banks' portfolios are not regulated, then in some future state of the world there are numerous bankruptcies.

Thus, you cannot simultaneously favor

- deposit insurance (to avoid bank runs)
- unregulated banking

Regulation of portfolios:

- Capital requirements
- direct restrictions on portfolio composition

Similar forces at work:

articles by Jeffrey Sachs and Joe Stiglitz

Diamond-Dybvig

Wednesday, March 10, 2010
7:24 PM

Diamond-Dybvig model of bank panics

Themes and ideas:

- benefits of societal risk-sharing achieved through financial intermediaries
- private information that gives rise to incentive compatibility constraints ("truth telling")
- individuals' best responses to their conjectures about aggregate behavior
- multiple equilibria (interpretable as a "bank run")
- government policy as a device to select unique (good) equilibrium (deposit insurance)

Model thus has many good features not present in the Kareken-Wallace model. But . . .

it neglects moral hazard.

Model justifies many of Bernanke's actions.

There is a large number D of potential depositors endowed with 1 unit of a good at time 0. The consumers live at $t = 0, 1, 2$; and are *ex ante* identical.

With probability $\pi \in (0,1)$, a consumer wants to consume at $t=1$ and gets utility $u(c_1)$; and gets consume at $t=1$ and gets utility $u(c_1)$; and gets no utility from consumption c_2 at $t=2$;

With prob $(1-\pi)$, the consumer wants to consume at date $t=2$ and receive utility $u(c_2)$ from consuming c_2. He gets no utility from c_1 at $t=1$.

Assume that $u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$, $\gamma > 1$.

A fraction π of the population of size D is the "early" type of consumer, a fraction $(1-\pi)$ is the "late" type of consumer. } "liquidity" shocks

At $t=0$, consumers don't know their types. But at time $t=1$, all consumers learn their types. This knowledge is private - and known only to themselves } liquidity shock is private information

The 1 unit of the good at time 0 can be invested in a technology at time 0.

It will yield 1 unit of time 1 good if the project is terminated at $t=1$, and $(1+r)$ units of the time 2 good if left in the investment until time $t=2$. $r > 0$

If the good is withdrawn from the investment technology at time 1, it can be stored until time 2 without loss or gain - i.e. 1 unit of time 1 good is transformed into 1 unit of time 2 good outside the investment

1 unit of time 2 good outside the investment technology.

Technology

time	0	1	2
input, output	1 $\to$	1	
	1 $\longrightarrow$		$(1+r)$
		1 —	1

Remarks: (1) The preference structure is designed to capture "liquidity" shocks – random impulses for early withdrawal.

(2) The technology with $(1+r)$, $r>0$ gross return if left in for two period, but gross return 1 if "liquidated prematurely" is designed to mimic the situation faced by financial intermediaries forced to make premature liquidations.

(3) Because there is a large number – think of a continuum of measure D – of ex ante identical consumers, there is scope for a social insurance arrangement.

A bank can play this role.

Autarky

First, let's consider what would happen if the consumer were in autarky — isolated and alone. He would confront the investment technology on his own and choose c_1 and c_2 as follows:

a) at $t=0$, he would invest 1 in the investment technology.

b) at $t=1$, his type is realized. If he is an early type, he would withdraw the entire amount and set $c_1 = 1$. If he is a late type, he leaves the investment in until maturity and enjoys $c_2 = (1+r)$ at $t=2$.

Note: it follows that

$$(*) \quad \frac{u'(c_1)}{u'(c_2)} = \frac{1}{(1+r)^{-\gamma}}$$

at the autarky allocation $c_1 = 1$, $c_2 = 1+r$

A benevolent planner:

Let c_1 = consn of an early type at $t=1$
c_2 = " " a late " at $t=2$

Consider a benevolent planner who chooses c_1 & c_2 to maximize the ex-ante expected utility of a representative consumer

$$\pi u(c_1) + (1-\pi) u(c_2)$$

subject to the resource constraint

$$(1-\pi)c_2 = (1+r)\,[\underbrace{1-\pi c_1}_{\text{amount left in the investment technology}}]$$

$$\pi c_1 + (1-\pi)\frac{c_2}{1+r} = 1$$

fraction of type 1

fraction of type 2

and

(*) $\quad c_2 \geq c_1$

"truthtelling" constraint - needed to get late types to wait & leave the investment in the technology until $t=2$. - otherwise, they would have incentive to withdraw and store on own with return 0.

because of private information

Note: we could put a Lagrange multiplier on $c_2 \geq c_1$ too. But we won't. Instead we'll...

Assume (*) satisfied, then check it later.

$c_2 \geq c_1$ is called an incentive compatibility constraint. (due to private information)

Formulate Lagrangian:

$$L = \pi u(c_1) + (1-\pi)u(c_2)$$
$$+ \theta\left[1 - \pi c_1 - \frac{(1-\pi)}{(1+r)}c_2\right]$$

θ is Lagrange multiplier

FONC: $\quad u'(c_1) = \theta$

$$u'(c_2) = \theta\frac{1}{1+r}$$

$$\Rightarrow \frac{u'(c_1)}{u'(c_2)} = (1+r),$$

Note: this differs from the autarky allocation (**) when $\gamma \neq 1$

$$u'(c) = c^{-\gamma} \Rightarrow$$

[but incentive constraint

$$\left(\frac{c_2}{c_1}\right)^{\gamma} = (1+r)$$

satisfied only when $\gamma \geq 1$]

$$c_2 = c_1 (1+r)^{\frac{1}{\gamma}}$$

Substitute into feasibility constraint to get

$$1 = \pi c_1 + \frac{(1-\pi)}{(1+r)} (1+r)^{\frac{1}{\gamma}} c_1$$

so

$$c_1 = \frac{1}{\pi + \frac{(1-\pi)}{(1+r)^{1-1/\gamma}}}$$

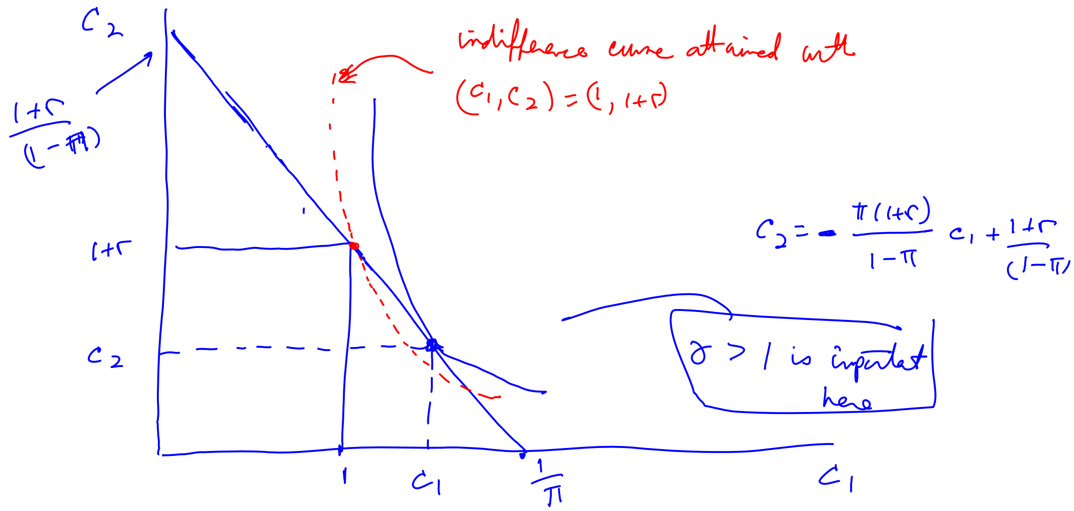
$\gamma > 1 \Rightarrow c_1 > 1$ (where $(c_1, c_2) = (1, 1+r)$ under "autarky")

$$c_2 = c_1 (1+r)^{\frac{1}{\gamma}}$$

$$= \frac{(1+r)^{\frac{1}{\gamma}}}{\pi + \frac{(1-\pi)}{(1+r)^{1-1/\gamma}}}$$

$$c_2 = \frac{(1+r)^1}{(1+r)^{1-\frac{1}{\gamma}}\pi + (1-\pi)} < 1+r$$

See following indifference curve for welfare gain with $\gamma > 1$:



social intertemporal transformation rate: $\frac{\pi}{1-\pi}(1+r)$

$$c_2 = -\frac{\pi(1+r)}{1-\pi} c_1 + \frac{(1+r)}{(1-\pi)}$$

$c_1 = 1 \Rightarrow$

$$c_2 = -\frac{\pi(1+r)}{(1-\pi)} + \frac{(1+r)}{(1-\pi)} = \frac{(1-\pi)}{1-\pi}(1+r) = 1+r$$

$\therefore (1, (1+r))$ is on the feasibility line

Note: the "truth telling" constraint

$c_2 > c_1$ is satisfied for us here when $\gamma > 1$

because:

$$c_2 = (1+r)^{\frac{1}{\gamma}} c_1 > c_1 \checkmark$$

Thus: autarky $c_1 = 1$, $c_2 = (1+r)$

risk-sharing: $c_1 > 1$, $c_2 = (1+r)^{\frac{1}{\gamma}} c_1 < (1+r)$

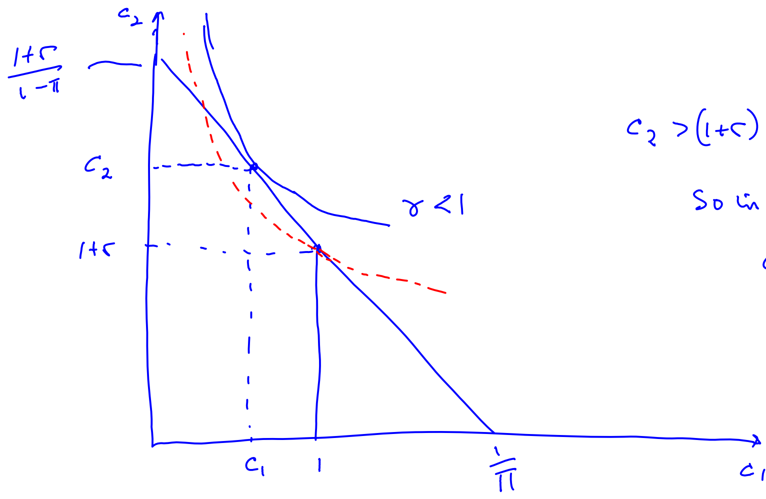
when $\gamma > 1$

when $\gamma < 1$, we have $c_1 < 1$, $c_2 = (1+r)^{\frac{1}{\gamma}} c_1 > c_1$

so long as $r > 0$, $\gamma > 0$

so incentive constraint still satisfied

When $\gamma < 1$ the picture is



$c_2 > (1+r) > 1 > c_1$

So incentive constraint

$c_2 > c_1$ is satisfied

Remark: $c_2 = (1+r)^{\frac{1}{\gamma}} c_1$ or $\frac{c_2}{c_1} = (1+r)^{\frac{1}{\gamma}}$

$\Rightarrow$ bigger "spread" when $\gamma < 1$.

Implement this risk-sharing arrangement with a banking deposit contract. The contract imposes a sequential service constraint - "first come, first serve".

Let $q = c_1$ from the planning problem.

$q = c_1 > 1$ $\boxed{q = c_1}$

Let D = deposits

W = withdrawals at $t=1$.

Bank offers q to early withdrawers,

$\max(D - qW, 0)(1+r)$ to late withdrawers.

If $W = \pi D$ if only early types withdraw then payout to late withdrawers is $(1+r)[D - qD]$, so return is

$$(1+r)\frac{D - qW}{(1-\pi)D} = \frac{D - D\pi q}{(1-\pi)D}(1+r)$$

[$(1-\pi)D$ = late types' deposits]

$$= \frac{(1-\pi q)(1+r)}{(1-\pi)}$$

$$= (1 - \pi c_1)\left(\frac{1+r}{1-\pi}\right) = c_2 \text{ in optimum of planning problem.}$$

If you think no one else will run, this will be your expected return $c_2 \equiv q > c_1$, so you won't want to run.

gross return in period 1 withdrawal

$$q \text{ if } D - qW \geq 0 ; \quad [W = \text{withdrawers}]$$

but if $qW > D$, can't pay everyone q; then pay

- q to the first x people in line, where $qx = D \Rightarrow x = \frac{D}{q} < D$ (because $q > 1$)

$\Rightarrow$

$$\frac{1}{q} = \frac{x}{D} = \text{fraction of people getting paid}$$

- 0 to fraction $\left(1 - \frac{1}{q}\right)$

If a late type person expects everyone to run to the bank in period 1, the expected return to you from running is

$$q \cdot \frac{1}{q} + 0 \cdot \left(1 - \frac{1}{q}\right) = 1$$

If one expects everyone to run to the bank in period 1, the expected return from your not running is 0.

$\therefore$ if you expect everyone to run, you run too.

How to cure the threat of a bank run.

Government deposit insurance: if government promises to pay everyone who withdraws early $q > 1$, then there will be no runs. The government will have to pay nothing out in equilibrium — (though it achieves this equilibrium by promising to pay it out off the equilibrium).

Diamond Dybvig mention one big caveat:

moral hazard. — not included in the analysis

Remarks about $\gamma < 1 \Rightarrow c_1 < 1$, $c_2 = (1+r)^{\frac{1}{\gamma}} c_1$ case.

Bank contract promises to pay $q < 1$ early withdrawers. Two observations

- If $W = D$ (everyone withdraws,
  $$D - qW = D(1-q) > 0$$
  the bank can pay everyone.
- If everyone a fraction $1-\varepsilon$ withdraws and ε don't, the return to the late types who leave in is

$$(1+r)\frac{D - q(1-\varepsilon)D}{\varepsilon} = \frac{(1+r)D[1-q(1-\varepsilon)]}{\varepsilon}$$

becomes arbitrarily large as $\varepsilon \to 0$

$\therefore$ no late type person ever wants to withdraw.

conclusion: with $\gamma < 1$ (and $\therefore$ $c_1 < 1$), there can be no runs with or without deposit insurance.

Homework 3

Sunday, March 28, 2010
11:47 AM

1. Please read the op ed article by Jeffry Sachs.

Jeff Sachs calculation (his article criticizing the Treasury plan)

Consider an asset that pays off

$X > 0$ with probability $\pi > 0$

and

$Y = 0$ with probability $(1-\pi)$

A risk neutral investor would be willing to pay

$\pi \cdot X + (1-\pi) Y = \pi X$ for the asset

Now suppose that the government offers to lend the buyer α. what he pays for the asset. The loan has interest rate $r=0$ and is "non-recourse" - the borrower can walk away from the loan costlessly if the asset does not pay off. Here $\alpha \in (0,1)$

How much would a risk-neutral investor be willing to pay for the asset?

Let P = amount he is willing to pay

he borrows αP and puts up $(1-\alpha)P$ of his own money.

His expected payoff is

$$\underbrace{\pi(X - \alpha P)}_{\text{payout in good state}} + (1-\pi)\max(0, \underbrace{Y - \alpha P}_{\text{repay loan}})$$

"0"

$$= \pi(X - \alpha P)$$

In terms of his own money, the expected payoff $\pi(X - \alpha P)$ costs him $(1-\alpha)P$

So to equate the expected payoff to the cost, set

$$\pi(X - \alpha P) = (1-\alpha)P$$

$$\Rightarrow$$

$$(*) \quad P = \frac{\pi}{\pi\alpha + (1-\alpha)} \cdot X$$

Note that $\lim_{\alpha \nearrow 1} P(\alpha) = X$

a. Use these formulas to replicate Jeff Sachs' calculations.

b. Without the government-supplied leverage, the asset is worth

πX , while with the leverage it is worth

$$\frac{\pi}{\pi\alpha + (1-\alpha)} X .$$

The expected value of the loan to the government is

$$\underbrace{\pi \cdot \alpha P}_{\text{payoff in good states}} - \alpha P \quad \uparrow \text{amount lent}$$

$$= -\alpha P(1-\pi) = -\frac{\alpha\pi(1-\pi)}{\pi\alpha + (1-\alpha)} X < 0$$

Note that the value paid by the investor + the value of the loan to the government is

$$\frac{\pi X}{\pi\alpha + (1-\alpha)} - \frac{\alpha\pi(1-\pi)}{\pi\alpha + (1-\alpha)} X$$

$$= \frac{\pi[1-\alpha(1-\pi)]}{\pi\alpha + (1-\alpha)} X = \pi X$$ = price of asset without the government loan.

$$1 - \alpha(1-\pi)$$ //

Remark: Sachs' point is that the Treasury plan transfers $\frac{\alpha\pi(1-\pi)}{\pi\alpha + (1-\alpha)}$ from the Treasury to the original owner of the asset that offers payoff $(X, 0)$ in the good state, bad state.

2. Consider the Kareken-Wallace model of moral hazard.

Recall that when the bank takes maximal risk by investing all of its deposits in the state A claim, it exposes the insurance agency to a lot of risk.

When $Q_A = \frac{D}{P_A}$ and $Q_B = 0$, the expected value of the losses of the insurance agency are

$$\pi_B \underbrace{(r+1)}_{\text{payment in state B}} D \quad \text{because } Q_B = 0.$$

Can you think of a scheme for pricing deposit insurance that

a. deters the bank from excessive risk taking, and

b. protects the deposit insurance agency from expected losses?

The idea is that the deposit insurance agency will charge the bank a fee for deposit insurance with the fee depending on (i) its total deposits, and (ii) its investment portfolio.

Growth model, 1

Sunday, March 28, 2010
3:20 PM

Some useful background

Euler's homogeneous function theorem

$$t^n F(x,y) = F(tx, ty)$$

F is homogeneous of order n

Let $x' = tx$, $y' = ty$

differentiate both sides with respect to t =>

$$n t^{n-1} F(x,y) = \frac{\partial F}{\partial x'} \frac{\partial x'}{\partial t} + \frac{\partial F}{\partial y'} \frac{\partial y'}{\partial t}$$

$$= x \frac{\partial F}{\partial x'} + y \frac{\partial F}{\partial y'}$$

$$n t^{n-1} F(x,y) = x \frac{\partial F}{\partial (xt)} + y \frac{\partial F}{\partial (yt)}$$

set $t = 1$ =>

$$n F(x,y) = x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y}$$

If $n = 1$ ("homogeneous of degree 1" or "constant returns to scale")

$$F(x,y) = x \underbrace{\frac{\partial F}{\partial x}}_{\text{marginal product of } x} + y \underbrace{\frac{\partial F}{\partial y}}_{\text{marginal product of } y}$$

$x \frac{\partial F}{}$ is competitive factor payments to x

$x \frac{\partial F}{\partial x}$ is competitive factor payments to x

---

Let $F(x,y)$ be homogeneous of degree n.

Then $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are homogeneous of degree $n-1$

Proof:

(*) $t^n F(x,y) = F(tx, ty)$

$$x' = tx, \quad y' = ty$$

differentiate both sides of (*) w.r.t. x

$$t^n \frac{\partial F}{\partial x}(x,y) = \frac{\partial F}{\partial x'} \cdot \frac{\partial x'}{\partial x} = \frac{\partial F}{\partial x'} t$$

(**) $t^{n-1} \frac{\partial F}{\partial x}(x,y) = \frac{\partial F}{\partial (xt)}(tx, ty)$

(**) is homogeneous of degree $n-1$. q.e.d.

Now

$$\frac{\partial F}{\partial x}(x,y) = \frac{\partial F}{\partial (xt)}(tx, ty)$$

set $t = \frac{1}{y}$

$$\frac{\partial F}{\partial x}(x,y) = \frac{\partial F}{\partial \left(\frac{x}{y}\right)}\left(\frac{x}{y}, 1\right)$$

let $F\left(\frac{x}{y}, 1\right) \equiv f\left(\frac{x}{y}\right)$

$\Rightarrow \quad \partial F \quad (\quad) \quad f'\left(\frac{x}{y}\right)$

$$\Rightarrow \quad \frac{\partial F}{\partial x}(x,y) = f'\left(\frac{x}{y}\right)$$

$$F(x,y) = y f\left(\frac{x}{y}\right)$$

$$\frac{\partial F}{\partial x} = y f'\left(\frac{x}{y}\right) \cdot \frac{1}{y} = f'\left(\frac{x}{y}\right) \qquad (+)$$

$$\frac{\partial F}{\partial y} = f\left(\frac{x}{y}\right) - y f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)$$

$$= f\left(\frac{x}{y}\right) - f'\left(\frac{x}{y}\right)\left(\frac{x}{y}\right) \qquad (++)$$

we'll use (+) and (++) later.

Malthus & Darwin

Malthusian growth model.

$$Y_t = z F(L_t, N_t)$$

output ↑ Y_t, land ↑ L_t, labor ↑ N_t

- F is homogeneous of degree 1
- $L_t = L$ is fixed over time

$C_t = Y_t$ (all production is consumed)

$$N_{t+1} = N_t \cdot g\left(\frac{C_t}{C^*}\right)$$

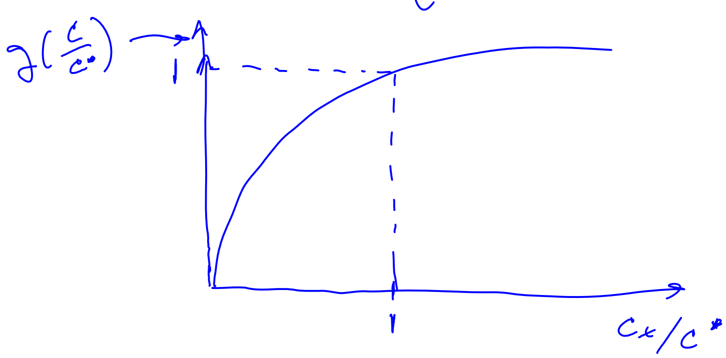
$$c_t = \frac{C_t}{N_t} = \text{consumption per capita}$$

c^* a constant

$g(1) = 1$, $g' >$, $g'' < 0$

we'll assume that

$$g\left(\frac{c}{c^*}\right) = \left(\frac{c_t}{c^*}\right)^\gamma \quad \text{where } 0 < \gamma < 1$$



$$\frac{N_{t+1}}{N_t} = \left(\frac{c_t}{c^*}\right)^\gamma$$

interpretation: c^* is subsistence consumption.

$c_t > c^* \Rightarrow \frac{N_{t+1}}{N_t} > 1$ population grows

$c_t < c^*$ $\frac{N_{t+1}}{N_t} < 1$ population shrinks

Note: $C_t = Y_t = z F(L, N_t)$

$$\Rightarrow c_t = \frac{C_t}{N_t} = z F\left(\frac{L}{N_t}, 1\right) \equiv z f(l_t)$$

$l_t = \frac{L}{N_t}$ land / labor ratio

Parameterize f :

$$z f(l) = z l^{\alpha}, \alpha \in (0,1) \quad \left(= z \left(\frac{L}{N}\right)^{\alpha} \right.$$

$$g\left(\frac{c_t}{c^*}\right) = \left(\frac{c_t}{c^*}\right)^{\gamma}, \quad \gamma \in (0,1)$$

$$\frac{N_{t+1}}{N_t} = \left(\frac{c_t}{c^*}\right)^{\gamma} = \left(\frac{z\left(\frac{L}{N_t}\right)^{\alpha}}{c^*}\right)^{\gamma}$$

$$\Rightarrow \quad \frac{N_{t+1}}{N_t} = \frac{z^{\gamma}\left(\frac{L}{N_t}\right)^{\alpha\gamma}}{(c^*)^{\gamma}}$$

$$\Rightarrow (*) \quad N_{t+1} = \left(\frac{z}{c^*}\right)^{\gamma} L^{\alpha\gamma} N_t^{1-\alpha\gamma}$$

look for a steady state $N_{t+1} = N_t = \bar{N}$

$$\bar{N} = \left(\frac{z}{c^*}\right)^{\gamma} L^{\alpha\gamma} \bar{N}^{1-\alpha\gamma}$$

$$\Rightarrow$$

$$(+) \quad \bar{N} = L \Big/ \left(\frac{c^*}{z}\right)^{1/\alpha}$$

(++) and $c_t = c^*$.

Effect of an increase in productivity z from

(+) & (++) $\Rightarrow$ $\bar{N}$ increases, but $c_t = \frac{C_t}{N_t} = c^*$

stays at subsistence.

Pessimistic? Realistic?

To get dynamics – take logs

$$\log N_{t+1} = \gamma(\log z + \alpha \log L) + (1-\alpha\gamma)\log N_t$$

or

$$n_{t+1} = \phi_0 + \phi_1 n_t \quad , \; n_t = \log N_t$$

this is of the simple form we've analyzed often before

$$x_{t+1} = A x_t \quad , \quad x_t = \begin{pmatrix} 1 \\ n_t \end{pmatrix} .$$

$\Rightarrow$

$$x_t = A^t x_0 \quad , \quad x_t \text{ converges}$$

because $\alpha\gamma \in (0,1)$.

Solow growth model

- Make N_t exogenous: $N_{t+1} = (1+n) \cdot N_t$, N_0 given
- Other factor of production is now capital – K_t not land.

$$K_{t+1} = (1-\delta)K_t + X_t \quad , \quad \delta \in (0,1)$$

$$Y_t = zF(K_t, N_t)$$ constant returns to scale

(*) $$C_t + X_t = zF(K_t, N_t)$$

$$\frac{K_{t+1}}{N_t} = (1-\delta)\frac{K_t}{N_t} + \frac{X_t}{N_t}$$

or

$$\frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} = (1-\delta)\frac{K_t}{N_t} + \frac{X_t}{N_t}$$

(+) $$k_{t+1} \cdot (1+n) = (1-\delta) k_t + x_t \quad , \quad x_t = \frac{X_t}{N_t}$$

Solow <u>assumes</u> $C_t = (1-s)\, zF(K_t, N_t)$

or $$c_t = \frac{C_t}{N_t} = (1-s)\, z f(k_t)$$

where $$f(k_t) = F\left(\frac{K_t}{N_t}, 1\right)$$

(*) $\Rightarrow$ $$X_t = zF(K_t, N_t) - C_t$$

$$= s\, zF(K_t, N_t)$$

$$\Rightarrow x_t = s\, z f(k_t)$$

thus (+) becomes

$$k_{t+1}(1+n) = (1-\delta) k_t + s\, z f(k_t)$$

or

(**) $$k_{t+1} = \frac{s}{1+n} z f(k_t) + \frac{(1-\delta)}{1+n} k_t$$

$$(**) \quad k_{t+1} = \frac{s}{1+n} z f(k_t) + \frac{(1-\delta)}{1+n} k_t$$

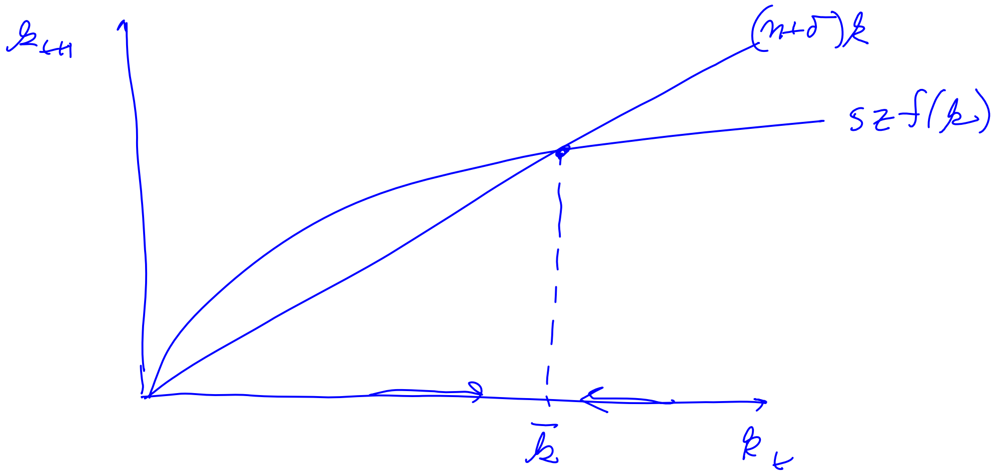
look for a fixed point

$$\bar{k} = \frac{s}{1+n} z f(\bar{k}) + \frac{(1-\delta)}{(1+n)} \bar{k}$$

$$\left[\frac{1+n}{1+n} - \frac{(1-\delta)}{1+n} \right] \bar{k} = \frac{s}{1+n} f(\bar{k})$$

or

$$(n+\delta)\bar{k} = s z f(\bar{k})$$



Use this as a machine to study effects on $\bar{k}$ of (a) increase in z

(b) increase in s

(c) increase in n

Return to dynamics:

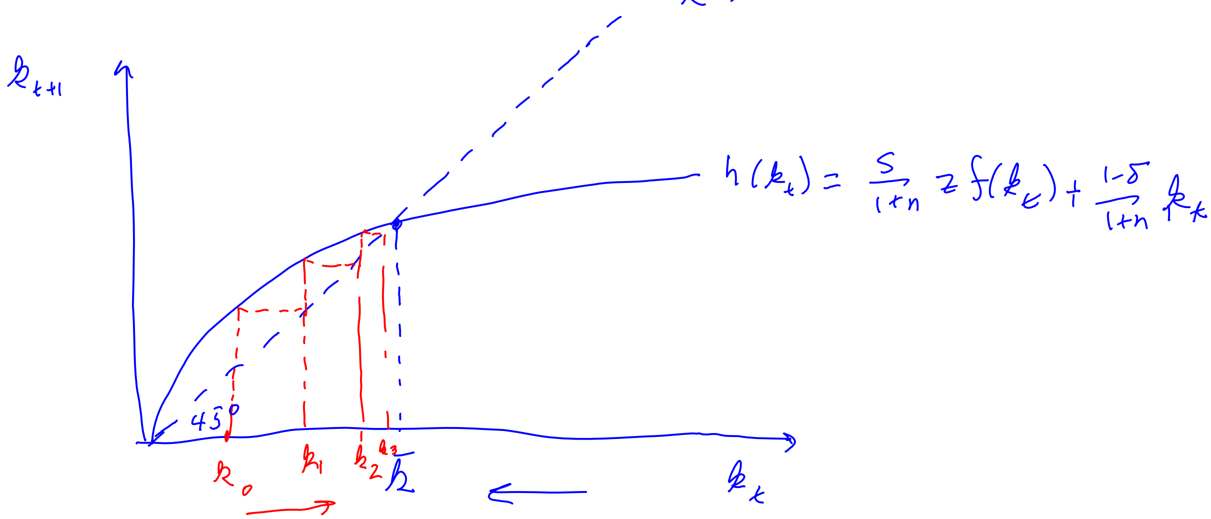
$$k_{t+1} = \frac{s}{1+n} z f(k_t) + \frac{(1-\delta)}{1+n} k_t$$

$$\equiv h(k_t) \quad ; \quad h(0) = \frac{s}{1+n} z f(0) = 0$$

$$h'(k_t) = \frac{s}{1+n} z f'(k_t) + \frac{(1-\delta)}{1+n}$$

$$h'(0) = +\infty \quad \text{if} \quad f'(0) = +\infty$$

$$\lim_{k \to \infty} h'(k) = \frac{(1-\delta)}{1+n} \quad \text{if} \quad \lim_{k \to \infty} f'(k) = 0$$



this graph shows the dynamics - convergence to steady state.

Mention real business cycle theory. - z_t stochastic.

Growth model, 2

Sunday, March 28, 2010
8:23 AM

Growth model

Solow model took savings rate s to be a constant - let's change that - make it the consequence of a purposeful choice of a representative consumer.

Growth model: Technology

$Y = F(K, N)$ linearly homogeneous e.g. $Ak^{\alpha} N^{1-\alpha}$

$$\lambda Y = F(\lambda K, \lambda N)$$

$$\Rightarrow F(k, 1) = f(k) \quad \text{where } k = K/N$$

$$F(K, N) = N f(k)$$

$$\frac{\partial F}{\partial K}(K,N) = f'(k)$$

$$\frac{\partial F}{\partial N}(K,N) = f(k) - k f'(k)$$

$$c_t + k_{t+1} = f(k_t) + (1-\delta) k_t$$ gross o/p per person

Preferences.

$u(c, 1-n) = u(c)$ (no disutility of labor)

$$u'(c) > 0, \quad u''(c) < 0, \quad u'(0) = +\infty$$

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

Planning problem:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.} \quad c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

$$k_0 \text{ given}$$

Digression: purely optional reading: we can solve the consumer's problem using dynamic programming. Let $v(k_0)$ be the optimal value function – i.e.

$$v(k_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

The Bellman equation is

$$v(k_t) = \max_{\{c_t, k_{t+1}\}} \left\{ u(c_t) + \beta v\left(f(k_t) + (1-\delta)k_t - c_t\right) \right\}$$

$$\text{FONC:} \quad u'(c_t) - \beta v'(k_{t+1}) \le 0 \qquad = 0 \text{ if } k_{t+1} > 0$$

Benveniste-Scheinkman:

$$v'(k) = u'(c)\left(f'(k) + (1-\delta)\right)$$

$$\Rightarrow \quad u'(c_t) = \beta u'(c_{t+1})\left[f'(k_{t+1}) + (1-\delta)\right]$$

Euler eqn. End of digression

Formulate as Lagrangian, finite T :

$$L = \sum_{t=0}^{T} \beta^t \left\{ u(c_t) + \lambda_t \left[f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \right] \right\}$$

FONC: $\{c_t, k_{t+1}\}$, λ_t

$$c_t : \quad u'(c_t) - \lambda_t = 0$$

$$k_{t+1} : \quad -\lambda_t + \beta \lambda_{t+1}\left[f'(k_{t+1}) + (1-\delta)\right], \quad t = 0, \dots, T-1$$

k_{T+1}: $\; -\beta^T \lambda_T \le 0 \qquad = 0$ if $k_{T+1} > 0$.

or $\; \beta^T \lambda_T k_{T+1} = 0$

infinite horizon $\; \lim_{T \to \infty} \beta^T \lambda_T k_{T+1} = 0$

Rearrange the FONC's

$$u'(c_t) = \lambda_t$$

$$(*) \qquad \lambda_t = \beta \lambda_{t+1} \left[f'(k_{t+1}) + (1-\delta) \right]$$

$$k_{t+1} + c_t = f(k_t) + (1-\delta) k_t$$

Look for steady state

$$(*) \Rightarrow \quad \beta^{-1} \equiv (1+\rho) = f'(k) + (1-\delta)$$

SS

$$\text{or } (\rho + \delta) = f'(k) \qquad \text{modified golden rule:}$$

Talk about in terms of preferences, technology

$$c + \delta k = f(k)$$

Solving by shooting: take finite horizon

$$u'(c_t) = \lambda_t \qquad (1)$$

$$\lambda_t = \beta \lambda_{t+1} \left[f'(k_{t+1}) + (1-\delta) \right] \qquad (2)$$

$$k_{t+1} + c_t = f(k_t) + (1-\delta) k_t \qquad (3)$$

$$\beta^T \lambda_T k_{T+1} = 0 \qquad (4)$$

Boundary conditions: k_0 given, $\beta^T \lambda_T k_{T+1} = 0 \Rightarrow k_{T+1} = 0$

Shooting method:

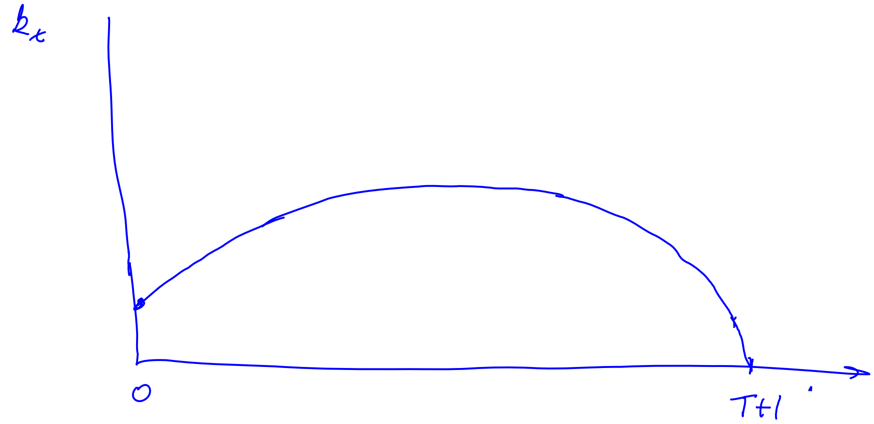
Set $\lambda_0 \Rightarrow c_0$. then (3) $\Rightarrow k_1$. Then (2) $\Rightarrow$

$\lambda_1 \Rightarrow c_1$. & so on

Check to see if "target" $k_{T+1} = 0$ is hit.

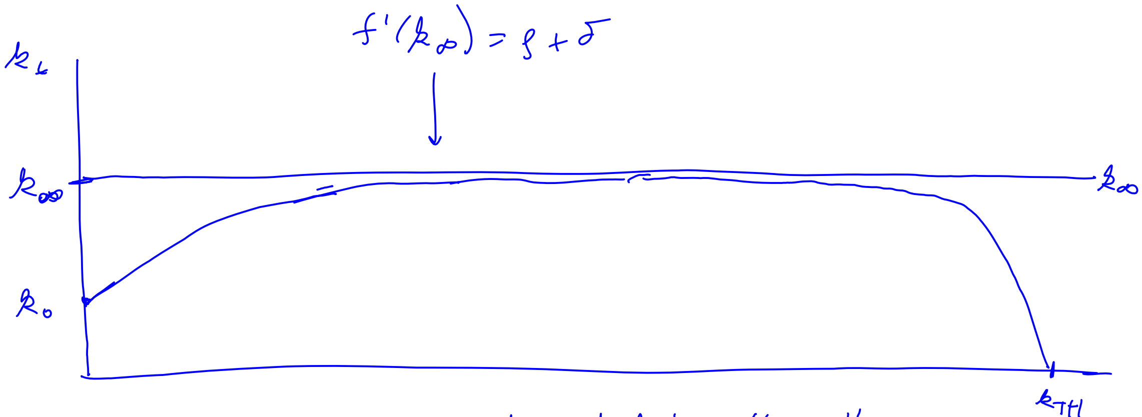
If $k_{T+1} > 0$, "eat more" by lowering λ_0.

try again.



now lengthen T. $k_t \to k_\infty$ where $f'(k_\infty) = \rho + \delta$

Digression: "turnpike" theorem idea.



For large T+1, - spend most of time "near" the "turnpike" k_∞

— end of digression

- Interpretation of solution in terms of forces impeding consumption smoothing. or ...
  "more about the shooting algorithm"

Write (∗) as

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta) \right]$$

or

$$1 = \beta \frac{u'(c_{t+1})}{u'(c_t)} \Phi_{t+1}$$

$$\Phi_{t+1} \equiv f'(k_{t+1}) + (1-\delta)$$ (just a definition of Φ_{t+1})

For example, let

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1 \\ \ln c & \text{if } \gamma = 1 \end{cases}$$

$$\frac{u'(c_{t+1})}{u'(c_t)} = \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}$$

$$= \left(\frac{c_t}{c_{t+1}} \right) \quad \text{if } \gamma = 1$$

Then we have, if $\gamma = 1$,

$$1 = \beta \left(\frac{c_t}{c_{t+1}} \right) \left[f'(k_{t+1}) + (1-\delta) \right]$$

or

$$c_{t+1} = \left(\beta \left[f'(k_{t+1}) + (1-\delta) \right] \right) c_t$$

where $f' > 0$, $f'' < 0$

$$f'(\bar{k}) = \rho + \delta$$

$$\therefore \quad \beta \left[f'(\bar{k}) + (1-\delta) \right] = \beta \left[\rho + \delta + 1 - \delta \right]$$
$$= \beta \left[1 + \rho \right] = 1$$

so $c_{t+1} = c_t$ when $k_{t+1} = \bar{k}$ (steady state)

Also, $k_{t+1} < \bar{k} \Rightarrow$

$$\beta\left[f'(k_{t+1}) + (1-\delta)\right] > 1 \quad \text{because } f'' < 0$$

(lower k, higher f')

$k_{t+1} > \bar{k} \Rightarrow$

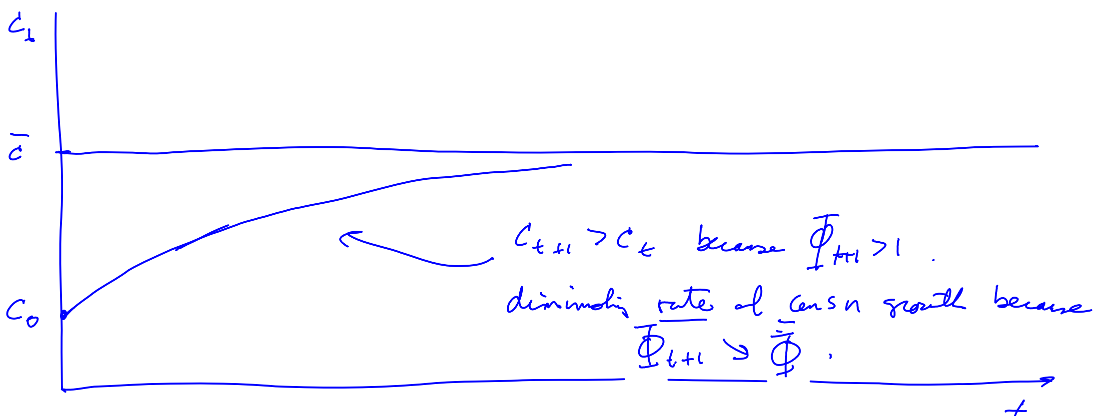
$$\beta\left[f'(k_{t+1}) + (1-\delta)\right] < 1$$

steady state consn:

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \quad \Rightarrow$$

$$\bar{c} + \bar{k} = f(\bar{k}) + (1-\delta)\bar{k}$$

$$\bar{c} = f(\bar{k}) - \delta\bar{k} \quad \text{or} \quad \bar{c} + \delta\bar{k} = f(\bar{k})$$



$$\left(\Phi_{t+1} \equiv f'(k_{t+1}) + (1-\delta)\right)$$

$c_0 < \bar{c}$ because $k_0 < \bar{k}$

$$c_{t+1} = \left(\beta\left[f'(k_{t+1}) + (1-\delta)\right]\right) c_t \quad \text{when } \gamma = 1$$

or

$$1 = \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \beta\left[f'(k_{t+1}) + (1-\delta)\right]$$

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta\left[f'(k_{t+1}) + (1-\delta)\right]$$

$\Leftarrow$

$$c_{t+1} = \left[\beta\left(f'(k_{t+1}) + (1-\delta)\right)\right]^{\frac{1}{\gamma}} c_t \quad \text{for } \gamma = 1$$

$$c_{t+1} = \left(\phi_{t+1}\right)^{\frac{1}{\gamma}} c_t$$

$$c_{t+1} = \left(\Phi_{t+1}\right)^{\frac{1}{\gamma}} c_t$$

γ governs the elasticity of consn growth with respect to Φ_{t+1}.

$c_{t+1} > c_t$: because $\Phi_{t+1} > 1$

diminishing <u>rate</u> of consumption growth because

$\Phi_{t+1} \searrow$ over time $\Phi_{t+1} \searrow \beta\left[f'(k) + (1-\delta)\right]$

<u>Summary:</u> Why not complete consumption smoothing:

because $\Phi_{t+1} \neq 1$ always - only in a steady state

<u>Decentralization</u>: <u>Price system</u> $\{q_t, w_t, r_t\}_{t=0}^{\infty}$ at date 0.

Household problem:

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t, 1-n_t) \quad , \quad 0<\beta<1$$

where $U(c_t, l_t)$, $u_c > 0, u_{cc} < 0, u_l > 0, u_{ll} < 0$

leisure $l_t = 1 - n_t$

$0 \leq l_t \leq 1$

special case $U(c_t, 1-n_t) = u(c_t) + v(1-n_t)$

Budget constraint:

$$\sum_{t=0}^{\infty} q_t (c_t + x_t) = \sum_{t=0}^{\infty} w_t n_t + \sum_{t=0}^{\infty} r_t k_t$$

rental on labor

rental on capital

$k \quad = (1 \quad \delta) k \quad + x$

$$k_{t+1} = (1-\delta) k_t + \overset{\text{labor}}{x_t}$$

or

$$\sum_{t=0}^{\infty} (w_t n_t + r_t k_t) = \sum_{t=0}^{\infty} q_t \left(c_t + \underbrace{k_{t+1} - (1-\delta) k_t}_{x_t}\right)$$

Form Lagrangian

Lagrange multiplier on $1-n_t \geq 0$

Lagrange multiplier on $n_t \geq 0$

$$L = \sum_{t=0}^{\infty} \beta^t U(c_t, 1-n_t) + \sum_{t=0}^{\infty} \left(\lambda_t (1-n_t) + \phi_t n_t\right)$$

$$+ \mu \left[\sum_{t=0}^{\infty} (w_t n_t + r_t k_t) - \sum_{t=0}^{\infty} q_t (c_t + k_{t+1} - (1-\delta) k_t) \right]$$

FONC:

c_t: $\beta^t U_c(c_t, 1-n_t) - \mu q_t = 0 \qquad , t = 0, 1, \ldots$

n_t: $-\beta^t U_\ell(c_t, 1-n_t) + \mu w_t + \phi_t - \lambda_t = 0 \qquad , t = 0, 1, \ldots$

Complementary slackness: $\lambda_t (1-n_t) = 0$

$\phi_t n_t = 0$

$\Rightarrow \beta^t U_\ell(t) = \mu w_t + \phi_t - \lambda_t$

$\Rightarrow \beta^t U_\ell(t) = \mu w_t \quad$ if $\quad 0 < n_t < 1$

$\beta^t U_\ell(t) < \mu w_t \quad$ if $\quad n_t = 1$

$\beta^t U_\ell(t) > \mu w_t \quad$ if $\quad n_t = 0$

k_t: $r_t + q_t(1-\delta) - q_{t-1} \qquad , t = 1, 2, \ldots$

or

$$q_t = q_{t+1}(1-\delta) + r_{t+1}$$

Summarizing:

$$q_t = q_{t+1}(1-\delta) + r_{t+1}$$ marg cond'n for k_t

$$\mu\, q_t = \beta^t U_c(t)$$ marg cond for c_t

$$\mu\, w_t = \beta^t U_\ell(t) \quad \text{if} \quad 0 < n_t < 1$$

$$> \beta^t U_\ell(t) \quad \text{if} \quad n_t = 1$$

$$< \beta^t U_\ell(t) \quad \text{if} \quad n_t = 0$$

we'll assume we have the $n_t = 1$ condition prevailing

– e.g. $U_\ell(c_t, \ell_t) = 0$ because, e.g.

$$U(c_t, \ell_t) = u(c_t)$$

Firm: Maximize

$$\sum_{t=0}^{\infty}\Big(q_t\big[\underbrace{(c_t + x_t)}_{\text{sales}}\big] - r_t K_t - w_t N_t\Big)$$

s.t. $c_t + x_t = F(K_t, N_t)$ — N_t: labor fixed by firm; K_t: capital hired by firm

$$\lambda F(K,L) = F(\lambda K, \lambda L)$$ constant returns to scale

$\Longrightarrow$ (Euler's theorem)

$$F(K,L) = \frac{\partial F}{\partial K} K + \frac{\partial F}{\partial N} N$$

Thus firm problem is

$$\max_{\{K_t, L_t\}} \sum_{t=0}^{\infty}\Big\{(q_t F_K - r_t)K_t + (q_t F_N - w_t)N_t\Big\}$$

Firm hires K_t & N_t as a price taker –

takes q_t, r_t, w_t as given.

Zero profit condition for firm interior position K_t, N_t:

$$q_t F_K - r_t = 0$$

$$q_t F_N - w_t = 0$$

constant returns to scale $\Rightarrow$

$$F_K(K,N) = F_K\left(\frac{K}{N}, 1\right) = f'\left(\frac{K}{N}\right)$$

$$F_N(K,N) = F_N\left(\frac{K}{N}, 1\right) = f\left(\frac{K}{N}\right) - f'\left(\frac{K}{N}\right) \cdot \frac{K}{N}$$

Note: size of firm is <u>indeterminate</u> - double size of firm leaves those marginal conditions intact.

The "number" of firms is thus indeterminate.

Defns: A price system $\{q_t, w_t, r_t\}_{t=0}^{\infty}$.

A feasible allocation $\{k_t, c_{t+1}\}_{t=0}^{n_t}$, $c_t \geq 0$, $k_{t+1} \geq 0$

<u>Defn</u>: A c.e. is a price system & feasible allocation such that

(a) given price system, allocation solves firm problem

(b) given price system, allocation solves HH problem

To compute equilibrium: guess & verify.

(1) Solve planning problem for $\{\tilde{c}_t, \tilde{k}_{t+1}\}_{t=0}^{\infty}$

(2) Guess $q_t = \beta^t u'(c_t) \left(= \beta^t U_c(c_t, 1-n_t)\right)$

$$w_t = \left(f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)\right) q_t$$

$$r_t = \left(f'(\tilde{k}_t)\right) . q_t$$

(3) Verify that the firm's & HH's FONC's are satisfied..

<u>Key conclusion</u>: For this economy, the allocation that solves the planning problem is a competitive equilibrium.

This is the end of the lecture. The below is entirely optional advanced material.

To formulate a <u>recursive competitive equilibrium</u>, first write soln of planning problem as

$$\tilde{k}_{t+1} = h(\tilde{k}_t) \qquad k_0 = \tilde{k}_0$$

$$w_t = w(\tilde{k}_t)$$

$$r_t = r(\tilde{k}_t)$$

Formulate Bellman eqn for HH

$$v(k_t, \tilde{k}_t) = \max_{c_t} \left\{ u(c_t) + \beta v(k_{t+1}, \tilde{k}_{t+1}) \right\}$$

s.t. $c_t + k_{t+1} = w(\tilde{k}_t) \cdot 1 + r(\tilde{k}_t) k_t$

$\tilde{k}_{t+1} = h(\tilde{k}_t)$,

$k_0, \tilde{k}_0 = k_0$ given

Firm problem - sequence of static problems (no forecasting)

$$\max_{n_t, k_t} \left\{ f(k_t) n_t - w(\tilde{k}_t) n_t - r(\tilde{k}_t) k_t \right\}$$

— end of optional material.

Taxes and government expenditures

Sunday, March 28, 2010
5:24 PM

Government expenditures in the growth model.

Feasibility becomes

$$c_t + x_t + g_t = f(k_t)$$

↑ government expenditures per capita

$$k_{t+1} = (1-\delta) k_t + x_t$$

or

$$c_t + k_{t+1} + g_t = f(k_t) + (1-\delta) k_t$$

$\{g_t\}_{t=0}^{\infty}$ an exogenous sequence.

Government budget constraint (at time 0)

$$\sum_{t=0}^{\infty} q_t g_t = \sum_{t=0}^{\infty} q_t \tau_{ct} c_t + \sum_{t=0}^{\infty} \tau_{kt} r_t k_t$$

$$+ \sum_{t=0}^{\infty} w_t \tau_{nt} n_t + \sum_{t=0}^{\infty} q_t \tau_{ht}$$

τ_{ct} - consumption tax
τ_{kt} - tax on earnings from capital
τ_{nt} - tax on earnings from labor

} potentially distorting taxes

τ_{ht} - lump sum tax (non distorting)

<u>Definition</u>: The household partially affects the amount of

taxes he pays for a <u>distorting tax</u> by possibly altering his choice of $\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}$.

The household's choice of $\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}$ leaves unaffected the amount it pays in non distorting taxes.

Household problem:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$\sum_{t=0}^{\infty} q_t \left[(1+\tau_{ct}) c_t + k_{t+1} - (1-\delta) k_t \right]$$

$$= \sum_{t=0}^{\infty} w_t (1-\tau_{nt}) \overset{1}{\overset{\shortparallel}{n_t}} + \sum_{t=0}^{\infty} r_t (1-\tau_{kt}) k_t - \sum_{t=0}^{\infty} q_t \tau_{ht}$$

s.t k_0 given

Here we are simplifying by having $n_t = 1$ inelastically supplied.

Formulate Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$+ \mu \Big[\sum_{t=0}^{\infty} w_t (1-\tau_{nt}) \cdot 1 + \sum_{t=0}^{\infty} r_t (1-\tau_{kt}) k_t$$

$$- \sum_{t=0}^{\infty} q_t \tau_{ht} - \sum_{t=0}^{\infty} q_t \left((1+\tau_{ct}) c_t + k_{t+1} - (1-\delta) k_t \right) \Big]$$

FONC: (we are free to set $\mu = 1$ by scaling prices)

c_t: $\beta^t u'(c_t) - \mu q_t (1+\tau_{ct}) = 0$, $t = 0, 1, \ldots$

k_t: $r_t(1-\tau_{kt}) + q_t(1-\delta) - q_{t-1} = 0$, $t = 1, 2, \ldots$

$$\Rightarrow \quad \frac{\beta^t u'(c_t)}{1+\tau_{ct}} = q_t \tag{1}$$

$$q_t = q_{t+1}(1-\delta) + (1-\tau_{kt+1}) r_{t+1} \tag{2}$$

<u>Firm problem:</u> Same as before – firms pay no taxes
households pay all taxes.

$$\Rightarrow \quad r_t = q_t f'(k_t)$$
$$w_t = q_t [f(k_t) - f'(k_t) k_t]$$

where k_t is $\frac{K}{L}$ for firm + for economy in equilibrium.

<u>Defn:</u> A <u>government policy</u> is a $\{g_t\}_{t=0}^{\infty}$ and tax sequences $\{\tau_{ct}, \tau_{nt}, \tau_{kt}, \tau_{ht}\}_{t=0}^{\infty}$.

A <u>feasible allocation</u> is a $\{c_t, k_{t+1}, g_t\}_{t=0}^{\infty}$ that satisfies

$$c_t + k_{t+1} + g_t = f(k_t) + (1-\delta) k_t \quad , \quad t \geq 0$$

k_0 given.

A competitive equilibrium (with taxes) is a government policy, a price system $\{q_t, r_t, w_t\}_{t=0}^{\infty}$, and a feasible allocation such that

a) Given the allocation and the price system, the government policy satisfies its budget constraint.

b) Given the price system and the government policy, the allocation solves the household's problem.

c) Given the price system and the govt policy, the allocation solves the firm's problem.

Computing an equilibrium:

Remark: in general, can't solve a planning problem for the allocation.

Instead - impose equilibrium conditions in the form of govt budget constraint & FONC's.

for the household & firm and attack them with the "shooting" algorithm:

We have

$$q_t = \frac{\beta^t u'(c_t)}{1 + \tau_{ct}}$$

$$q_t = q_{t+1} \quad (1-\delta) + (1-\tau_{k\,t+1}) r_{t+1}$$

or

$$q_t = q_{t+1} \left[(1-\delta) + (1-\tau_{k\,t+1}) f'(k_{t+1}) \right]$$

$$\Rightarrow \frac{\beta^t u'(c_t)}{(1+\tau_{ct})} = \frac{\beta^{t+1} u'(c_{t+1})}{(1+\tau_{c\,t+1})} \left[(1-\delta) + (1-\tau_{k\,t+1}) f'(k_{t+1}) \right]$$

or

$$(*) \qquad u'(c_t) = \beta\, u'(c_{t+1}) \frac{(1+\tau_{ct})}{(1+\tau_{c\,t+1})} \left[(1-\delta) + (1-\tau_{k\,t+1}) f'(k_{t+1}) \right]$$

$$(**) \qquad c_t + k_{t+1} + g_t = f(k_t) + (1-\delta) k_t$$

k_0 given

$\{g_t, \tau_{ct}, \tau_{kt}\}_{t=0}^{\infty}$ given

To solve $(*), (**)$ for $\{c_t, k_{t+1}\}_{t\geq 0}^{\infty}$

Assume that $g_t \to \bar{g}$, $\tau_{ct} \to \bar{\tau}_c$, $\tau_{kt} \to \bar{\tau}_k$ and

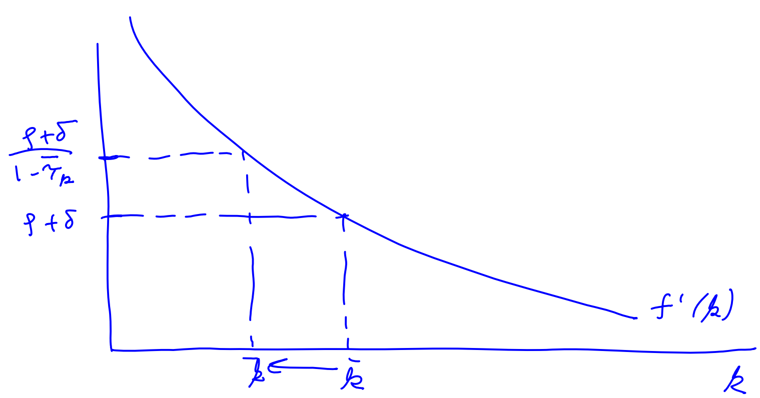
look for steady state $-$ $(*) \Rightarrow$

$c_t = c_{t+1} = \bar{c}$

$(*) \Rightarrow$

$$1 = \beta \left[(1-\delta) + (1-\bar{\tau}_k) f'(\bar{k}) \right]$$

$$\Rightarrow \qquad \frac{\rho + \delta}{1-\bar{\tau}_k} = f'(\bar{k})$$



raising $\bar{\tau}_k > 0$ causes steady state $\bar{k}$ to be lower

$$(*) \qquad u'(c_t) = \beta\, u'(c_{t+1}) \frac{(1+\tau_{ct})}{(1+\tau_{c\,t+1})} \left[(1-\delta) + (1-\tau_{k\,t+1}) f'(k_{t+1}) \right]$$

(**) $c_t + k_{t+1} + g_t = f(k_t) + (1-\delta)k_t$

Solve (*), (**) by "shooting" to $\bar{k}$ steady state.

guess at c_0, solve (**) for k_1, solve (*) for c_1, ...

continue & hope you hit $\bar{k}$.

More general utility specification

$$U(c, 1-n) = \frac{c^{1-\gamma}}{1-\gamma} + B\frac{(1-n)^{1-\varphi}}{1-\varphi}$$

$\log c$ if $\gamma = 1$ $\qquad$ $\ln(1-n)$ if $\varphi = 1$

$$\sum_{t=0}^{\infty}\beta^t\Big\{U(c_t, 1-n_t) + \mu\Big[\sum_{t=0}^{\infty} r_t(1-\tau_{kt})k_t$$

$$+ w_t(1-\tau_{nt})n_t - q_t\tau_{ht} - q_t(1+\tau_{ct})c_t$$

$$+ (1-\tau_{it})q_{bt}\left[k_{t+1} - (1-\delta)k_t\right]\Big]\Big\}$$

c_t: $\quad \beta^t U_{1t} = \mu q_{bt}(1+\tau_{ct})$

n_t: $\quad \beta^t U_{2t} \le \mu w_t(1-\tau_{nt})$, $=$ if $0 < n_t < 1$

$$\begin{cases} U_{1t} = c_t^{-\gamma} \\ U_{2t} = B(1-n_t)^{-\varphi} \end{cases}$$

To solve:

$$\frac{(1-\tau_{it})}{(1+\tau_{ct})} U_1(c_t, 1-n_t) = \beta(1+\tau_{ct+1})^{-1} U_1(c_{t+1}, 1-n_{t+1})$$

$$(1+\tau_{ct})$$

$$* \left[(1-\tau_{i\,t+1})(1-\delta) + (1-\tau_{k\,t+1}) F_k(k_{t+1}, n_{t+1}) \right]$$

$$\frac{U_2(c_t, 1-n_t)}{U_1(c_t, 1-n_t)} = \frac{(1-\tau_{nt})}{(1+\tau_{ct})} F_n(k_t, n_t)$$

To get steady-state :

$\bar{k}$ capital per person

$\bar{n}$ - work per person

solve

$$1 = \beta \left[(1-\delta) + \frac{(1-\tau_k)}{(1-\tau_i)} F_k(\bar{k}, \bar{n}) \right]$$

c = Consn per capita

$$\frac{U_2}{U_1} = \frac{(1-\tau_n)}{(1+\tau_c)} F_n(\bar{k}, \bar{n})$$

n = hours per person

$$\tilde{k} = \frac{\bar{k}}{\bar{n}}$$

$$1 = \beta \left[(1-\delta) + \frac{(1-\tau_k)}{(1-\tau_i)} f'(\tilde{k}) \right]$$

$$\Rightarrow \tilde{k}$$

$$\frac{U_2}{U_1} = \frac{(1-\tau_n)}{(1+\tau_c)} \left[f(\tilde{k}) - \tilde{k} f'(\tilde{k}) \right]$$

or

$$(*) \quad \frac{B(1-\bar{n})^{-\varphi}}{\bar{c}^{-\gamma}} = \frac{(1-\tau_n)}{(1+\tau_c)} \left[f(\tilde{k}) - \tilde{k} f'(\tilde{k}) \right]$$

$$(**) \quad \bar{c} + \bar{g} + \delta \underbrace{\bar{k}}_{\tilde{k}\bar{n}} = \bar{n} f(\tilde{k})$$

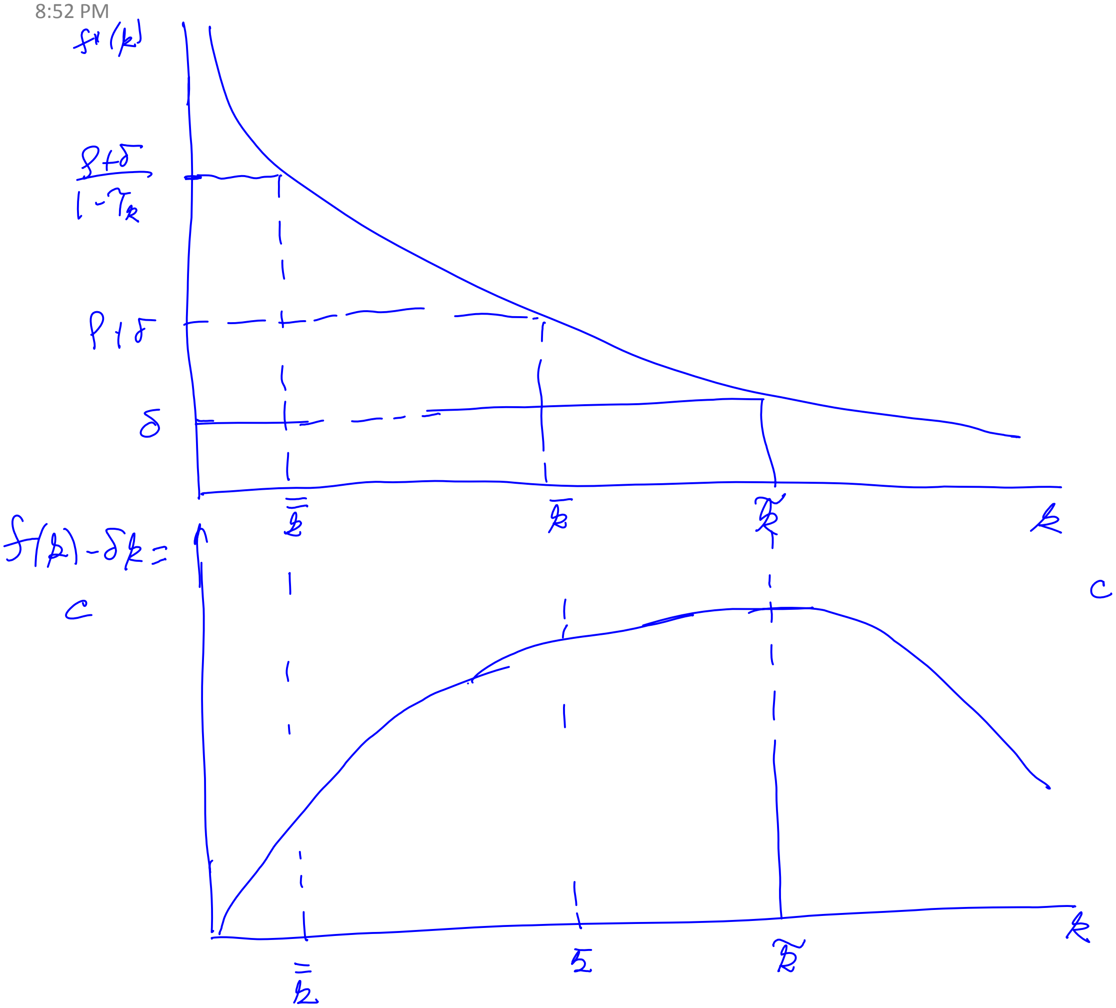
given $\tilde{k}$, there are two fns in $\bar{n}$ and $\bar{c}$

because $\bar{k} = \tilde{k} \cdot \bar{n}$

Steady states

Sunday, March 28, 2010

8:52 PM



max steady state c

$\bar{k}$ optimal steady state

steady state with $\tau_k > 0$

Use these graphs to prove that steady state c is lower for steady state with $\tau_k > 0$ than with $\tau_k = 0$.

Foresight versus surprise

Sunday, March 28, 2010
8:45 AM

Recall the optimal growth model with an infinitely lived consumer who orders consumption streams according to

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , u' > 0, u'' < 0$$

for example $u(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$, $\gamma > 0$

The technology is

$$c_t + g_t + k_{t+1} = f(k_t) + (1-\delta) k_t \quad , \quad \delta \in (0,1)$$

c_t = consumption per capita

g_t = government purchases per capita

k_t = capital per unit labor

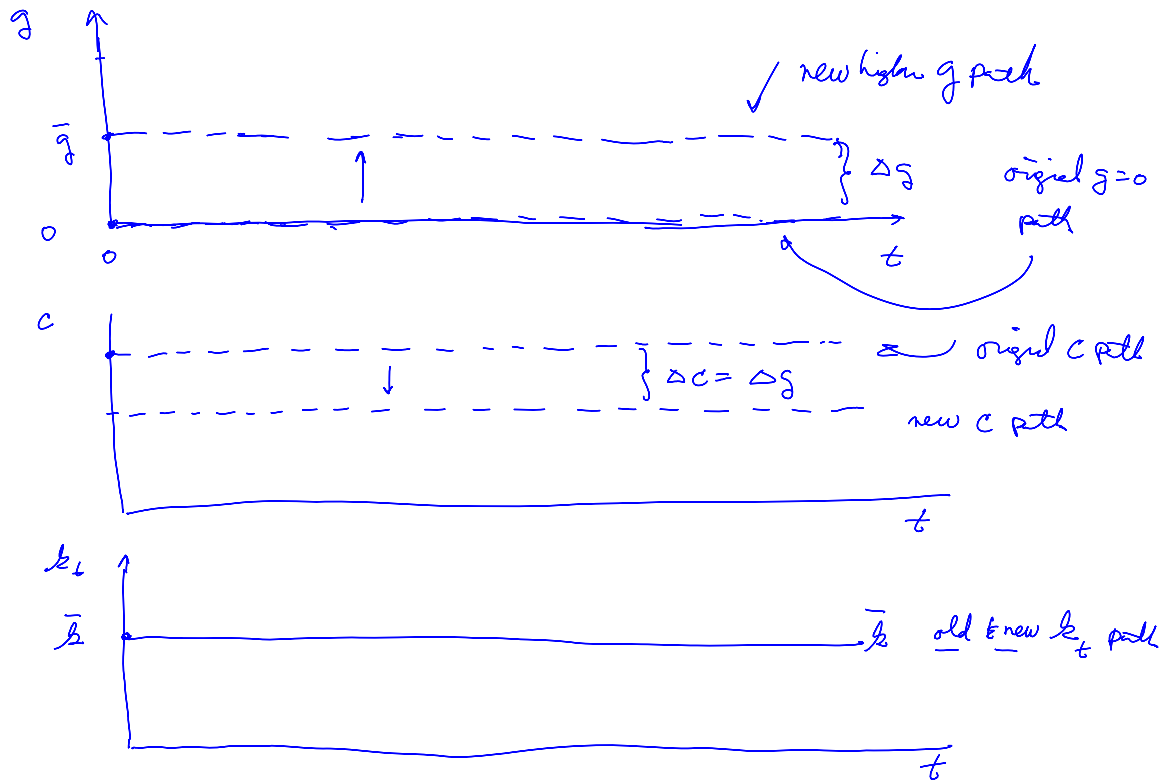
In class, we studied the following two scenarios.

(1) Before $t=0$, the economy was resting in the steady state k_t appropriate for a situation in which g had always been $= 0$. , i.e.

$$k_0 = \bar{k} \quad \text{where} \quad f'(\bar{k}) = \rho + \delta$$

At $t=0$, government expenditures per capita suddenly and unexpected. rise to $g_t = \bar{g} > 0 \;\forall\; t \geq 0$.

Assume that g is entirely financed by lump sum taxes. We saw that the effects of the increase taxes. We saw that the effects of the increase in g were as follows:

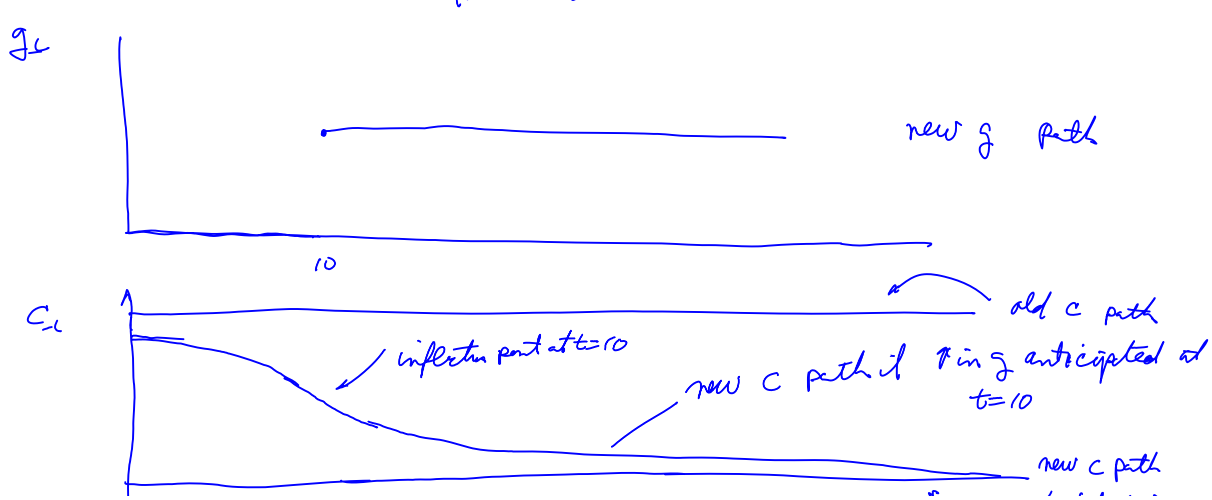


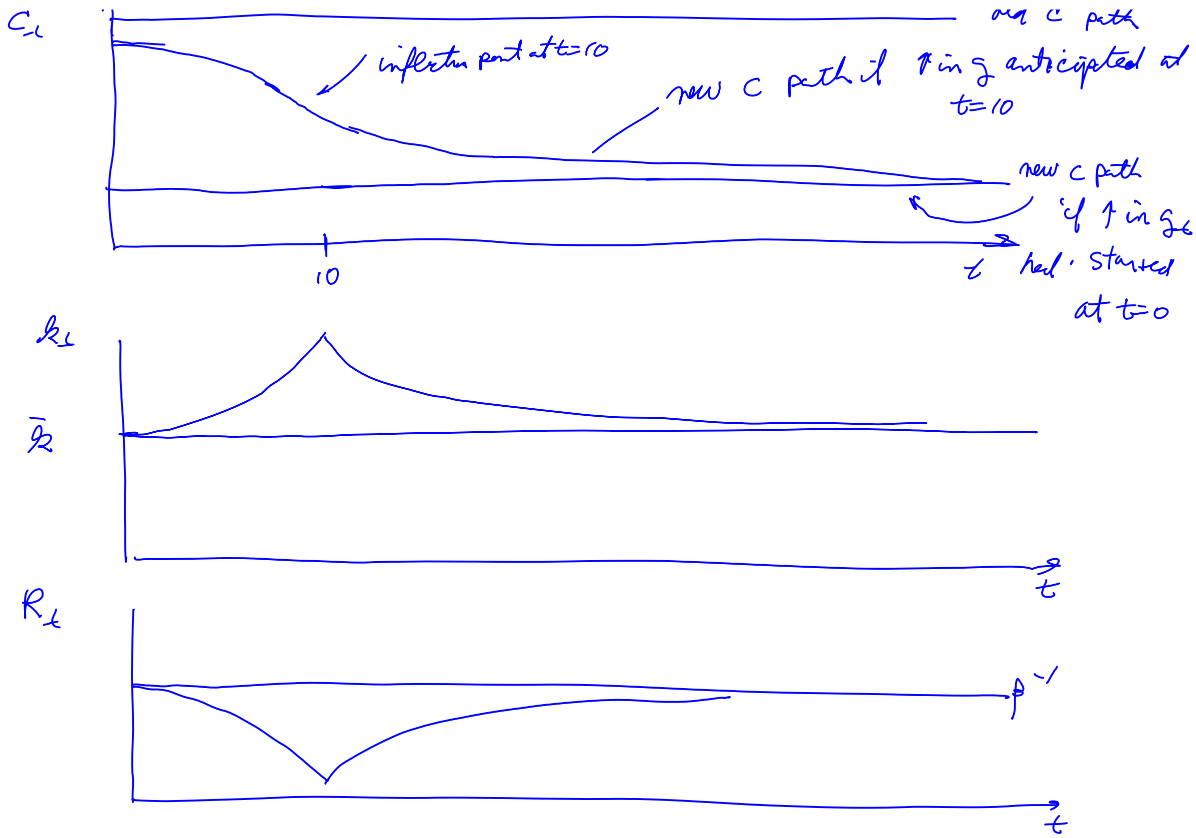
But if instead, we model the increase in g not as a surprise but as foreseen, the path is different.

If at $t=0$ the path of g_t changes unexpectedly from 0 for all t to

$$g_t = \begin{cases} 0 & t = 0, \dots, 9 \\ \bar{g} > 0 & t = 10, \dots \end{cases}$$

we saw that the consequence was as follows:





The above graph shows the path of c_t, $t \geq 0$, when the ↑ in g_t is anticipated to occur 10 periods in the future. Here c_t drops immediately at $t=0$ and then declines smoothly to the same steady state level it would have had the increase in g_t occurred at $t=0$ instead of $t=10$.

A general principle is illustrated by this example: the effects of an increase in g_t at t depend on whether the change is unanticipated or whether it had been anticipated in advance.

Homework 4

Tuesday, April 06, 2010
10:09 AM

Homework 4.

The first several problems assume the following environment. A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t \log c_t \quad , \quad 0 < \beta < 1$$

$$\beta = \frac{1}{1+\rho} \quad , \rho > 0$$

where c_t is consumption per worker.

The technology is

$$y_t = f(k_t) = z k_t^{\alpha} \quad , \quad 0 < \alpha < 1, \; z > 0$$

where y_t is output per unit labor and k_t is capital per unit labor

$$y_t = c_t + x_t + g_t$$

$x_t =$ gross investment per unit labor

$g_t =$ government expenditures per unit of labor

$$k_{t+1} = (1-\delta) k_t + x_t \quad , \quad 0 < \delta < 1$$

The government finances its expenditures by levying a flat rate tax τ_{ct} on the value of consumption goods purchased at t, a flat rate tax of τ_{nt} on the value of labor earnings at t, a flat rate tax τ_{kt} on earnings from capital a flat rate tax τ_{kt} on earnings from capital at t, and a lump sum tax of τ_{ht} in the t consumption goods per worker at time t.

Let $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ be a price system.

1. Define a competitive equilibrium with taxes and government purchases.

2. Assume that $g_t = 0$ for all $t \geq 0$. Assume that all taxes are zero for all $t \geq 0$. Find a steady state level of the capital stock per worker. Find the level of consumption per worker c at the steady state level of capital stock per worker.

3. Find the level of steady state c as a function of steady state k. Suppose now that even though $g_t = 0 \ \forall t \geq 0$, the government can tax ($\tau_k > 0$) or subsidize ($\tau_k < 0$) returns on capital by setting a constant over time tax $\tau_{kt} = \tau_k \ \forall t \geq 0$ on capital. The government rebates the revenues it raises (if $\tau_k > 0$) to the public by setting lump sum taxes negative ($\tau_h < 0$);

public by setting lump sum taxes negative ($\tau_h < 0$); or the government raises the revenues to subsidize capital (if $\tau_k < 0$), by setting a positive lump sum tax $\tau_h > 0$. Please find a level of the tax or subsidy on capital τ_k that maximizes the steady state level of consumption per unit labor. Is τ_k positive, negative, or zero?

4. Suppose that you live in an economy where in the past g_t was always $= 0$ and where for a long time the tax or subsidy on capital was set at the problem 3 rate that maximizes steady state c = consumption per worker. Suppose that τ_k had been set at this rate for such a long time that you are at a steady state at time 0, so that k_0 = steady state k you computed in problem 3. Suppose that $g_t = 0 \;\forall\, t \geq 0$.

a. Starting from k_0, describe the allocation that would be chosen by a planner who solves the problem

$$\max \sum^{\infty} \beta^t \log c_t$$

$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0} \ldots$

$$c_t + k_{t+1} = z k_t^{\alpha} + (1-\delta) k_t .$$

Find the steady state level of capital per worker and consumption per worker. Compare with the initial level of k_0.

b. Use your answer to (a) to discuss whether the τ_k set in problem 3 is too high or too low.

5. Suppose now that $g_t = 0 \ \forall t$ and that all taxes are zero for all time. Let $\bar{k}(z=1)$ be the steady state level of capital associated with a given level of $z=1$. Let $k_0 = \bar{k}(z=1)$. Suppose that at time 0 z suddenly and unexpectedly jumps to $z = 1.5$.

a. Find the new steady state level of capital.

b. Describe the transition path for $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ as k moves from $\bar{k}(z=0)$ to $\bar{k}(z=1)$. Describe the behavior of

$$R_{t+1} \equiv f'(k_{t+1}) + (1-\delta)$$

along the transition path, and how c_{t+1}/c_t is associated with R.

is associated with R_{t+1}.

6. Suppose that at time $t=0$ the economy is in a steady state associated with a level of $g=0$ always and no taxes of any kind. Let $\bar{k}$ be the level of capital per worker and $\bar{c} = f(\bar{k}) - \delta\bar{k}$ the level of consumption per worker in this initial steady state.

Let $k_0 = \bar{k}$. Now assume that at time $t=0$ the government begins spending $g_t = \bar{g}$ $\forall t \geq 0$ where $\bar{g} = \frac{1}{4}\left[f(k_0) - \delta k_0\right]$.

The government finances its purchase by imposing lump sum taxes.

(a) Find the new steady state level of capital and consumption per worker.

(b) Describe the transition path of consumption and capital to the new steady state

(c) Explain the behavior of c_{t+1}/c_t and $R_{t+1} \equiv f'(k_{t+1}) + (1-\delta)$ along a transition path to a new steady state

(d) Prove that the timing of lump sum taxes is irrelevant.

Term structure

Saturday, April 11, 2009
11:39 AM

Term structure of interest rates.

<u>Determinate model:</u>

q_t^0 = price of time t consumption good in terms of time 0 consumption good.

$\sim \frac{\text{time 0 goods}}{\text{time } t \text{ good}} \sim$ units

$$q_t^0 = q_0^0 \cdot \frac{q_1^0}{q_0^0} \cdot \frac{q_2^0}{q_1^0} \cdot \ldots \cdot \frac{q_t^0}{q_{t-1}^0}$$

$$= 1\, m_{0,1}\, m_{1,2} \cdots m_{t-1,t} \qquad , q_0^0 = 1$$

where $m_{t-1,t} = \frac{q_t^0}{q_{t-1}^0} \sim \frac{\text{time } t-1 \text{ goods}}{\text{time } t \text{ goods}}$

numeraire

$$\frac{\frac{\text{time 0 good}}{\text{time } t \text{ good}}}{\frac{\text{time 0 good}}{\text{time } t-1 \text{ good}}} = \frac{\text{time } t-1 \text{ good}}{\text{time } t \text{ good}}$$

$$m_{t-1,t} \equiv \exp(-r_{t-1,t}) \approx \frac{1}{1+r_{t-1,t}}$$

$$-\log m_{t-1,t} = r_{t-1,t} \qquad \left(\log \frac{1}{1+r_{t-1,t}} \approx -r_{t-1,t}\right)$$

$r_{t-1,t}$ = one period net yield between $t-1, t$.

Time 0
Price of a t-period pure discount bond promising 1 unit of consumption at t,

$$q_t^0 = m_{0,1}\, m_{1,2} \cdots m_{t-1,t}$$

$$= \exp(-r_{0,1}) \exp(-r_{1,2}) \cdots \exp(-r_{t-1,t})$$

(*) $$q_t^0 = \exp\left(-\left(r_{0,1} + r_{1,2} + \cdots + r_{t-1,t}\right)\right)$$

definition of t period yield: $r_{0,t}$ "from" "to"

$$q_t^0 = \exp(-t\, r_{0,t})$$

or

(**) $$-\frac{1}{t} \log q_t^0 = r_{0,t}$$

(*) and (**) =>

$$\underset{\text{long yield}}{r_{0,t}} = \frac{1}{t} \underbrace{\left(r_{0,1} + r_{1,2} + r_{2,3} + \cdots + r_{t-1,t}\right)}_{\text{average of } t \text{ one-period yields over the horizon.}}$$

$r_{t-1,t}$ a "forward" one period yield for $s < t-1$.

More generally,

$$\underset{\text{"long" } \tau\text{-period yield at } t}{r_{t,t+\tau}} = \frac{1}{\tau} \underbrace{\left(r_{t,t+1} + r_{t+1,t+2} + \cdots + r_{t+\tau-1,t+\tau}\right)}_{\text{average of short yields from } t \text{ to } t+\tau}$$

"expectation theory" of the term structure of interest rates

Specific theory about q_t^0 - from growth model

$$q_t^0 = \mu \beta^t \frac{u'(c_t)}{1+\tau_{ct}}$$

then

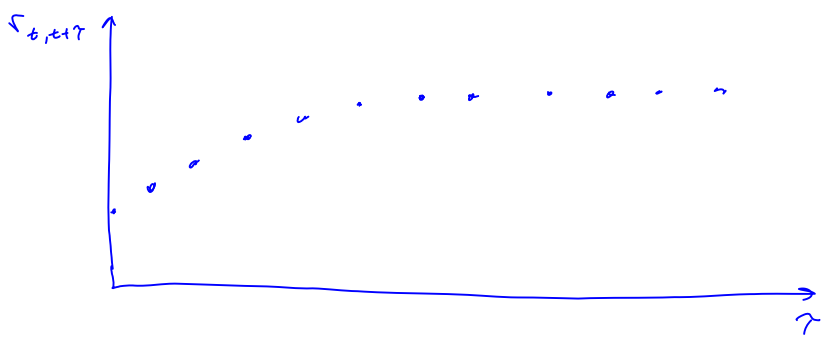
$$m_{t-1,t} = \beta \frac{u'(c_t)}{u'(c_{t-1})} \cdot \left(\frac{1+\tau_{c\,t-1}}{1+\tau_{ct}}\right) = \exp(-r_{t-1,t})$$

specific theory about "discount factor" $m_{t-1,t} \equiv \exp(-r_{t-1,t})$

Remark: Consider the "expectations theory" formula

$$r_{t,t+\tau} = \frac{1}{\tau}\left(r_{t,t+1} + r_{t+1,t+2} + \cdots + r_{t+\tau-1,t+\tau}\right)$$

The yield curve at t is a plot of $r_{t,t+\tau}$ against horizon τ.



Reported daily on Bloomberg

yield curve at a particular date

Here the yield curve tells us that the market expects short rates $r_{s-1,s}$ to rise over time –

why? – well, note, for example, that

$$r_{t,t+2} = \frac{1}{2}\left[r_{t,t+1} + r_{t+1,t+2}\right]$$

$$r_{t,t+1} \equiv \frac{1}{2}\left[r_{t,t+1} + r_{t,t+1}\right]$$

$$\Rightarrow r_{t,t+2} - r_{t,t+1} = \frac{1}{2}\left[r_{t+1,t+2} - r_{t,t+1}\right]$$

two year yield – one year yield

$\therefore$

$\therefore$

$$r_{t,t+2} - r_{t,t+1} > 0 \implies r_{t+1,t+2} > r_{t,t+1}$$

upward sloping yield curve

one period yields are expected to rise

Link to growth model:

remember

$$(*) \quad m_{t-1,t} = \beta \frac{u'(c_t)}{u'(c_{t-1})} \cdot \left(\frac{1+\tau_{ct-1}}{1+\tau_{ct}}\right) = \exp\left(-r_{t-1,t}\right)$$

Recall the FONC from the growth model

$$u'(c_t) = \beta u'(c_{t+1}) R_{t+1}$$

where $R_{t+1} = \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} \left[(1-\delta) + (1-\tau_{kt+1}) f'(k_{t+1}) \right]$

Evidently $(*)$ (shift forward one period)

$$u'(c_t) = \beta\, u'(c_{t+1}) \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} \exp\left(+r_{t,t+1}\right)$$

So evidently

$$\exp(r_{t,t+1}) = (1-\delta) + (1-\tau_{kt+1}) f'(k_{t+1})$$

$$\approx (1+r_{t,t+1})$$

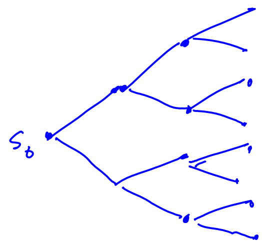
and $R_{t+1} = \frac{1+\tau_{ct}}{1+\tau_{ct+1}} \exp(r_{t,t+1})$

Stochastic model:

Recall the "pricing in trees" model

where $s_t \in S \sim$ "state space", $t = 0, 1, 2, \ldots$

$s^t = (s_t, s_{t-1}, \ldots, s_0)$ a history



each "node" is a history s^t

we formulated a model with a competitive price

$q_t^0(s^t)$ = price of one unit of consumption at date t contingent on history s^t — measured in units of time $\frac{0 \text{ good}}{\text{time } t, \text{ history } s^t \text{ good}}$

Write

"1" — choice of time 0 goods as numeraire

$$q_t^0(s^t) = q_0^0(s_0) \frac{q_1^0(s^1)}{q_0^0(s_0)} \frac{q_2^0(s^2)}{q_1^0(s^1)} \cdots \frac{q_t^0(s^t)}{q_{t-1}^0(s^{t-1})}$$

$$= q_0^0(s_0)\, m_{0,1}(s^1)\, m_{1,2}(s^2) \cdots m_{t-1,t}(s^t)$$

where $\dfrac{q_t^0(s^t)}{q_{t-1}^0(s^{t-1})} = m_{t-1,t}(s^t)$

$$= \frac{\text{goods at } t-1, s^{t-1}}{\text{goods at } t, s^t}$$

= price of a unit of cons'n at t, s^t in units of time $t-1$, history s^{t-1} goods

One period risk-free yield $r_{t-1,t}(s^{t-1})$:

$$\sum_{s_t \in S} m_{t-1,t}(s_t, s^{t-1}) \equiv \exp\left(-r_{t-1,t}(s^{t-1})\right)$$

cost to buy one unit of consumption in all states $s_t \in S$

$$\approx \frac{1}{1 + r_{t-1,t}}$$

in all states $s_t \in S$
next period starting from
history s^{t-1} at time $t-1$.

- - - - - - - - - - - - - - - - -

Remark: relationship between risk-free yields of different horizons is now more complicated than given by the "pure expectations" theory that we derived for the non-stochastic case.

For example, the two period risk-free yield $r_{t-1,t+1}(s^{t-1})$

$$\exp(-2\, r_{t-1,t+1}) = \underbrace{\sum_{s_t \in S}}_{\substack{\text{sum over all}\\\text{"intermediate"}\\\text{states}}} \underbrace{\sum_{s_{t+1} \in S}}_{\substack{\text{sum over these}\\\text{to get}\\\text{consumption for}\\\text{sure at date } t}} m_{t-1,t}(s_t \mid s^{t-1})\, m_{t,t+1}(s_{t+1} \mid s_t, s^{t-1})$$

Remark: formulas must be modified for inflation.

- The simplest way is to reinterpret q^0_t as the price of one dollar at t in time 0 dollars and $m_{t-1,t}$ as the price of one dollar at t in time $t-1$ dollars. Then the above formulas apply directly to the yields reported on Bloomberg.
- Interpreting q^0_t as a "nominal" (dollar) pricing function in this way breaks the link $q^0_t = \mu \beta^t \dfrac{u'(c_t)}{1+\tau_{ct}}$

that we established, for example, in the growth model. Instead, we would have something like

"nominal" price function

$$\tilde{q}^0_t = \underbrace{\mu \beta^t u'(c_t)}_{q^0_t = \text{"real" price function}} \frac{p_0}{p_t}$$

where p_t = price level at time t

$$\sim \frac{\text{time } t \text{ dollars}}{\text{time } t \text{ goods}}$$

Note that inflation $\left(\frac{p_t}{p_0} > 1\right)$

causes the price of time t dollars in terms of time 0 dollars to <u>decrease</u>

q^0_t = real price system

$\tilde{q}^0_t = \frac{p_0}{p_t} q^0_t$ = nominal price system

represent $\tilde{q}^0_t$ as

$$\tilde{q}^0_t = q^0_t \frac{p_0}{p_1} \frac{p_1}{p_2} \cdots \frac{p_{t-1}}{p_t}$$

$$= q^0_0 \left(m_{0,1} \frac{p_0}{p_1}\right)\left(m_{1,2} \frac{p_1}{p_2}\right) \cdots \left(m_{t-1,t} \frac{p_{t-1}}{p_t}\right)$$

represent $m_{t-1,t} = \exp(-r_{t-1,t})$

$$\frac{p_{t-1}}{p_t} = \exp(-\pi_{t-1,t})$$

$$\Updownarrow$$

$$P_t = P_{t-1} \exp(\pi_{t-1,t})$$

or

$$\pi_{t-1,t} = \log\left(\frac{P_t}{P_{t-1}}\right)$$

Thus

$$\tilde{q}_t^0 = q_t^0 \exp\left(-(r_{0,1} + \pi_{0,1})\right) \exp\left(-(r_{1,2} + \pi_{1,2})\right) \cdots \exp\left(-(r_{t-1,t} + \pi_{t-1,t})\right)$$

$$= q_t^0 \exp\left[-t\, \tilde{r}_{0,t}\right]$$

where $\tilde{r}_{0,t} = \frac{1}{t}\left[\tilde{r}_{0,1} + \tilde{r}_{1,2} + \cdots + \tilde{r}_{t-1,t}\right]$

<u>nominal</u> yield to maturity on t-period zero coupon risk free nominal bond

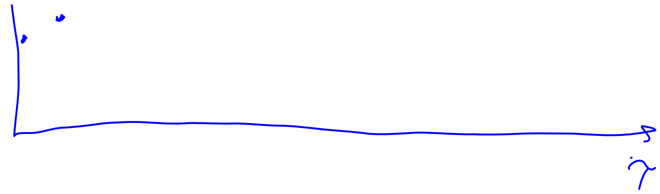
$$\tilde{r}_{t-1,t} = r_{t-1,t} + \pi_{t-1,t}$$

nominal one-period rate — real one-period rate — expected inflation from $t-1, t$

our <u>theory</u> of $r_{t-1,t} \leftarrow$ real yield.

$$\exp(-r_{t-1,t}) = \beta \frac{u'(c_t)}{u'(c_{t-1})}$$ when $1+\tau_{ct} = 1+\tau_{ct-1}$

Each day Bloomberg plots



nominal yield to maturity on τ-period bonds.

Remark: people read these to make inferences about

(a) expected inflation

(b) paths of expected <u>real</u> short rates of interest.

Irving Fisher theory:

$$\tilde{r}_{t,t+\tau} = r_{t,t+\tau} + \pi_{t,t+\tau}$$

$$\pi_{t,t+\tau} = \frac{1}{\tau}\left[\pi_{t,t+1} + \cdots + \pi_{t+\tau-1,t+\tau}\right]$$

Incomplete mkts

Friday, April 09, 2010
10:47 AM

Household problem: choose $c_0, c_1(1), c_1(2), b$

to maximize

$$u(c_0) + \beta u(c_1(1))\pi + \beta u(c_1(2))(1-\pi)$$

s.t. $c_0 + b \le y_0$

$$c_1(1) \le y_1(1) + R(1) b$$

$$c_1(2) \le y_1(2) + R(2) b$$

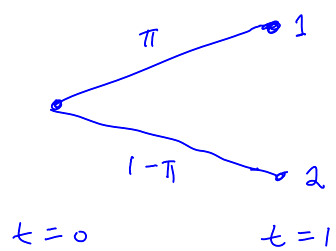
$$c_0 \ge 0, \; c_1(1) \ge 0, \; c_1(2) \ge 0$$

$R(1)$ is gross return on single asset in state 1
$R(2)$ " " " " " " " " 2

$$u' > 0, \; u'' < 0, \; u'(0) = +\infty$$

The asset is risky when $R(1) \ne R(2)$
The asset is risk free when $R(1) = R(2)$

There is only one asset available even though there are two states 1 and 2 at $t=1$



π = Prob state = 1

Form Lagrangian:

$$L = u(c_0) + \beta u(c_1(1))\pi + \beta u(c_1(2))(1-\pi)$$

$$+\theta_0[y_0 - b - c_0]$$

$$+\theta_1[y_1(1) + R(1)b - c_1(1)]$$

$$+\theta_2[y_1(2) + R(2)b - c_1(2)]$$

$$-\eta_1 c_1(1) - \eta_2 c_1(2) - \eta_0 c_0$$

max L w.r.t. $c_0, c_1(1), c_1(2), b$

minimize L w.r.t. $\theta_0, \theta_1, \theta_2, \eta_1, \eta_2, \eta_0$

FONC:

c_0: $\quad u'(c_0) - \theta_0 - \eta_0 = 0, \qquad \eta_0 c_0 = 0$

$c_1(1)$: $\quad \beta u'(c_1(1))\pi - \theta_1 - \eta_1 = 0, \quad \eta_1 c_1(1) = 0$

$c_1(2)$: $\quad \beta u'(c_1(2))(1-\pi) - \theta_2 - \eta_2 = 0, \quad \eta_2 c_1(2) = 0$

b: $\quad -\theta_0 + \theta_1 R(1) + \theta_2 R(2) = 0$

seek an interior soln (use $u'(0) = +\infty$ to justify this)

$$\Rightarrow \eta_0 = \eta_1 = \eta_2 = 0$$

FONC's imply:

$$\theta_0 = u'(c_0)$$

$$\theta_1 = \beta u'(c_1(1))\pi$$

$$\theta_2 = \beta u'(c_1(2))(1-\pi)$$

$$u'(c_0) = \beta u'(c_1(1)) R_1 \pi + \beta u'(c_1(2)) R_2 (1-\pi)$$

or

$$1 = \beta E \frac{u'(c_1)}{} R$$

random variables

$$1 = \beta E \frac{u'(c_1)}{u'(c_0)} R$$

random variables

mathematical expectation w.r.t. distribution of (c_1, R)

or

$$(*) \quad 1 = \beta \left[\frac{u'(y_1(1) + R(1)\, b) R(1)\, \pi + u'(y_1(2) + R(2)) R(2) (1-\pi)}{u'(c_0)} \right]$$

given $y_0, y_1(1), y_1(2), R(1), R(2)$

$(*)$ is one equation in the unknown b.

solving $(*)$ for $b = b(y_0, y_1(1), y_1(2), R(1), R(2))$

gives the consumer's demand curve for b.

Note: we seek a soln for $(*)$ s.t.

$$y_1(1) + R(1)\, b \geq 0$$

$$y_1(2) + R(2)\, b \geq 0$$

$$y_0 - b \geq 0$$

A popular special case: the asset is risk-free, meaning $R(1) = R(2)$ (same return in all states)

Then $(*)$ becomes

$$(**) \quad 1 = \beta R \frac{[u'(y_1(1) + Rb)\pi + u'(y_1(2) + Rb)(1-\pi)]}{u'(c_0)}$$

$$(*) \quad 1 = \beta R \frac{\ldots}{u'(y_0 - b)}$$

to be solved for b s.t.

$$y_1(1) + Rb \geq 0$$

$$y_1(2) + Rb \geq 0$$

$$y_0 - b \geq 0$$

General Equilibrium version. Suppose there are two consumers, one whose demand b solves $(*)$ and a second whose demand $\hat{b}$ solves

$$(**) \quad 1 = \beta R \frac{[u'(\hat{y}_1(1) + R\hat{b})\pi + u'(\hat{y}_1(2) + R\hat{b})(1-\pi)]}{u'(\hat{y}_0 - \hat{b})}$$

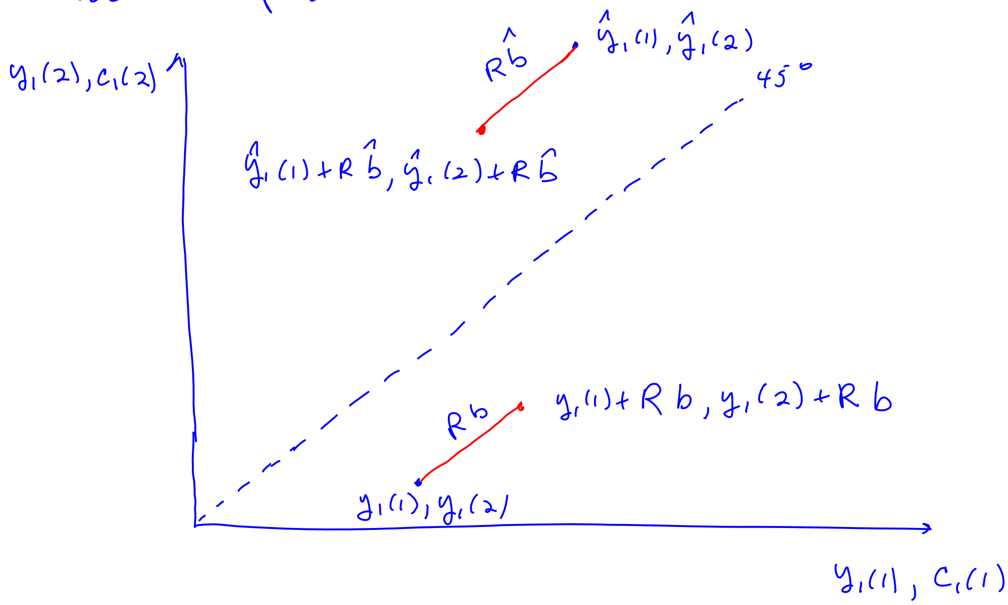
where $\hat{y}_0, \hat{y}_1(1), \hat{y}_1(2)$ is person ^'s endowment, and R is the <u>same</u> gross-return on a risk free asset as in $(*)$.

An incomplete market equilibrium is an $R, b, \hat{b}$ that satisfy $(*)$, $(**)$, and

$$(***) \quad b + \hat{b} = 0 .$$

Remark: $(***)$ says that the asset is "in zero net supply", so this is an economy in which one person's <u>asset</u> is the other person's debt.

Graphical depiction of what trades in the risk-free asset accomplish:



$$\boxed{\hat{b} = -b}$$

remark: trading the risk free bond allows the households to trade consumption between times 0 and 1 but not across states at time 1.

The consumers are restricted to moving along lines parallel to the 45° line with length Rb, $R\hat{b}$ governed by $b, \hat{b}$.

Note: In general case (we allow $R(1) \neq R(2)$)

we have

(+) $1 = E\, m_j R$

where $m_j = \dfrac{\beta\, u'(c_1^j)}{u'(c_0^j)}$

is the (random) marginal rate of substit'n for consumer j.

Remark: m_j is called consumer j's stochastic discount factor.

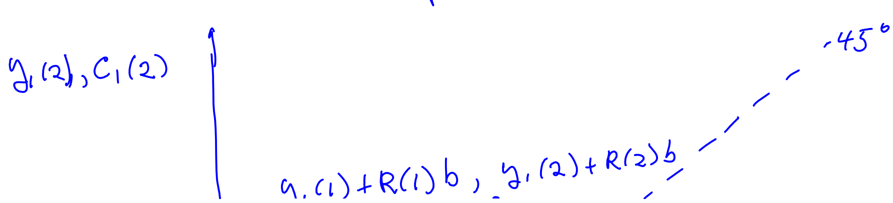
But we do not equate

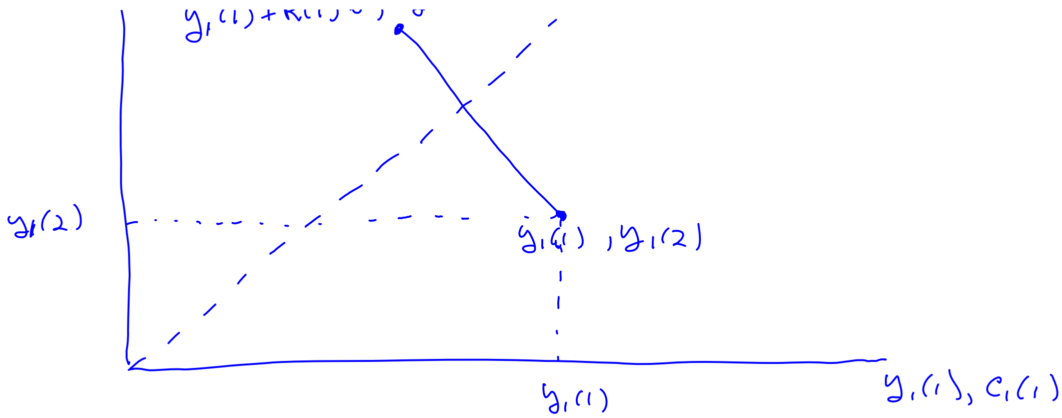
m^j across consumers, as does occur with complete markets and common probabilities across agents.

Remark: If we have as many assets as states, we might be able to conclude that

$m_j = m_i$ for all i, j. (complete markets)

Redraw the above graph when $R(1) \neq R(2)$





Here $b > 0$ and $R(1) < 0$, $R(2) > 0$

or $b < 0$ and $R(1) > 0$; $R(2) < 0$

$$d\, c_1(1) = R(1)\, d b$$

$$d\, c_1(2) = R(2)\, d b$$

$$\Rightarrow \quad \frac{d\, c_1(2)}{d\, c_1(1)} = \frac{R(2)}{R(1)}$$

this is the direction in the $c_1(1), c_1(2)$ plane that trades of b attain.

Remark: Consider the key asset pricing equation

$$E\, m\, R = 1$$

or

$$\int_{\omega \in \Omega} m(\omega)\, R(\omega)\, f(\omega) = 1 \qquad (*)$$

where $m(\omega)$ is the s.d.f. , $R(\omega)$ is the return in state $\omega \in \Omega$, and $f(\omega) \geq 0$ is the probability density of ω, $\int f(\omega) = 1$.

Rewrite $(*)$ as

$$R_f^{-1} \int R(\omega) \left[\frac{m(\omega) f(\omega)}{\int m(\tilde{\omega}) f(\tilde{\omega}) \, d\tilde{\omega}} \right] d\tilde{\omega}$$

or

$$R_f^{-1} \int R(\omega) \, \hat{f}(\omega) \, d\omega = 1 \qquad (**)$$

where $\int m(\tilde{\omega}) f(\tilde{\omega}) \, d\tilde{\omega} \equiv E m \equiv R_f^{-1}$

$\sim$ reciprocal of risk free rate of return.

because

$$1 = \int R_f \, m(\omega) f(\omega) \, d\omega = R_f \int m(\omega) f(\omega) \, d\omega$$

$$\Rightarrow R_f^{-1} = \int m(\omega) f(\omega) \, d\omega$$

Thus $(**)$ becomes

$$(+) \qquad \int R(\omega) \, \hat{f}(\omega) \, d\omega = R_f$$

"risk neutral" probability:

$$\hat{f}(\omega) = \frac{m(\omega)}{\int m(\tilde{\omega}) f(\tilde{\omega}) \, d\tilde{\omega}} f(\omega)$$

note: $\hat{f}(\omega) \geq 0$ and $\int \hat{f}(\omega) = 1$

so $\hat{f}(\omega)$ is a probability density

(+) says that expected return of <u>any</u> asset evaluated at the "risk-neutral" probabilities equals the risk-free return.

SDF

Friday, April 09, 2010
12:19 PM

$\omega \in \Omega$ state of the world

$f(\omega) \sim$ prob of ω (density)

$$\int f(\omega)\, d\omega = 1 \quad , \; f(\omega) \geq 0$$

$x(\omega)$ - random payoff (next period)

household problem: choose c_0, $c_1(\omega)$, b to

maximize

$$u(c_0) + \beta \int u(c_1(\omega)) f(\omega)\, d\omega$$

s.t. $c_0 + p \cdot b \leq y$

$$c_1(\omega) \leq x(\omega) \cdot p$$

$u' > 0, \; u'' < 0, \; u'(0) = +\infty, \; y_0 > 0$

Lagrangian L:

$$L = u(c_0) + \beta \int u(c_1(\omega)) f(\omega)\, d\omega$$

$$+ \theta \left[y - c_0 - p \cdot b \right]$$

$$+ \int \lambda(\omega) \left[x(\omega)\, b - c_1(\omega) \right] d\omega$$

θ, $\lambda(\omega)$ are Lagrange multipliers.

FONC:

c_0: $\quad u'(c_0) - \theta = 0$

$$c_1(\omega): \quad \beta u'(c_1(\omega)) f(\omega) - \lambda(\omega) = 0$$

$$b: \quad -\theta p + \int \lambda(\omega) x(\omega) d\omega = 0$$

$$\theta p = \int \lambda(\omega) \, x(\omega) \, d\omega$$

$$u'(c_0) \, p = \beta \int u'(c_1(\omega)) x(\omega) \, f(\omega) \, d\omega$$

$$p = \int \underbrace{\beta \frac{u'(c_1(\omega))}{u'(c_0)}}_{m(\omega)} \; x(\omega) \, f(\omega) d\omega$$

$m(\omega)$ – one for each person who holds (or issues) the asset.

(*) $$p = \int m(\omega) \, x(\omega) \, f(\omega) \, d\omega$$

price — s.d.f. — payoff — probability

<u>Remark</u>: There are as many s.d.f.'s as there are

$$m(\omega) = \frac{\beta u'(c_1(\omega))}{u'(c_0)} \quad \text{across agents.}$$

a random variable.

how many & how far apart the s.d.f.'s are depends on whether markets are complete or incomplete and how incomplete they are.

note: (*) is

$$p = \int m(\omega)\, x(\omega)\, f(\omega)\, d\omega$$

or

$$p = E\, m x$$

or

$$1 = E\, m R \qquad \text{where } R \equiv \frac{x}{p} \quad \text{(a gross } \underline{\text{return}}\text{)}$$

Remark: if $x(\omega) = 1 \quad \forall\ \omega$,

$p = E m$ — price of a risk-free asset

Remark: if we have discrete states $j = 1, \ldots, J$

formula (*) becomes

$$p = \sum_j \underset{\substack{\uparrow \\ \text{s.d.f.}}}{m_j}\ \underset{\substack{\uparrow \\ \text{payoff in state } j}}{x_j}\ f_j$$

f_j — probability of state j

Remark: If we have multiple securities $i = 1, \ldots, I$, we have

$$p_i = \int m(\omega)\, \underset{\substack{\uparrow \\ \text{payoff on security } i}}{x_i(\omega)}\, f(\omega)\, d\omega, \quad i = 1, \ldots, I$$

$$\overset{I\times 1}{p} = \int m(\omega)\, \overset{1\times1}{x(\omega)}\, \overset{I\times1}{f(\omega)}\, d\omega$$

or

$$\text{or}$$

(*) $$p = E\, m\, x$$

$$p = \begin{pmatrix} p_1 \\ \vdots \\ p_I \end{pmatrix}, \quad x = \begin{pmatrix} x_1(\omega) \\ x_2(\omega) \\ \vdots \\ x_n(\omega) \end{pmatrix}$$

Multiple m's satisfy (*) in general with incomplete markets. Goal: deduce some restrictions that (*) puts on data on p and x for all possible m's satisfying (*). This will guide us about what is a good or acceptable theory of a s.d.f.

One way to proceed: we don't have data on m, but we do on x.

Imagine constructing linear least squares projection

(†) $$\underset{1\times1}{m} = \underset{1\times1}{a} + \underset{1\times I}{x'}\, \underset{I\times1}{b} + \underset{1\times1}{e}$$

where $E\, e\, x = 0_{I\times1}$ "$e \perp x$",
$E\, e = 0$ "$e \perp 1$"

$e \perp x$ identifies $a + x'b$ as projection

$e \perp x$ identifies $a + x'b$ as projection of m on $[1, x]$.

$(+) \Rightarrow \quad Em = a + Ex'b$

$\Rightarrow \quad m - Em = (x' - Ex')b + e$

$$(x - Ex)(m - Em) = (x - Ex)(x' - Ex')b' + (x - Ex)e$$

Take E on both sides

$$E(x - Ex)(m - Em) = E(x - Ex)(x' - Ex')b + 0$$

$$cov(x, m) = cov(x, x)\, b$$

or

$$b = [cov(x, x)]^{-1} cov(x, m)$$

$$a = Em - Ex'b$$

we have no data on m. but we have data on $\rho = E\, m x$

Note:

$$cov(x, m) = E\, mx - Em\, Ex$$

$$= \rho - Em\, Ex$$

"know" ↗ ρ, ↑ Em don't, ↑ Ex "know"

Know — (unless risk free rate is observed)

$\therefore$

$$b = [\operatorname{cov}(x,x)]^{-1}[p - Em\,Ex]$$

For a given guess about Em, we can estimate b from $\operatorname{cov}(x,x)$ & Ex & p.

Note:

$$m = a + x'b + e$$

$$(**) \quad \operatorname{var} m = \operatorname{var}(x'b) + \operatorname{var} e \geq \operatorname{var}(x'b)$$

because $e \perp x$ & $\operatorname{var} e \geq 0$

$(**)$ is a version of the Cauchy-Schwartz inequality

$(**) \Rightarrow$

$$(**) \quad \sigma(m) \geq [\operatorname{var}(x'b)]^{1/2}$$

where $b = [\operatorname{cov}(x,x)]^{-1}[p - Em\,Ex]$

$(**)$ gives us the Hansen-Jagannathan bound on $\sigma(m)$

a lower bound on $\sigma(m)$'s for a given Em.

If instead of payoffs, we use data on

If instead of payoffs, we use data on

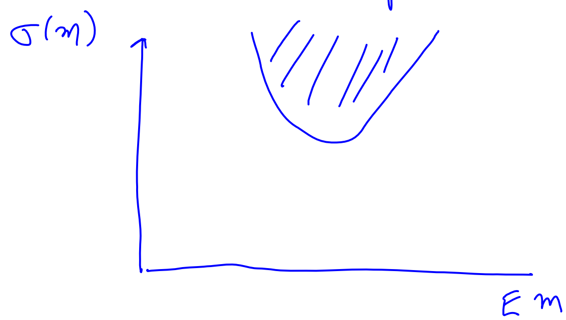
<u>returns</u> $R_i = \dfrac{x_i}{p_i}$, $R = \begin{pmatrix} R_1 \\ \vdots \\ R_I \end{pmatrix}$

we would have

(1) $$b = [\mathrm{cov}(R,R)]^{-1}[1 - E m E R]$$

(2) $$\sigma(m) \geq [\mathrm{var}(R'b)]^{1/2} = \sqrt{b' \mathrm{cov}(R,R)\, b}$$

we can use this to compute a set of m's



← pairs $E(m), \sigma(m)$ in the shaded area satisfy the HJ bound.

Example of a theoretical s.d.f.

$$m_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)} \quad , \; C_t \text{ - Consumption}$$

$$c_t = \log C_t , \quad C_t = \exp(c_t)$$

$$u'(C_{t+1}) = C_{t+1}^{-\gamma} \qquad\qquad u(C) = \frac{1}{1-\gamma} C^{1-\gamma}$$

$$= (\exp c_{t+1})^{-\gamma} = \exp(-\gamma c_{t+1})$$

$\therefore$ $\quad \cap \; \exp(-\gamma c_{t+1})$

$\therefore$

$$m_{t+1} = \beta \frac{\exp(-\gamma c_{t+1})}{\exp(-\gamma c_t)}$$

$$m_{t+1} = \beta \exp(-\gamma(c_{t+1} - c_t))$$

Common statistical model:

$$c_{t+1} = c_t + \mu_c + \sigma_\varepsilon \varepsilon_{t+1}$$

$$\varepsilon_{t+1} \sim N(0,1)$$

log of consumption is a random walk with drift μ

C_t consumption is a "geometric random walk with drift"

Then

$$(*) \quad m_{t+1} = \beta \exp(-\gamma(\mu_c + \sigma_\varepsilon \varepsilon_{t+1}))$$

$$(**) \quad \log m_{t+1} = \log \beta - \gamma(\mu_c + \sigma_\varepsilon \varepsilon_{t+1})$$

Wikipedia remark:

a random variable X is said to be log normal

if $\log X \sim N(\mu, \sigma^2)$

mean variance of $\log X$

Fact:

$$(+) \quad E X = \exp(\mu + \tfrac{1}{2}\sigma^2)$$

$$\operatorname{std}(X) = \exp(\mu + \tfrac{1}{2}\sigma^2)\sqrt{e^{\sigma^2} - 1}$$

$\therefore$

(‡) $$\frac{\operatorname{std}(X)}{E(X)} = \sqrt{e^{\sigma^2} - 1}$$

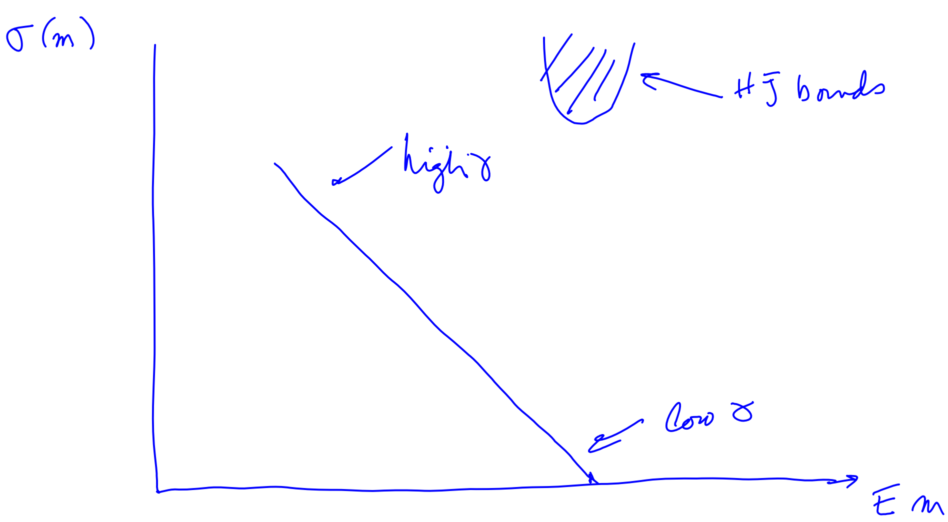
Apply (†) and (‡) to (*), (**)

$$\log m \sim N\left(\log \beta - \gamma \mu_c,\ \gamma^2 \sigma_\varepsilon^2\right)$$

$$\Rightarrow \quad E\, m = \beta \exp\left[\gamma\left(-\mu_c + \tfrac{1}{2}\gamma \sigma_\varepsilon^2\right)\right]$$

$$\frac{\sigma(m)}{E(m)} = \sqrt{e^{\gamma^2 \sigma_\varepsilon^2} - 1}$$

Tallarini: Plot these for various values of γ with estimated μ_c, σ_ε from U.S. time series.



Remark: recall

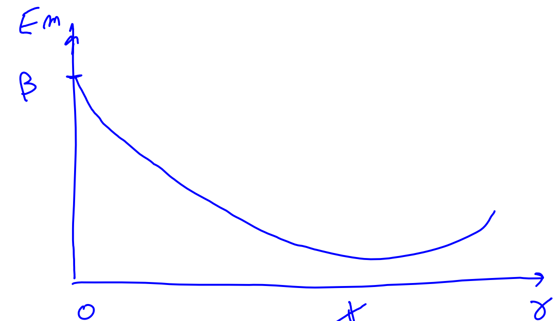
$$E\, m = \beta \exp\left[\gamma\left(-\mu_c + \gamma \frac{\sigma_\varepsilon^2}{2}\right)\right]$$

the $\exp(-\gamma \mu_c)$ term decreases as $\gamma \uparrow$

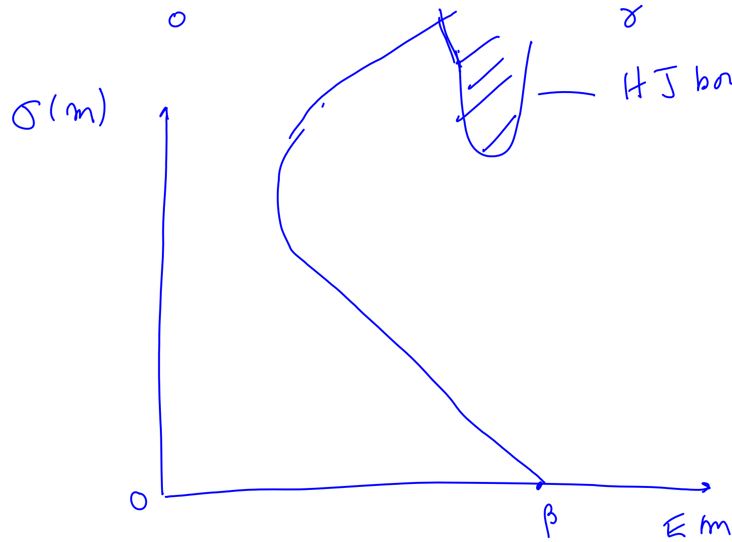
the $\exp\left(\gamma^2 \frac{\sigma_\varepsilon^2}{2}\right)$ term <u>increases</u> as $\gamma \uparrow$

μ_c & σ_ε^2 empirically take values with aggregate consumption data for U.S. that make

Em fall at first, then rise as $\gamma \uparrow$



$\Rightarrow$



so you can get inside the HJ bounds with an "astronomical" value of γ

<u>Remark:</u> risk free rate puzzle is the tendency of Em to fall (R_f to <u>rise</u>) as $\gamma \uparrow$ for "low" values of γ

Remark: the equity premium puzzle is the high measured "market price of risk"

$$\frac{\sigma(m)}{Em}$$

HJ,2

Sunday, April 25, 2010
8:17 PM

$c_t = \log C_t$

$c_{t+1} - c_t = \mu_c + \sigma_\varepsilon \varepsilon_{t+1}$, $\varepsilon_{t+1} \sim \mathcal{N}(0,1)$

geometric random walk with drift

we established that with

$$U(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$$

(*) $$E[m] = \beta \exp\left(\gamma\left(-\mu_c + \frac{\gamma\sigma_\varepsilon^2}{2}\right)\right)$$

discounting, intertemporal smoothing, precautionary saving because of risk

(**) $$\frac{\sigma(m)}{E(m)} = \left\{\exp\left(\gamma^2\sigma_\varepsilon^2\right) - 1\right\}^{\frac{1}{2}}$$

risk aversion

Problem: the same parameter γ pushes up $\sigma(m)$ while pushing down $E\,m$ (because of preference for intertemporal smoothing)

Focus on

$$E\,m = \beta \exp\left(-\gamma\mu_c + \gamma^2\frac{\sigma_\varepsilon^2}{2}\right)$$

intertemporal substitution

$$\frac{std(m)}{\ldots} = \sqrt{\exp(\gamma^2\sigma_\varepsilon^2) - 1}$$

$\overline{E(m)}$

risk aversion (across states)

Idea of Epstein-Zin-Weil: separate these two - set γ in 1 in IES, but γ large for risk aversion.

Solution: Epstein-Zin-Weil-Tallarini

Change utility function to

$$U_t = c_t - \beta\theta\, E_t\left[\exp\left(U_{t+1}/-\theta\right)\right]$$

where $$\theta \equiv \frac{-1}{(1-\beta)(1-\gamma)}$$

or $$\hat{\gamma} = 1 + \frac{1}{(1-\beta)\theta} \qquad (>1 \text{ when } \theta < +\infty)$$

Then

no γ here

this effect pushes $E\,m$ up slightly

$$E[m] = \beta \exp\left[-\mu_c + \frac{\sigma_\varepsilon^2}{2}(2\hat{\gamma}-1)\right]$$

$$\frac{\sigma(m)}{E(m)} = \left\{\exp\left(\sigma_\varepsilon^2\hat{\gamma}^2 - 1\right)\right\}^{\frac{1}{2}} \qquad \hat{\gamma} \text{ here}$$

Reinterpretation of Tallarini.

Replace random walk model with "pessimistic" model

$$c_{t+1} - c_t = \mu_c + \sigma_\varepsilon\left(\varepsilon_{t+1} + w(\theta)\right) \qquad w(\theta) = \frac{-\sigma_\varepsilon}{}$$

$(1-\beta)\theta$

$$= \hat{\mu}_c + \sigma_\varepsilon \varepsilon_{t+1}$$

$$\hat{\mu}_c = \mu - \sigma_\varepsilon^2 (\hat{\gamma} - 1) \qquad \hat{\gamma}$$

Plug in the mean $\hat{\mu}_c$ in formula $(*)$, putting $\gamma = 1$
(low risk aversion)

$$E\,m = \beta \exp\left[-\mu + \sigma_\varepsilon^2 (\hat{\gamma} - 1) + \frac{\sigma_\varepsilon^2}{2} \right]$$

$$= \beta \exp\left[-\mu + \frac{\sigma_\varepsilon^2 (2\hat{\gamma} - 1)}{2} \right]$$

which is Tallarini's formula

give the two interpretations of Tallarini's.

(1) Separate intertemporal substn from risk-aversion

(2) Risk aversion vs. model uncertainty

SDF, 2

Sunday, April 25, 2010
7:40 PM

More implications of $E\, m_{t+1} R_{t+1} = 1$

Let $R_f^{-1} = E\, m$ be the risk-free rate's reciprocal.

An excess return on asset i is defined as

$$\xi^i_{t+1} \equiv R^i_{t+1} - R_f$$

Evidently, because $E_t\, m_{t+1} R_f = 1$ and $E_t\, m_{t+1} R^i_{t+1} = 1$ it follows that

$$(*) \qquad E_t\, m_{t+1} \xi^i_{t+1} = 0 \qquad \forall i$$

Defn of (conditional) covariance:

$$\mathrm{cov}_t(m_{t+1}, \xi^i_{t+1}) = E_t\, m_{t+1} \xi^i_{t+1} - E_t m_{t+1} E_t \xi^i_{t+1}$$

$$\Rightarrow \quad \underbrace{E_t\, m_{t+1} \xi^i_{t+1}}_{\substack{\parallel \\ 0 \text{ by } (*)}} = \mathrm{cov}_t(m_{t+1}, \xi^i_{t+1}) + E_t m_{t+1} E_t \xi^i_{t+1}$$

$$(**) \Rightarrow \quad E_t m_{t+1} E_t \xi^i_{t+1} = - \mathrm{cov}(m_{t+1}, \xi^i_{t+1})$$

Now remember fact about (population) least squares regression

$$\xi^i_{t+1} = \underbrace{\frac{\mathrm{cov}(m_{t+1}, \xi^i_{t+1})}{\mathrm{var}(m_{t+1})}}_{\sim} m_{t+1} + \epsilon_{t+1}$$

$$\epsilon_{t+1} \perp m_{t+1}$$

β regression coefficient

$\uparrow$

regress ξ^i_{t+1} against m_{t+1}

$\therefore$

$$var(\xi^i_{t+1}) = \frac{|cov(m_{t+1}, \xi^i_{t+1})|^2}{var(m_{t+1})^2} \, var\, m_{t+1} + \underset{>0}{var(\varepsilon_{t+1})}$$

$\Rightarrow$

$$var(\xi^i_{t+1}) \geq \frac{|cov(m_{t+1}, \xi^i_{t+1})|^2}{var(m_{t+1})}$$

$\Rightarrow$

$$(+) \qquad 1 \geq \frac{|cov(m_{t+1}, \xi^i_{t+1})|^2}{var(m_{t+1})\, var(\xi^i_{t+1})} \qquad \text{"} 1 \geq R^2 \text{"}$$

Now use (**) and (+)

$$|E_t\, m_{t+1}\, E_t\, \xi^i_{t+1}|^2 = |cov(m_{t+1}, \xi^i_{t+1})|^2 \qquad \text{(this is (**) squared)}$$

$$\leq var(m_{t+1}) \cdot var(\xi^i_{t+1})$$

$\Rightarrow$

$$E_t\, m_{t+1}\, E_t\, \xi^i_{t+1} \leq \sigma_t(m_{t+1}) \cdot \sigma_t(\xi^i_{t+1}) \qquad (\sigma_t \text{ is std}_t)$$

$$(++) \qquad \frac{E_t\, \xi^i_{t+1}}{\sigma_t(\xi^i_{t+1})} \leq \frac{\sigma_t(m_{t+1})}{E_t(m_{t+1})} = \text{"market price of risk"}$$

$\uparrow$ "Sharpe ratio" $\qquad \uparrow$ " maximum Sharp ratio

Remark The HJ bounds are bounds on the RHS of (++)

Remark: Write (**) as

$$E_t m_{t+1} E_t \xi^i_{t+1} = - \operatorname{cov}_t(m_{t+1}, \xi^i_{t+1})$$

or

$$E_t m_{t+1} E_t \xi^i_{t+1} = - \frac{\operatorname{cov}_t(m_{t+1}, \xi^i_{t+1})}{\operatorname{var}(m_{t+1})} \cdot \operatorname{var} m_{t+1}$$

$$E_t \xi^i_{t+1} = - \underbrace{\frac{\operatorname{cov}(m_{t+1}, \xi^i_{t+1})}{\operatorname{var} m_{t+1}}}_{\hat{\beta}} \cdot \underbrace{\operatorname{var} m_{t+1} \cdot R_{f\,t+1}}_{\text{"undiversifiable risk.}}$$

$\hat{\beta}$ = regression coefficient of excess return on SDF.

Recall consumption based model of m_{t+1} :

$$m_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}, \quad u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$$

$$c_t = \log C_t, \quad C_t = \exp(c_t), \quad \beta = \exp(-\rho)$$

$\Rightarrow$

$$c_{t+1} - c_t = \mu_c + \sigma_\varepsilon \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}(0,1)$$

$\Rightarrow$

$$m_{t+1} = \beta \exp(-\gamma(c_{t+1} - c_t))$$

$$= \beta \exp(-\gamma(\mu_c + \sigma_\varepsilon \varepsilon_{t+1}))$$

(*) $$\log m_{t+1} = \underbrace{-\rho}_{\text{discounting}} \underbrace{- \gamma\mu_c}_{\text{growth}} \underbrace{- \gamma\sigma_\varepsilon \varepsilon_{t+1}}_{\text{risk}}$$

$\gamma\sigma_\varepsilon$ is "risk exposure"

Remark: $E_t m_{t+1} = \exp(-\rho)\exp(-\gamma\mu_c + \frac{1}{2}\gamma^2\sigma_\varepsilon^2)$

$= \exp(-r_t)$ where r_t = one period risk free <u>rate</u> (net)

thus $r_t = \rho + \gamma\mu_c \underbrace{- \frac{1}{2}\gamma^2\sigma_\varepsilon^2}_{\text{correction for uncertainty}}$

Thus, we can write

$$\log m_{t+1} = \underbrace{-r_t}_{\text{"short rate"}} \underbrace{- \frac{1}{2}\gamma^2\sigma_\varepsilon^2}_{\text{"correction"}} \underbrace{- \gamma\sigma_\varepsilon \varepsilon_{t+1}}_{\text{risk exposure}}$$

Because of failure of this model with reasonable γ to attain HJ bounds, we want to

(1) generalize the model by allowing more sources of

macroeconomic risk ε_{t+1} to be priced

(2) divorce ε_{t+1} from consumption alone.

"Bond price model"

$$\overset{n\times 1}{z_{t+1}} = \overset{n\times 1}{\mu} + \overset{n\times n}{\Phi}\,\overset{n\times 1}{z_t} + \overset{n\times n}{C}\,\overset{n\times 1}{\varepsilon_{t+1}}, \quad \varepsilon_{t+1} \sim N(0, I)$$

$$r_t = \overset{1\times 1}{\delta_0} + \overset{1\times n}{\delta_z'}\, z_t \qquad \text{"short rate"}$$

$$\log m_{t+1} = -r_t - \frac{1}{2}\overset{1\times n}{\Lambda_t'}\,\overset{n\times 1}{\Lambda_t} - \Lambda_t'\,\varepsilon_{t+1}$$

(same ε_{t+1} as above)

$$\overset{n\times 1}{\Lambda_t} = \overset{n\times 1}{\Lambda_0} + \overset{n\times n}{\Lambda_z}\,\overset{n\times 1}{z_t}$$

$\Rightarrow$

$$\log m_{t+1} = -(\delta_0 + \delta_z' z_t) - \frac{1}{2}(\Lambda_0 + \Lambda_z z_t)'(\Lambda_0 + \Lambda_z z_t) - (\Lambda_0 + \Lambda_z z_t)'\varepsilon_{t+1}$$

$$m_{t+1} = \exp(\quad '' \quad)$$

Objects to estimate: $\overset{n\times 1}{\mu}, \overset{n\times n}{\Phi}, \overset{n\times n}{C}$ $\leftarrow$ (can estimate from VAR for z_t)

$$\underset{1\times 1}{\delta_0}, \underset{1\times n}{\delta_z'}, \underset{n\times 1}{\Lambda_0}, \underset{n\times n}{\Lambda_z}$$

estimate from moment conditions

$$E_t\left[m_{t+1} R^i_{t+1}\right] = 1$$

for a number of returns R^i_{t+1} and dates $i = 1, \dots, I$; $t = 1, \dots, T$

The system we'll estimate is

$$(*)\quad E_t\left[\left(\exp\left[-(\delta_0+\delta_z' z_t)-\tfrac{1}{2}(\Lambda_0+\Lambda_z z_t)'(\Lambda_0+\Lambda_z z_t) - (\Lambda_0+\Lambda_z z_t)'\underbrace{\left(C^{-1}(z_{t+1}-\Phi z_t)\right)}_{\uparrow\ \varepsilon_{t+1}}\right]\right) R^i_{t+1}\right]=1$$

$$(**)\quad z_{t+1} = \mu + \Phi z_t + C\,\varepsilon_{t+1}\ ,\quad \varepsilon_{t+1}\perp z_t$$

Remark: This is an "affine model of the yield curve".

Let $y_t(\tau)$ be the yield on a zero coupon bond of maturity τ at date t, $\tau = 1, 2, \ldots$

Then

$$y_t(\tau) = a(\tau) + b(\tau) z_t$$

where $a(1) = \delta_0\ ,\ b(1) = \delta_z$

and $a(\tau), b(\tau)$ can be computed recursively as functions of the δ's & Λ's.

This is called the "affine" term structure model.

Remark: Objects in HJ bounds:

$$E_t\, m_{t+1} = \exp\left(-(\delta_0 + \delta_z' z_t)\right)$$

$$\text{std}_t\, m_{t+1} = \exp\left(-(\delta_0+\delta_z' z_t)\right)\cdot\sqrt{e^{(\Lambda_0+\Lambda_z z_t)'(\Lambda_0+\Lambda_z z_t)}-1}$$

Portfolio choice

Monday, April 26, 2010
9:12 AM

Basic portfolio choice model based on "fundamentals"

$$\max_{c_0, c_1(\omega), b} \quad u(c_0) + \beta \int u(c_1(\omega)) f(\omega)\, d\omega$$

$$\text{s.t.} \quad c_0 + \overset{n\times 1}{p} \cdot \overset{n\times 1}{b} \leq y_0$$

$$c_1(\omega) \leq b \cdot x(\omega) + y_1(\omega)$$

Lagrangian:

$$u(c_0) + \beta \int u(c_1(\omega)) f(\omega)\, d\omega$$

$$+ \theta \left[y_0 - c_0 - p \cdot b \right]$$

$$+ \int \lambda(\omega) \left[b \cdot x(\omega) - c_1(\omega) \right] d\omega$$

FONC: $(*)$ $\quad \overset{n\times 1}{p} = \int \beta \dfrac{u'(c_1(\omega))}{u(c_0)} x(\omega) f(\omega)\, d\omega$

$$c_1(\omega) = b \cdot x(\omega) + y_1(\omega)$$

substitute for $c_1(\omega)$ into $(*)$ to get

$$(+) \qquad \overset{n\times 1}{p} = \beta \int \frac{u'(b \cdot x(\omega) + y_1(\omega))}{u'(y_0 - p \cdot b)} x(\omega) f(\omega)\, d\omega$$

$(+)$ are n equations in n unknowns. $(\overset{n\times 1}{b})$

More superficial "Wall Street" formulation

vector of random returns $\underset{n\times 1}{R} \sim N(\underset{n\times 1}{\mu}, \underset{n\times n}{\Sigma})$

portfolio b

end of period wealth: $\underset{1\times 1}{W} = \underset{n\times 1}{R} \cdot \underset{n\times 1}{b}$ $\quad \cdot$ inner product

mean variance preferences: rank portfolios by

$E\, W - \lambda \operatorname{var} W$, $\lambda > 0$ measures "risk tolerance"

$= E\, R\cdot b - \lambda \operatorname{var}(R\cdot b)$

$= E\, R\cdot b - \lambda [b' (\operatorname{cov}(R,R))\, b]$

$= \mu \cdot b - \lambda [b' \Sigma b]$

Problem: $\max\limits_{b} \ \mu' b - \lambda [b' \Sigma b]$

FONC: $\mu - \lambda \Sigma b = 0$

$\Rightarrow \boxed{\hat{b} = \lambda^{-1} \Sigma^{-1} \mu}$ optimal portfolio

$\operatorname{var}(R\cdot b) = \lambda^{-2} \mu' \Sigma^{-1} \mu = \eta$

$\lambda^{-2} = \frac{\eta}{\quad}$

$$\mu' \Sigma^{-1} \mu$$

$$\lambda^{-1} = \frac{\sqrt{\eta}}{\sqrt{\mu' \Sigma^{-1} \mu}}$$

mean $\quad E\, R \cdot b = \sqrt{\frac{\eta}{\mu \Sigma^{-1} \mu}} \cdot \mu \Sigma^{-1} \mu = \sqrt{\eta} \cdot \sqrt{\mu \Sigma^{-1} \mu}$

Asset pricing, 2

Wednesday, April 07, 2010
10:32 AM

$$1 = E\, m_{t+1} R_{t+1}, \qquad \text{cov}(R_{t+1}, m_{t+1}) = E\, m_{t+1} R_{t+1} - E\, m_{t+1} E R_{t+1}$$

$$\Rightarrow \quad E\, m_{t+1} R_{t+1} = E\, m_{t+1} E R_{t+1} + \text{cov}(R_{t+1}, m_{t+1})$$

$$\Downarrow$$

$$1 = E\, m_{t+1} E R_{t+1} + \text{cov}(R_{t+1}, m_{t+1})$$

$$E\, m_{t+1} = R_{f\,t+1}^{-1} \quad \text{risk free rate}$$

$$(E\, m_{t+1})^{-1} = E R_{t+1} + \text{cov}(R_{t+1}, m_{t+1})(E\, m_{t+1})^{-1}$$

or

$$(*) \quad E R_{t+1} - R_{f\,t+1} = -\text{cov}(R_{t+1}, m_{t+1}) R_{f\,t+1}$$

(*) applies to any asset with return R^i_{t+1}, and also conditional on any information at t $\Rightarrow$ replace E and cov with E_t, cov_t.

$$(*) \quad E_t R^i_{t+1} - R_{f\,t+1} = -\text{cov}_t(R^i_{t+1}, m_{t+1}) \cdot R_{f\,t+1}$$

$$= \underbrace{-\frac{\text{cov}_t(R^i_{t+1}, m_{t+1})}{\text{var}_t\, m_{t+1}}}_{\text{"}\tilde{\beta}_i\text{"}} \cdot \underbrace{(\text{var}\, m_{t+1}) R_{f,t+1}}_{\text{"price of risk"}}$$

risk

note: $\tilde{\beta}_i = \dfrac{-\text{cov}_t(R^i_{t+1}, m_{t+1})}{\text{var}_t\, m_{t+1}}$

is regression coefficient of R^i_{t+1} on m_{t+1}

$\tilde{\beta}$ is regression coefficient (not to be confused with our discount factor β below)

examples:

example:

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$$

[β here, not $\tilde{\beta}$ above]

$\Rightarrow$

$$\ln m_{t+1} = \ln \beta - \gamma(\ln c_{t+1} - \ln c_t)$$

$$= \ln \beta - \gamma(\mu + \sigma \varepsilon_{t+1})$$

$(\ln c_{t+1} - \ln c_t = \mu + \sigma \varepsilon_{t+1})$

$\varepsilon_{t+1} \sim N(0,1)$

"Consumption is a geometric random walk with drift μ"

(*) $m_{t+1} = \exp[\ln \beta - \gamma\mu - \gamma\sigma\varepsilon_{t+1}]$

apply log normal formulas to compute

$\mathrm{var}_t\, m_{t+1} =$

$E_t\, m_{t+1} =$

$$\ln m_{t+1} = \ln \beta - \gamma(\mu + \sigma \varepsilon_{t+1})$$

(+) $$\sim N(\underbrace{\ln \beta - \gamma\mu}_{\mu_m}, \underbrace{\gamma^2\sigma^2}_{\sigma_m^2})$$

log normal formulas:

$\log m \sim N(\mu_m, \sigma_m^2) \Rightarrow$

$$E\, m = \exp\left(\mu_m + \tfrac{1}{2}\sigma_m^2\right)$$

$$\mathrm{var}(m) = \left(e^{\sigma_m^2} - 1\right) e^{2\mu_m + \sigma_m^2}$$

$$\mathrm{std}(m) = \left(e^{\mu_m + \frac{\sigma_m^2}{2}}\right)\sqrt{e^{\sigma_m^2} - 1}$$

$\beta = e^{-\rho}$

$\ln \beta = -\rho$

so (+) $\Rightarrow$

$\mu_m = -\rho - \gamma\mu$

$\sigma_m = \gamma^2\sigma^2$

$$E(m) = \exp\left(\mu_m + \tfrac{1}{2}\sigma_m^2\right)$$

$$= \exp\left(-\rho - \gamma\mu + \tfrac{1}{2}\gamma^2\sigma^2\right)$$

$$\sigma_m \equiv \mathrm{std}\, m = \left[e^{(-\rho-\gamma\mu) + \frac{1}{2}\gamma^2\sigma^2}\right]\left(\sqrt{e^{\gamma^2\sigma^2} - 1}\right)$$

$$\frac{\sigma_m}{E\, m} \quad \text{is Hansen's market price of risk.}$$

Stop reading here

Use these formulas to "fill in" the

price of risk: $- \dfrac{cov(R^i_{t+1}, m_{t+1})}{var_t\, m_{t+1}}$

amount of risk: $var_t\, m_{t+1} R_{f\,t+1}$

Link to continuous time formulation:

$m_t = e^{-\rho t} c_t^{-\gamma}$ Lucas/Merton model

$$\frac{dm}{m} = -r\,dt - \eta\, dB_t \quad ,$$

η = price of risk

$$\eta = \frac{\mu - r}{\sigma}$$

eg $\eta = \gamma \sigma$ and $\mu - r = \gamma \sigma^2$

$\dfrac{\sigma(m)}{E\, m} = \eta$ = price of risk in the Lucas model –

seemingly different notion of price of risk in the

$$E_t R^i_{t+1} - R_{f\,t+1} = \beta \cdot \underbrace{(\lambda_0 + \lambda_1 X_t)}_{\frac{var\, m_{t+1}}{E_t\, m_{t+1}}}$$

price of risk

" risk

" risk

risk is that the s.d.f. is moving around.

Asset pricing, 3

Monday, March 01, 2010
9:15 AM

Extract implications of

$$1 = \beta R \frac{u'(c_{t+1})}{u'(c_t)}$$

$$R = (1+r), \quad \beta = \frac{1}{1+\rho}$$

in non stochastic case

[later we'll consider implications of

$$1 = \beta R E_t \frac{u'(c_{t+1})}{u'(c_t)}$$

when R is risk-free but c_{t+1} is stochastic.]

Form of utility: power or constant relative risk aversion

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad \gamma \neq 1$$

$$= \log_e c \quad \text{if } \gamma = 1$$

$$u'(c) = c^{-\gamma}. \quad \Rightarrow$$

$$1 = \beta R \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$$

or

$$R = \beta^{-1} \left(\frac{c_{t+1}}{c_t}\right)^{\gamma}$$

Suppose

$$c_{t+1} = \delta c_t$$

where

$$\delta = 1 + g$$, g is net growth rate of consumption

$$\log \delta \approx g$$

$$\log \beta^{-1} \approx \rho$$

$$\log R \approx r$$

$$R = \beta^{-1} \delta^{\gamma}$$

$$\log R = -\log \beta + \gamma \log \delta$$

or

$$r \approx \rho + \gamma g$$

unsmoothness imposed by growth

- higher g raises r
- higher γ raises r

distaste for unsmoothness

need higher r to reconcile the consumer to the growth.

Stochastic counterpart with log normal consumption growth

$\varepsilon \sim N(0,1)$

Fact: $\log x_{\ldots} \sim N(\mu, \sigma^2) \Rightarrow x_{\ldots} = \exp(\mu + \sigma \varepsilon)$

Fact: $\log x_{t+1} \sim N(\mu, \sigma^2) \Rightarrow x_{t+1} = \exp(\mu + \sigma \varepsilon)$

$\log E x_{t+1} = \mu + \frac{\sigma^2}{2}$ $\qquad E x_{t+1} = \exp^{\left(\mu + \frac{\sigma^2}{2}\right)}$

e.g.,

$$\log x = \sigma_x \varepsilon_{t+1} - \frac{\sigma_x^2}{2}$$

or

$$x = \exp\left(\sigma_x \varepsilon_{t+1} - \frac{\sigma_x^2}{2}\right)$$

where $\varepsilon_{t+1} \sim N(0,1)$

$$E \log x = -\frac{\sigma_x^2}{2}$$

$$\operatorname{var} \log x = \sigma_x^2 \qquad (\text{coeff on } \varepsilon_{t+1}^2)$$

$$\Rightarrow \log E x = 0.$$

application

$$\frac{c_{t+1}}{c_t} = \delta \exp\left\{\sigma_c \varepsilon_{t+1} - \frac{\sigma_c^2}{2}\right\}, \quad \varepsilon_{t+1} \sim N(0,1)$$

$$\log \frac{c_{t+1}}{c_t} = \log \delta + \sigma_c \varepsilon_{t+1} - \frac{\sigma_c^2}{2}$$

$$\sim N\left(\log \delta - \frac{\sigma_c^2}{2}, \sigma_c^2\right)$$

$\Rightarrow$

$$E\left(\frac{c_{t+1}}{c_t}\right) = \exp\left(\log \delta - \frac{\sigma_c^2}{2} + \frac{\sigma_c^2}{2}\right)$$

$$= \exp \log \delta = \delta \quad \checkmark$$

Now evaluate $\qquad \delta$

$$\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} = \delta^{-\gamma} \exp\left(-\gamma\left(\sigma_c \varepsilon_{t+1} - \frac{\sigma_c^3}{2}\right)\right)$$

$$= \exp\left[\log \delta^{-\gamma} - \gamma\left(\sigma_c \varepsilon_{t+1} - \frac{\sigma_c^3}{2}\right)\right]$$

$$= \exp\left[\log \delta^{-\gamma} + \gamma \frac{\sigma_c^3}{2} - \gamma \sigma_c \varepsilon_{t+1}\right]$$

$\Rightarrow$

$$\log\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \sim N\Big(\underset{\uparrow \atop \text{mean}}{\log \delta^{-\gamma} + \gamma \frac{\sigma_c^3}{2}},\ \underset{\uparrow \atop \text{variance}}{\gamma^2 \sigma_c^2}\Big)$$

$\Rightarrow$

$$E\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} = \exp\left(\log \delta^{-\gamma} + \gamma \frac{\sigma_c^3}{2} + \gamma^2 \frac{\sigma_c^3}{2}\right)$$

$$= \exp\left(\log \delta^{-\gamma} + \frac{\sigma_c^2}{2} \gamma (1+\gamma)\right)$$

$$= \delta^{-\gamma} \exp \frac{(1+\gamma)\gamma \sigma_c^2}{2}$$

Apply this to

$$1 = \beta R\ E_t \frac{u'(c_{t+1})}{u'(c_t)}$$

$$\log(1+r) = -\log \beta + \gamma \log \delta - (1+\gamma)\gamma \frac{\sigma_c^2}{2}$$

or

$$r \approx \rho + \gamma \underset{\uparrow \atop \text{growth}}{g} - \underbrace{(1+\gamma)\gamma \frac{\sigma_c^3}{2}}, \quad \delta = 1 + g$$

↑ discounting ↑ growth precautionary savings

Asset pricing 4

Wednesday, April 07, 2010
3:27 PM

Expected utility two states

Background model is

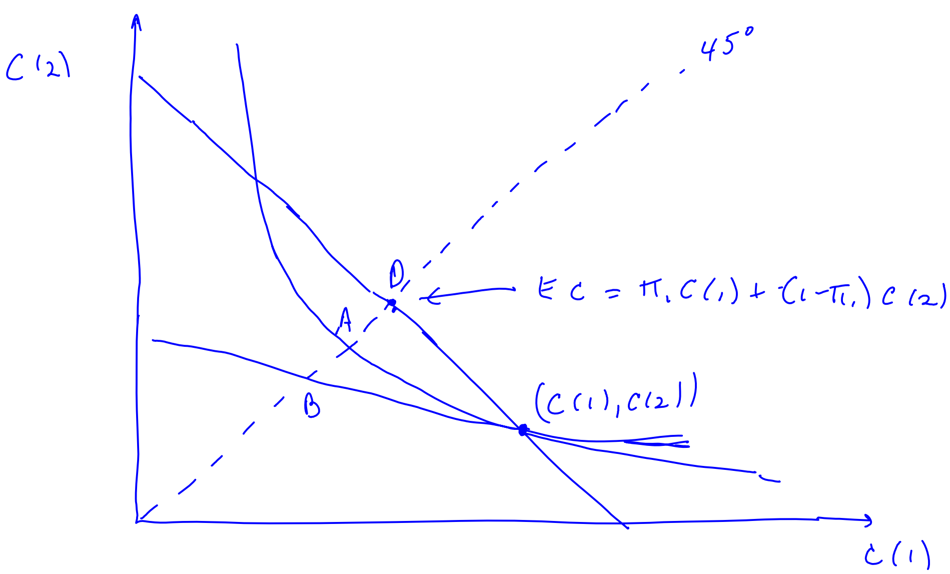
$$C_0 \quad + \quad \pi_1 u(C_1(1)) + (1-\pi_1) u(C_1(2))$$

$\uparrow$ date $\qquad C_1(i)$

$\uparrow$ date $\leftarrow$ state $i = 1, 2$

Simplify & draw indifference curves for utility time

$$U = \pi_1 u(c(1)) + (1-\pi_1) u(c(2))$$



D: $(\bar{c}, \bar{c})$ where $\bar{c} = \pi_1 c(1) + (1-\pi_1) c(2)$

A: $(\hat{c}, \hat{c})$ where $u(\hat{c}) = \pi_1 u(c(1)) + (1-\pi_1) u(c(2))$

B

$$U = \pi_1 u(c(1)) + \pi_2 u(c(2))$$

$$dU = \pi_1 u'(c(1))\,dc(1) + (1-\pi_1) u'(c(2))\,dc(2)$$

$= 0 \implies$

$$\frac{dc(2)}{dc(1)} = \frac{-\pi_1}{(1-\pi_1)} \frac{u'(c(1))}{u'(c(2))} =$$

Budget constraint:

$$I_1 = P_1 C(1) + P_2 C(2) \quad (+ c_0)$$

$$P_1 \sim \frac{\text{time 0 good}}{\text{unit of } C \text{ in state 1 at time 1}}, \quad P_2 = \frac{\text{time 0 good}}{}$$

$$dI = P_1\,dC(1) + P_2\,dC(2)$$

$$\frac{dc(2)}{dc(1)} = -\frac{P_1}{P_2}$$

$\therefore$ we'll equate $\frac{P_1}{P_2} = + \frac{\pi_1 u'(c(1))}{(1-\pi_1) u'(c(2))}$

eg $\frac{P_1}{P_2} = \frac{\pi_1}{1-\pi_1} \cdot \frac{\frac{1}{c(1)}}{\frac{1}{c(2)}} = \frac{c(2)}{c(1)}$

with $\pi_1 = .5$, $c(2) < c(1)$, $P_1/P_2 < 1$

Point B:

$$(p_1 + p_2)\check{c} = p_1 c(1) + p_2 c(2)$$

$$\left(\frac{p_1}{p_2} + 1\right)\check{c} = \frac{p_1}{p_2} c(1) + c(2)$$

or

$$\check{c} = \frac{p_1}{p_1+p_2} c(1) + \frac{p_2}{p_1+p_2} c(2)$$

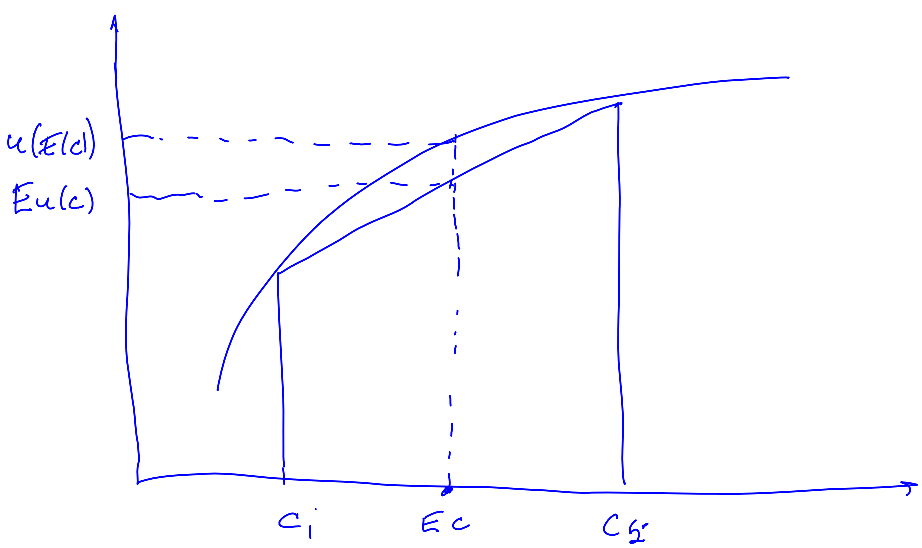
risk neutral probabilities

$$\check{c} = \frac{\pi_1 u'(c_1)}{\pi_1 u'(c(1)) + (1-\pi_1)u'(c(2))} c(1) + \frac{(1-\pi_1)u'(c_2)}{\pi_1 u'(c(1)) + (1-\pi_1)u'(c(2))} c(2)$$

B: $\check{c}$ = expected consumption at risk neutral probabilities.

$(p_1 + p_2)$ = price of a sure claim on cons^n at $t=1$

$$\frac{1}{p_1+p_2} = \text{gross risk free interest rate.}$$



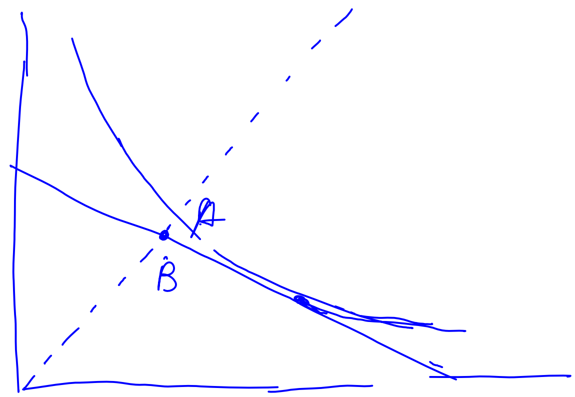
concavity of u $\Rightarrow$ $E\,u(c) < u(E\,c)$

Asset pricing:

price the endowment.

B is asset price

is certainty equivalent



let α be asset price:

$$\alpha = \frac{p_1}{p_1+p_2} c(1) + \frac{p_2}{p_1+p_2} c(2)$$

risk neutral probabilities

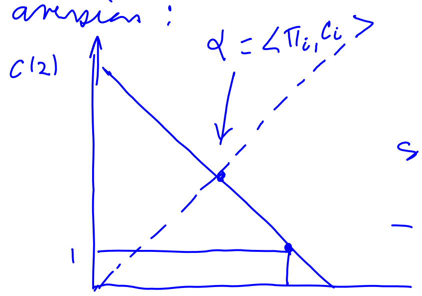
$$\alpha = \frac{\pi_1 u'(c_1)}{\pi_1 u'(c(1)) + (1-\pi_1)u'(c(2))} c(1) + \frac{(1-\pi_1)u'(c_2)}{\pi_1 u'(c(1)) + (1-\pi_1)u'(c(2))} c(2)$$

$$\alpha = \langle m(i), c(i) \rangle$$

$m(i)$ is a risk-neutral probability

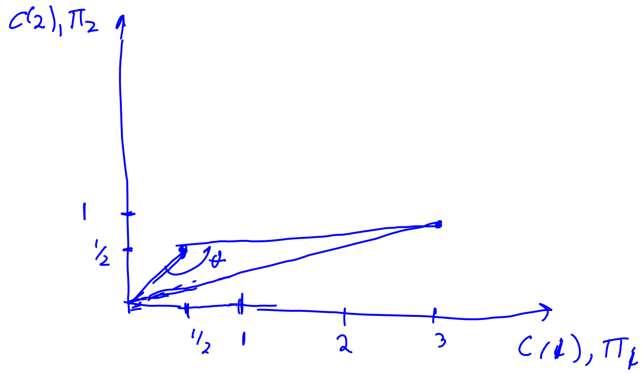
risk-neutral pricing - no risk aversion:

$$\alpha = \langle \pi_i, c_i \rangle$$

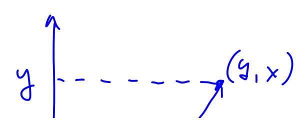


slope $= -\frac{\pi_1}{\pi_2} = -1$

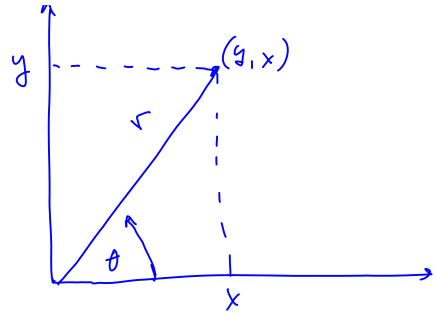
$\alpha = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 3 = 2$



$\cos\theta = \langle \pi_i, c_i \rangle$



$\cos\theta = \langle \pi_i, c_i \rangle$



$\cos\theta = \frac{x}{r}$, $r = \sqrt{x^2 + y^2}$

Re do

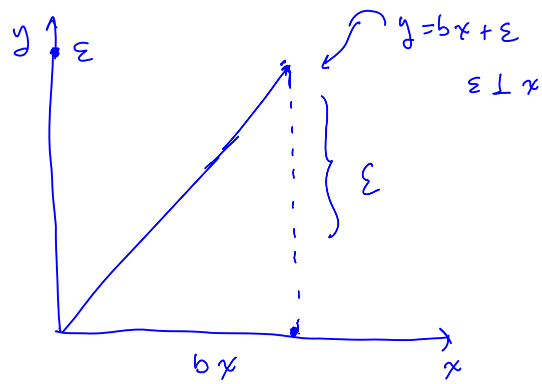
To specify graphically regression coefficient, correlation coefficient, covariance

$b = \frac{\langle x, y \rangle}{\langle x, x \rangle}$

$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}$

$\rho \equiv \frac{\langle x, y \rangle}{\langle x, x \rangle \langle y, y \rangle}$

Get Luenberger



$y = b\, x + \varepsilon$

$\langle y, x \rangle = b \langle x, x \rangle + \underbrace{\langle \varepsilon, x \rangle}_{= 0}$

$b = \frac{\langle y, x \rangle}{\langle x, x \rangle}$

↑ projection on a subspace

Start here:

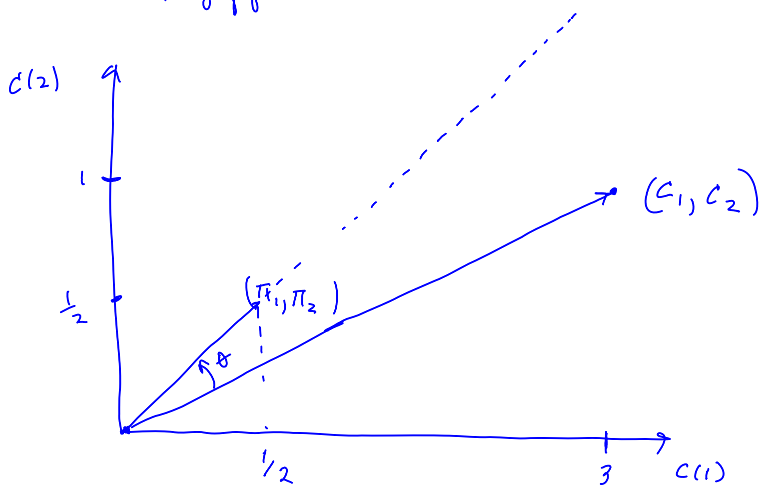
norm : $|\langle x, x \rangle|^{1/2}$ normed linear space

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\langle x, x \rangle = \left(\sum x_i^2\right)$

vector 1: $c = (c_1, c_2)$

$\pi = (\pi_1, \pi_2)$

expected value of payoff: $E\, c = \sum \pi_i c_i = \langle \pi_i, c_i \rangle$



$\langle c, \pi \rangle = \sum c_i \pi_i$

geometric interpretation of Cauchy-Schwartz inequality: $|\cos^2\theta| \le 1$

$$\cos\theta = \frac{\langle \pi, c \rangle}{\langle \pi, \pi \rangle^{1/2} \langle c, c \rangle^{1/2}}$$

check out ↗

market price of risk:

$$\frac{\langle m, m \rangle^{1/2}}{\langle \pi, m \rangle}$$

be careful of defn of inner product

$\langle c, m \rangle = \sum \pi_i c_i m_i$

$$\sum_{i=1}^{I} u(c_i)\, \pi_i$$

$$h^{-1} \sum_{i=1}^{I} \pi_i\, h(u(c_i))$$

extra adjustment for eg. risk

eg. $-\theta \log \sum_{i=1}^{I} \pi_i \exp\left(-\frac{u(c_i)}{\theta}\right) \qquad \theta > 0$

I

e.g. $-\theta \log \sum_{i=1}^{I} \pi_i \exp\left(-\frac{u(c_i)}{\theta}\right) \qquad \theta > 0$

$$U = h^{-1} \sum_{i=1}^{I} \pi_i h(u(c_i))$$

$$h(U) = \sum_{i=1}^{I} \pi_i h(u(c_i))$$

$$h'(U)\, dU = \sum_i \pi_i h'(u(c_i)) u'(c_i)\, dc_i$$

A preference specification.

$$U = \ln c - B h$$

$$wh \le c$$

h - hours
c - consumption.

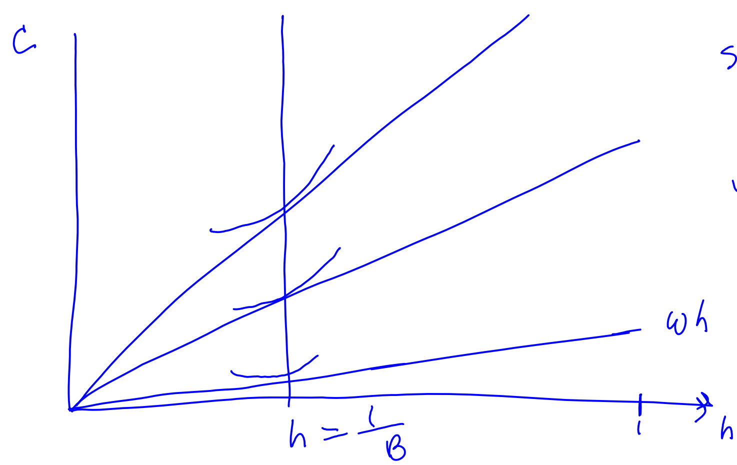
$$\max_{c,h} \ \ln c - Bh + \lambda [wh - c]$$

$$c: \ \frac{1}{c} - \lambda = 0$$

$$h: \ -B + \lambda w = 0 \quad \Rightarrow \quad h = \frac{1}{B}$$

$$c = \frac{w}{B}$$

Remark: income & substitution effects cancel:



Substitution effect:
work more;
income effect: work
less

Experiment 1: govt levies a tax τ_h on labor income

& uses it to finance G - govt expenditures.

HH problem becomes:

$$\max_{c,h} \left\{ \ln c - Bh + \lambda [w(1-\tau_h)h - c] \right\}$$

FONC:

$$c: \frac{1}{c} - \lambda = 0$$

$$h: -B + \lambda w(1-\tau_h) = 0$$

$$\Rightarrow \quad B = \frac{1}{c} w(1-\tau_h)$$

But $c = w(1-\tau_h)h$

$$\Rightarrow \quad B = \frac{1}{h} \quad \text{or} \quad h = \frac{1}{B}$$

Here the effect of the tax & spend policy is to leave h unaffected. government purchases crowd out private consumption — c goes down by the increase in G.

Experiment 2: government imposes labor tax rate τ_h but rebates the proceeds lump sum to the household.

HH optimum problem $\Rightarrow$

$$B = \frac{1}{c} w(1-\tau_h)$$

just as before. But now c is given by

$$c = w(1-\tau_h)h + T$$

where T is the rebate.

The rebate is treated as exogenous (beyond influence) by the household.

But $T = w\tau_h h$ after all

$$\therefore \quad c = w(1-\tau_h)h + w\tau_h h$$

$$= wh$$

$$\therefore \quad B = \frac{1}{c} w(1-\tau_h) = \frac{1}{wh} w(1-\tau_h)$$

$$\Rightarrow \quad h = \frac{(1-\tau_h)}{B}$$

So now, taxes discourage work.

Here we have isolated the substitution effect and neutralized the income effect.

Remark: Describe Prescott's analysis.

Lect 4

Sunday, February 08, 2009
8:18 PM

Models of indivisible labor

1. employment lotteries model.

Hansen - Rogerson & Prescott

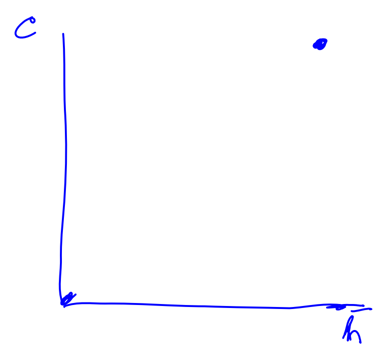
preferences

$u(c) - v(h)$

s.t. $c \le wh$ but now

$h \in \{0, 1\}$

don't work, work.



Two points are in the choice set: $(c, h) = (w, 1)$ or $(0, 0)$

choose to work if

$u(w) - v(1) > u(0) - v(0)$

reservation wage, if it exists, satisfies

$u(\bar{w}) - v(1) = u(0) - v(0)$

work if $w \ge \bar{w}$.

Example:

$\ln c - Bh$, $B > 0$ $(B = v(1))$

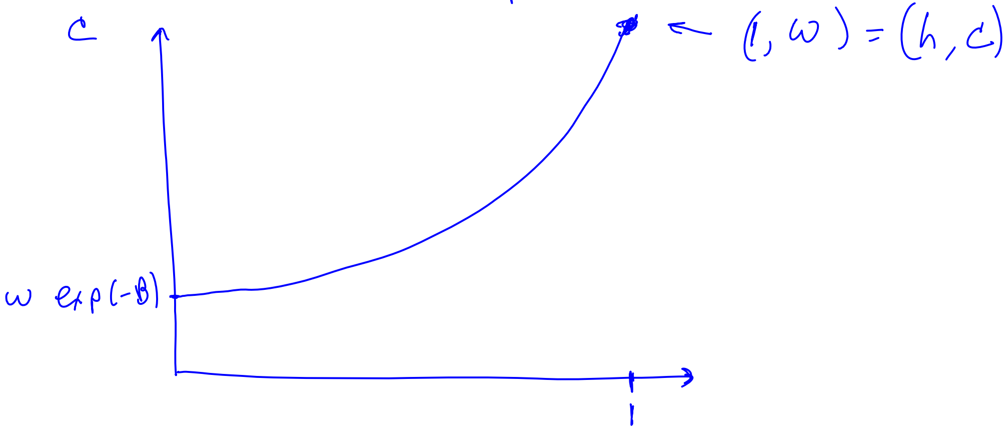
Solve for indifference curve through $(c, h) = (w, 1)$

$$\bar{U} = \ln w - B$$

Solve for $\tilde{c}$ that sets

$$\ln \tilde{c} - B \overset{=0}{\tilde{h}} = \ln \tilde{c} = \ln w - B$$

$$\Rightarrow \tilde{c} = w \exp(-B) \quad > 0$$



Note: By himself, our worker would always work for these preferences, $\ln c - B h$.

Employment lotteries:

There is now a continuum of ex ante identical workers indexed by $j \in [0, 1]$, each of whom has identical preferences given by

$$u(c^j) - v(h^j), \quad h^j \in \{0, 1\}$$

don't

don't work $\nearrow$ $\uparrow$ work

A benevolent planner choose $\{c^j\},\{h^j\}$ $\forall j \in [0,1]$

to maximize

$$\int_0^1 [u(c^j) - v(h^j)]\,dj = \int_0^1 u(c^j)\,dj - v(1)\int_0^1 h^j\,dj$$

$$\text{s.t.} \quad \int_0^1 c^j\,dj \le \omega \int_0^1 h^j\,dj$$

let $\Phi = \int_0^1 h^j\,dj$ = fraction of j's working

Key step

Write the planning problem:

$$L = \int_0^1 u(c^j)\,dj - v(1)\,\Phi$$

$$+ \lambda \left[\omega \phi - \int_0^1 c^j\,dj\right]$$

FONC:

$$c_j: \quad u'(c^j) - \lambda = 0$$

$$\phi: \quad -v(1) + \lambda\omega = 0$$

$\Rightarrow$ $u'(c^j) = \lambda$ independent of j $\Rightarrow$ $c^j = c \ \forall j$

$$\omega\varphi = c = \int_0^1 c^j\,dj = c\int_0^1 dj$$

$v(1) = \lambda w$

$v(1) = u'(c) w$

$B = u'(w\varphi) w$

or $u'(w\varphi) = \frac{B}{w}$

in log case: $\frac{1}{\varphi w} = \frac{B}{w} \implies \varphi = \frac{1}{B}$

now φ is fraction of people working.

This is how some people "calibrate" B.

ex post utility of people who work:

$u(\bar{c}) - v(1)$ $\qquad \bar{c} = c^j \ \forall j$.

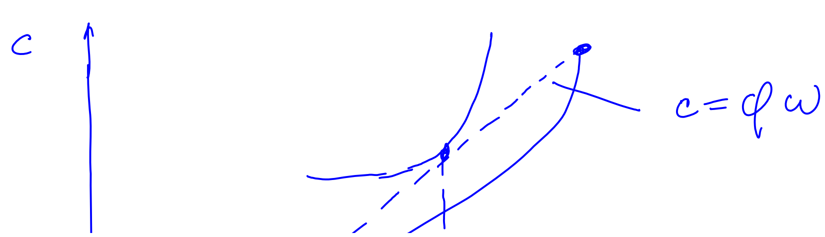
ex post utility of people who don't work

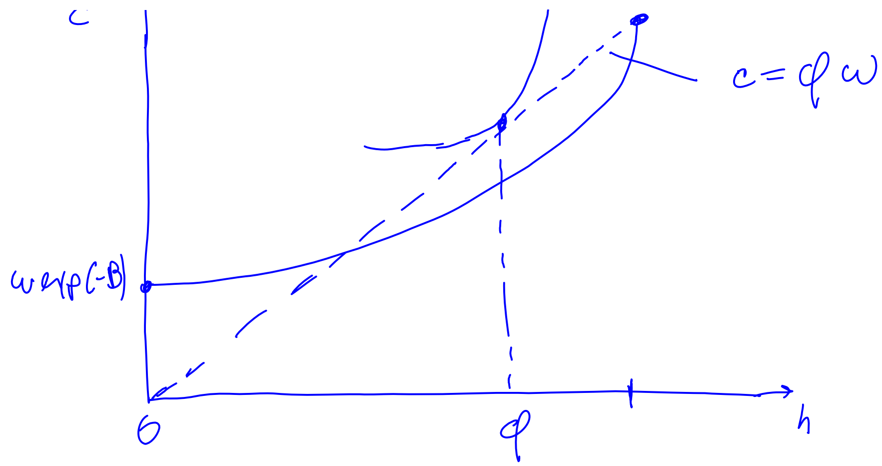
$u(\bar{c})$

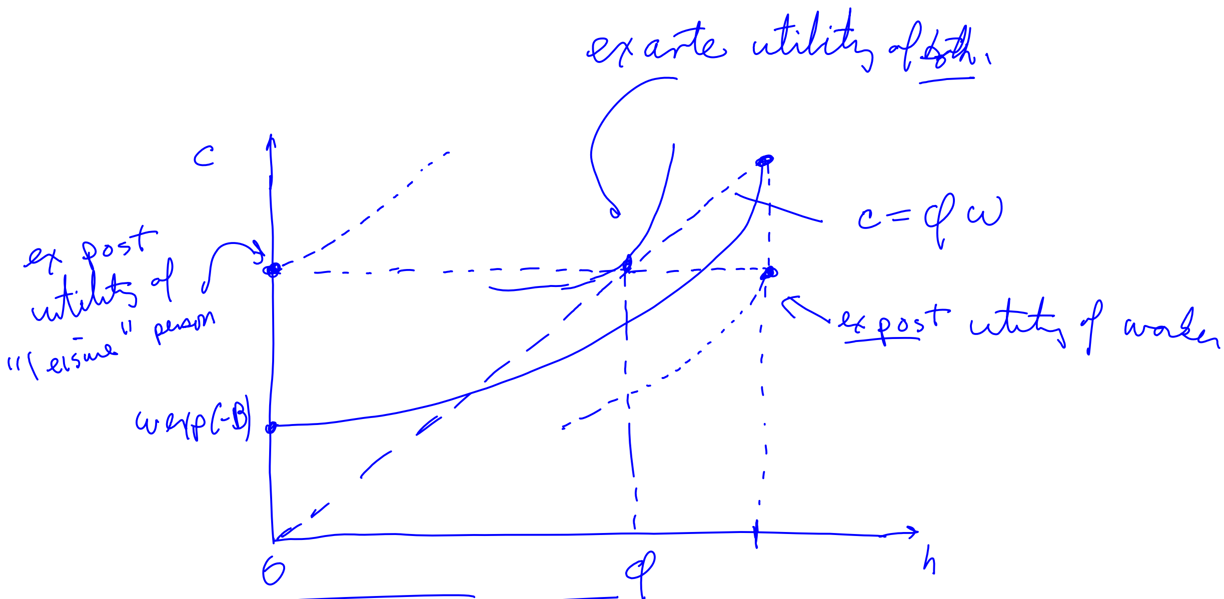
ex ante utility is

$$(1-\varphi) u(\bar{c}) + \varphi (u(\bar{c}) - B) = u(\bar{c}) - \varphi B$$

Planner runs a lottery (φ) to assign people to work, gives everyone c.







Other way to calibrate B: fraction of life time worked

No lotteries Instead - time averaging.

Consumer on own faces indivisibility.
intertemporal problem

$$\max_{\{c_t, h_t\}_0^1} \int_0^1 e^{-\delta t}\left[\ln c_t - B h_t\right] dt, \quad h_t \in \{0,1\}$$

s.t. $\int_0^1 e^{-rt}[w h_t - c_t]\,dt$

Focus on case where $\delta = r$

Let $\int_0^1 e^{-rt} h_t\,dt = H_t$ = present value of time worked

$$L = \int_0^1 e^{-rt} \ln c_t\,dt - B\,H_t$$
$$+ \lambda\left[w H_t - \int_0^1 e^{-rt} c_t\,dt \right]$$

FONC:

c_t: $e^{-rt}\frac{1}{c_t} - \lambda e^{-rt} = 0$

$\Rightarrow \frac{1}{c_t} = \lambda \quad \forall t$

$\Rightarrow$ constant consumption through time

H_t: $-B + \lambda w = 0$

$w = \frac{B}{\lambda} = Bc$ or $\bar{c} = \frac{w}{B}$

$$w \cdot H = \bar{c}\int_0^1 e^{-rt}\,dt$$
$$= \bar{c}\left[-\frac{1}{r}e^{-rt}\right]_0^1$$
$$= \bar{c}\left[\frac{1-e^{-r}}{r}\right]$$

$\Rightarrow$ $w H = \frac{w}{r}\left[\frac{1-e^{-r}}{\ }\right]$

$$w H = \frac{w}{B}\left[\frac{1-e}{r}\right]$$

wage · PV of work

PV of cons^n

$$H = \frac{1}{B}\left[\frac{1-e^{-r}}{r}\right]$$

$$\lim_{r \to 0} \frac{1-e^{-r}}{r} = \lim_{r \to 0} \frac{e^{-r}}{1} = 1$$

agrees with earlier formula.

how calibrate B? — fraction of life time worked.

Labor SY

Saturday, February 14, 2009
4:37 PM

Consider a preference specification

$$u(c, 1-h) = \frac{c^{1-m}}{1-m} + \frac{(1-h)^{1-m}}{1-m}$$

c = Consumption

h = time worked

Remark $m \to 1 \Rightarrow \ln c + \ln(1-h)$

$$u_c = c^{-m}$$
$$u_{1-h} = (1-h)^{-m}$$

$c = wh$ (constraint)

Problem: $\max_{c,h} \left\{ \frac{c^{1-m}}{1-m} + \frac{(1-h)^{1-m}}{1-m} \right.$

$$+ \lambda \{ wh - c \}$$

FONC: c: $c^{-m} - \lambda = 0$

assume interior soln

h: $-(1-h)^{-m} + \lambda w = 0$

$$\Rightarrow \quad -(1-h)^{-m} + c^{-m} w = 0$$

$$\Rightarrow \quad w = \frac{(1-h)^{-m}}{c^{-m}} \quad \text{or}$$

$\ell = 1-h$

$\boxed{u_\ell = (1-h)^{-m}} = u_{1-h}$

$$w = \frac{u_\ell}{u_c} = \frac{(1-h)^{-m}}{c^{-m}}$$

marginal condition

to solve:

$$c = wh \quad \text{and} \quad w = \frac{(1-h)^{-m}}{c^{-m}}$$

for c & h..

$$(1-h)^{-m} = w c^{-m}$$

raise both sides to $-\frac{1}{m}$ $\Rightarrow$

$$(1-h) = w^{-1/m} c$$

$\Rightarrow$ [illegible]

$$\Rightarrow \quad h = 1 - w^{-\frac{1}{m}} c \qquad (+)$$

but $c = wh \Rightarrow$

$$c = w\left[1 - w^{-1/m} c\right]$$

$$\Rightarrow \quad \left[1 + w^{1-\frac{1}{m}}\right] c = w$$

$$(*) \quad c = \frac{w}{1 + w^{1-\frac{1}{m}}}$$

$$(+) \Rightarrow \quad h = 1 - w^{-1/m} \left(\frac{w}{1 + w^{1-\frac{1}{m}}} \right)$$

$$(**) \quad h = 1 - \frac{w^{1-\frac{1}{m}}}{1 + w^{1-\frac{1}{m}}}$$

Note: $(*)$ is upward sloping if $0 < m < 1$.
downward sloping if $m > 1$
constant if $m = 1$

To plot this & differentiate: use cobbdouglaspref.m in matlab programs

---

Alternative "business cycle" type of specification

$$u(c, 1-n) = \frac{c^{1-\gamma}}{1-\gamma} - B\, n$$

$$\max \frac{c^{1-\gamma}}{1-\gamma} - B\, n \quad , \text{ s.t. } \quad n \geq 0 \; : \; \mu$$
$$1 - n \geq 0 \; : \; \theta$$

$$\mathcal{l} = \frac{c^{1-\gamma}}{} - B n + \mu n + \theta(1-n) + \lambda [w n - c]$$

$$L = \frac{c^{1-\gamma}}{1-\gamma} - Bn + \mu n + \theta(1-n) + \lambda[wn - c]$$

FONC:

c: $c^{-\gamma} - \lambda = 0$

n: $-B + c^{-\gamma} w + \mu - \theta = 0$

$\mu n = 0, \quad \theta(1-n) = 0$

$c = wn \Rightarrow$

$-B + (wn)^{-\gamma} w = 0$ at interior soln.

$w^{1-\gamma} \cdot n^{-\gamma} = B$

$\frac{w^{1-\gamma}}{B} = n^{\gamma}$

$$n = \left(\frac{w^{1-\gamma}}{B}\right)^{\frac{1}{\gamma}} \qquad (*)$$

Special case: $\gamma = 1$

$$L = \ln c - Bn + \lambda[wn - c] + \mu n + \theta(1-n)$$

FONC: c: $\frac{1}{c} - \lambda = 0$

n: $-B + \frac{1}{c} w + \mu - \theta = 0$, $\mu n = 0$, $\theta(1-n) = 0$

$-B + \frac{w}{wn} + \mu - \theta = 0$

$\frac{1}{n} + \mu - \theta = B$, $\quad B = \frac{1}{n}$, $\theta = 0, \mu = 0$

at interior soln

If $B^{-1} > 1$, $n = 1$, i.e. if $B < 1$, $n = 1$

$\mu = 0$ (by $\log c$)

$\theta > 0$ if $B < 1$

$-B + \frac{1}{n} - \theta = 0 \Rightarrow \boxed{\theta = 1 - B}$ if $n = 1$.

I ignore this section ↓ "too ugly" because of polynomial.

Redo with preferences

$$u(c, 1-h) = \frac{c^{1-\gamma}}{1-\gamma} + B \frac{(1-h)^{1-\varphi}}{1-\varphi}$$

$$\max_h = \frac{c^{1-\gamma}}{1-\gamma} + B \frac{(1-h)^{1-\varphi}}{1-\varphi}$$

$$+ \lambda [\omega h - c]$$

FONC:

$$c: \quad c^{-\gamma} - \lambda = 0$$

$$h: \quad -B(1-h)^{-\varphi} + \lambda \omega = 0$$

$$\Longrightarrow \quad B(1-h)^{-\varphi} = \omega c^{-\gamma} \quad , \quad c = \omega h$$

$$B(1-h)^{-\varphi} = \omega (\omega h)^{-\gamma} = \omega^{1-\gamma} \cdot h^{-\gamma}$$

$$(1-h)^{-\varphi} = \frac{\omega}{B} (\omega h)^{-\gamma}$$

$$1 - h = \left(\frac{\omega}{B}\right)^{-\frac{1}{\varphi}} (\omega h)^{\frac{\gamma}{\varphi}}$$

a nasty polynomial

Remark: Box - Cox transformation:

$$y(\lambda) = \left\{ \frac{y^{\lambda} - 1}{\lambda} \quad \text{if } \lambda \neq 0 \right.$$

$$y(\lambda) = \begin{cases} \qquad \lambda & \\ \log y & \text{if } \lambda = 0 \end{cases}$$

l'hospital's rule:

$$\lim_{\lambda \to 0} \frac{y^{\lambda} - 1}{\lambda} = \lim_{\lambda \to 0} \frac{\lambda y^{\lambda - 1}}{1} = \lim_{\lambda \to 0} \cancel{\lambda}\, y^{\lambda - 1} = \log y$$

Permanent income, 4

Tuesday, February 17, 2009
4:28 PM

A person's income stream is random. In particular, at time 0, nature flips a "coin" that determines whether the consumer is "lucky" or not. The consumer's income stream is

$$y_t = \begin{cases} 0 & \text{for all } t \geq 0 \text{ if unlucky} \\ \bar{y} & \text{for all } t \geq 0 \text{ if lucky} \end{cases}$$

The probability of the consumer being "lucky" is $p \in (0,1)$.

Consumer's preferences - when outcomes are contingent on coin flip.

let $c_t(\omega)$ = cons'n in "state" ω

$\omega = 0$ if unlucky

$\omega = 1$ if lucky

$u' > 0, u'' < 0$

$$U = p \sum_{t=0}^{\infty} \beta^t u(c_t(1)) + (1-p) \sum_{t=0}^{\infty} \beta^t u(c_t(0))$$

lucky   unlucky

Let $q_t(1)$ = price of 1 unit of consumption in date t, contingent on being lucky

$q_t(0)$ = price of 1 unit of cons^n, contingent on being unlucky.

budget constraint:

$$\sum_{t=0}^{\infty} q_t(1)\, c_t(1) + \sum_{t=1}^{\infty} q_t(0)\, c_t(0)$$

"state contingent" prices of future consumption.

"coin flip" is realized just before time 0.

$$\leq \sum_{t=0}^{\infty} q_t(1)\cdot y + \sum_{t=0}^{\infty} q_t(0)\cdot 0$$

unlucky

So Lagrangian is:

$$L = \rho \sum_{t=0}^{\infty} \beta^t u(c_t(1)) + (1-\rho)\sum_{t=0}^{\infty}\beta^t u(c_t(0))$$

$$+ \lambda\left[\sum_{t=0}^{\infty} q_t(1)\, y - \sum_{t=0}^{\infty} q_t(1)\, c_t(1) - \sum_{t=0}^{\infty} q_t(0)\, c_t(0)\right]$$

FONC:

$c_t(0)$: $(1-\rho)\beta^t u'(c_t(0)) - \lambda q_t(0) = 0$

$c_t(1)$: $\rho\beta^t u'(c_t(1)) - \lambda q_t(1) = 0$

$c_t(1)$: $\quad p\beta^t u'(c_t(1)) - \lambda\, q_t(1) = 0$

take ratio:

$$\frac{(1-p)\, u'(c_t(0))}{p\, u'(c_t(1))} = \frac{q_t(0)}{q_t(1)}$$

ratio probabilities * marginal utilities = ratio of state contingent prices.

options

Saturday, February 21, 2009
10:35 AM

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languages

- Français

Definition [edit]

Suppose $X \sim N(\mu,\sigma^2)$ has a normal distribution and lies within the interval $X \in (a,b),\ -\infty \le a < b \le \infty$. Then X conditional on $a < X < b$ has a truncation normal distribution with probability density function

$$f(x;\mu,\sigma,a,b) = \frac{\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})},$$

where $\phi(\cdot)$ is the probability density function of the standard normal distribution, $\Phi(\cdot)$ its cumulative distribution function, with the understanding that if $b=\infty$, then $\Phi(\frac{b-\mu}{\sigma})=1$. (And if $a=-\infty$, then $\Phi(\frac{a-\mu}{\sigma})=0$).

Moments [edit]

Two sided truncation:

$$E(X|a<X<b) = \mu + \frac{\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}\sigma$$

$$Var(X|a<X<b) = \sigma^2\left[1 + \frac{\frac{a-\mu}{\sigma}\phi(\frac{a-\mu}{\sigma}) - \frac{b-\mu}{\sigma}\phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} - \left(\frac{\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}\right)^2\right],$$

One sided truncation:

$$E(X|a<X) = \mu + \sigma\lambda(\alpha),\quad Var(X|X>a) = \sigma^2[1-\delta(\alpha)],$$

where $\alpha = (a-\mu)/\sigma,\ \lambda(\alpha) = \phi(\alpha)/[1-\Phi(\alpha)]$ and $\delta(\alpha) = \lambda(\alpha)[\lambda(\alpha)-\alpha]$.

See also [edit]

- Normal distribution
- Truncated distribution

$\phi(x) =$ normal density

$\Phi(x) =$ normal c.d.f. $\quad \Phi(x) = \int_{-\infty}^{x} \phi(s)\,ds$

References [edit]

Done

100% 10:37 AM

Option pricing:

Let p be the price of a stock and assume that p is a random variable with probability density $f(p)$

A call option is the right to purchase the stock tomorrow at strike price $\bar{p}$.

The value tomorrow of the call option is

0 if $p < \bar{p}$

and $p - \bar{p}$ if $p > \bar{p}$

or $\max(p - \bar{p}, 0)$

The expected value of the option is then

$$\int_{\bar{p}}^{\infty} (p - \bar{p}) f(p)\, dp$$

$$= E(p \mid p \geq \bar{p})$$

Suppose p has a Gaussian or normal distribution

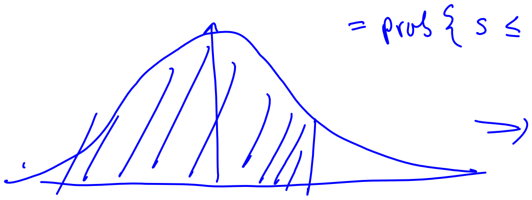
$$f(p) = \phi(p) \propto \exp\left(-\tfrac{1}{2}(p-\mu)^2/\sigma^2\right)$$

proportional to (for $\propto$), mean (for μ), standard deviation (for σ)

$\phi(p)$ is the probability density

$$\Phi(p) = \int_{-\infty}^{p} \phi(s)\, ds = \text{Cumulative density function (CDF)}$$

$$= \text{prob}\{s \leq p\}$$



Formula for the mean of a truncated normal:

$$E(p \mid p \geq \bar{p}) = \mu + \sigma \lambda(\alpha)$$

$$\alpha = \frac{(\bar{p} - \mu)}{\sigma}$$

$$\lambda(\alpha) = \frac{\phi(\alpha)}{1 - \Phi(\alpha)}$$

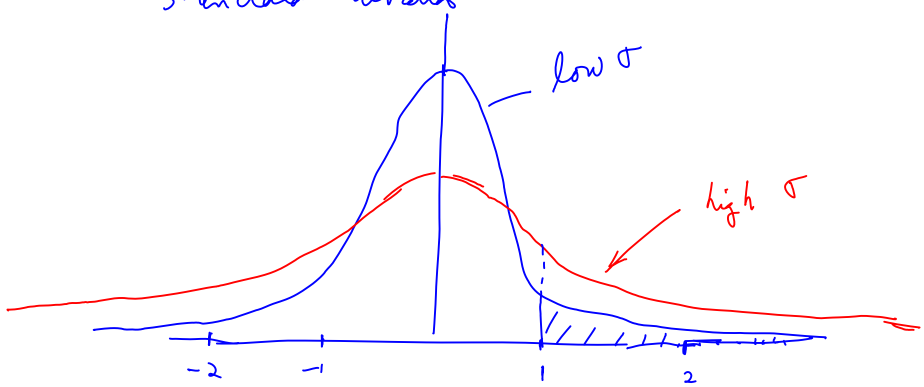
density (for $\phi(\alpha)$), CDF (for $\Phi(\alpha)$)

Example: $\mu = 1$

$\bar{p} = 2$

σ	$E\, p \mid p \geq \bar{p}$
1	1.8
2	2.6
3	3.4
4	4.2
5	5.8

standard deviation –



Finally, value of option ~~today~~ is

$$A = \beta \int_{\bar{p}}^{\infty} (p - \bar{p})\, f(p)\, dp$$

Note:

$$\int_{\bar{p}}^{\infty} p \left[\frac{f(p)}{\int_{\bar{p}}^{\infty} p(s)\, ds} \right] dp = E[p \mid p \geq \bar{p}]$$

(given by formula above)

↑

density of $p \mid p \geq \bar{p}$ – it integrates to 1 $\left(\int [\;]\, dp = 1 \right)$

$$A = \beta \left[\int_{\bar{p}}^{\infty} p\, f(p)\, dp - \bar{p} \int_{\bar{p}}^{\infty} f(p)\, dp \right]$$

$$= \beta \left[\int_{\bar{p}}^{\infty} p \left[\frac{f(p)}{\int^{\infty}} \right] \int_{\bar{p}}^{\infty} f(s)\, ds - \bar{p} \int_{\bar{p}}^{\infty} f(p)\, dp \right.$$

$$A = \beta \left[\int_{\bar{p}}^{\infty} p\, f(p)\,dp - \bar{p} \int_{\bar{p}}^{\infty} f(p)\,dp \right]$$

$$= \beta \left[\int_{\bar{p}}^{\infty} p \left[\frac{f(p)}{\int_{\bar{p}}^{\infty} f(s)\,ds} \right] \underbrace{\int_{\bar{p}}^{\infty} f(s)\,ds}_{1-\Phi(\bar{p})} - \bar{p} \underbrace{\int_{\bar{p}}^{\infty} f(p)\,dp}_{1-\Phi(\bar{p})} \right]$$

$$= \beta \left[E\left[p \mid p \geq \bar{p}\right] \cdot \left(1 - \Phi(\bar{p})\right) - \bar{p}\left(1 - \Phi(\bar{p})\right) \right]$$

$$\boxed{A = \beta \left[E\left[(p - \bar{p}) \mid p \geq \bar{p}\right] \left(1 - \Phi(\bar{p})\right) \right]}$$ option price formula.

A put option: the right to sell tomorrow at price $\underline{p}$. Value of a put today

$$\beta \int_{-\infty}^{\bar{p}} (\underline{p} - p)\, f(p)\, dp$$

you can sell it at this price

you can buy it at this price

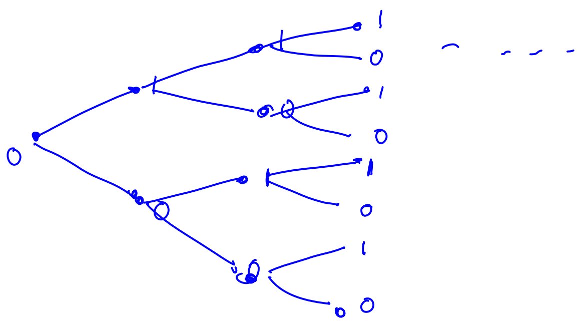
Complete markets

Saturday, March 21, 2009
6:02 PM

Asset pricing in complete markets

A state $s_t \in S$ for $t = 0, 1, 2, \ldots, +\infty$

eg. $S = \{0, 1\}$



an event tree

Let $s^t = (s_t, s_{t-1}, \ldots, s_1, s_0)$ be a history up to t.

s^∞ is an infinite history

note $s^t \in S \times S \times \cdots \times S = S^t$

$s^\infty \in S^\infty$

Remark histories are huge objects

Let $c_t(s^t)$ denote the amount a person consumes at date t contingent on history s^t.

note in general $c_t(\bar{s}^t) \neq c_t(\tilde{s}_t^t)$

for different histories $\bar{s}^t \neq \tilde{s}_t$

eg. $\bar{s}_t = (0, 0, 0, 1, 1, 0)$

$\tilde{s}_t = (1,1,1,1,1,1)$ (different luck)

Let $\pi_t(s^t)$ be the probability of history s^t.

$$\sum_{s^t} \pi_t(s^t) = 1 \; ; \; \pi_t(s^t) \geq 1 \quad \forall t, \forall s^t$$

probabilities are non negative and sum to 1

To make sure that the probabilities for different t's are consistent, we can just posit a single joint probability for s^∞

$\pi_\infty(s^\infty)$ where $\pi_\infty(s^\infty) \geq 0 \quad \forall s^\infty$

and $\sum_{s^\infty} \pi_\infty(s^\infty) = 1$

sum over all infinite histories.

Then $\forall \bar{s}^t \in S^t$

$$\pi_t(\bar{s}^t) = \sum_{s^\infty : s^t = \bar{s}^t} \pi_\infty(s^\infty)$$

↑ all '∞ histories such that partial history $s^t = \bar{s}^t$.

Preferences of a household over risky consumption streams:

$$\sum_{t=0}^{\infty} \sum_{s^t} \underbrace{\beta^t u(c_t(s^t))} \pi_t(s^t)$$

↑ ↑ ↑ ↑

time   all histories   utility at history s^t probability of history s^t

"expected utility"

Complete markets: at time 0, the household can buy or sell one unit of consumption at t contingent on history s^t at price $q_t(s^t)$. (lots of prices!) Let the household have initial wealth W_0 at $t=0$. The household choose $\{c_t(s^t)\}_{t=0, s^t \in S^t}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t)$$

subject to

$$(**) \quad \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) c_t(s^t) \leq W_0$$

Assume $u' > 0$, $u'' < 0$, $u'(0) = +\infty$

Form

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t)) \pi_t(s^t)$$
$$+ \lambda \left[W_0 - \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) c_t(s^t) \right]$$

FONC:

$c_t(s^t)$: $\beta^t u'(c_t(s^t))\pi_t(s^t)$

$$-\lambda q_t(s^t) = 0$$

$\Rightarrow$

(*) $$\lambda q_t(s^t) = \beta^t u'(c_t(s^t))\pi_t(s^t)$$

(*) and the budget constraint (**) are to be solved for $c_t(s^t)$ as a function of $\pi_t(s^t)$, $q_t(s^t)$, and W_0.

Arrow & Debreu & Hicks idea $\Uparrow$

Lucas-Breeden asset pricing model (an extension of the midterm question!)

Consider a "representative consumer" with endowment in the form of income $y_t(s^t)$ in history s^t. His initial wealth is evidently

$$W_0 = \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t)\, y_t(s^t).$$

Lucas-Breeden idea: find prices $q_t(s^t)$ that cause the household to want not to trade - & to consume $y_t(s^t)$.

These prices evidently solve (*), $y_t(s^t) = c_t(s^t)$

These prices evidently solve $\quad y_t(s^t) = c_t(s^t)$

$$\lambda q_t(s^t) = \beta^t u'(y_t(s^t)) \pi_t(s^t)$$

we can set λ = to any positive number, e.g. 1

(Multiplying all prices by the same positive scalar leaves budget sets and $\therefore$ optimal consumption unchanged).

Thus, the Lucas-Breeden theory of history-contingent prices is

$$q_t(s^t) = \lambda^{-1} \beta^t u'(y_t(s^t)) \pi_t(s^t)$$

any positive #

$q_t(s^t)$ are sometimes called state-price deflators

$$q_t(s^t) = \lambda^{-1} \beta^t u'(y_t(s^t)) \pi_t(s^t)$$

discounting for remoteness of time. — risk aversion — amount of good available — probabilities of s^t

Remark: The time 0 wealth of an agent endowed with a stream $y_t(s^t)$ is

$$W_0 = \sum^{\infty} q_t(s^t) y_t(s^t)$$

$t=0$

Suppose there are I households $i=1,\dots,I$
with preferences ordered by

$$\sum_t \sum_{s^t} \beta^t u(c_t^i(s^t))\,\pi_t(s^t) \qquad u'>0,\ u''<0$$

note: u and $\pi_t(s^t)$ are not indexed by i here - in principle they could be

but here, household $i=1,\dots,I$ have the <u>same</u> preference ordering over consumption plans $c_t^i(s^t)$

Then the prices evidently satisfy

$$\lambda^i q_t(s^t) = \beta^t u'(c_t^i(s^t))\,\pi_t(s^t)$$

for some λ^i - lagrange multiplier
$i=1,\dots,I$

where λ^i is chosen to make sure that

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t)\, c_t^i(s^t) = W_0$$

<u>Simple example</u>: $I=2$, $s_t \in \{0,1\}$

$$y_t^1(s^t) = \begin{cases} 1 & \text{if } s_t=1 \\ 0 & \text{if } s_t=0 \end{cases}, \quad t\geq 0$$

$$y_t^2(s^t) = \begin{cases} 0 & \text{if } s_t=1 \\ 1 & \text{if } s_t=0 \end{cases} \quad t\geq 0$$

Note: $y^1(s^t) + y^2(s^t) = 1 \quad \forall t \quad \forall s^t$

Note: $y_t^1(s^t) + y_t^2(s^t) = 1 \quad \forall t \quad \forall s^t$

Guess: $c_t^1(s^t) = \bar{c}_1 \quad \forall t \quad \forall s^t$

$c_2^2(s^t) = \bar{c}_2 \quad \forall t, \forall s^t$

$\Rightarrow$

$$\lambda^1 q_t(s^t) = \beta^t u'(\bar{c}^1) \pi_t(s^t)$$

set $\lambda^1 = u'(\bar{c}^1)$ (a decision that scales prices)

$$q_t(s^t) = \underbrace{\beta^t}_{\text{discount factor}} \underbrace{\pi_t(s^t)}_{\text{probability}}$$

To compute

$$W^i = \sum_{t=0}^{\infty} \sum_{s^t} q_t(s^t) y_t^i(s^t)$$

$$= \sum_{t=0}^{\infty} \sum_{s^t} \beta^t y_t^i(s^t) \pi_t(s^t)$$

Suppose now that

$$\pi_t(s^t) = \pi(s_t)\pi(s_{t-1}) \cdots \pi(s_0)$$

independent, identically distributed s_t's.

and $\pi(1) = .5$

$\pi(0) = .5$

then $$W^i = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t y_t^i(s^t) \pi_t(s^t)$$

$t=0 \quad s^t$

$$= \sum_{t=0}^{\infty} \beta^t \cdot .5 = \frac{.5}{1-\beta} \qquad i=1,2.$$

PV of expenditures:

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \, \bar{c}_i \, \pi_t(s^t)$$

$$= \bar{c}_i \sum_{t=0}^{\infty} \beta^t \underbrace{\sum_{s^t} \pi_t(s^t)}_{=1} = \bar{c}_i \sum_{t=0}^{\infty} \beta^t = \frac{\bar{c}_i}{1-\beta}$$

to solve $\dfrac{\bar{c}_i}{1-\beta} = \dfrac{.5}{1-\beta} \Rightarrow \bar{c}_i = .5, \; i=1,2$

Note: here we have used different notation than in the growth model with infinitely lived agents.

Overlapping generations

Sunday, March 29, 2009
5:35 PM

The overlapping generations model:

Use to study

growth

- government debt & taxes
- social security schemes

Components: time : discrete $t = 1, 2, \ldots, +\infty$.

Demography: at each $t \geq 1$, there are born N (no population growth) two period lived workers.

At $t = 1$, there are alive N old people who want to consume but do not work. At each $t \geq 1$, young agents work but consume nothing. Old agents consume, but do no work. (This is a simplification designed to ease our computational burden while leaving in some essential forces.)

Technology: Aggregate:

$$Y_t = F(K_t, L_t) \quad \text{(constant returns to scale)}$$

$$Y_t = C_t + I_t$$

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Technology: Aggregate:

$$Y_t = F(K_t, L_t) \quad \text{(constant returns to scale)}$$

$$Y_t = C_t + I_t$$

$$K_{t+1} = (1-\delta) K_t + I_t, \quad \delta \in (0,1)$$

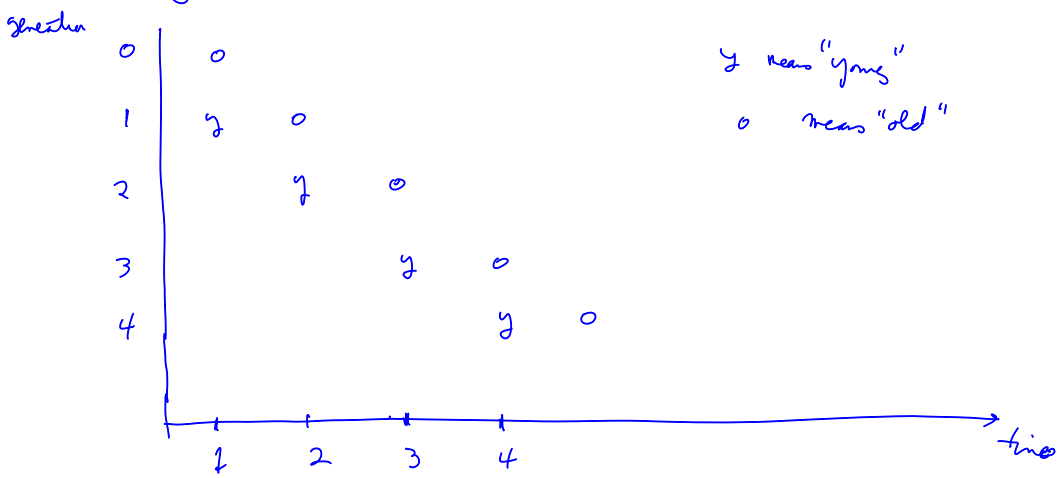
$$C_t = N \cdot c_{yt} + N c_{ot} \qquad \text{feasibility}$$

$$C_t = N \cdot c_{yt} + N c_{ot} \qquad \underline{\text{feasibility}}$$

c_{yt} = cons^n of a <u>young</u> person at t

c_{ot} = cons^n of an old person at t.

Demography:



Demography - you are either young, or old, or not alive.

c_{ot} = cons^n of an old person at t

c_{yt} = " " a young person at t

So, the consumption profile for a person of generation born in t is (c_{yt}, c_{ot+1})

young at t old at $t+1$

<u>"Command" economy</u> (Solow)

$L_t = N$ (all young work)

$I_t = s F(K_t, L_t)$ s = saving rate commanded

$\Rightarrow$
$$\frac{K_{t+1}}{N} = \frac{K_t}{N} + s F\left(\frac{K_t}{N}, 1\right) - \delta \frac{K_t}{N}$$

or
$$k_{t+1} = (1-\delta) k_t + s f(k_t) \qquad \text{(Solow model)}$$

Now we'll consider a market economy

Agents – have preferences over work, consumption

labor market (L_t, w_t)

goods market Y_t, c_{ot}, c_{yt}, I_t

capital market: r_t - net interest rate from t to $t+1$

Firms: rent labor and capital at competitive prices.

w_t - wage rate

ρ_t - rental rate for capital

Assume $Y_t = F(K_t, L_t) = K_t^{\beta} L_t^{1-\beta}, \quad \beta \in (0,1)$

β is share of capital

$1-\beta$ is share of labor

$$\rho_t = \frac{\partial F}{\partial K_t}(K_t, L_t) = \beta K_t^{\beta-1} L_t^{1-\beta}$$

$$= \beta k_t^{\beta-1}$$

so $\rho_t = \beta k_t^{\beta-1}$

$$w_t = \frac{\partial F(K_t, L_t)}{\partial L_t} = (1-\beta) K_t^{\beta} L_t^{-\beta}$$

$$= (1-\beta)\left(\frac{K_t}{L_t}\right)^{\beta}$$

$$= (1-\beta) k_t^{\beta}$$

$[w_t = (1-\beta) k_t^{\beta}]$

$$Y_t - \underbrace{\left(\beta K_t^{\beta-1} L_t^{1-\beta}\right)}_{\rho_t} K_t - \underbrace{\left((1-\beta) K_t^{\beta} L_t^{1-\beta}\right)}_{w_t} L_t$$

$$= Y_t - \beta K_t^{\beta} L_t^{1-\beta} - (1-\beta) K_t^{\beta} L_t^{1-\beta}$$

$$= Y_t - \beta Y_t - (1-\beta) Y_t = \underline{\underline{0}}$$

Factor payments exhaust the product - zero profits.

Simplifying assumptions:

- Agents don't value leisure
- Agents are only endowed with useful labor when young
- Agents only value consumption when old

eg utility fn is $u(c_{ot})$

Timing: young workers

saving at t earnings at t
$a_{t+1} = w_t$

- supply $L_t = N$,
- earn $w_t \cdot L_t$
- consume $c_{yt} = 0$ and deposit $a_{t+1} = w_t$ per worker
- deposit $N \cdot a_{t+1}$ with an "intermediary"

old workers:

- withdraw $(1+r_{t+1}) N a_{t+1}$, $(r_{t+1} = \rho_{t+1} - \delta)$

[Note: It will turn out that $(1+r_{t+1}) N a_{t+1} = (1-\delta) K_{t+1} + \rho_{t+1} K_{t+1}$]

- consume $c_{ot+1} = (1+r_{t+1}) a_{t+1}$ $(= w_t \cdot (1+r_{t+1}))$

t = ... intermediary:

Financial intermediary:

at t gets deposits $N a_{t+1}$

- Uses the deposits to purchase $K_{t+1} = N a_{t+1}$
- At $t+1$ - rents K_{t+1} to firms and

  receives $\rho_{t+1} K_{t+1}$

- Pays out $\underbrace{(1-\delta) K_{t+1}}_{\text{remaining capital}} + \underbrace{\rho_{t+1} K_{t+1}}_{\text{rentals}} \equiv (1+r_{t+1}) \overbrace{N a_{t+1}}^{K_{t+1}}$

  to depositors. so $[(1-\delta) + \rho_{t+1}] K_{t+1} = (1+r_{t+1}) K_{t+1}$

Note: $(1-\delta)K_{t+1}$ of capital survives & is sold by the time $(t+1)$ old to the time $(t+1)$ young, through intermediary.

For our specification:

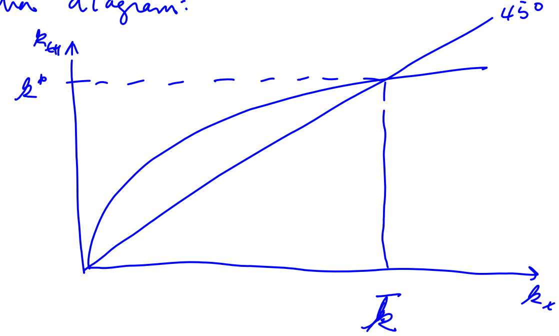
young at t earn

$$w_t N = N a_{t+1} = K_{t+1}$$

$$K_{t+1} = w_t N = (1-\beta) k_t^{\beta} N = K_{t+1} = N k_{t+1}$$

$$\Rightarrow k_{t+1} = (1-\beta) k_t^{\beta}.$$

$\Rightarrow$ transition diagram:



$$\bar{k} = (1-\beta) \bar{k}^{\beta} \qquad \bar{k}^{1-\beta} = (1-\beta)$$

$$(1-\beta)\log \bar{k} = \log(1-\beta)$$

$$(1-\beta)\log \bar{k} = \log(1-\beta)$$

$$\log \bar{k} = \frac{\log(1-\beta)}{1-\beta}$$

or $\bar{k} = (1-\beta)^{\frac{1}{1-\beta}}$

steady state factor payments:

$$\bar{\rho} = \beta \left[(1-\beta)^{\frac{1}{1-\beta}} \right]^{\beta-1}$$

$$\bar{\rho} = \frac{\beta}{1-\beta} \qquad\qquad \bar{\rho} = \frac{\beta}{1-\beta}$$

$$\bar{\omega} = (1-\beta)k^{\beta} = (1-\beta)(1-\beta)^{\frac{\beta}{1-\beta}}$$

$$= (1-\beta)^{\frac{1-\beta}{1-\beta}} (1-\beta)^{\frac{\beta}{1-\beta}}$$

$$\bar{\omega} = (1-\beta)^{\frac{1}{1-\beta}} \qquad\qquad \bar{\omega} = (1-\beta)^{\frac{1}{1-\beta}}$$

Please recall that $r_{t+1} = \rho_{t+1} - \delta$

$\bar{r} = \frac{\beta}{1-\beta} - \delta$ may be bigger or less than 0 depending on β & δ.

Budget constraints in the above problem:

$$c_{yt} + a_{t+1} = \omega_t$$

$$c_{o\,t+1} = (1+r_{t+1})\, a_{t+1}$$

we imposed: $c_{yt} = 0 \implies$

$$c_{o\,t+1} = (1+r_{t+1})\, a_{t+1} = (1+r_{t+1})\,\omega_t$$

$a_{t+1} = k_{t+1}$ all saving is in form of capital.

return on capital:

$$\underbrace{(1-\delta)k_{t+1}}_{\text{amount left over after production}} + \underbrace{\rho_{t+1} k_{t+1}}_{\text{rental on capital}}$$

$$(1+r_{t+1}) = (1-\delta) + \rho_{t+1}$$

$$(1+r_{t+1}) = (1-\delta) + \rho_{t+1}$$

$$\text{or } r_{t+1} = \rho_{t+1} - \delta$$

Feasibility: $k_{t+1} = (1-\delta) k_t + x_t$

$$c_{ot} + c_{yt} + x_t = f(k_t)$$

$$\overset{\shortparallel}{} \quad \overset{\shortparallel}{0} \quad \overset{\shortparallel}{} + (k_{t+1} - (1-\delta)k_t) = k_t^{\beta}$$

$$* \quad c_{ot} + (k_{t+1} - (1-\delta)k_t) = k_t^{\beta} \qquad \text{feasibility}$$

$$c_{ot} = k_t (1+r_t) = k_t (1-\delta + \beta k_t^{\beta-1}) \qquad \text{budget constraint}$$

(w_{t-1} carried over from last period) $\qquad (w_{t-1} = (1-\beta) k_{t-1}^{\beta})$

(*) $\Rightarrow$ $\quad \rho_t \shortparallel$

$$k_t (1-\delta + \beta k_t^{\beta-1}) + (k_{t+1} - (1-\delta)k_t) = k_t^{\beta}$$

$$\beta k_t^{\beta} + k_{t+1} = k_t^{\beta}$$

$$\boxed{k_{t+1} = (1-\beta) k_t^{\beta}} \qquad \checkmark \qquad \text{Fits together.}$$

Now do a social security scheme:

Tax each young agent e

Transfer each old agent e

Budget constraints:

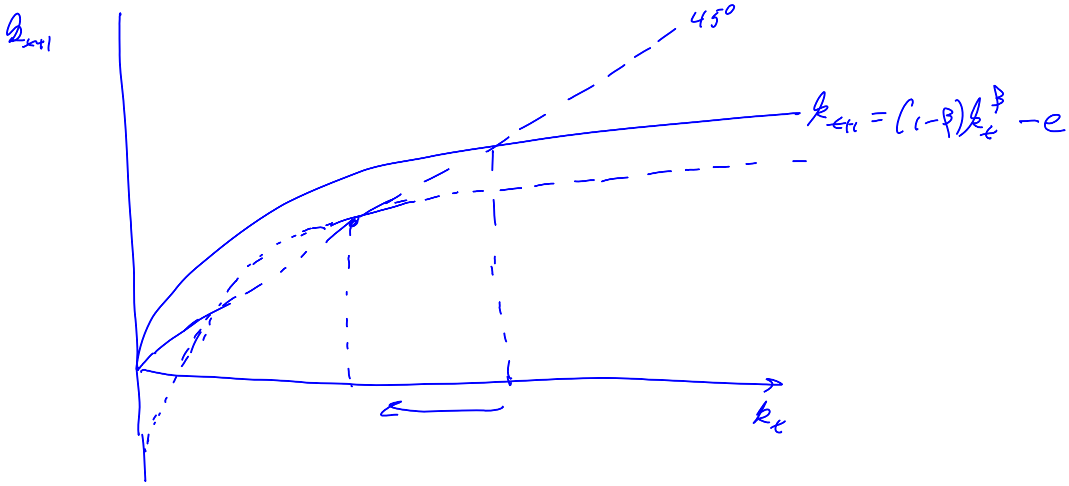
$$c_{yt} + a_{t+1} = w_t - e$$

$$c_{ot+1} = a_{t+1}(1+r_{t+1}) + e.$$

Same preferences as before:

Same preferences as before:

$$\underset{\underset{\text{savings}}{\parallel}}{a_{t+1}} = k_{t+1} = (1-\beta)k_t^{\beta} - e$$



Who gains, who loses? It depends.

- initial time 0 old gain always - because they get transfer of e when old.

- young for generations born at $t \geq 0$ lose e at time t and gain e at time $t+1$. The present value of the "gain" at t is:

$$-e + \frac{e}{1+r_{t+1}}$$

gain if $\frac{1}{1+r_{t+1}} > 1$ or $r_{t+1} < 0$

otherwise, a loss.

Key equation:

$$\underset{\substack{\uparrow \\ \text{savings}}}{(w_t - e)} = \underbrace{(1-\beta)k_t^{\beta}}_{\substack{\uparrow \\ \text{falls with} \\ k}} - e = k_{t+1}$$

- wage fall
- e rises (taxes ↑)

$\Rightarrow$ less capital accumulation

- wage falls
- e rises (taxes ↑) $\Rightarrow$ less capital accumulation

$$c_{0t} = e + (r_t + 1) k_t \qquad \text{at } t=0,\ k_0 \text{ unchanged}$$

so c_{0t} ↑ at $t=0$.

but

$$c_{0,1} = e + (r_1 + 1)\big[\overbrace{(1-\beta) k_0^{\beta} - e}^{k_1}\big]$$

$$= (r_1+1)(1-\beta) k_0^{\beta} - r_1 e$$

notice key role of sign of r_1

Summary of dynamic equations:

$$k_{t+1} = (1-\beta) k_t^{\beta} - e = w_t - e$$

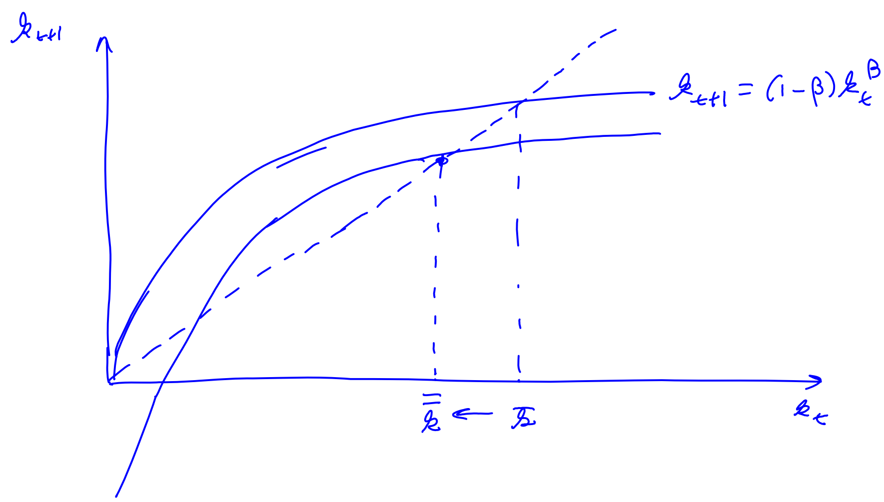
$$\rho_t = \beta k_t^{\beta-1}$$

$$w_t = (1-\beta) k_t^{\beta}$$

$$r_t = \rho_t - \delta$$

$$c_{0t+1} = (1+r_{t+1}) k_{t+1} + e = (1+r_{t+1})\left[w_t - e\right] + e$$

start from steady state with $e=0$ and trace what happens to c_{0t+1} over time.



Starting from $\bar{k}$, $k_{t+1} \searrow \bar{\bar{k}} < \bar{k}$

What happens to c_{0t} compared to the no transfer $e=0$ case? Which generations gain? Which, if any, lose?

The answers depend on the value of $\bar{r} = \bar{\rho} - \delta$ in the initial steady state, which in turn depends on (β, δ), as well as on the size of the transfer.

The following propositions are true:

Proposition 1: Suppose $\bar{r} < 0$ in the $e=0$ steady state. Then there exists a positive social security transfer $e>0$ that makes every generation better off, i.e. c_{0t} increases for all $t \geq 0$. Further, by driving k downward, the social security system increases the eventual steady state r.

Proposition 2: Suppose that (β, δ) are such that $r>0$ in an initial steady state with $e=0$. Then a social security transfer $e>0$ makes the initial old generation better off by increasing c_{00} (c old at $t=0$) but makes all subsequent

$t = 0$)), but makes <u>all</u> subsequent generations worse off.

<u>Remark:</u> The little matlab program olg.m can be used to illustrate these propositions.

<u>Remark:</u> A steady state equilibrium in which $\bar{r} < 0$ is said to be <u>dynamically inefficient</u>. One with $\bar{r} > 0$ is said to be <u>dynamically efficient</u>.

Summary lecture

Wednesday, April 29, 2009
9:29 AM

Last lecture in class we showed

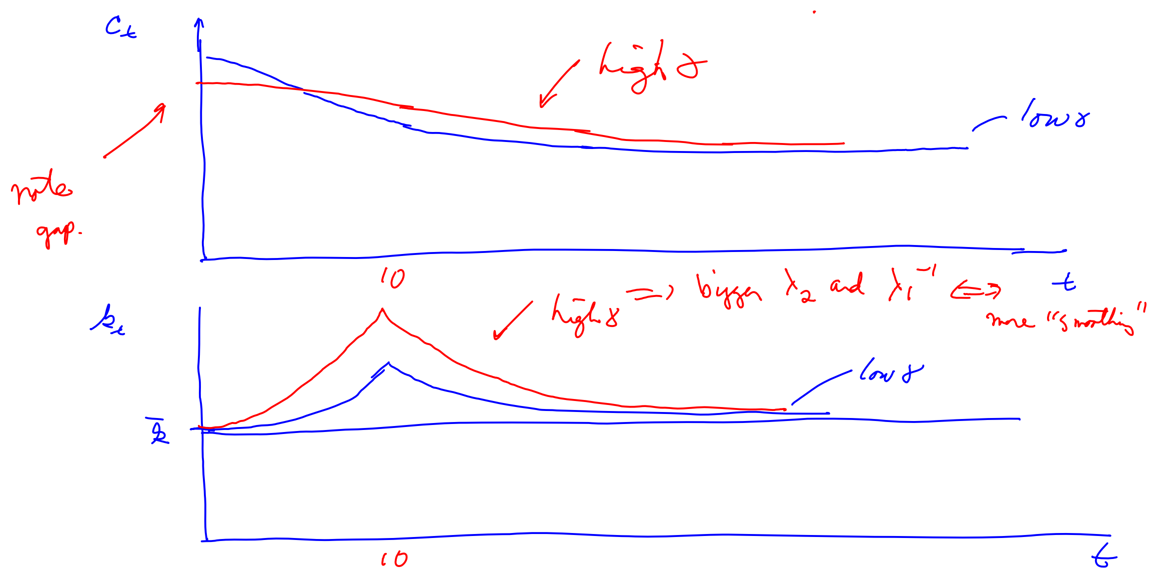
$$k_{t+1} = \lambda_2 k_t - \lambda_2^{-1} \phi_2^{-1} \sum_{j=0}^{\infty} (\lambda_1)^{-j} \left[A_0 + A_1 z_{t+j} + A_2 z_{t+j+1} \right]$$

λ_2 and λ_1 are functions of a number of parameters including γ.

λ_1^{-1} & λ_2 increase when γ increases.

Two economies - everything the same except γ

high γ & low γ economies



response to an foreseen ⇑ in g at $t=10$ for a high & low γ economy

A preference for more smoothness in c_t — higher γ — surfaces in higher value of λ — from last lecture.

— — — —

Asset pricing:

q_t^0 lets us price anything - eg. $\{d_t\}_{t=0}^{\infty}$

$$a_0 = \sum_{t=0}^{\infty} q_t^0 \, d_t$$

for much of course, we exogenously set q_t^0, eg. $q_t = \beta^t$. Later we came up with a theory $q_t = \beta^t \frac{u'(c_t)}{1 + \tau_{ct}}$

"consumption theory of asset pricing"

To add risk - $s_t \in S$, s^t, $\pi_t(s^t) \Rightarrow q_t^0(s^t)$

$$a_0 = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) \, d_t(s^t)$$

eg. $q_t^0(s^t) = \beta^t u'(c_t(s^t)) \, \pi_t(s^t)$

- Term structure of interest rates a special case

Asset pricing examples from early in course:

$$p_t = \beta \, p_{t+1} + d_{t+1}$$

$$d_t = G x_t$$

$$x_{t+1} = A x_t \quad \Rightarrow$$

$$p_t = G (I - \beta A)^{-1} x_t \; + \text{ bubble term}$$

random version

$$x_{t+1} = A x_t + C \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}(0, I)$$

$$p_t = \beta E_t p_{t+1} + d_{t+1}$$

solve Bellman equation $p_t = H x_t$ = (value function)

$$H x_t = \beta E_t H x_{t+1} + G x_t \implies$$

$$H = G(I - \beta A)^{-1}$$

This is an example in which

$$q_t^0 = \beta^t \pi_t(\varepsilon^t)$$

---

labor income as an asset:

Friedman model with stochastic labor income

$$x_{t+1} = A x_t + C \varepsilon_{t+1}$$

$$y_t = G x_t$$

$$F_{t+1} = \beta^{-1}[F_t + y_t - c_t]$$

F_t = financial assets

y_t = labor income

c_t = consumption

$$c_t = (1-\beta)\Big[F_t + \underbrace{E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}}_{\text{"human wealth"}}\Big]$$

approximation is exact when

(a) $q^0_t = \beta^t$, and

(b) $u(c_t)$ is quadratic $\Rightarrow$ $u'(c_t)$ is linear in c_t

We deduced

$$c_{t+1} - c_t = (1-\beta) \underbrace{\sum_{j=0}^{\infty} \beta^j \left(E_{t+1}\, y_{t+j+1} - E_t\, y_{t+j} \right)}_{\text{innovation to present value of labor income}}$$

$\Rightarrow$

$$c_{t+1} - c_t = (1-\beta)\, G (I - \beta A)^{-1} C\, \varepsilon_{t+1} \equiv M \varepsilon_{t+1} \equiv \varepsilon^c_{t+1}$$

consumption is a random walk $\qquad \varepsilon^c_{t+1} = M \varepsilon_{t+1}$,

$\sigma_c^2 = \operatorname{var} \varepsilon^c_{t+1}$

$$E_t \left(c_{t+j} - c_t \right)^2 = \operatorname{var}_t \left(\varepsilon_{t+j} + \varepsilon_{t+j-1} + \cdots + \varepsilon_{t+1} \right)$$

$$= j\, \sigma_c^2$$

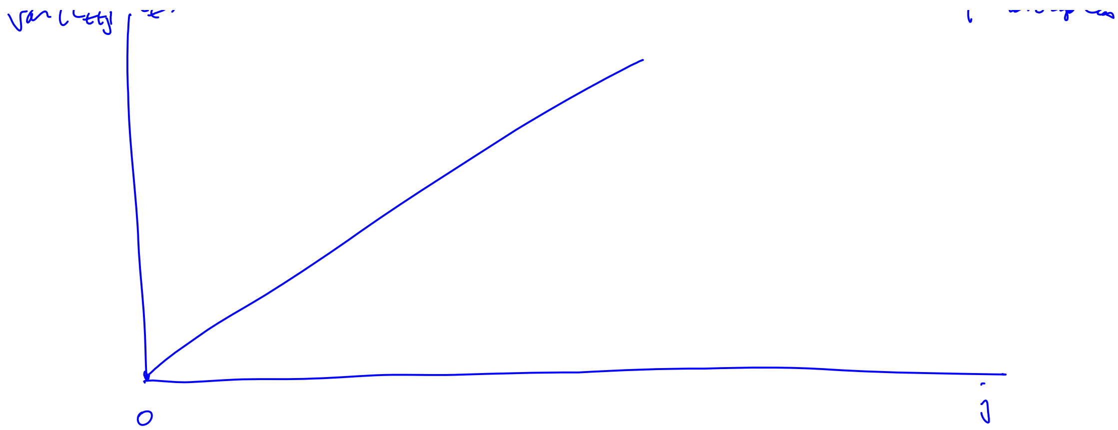
$\Rightarrow$ variance increases linearly with horizon j.

If put i subscripts on c^i_t, F^i_t, y^i_t

this becomes a theory of a spreading out

cross section distribution of consumption & wealth (F^i_t)

as j increases ; $\operatorname{var}(c^i_{t+j} - c^i_t)$ = cross section variance of consumption

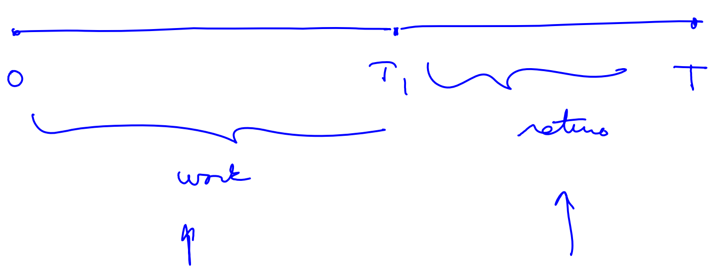


Use as the basis of a theory to explain Deaton-Paxton & Kaplan-Violante observations on consumption & wealth inequality

To fit the data - change to a finite horizon model

$$E_0 \sum_{t=0}^{T} \beta^t u(c_t)$$

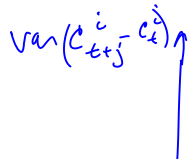
$$F_{t+1} = \beta^{-1} \left[F_t + y_t - c_t \right]$$

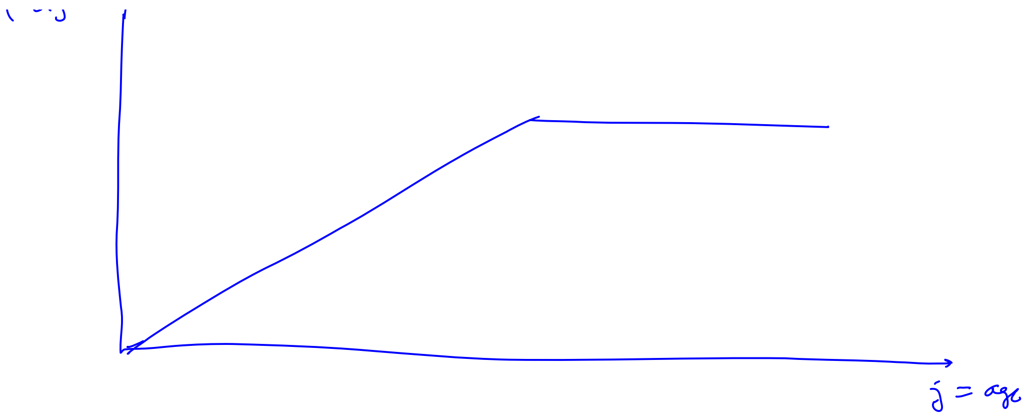


$$x_{t+1} = A x_t + C \varepsilon_{t+1}$$

$$y_t = G x_t$$

$$y_t = 0$$





Cross section variance of consumption 's until date of retirement.

Homework 1

Tuesday, February 10, 2009
11:10 AM

Homework 1

Instructions: Feel free to work in groups.

Please write up your own homework even if you have worked in a group.

Due: Feb 18

1. Consider a dividend stream described by the difference equation

$$d_{t+1} = \bar{d} + \rho_1 d_t \quad \text{for } t \geq 0$$

where $\bar{d}$ is a constant, d_0 is given, and $t = 0, 1, 2, \ldots$

At time t, a stock offering a perpetual claim to the stream d_t is valued at price p_t where p_t satisfies

$$p_t = \beta p_{t+1} + d_t \quad , \quad t \geq 0 \quad .$$

a. Find a formula for p_t ~~expressed~~ in terms of d_t, d_{t-1}, d_{t-2} and possibly calendar time t.

b. Does your p_t formula display a "bubble"? Explain.

2. A worker's labor income is given by

$$y_{t+1} = \delta\, y_t \quad , \qquad \delta > 1$$

for $t = 0, 1, \ldots, +\infty$

The worker's consumption c_t satisfies

$$c_{t+1} = (\beta R)\, c_t \quad , \ c_0 \text{ to be determined}$$

where $\beta < 1$, $R > 1$ and $(\beta R) < 1$.

Assume that $\beta\delta < 1$

The worker's consumption path must satisfy the intertemporal budget constraint

$$(*) \quad \sum_{t=0}^{\infty} R^{-t} (c_t - y_t) = 0 \ .$$

a. Solve equation (*) for c_0 as a function of y_0.

b. Now assume that

$$y_{t+1} = \delta_{11}\, y_t + \delta_{12}\, z_t$$

$$z_{t+1} = \delta_{21}\, y_t + \delta_{22}\, z_t$$

show how to solve for c_0 as a function show how to solve for c_0 as a function of (y_0, z_0).

3. The unemployment rate at month t, u_t, satisfies the difference equation

$$(*) \quad u_{t+1} = \alpha(1-u_t) + (1-\varphi)u_t$$

where α = rate of inflow into unemployment

φ = rate of exit from unemployment.

Definition: A steady state unemployment rate $\bar{u}$ satisfies $(*)$ with $\bar{u} = u_{t+1} = u_t$.

a. Compute a formula for the steady state unemployment rate $\bar{u}$ as a function of α and φ.

b. In 2007 in the U.S., $\alpha = .0127$ and $\varphi = .2579$. Compute the implied steady state unemployment rate.

c. Suppose that in a recession year 2xxx, $\varphi = .15$ and $\alpha = .0175$. These are monthly rates.
Compute the steady state unemployment rate in year 2xxx

in year 2XXX.

d. Extra credit. Compute the average duration of a job in 2XXX. Compute the average duration of unemployment in 2XXX.

4. Answer question 7 on page 179 of Williamson. After answering the question, say what would change if household preferences were to take the special form

$$u(c, 1-h) = \ln c - B h \quad , \quad B > 0 .$$

Homework 2

Wednesday, February 18, 2009
3:03 PM

Homework 2: due March 4.

1. An economy consists of two consumers, $i=1,2$, each of whom orders consumption streams according to the intertemporal utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i) \quad \text{when } 0<\beta<1 \text{ and}$$
$$u'>0 \text{ and } u''<0$$

each person has intertemporal budget constraint

$$\sum_{t=0}^{\infty} \beta^t c_t^i = \sum_{t=0}^{\infty} \beta^t y_t^i .$$

Consumer 1 has endowment stream

$$\begin{array}{lcccccccccc} t= & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots \\ y_t^1 = \{ & 1, & 1, & 0, & 1, & 1, & 0, & 1, & 1, & 0, & \ldots \} \end{array}$$

while consumer 2 has stream

$$y_t^2 = \{0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots$$

a. Find the optimal consumption paths $\{c_t^i\}_{t=0}^{\infty}$ for $i=1,2$.

b. Compute $c_t^1 + c_t^2$ for $t=0,1,2,\ldots$

c. For $t = 0, 1, 2, \ldots$, please compute

A^1_{t+1} where

$$A^1_{t+1} = \beta^{-1}\left[A^1_t + y^1_t - c^1_t\right], \quad A^1_0 = 0$$

d. For $t = 0, 1, 2, \ldots$, please compute

A^2_{t+1} where

$$A^2_{t+1} = \beta^{-1}\left[A^2_t + y^2_t - c^2_t\right].$$

---

2. Consider an economy consisting of two people, each of whom orders consumption streams according to

$$\sum_{t=0}^{\infty} \beta^t u(c^i_t), \quad i = 1, 2, \quad u' > 0, u'' < 0$$

$$0 < \beta < 1$$

Person 1 has income stream

$$y^1_t = \delta^t, \quad t \geq 0, \quad \delta \in (0,1).$$

Person 2 has income stream

$$y^2_t = 1 - \delta^t, \quad t \geq 0, \quad \delta \in (0,1).$$

Each person has the intertemporal budget constraint

$$\sum_{t=0}^{\infty} \beta^t \left[c^i_t - y^i_t\right] = 0$$

a. Find the optimal consumption paths $\{c_t^i\}_{t=0}^{\infty}$ for $i=1,2$.

b. Compute $c_t^1 + c_t^2$ for $t=0,1,2,\ldots$

c. For $t=0,1,2,\ldots$, please compute A_{t+1}^1 where

$$A_{t+1}^1 = \beta^{-1}\left[A_t^1 + y_t^1 - c_t^1\right], \quad A_0^1 = 0.$$

d. For $t=0,1,2,\ldots$, please compute A_{t+1}^2 where

$$A_{t+1}^2 = \beta^{-1}\left[A_t^2 + y_t^2 - c_t^2\right].$$

e. Given β with $0<\beta<1$, please find a value of δ that makes consumers 1 and 2 consume identical amounts for $t=0,1,2,\ldots$.

e. Assume the same two consumers with the same income processes mentioned above. But now assume that each consumer has the <u>sequence</u> of budget constraints

$$A_{t+1}^i = \beta^{-1}\left[A_t^i + y_t^i - c_t^i\right],$$

where $A^i = 0$ and

where $A_0^i = 0$ and

$$(*) \qquad A_{t+1}^i \geq 0 \quad \text{for } t = 0, 1, \ldots$$

Here $(*)$ is a <u>no-borrowing</u> constraint.

Please find the optimal consumption path $\{c_t^i\}_{t=0}^{\infty}$ for $i = 1, 2$.

---

3. A stock price obeys the difference equation

$$p_t = \beta E_t p_{t+1} + d_t$$

where $\beta \in (0,1)$, p_t is the stock price today, $E_t p_{t+1}$ is the best forecast of the price tomorrow, and d_t is the dividend process; d_t obeys the random difference equation

$$(*) \qquad d_{t+1} = \bar{d} + \rho_1 d_t + \rho_2 d_{t-1} + c\,\varepsilon_{t+1},$$

$$d_0,\ d_{-1} \text{ given, and}$$

where $\{\varepsilon_{t+1}\}_{t=0}^{\infty}$ is an independently and identically distributed sequence of normal random variables with mean 0 $(E_t \varepsilon_{t+1} = 0)$ and standard deviation 1 $\left(\sqrt{E_t(\varepsilon_{t+1} - 0)^2} = 0\right)$.

[illegible]

a. Find a <u>fundamental</u> solution of (*) of the form

$$p_t = f(d_t, d_{t-1}),$$ giving an explicit expression for the function f.

b. Find a solution of (*) that displays a <u>bubble</u>.

4. A government wants to <u>minimize</u> the following measure of <u>tax distortions</u>:

$$\sum_{t=0}^{\infty} \beta^t D(T_t)$$

where $D(T_t)$ is a measure of the costs of distortions at date t, T_t are total tax revenues, and $D' > 0$, $D'' > 0$.

The government has an exogenous stream of expenditures $\{G_t\}_{t=0}^{\infty}$ and faces the sequence of government budget constraints

$$B_{t+1} = R\left[B_t + G_t - T_t\right], \quad B_0 = 0$$

where B_{t+1} is government <u>debt</u> issued at t and due at $t+1$. The government can borrow or lend. $R := 1+r > 1$ is the gross rate of return on government debt. Assume that $R = \beta^{-1}$.

Assume that $\lim_{T\to\infty} \beta^{t} B_{t+1} = 0$ as a condition to rule out "Ponzi schemes"

a. Consider the government expenditure process

$$G_t = \begin{cases} 1 & t \text{ even} \\ 0 & t \text{ odd} \end{cases}$$

Find the optimal setting for taxes $\{T_t\}_{t=0}^{\infty}$ for $t = 0, 1, 2, \ldots$

b. Consider the expenditure process

$$G_t = \begin{cases} 1 & t \neq 10j, \ j = 0, 1, 2, \ldots; t = 0, 1, 2 \\ 10 & t = 10j, \ j = 0, 1, 2, \ldots \end{cases}$$

here a "war" happens in periods 0, 10, 20, ... and "peace" occurs the rest of the time.

Find the optimal level of $\{T_t\}_{t=0}^{\infty}$.

c. Does the analysis in this problem remind you of any other problem or model you have seen? If so, explain.

Homework 3
Tuesday, March 24, 2009
2:51 PM

Homework 3 :

Due Monday April 6.

1. Each period, a previously unemployed worker draws <u>two</u> offers to work forever from the c.d.f.

$$F(\omega) = \sqrt{\frac{\omega}{B}}, \quad 0 \leq \omega \leq B.$$

Successive draws within a period and across periods are identically and independently distributed. The offers are to work forever. There is no quitting, no firing, and no recall of offers from past periods.

The worker wants to maximize

$$E_{-1} \sum_{t=0}^{\infty} \beta^t y_t, \quad 0 < \beta < 1$$

where

$$y_t = \begin{cases} \omega & \text{if employed at wage } \omega \\ c & \text{if unemployed} \end{cases}$$

where w is the wage and c is unemployment compensation, and E_{-1} is an expected value (mean) before the two offers are drawn in period 0.

a. Verify that $F(\omega) = \sqrt{\frac{\omega}{B}}, \quad 0 \leq \omega \leq B$ is a legitimate c.d.f. (cumulative distribution function).

c.d.f. (cumulative distribution function).

b. Argue rigorously that the worker's optimal strategy has a reservation wage strategy:

accept $\tilde{\omega}$ if $\tilde{\omega} > \bar{\omega}$

reject $\tilde{\omega}$ if $\tilde{\omega} < \bar{\omega}$

where $\tilde{\omega} = \max(\omega_1, \omega_2) =$ maximum of two offers ω_1, ω_2 drawn this period. (i.e. find a Bellman equation in an optimal value function).

c. Find a formula for $\bar{\omega}$ as a function of (B, β, c). Hint: you will have to compute a definite integral.

d. Find a formula for

$\Psi \equiv$ Probability of leaving unemployment this period

as a function of (B, β, c).

e. Give the value of $\bar{\omega}$ when $\beta = .9$, $B = 10$, $c = 0$. Find the value of Ψ.

f. Give the value of $\bar{\omega}$ when $\beta = .9$, $B = 10$, $c = 6$. Find the value of Ψ.

2. Consider Solow's growth model in the special case $\delta = 1$ and $f(k) = zk^{\alpha}$, $\alpha \in (0,1)$.

a. Find a difference equation expressing $\log k_{t+1}$ as a function of $\log k_t$.

b. Find the steady state value of $\log k_t$.

c. Argue that the steady state is "stable" in the sense that

$$\lim_{t \to \infty} \log k_t = \log \bar{k}$$

starting from any initial condition $\log k_0 > -\infty$.

3. Please answer question 10 on page 226 of Williamson, 3rd edition.

4. Please answer question 11 on page 226 of Williamson, 3rd edition.

Note: the following problem is challenging.

5. Each period, a previously unemployed worker draws one and only one offer to work from a known c.d.f. There is no quitting, no firing, and no recall of offers

no quitting, no firing, and no recall of offers drawn in previous periods. The worker wants to maximize

$$E_{-1} \sum_{t=0}^{\infty} \beta^t y_t \quad , \quad 0 < \beta < 1$$

$$y_t = \begin{cases} w & \text{if employed at } t \\ c & \text{if unemployed at } t \end{cases}$$

Here c is unemployment compensation and w is the wage.

For each t odd (i.e., $t = 1, 3, 5, \ldots$) the worker draws one offer from a c.d.f. $F_o(w)$; $F_o(0) = 0$, $F_o(B) = 1$

For each t even (i.e., $t = 0, 2, 4, \ldots$) the worker draws one offer from a c.d.f. $F_e(w)$, $F_e(0) = 0$, $F_e(B) = 1$.

Successive draws are independent, but not identically distributed.

a. Describe how to solve the worker's problem using "dynamic programming."

b. Find the form of the worker's optimal strategy. Is it of the "reservation wage" form? Explain.

Hint: Let $v_0(w)$ be the optimal value of the problem for a previously unemployed worker who has just drawn wage w in an odd period. Let ...

Practice problems 1

Friday, February 26, 2010
9:49 AM

Here are some practice problems. They are not assigned but are designed to help you learn.

1. A consumer has multi-period utility function

$$(1) \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad 0 < \beta < 1$$

where $u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma} \quad , \gamma \geq 0$

The consumer chooses $\{c_t, A_{t+1}\}_{t=0}^{\infty}$ to maximize (1) subject to $c_t \geq 0$

$$A_{t+1} = R_t [A_t + y_t - c_t]$$

$$\lim_{T \to \infty} \beta^T u'(c_T) A_{T+1} = 0 \quad , \quad A_0 = 0$$

where R_t is the gross interest rate on financial assets between t and $t+1$, $y_t > 0$ is the consumer's labor income at t, and A_t is the consumer's financial assets at the beginning of period t.

a. Verify that $u' > 0$, $u'' < 0$.

b. It just happens that $\{R_t\}_{0}^{\infty}$ is such that the consumer want to set $c_t = y_t \quad \forall t \geq 0$

and $A_{t+1} = 0 \quad \forall t \geq 0$, where

$$y_t = \begin{cases} y_0 \delta^t & , t = 0, \ldots, T \\ y_0 \delta^T \phi^{t-T} & , t = T+1, \ldots, \infty \end{cases}$$

Find a formula for R_t at each $t \geq 0$.

Justify your formula.

Interpret your formula for R_t in terms of (i) the consumer's impatience, and (ii) the consumer's income growth.

2. A consumer has preferences ordered by

$$(1) \quad \sum_{t=0}^{T} \beta^t c_t \quad , \quad\quad , \quad 0 < \beta < 1 .$$

She maximizes (1) subject to

$$A_{t+1} = R\left[A_t - c_t\right], \quad c_t \geq 0, \quad A_{T+1} \geq 0$$

where $A_0 > 0$ and $R > 0$ is a constant gross rate of interest on financial assets. The consumer can borrow or lend.

Find the consumer's optimal plan for $\{c_t, A_{t+1}\}_{t=0}^{T}$.

Long Hint: View this as an exercise in using

- the Kuhn-Tucker conditions.

Form the Lagrangian

$$L = \sum_{t=0}^{T} \beta^t \{ c_t + \lambda_t [R(A_t - c_t) - A_{t+1}] \}$$

where $\lambda_t \geq 0$ is a Lagrange multiplier.

FONC:

c_t: $\beta^t [1 - \lambda_t R] \leq 0$, $= 0$ if $c_t > 0$, $(**)$
(because $c_t \geq 0$) $t=0,\dots,T$

A_t: $\lambda_t R - \beta^{-1} \lambda_{t-1} = 0$, $t=1,\dots,T$ $(*)$
(because A_t can be + or −)

A_{T+1}: $-\beta^T \lambda_T \leq 0$, $= 0$ if $A_{T+1} > 0$

$\Rightarrow \beta^T \lambda_T A_{T+1} = 0$

$\Rightarrow A_{T+1} = 0$ if $\lambda_T > 0$.

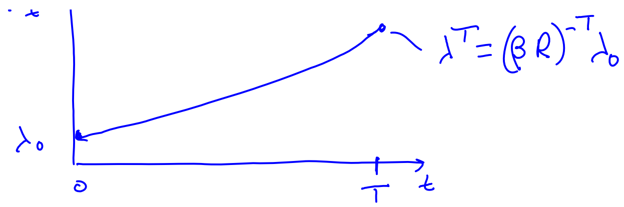
$(*) \Rightarrow \lambda_t = \beta R \lambda_{t+1}$, $t = 0, \dots, T-1$

or $\lambda_{t+1} = (\beta R)^{-1} \lambda_t$ $(**)$

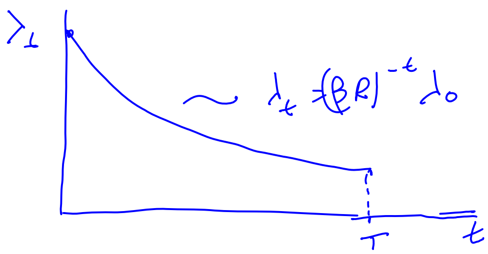
Three cases:

(i) $\beta R < 1 \Rightarrow (\beta R)^{-1} > 1$

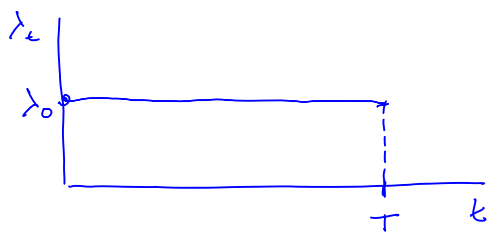
λ_1



(ii) $\beta R > 1 \Rightarrow (\beta R)^{-1} < 1 \Rightarrow$



(iii) $\beta R = 1 \Rightarrow (\beta R)^{-1} = 1 \Rightarrow \lambda_t = \lambda_0$



Now return to (***) $\Rightarrow$

$$1 - \lambda_t R \le 0, \; = 0 \text{ if } c_t > 0.$$

$$1 \le \lambda_t R, \; = \text{ if } c_t > 0$$

or

$$\frac{1}{R} \le \lambda_t, \; = \text{ if } c_t > 0.$$

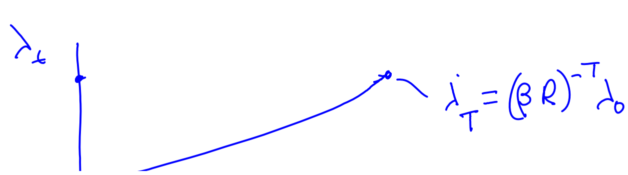
Evidently, the figures for case (i) and (ii) are

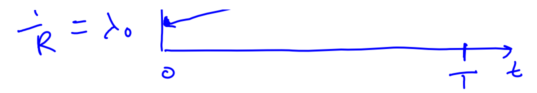
(i) $\beta R < 1 \Rightarrow (\beta R)^{-1} > 1$

$\lambda_0 = \frac{1}{R}, \; c_0 > 0$

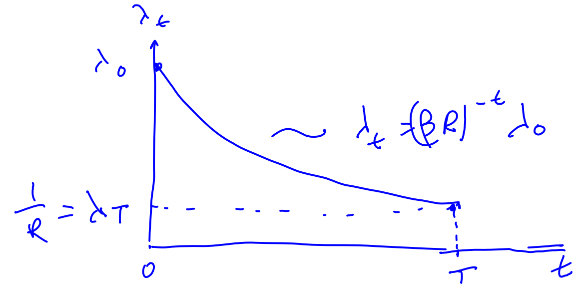
$\lambda_t > \frac{1}{R} \Rightarrow c_t = 0$

$t \geq 1$





(ii) $\beta R > 1 \Rightarrow (\beta R)^{-1} < 1 \Rightarrow$



$\lambda_T = \frac{1}{R}, \Rightarrow c_T > 0$

$\lambda_t > \frac{1}{R}, \; t = 0, \ldots, T-1$

$\Rightarrow c_t = 0, \; t < T.$

So when $(\beta R) < 1$, the consumer sets $c_0 = A_0$, $A_{t+1} = 0$, $t = 0, \ldots, T$, and $c_t = 0$ for $t = 1, \ldots, T$.

When $(\beta R) > 1$, the consumer sets $A_t = (\beta R)^t A_0$ and $c_T = A^T$, $c_t = 0$, $t = 0, 1, \ldots, T-1$.

In case (iii), $\{c_t\}_{t=0}^{T}$, $\{A_{t+1}\}_{t=0}^{T-1}$ are indeterminate, there being many solutions. Please find one of them and explain. (Hint: impatience and positive interest just offset each other in case (iii).)

Further explanation and intuition.

Let us interpret the condition

$$(*) \qquad \beta^t \cdot 1 - \beta^t \lambda_t R \le 0, \quad = 0 \text{ if } c_t > 0$$

or

$$\beta^t \le \beta^t \lambda_t R, \quad = \text{ if } c_t > 0$$

First consider the time 0 utility functional

First consider the time 0 utility functional

$$U_0 = \sum_{t=0}^{\infty} \beta^t c_t$$

evidently

$$\frac{\partial U_0}{\partial c_t} = \beta^t .$$

so the term $\beta^t \cdot 1$ in (*) is the marginal utility, as measured by U_0, of c_t.

Now, what about the term $\beta^t \lambda_t R$ in (*)?

well, if we differentiate the Lagrangian w.r.t. A_t we get $\beta^t \lambda_t R$. And at the optimum, differentiating the Lagrangian L w.r.t. A_t is "like" differentiating U_0 w.r.t. A_t (why?, because at the optimum,

$$\sum \beta^t \lambda_t \left[R(A_t + y_t - c_t) - A_{t+1} \right] = 0$$

so $L = U_0$ at the optimum.)

Thus, $\beta^t \lambda_t R$ is the marginal utility, in the sense of U_0, of A_t.

The condition (*) says

$$\underbrace{\beta^t \cdot 1}_{\substack{\text{marginal}\\\text{utility of}\\\text{cons}^n\text{ at } t}} < \underbrace{\beta^t \lambda_t R}_{\substack{\text{marginal}\\\text{utility}\\\text{of assets at } t}} \Rightarrow c_t = 0$$

cons'n at t   of assets at t

i.e., only consume when marginal utility of consumption = marginal utility of assets; otherwise, just save.

3. A consumer wants to maximize the expected value of

$$\sum_{t=0}^{\infty} \beta^t \left\{ -\frac{1}{2} (c_t - b)^2 \right\} \quad , 0 < \beta < 1$$

where b is a very big positive number representing "bliss" consumption,

subject to the budget constraints

$$F_{t+1} = \beta^{-1} \left[F_t + y_t - c_t \right] \quad , F_0 = 0$$

where F_t is financial assets at t,

β^{-1} is the gross interest rate from t to $t+1$

and y_t is labor income.

Labor income obeys

$$y_{t+1} = \bar{\delta} + \delta_1 y_t + \delta_2 y_{t-1} + \sigma_\varepsilon \varepsilon_{t+1}$$

where ε_{t+1} is an i.i.d. (independently and identically distributed) sequence of scalar random variables with

$$\varepsilon_{t+1} \sim N(0, 1)$$

(normal with mean 0 and variance 1)

The consumer's optimal decision rule satisfies

$$(*) \quad c_t = (1-\beta)\left[F_t + E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}\right]$$

here E_t is the expectation conditioned on information, known at t.

a. Find an expression for the consumer's decision rule of the form

$$c_t = (1-\beta)\left[F_t + \alpha_0 + \alpha_1 y_t + \alpha_2 y_{t-1}\right],$$

please describe how to find formulas for $\alpha_0, \alpha_1, \alpha_2$.

b. The growth rates of consumption for this consumer over the last year were as follows:

quarter	$c_t - c_{t-1}$
2008 I	1000
2008 II	0
2008 III	0
2008 IV	-4000

What can you infer from these numbers about the consumer's past and future labor income?

4. (Another exercise in Kuhn-Tucker)

A consumer lives for two periods and has multi-period utility function

$$\ln c_1 - \bar{B} h_1 + \beta [\ln c_2 - \bar{B} h_2]$$

$$0 < \beta < 1 \quad , \bar{B} > 0$$

$$c_1 \geq 0, \; c_2 \geq 0, \; h_1 \geq 0, \; h_2 \geq 0 \qquad (**)$$

c_i is consumption in period i

h_i is labor supplied in period i

The consumer's intertemporal budget constraint is

$$c_1 + \beta c_2 = w h_1 + \beta (w_2 h_2)$$

where here β^{-1} is the one period gross interest rates.

and $w_1 > 0$ is the first period real wage and $w_2 > 0$ is the second period real wage.

a. Suppose that $w_1 > w_2$. Find the

consumer's optimal plan for c_1, c_2, h_1, h_2.

(please remember the inequalities in (**)).

b. Suppose that $\omega_2 > \omega_1$. Find the consumer's optimal plan for c_1, c_2, h_1, h_2.

c. Explain why your answers in (a) and (b) differ.

Practice problems 2

Friday, February 26, 2010

8:31 PM

Here are some sample problems for practice.

1. "Equalizing differences."

A person who has just graduated from high school can enter the work force now (at time t) and earn $w^h_{t+j} = \delta_h^j w_t$, $j = 0, 1, \ldots, T$; where $\delta_h > 0$. (He/she must retire after $t+T+1$.) The present value of his life time earnings if he goes to work right away is

$$PV^h_t = \sum_{j=0}^{T} \beta^j w^h_{t+j} . \qquad \beta = \frac{1}{1+r},\ r>0$$

If the high school graduate instead enters college (then graduate school) and starts working in period $t+k$, he/she earns w^c_s in period $s = t+k, t+k+1, \ldots, t+T$

where $w^c_{t+j} = \delta_c w^c_t$. , $\delta_c > 0$

The present value of going to college during $t, t+1, \ldots, t+k-1$ (during which he/she earns nothing) is

$$PV^c_t = \sum_{j=k}^{T} \beta^j w^c_{t+j}$$

a. Find a formula for PV^h_t as a function of $\beta, w^h_t, T, \delta_h$

$\beta, w_t^h, T, \delta_h$

b. Find a formula for PV_t^c as a function of $\beta, w_t^c, T, \delta_c$, and k.

c. Given $\beta, w_t^h, T, \delta_h, \delta_c, w_t^h$, find a value of w_t^c that equates the present values of going to college and not going to college.

2. Equalizing differences, again.

A truck driver works every period $t \geq 0$ at wage $w_{dt} = \delta_d^t w_{d0}$, $0 < \delta_d < \frac{1}{\beta}$, $\beta = \frac{1}{1+r}$, $r > 0$

A construction worker works only every even period at wage

$$w_{ct} = \begin{cases} \delta_c^t w_{c0} & \text{if } t = 0, 2, 4, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Each worker lives forever.

a. Compute the present value of the wages of a truck driver

$$PV_{d,0} = \sum_{t=0}^{\infty} \beta^t w_{dt}$$

b. Compute the present value of the wages of a construction worker

$$PV_{c,0} = w_{c,0} + \beta^2 w_{c,2} + \beta^4 w_{c,4} + \ldots$$

c. Find a formula for $w_{i,0}$ that equates the present value of a construction worker to the present value of a truck driver.

3. Consider an economy consisting of two people, each of whom orders consumption streams according to

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i) \quad , i=1,2, \; u'>0, u''<0$$

$$0<\beta<1$$

Assume that $u(c) = \ln c$

Person 1 has income stream

$y_t^1 = \delta^t \quad , t \geq 0 \quad , \quad \delta \varepsilon (0,1).$

Person 2 has income stream

$y_t^2 = 1 - \delta^t \quad , t \geq 0 , \quad \delta \varepsilon (0,1) .$

Each consumer faces the sequence of budget constraints

$$A_{t+1}^i = R_t \left[A_t^i + y_t^i - c_t^i \right],$$

$$A_0^i = 0 \qquad i=1,2$$

$A_{t+1}^i \quad , \; i=1,2, \;\; t \geq 0,$ can be either positive or negative but must obey

$$\lim_{T \to \infty} \beta^T u'(c_T) A_T = 0 .$$

a. Assume that $R_t = \beta^{-1}$. (There is an outside lender who supplies loans at this rate). Find optimal consumption, asset plans $(c_t^i, A_{t+1}^i)_{t=0}^{\infty}$ for $i = 1, 2$.

- From now on assume that the consumer cannot borrow so that $\underline{A_{t+1}^i \geq 0}$ for all $t \geq 0$ for $i = 1, 2$.

b. Now relax the part (a) assumption that $R_t = \beta^{-1}$. Instead, find a sequence of gross interest rates $\{R_t\}_{t=1}^{\infty}$ that makes consumer 1 want to consume his endowment each period so that

$$c_t^1 = \delta^t \quad , t \geq 0$$

Hint: get the FONC's for the consumer's problem, set $c_t^1 = \delta^t, t \geq 0$, stares, and solve for $\{R_t\}_{t=0}^{\infty}$

c. At the $\{R_t\}_{t=0}^{\infty}$ that you found in part b, verify that consumer 2 will not want to lend.

d. Assume that $\{R_t\}_{t=1}^{\infty}$ that you found in part (b)

d. Argue that $\{\bar{R}_t\}_{t=1}^{\infty}$ that you found in part (b) is a competitive equilibrium $\{R_t\}_{t=1}^{\infty}$ for the closed economy in which $c_t^1 + c_t^2 = y_t^1 + y_t^2$ and in which there is no outside lender.

e. Now relax the assumption that $R_t = \beta^{-1}$. Instead, find a sequence of gross interest rates $\{R_t\}_{t=1}^{\infty}$ that makes consumer 2 want to consume his endowment each period so that

$c_t^2 = 1 - \delta^t, \; t \geq 0$

Is R_t higher or lower than what you found in part b? At this R_t, will consumer 1 want to borrow or lend?

Practice problems 3

Friday, February 26, 2010
11:11 AM

1. Consumer $i=1,2$ orders consumption streams $\{c_t^i\}_{t=0}^{\infty}$ according to

$$(1) \quad \sum_{\tau=0}^{\infty} \beta^t u(c_t) \quad \text{where } u'>0,\ u''<0$$

$$c_t \geq 0, \quad 0<\beta<1$$

The consumer has exogenous labor income y_t^i and must pay a constant lump sum tax τ (the same for both $i=1,2$) each period.

The income of consumer 1 is

$$y_t^1 = \begin{cases} \bar{Y}_0, & 0 \le t \le T \\ \bar{Y}_1, & T+1 \le t \end{cases} \qquad \boxed{\text{, where } \bar{Y}_1 > \bar{Y}_0}$$

The income of consumer 2 is

$$y_t^2 = \begin{cases} \bar{Y}_1, & 0 \le t \le T \\ \bar{Y}_0, & T+1 \le t \end{cases} \qquad , t=0,1,2,\ldots$$

The tax rate $\tau_t = \bar{\tau}$ for all $t \geq 0$ where $0<\bar{\tau}<\bar{Y}_0<\bar{Y}_1$. (Please notice this restriction on $\bar{\tau}$.)

Consumer i faces a sequence of budget constraints

$$A_{t+1}^i = \beta^{-1}\left[A_t^i + y_t^i - \tau_t - c_t^i\right]$$

$$c_t^i \geq 0 \quad , A_0^i = 0$$

$$A_{t+1}^i \geq 0 . \quad t = 0, 1, 2, \ldots$$

Notice that $A_{t+1}^i \geq 0$ is a no-borrowing constraint.

Notice that both consumers pay the same tax $\tau_t = \bar{\tau}$ each period.

a. Please give formulas for the optimal consumption paths of consumers $i = 1, 2$ for each t, $\{c_t^i\}_{t=0}^{\infty}$, $i = 1, 2$.

b. Now suppose that the tax for each person is changed to

$$\tau_t = \begin{cases} 0 , & 0 \leq t \leq T \\ \tau^* , & T+1 \leq t \end{cases}$$

where $$\tau^* = \frac{\bar{\tau}}{\beta^{T+1}} \qquad (+)$$

Find optimal levels of $\{c_t^i\}_{t=0}^{\infty}$, $i = 1, 2$.

c. Compare your solutions from parts (a) and (b) and discuss why they differ (if they do differ).

d. Interpret condition (†).

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2. A consumer wants to choose $\{c_t, A_{t+1}\}_{t=0}^{\infty}$

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , 0 < \beta < 1, \ u' > 0, \ u'' < 0$$

subject to the sequence of budget constraints

$$F_{t+1} = \beta^{-1} [\underbrace{F_t}_{\text{financial assets}} + \underbrace{y_t}_{\text{endowment}} - \underbrace{c_t}_{\text{consumption}}] , F_0 = 0 .$$

$R = \beta^{-1}$

$R = (1 + \underbrace{r}_{\text{net interest rate}})$

$F_0 = 0$ is given.

we also require

$$\lim_{T \to \infty} \beta^T u'(c_T) F_T = 0$$

the consumer can borrow <u>and</u> lend, so

F_{t+1} can be + or −.

There is no uncertainty, so the consumer has perfect foresight about $\{y_t\}_{t=0}^{\infty}$.

Consider the endowment path

$$y_t = \begin{cases} y_0 \delta^t & , t = 0, 1, \dots, T \\ y_0 \delta^T & , \text{for } t \geq T \end{cases}$$

where $1 < \delta < \frac{1}{\beta}$.

a. Graph the endowment path

b. Compute the optimal path for $\{c_t\}_{t=0}^{\infty}$ – i.e. give a formula for c_t.

c. Compute F_t for $t \geq T$.

Now assume that the consumer cannot borrow but can lend. Suppose $\delta < 1$.

d. Compute the optimal path for $\{c_t\}_{t=0}^{\infty}$.

e. Compute F_t for $t \geq T$.

Now assume that the consumer cannot borrow but can lend and that $\delta > 1$.

f. Compute the optimal path for $\{c_t\}_{t=0}^{\infty}$.

g. Compute F_t for $t \geq T$.

3. A consumer chooses $\{c_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , u' > 0, u'' < 0$$

subject to $\sum_{t=0}^{\infty} q_t^0 c_t \leq \sum_{t=0}^{\infty} q_t^0 y_t$

where $\{y_t\}_{t=0}^{\infty}$ is an exogenous endowment

and q_t^0 = the price of one unit of consumption at time t, measured in units of time 0 consumption,

so

$$q_0^0 = 1 \qquad \text{(a normalization)}$$

Assume that

$$u(c) = \frac{1}{1-\gamma} c^{1-\gamma} \quad , \gamma > 0 \quad , \gamma \neq 1$$

$$= \ln c \qquad \text{if } \gamma = 1$$

(Remember that $\quad q_t^0 \equiv \dfrac{1}{(1+\rho_{0t})^t}$

where ρ_{0t} is the yield to maturity on a t-period zero-coupon bond)

Consider a "Lucas model" where there is a representative consumer and prices $\{q_t^0\}_{t=0}^{\infty}$ must adjust to induce the consumer to set $c_t = y_t \ \forall t \geq 0$.

a. Suppose $\quad y_t = y_0 \delta^t \quad , \quad 0 < \delta < \frac{1}{\beta}$.

Compute q_t^0. Compute ρ_{0t}.

b. In the special case of $\gamma = 1$ (log utility),

b. In the special case of $\gamma = 1$ (log utility), compute q_{0t}^{0} and ρ_{0t}.

c. When $\delta = 1$, compute q_{0t}^{0}.

d. Is the one-period interest rate ρ_{01} "high" or "low" when $\delta > 1$?

e. Is the one-period interest rate ρ_{01} "high" or "low" when $\delta < 1$?

Practice problems 4

Saturday, February 27, 2010

3:36 PM

1. A consumer receives an exogenous endowment $\{y_t\}_{t=0}^{\infty}$ that evolves according to the law of motion

$$y_{t+1} = y_t + \sigma_t \varepsilon_{t+1}$$

where $y_0 > 0$ is given, $\{\varepsilon_{t+1}\}_{t=0}^{\infty}$ is an independently and identically distributed normal random variable with mean 0 and variance 1, and

$$\sigma_t = \begin{cases} \bar{\sigma} > 0 & \text{for } t = 0, 1 \\ \sigma_t = 0 & \text{for } t \geq 2 \end{cases}$$

At time t, the consumer knows the history $y^t = (y_t, y_{t-1}, \ldots, y_0)$ but not the future values of y_s, $s > t$.

The consumer chooses $\{c_t, F_{t+1}\}_{t=0}^{\infty}$ subject to the sequence of budget constraints

$$F_{t+1} = \beta^{-1}\left[F_t + y_t - c_t\right]$$

$$F_0 \geq 0 \text{ given}$$

where $\beta^{-1} = R = (1+r)$ is the gross one-period interest rate on risk-free borrowing or lending. The consumer can borrow or lend, but his/her plan must satisfy

$$E_0 \beta^T u'(c_T) F_T = 0$$

(no Ponzi scheme)

The consumer wants to choose $\{c_t, F_{t+1}\}_{t=0}^{\infty}$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

mathematical expectation given time 0 information

where $u(c_t) = \alpha_0 - \frac{\alpha_1}{2} c_t^2$, $\alpha_0 > 0$, $\alpha_1 > 0$

(quadratic utility)

The consumer's first-order condition with respect to $c_{t+1}, F_{t+1} \Rightarrow$

$$u'(c_t) = E_t u'(c_{t+1})$$

which in turn implies, through quadratic utility,

$E_t \, c_{t+1} = c_t$

expectation conditional on time t information

This condition and the budget constraints leads the consumer to set c_t so that

$$c_t = (1-\beta)\left[F_t + E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$$

financial wealth

"human wealth

a. Draw 5 "sample paths" for $\{y_t\}_{t=0}^{5}$

b. Find a "consumption function" of the form

$$c_t = \delta_0 + \delta_1 F_t + \phi_0 y_t + \phi_1 y_{t-1}$$

Please give formulas for $\delta_0, \delta_1, \phi_0, \phi_1$ in terms of β and the other fundamental parameters of the model.

c. Please draw 5 sample paths of $\{c_t\}_{t=0}^{5}$

Parts d. and e. are extra credit:

d. Find a formula for $\{F_t\}$ of the form

$$F_{t+1} = F_t + \alpha_t \varepsilon_{t+1}$$

and give a formula for α_t, $t \geq 0$

e. Find a formula for $\{c_t\}$ of the form

$$c_{t+1} = c_t + \eta_t \varepsilon_{t+1}$$

and give a formula for η_t, $t \geq 0$.

Note: we allow α_t and η_t to depend on t, so you have to find an α_t and η_t for each $t \geq 0$.

2. A consumer's endowment $\{y_t\}_{t=0}^{\infty}$ follows the law of motion

$$y_{t+1} = \rho\, y_t + \sigma \varepsilon_{t+1}, \quad \sigma > 0$$

$$y_0 > 0 \text{ given}, \quad |\rho| \leq 1$$

$$\varepsilon_{t+1} \sim N(0,1) \quad (\text{Gaussian i.i.d.})$$

the consumer knows $y^t = y_t, y_{t-1}, \dots, y_0$ at time $t \geq 0$.

The consumer sets $\{c_t, F_{t+1}\}_{t=0}^{\infty}$ according to

$$c_t = (1-\beta)\left[F_t + E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right],$$

$$F_{t+1} = \beta^{-1}\left[F_t + y_t - c_t \right]$$

$$0 < \beta < 1$$

a. Find an expression for c_t of the form

$$c_t = \delta_0 + \delta_1 F_t + \phi_0 y_t + \phi_1 y_{t-1}$$

giving formulas for ϕ_0 and ϕ_1 as functions of β and ρ and σ.

b. For $\beta = .95$, $\sigma = .05$, please fill in the following table:

ρ	ϕ_0	ϕ_1
-1	$\boxed{}$	$\boxed{}$
$-.5$	$\boxed{}$	$\boxed{}$
0	$\boxed{}$	$\boxed{}$
$.5$	$\boxed{}$	$\boxed{}$
1	$\boxed{}$	$\boxed{}$

that is, fill in the missing entries for ϕ_0 and ϕ_1.

Practice problems 5a

Monday, April 12, 2010
9:05 AM

The first several problems assume the following environment. A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t \log c_t \quad , \quad 0 < \beta < 1$$

$$\beta = \frac{1}{1+\rho} \quad , \rho > 0$$

where c_t is consumption per worker.

The technology is

$$y_t = f(k_t) = z k_t^{\alpha} \quad , \quad 0 < \alpha < 1, \; z > 0$$

where y_t is output per unit labor and k_t is capital per unit labor

$$y_t = c_t + x_t + g_t$$

x_t = gross investment per unit labor

g_t = government expenditures per unit of labor

$$k_{t+1} = (1-\delta) k_t + x_t \quad , \quad 0 < \delta < 1$$

Assume a competitive equilibrium with a price system $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ and a government policy $\{g_t, \tau_{ht}\}_{t=0}^{\infty}$.

1. Assume that the government finances its expenditures by levying lump sum taxes. There are no

distorting taxes. Assume that at time 0, the economy starts out with a capital per unit labor k_0 that equals the steady state value appropriate for an economy in which g_t had been zero forever.

a. Find a formula for the steady state capital stock when $g_t = 0 \ \forall t$.

b. Compare the steady state capital labor ratio $\bar{k}$ in the competitive equilibrium with the capital labor ratio $\tilde{k}$ that maximizes steady state consumption per capita; i.e., $\tilde{k}$ solves

$$\tilde{c} = \max_k f(k) - \delta k$$

Is $\tilde{k}$ greater than or less than $\bar{k}$? If they differ, why? Is $\tilde{c}$ greater or less than $\bar{c} = f(\bar{k}) - \delta \bar{k}$? Explain why.

c. Now assume that at time 0, g_t suddenly jumps to the value $g = \frac{1}{2}\bar{c}$ where $\bar{c}$ is the value of consumption per capita in the initial steady state in which g was zero forever. Starting from $k_0 = \bar{k}$ for the old $g = 0$ steady state, find the time paths of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ associated with the new path

$g_t = g > 0$ for government expenditures per capita.

Also show the time path for $R_{t+1} \equiv (1-\delta) + f'(k_{t+1})$.

Explain why the new time paths are as they are.

2. Trade and growth

Consider the problem of a planner in a small economy. When the economy is <u>closed to trade</u>, the planner chooses $\{c_t, k_{t+1}\}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad 0 < \beta < 1 \, , \quad \beta = \frac{1}{1+\rho}$$

$$\rho > 0$$

subject to

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \quad , \quad \delta \in (0,1)$$

where $u(c_t) = \dfrac{c_t^{1-\gamma}}{1-\gamma} \quad , \gamma > 0$

$$f(k_t) = z k_t^{\alpha} \quad , \quad 0 < \alpha < 1$$

Let $\bar{k}$ be the steady state value of k_t under the optimal plan.

a. Find a formula for $\bar{k}$.

b. Assume that $k_0 < \bar{k}$. Describe time paths for $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ and $R_{t+1} = (1-\delta) + f'(k_{t+1})$.

c. What is the steady state value of R_{t+1}?

d. Is R_{t+1} less or greater than its steady state value when $k_{t+1} < \bar{k}$?

Part II.

Now assume that the economy is open to international trade in capital and financial assets. Assume that there is a fixed world gross rate of return $R = \beta^{-1}$ at which the planner can borrow or lend.

The planner can use the proceeds of borrowing to purchase goods on the international market. These goods can be used to augment capital or to consume.

Let k_0^- be the level of initial capital $(k_0^- < \bar{k}_0)$ just before the country opens up to trade just before time 0. Let k_0^- be the same initial capital per capita $k_0^- < \bar{k}$ studied in parts a. – d.

At time $t = -1$, after k_0^- was set, trade opens up. At $t = -1$, the planner can issue IOU'S or bonds in amount B_{-1} and use the proceeds to purchase capital, thereby setting

$$k_0 = k_0^- + B_{-1}$$

where B_{-1} is denominated in time -1 consumption goods. The bonds are one-period in duration and bear the constant world gross interest rate $R = \beta^{-1}$.

For $t \geq 0$, the planner faces the constraints

$$c_t + k_{t+1} + R B_{t-1} = f(k_t) + (1-\delta) k_t + B_t ,$$

Here $R B_{t-1}$ is the interest and principal on bonds issued in $t-1$, and B_t is the amount of one-period bonds issued at t.

The planning problem is now to choose $\{c_t, k_t, B_t\}_{t=0}^{\infty}$ and B_{-1}, subject to k_0^- given. (Notice that k_0 is now a choice variable and that k_0^- is an initial condition.)

Solve the planning problem in this small open economy and compare the solution to the solution in the closed economy.

<u>Hint</u>: Form the Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t \Big\{ u(c_t)$$

$$+ \lambda_t \left[f(k_t) + B_t - c_t - k_{t+1} + (1-\delta) k_t - R B_{t-1} \right] \Big\}$$

$$+ \phi \left[k_0^- + B_{-1} - k_0 \right]$$

where $\{\lambda_t\}_{t=0}^{\infty}$, ϕ are Lagrange multipliers.

FONC:

$c_t: \quad u'(c_t) - \lambda_t = 0 \qquad , t = 0, 1, \ldots$

$k_t: \quad \lambda_t [f'(k_t) + (1-\delta)] - \beta^{-1}\lambda_{t-1} = 0, \quad t = 1, 2, \ldots$

$k_0: \quad \lambda_0 [f'(k_0) + (1-\delta)] - \phi = 0$

$B_t: \quad \lambda_t - \lambda_{t+1} R\beta = 0, \quad t = 0, 1, \ldots$

$B_{-1}: \quad \phi - \lambda_0 R = 0$

$\Rightarrow \quad \phi = \lambda_0 R$

$\lambda_{t+1} = \lambda_t \quad , t \geq 0 \Rightarrow \lambda_t = \lambda \qquad \forall t$

$\lambda_0 [f'(k_0) + (1-\delta)] = \lambda_0 R$

or

$f'(k_0) + (1-\delta) = (1+\rho)$

or

$f'(k_0) = \rho + \delta$

$\Rightarrow k_0 = \bar{k}$ for optimal steady state in the closed economy.

also $\lambda_t = \lambda \quad \forall t \Rightarrow$

$f'(k_t) + (1-\delta) = 1 + \rho \quad \Rightarrow$

$f'(k_t) = \rho + \delta \qquad \forall t$

$\Rightarrow k_t = \bar{k} \qquad \forall t.$

Thus the soln is

$B_{-1} = \bar{k} - k_0 \qquad (k_0 = \bar{k})$

$B_t = B_{t-1} \qquad \forall t \geq 0$

roll over the debt

$$B = (\bar{k} - k_0^-)$$

Consumption c_t:

$$c_t + b_{t+1} + R B_{t-1} = f(k_t) + (1-\delta) k_t + B_t$$

with $k_t = \bar{k}$ and $B_t = \bar{B}$ $\Rightarrow$

$$\bar{c} + \delta\bar{k} + (R-1) B = f(\bar{k})$$

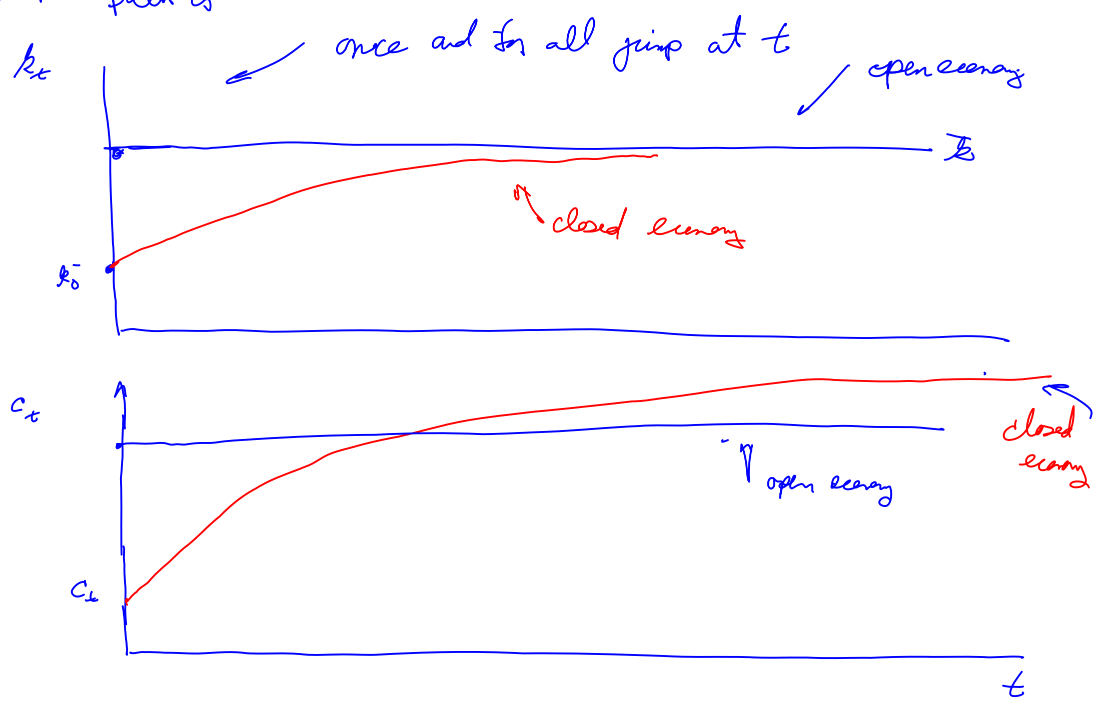
or

$$\bar{c} = f(\bar{k}) - \delta\bar{k} - (R-1)[\bar{k} - k_0^-]$$

or

$$\bar{c} = f(\bar{k}) - \delta\bar{k} - \underbrace{\rho \underbrace{[\bar{k} - k_0^-]}_{\text{initial borrowing}}}_{\text{net interest on initial borrowing.}}$$

So the path is



difference in consumption asymptotically closed economy: $\bar{c} = f(\bar{k}) - \delta \bar{k}$

open economy: $\bar{c} = f(\bar{k}) - \delta \bar{k} - \rho [\bar{k} - k_0]$

Question: Is welfare $\sum_{t=0}^{\infty} \beta^t u(c_t)$ higher in the "open" or "closed" economy.

Hint: (The planner is free to set $B_t = 0$ $\forall\, t \geq -1$ but chooses not to)

Practice problems 5

Sunday, April 11, 2010
3:03 PM

The next several problems assume the following environment. A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad 0 < \beta < 1$$

$$\beta = \frac{1}{1+\rho} \quad , \rho > 0$$

where c_t is consumption per worker and

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad , \gamma > 0 \quad , \gamma = 1$$

$$u(c) = \log c \quad \text{if } \gamma = 1$$

The technology is

$$y_t = f(k_t) = z k_t^{\alpha} \quad , \quad 0 < \alpha < 1, \; z > 0$$

where y_t is output per unit labor and k_t is capital per unit labor

$$y_t = c_t + x_t + g_t$$

x_t = gross investment per unit labor

g_t = government expenditures per unit of labor

$$k_{t+1} = (1-\delta) k_t + x_t \quad , \quad 0 < \delta < 1$$

The government finances its expenditures by levying some combination of a flat rate tax τ_{ct} on the value of consumption goods purchased at t, a flat rate tax of τ_{nt} on the value of labor earnings at t, a flat rate tax τ_{kt} on earnings from capital at t, and a lump sum tax of τ_{ht} in time t consumption goods per worker at time t.

Let $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ be a price system.

3. Consider an economy of the type described above in which $g_t = \bar{g} > 0$ for all $t \geq 0$ and in which initially the government finances all expenditures by lump sum taxes.

a. Find a formula for the steady state capital labor ratio k_t for this economy. Find formulas for the steady state levels of c_t and

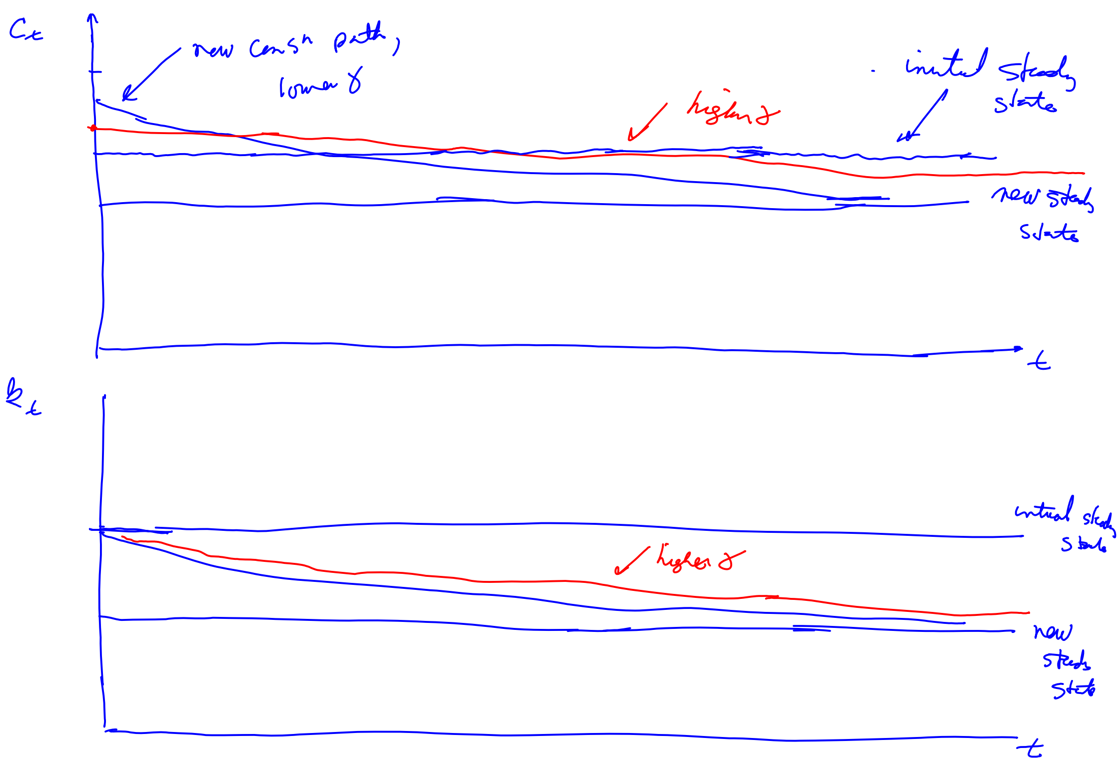
$$R_t = [(1-\delta) + (1-\tau_{k\,t+1}) f'(k_{t+1})].$$

b. Now suppose that starting from $k_0 = \bar{k} =$ steady state that you computed in part a, the government suddenly increases the tax on earnings on capital to a constant level $\tau_k > 0$. The government adjusts lump sum taxes to keep the government budget balanced. Describe competitive equilibrium time paths for c_t, k_{t+1}, R_t and their relationship to corresponding values in the old steady state that you described in part a.

c. Describe how the shapes of the paths that you found in part b. depend on the curvature parameter γ in the utility function $u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$. Higher values of γ imply higher curvature and more aversion to consumption paths that fluctuate.

Higher values of γ imply that the consumer values smooth consumption paths even more.

Thus, here is what we expect:



—— higher γ adjustment path

—— lower γ adjustment path

Things look this way because of the consumer's

Things look this way because of the consumer's attitude about consumption variability. The interest rate and k_t adjust to make this happen.

Remark: formulas in section 11.7.3 of the typed chapter verifies this by giving a formula for
λ as a function of γ where
λ was the decay rates discussed in class.

d. Starting from the steady state $\bar{k}$ that you computed in part a, now consider a situation in which the government announces at time 0 that starting in period 10 the tax on earnings from capital τ_k will rise permanently to $\bar{\tau}_k > 0$. The government adjusts its lump sum taxes to balance it's budget.

i) find new steady values for k_t, c_t, R_t.

ii) Describe the shapes of the transition paths from the initial steady state to the new one for k_t, R_t, c_t.

iii) decide how the shapes of these transition paths depend on the curvature parameter γ in the utility function $u(c)$.

Hint: when γ is bigger, consumers more strongly prefer smoother consumption paths

4. Trade and growth, version II

Consider the problem of a planner in a small economy. When the economy is closed to trade, the planner chooses $\{c_t, k_{t+1}\}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad 0 < \beta < 1 \ , \ \beta = \frac{1}{1+\rho}$$

$$\rho > 0$$

subject to

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \quad , \quad \delta \in (0,1)$$

where $u(c_t) = \dfrac{c_t^{1-\gamma}}{1-\gamma} \quad , \gamma > 0$

$$f(k_t) = z k_t^{\alpha} \quad , \quad 0 < \alpha < 1$$

Let $\bar{k}$ be the steady state value of k_t under the optimal plan.

a. Find a formula for $\bar{k}$.

b. Assume that $k_0 > \bar{k}$. Describe time paths for $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ and $R_{t+1} = (1-\delta) + f'(k_{t+1})$.

c. What is the steady state value of R_{t+1}?

d. Is R_{t+1} less or greater than its steady state value when $k_{t+1} > \bar{k}$?

Part II.

e. Now assume that the economy is open to international trade in capital and financial assets. Assume that there is a fixed world gross rate of return $R = \beta^{-1}$ at which the planner can borrow or lend. In particular, the planner is free to use the following plan. The planner can sell all of its capital k_0^- and simply consume the interest payments.

Let k_0^- be the level of initial capital $(k_0^- > \bar{k})$ just before the country opens up to trade just before time 0. Let k_0^- be the same initial capital per capita $k_0^- > \bar{k}$ studied in parts a. – d.

At time $t=-1$, after k_0^- was set, trade opens up. At $t=-1$, the planner sells k_0^- in exchange for IOU's from the rest of the world in the amount $A_{-1} = k_0^-$.

The bonds A_{-1} are one-period in duration and bear the constant world gross interest rate $R = \beta^{-1}$.

After the sale of $k_0^{-1} = A_{-1}$, the planner has zero capital and so shuts down the technology. Instead, the planner uses the asset market to smooth consumption. The planner thus chooses (A_{t+1}, c_t) to maximize

$$\sum^{\infty} \beta^t u(c_t)$$

$\beta = \frac{1}{1+\rho}$

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \qquad \beta = \frac{1}{1+\rho}$$

$$c_t + \beta A_{t+1} = A_t , \qquad , A_0 = \bar{k}_0$$

or

$$A_{t+1} = \beta^{-1}[A_t - c_t]$$

Form the Lagrangian:

$$\sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t \left[\beta^{-1}(A_t - c_t) - A_{t+1} \right] \right\}$$

FONC:

$$c_t : \quad u'(c_t) - \lambda_t \beta^{-1} = 0$$

$$A_t : \; -\beta^{-1} \lambda_{t-1} + \beta^{-1} \lambda_t = 0$$

$$\lim_{T\to\infty} \beta^T \lambda_T A_{T+1} = 0$$

$$\Rightarrow \quad \lambda_t = \lambda_0 \quad \forall t$$

$$u'(c_t) = u'(c_0) \quad \forall t$$

$$c_t = c_0 \quad \forall t .$$

then roll over assets & just consume interest:

$$A_{t+1} = \beta^{-1}[A_t - c_t] \quad \Rightarrow$$

$$\bar{A} = \beta^{-1}[\bar{A} - c_0] \quad \Rightarrow$$

$$(1 - \beta^{-1})\bar{A} = -\beta^{-1} c_0$$

or

$$\beta^{-1} c_0 = (\beta^{-1} - 1)\bar{A}$$

$$c_0 = (1-\beta)\bar{A} \qquad 1 - \beta = 1 - \frac{1}{1+\rho} = \frac{\rho}{1+\rho} = \beta\rho$$

$$c_0 = \rho\beta A \quad , \quad \bar{A} = A_0 = k_0^{-1}$$

Thus, the optimal path for consumption is

$$c_t = \rho \bar{A} = \rho k_0^{-1}$$

f. Compare the time path of (c_t, k_t, R_t) that you computed in parts a, b, c, d with "no trade" with the part e one "with trade" in which the government "shuts down the home technology" and lives entirely from returns on foreign assets. Can you say which path the representative consumer would prefer?

- Hint: I think it could go either way for reasons that might be easier to appreciate after you look at part g.

g. Now return to the economy in part e with $k_0^- > \bar{k}$ from part d. Assume that the planner is free to borrow or lend capital at the fixed gross interest rate of $\beta^{-1} = 1+\rho$, as before. But now assume the planner chooses the optimal amount of k_0^{-1} to sell off and so not not necessarily sell off the entire k_0^{-1} and possibly continues to operate the technology.

i) Find the solution of the planning problem.

Hint: you might want to review your solution to problem 2 of practice problems 4.

ii) Explain why it is optimal not to shut down the technology. Hint: Starting from having shut the technology down, think of putting a small amount ε of capital into the technology – this earns $z\varepsilon^{\alpha}$ and costs $\rho\varepsilon$ in terms of foregone interest. because $0 < \alpha < 1$ $\rho\varepsilon < z\varepsilon^{\alpha}$ for small ε. – The technology is very productive for small ε – so use it.

3. Term structure of interest rates.

This problem assumes the following environment. A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

$$\beta = \frac{1}{1+\rho}, \quad \rho > 0$$

where c_t is consumption per worker and

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad \gamma = 1$$

$$u(c) = \log c \quad \text{if } \gamma = 1$$

$u(c) = \log c \quad$ if $\gamma = 1$

The technology is .

$$y_t = f(k_t) = z k_t^{\alpha}, \quad 0 < \alpha < 1, \; z > 0$$

where y_t is output per unit labor and

k_t is capital per unit labor

$$y_t = c_t + x_t + g_t$$

x_t = gross investment per unit labor

g_t = government expenditures per unit of labor

$$k_{t+1} = (1-\delta) k_t + x_t, \quad 0 < \delta < 1$$

c_t = consumption per worker

The government finances its expenditures by levying some combination of a flat rate tax τ_{ct} on the value of consumption goods purchased at t, a flat rate tax of τ_{nt} on the value of labor earnings at t, a flat rate tax τ_{kt} on earnings from capital at t, and a lump sum tax of τ_{ht} in time t consumption goods per worker at time t.

Let $\{q_t^0, r_t^0, w_t^0\}_{t=0}^{\infty}$ be a price system.

Here the superscript 0 means the time 0 price system.

a. Recall that we can represent

$$q_t^0 = q_0^0 \, m_{0,1} \, m_{1,2} \cdots m_{t-1,t}$$

where $m_{t-1,t} = \dfrac{q_t^0}{\;}$

where $m_{t-1,t} = \dfrac{q_t^0}{q_{t-1}^0}$

and $m_{t-1,t} \equiv \exp(-r_{t-1,t}) \approx \dfrac{1}{1+r_{t-1,t}}$

Further recall that the t period long yield is satisfies

$$q_t^0 = \exp(-t\, r_{0,t})$$

or

$$r_{0,t} = \frac{1}{t}\left[r_{0,1} + r_{1,2} + \cdots + r_{t-1,t} \right]$$

Now suppose that at $t=0$, $k_0 = \bar{k}$, where $\bar{k}$ is the steady state appropriate for an economy with constant $g_t = \bar{g} > 0$ and all expenditures financed by lump sum taxes. Find q_t^0 for this economy.

b. Plot $r_{t-1,t}$ for this economy for $t = 1, 2, \ldots, 10$.

c. Plot $r_{0,t}$ for this economy for $t = 1, \ldots, 10$. (this is what Bloomberg plots)

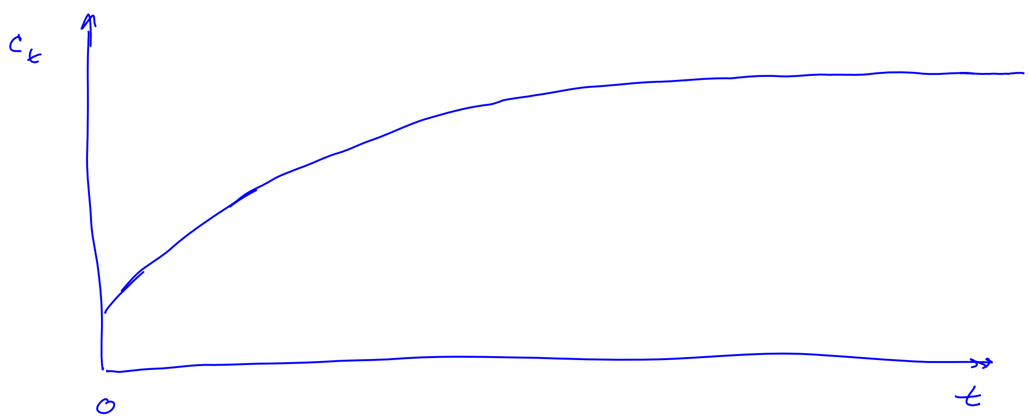
d. Now assume that at time 0, starting from $k_0 = \bar{k}$ for the steady state you computed in part a, the government unexpectedly and permanently raises the tax rate on income from capital $\tau_{kt} = \tau_k > 0$ to a positive rate.

i) Plot $r_{t-1,t}$ for $t=1,...,10$ for this economy. Explain how you got this outcome.

ii) Plot $r_{0,t}$ for $t=1,...,10$ for this economy. Explain how you got this outcome.

4. The following problem assumes the same economic environment as the previous problem - i.e., the "growth model" with fiscal policy.

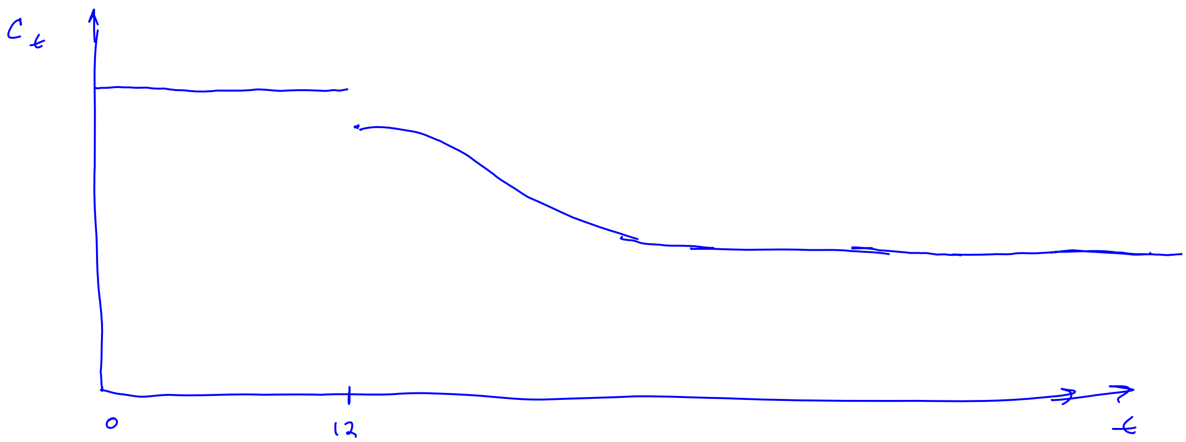
Suppose that you observe the following path for consumption per capita



Say what you can about likely behavior over time of k_t, R_t. $[(1-\delta)+(1-\tau_{kt})f'(k_t)]$, g_t, and τ_{kt}. (You are free to make up any story you like that is consistent with the model.)

5. Assume the same economic environment as in the previous

two problems. Assume that someone has observed the following time path for c_t:



a. Describe a consistent set of assumptions about fiscal policy that explains this time path for c_t. In doing so, please distinguish carefully between changes in taxes and expenditures that are <u>foreseen</u> versus <u>unforeseen</u>.

b. Describe what is happening to k_t, R_t, and g_t over time. Make whatever assumptions you need to get a complete but consistent story — here "consistent" means "consistent with the economic environment we have assumed."

Practice problems 6

Sunday, April 11, 2010
3:19 PM

1. Consider the optimal growth model with a representative consumer with preferences

$$\sum_{t=0}^{\infty} \beta^t c_t \,, \qquad 0 < \beta < 1 \,,$$

with technology

$$c_t + k_{t+1} = f(k_t) + (1-\delta) k_t \,, \qquad \delta \in (0,1)$$

$$c_t \geq 0 \,, \quad k_0 > 0 \text{ given}$$

$$f' > 0, \; f'' < 0 \,, \quad \lim_{k \to 0} f'(k) = +\infty \,, \; \lim_{k \to +\infty} f'(k) = 0$$

Here c_t = Consumption / person

k_t = capital / worker

Let $\bar{k}$ solve $f'(\bar{k}) = \rho + \delta$.

a. For k_0 given formulate and solve the optimal planning problem.

b. For $k_0 > \bar{k}$, describe the optimal time path of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$.

c. For $k_0 < \bar{k}$, describe the optimal time path of (c_t, k_{t+1})

d. Let the saving rate s_t be defined as the s_t that satisfies

$$k_{t+1} = s_t f(k_t) + (1-\delta) k_t$$

(in the Solow model, we assumed that $s_t = s$, an exogenous constant.) Here s_t in general varies along a $\{c_t, k_{t+1}\}$. Say what you can about how s_t varies as a function of k_t.

Hint: Form the Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t \{ c_t + \lambda_t [f(k_t) + (1-\delta) k_t - c_t - k_{t+1}] \}$$

FONC:

(1) c_t: $1 - \lambda_t \le 0$, $= 0$ if $c_t > 0$, $t \ge 0$

(2) k_t: $\lambda_t [f'(k_t) + (1-\delta)] - \beta^{-1} \lambda_{t-1} = 0$, $t \ge 1$

(1) $\Rightarrow$ $\lambda_t \ge 1$, $\lambda_t = 1 \Rightarrow c_t > 0$

$\lambda_t > 1 \Rightarrow c_t = 0$

(2) $\Rightarrow$ $\lambda_t = \left(\dfrac{\beta^{-1}}{f'(k_t) + (1-\delta)} \right) \lambda_{t-1}$, $\beta = \dfrac{1}{1+\rho}$

$$\Rightarrow \quad \lambda_t = \frac{(1+\rho)}{f'(k_t) + (1-\delta)} \lambda_{t-1}$$

thus, $\dfrac{\lambda_t}{\lambda_{t-1}} < 1$ when $f'(k_t) + (1-\delta) > (1+\rho)$

or

$f'(k_t) > \rho + \delta$

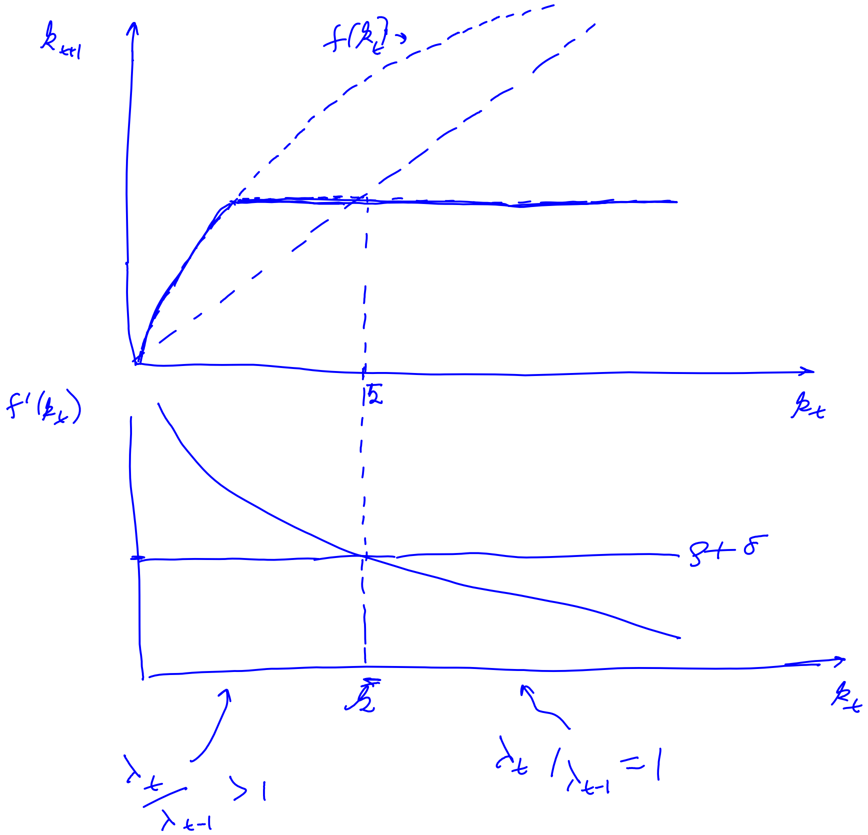
$\dfrac{\lambda_t}{1} = 1$ when $f'(k_t) = \rho + \delta$

λ_{t-1}

$\frac{\lambda_t}{\lambda_{t-1}} > 1$ when $f'(k_t) < \rho + \delta$

These outcomes indicate the following optimal path.

when $f'(k_t) > \rho + \delta$, $\lambda_t > \lambda_{t-1}$.



Optimal policy is:

if $k_0 > \bar{k}$,

set c_0 to solve

$$\bar{k} + c_0 = f(k_0) + (1-\delta)k_0$$

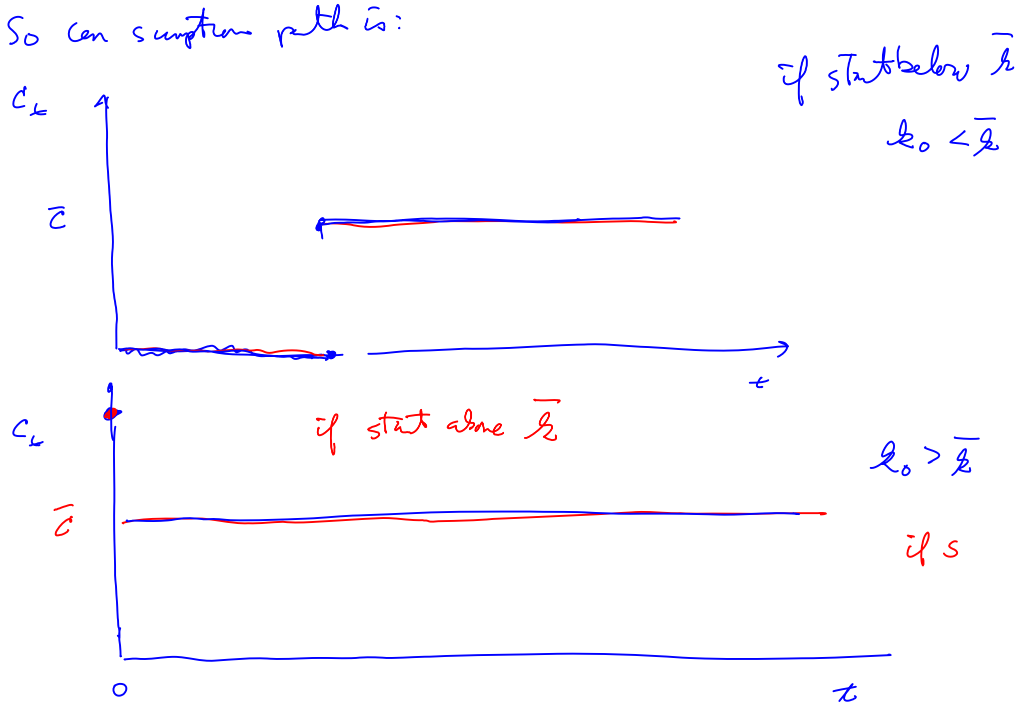
so that you jump to $\bar{k}$ in one step.

if $k_0 < \bar{k}$,

save everything until you get to $\bar{k}$ $\Rightarrow$

get to $\bar{k}$ as fast as you can.

So consumption path is:



Note: with preference $\sum_{t=0}^{\infty} \beta^t c_t$ - consumer

is willing to substitute consumption across time easily —

leads to a very unsmooth consumption path

under the optimal plan.

Compare above optimal growth model with the Solow model. The Solow model has the same technology

$$c_t + k_{t+1} = (1-\delta)k_t + f(k_t)$$

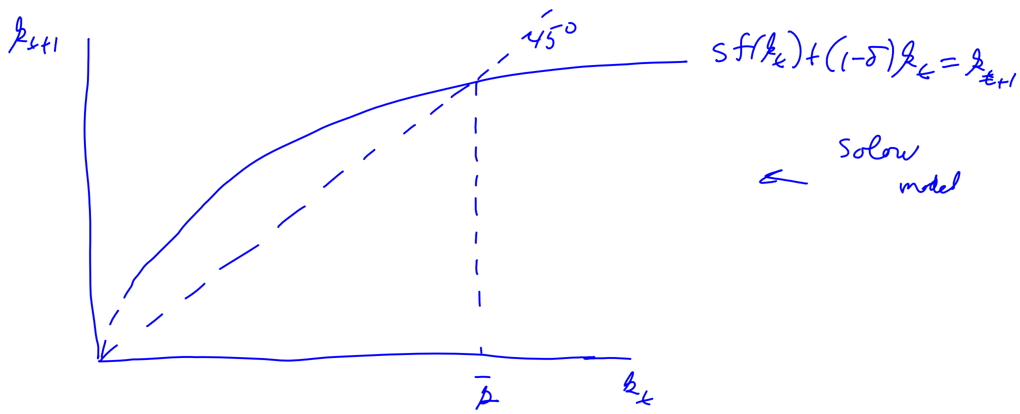
or

$$k_{t+1} = (1-\delta)k_t + (f(k_t) - c_t)$$

The Solow model assumes

$$\text{saving} \equiv f(k_t) - c_t = s f(k_t)$$

where s is a $\underline{\text{constant}}$ saving rate.



Steady state:

(*) $\quad s f(\bar{k}) = \delta \bar{k} \qquad$ at steady state of Solow model

(*) is eqn that determines steady state $\bar{k}$ of Solow model as function of assumed constant saving rate s.

In the above optimal growth model with preferences

$$\sum_{t=0}^{\infty} \beta^t c_t \qquad (\text{i.e. } u(c_t) = c_t)$$

the saving rate is $\underline{\text{not}}$ constant. Instead of (*), the steady state capital stock is determined by the usual equation

$$f'(\bar{k}) = \rho + \delta$$

the saving rate (s) adjust to make this happen.

Of course, at the steady state of the optimal growth model

$$k_{t+1} = (1-\delta)k_t + (f(k_t) - c_t) \Rightarrow$$

$$\delta \bar{k} = f(\bar{k}) - \bar{c} \equiv \bar{s} f(\bar{k})$$ [where $\bar{s}$ is saving rate at the steady state]

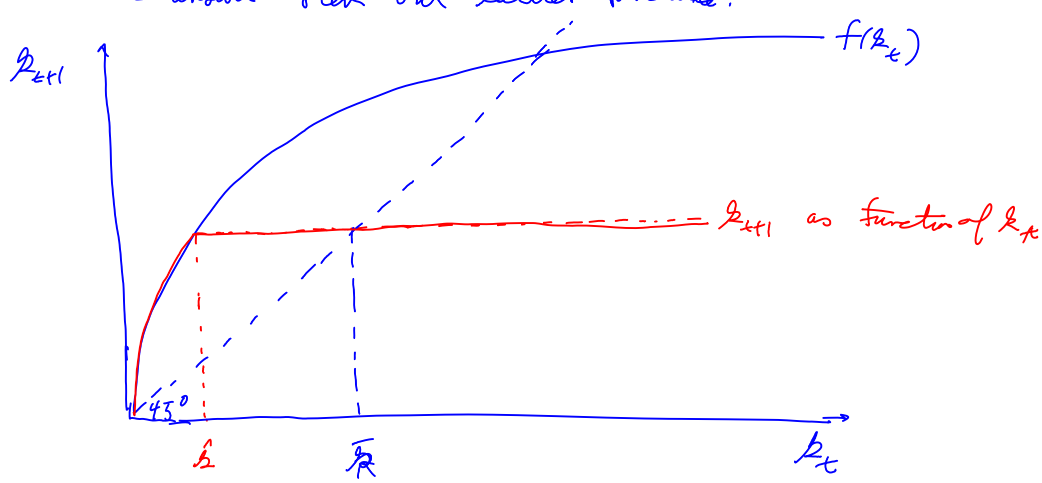
or

$$\bar{s} = \frac{\delta \bar{k}}{f(\bar{k})}$$

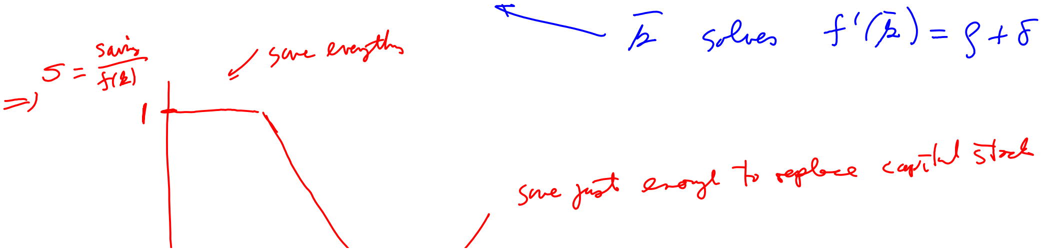
but here $\bar{s}$ is an outcome - to be solved for as a function of $\bar{k}$ - and not a determinant of $\bar{k}$ as it is in the Solow model.

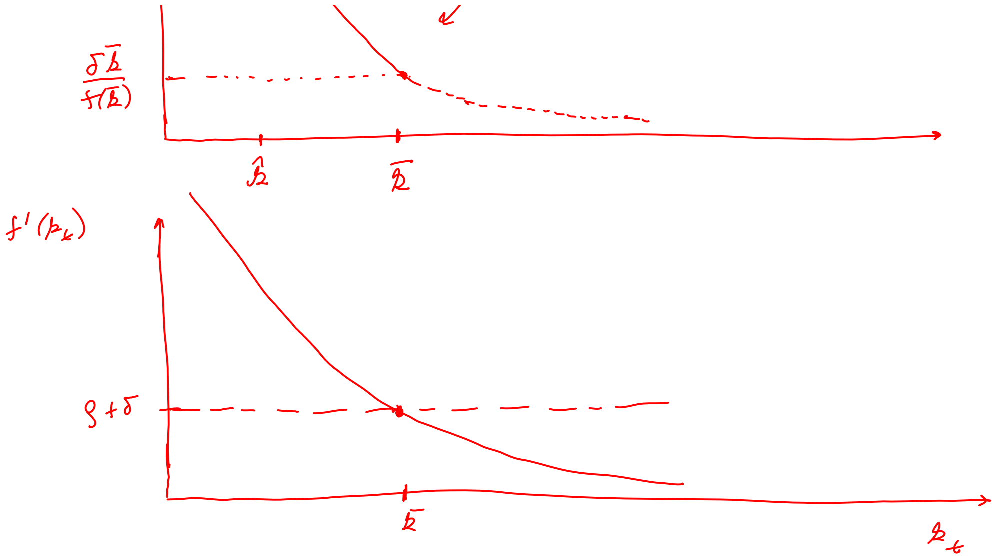
What about the saving rate $\frac{f(k) - c}{f(k)}$

away from the steady state? We can deduce the answer from our earlier picture:



$\bar{k}$ solves $f'(\bar{k}) = \rho + \delta$





Note how the rate of return on capital co varies with the saving rate here. Compare this with the constant saving rate assumed in the Solow model.

Sunday, April 11, 2010
3:20 PM

Sunday, April 11, 2010
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Practice problems 3

Sunday, March 08, 2009
7:54 PM

1. A consumer-worker lives for two periods and has utility function

$$c_1 + \beta c_2 + \ln(1-h_1) + \beta \ln(1-h_2) \quad , \quad 0<\beta<1$$

where $c_t \geq 0$ is consumption in period $t=1,2$;

$h_t \in [0,1]$ is time spent working in period $t=1,2$.

The worker's intertemporal budget constraint is

$$c_1 + \beta c_2 = w_1 + \beta w_2$$

where β^{-1} is the interest rate and $w_1 \geq 1$, $w_2 \geq 1$ are the real wages in period 1 and 2.

We require that $c_t \geq 0$ and $h_t \in [0,1]$, $t=1,2$

a. Find the consumer's optimal level of c_1, c_2, h_1, h_2 as functions of w_1 and w_2.

b. Find the optimal level of c_1, c_2, h_1, h_2 when $w_1 = w_2 = 1$.

Practice problems 6

Tuesday, April 14, 2009
7:20 PM

1. Jeff Sachs calculation (his article criticizing the Treasury plan)

Consider an asset that pays off

$X > 0$ with probability $\pi > 0$

and

$Y = 0$ with probability $(1-\pi)$

A risk neutral investor would be willing to pay

$$\pi \cdot X + (1-\pi) Y = \pi X$$ for the asset

Now suppose that the government offers to lend the buyer α. what he pays for the asset. The loan has interest rate $r = 0$ and is "non-recourse" - the borrower can walk away from the loan costlessly if the asset does not pay off. Here $\alpha \in (0,1)$

How much would a risk-neutral investor be willing to pay for the asset?

Let P = amount he is willing to pay

he borrows αP and puts up
$(1-\alpha)P$ of his own money.

His expected payoff is

$$\pi(X - \alpha P) + (1-\pi)\max(0, Y - \alpha P)$$

payout in good state, repay loan, $= 0$

$$= \pi(X - \alpha P)$$

In terms of his own money, the expected payoff
$\pi(X - \alpha P)$ costs him $(1-\alpha)P$

So to equate the expected payoff to the cost, set

$$\pi(X - \alpha P) = (1-\alpha)P$$

$$\Rightarrow$$

$$(*) \quad P = \frac{\pi}{\pi\alpha + (1-\alpha)} \cdot X$$

Note that $\lim_{\alpha \nearrow 1} P(\alpha) = X$ —

a. Use these formulas to replicate Jeff Sachs' calculations.

b. Without the government-supplied leverage, the asset is worth

πX, while with the leverage it is worth

πX, while with the leverage it is worth

$$\frac{\pi}{\pi\alpha + (1-\alpha)} X .$$

The expected value of the loan to the government is

$$\pi \cdot \underbrace{\alpha P}_{\substack{\text{payoff in} \\ \text{good states}}} - \underbrace{\alpha P}_{\uparrow \text{ amount lent}}$$

$$= -\alpha P (1-\pi) = -\frac{\alpha \pi (1-\pi)}{\pi\alpha + (1-\alpha)} X < 0$$

Note that the value paid by the investors + the value of the loan to the government is

$$\frac{\pi X}{\pi\alpha + (1-\alpha)} - \frac{\alpha\pi(1-\pi)}{\pi\alpha + (1-\alpha)} X$$

$$= \frac{\pi[1-\alpha(1-\pi)]}{\pi\alpha + (1-\alpha)} X = \pi X \quad = \text{price of asset without the government loan.}$$

$1-\alpha(1-\pi)$

Remark: Sachs' point is that the Treasury plan transfers $\frac{\alpha\pi(1-\pi)}{\pi\alpha + (1-\alpha)}$ from the Treasury to the original owner of the asset that offers payoff $(X, 0)$ in the good state, bad state.

---

2. Consider the overlapping generations model with capital discussed in class.

At each date $t \geq 1$ there are born N identical young agents who work when young, consume when old. The consumption stream of a young agent born in t is (c_{yt}, c_{ot+1})

cons'n when young

cons'n when old

The young person's utility function is $u(c_{ot+1})$ — he/she doesn't value consumption when young.

Each young person is endowed with one unit of labor that he/she supplies inelastically.

The technology and financial structure are as described in class. In particular,

for $t \geq 1$, $y_t = k_t^{\beta}$, y_t = output per young worker

k_t = capital per young worker

wage = $w_t = (1-\beta)k_t^{\beta}$

rental rate on capital $\rho_t = \beta k_t^{\beta-1}$

N = # of young workers, constant over time

$\beta \in (0,1)$

$\delta \in (0,1)$ depreciation rate

$r_t \equiv \rho_t - \delta$ return on savings from $t-1$ to t

per capita savings $= w_t$

e = social security tax paid when young

= social security payment received when old

$k_{t+1} = a_{t+1}$ savings decided at t

$k_{t+1} = w_t - e$

(*) $k_{t+1} = (1-\beta)k_t - e$

consumption when old , $t \geq 0$ (includes initial old

$c_{0t+1} = (r_{t+1} + 1)k_t + e$ at $t = 1$)

a. Suppose that $e = 0$ for all $t \geq 0$.

Find formulas for the steady state values of

k_t, w_t, ρ_t, r_t, c_{0t+1}.

b. Suppose that $\beta = 1/4$, $\delta = 1/3$. Find

steady state values of k, w, ρ, c_0.

consumption of old.

c. Now suppose that $\beta = 1/4, \delta = 1/3$, and

k_1 = steady state value computed in part b.

Starting at time $t = 1$, suppose that we institute

a social security scheme with $e = .1$ —

this is a transfer to the old at t financed

by an equal tax on the young. Find the

path of k_{t+1}, c_{0t}, r_t, w_t associated with this new policy for $t = 1, 2, ..., 10$.

Which generations are better off with $e = .1$ than with $e = 0$? Which are worse off?

Hint: you will need to write a matlab or excel program to iterate on (*) starting from the given value of k_1.

d. Now suppose that $\beta = 1/5$ and $\delta = 1/3$ and $e = 0$

Find steady state values of k, w, r, and c_0.

e. Starting from the steady state k that you found in d as the initial capital owned by the initial old, on which they earn payments of $r_1 k_1$, suppose there is started a social security transfer system at time 1 with $e = .1$.

Find values of c_{0t}, r_t, k_{t+1} for $t = 1, ..., 10$.

Compared to the steady states you computed in part d, which generations are better / worse off?

Hint: you will have to write an excel or matlab program to iterate on (*).

The next several problems assume the following environment. A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad 0 < \beta < 1$$

$$\beta = \frac{1}{1+\rho} \quad , \rho > 0$$

where c_t is consumption per worker and

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad , \gamma > 0 \quad , \gamma \neq 1$$

$$u(c) = \log c \quad \text{if } \gamma = 1$$

The technology is

$$y_t = f(k_t) = z k_t^{\alpha} \quad , \quad 0 < \alpha < 1, \; z > 0$$

where y_t is output per unit labor and k_t is capital per unit labor

$$y_t = c_t + x_t + g_t$$

x_t = gross investment per unit labor

g_t = government expenditures per unit of labor

$$k_{t+1} = (1-\delta) k_t + x_t \quad , \quad 0 < \delta < 1$$

The government finances its expenditures by levying some combination of a flat rate tax τ_{ct} on the value of consumption goods purchased at t, a flat rate tax of τ_{nt} on the value of labor earnings at t, a flat rate tax τ_{kt} on earnings from capital at t, and a lump sum tax of τ_{ht} in time t consumption goods per worker at time t.

Let $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ be a price system.

3. Consider an economy of the type described above in which $g_t = \bar{g} > 0$ for all $t \geq 0$ and in which initially the government finances all expenditures by lump sum taxes.

a. Find a formula for the steady state capital labor ratio k_t for this economy. Find formulas for the steady state levels of c_t and

$$R_t = \left[(1-\delta) + (1-\tau_{k\,t+1}) f'(k_{t+1}) \right].$$

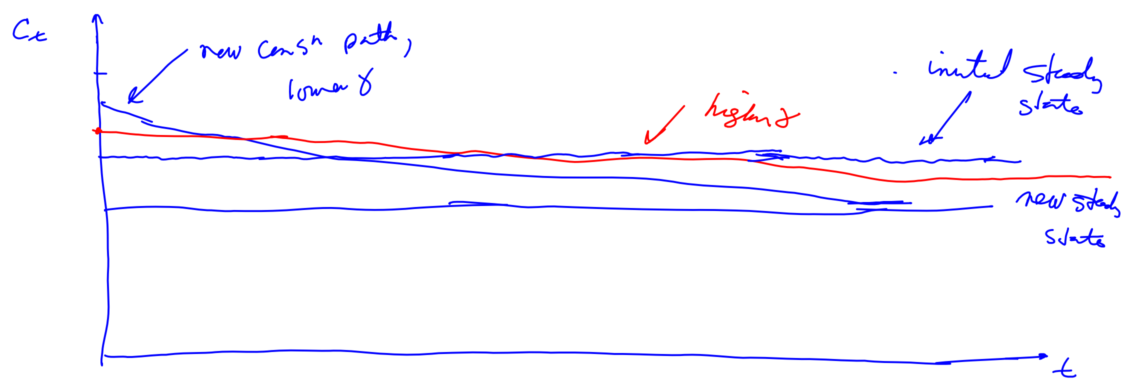
b. Now suppose that starting from $k_0 = \bar{k} =$ steady state that you computed in part a, the government suddenly increases the tax on earnings on capital to a constant level $\tau_k > 0$. The government adjusts lump sum taxes to keep the government budget balanced. Describe competitive equilibrium time paths for c_t, k_{t+1}, R_t and their relationship to corresponding values in the old steady state that you derived in part a.

steady states that you described in part a.

c. Describe how the shapes of the paths that you found in part b. depend on the curvature parameter γ in the utility function $u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$. Higher values of γ imply higher curvature and more aversion to consumption paths that fluctuate.

Higher values of γ imply that the consumer values smooth consumption paths even more.

Thus, here is what we expect:



—— higher γ adjustment path

—— lower γ adjustment path

Things look this way because of the consumer's attitudes about consumption variability. The interest rate and k_t adjust to make this happen.

Remark: formulas in section 11.7.3 of the typed chapter verifies this by giving a formula for

λ as a function of γ where

λ was the decay rate discussed in class.

d. Starting from the steady state $\bar{k}$ that you computed in part a, now consider a situation in which the government announces at time 0 that starting in period 10 the tax on earnings from capital τ_k will rise permanently to $\bar{\tau}_k > 0$. The government adjusts its lump sum taxes to balance it's budget.

i) find new steady values for k_t, c_t, R_t.

ii) Describe the shapes of the transition paths from the initial steady state to the new one for k_t, R_t, c_t.

iii) decide how the shapes of these transition paths depend on the curvature parameter paths depend on the curvature parameter γ in the utility function $u(c)$.

<u>Hint</u>: when γ is bigger, consumers more strongly prefer smoother consumption paths

4. <u>Trade and growth, version II</u>

Consider the problem of a planner in a small economy. When the economy is <u>closed to trade</u>, the planner chooses $\{c_t, k_{t+1}\}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad 0 < \beta < 1 \ , \quad \beta = \frac{1}{1+\rho}$$

$$\rho > 0$$

subject to

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \quad , \quad \delta \in (0,1)$$

where $u(c_t) = \dfrac{c_t^{1-\gamma}}{1-\gamma}$, $\gamma > 0$

$$f(k_t) = z k_t^{\alpha} \quad , \quad 0 < \alpha < 1$$

Let $\bar{k}$ be the steady state value of k_t under the optimal plan.

a. Find a formula for $\bar{k}$.

b. Assume that $k_0 > \bar{k}$. Describe time paths for $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ and $R_{t+1} = (1-\delta) + f'(k_{t+1})$.

c. What is the steady state value of R_{t+1}?

d. Is R_{t+1} less or greater than its steady state value when $k_{t+1} > \bar{k}$?

Part II.

e. Now assume that the economy is open to international trade in capital and financial assets. Assume that there is a fixed world gross rate of return $R = \beta^{-1}$ at which the planner can borrow or lend. In particular, the planner is free to use the following plan. The planner can sell all of its capital k_0^- and simply consume the interest payments.

Let k_0^- be the level of initial capital ($k_0^- > \bar{k}$) just before the country opens up to trade just before time 0. Let k_0^- be the same initial capital per capita $k_0^- > \bar{k}$ studied in parts a. - d.

At time $t=-1$, after k_0^- was set, trade opens up. At $t=-1$, the planner sells k_0^- in exchange for IOU's from the rest of the world in the amount $A_{-1} = k_0^-$.

The bonds A_{-1} are one-period in duration and bear the constant world gross interest rate $R = \beta^{-1}$.

After the sale of $k_0^{-1} = A_{-1}$, the planner has zero capital and so shuts down the technology. Instead, the planner uses the asset market to

Smooth consumption. The planner thus chooses (A_{t+1}, c_t) to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \qquad \beta = \frac{1}{1+\rho}$$

$$c_t + \beta A_{t+1} = A_t, \qquad , A_0 = \bar{k}_0$$

or

$$A_{t+1} = \beta^{-1}[A_t - c_t]$$

Form the Lagrangian:

$$\sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t \left[\beta^{-1}(A_t - c_t) - A_{t+1} \right] \right\}$$

FONC:

$$c_t: \quad u'(c_t) - \lambda_t \beta^{-1} = 0$$

$$A_t: \quad -\beta^{-1}\lambda_{t-1} + \beta^{-1}\lambda_t = 0$$

$$\lim_{T\to\infty} \beta^T \lambda_T A_{T+1} = 0$$

$$\Rightarrow \quad \lambda_t = \lambda_0 \quad \forall t$$

$$u'(c_t) = u'(c_0) \quad \forall t$$

$$c_t = c_0 \quad \forall t.$$

then roll over assets & just consume interest:

$$A_{t+1} = \beta^{-1}[A_t - c_t] \quad \Rightarrow$$

$$\bar{A} = \beta^{-1}[\bar{A} - c_0] \quad \Rightarrow$$

$$(1 - \beta^{-1})\bar{A} = -\beta^{-1} c_0$$

or

$$\beta^{-1} c_0 = (\beta^{-1} - 1)\bar{A}$$

$$c_0 = (1-\beta)\bar{A} \qquad 1-\beta = 1 - \frac{1}{1+\rho} = \frac{\rho}{1+\rho} = \beta\rho$$

$$c_0 = \rho\beta A \quad , \quad \bar{A} = A_0 = k_0^{-1}$$

Thus, the optimal path for consumption is

$$c_t = \rho\,\bar{A} = \rho k_0^{-1}$$

f. Compare the time path of (c_t, k_t, R_t) that you computed in parts a, b, c, d with "no trade" with the part e one "with trade" in which the government "shuts down the home technology" and lives entirely from returns on foreign assets. Can you say which path the representative consumer would prefer?

- Hint: I think it could go either way for reasons that might be easier to appreciate after you look at part g.

g. Now return to the economy in part e with $k_0^- > \bar{k}$ from part d. Assume that the planner is free to borrow or lend capital at the fixed gross interest rate of $\beta^{-1} = 1+\rho$, as before. But now assume the planner chooses the <u>optimal</u> amount of k_0^{-1} to sell off and so not

not necessarily sell off the entire k_0^{-1} and possibly continues to operate the technology.

i) Find the solution of the planning problem.
(hint: you might want to review your solution to problem 2 of practice problems 4.

ii) Explain why it is optimal not to shut down the technology. Hint: Starting from having shut the technology down, think of putting a small amount ε of capital into the technology — this earns $z\varepsilon^{\alpha}$ and costs $\rho\varepsilon$ in terms of foregone interest. because $0 < \alpha < 1$
$\rho\varepsilon < z\varepsilon^{\alpha}$ for small ε. — The technology is very productive for small ε — so use it.

Practice problems 7

Monday, April 27, 2009
11:16 AM

1. Consider a consumer who wants to choose $\{c_t\}_{t=0}^{T}$ to maximize

$$\sum_{t=0}^{T} \beta^t c_t \quad , \quad 0 < \beta < 1$$

subject to the intertemporal budget constraint

$$\sum_{t=0}^{T} R^{-t} [c_t - y_t] = 0 \quad , \quad R > 1$$

where $c_t \geq 0$ and $y_t > 0$ for $t = 0, \dots, T$
Here $R > 1$ is the gross interest rate $(1+r)$, $r > 0$.

Assume R is constant. Here $\{y_t\}_{t=0}^{T}$ is an exogenous sequence of "labor income".

a. Assume that $\beta R < 1$. Find the optimal path $\{c_t\}_{t=0}^{T}$.

b. Assume that $\beta R > 1$. Find the optimal path $\{c_t\}_{t=0}^{T}$.

c. Assume that $\beta R = 1$. Find an optimal path $\{c_t\}_{t=0}^{T}$.

2. Consider the Kareken-Wallace model of moral

Recall that when the bank takes maximal risk by investing all of its deposits in the state A claim, it exposes the insurance agency to a lot of risk.

When $Q_A = \frac{D}{P_A}$ and $Q_B = 0$, the expected value of the losses of the insurance agency are

$$\Pi_B \underbrace{(r+1)}_{\text{payout in state } B \text{ because } Q_B = 0.} D$$

Can you think of a scheme for pricing deposit insurance that

a. deters the bank from excessive risk taking, and

b. Protects the deposit insurance agency from expected losses?

The idea is that the deposit insurance agency will charge the bank a fee for deposit insurance with the fee depending on (i) its total deposits, and (ii) its investment portfolio.

3. Term structure of interest rates.

This problem assumes the

following environment. A representative consumer has preferences ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , \quad 0 < \beta < 1$$

$$\beta = \frac{1}{1+\rho} \quad , \rho > 0$$

where c_t is consumption per worker and

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad , \gamma > 0 \quad , \gamma \neq 1$$

$$u(c) = \log c \quad \text{if } \gamma = 1$$

The technology is

$$y_t = f(k_t) = z k_t^{\alpha} \quad , \quad 0 < \alpha < 1, \; z > 0$$

where y_t is output per unit labor and k_t is capital per unit labor

$$y_t = c_t + x_t + g_t$$

x_t = gross investment per unit labor

g_t = government expenditures per unit of labor

$$k_{t+1} = (1-\delta) k_t + x_t \quad , \quad 0 < \delta < 1$$

c_t = consumption per worker

The government finances its expenditures by levying some combination of a flat rate tax τ_{ct} on the value of consumption goods purchased at t, a flat rate tax of τ_{nt} on the value of labor earnings at t, a flat rate tax τ_{kt} on earnings from capital at t, and a lump sum tax of τ_{ht} in time t

consumption goods per worker at time t.

Let $\{q_t^0, r_t^0, w_t^0\}_{t=0}^{\infty}$ be a price system.

Here the superscript 0 means the time 0 price system.

a. Recall that we can represent

$$q_t^0 = q_0^0 \, m_{0,1} \, m_{1,2} \cdots m_{t-1,t}$$

where $$m_{t-1,t} = \frac{q_t^0}{q_{t-1}^0}$$

and $$m_{t-1,t} \equiv \exp(-r_{t-1,t}) \approx \frac{1}{1 + r_{t-1,t}}$$

Further recall that the t period long yield is

satisfies

$$q_t^0 = \exp(-t \, r_{0,t})$$

or

$$r_{0,t} = \frac{1}{t}\left[r_{0,1} + r_{1,2} + \cdots + r_{t-1,t} \right]$$

Now suppose that at $t=0$, $k_0 = \bar{k}$, where $\bar{k}$ is the steady state appropriate for an economy with constant $g_t = \bar{g} > 0$ and all expenditures financed by lump sum taxes. Find q_t^0 for this economy.

b. Plot $r_{t-1,t}$ for this economy for $t = 1, 2, \dots, 10$.

c. Plot $r_{0,t}$ for this economy for $t = 1, \dots, 10$.

(this is what Bloomberg plots)

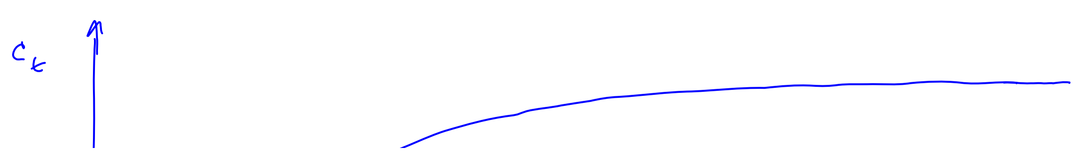
d. Now assume that at time 0, starting from $k_0 = \bar{k}$ for the steady state you computed in part a, the government unexpectedly and permanently raises the tax rate on income from capital $\tau_{kt} = \tau_k > 0$ to a positive rate.

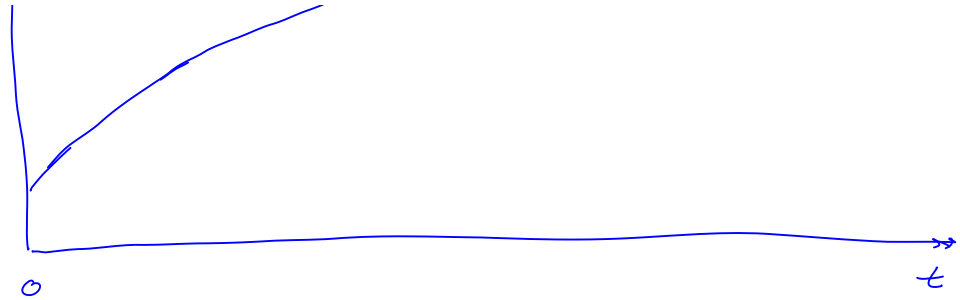
i) Plot $r_{t-1,t}$ for $t = 1, \dots, 10$ for this economy. Explain how you got this outcome.

ii) Plot $r_{0,t}$ for $t = 1, \dots, 10$ for this economy. Explain how you got this outcome.

4. The following problem assumes the same economic environment as the previous problem – i.e., the "growth model" with fiscal policy.

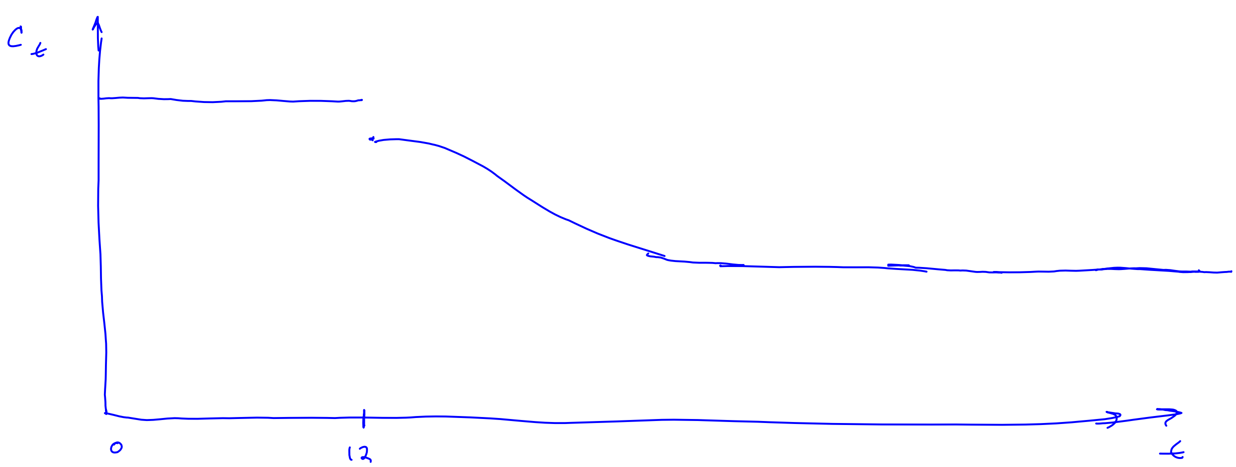
Suppose that you observe the following path for consumption per capita





Say what you can about likely behavior over time of k_t, $R_t \; [(1-\delta) + (1-\tau_{kt}) f'(k_t)]$, g_t, and τ_{kt}. (You are free to make up any story you like that is consistent with the model.)

5. Assume the same economic environment as in the previous two problems. Assume that someone has observed the following time path for c_t:



a. Describe a consistent set of assumptions about fiscal policy that explains this time path for c_t. In doing so, please distinguish carefully between changes in taxes and expenditures that are foreseen versus unforeseen.

versus <u>unforeseen</u>.

b. Describe what is happening to k_t, R_t, and q_t over time. Make whatever assumptions you want to get a complete but consistent story — here "consistent" means "consistent with the economic environment we have assumed."

Practice problems 8

Monday, May 04, 2009
8:45 AM

1. Consider the optimal growth model with a representative consumer with preferences

$$\sum_{t=0}^{\infty} \beta^t c_t \quad , \qquad 0 < \beta < 1 \, ,$$

with techology

$$c_t + k_{t+1} = f(k_t) + (1-\delta) k_t \quad , \qquad \delta \in (0,1)$$

$$c_t \geq 0 \qquad , \; k_0 > 0 \text{ given}$$

$$f' > 0, \; f'' < 0 \quad , \quad \lim_{k \searrow 0} f'(k) = +\infty \, , \lim_{k \to +\infty} f'(k) = 0$$

Here: c_t = consumption / person

k_t = capital / worker

Let $\bar{k}$ solve $f'(\bar{k}) = \rho + \delta$.

a. For k_0 given formulate and solve the optimal planning problem.

b. For $k_0 > \bar{k}$, describe the optimal time path of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$.

c. For $k_0 < \bar{k}$, describe the optimal time path of (c_t, k_{t+1})

d. Let the <u>saving rate</u> s_t be defined as the s_t that satisfies

$$k_{t+1} = s_t f(k_t) + (1-\delta) k_t$$

(in the Solow model, we assumed that $s_t = s$, an exogenous constant.) Here s_t in general varies along a $\{c_t, k_{t+1}\}$. Say what you can about how s_t varies as a function of k_t.

Hint: Form the Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t \{ c_t + \lambda_t [f(k_t) + (1-\delta) k_t - c_t - k_{t+1}] \}$$

FONC:

(1) c_t: $1 - \lambda_t \leq 0$, $= 0$ if $c_t > 0$, $t \geq 0$

(2) k_t: $\lambda_t [f'(k_t) + (1-\delta)] - \beta^{-1} \lambda_{t-1} = 0$, $t \geq 1$

(1) $\Rightarrow$ $\lambda_t \geq 1$, $\lambda_t = 1 \Rightarrow c_t > 0$

$\lambda_t > 1 \Rightarrow c_t = 0$

(2) $\Rightarrow$ $\lambda_t = \left(\dfrac{\beta^{-1}}{f'(k_t) + (1-\delta)} \right) \lambda_{t-1}$, $\beta = \dfrac{1}{1+\rho}$

$\Rightarrow$ $\lambda_t = \dfrac{(1+\rho)}{f'(k_t) + (1-\delta)} \lambda_{t-1}$

thus, $\dfrac{\lambda_t}{\lambda_{t-1}} < 1$ when $f'(k_t) + (1-\delta) > (1+\rho)$

or

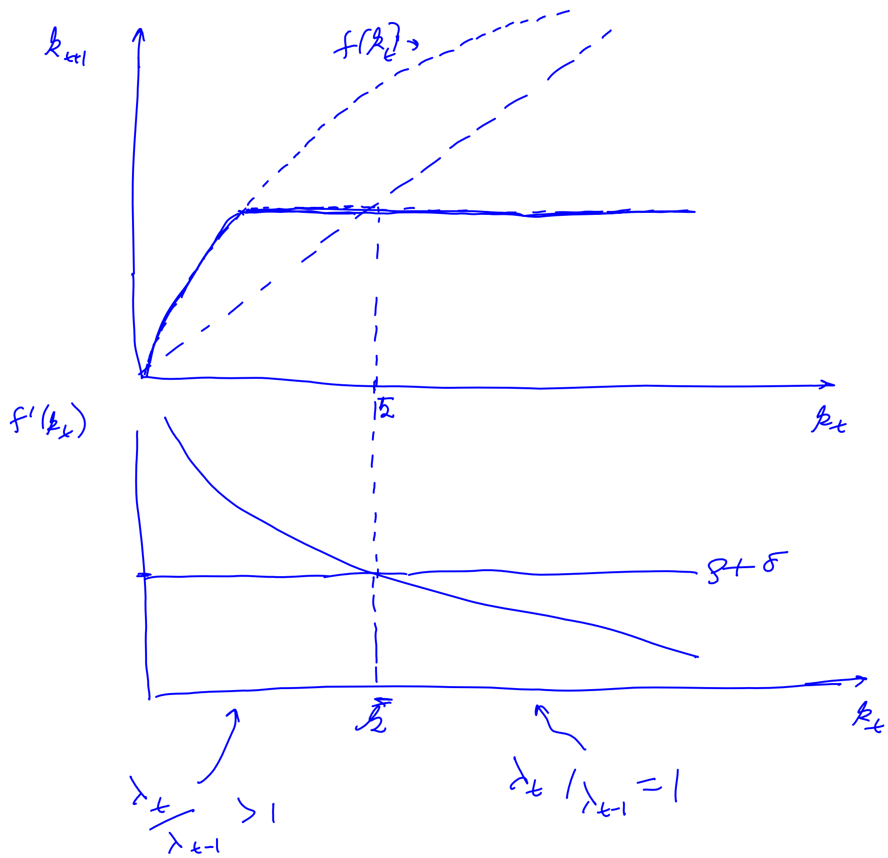
$f'(k_t) > \rho + \delta$

$$\frac{\lambda_t}{\lambda_{t-1}} = 1 \quad \text{when} \quad f'(k_t) = \rho + \delta$$

$$\frac{\lambda_t}{\lambda_{t-1}} > 1 \quad \text{when} \quad f'(k_t) < \rho + \delta$$

These outcomes indicate the following optimal path.

when $f'(k_t) > \rho + \delta$, $\lambda_t > \lambda_{t-1}$.



Optimal policy is:

if $k_0 > \bar{k}$,

set c_0 to solve

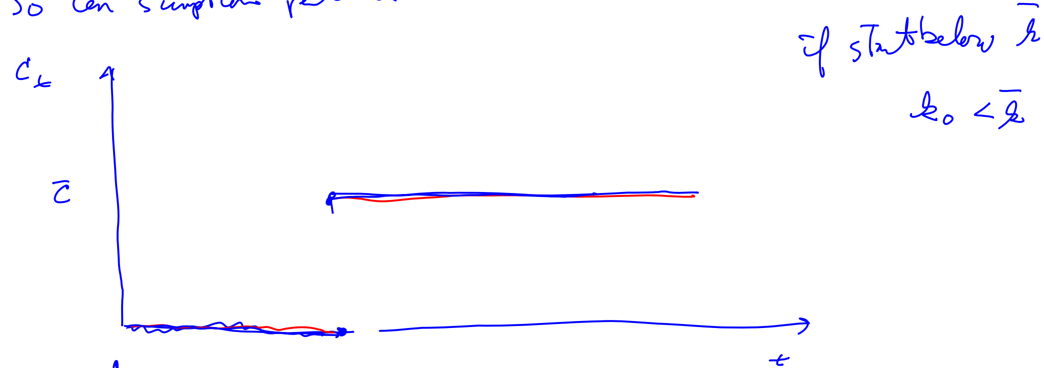
$$\bar{k} + c_0 = f(k_0) + (1-\delta) k_0$$

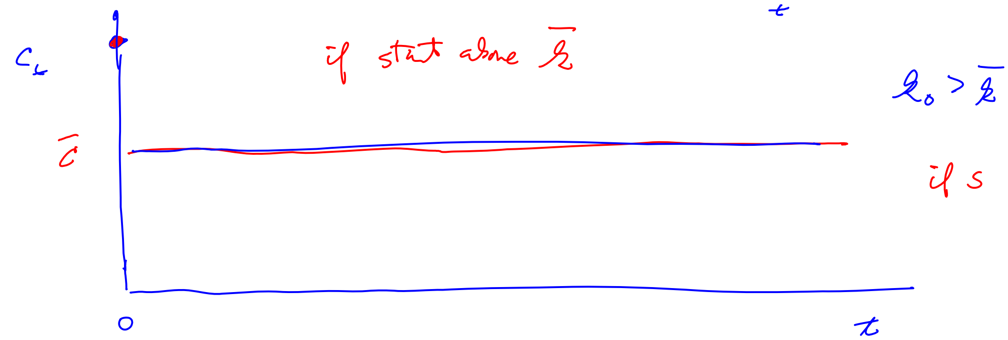
so that you jump to $\bar{k}$ in one step.

if $k_0 < \bar{k}$,

save everything until you get to $\bar{k}$ $\Longrightarrow$ get to $\bar{k}$ as fast as you can.

So consumption path is:





Note: with preference $\sum_{t=0}^{\infty} \beta^t c_t$ - consumer

is willing to substitute consumption across time easily —

leads to a very unsmooth consumption path

under the optimal plan.

Compare above optimal growth model with the Solow model. The Solow model has the same technology

$$c_t + k_{t+1} = (1-\delta)k_t + f(k_t)$$

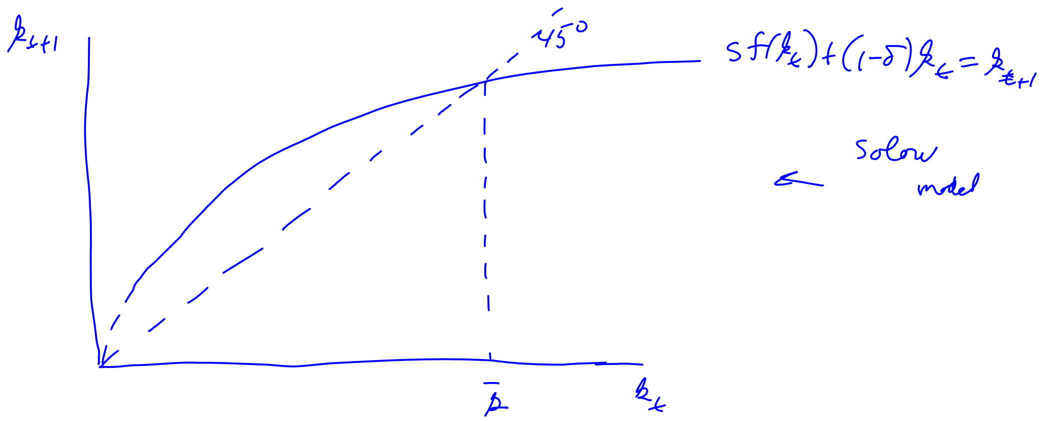
or

$$k_{t+1} = (1-\delta)k_t + (f(k_t) - c_t)$$

The Solow model assumes

$$\text{saving} \equiv f(k_t) - c_t = s f(k_t)$$

where s is a <u>constant</u> saving rate.



Steady state:

(*) $\quad s f(\bar{k}) = \delta \bar{k}$ $\quad$ at steady state of Solow model

(*) is eqn that determines steady state $\bar{k}$ of Solow model as function of assumed constant saving rate s.

In the above optimal growth model with preferences

$$\sum_{t=0}^{\infty} \beta^t c_t \qquad (\text{i.e. } u(c_t) = c_t)$$

the saving rate is <u>not</u> constant. Instead of (*), the steady state capital stock is determined by the usual equation

$$f'(\bar{k}) = \rho + \delta$$

the saving rate (s) adjust to make this happen.

Of course, at the steady state of the optimal growth model

$$k_{t+1} = (1-\delta)k_t + (f(k_t) - c_t) \implies$$

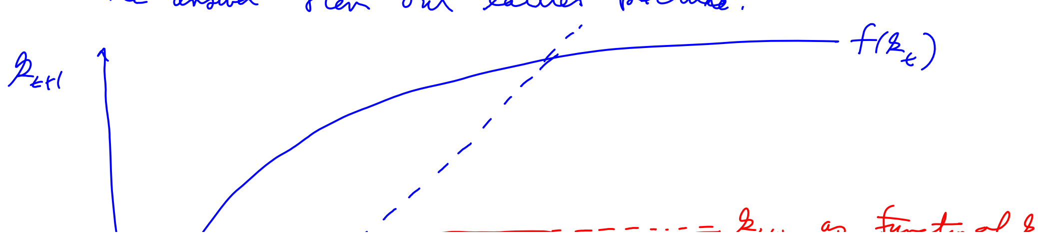
$$\delta\bar{k} = f(\bar{k}) - \bar{c} \equiv \bar{s}\, f(\bar{k})$$ [where $\bar{s}$ is saving rate at the steady state]

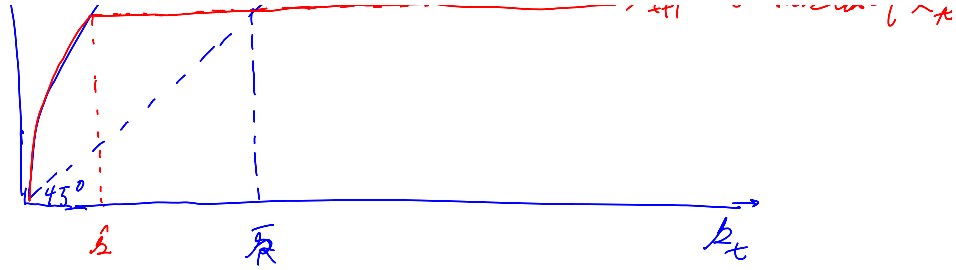
or

$$\bar{s} = \frac{\delta\bar{k}}{f(\bar{k})}$$

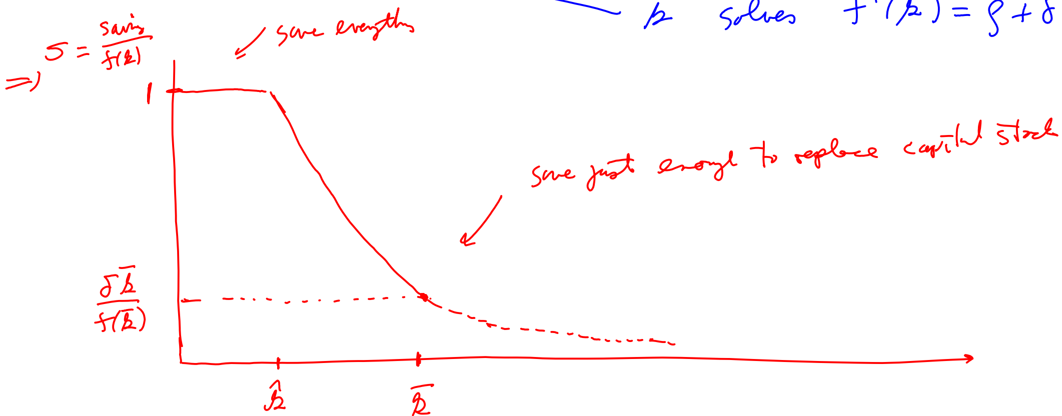
but here $\bar{s}$ is an outcome – to be solved for as a function of $\bar{k}$ – and not a determinant of $\bar{k}$ as it is in the Solow model.

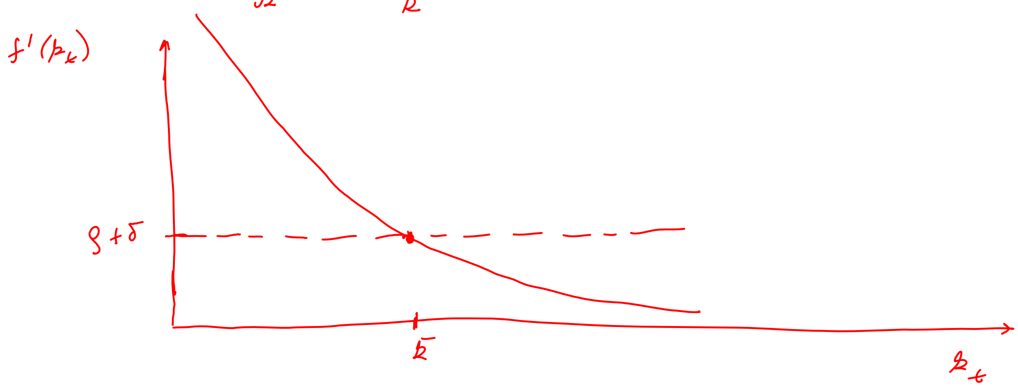
What about the saving rate $\dfrac{f(k)-c}{f(k)}$ away from the steady state? We can deduce the answer from our earlier picture:





$\bar{k}$ solves $f'(\bar{k}) = \rho + \delta$





Note how the rate of return on capital co varies with the saving rate here. Compare this with the constant saving rate assumed in the Solow model.

Practice problems 9

Tuesday, May 05, 2009
9:28 AM

1. Consider a competitive equilibrium version of the practice problems 8, problem 1 economy. There is a government that chooses sequences $\{g_t, \tau_{kt}, \tau_{ht}\}_{t=0}^{\infty}$

govt expenditures — capital tax — lump sum tax

and a price system $\{q_t, w_t, r_t\}_{t=0}^{\infty}$.

Suppose we start from a prior steady state $k_0 = \bar{k}$ in which the government financed a constant $g_t = \bar{g}$ with lump sum taxes and $\tau_k = 0$. At $t=0$ the government announces a one-and-for all increase in τ_k to a constant level $\tau_{kt} = \bar{\tau}_k > 0$. It adjusts lump sum taxes to leave the government budget balances. It leaves $g_t = \bar{g}$ as before. Describe equilibrium paths of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ and $\{q_t\}_{t=0}^{\infty}$.

Hint: The household problem is

$$L = \sum_{t=0}^{\infty} \beta^t c_t + \mu \left[\sum_{t=0}^{\infty} \Big(r_t (1-\tau_{kt}) k_t + w_t n_t - q_t \tau_{ht} - q_t \big(c_t + k_{t+1} - (1-\delta) k_t \big) \Big) \right]$$

FONC:

FONC:

c_t: $\beta^t - \mu q_t \le 0$, $= 0$ if $c_t > 0$, $t \ge 0$

k_t: $r_t(1-\tau_{kt}) + q_t(1-\delta) - q_{t-1} = 0$, $t \ge 1$

$\Rightarrow$ (+) $q_t = r_{t+1}(1-\tau_{kt+1}) + q_{t+1}(1-\delta)$

Firm problem: $\Rightarrow$

(++) $q_t f'(k_t) = r_t$

$q_t [f(k_t) - k_t f'(k_t)] = w_t$

Substitute (++) into (+) $\Rightarrow$

$$q_t = q_{t+1} f'(k_{t+1})(1-\tau_{kt+1}) + q_{t+1}(1-\delta)$$

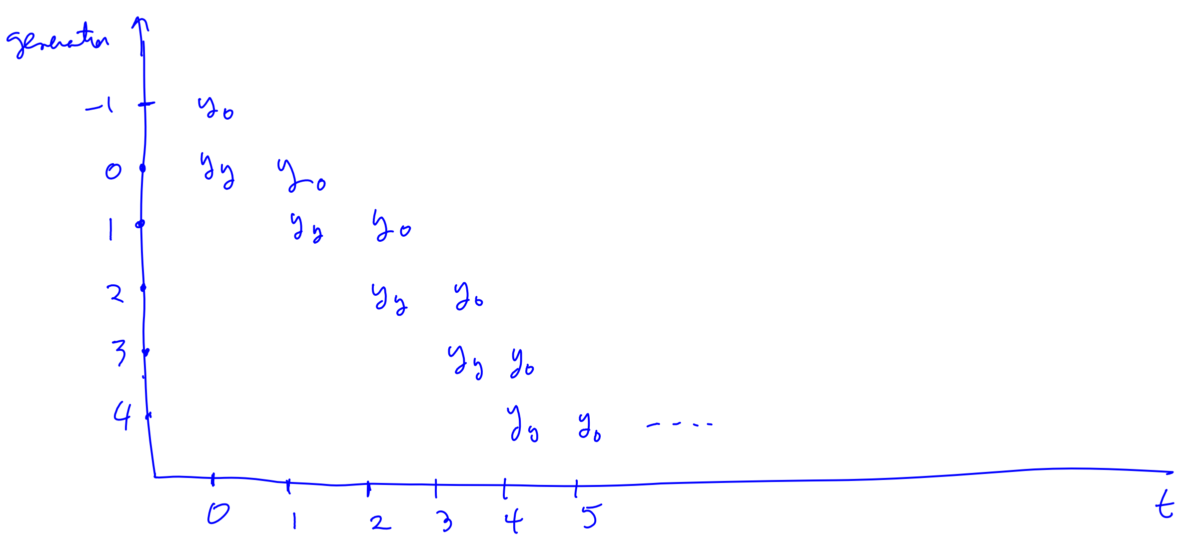
$$= q_{t+1}[f'(k_{t+1})(1-\tau_{kt+1}) + (1-\delta)]$$

Practice problems 5

Tuesday, April 14, 2009

5:08 PM

1. Consider the following simple overlapping generations model. At each date $t \geq 0$ there are born N two period lived agents. At $t = 0$ there are also N old people - these are the "initial old". This is a "pure exchange" - "no production" economy. The good arrives exogenously ("from heaven") each period. Each old person receives $y_o > 0$ of the consumption good and each young person recieves $y_y > y_o > 0$ units of the consumption good. The good cannot be <u>stored</u>, but it can be <u>exchanged</u>. The endowment pattern is thus



The aggregate endowment at $t=0$ is then

$N y_y + N y_o$.

For $t \geq 0$, each young person has preference described by the following utility function:

$$\ln(c_{yt}) + \ln(c_{ot+1})$$

where c_{yt} = consumption of a young person at t

c_{ot+1} = consumption of an old person at t.

If a time t young person could save a_{t+1} and earn $a_{t+1}(1+r_{t+1})$ next period by doing so, this person would face the following intertemporal optimum problem:

$$\max_{c_{yt},\, c_{ot+1}} \ln c_{yt} + \ln c_{ot}$$

subject to

$$c_{yt} + a_{t+1} = y_y \qquad (1)$$

$$c_{ot+1} = (1+r_{t+1})\, a_{t+1} + y_o \qquad (2)$$

If $a_{t+1} > 0$, the person is a lender or saver

If $a_{t+1} < 0$, the person is a borrower.

Form the Lagrangian -

$$L = \ln c_{yt} + \ln c_{ot+1} + \lambda \left[y_y + \frac{y_o}{1+r_{t+1}} - c_{yt} - \frac{c_{ot+1}}{1+r_{t+1}} \right]$$

$$\quad 1+r_{t+1} \qquad\qquad 1+r_{t+1}$$

Note: we have solved (2) for a_{t+1} and substituted into (1) to get the single intertemporal constraint
$$\underbrace{y_y + \frac{y_o}{1+r_{t+1}}}_{\text{PV of endowment}} = \underbrace{c_{yt} + \frac{c_{o\,t+1}}{1+r_{t+1}}}_{\text{PV of consumption}}$$

The first order conditions are:

$$\left.\begin{aligned} \frac{1}{c_{yt}} &= \lambda \\ \frac{1}{c_{o\cdot t+1}} &= \lambda \frac{1}{1+r_{t+1}} \end{aligned}\right\} \Rightarrow \quad \frac{1}{c_{o\,t+1}} = \frac{1}{c_{yt}}(1+r_{t+1})^{-1}$$

or

$$(*) \quad c_{o\,t+1} = c_{yt}(1+r_{t+1})$$

and the intertemporal constraint.
Substitute (*) into the intertemporal constraint

$$c_{yt} + c_{yt} = y_y + \frac{y_o}{1+r_{t+1}}$$

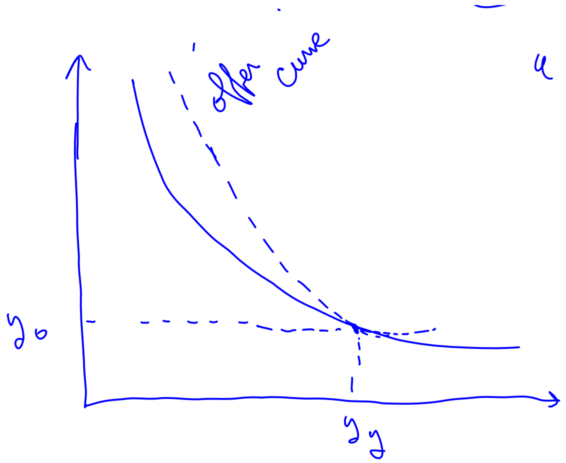
$$c_{yt} = \frac{1}{2}\left[y_y + \frac{y_o}{1+r_{t+1}}\right]$$

$\Rightarrow$

$$\text{savings} \equiv a_{t+1} = y_y - c_{yt} = \frac{1}{2}\left[y_y - \frac{y_o}{(1+r_{t+1})}\right]$$

Aggregate savings of the young:

$$N(y_y - c_{yt}) = \frac{N}{2}\left[y_y - \frac{y_o}{(1+r_{t+1})}\right]$$



a function of the interest rate r_{t+1}

offer curve depicts chosen c_{yt}, c_{ot+1} as $(1+r_{t+1})$ increases

Note: the interest rate that reconciles the consumer to consume his endowment, i.e., to set $(c_{yt}, c_{ot+1}) = (y_y, y_o)$ is found by setting

$$a_{t+1} = \frac{1}{2}\left(y_y - \frac{y_o}{1+r_{t+1}}\right) = 0 \qquad (\text{saving} = 0)$$

$$\Rightarrow \quad y_y = \frac{y_o}{1+r_{t+1}}$$

$$\Rightarrow \quad 1 + r_{t+1} = \frac{y_o}{y_y} < 1$$

$$\Rightarrow \quad r_{t+1} < 1$$

the interest rate $1+r_{t+1} = \frac{y_o}{y_y}$ makes the consumer content to consume his endowment

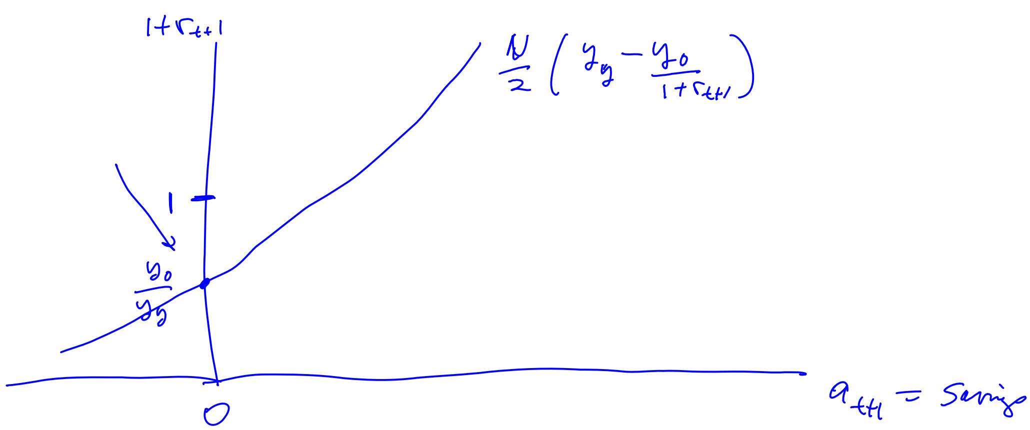
$$1+r_{t+1} = \frac{y_o}{y_y} = \text{slope of indifference curve}$$

through the endowment point.

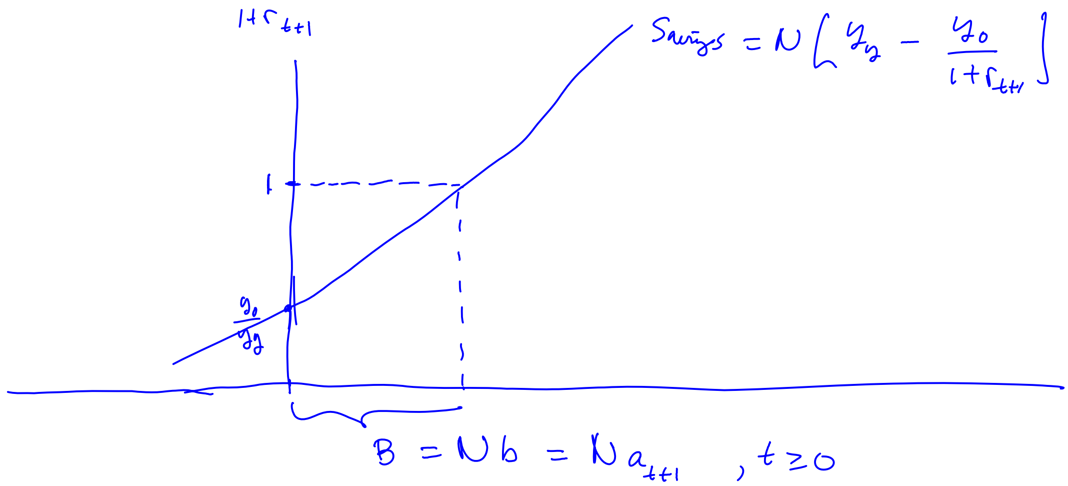
(this is another version of the midterm question!)

Remark: In the economy without a government — no social security or government debt, there is no one for a young person to borrow or lend with — why — because the only people who could repay are other young people — same cohort — and all of them have the same preferences & the same endowments. (If we introduced within-generation heterogeneity either of preferences or endowments, then we could generate non zero saving for some and non zero borrowing for others. Aggregate saving would still be zero).

So the situation looks like this:



$(1+r_{t+1}) = \frac{y_o}{y_y} < 1$ is a competitive equilibrium interest rate with zero interest rate with zero saving.



Now consider the following scheme. At time $t=0$ the government gives each old person b units of a bond where $b = \frac{1}{2}\left[y_y + y_0\right] - y_0$

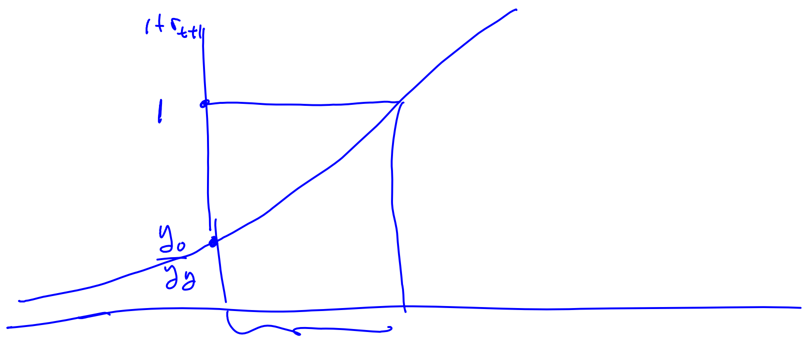
$$= \frac{1}{2}\left[y_y - y_0\right]$$

That is, An old person at $t=0$ sells the bond to a young person at time 0 in exchange for $\frac{1}{2}\left[y_y - y_0\right]$, goods. So the old person ends up consuming $\frac{1}{2}\left[y_y + y_0\right] > y_0$ at $t=0$.

The young person at $t=0$ then sets $a_1 = \frac{1}{2}\left[y_y - y_0\right] = b$, carries it into period 1, and sells it to a time 1 young person for $(1+r_{t+1})b = b$ (so r_{t+1}) time 1 consumption goods. That enables the time 1 consumption goods. That enables the time 0 young person to consume

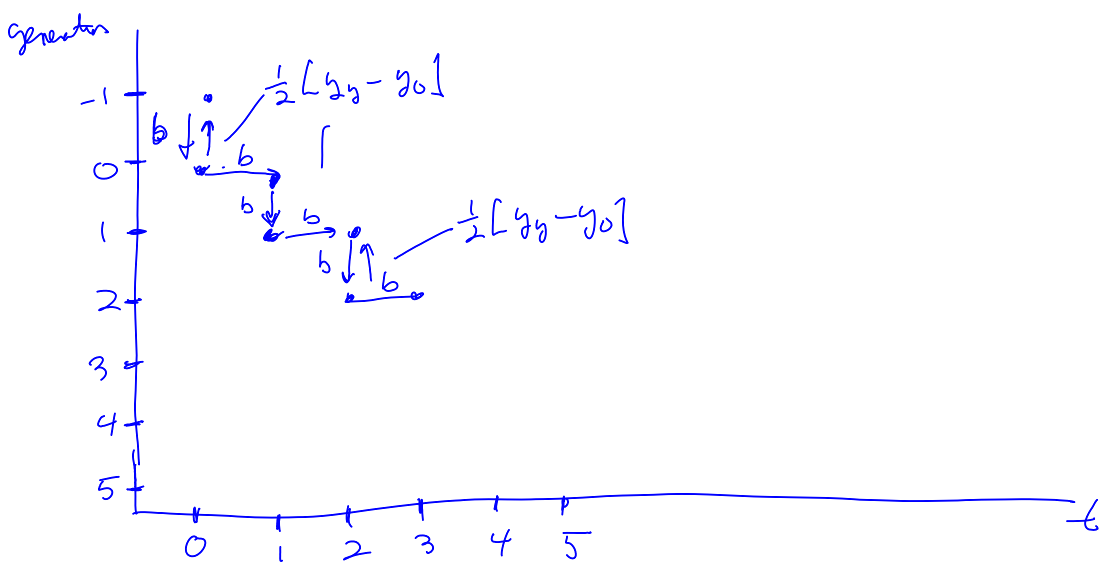
$$(c_{y0}, c_{01}) = \left(\tfrac{1}{2}[y_y + y_0],\ \tfrac{1}{2}[y_y + y_0]\right)$$

and thus completely smooth consumption over his lifetime. Thus we have



B = Nb = dissaving of old = saving of young

Each new generation of young people purchases government debt from the old people. So the picture is



The government debt gets passed from generation to generation in exchange for goods.

This government debt scheme is equivalent to a social security scheme. Explain.

Dynamics –

generalization of model to allow time-varying (r_{t+1}, a_{t+1}). Previous allocation will be a special case – a steady state with $r_{t+1} = 0$.

Saving Function:

(1) $a_{t+1} = \frac{1}{2}\left[y_y - \frac{1}{1+r_{t+1}} y_o\right]$

government budget

(2) $a_{t+1} = a_t(1+r_t)$, $t \geq 0$

a_0 = transfer to initial old.

define $R_t = (1+r_t)$

(1) & (2) =>

$$\frac{1}{2}\left[y_y - \frac{y_o}{R_{t+1}}\right] = \frac{1}{2}\left[y_y - \frac{y_o}{R_t}\right] R_t$$

$$= \frac{1}{2}\left[y_y R_t - y_o\right]$$

Notice: there are two steady states:

(1) $R_{t+1} = R_t = 1$ and $a_{t+1} = a_t = \bar{a} = \frac{1}{2}[y_y - y_o]$

(2) $R_{t+1} = R_t = \frac{y_o}{y_y}$ and $a_{t+1} = a_t = \bar{a} = 0$

above we "chose" the first one. Let's study the robustness of that choice and also we whether there are also non-steady state equilibria.

To study: the dynamic system

$$\left[y_y - \frac{y_0}{R_{t+1}} \right] = \left[y_y R_t - y_0 \right] \quad \Rightarrow$$

after several lines of algebra

$$R_{t+1} = \frac{1}{\left(1 - \frac{y_y}{y_0}(R_t - 1)\right)}$$

$$= f(R_t)$$

note: this has only two steady states

$$\bar{R} = f(\bar{R}) \Rightarrow \bar{R} = 1 \quad \text{or} \quad \frac{y_0}{y_y} < 1$$

Compute:

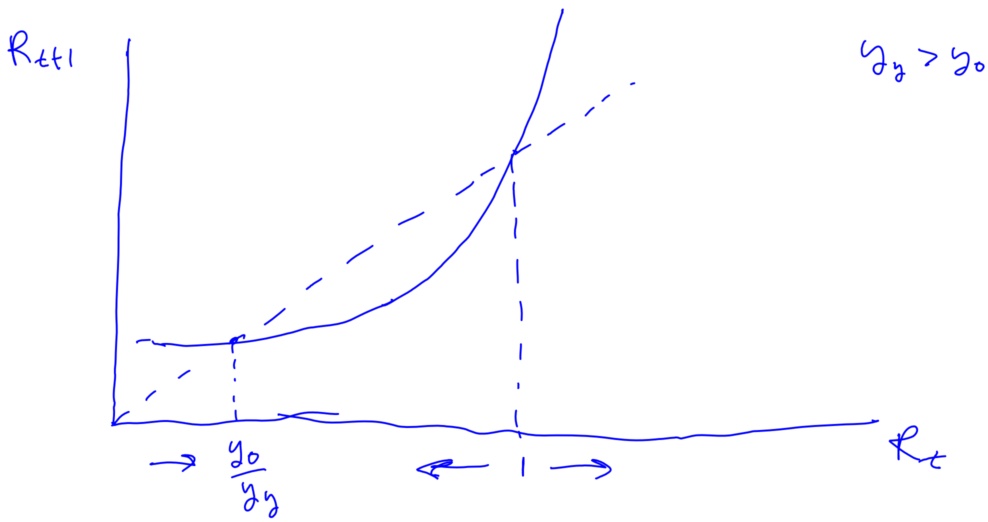
$$\frac{d\,f(R)}{dR} = \frac{y_y}{\left(y_0\left(\left(y_y \cdot (R-1)\right)/y_0 - 1\right)^2\right)}$$

we can calculate that:

$$\left.\frac{d\,f(R)}{dR}\right|_{R=1} = \frac{y_y}{y_0} > 1$$

$$\left.\frac{d\,f(R)}{dR}\right|_{R=\frac{y_0}{y_y}} = \frac{y_0}{y_y} < 1$$

$\therefore$ the picture is like this.



$\frac{y_0}{y_y}$ is stable

1 is unstable

If government sets $a_1 = \frac{1}{2}[y_y - y_0]$

and transfers $\frac{1}{2}[y_y - y_0]$ to the initial old,

the equilibrium has $a_{t+1} = a_1 \quad \forall \quad t \geq 0$

and $R_{t+1} = 1$ for all $t \geq 0$.

government "rolls over" a constant debt.

If $0 < \bar{a}_1 < \frac{1}{2}[y_y - y_0]$,

savings at 1 must equal

$$\bar{a}_1 = \frac{1}{2}\left[y_y - \frac{y_0}{R_1}\right] < \frac{1}{2}[y_y - y_0]$$

$$\Rightarrow -\frac{y_0}{R_1} < -y_0$$

$$\Rightarrow \frac{y_0}{R_1} > y_0$$

$$\Rightarrow \quad \frac{1}{R_1} > 1 \quad \Rightarrow \quad \underline{\underline{R_1 < 1}}.$$

$$\therefore \quad a_2 = R_1 \bar{a}_1 < \bar{a}_1$$

and so on $\Rightarrow$ $a_{t+1} \searrow 0$

and $R_{t+1} \searrow \dfrac{y_o}{y_y}$.

Thus, not issuing enough government debt at $t=0$ causes the value of debt to fall over time toward zero and "undo" the "social security" like transfer scheme.

Why not try setting

$$a_1 > \tfrac{1}{2}[y_y - y_o] ?$$

It won't work. It would require an $R_1 > 1$ and $R_2 > R_1$, ---- which would lead $a_{t+1} \rightarrow +\infty$.

that is not feasible. Eventually the value of government debt would exceed the total endowments in the economy. This is not possible because each period, the current old are selling the entire government debt to the young and consuming the entire proceeds. These proceeds are bounded above by $[y_y + y_o]$ = total endowment of economy at time t.

Remark: an isomorphism –

we can set

$$a_{t+1} = \frac{M}{p_t} \quad , \quad R_{t+1} = \frac{p_t}{p_{t+1}}$$

p_t = price level at t – $\frac{\$}{\text{time } t \text{ goods}}$

M = unbacked fiat money – $ the government gives to the old

$\frac{M_t}{p_t}$ = real balances ~ $\frac{\$}{\$/\text{time } t \text{ good}}$ ~ time t goods

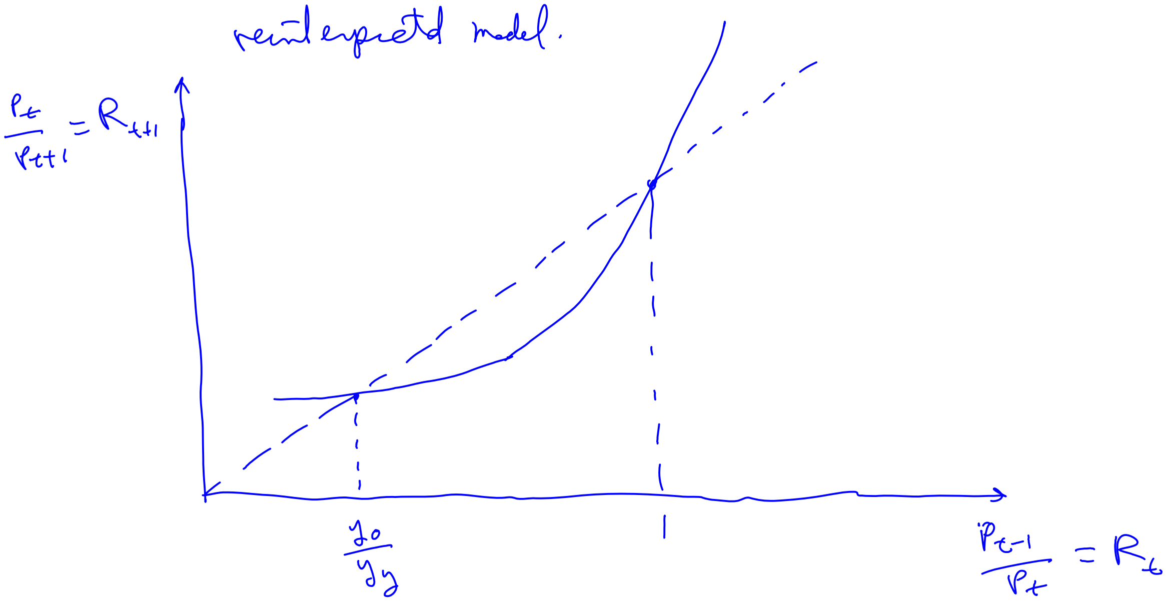
= saving

$R_{t+1} = \frac{p_t}{p_{t+1}}$ ~ $\frac{\text{time } t{+}1 \text{ goods}}{\text{time } t \text{ goods}}$ = gross real return on money

$R_{t+1} > 1$ if there is deflation

< 1 if there is inflation.

The above analysis applies to this reinterpreted model.



saving function can be written:

$$\frac{M}{P_t} = \frac{1}{2}\left[y_y - \frac{y_o}{R_{t+1}} \right]$$

$$= \frac{1}{2}\left[y_y - y_o \frac{P_{t+1}}{P_t} \right]$$

$\Longrightarrow$

$$M = \frac{1}{2}\left[P_t y_y - y_o P_{t+1} \right]$$

$$2M = y_y P_t - y_o P_{t+1}$$

$\Longrightarrow$

$$(*) \quad P_t = \frac{2}{y_y} M + \frac{y_o}{y_y} P_{t+1} \qquad , t \geq 0$$

Solution of this difference equation is

$$P_t = \underbrace{\frac{2}{y_y}\left[\frac{1}{1 - \frac{y_o}{y_y}} \right] M}_{\text{fundamental solution}} + \underbrace{c\left(\frac{y_y}{y_o}\right)^t}_{\text{bubble}}$$

for any $c > 0$

note: if $c = 0 \Longrightarrow$

$$\frac{M}{P_t} = \frac{y_y}{2}\left[1 - \frac{y_o}{y_y} \right]$$

$$= \frac{1}{2}\left[y_y - y_o \right] = \bar{a}_1 \quad \text{in our}$$

government debt model.

and P_t = constant

$$P_t = M \cancel{/} \left(\frac{1}{2}(y_y - y_0)\right)$$

↑ quantity theory of money.

$\uparrow M \Rightarrow \uparrow P$ proportionately

But if $c > 0$

$$\frac{M}{P_t} \to 0 \quad \text{and}$$

$$\frac{P_{t+1}}{P_t} \to \frac{y_y}{y_0} > 1$$

$\Rightarrow$ exploding prices drive value of money to zero.

Explain in terms of "fiat" character of money.

- describe how equation

$$P_t = \frac{2}{y_y} M + \frac{y_0}{y_y} P_{t+1} \qquad , t \geq 0$$

expresses how current price level depends on expectation of future price level (value of money)

$$\frac{M}{P_t} = \frac{1}{2}\left[y_y - \frac{y_0}{(P_t/P_{t+1})}\right]$$

demand for real balances vary inversely with expected inflation.

Scratch

Sunday, March 08, 2009
7:28 PM

Homework problem :- answer to problem

$$c_0 + \beta c_1 + \frac{A}{1-\gamma}(1-h_0)^{1-\gamma} - \beta \frac{A}{1-\gamma}(1-h_1)^{1-\gamma}$$

$h \in (0,1)$

$$\text{s.t} \quad c_0 + \beta c_1 \leq \omega_0 h_0 + \beta \omega_1 h_1$$

$$L = c_0 + \beta c_1 + \frac{A}{1-\gamma}(1-h_0)^{1-\gamma} + \frac{\beta A}{1-\gamma}(1-h_1)^{1-\gamma}$$
$$+ \lambda \left[\omega_0 h_0 + \beta \omega_1 h_1 - c_0 - \beta c_1 \right] \qquad , c_0, c_1 > 0$$

FONC:

$c_0: \quad 1 - \lambda \leq 0 \quad = 0 \text{ if } c_0 > 0$

$c_1: \quad 1 - \lambda \leq 0 \quad , = 0 \text{ if } c_1 > 0$

$\Rightarrow \} \quad 1 = \lambda$ because consume same time

$h_0: \quad A(1-h_0)^{-\gamma} + \lambda \omega_0 = 0$ interior

$h_1: \quad \beta A (1-h_1)^{-\gamma} + \lambda \beta \omega_1 = 0$ "

$$\Rightarrow \quad A(1-h_0)^{-\gamma} = -\omega_0$$
$$A(1-h_1)^{-\gamma} = -\omega_1$$

$$(1-h_0)^{-\gamma} = -\frac{\omega_0}{A}$$

$$(1-h_0) = \left(-\frac{\omega_0}{A}\right)^{-\frac{1}{\gamma}}$$

eg. $\gamma = 1$

$$(1-h_0) = \left(\frac{-\omega_0}{A}\right)^{-1} = -\frac{A}{\omega_0}$$

or

$$h_0 = 1 + \frac{A}{\omega_0}$$

$$A(1-h_0)^{-\gamma} = -\omega_0$$

$$(1-h_0)^{-\gamma} = -\frac{\omega_0}{A}$$

$$(1-h_0) = \left(\frac{-\omega_0}{A}\right)^{-\frac{1}{\gamma}}$$

eg. $\gamma = 1$

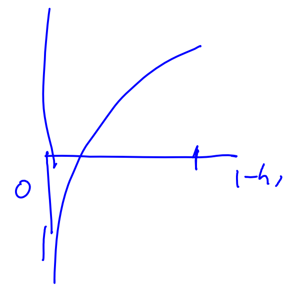
$$(1-h_0) = \left(\frac{-\omega_0}{A}\right)^{-1}$$

$$1 - h_0 = \frac{-A}{\omega_0}$$

$$1 + \frac{A_1}{\omega_0} = h_0$$

$\ln(1-h_0)$, $h_0 = 1 \Rightarrow \log(0) = -\infty$.

$$L = c_1 + \beta c_2 + \ln(1-h_1) + \beta \ln(1-h_2) + \lambda[w_1 h_1 + \beta w_2 h_2 - c_1 - \beta c_2]$$



FONC:

$c_1: \ 1 - \lambda \le 0 \ = \text{ if } c_1 > 0$

$c_2: \ 1 - \lambda \le 0 = \text{ if } c_2 > 0$

$h_1: \ \frac{-1}{1-h_1} + w_1 = 0 \Rightarrow \quad w_1 = \frac{1}{1-h_1}$

$1 - h_1 = \frac{1}{w_1}$

$h_2: \ \frac{-1}{1-h_2} + w_2 = 0$

$h_1 = 1 - \frac{1}{w_1}$

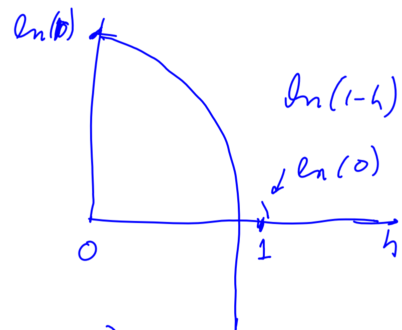
$h_2 = 1 - \frac{1}{w_2}$

$$c_1 + \beta c_2 = w_1\left(1 - \frac{1}{w_1}\right) + \beta w_2\left(1 - \frac{1}{w_2}\right) \quad , h \ge 0$$

$$= (w_1 - 1) + \beta(w_2 - 1)$$

$w_1 = 1$, $w_2 = 1$. $\Rightarrow$

$h_1 = 1 - \frac{1}{w_1} = 0$.



$w_1 \ge 1$, $w_2 \ge 1$.

$\ln(1-h_1) = g(h)$

$g'(h) = \frac{-1}{1-h_1} < 0$, $g''(h) = \frac{1(-1)}{(1-h_1)^2} < 0$

Scratch

Tuesday, March 24, 2009
3:41 PM

Williamson problem 10.

$$Y_t = z K_t$$

$$\frac{Y_t}{N_t} = z \frac{K_t}{N_t}$$

$$\frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} = (1-\delta) \frac{K_t}{N_t} + \frac{X_t}{N_t}$$

$$k_{t+1}(1+n) = (1-\delta) k_t + x_t$$

$$f(k_t) = \frac{Y_t}{N_t} = z\frac{K_t}{N_t} = z k_t$$

$$z f(k_t) = z k_t \qquad \Rightarrow$$

$$k_{t+1} = \frac{s}{(1+n)} z k_t + \frac{(1-\delta)}{1+n} k_t$$

$$k_{t+1} = \left[\frac{sz}{1+n} + \frac{(1-\delta)}{1+n}\right] k_t$$

$$k_{t+1} = \left[\frac{sz + (1-\delta)}{1+n}\right] k_t$$

New Problem

Wednesday, March 10, 2010

5:57 PM

1. (15 points)

A consumer chooses $\{c_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad , u'>0, u''<0$$

subject to $$\sum_{t=0}^{\infty} q_t^0 c_t \leq \sum_{t=0}^{\infty} q_t^0 y_t$$

where $\{y_t\}_{t=0}^{\infty}$ is an exogenous endowment

and q_t^0 = the price of one unit of consumption at time t, measured in units of time 0 consumption,

so

$q_0^0 = 1$ (a normalization)

Assume that

$$u(c) = \frac{1}{1-\gamma} c^{1-\gamma} \quad , \gamma > 0 \quad , \gamma \neq 1$$

$$= \ln c \quad \text{if } \gamma = 1$$

(Remember that $$q_t^0 = \frac{1}{(1+\rho_{0t})^t}$$

where ρ_{0t} is the yield to maturity on a t-period zero-coupon bond)

Consider a "Lucas model" where there is a representative consumer and prices $\{q_t^0\}_{t=0}^{\infty}$ must adjust to induce the consumer to set $c_t = y_t \ \forall t \geq 0$.

a. Suppose $y_t = y_0 \delta^t$, $0 < \delta < \frac{1}{\beta}$.

Compute q_t^0. Compute ρ_{0t}.

b. In the special case of $\gamma = 1$ (log utility), compute q_t^0 and ρ_{0t}.

c. When $\delta = 1$, compute q_t^0.

d. Is the one-period interest rate ρ_{01} . "high" or "low" when $\delta > 1$?

e. Is the one-period interest rate ρ_{01} "high" or "low" when δ