Risk & Risk Aversion
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Risk and risk aversion are fundamental building blocks for thinking about how economic risk is priced. We say people are *risk averse* if they prefer a sure thing to a risky outcome with the same mean. The question is what form this risk aversion takes. We'll see that the entire distribution matters. If we think about this in terms of moments, it's not just the variance that's relevant, but skewness and kurtosis, too.

A quick review follows. We start off abstractly, then add more detail as we go. Some of the math is unnecessary, but I thought looking at the issue from several different perspectives would give us deeper insight. We end up with what will become our workhorse: expected utility with a power utility function.

Risk

The risk we care about is the risk of outcomes, which we'll describe with our basic setup for random variables: states z drawn from a set $\mathcal{Z}$ and probabilities $p(z)$. Suppose the outcome we care about is a random variable $c(z)$, which assigns an outcome "consumption in state z" to every state z. We say the outcome is riskless or riskfree if $c(z)$ is the same for all z. Otherwise, it's risky. What kinds of risk are we concerned with? That depends on our preferences, which we turn to next.

Utility functions and certainty equivalents

Utility functions are mathematical devices that we use to describe the preferences of an "economic agent," a hypothetical individual who follows whatever rules we lay down. The function tells us the amount of utility ("happiness") the agent gets from any combination of consumption quantities. For example, the function $U(a, b)$ might express preferences over quantities of apples and bananas. We assume without fail that more is always better: increasing the quantities a or b increases utility U. Other properties of U it might indicate whether the agent prefers apples or bananas, and how many of one it takes to compensate for the loss of the other.

We use the same tool to describe preferences over risky consumption. Let us say that the set of possible states is finite: $\mathcal{Z} = \{1, 2, \ldots, Z\}$. Nothing important depends on this, but it makes the notation simpler. The vector of possible consumption outcomes is therefore $[c(1), c(2), \ldots, c(Z)]$. Utility over these outcomes might be written $U[c(1), c(2), \ldots, c(Z)]$. Again we would say that U is increasing in each of its arguments (more is better). Depending on other properties of U, the agent might prefer to invest different amounts in risky securities or take jobs with more or less risky income outcomes — any decision that involves risk.

We've done this at a high level of abstraction, but it's helpful nonetheless to give labels to things, even if we're somewhat vague at this point about what they mean.

[Draw indifference curve for 2-state case.]

One way to assess the agent's attitude toward risk is to look at the utility of a given vector of consumption outcomes with that of a constant outcome. The *certainty equivalent* μ is the value of constant consumption that generates exactly the same utility:

$$U[c(1), c(2), \ldots, c(Z)] \quad = \quad U(\mu, \mu, \ldots, \mu). \tag{1}$$

The idea is to think about this as an equation that (implicitly) determines μ. If we knew the function U, we could presumably do the calculation; we'll do exactly that later on once we've been more explicit about U. In a sense, μ converts the utility of the risky outcome into handy consumption units. Note, too, that if consumption is constant, then the certainty equivalent equals that same constant.

[Back to 2-state case. Show 45-degree line and certainty equivalent.]

Even at this level of generality, we can talk about risk aversion. Consider these two objects related to a risky consumption outcome $\{c(z)\}$:

- The certainty equivalent μ.

- The mean consumption outcome. The mean is

$$\bar{c} \quad = \quad \sum_z p(z)c(z) \quad = \quad E(c).$$

Comparing these two objects tells us how much we dislike the risky outcome. If $\mu = \bar{c}$, then we don't care about risk at all. Why? Because risky consumption gives us the same amount of utility as constant consumption equal to $\bar{c}$. Evidently risk doesn't affect our utility in this case. More generally, we say the agent is risk averse if the certainty equivalent of the risky outcome is less than the mean: $\mu < \bar{c}$. The difference between them measures how much we dislike the risky outcomes. A useful measure of this dislike is the "risk penalty"

$$rp \quad = \quad \log(\bar{c}/\mu). \tag{2}$$

It tells us how much of a consumption penalty we'd be willing to accept relative to the mean to have riskless consumption. The log form is arbitrary, but will be convenient later on.

Here's an example. Let's say there are two states with probabilities $p(1) = p(2) = 1/2$. Suppose $c(1) = 2$ and $c(2) = 8$. The $\bar{c} = 5$. What is the certainty equivalent μ? You need to know your preferences to do this, but let's it's it's 4: you're willing to forgo one relative to the mean to eliminate the risk. The risk penalty is therefore $rp = \log(\bar{c}/\mu) = \log(5/4) = 0.2231$.

Concave and convex functions

Here's some terminology we'll use shortly — it comes up a lot in economics and finance. We say a function $f(x)$ is *concave* if for any two points x_1 and x_2, the function of a weighted average is less than the weighted average of the function:

$$f[wx_1 + (1-w)x_2] \quad \leq \quad wf(x_1) + (1-w)f(x_2)$$

for any number w between zero and one. If f is differentiable, that means $f''(x) < 0$. We say $f(x)$ is *convex* if the inequality goes the other way. Equivalently, f is convex if $-f$ is concave. Sometimes people use strong inequalities and distinguish between strict and weak concavity/convexity.

Examples. Which of the following are concave? Convex?

$$
\begin{aligned}
(a) \quad f(x) &= x - x^2 \\
(b) \quad f(x) &= \log x \\
(c) \quad f(x) &= e^x \\
(d) \quad f(x) &= x^{1-\alpha}/(1-\alpha) \\
(e) \quad f(x) &= \max\{0, x - b\}.
\end{aligned}
$$

Think about these, maybe draw them to see what they look like. The last one you might recognize as the payoff from a call option on x with strike price b.

Suppose, now, that x is a random variable and $f(x)$ is a function of it. *Jensen's inequality* says that if f is concave, then

$$Ef(x) \quad \leq \quad f[E(x)],$$

with equality if x is constant. The difference between the left and right sides reflects the impact of risk, so you can see that this might be an essential tool in quantifying risk. If f is convex, the inequality goes the other way.

Example. Suppose x is normal with mean κ_1 and variance κ_2. Then

$$E\left(e^x\right) \quad = \quad e^{\kappa_1 + \kappa_2/2} \quad \geq \quad e^{\kappa_1},$$

with equality if $\kappa_2 = 0$ (x is constant at μ). [You should ask yourself: what does this have to do with Jensen's inequality?]

Expected utility

Most work in economics and finance uses expected utility, a specific form of the function U that we'll define shortly. We'll be no different. But I think our long buildup is useful because it hints that other approaches to risk are possible. In fact, it's an active topic of research, both in economics and psychology.

Expected utility gives U the additive form

$$U[c(1), c(2), \ldots, c(Z)] \;\; = \;\; \sum_z p(z)u[c(z)] \;\; = \;\; E[u(c)]. \tag{3}$$

for some state-utility function u. This puts a lot of structure on preferences, but structure is good if it's sensible, it gives us more precise predictions.

Probabilities here are (in theory) subjective: they're positive numbers $p(z)$ that sum to one that reflect the agent's preferences. We go one step further and make the assumption of *rational expectations*: $p(z)$ is the true probability distribution over states z. There's been no end of debate about this, but it's incredibly useful when we implement expected utility and compare its predictions with evidence.

The action in expected utility is in the function u. It controls the certainty equivalent, the solution to

$$\sum_z p(z)u[c(z)] \;\; = \;\; \sum_z p(z)u(\mu) \;\; = \;\; u(\mu).$$

More on this shortly. Clearly u must be increasing (more is better): that is, $u'(c) > 0$. That means we can solve for the certainty equivalent:

$$\mu \;\; = \;\; u^{-1}\left(\sum_z p(z)u[c(z)] \right) \;\; = \;\; u^{-1}\left[Eu(c) \right].$$

The expression u^{-1} means the inverse function of u, so that $u^{-1}[u(x)] = x$. This is a little abstract at this point, but we'll put it to work in the next section.

Risk aversion is more complicated. We'll look at it a couple different ways. Both tell us that u is concave, which here means $u''(c) < 0$. The first is to look at the 2-state case. Suppose the two states occur with probability one-half each and consumption outcomes $c(1) < c(2)$. Mean consumption is therefore $\bar{c} = (1/2)c(1) + (1/2)c(2)$. Expected utility is

$$E[u(c)] \;\; = \;\; (1/2)u[c(1)] + (1/2)u[c(2)].$$

When is this less than $u(\bar{c})$? When the function u is concave. so we have $E[u(c)] < u(\bar{c})$ (risk aversion) if $u'' < 0$. And if we take u^{-1} of both sides, we have $\mu < \bar{c}$.

[As usual, it's easier if you draw a picture. Plot $u(c)$ v c. Show expected utility and certainty equivalent.]

Another approach is to do a Taylor series expansion, a trick devised by Paul Samuelson. Recall that the Taylor series expansion (if it exists) of a function f of a variable x around the point x^* is

$$f(x) \;\; = \;\; f(x^*) + f'(x^*)(x - x^*) + f''(x^*)(x - x^*)^2/2 + f'''(x^*)(x - x^*)^3/3! + \cdots.$$

The Taylor series expansion of u around mean consumption $\bar{c}$ is

$$u(c) \;\; = \;\; u(\bar{c}) + u'(\bar{c})(c - \bar{c}) + u''(\bar{c})(c - \bar{c})^2/2 + u'''(\bar{c})(c - \bar{c})^3/3! + \cdots.$$

Expected utility is therefore

$$
\begin{aligned}
Eu(c) &= u(\bar{c}) + u''(\bar{c})E(c - \bar{c})^2/2 + +u'''(\bar{c})E(c - \bar{c})^3/3! + \cdots \\
&= u(\bar{c}) + u''(\bar{c})\mu_2/2 + u'''(\bar{c})\mu_3/3! + u''''(\bar{c})\mu_4/4! + \cdots,
\end{aligned}
$$

where μ_j is the jth central moment. We'll refer to this as the Samuelson expansion. Note that the linear term disappears. It's sometimes said that expected utility agents don't care about small risks, because if the risks are small enough the other terms go to zero and $u(c) \approx u(\bar{c})$. Note too that increasing the variance reduces utility since $u'' < 0$, so variance is evidently a disliked component of risk. What about skewness or kurtosis, captured here by the third and fourth central moments, μ_3 and μ_4? The answer depends, apparently, on the third and fourth derivatives of u, whatever those might be.

Power utility

Since we have a practical applied interest in this subject, we'd like to have a specific function we can put to work. That function is the power function; applied to utility we would write

$$
u(c) = c^{1-\alpha}/(1 - \alpha) \tag{4}
$$

for some parameter $\alpha > 0$. The parameter α controls sensitivity to risks of all kinds and is often referred to (for reasons we won't go into) as the *coefficient of relative risk aversion*. Larger values of α are associated with stronger aversion to risk.

What about the derivatives? The first two are

$$
\begin{aligned}
u'(c) &= c^{-\alpha} &> 0 \\
u''(c) &= -\alpha c^{-\alpha-1} &< 0,
\end{aligned}
$$

which have the signs we want. The third and fourth derivatives are

$$
\begin{aligned}
u'''(c) &= \alpha(\alpha + 1)c^{-\alpha-2} &> 0 \\
u''''(c) &= -\alpha(\alpha + 1)(\alpha + 2)c^{-\alpha-2} &< 0.
\end{aligned}
$$

The Samuelson expansion (4) tells us then that power expected utility agents like positive skewness and dislike positive kurtosis.

With this functional form, the certainty equivalent is

$$
\mu = \left(\sum_z p(z)c(z)^{1-\alpha} \right)^{1/(1-\alpha)} = \left[E(c^{1-\alpha}) \right]^{1/(1-\alpha)}. \tag{5}
$$

[If that's not clear, go back to the previous versions.] It's a little ugly, but at least we have a formula we can use — and we'll do that right now.

One question we might ask ourselves is what values of α are reasonable. That's a tough question, but a useful starting point is to consider an example and apply introspection. Suppose you are given two choices for your career income, with the same income every year:

- Choice 1: probability one-half each of an income (= consumption) of 100k a year or 200k a year.

- Choice 2: a sure income (= consumption) of $\bar{c}$ (your certainty equivalent).

The question for you (here's where introspection comes in) is What value of μ generates the same utility as Choice 1? Given an answer, we can reverse engineer your risk aversion parameter α.

Here's a Matlab script that does the calculation:

```
% inputs
c1 = 100;
c2 = 200;
cbar = 125        % introspective estimate of your certainty equivalent
% guess alpha and see how close mu comes to cbar
alpha = 2         % comment: enter number of your choice
mu = (0.5*c1^(1-alpha)+0.5*c2^(1-alpha))^(1/(1-alpha))
```

If your certainty equivalent is $\bar{c} = 125$, then your implied value of α is about 3.25. You can figure this out by trying different values of α until you find one that gives you the same certainty equivalent: $\mu = \bar{c}$. Or you could do this for a bunch of values of α at the same time (think of α as a grid) using Matlab's vector capabilities. Or you could use a numerical equation-solver, which we'll do later in the course.

Cumulant expansion of certainty equivalents

Here's another expansion, this one expressing the risk penalty in terms of the cumulants of $\log c$. Let $x = \log c$ have cgf $k(s) = \log E(e^{sx})$. Note that $\bar{c} = E(c) = E(e^x) = \exp[k(1)]$. The certainty equivalent (5) follows from

$$
\begin{aligned}
E(c^{1-\alpha}) &= Ee^{(1-\alpha)x} = \exp[k(1-\alpha)] \\
\mu &= [E(c^{1-\alpha})]^{1/(1-\alpha)} = \exp[(1-\alpha)^{-1}k(1-\alpha)].
\end{aligned}
$$

The risk penalty (2) is therefore

$$
\begin{aligned}
rp &= k(1) - (1-\alpha)^{-1}k(1-\alpha) \\
&= [1-(1-\alpha)]\kappa_2/2 + [1-(1-\alpha)^2]\kappa_3/3! + [1-(1-\alpha)^3]\kappa_4/4! + \cdots,
\end{aligned}
$$

where κ_j is the jth cumulant of $x = \log c$. The one in each coefficient is simply the impact of the cumulant on the mean: increasing any cumulant of $\log c$ raises the mean of c. Beyond that, the magnitude of the impact of a cumulant on the risk penalty depends on the risk aversion parameter α. The coefficient of the variance (κ_2) is simply α. Higher skewness (κ_3) reduces the risk penalty if $\alpha > 2$. And so on.

Bottom line

Expected utility depends, in general, on the complete distribution of outcomes, including its moments and cumulants. The power utility version is particularly useful, since risk aversion is captured by a single parameter.