Lab Report #3: Two-Period Macro Models

(Started: February 2, 2012; Revised: February 20, 2012)

Due at the start of class. You may speak to others, but whatever you hand in should be your own work.

1. (consumption, investment, and state prices) Consider a two-period economy with a linear technology. What is equilibrium consumption growth? What are the state prices? What is a claim to one unit of capital worth?

   To address these questions, we use a variant of our two-period economy, with dates 0 and 1 and states z at date 1 that occur with probability $p(z)$. The representative agent has utility function

   $$u(c_0) + \beta \sum_z p(z) u[c_1(z)],$$

   with $u(c) = c^{1-\alpha}/(1-\alpha)$ for $\alpha > 0$ (power utility). She is endowed with y_0 units of the date-0 good, nothing at date 1. The technology is linear: k units of the date-0 good invested in capital generate zk units of the good in state z at date 1. The resource constraints are therefore

   $$\begin{aligned} c_0 + k &= y_0 \\ c_1(z) &= zk, \end{aligned}$$

   with one of the latter for each state z. Think of the productivity factor z as $a(z)$ with $a(z) = z$.

   (a) What are the classical "ingredients" of this economy?

   (b) What is the associated Pareto problem? What are its first-order conditions?

   (c) Suppose z is lognormal: that is, $\log z \sim \mathcal{N}(\kappa_1, \kappa_2)$. Use the properties of lognormal random variables to show that $E(z^a) = \exp(a\kappa_1 + a^2\kappa_2/2)$ for any real number a.

   (d) Use this result to find the optimal values of c_0 and k. Given these values, what is saving?

   (e) (extra credit) What is $c_1(z)$? What are the state prices?

   (f) (extra credit) What is the value of one unit of capital, that is, a claim to z units of output in each state z?

   Solution:

   (a) Commodities: the good at date 0 plus the good in all states at date 1
   Agents: one
   Preferences, endowment, and technologies: given above
   Resource constraints: ditto

(b) Pareto problem based on the Lagrangean:

$$\begin{aligned}\mathcal{L} &= \log c_0 + \beta \sum_z p(z) \log c_1(z) \\ &\quad + q_0(y_0 - c_0 - k) + \sum_z q_1(z)[zk - c_1(z)].\end{aligned}$$

We choose c_0, k, and $c_1(z)$ to maximize this. The first-order conditions are

$$\begin{aligned} c_0: \quad & c_0^{-\alpha} &=& \; q_0 \\ k: \quad & q_0 &=& \sum_z q_1(z) z \\ c_1(z): \quad & \beta p(z) c_1(z)^{-\alpha} &=& \; q_1(z) \end{aligned}$$

(c) Define $x = \log z$. Its mgf is $e^{sx} = E(e^{s \log z}) = E(z^s) = e^{s\kappa_1 + s^2\kappa_2/2}$. Setting $s = a$ gives us the answer.

(d) This is moderately demanding, but here's how it works. With apologies for mixing sums and integrals, we use the first and third first-order conditions to substitute for q_0 and $q_1(z)$ in the second:

$$\begin{aligned} 1 &= \sum_z \frac{q_1(z)}{q_0} z = \sum_z \frac{\beta p(z) c_1(z)^{-\alpha}}{c_0^{-\alpha}} z = \sum_z \beta p(z) \frac{(zk)^{-\alpha}}{(y_0 - k)^{-\alpha}} z \\ &= \beta (y_0 - k)^{\alpha} k^{-\alpha} \sum_z p(z) z^{1-\alpha} = \beta (y_0 - k)^{\alpha} k^{-\alpha} E(z^{1-\alpha}). \end{aligned}$$

To simplify, denote $E(z^{1-\alpha}) = Z$. Then consumption and capital are

$$k = \frac{(\beta Z)^{1/\alpha}}{1 + (\beta Z)^{1/\alpha}} \, y_0, \quad c_0 = \frac{1}{1 + (\beta Z)^{1/\alpha}} \, y_0$$

Using the lognormal result, we have

$$Z = E(z^{1-\alpha}) = e^{(1-\alpha)\kappa_1 + (1-\alpha)^2 \kappa_2/2}$$

(e) Evidently

$$c_1(z) = zk = z y_0 \frac{(\beta Z)^{1/\alpha}}{1 + (\beta Z)^{1/\alpha}}.$$

The big part at the end is a constant, so it's ugly but innocuous. The state prices are (more tedious substitution)

$$q(z) = p(z) \beta [c_1(z)/c_0]^{-\alpha} = p(z) z^{-\alpha}/Z.$$

As usual, the state price is the product of a pricing kernel [here $m(z) = z^{-\alpha}/Z$] and a probability [the normal density for $\log z$, which we haven't bothered to write out].

(f) This is claim to z next period, with value at date 0 of

$$q^e \;=\; E(z^{1-\alpha}/Z \;=\; 1.$$

Hmmmm... Why does this make sense? Because one unit of capital is one unit of the good at date 0, whose price is one since we're valuing assets in units of the date-0 good.

2. (pricing state-contingent claims) Consider a two-period exchange economy with power utility and endowment growth and $\log c_1 - \log c_0 = \log y_1 - \log y_0 = z \sim \mathcal{N}(\kappa_1, \kappa_2)$ and $y_0 = 1$. We'll compute the prices of equity and two equity derivatives using a representative agent's marginal rate of substitution as a pricing kernel.

   (a) What is the marginal rate of substitution for this economy?

   (b) What is the price q^1 of a bond that pays one in each state? Express it as a function of α, β, κ_1, and κ_2.

   (c) What is the price q^e of "equity," a claim to the date-1 endowment $y_1(z) = e^z$? What is its return? Expected return?

   (d) Consider an "upside" derivative that pays y_1 if $y_1 \geq b$ and a "downside" derivative that pays y_1 if $y_1 \leq b$. Show that the prices (call them q^u and q^d) sum to the price of equity.

   (e) (extra credit) What is the price of the downside derivative for the lognormal case? Hint: this involves integrating the normal distribution as we did when we computed the normal mgf. The difference is that we only integrate over the region in which the dividend is positive.

Solution:

(a) The mrs is

$$m(z) \;=\; \beta[c_1(z)/c_0]^{-\alpha} \;=\; \beta e^{-\alpha z},$$

so $\log m \sim \mathcal{N}(\log \beta - \alpha\kappa_1, \alpha^2\kappa_2)$.

(b) The lognormal result again:

$$q^1 \;=\; E(m) \;=\; \beta e^{-\alpha\kappa_1 + \alpha^2\kappa_2/2}.$$

The return is $r^1 = 1/q^1$.

(c) With our distributional assumptions, we have

$$\begin{aligned} q^e &= E(my_1) \;=\; \beta E(e^{-\alpha z} e^z) \;=\; \beta E(e^{(1-\alpha)z}) \\ &= \beta e^{(1-\alpha)\kappa_1 + (1-\alpha)^2\kappa_2/2}. \end{aligned}$$

This should look familiar.

(d) We're splitting the dividends in two — upside and downside — so a claim to one unit of each is equivalent to a claim to equity.

(e) This is moderately demanding, too, but it's exactly the kind of math we'll use later to price options. The price of the downside derivative is

$$q^d \;=\; \int_{-\infty}^{z^*} m(z) y_1(z) p(z) dz$$

with $z^* = \log b$ the upper bound on z corresponding to $y_1 = b$. Now we're back to the same math we used to compute the mgf for a normal random variable:

$$\begin{aligned} m(z)y_1(z)p(z) &= \beta e^{-\alpha z} e^z (2\pi\kappa_2)^{-1/2} e^{-(z-\kappa_1)^2/2\kappa_2} \\ &= \beta e^{(1-\alpha)\kappa_1 + (1-\alpha)^2\kappa_2/2} (2\pi\kappa_2)^{-1/2} e^{-(z+(1-\alpha)\kappa_2-\kappa_1)^2/2\kappa_2} \\ &= q^e (2\pi\kappa_2)^{-1/2} \exp[-(w-\kappa_1)^2/2\kappa_2], \end{aligned}$$

where $w = z + (1-\alpha)\kappa_2$. The integral is therefore

$$(2\pi\kappa_2)^{-1/2} \int_{-\infty}^{\log b + (1-\alpha)\kappa_2} \exp[-(w-\kappa_1)^2/2\kappa_2] dw.$$

More on this another time.

3. (risk-neutral probabilities) Consider yet another two-period exchange economy with power utility and Bernoulli endowment growth:

$$\log(c_1/c_0) \;=\; \begin{cases} 0 & \text{with probability } 1-\omega \quad \text{(state } z=1\text{)} \\ g > 0 & \text{with probability } \omega \quad \text{(state } z=2\text{)} \end{cases}$$

(a) What is the pricing kernel for this economy? Is the pricing kernel higher in state 1 or state 2? Why?

(b) What are the state prices? Which is more valuable, a claim to one in state 1 or a claim to one in state 2? Why?

(c) What is the price q^1 of a bond paying one in each state?

(d) What are the risk-neutral probabilities? Why are they different from the true probabilities?

Solution:

(a) Same pricing kernel as before, but in a two-state setting. In logs:

$$\log m \;=\; \begin{cases} \log\beta & \text{in state } z=1 \\ \log\beta - \alpha g & \text{in state } z=2. \end{cases}$$

Evidently m is lower in state 2. Why? There's more consumption in state 2, so its marginal utility is lower.

(b) State prices are connected to the pricing kernel by $q(z) = p(z)m(z)$:

$$q(z) \;=\; p(z)m(z) \;=\; \begin{cases} (1-\omega)\beta & \text{in state } z=1 \\ \omega\beta e^{-\alpha g} & \text{in state } z=2. \end{cases}$$

(c) The sum of state prices:

$$q^1 \;=\; q(1) + q(2) \;=\; \beta\left[(1-\omega) + \omega e^{-\alpha g}\right].$$

(d) Risk-neutral probabilities are defined by $q^1 p^*(z) = p(z)m(z)$. Here we have

$$p^*(2) \;=\; \frac{\omega e^{-\alpha g}}{1 - \omega + \omega e^{-\alpha g}}$$

and $p^*(1) = 1 - p^*(2)$. These "probabilities" are adjusted for risk. If there's no risk aversion ($\alpha = 0$), $p^*(z) = p(z)$. Otherwise, $p^*(2) < p(2)$: we put less weight on the good state (and more on the bad state), which is a way of taking into account the risk. Note, too, that β drops out: it affects discounting but not risk adjustment.