Quiz #3

April 2012

Please write your name below. Then complete the exam in the space provided. There are TWO questions. You may refer to one page of notes: standard paper, both sides, any content you wish.

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(Name and signature)

1. (moving averages) (50 points) Consider the MA(2),

$$x_t \;=\; \delta + w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2},$$

with $\{w_t\} \sim \text{NID}(0,1)$ (the w's are independent normals with mean zero and variance one). Our mission is to explore its properties.

(a) What is the mean of x? The variance? (10 points)

(b) What are the conditional means, $E_t(x_{t+1})$, $E_t(x_{t+2})$, and $E_t(x_{t+3})$? (10 points)

(c) What are the conditional variances, $\text{Var}_t(x_{t+1})$, $\text{Var}_t(x_{t+2})$, and $\text{Var}_t(x_{t+3})$? (10 points)

(d) What is the autocovariance function,

$$\gamma(k) \;=\; \text{Cov}(x_t, x_{t-k}),$$

for $k = 0, 1, 2, 3$? (10 points)

(e) What is the autocorrelation function? Under what conditions are $\rho(1)$ and $\rho(2)$ positive? (10 points)

Solution:

(a) The mean is δ,

$$E(x_t) \;=\; E(\delta + w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2}) \;=\; \delta.$$

The variance is

$$\text{Var}(x_t) \;=\; E(x_t - \delta)^2 \;=\; 1 + \theta_1^2 + \theta_2^2.$$

(b) The conditional means are

$$
\begin{aligned}
E_t(x_{t+1}) &= E_t(\delta + w_{t+1} + \theta_1 w_t + \theta_2 w_{t-1}) &= \delta + \theta_1 w_t + \theta_2 w_{t-1} \\
E_t(x_{t+2}) &= E_t(\delta + w_{t+2} + \theta_1 w_{t+1} + \theta_2 w_t) &= \delta + \theta_2 w_t \\
E_t(x_{t+3}) &= E_t(\delta + w_{t+3} + \theta_1 w_{t+2} + \theta_2 w_{t+1}) &= \delta.
\end{aligned}
$$

You can see that as we increase the forecast horizon, the conditional mean approaches the mean.

(c) The conditional variances are

$$
\begin{aligned}
\text{Var}_t(x_{t+1}) &= E_t[(w_{t+1})^2] &= 1 \\
\text{Var}_t(x_{t+2}) &= E_t[(w_{t+2} + \theta_1 w_{t+1})^2] &= 1 + \theta_1^2 \\
\text{Var}_t(x_{t+3}) &= E_t[(w_{t+3} + \theta_1 w_{t+2} + \theta_2 w_{t+1})^2] &= 1 + \theta_1^2 + \theta_2^2.
\end{aligned}
$$

You see here that as we increase the forecast horizon, the conditional variance approaches the variance.

(d) The autocovariance function is

$$
\text{Cov}(x_t, x_{t-k}) = \begin{cases} 1 + \theta_1^2 + \theta_2^2 & k = 0 \\ \theta_1 + \theta_1\theta_2 & k = 1 \\ \theta_2 & k = 2 \\ 0 & k \geq 3. \end{cases}
$$

(e) Autocorrelations are scaled autocovariances: $\rho(k) = \gamma(k)/\gamma(0)$. $\rho(2)$ is positive if θ_2 is. $\rho(1)$ is positive if $\theta_1(1+\theta_2)$ is. Both are therefore positive if θ_1 and θ_2 are positive.

2. (moving average bond pricing) (50 points) Consider the bond pricing model

$$
\begin{aligned}
\log m_{t+1} &= -\lambda^2/2 - x_t + \lambda w_{t+1} \\
x_t &= \delta + \sigma(w_t + \theta w_{t-1}).
\end{aligned}
$$

(a) What is the short rate f_t^0? (10 points)

(b) Suppose bond prices take the form

$$
\log q_t^n = A_n + B_n w_t + C_n w_{t-1}.
$$

Use the pricing relation to derive recursions connecting $(A_{n+1}, B_{n+1}, C_{n+1})$ to (A_n, B_n, C_n). What are (A_n, B_n, C_n) for $n = 0, 1, 2, 3$? (20 points)

(c) Express forward rates as functions of the state (w_t, w_{t-1}). What are f_t^1 and f_t^2? (10 points)

(d) What is $E(f^1 - f^0)$? What parameters govern its sign? (10 points)

Solution:

(a) The short rate is

$$f_t^0 \quad = \quad -\log E_t m_{t+1} \quad = \quad \lambda^2/2 + x_t - \lambda^2/2 \quad = \quad x_t.$$

The second equality is the usual "mean plus variance over two" with the sign flipped (as indicated by the first equality). In other words: the usual setup. In what follows, we'll kill off x_t by substituting.

(b) Bond prices follow from the pricing relation,

$$q_t^{n+1} \quad = \quad E_t(m_{t+1} q_{t+1}^n),$$

starting with $n = 0$ and $q_t^0 = 1$. The state in this case is (w_t, w_{t-1}), a simple example of a two-dimensional model, hence the extra term in the form of the bond price. We need

$$\log(m_{t+1} q_{t+1}^n) \quad = \quad A_n - (\lambda^2/2 + \delta) + (\lambda + B_n) w_{t+1} + (C_n - \sigma) w_t - \sigma\theta w_{t-1}.$$

The (conditional) mean and variance are

$$\begin{aligned} E_t[\log(m_{t+1} q_{t+1}^n)] &= A_n - (\lambda^2/2 + \delta) + (C_n - \sigma) w_t - \sigma\theta w_{t-1} \\ \mathrm{Var}_t[\log(m_{t+1} q_{t+1}^n)] &= (\lambda + B_n)^2. \end{aligned}$$

Using "mean plus variance over two" and lining up terms gives us

$$\begin{aligned} A_{n+1} &= A_n - (\lambda^2/2 + \delta) + (\lambda + B_n)^2/2 \\ &= A_n - \delta + \lambda B_n + (B_n)^2/2 \\ B_{n+1} &= C_n - \sigma \\ C_{n+1} &= -\sigma\theta \end{aligned}$$

for $n = 0, 1, 2, \ldots$. That gives us

n	A_n	B_n	C_n
0	0	0	0
1	$-\delta$	$-\sigma$	$-\sigma\theta$
2	$-2\delta - \lambda\sigma + \sigma^2/2$	$-\sigma(1+\theta)$	$-\sigma\theta$
3	X	$-\sigma(1+\theta)$	$-\sigma\theta$

with $X = -3\delta - \lambda(2+\theta) + [1 + (1+\theta)^2]\sigma^2/2$.

(c) In general, forward rates are

$$f_t^n \quad = \quad (A_n - A_{n+1}) + (B_n - B_{n+1}) w_t + (C_n - C_{n+1}) w_{t-1}.$$

That gives us

$$\begin{aligned} f_t^0 &= \delta + \sigma w_t + \sigma\theta w_{t-1} \\ f_t^1 &= \delta + \lambda\sigma - \sigma^2/2 + \sigma\theta w_t \\ f_t^2 &= \delta - (1+\theta)^2\sigma^2/2 + \lambda\sigma(1+\theta). \end{aligned}$$

(d) The means are the same with $w_t = w_{t-1} = 0$, their mean. Therefore

$$E(f^1 - f^0) = \lambda\sigma - \sigma^2/2.$$

Therefore we need $\lambda\sigma > \sigma^2/2$, so a necessary condition is that λ and σ have the same sign.