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Lecture 4

Two-Period Competitive Equilibrium Model: Part 1

In the last lecture we developed a static compeitive equilibrium model of output and employment. Firms in the economy only produced consumption goods. Now we consider a two period model with investment goods as well as consumption. In the competiive equilibrium, the allocation of output between consumption and investment is determined, as well as the overall level of output and employment. In this lecture, we consider an economy with capital as the only productive input. We subsequently add labor. I stress that what I am presenting now are models of long run equilibrium behavior (or trend behavior) as opposed to business cycles.

1 Environment

Assume:

(i) Two periods: 1 and 2.

(ii) One representative household that: consumes, saves and receives dividend income (from ownership of firms.)

(iii) One representative firm that produces output using capital, invests in new capital, and pays dividends to households.

(iv) The household and the firm act competitively, i.e., each takes market prices as given

(v) Physical capital is the only productive input..

We next characterize preferences, technology and resource constraints:

Preferences

The household consumes in both periods. Let C^i_k be household i's consumption in period k. Then household preferences are given by

$$\frac{1}{1-\sigma}(C^i_1)^{1-\sigma} + \frac{1}{1-\sigma}\beta(C^i_2)^{1-\sigma} \quad : \quad \text{if } \sigma \neq 1 \tag{1}$$

$$\log C^i_1 + \beta \log C^i_2 \quad : \quad \text{if } \sigma = 1 \tag{2}$$

with $\sigma > 0$ and $0 < \beta < 1$. The parameter β is known as the household's subjective discount factor. It reflects how the household weighs consuming in the future relative to consuming today.

Technology

Firm j produces output in period 1 using capital input, K^j_1, and also produces output in period 2 using capital.input K^j_2 Captial in period 2 depends upon the amount of period 1 investment, I. Let Y^j_k be output in period k and A_k total factor productivity. Then each period, production is given by

$$Y^j_1 = A_1(K^j_1)^\alpha \tag{3}$$

$$Y^j_2 = A_2(K^j_2)^\alpha \tag{4}$$

The link between capital and investment is given by

$$K^j_2 = (1-\delta)K^j_1 + I^j - \frac{c}{2}(\frac{I^j}{K^j_1})^2 K^j_1 \tag{5}$$

where δ is the rate of depreciation and where $\frac{c}{2}(\frac{I^j}{K^j_1})^2 K^j_1$ reflects costs to the firm of adjusting its capital stock. Capital in period 2 depends on capital leftover from period 1.plus period 1 investment net adjustment costs. Note that we take K^j_1 as given in period 1.

Economy-Wide Resource Constraints

In period 1, output is divided between consumption and investment: there is no government and no external sector.

$$Y_1 = C_1 + I \tag{6}$$

In period 2, all output is consumed, as is the capital stock that is left over

$$(1-\delta)K_2 + Y_2 = C_2 \tag{7}$$

(We don't literally mean that households eat capital - rather including $(1-\delta)K_2$ in the resource constraint means that in the final period a unit of capital has the same value to the household as a unit of consumption goods. This is a simple way of assigning of value to capital in the final period.)

2 Household and Firm Behavior

2.0.1 The household decision problem

Let S^i be household i's saving, Π_k^i dividends in period $_k$, R the gross real interest rate (equal to one plus the net interest rate), all in units of consumption goods. Further, we normalize the price of consumption goods at unity. The representative household chooses C_1^i, C_2^i, and S^i,to solve

$$\max \frac{1}{1-\sigma}(C_1^i)^{1-\sigma} + \frac{1}{1-\sigma}\beta(C_2^i)^{1-\sigma} \tag{8}$$

subject to:

$$C_1^i = \Pi_1^i - S^i \tag{9}$$

$$C_2^i = \Pi_2^i + RS^i \tag{10}$$

The household takes as given R, Π_1^i and Π_2^i. Note that S^i may be positive or negative. Negative values of S^i imply borrowing. Implicit in our formulation, however, is the assumption of perfect capital markets; i.e., the households is able to borrow at the same rate R for which it is able to lend.

It is instructive to combine the two period budget constraints given by (9) and (10) into a single intertemporal budget constraint given by

$$C_1^i + \frac{C_2^i}{R} = \Pi_1^i + \frac{\Pi_2^i}{R}. \tag{11}$$

According to equation (11), the households lifetime consumption plan must satisfy the constraint that the present value of consumption is equal to the present value of income. The latter is given by the sum of labor income in the first period and the discounted stream of dividend income. Accordingly, consumption depends on lifetime income as opposed to income in the current period. This implication, however, depends on the assumption of perfect capital markets. What permitted collapsing the period budget constraints into the single intertemporal constraint for negative as well as positive values of S is that the value of R is independent of the sign of S; i.e., the individual can freely borrow or lend at the gross rate R.

To solve the household's decision problem, it is simplest to turn the constrained problem into an unconstrained one by plugging (9) and (10) into (**??**). This is possible as long as the two one-period budget constraints are always binding, so that there is no unused income. This latter condition is ensured by the assumptions we made on the utility function..

The representative household accordingly chooses S to solve

$$\max(\frac{1}{1-\sigma}(\Pi_1 - S^i)^{1-\sigma} + \beta\frac{1}{1-\sigma}(\Pi_2 + RS^i)^{1-\sigma},$$

given R, Π_1, and Π_2.

The first order necessary condition with respect to saving is given by:

$$\underbrace{-(C_1^i)^{-\sigma}}_{\text{MC of Savings}} + \underbrace{R\beta(C_2^i)^{-\sigma}}_{\text{MB of Savings}} = 0 \tag{12}$$

The opportunity cost of saving a unit of goods is the marginal utility of consumption. The marginal benefit is the gross real rate times the marginal utility of future consumption (from the standpoint of today.) Again, restrictions on the utility function ensure that the first order condition characterizes an optimum.

Rearranging equation (12) yields a relation between marginal utilities of consumption across time, known as a consumption euler equation:

$$(C_1^i)^{-\sigma} = R\beta(C_2^i)^{-\sigma} \tag{13}$$

Given concavity of the utility function, the consumption euler equation implies that individuals should try to smooth consumption over time. In the limiting case where $R\beta = 1$, it is optimal for individuals to consume the same amount each period.

Note that the consumption euler equation combined with the intertemporal budget constraint determines the optimal lifetime consumption plan, i.e., the optimal values of C_1^i and C_2^i. Note that this simple model captures the key aspects of the life-cycle permanent income hypothesis developed by Friedman and Modigliani. The key aspects are consumption smoothing and the dependence of consumption on lifetime resources.

To obtain an expression in terms of consumption goods, rearrange the consumption euler equation as follows:

$$1/R = \frac{\beta(C_2^i)^{-\sigma}}{(C_1^i)^{-\sigma}}. \tag{14}$$

The left side is the price in units of current consumption of a unit of future consumption (i.e., to buy a unit of second period consumption, the individual must save $1/R$ units of current consumption). The right side is the household's intertemporal marginal rate of substitution - the value the household places on a an additional unit of future consumption in units of current consumption.

Finally, to obtain a saving supply curve, invert equation (14) and combine with the period budget constraints to obtain:

$$R = \frac{(\Pi_1 - S)^{-\sigma}}{\beta(\Pi_2 + RS)^{-\sigma}} \tag{15}$$

We define the saving supply curve as combinations of R and S that satisfy equation (15), given W and N^s. In analogy to the labor supply curve, there is both a substitution and an income effect of changes in R . A rise in R increases the opportunity cost of saving (the substitution effect). However, it also reduces the marginal value of second period consumption by reducing the marginal utility of second period consumption (the income effect). Given that the intertemporal marginal rate of substitution (the inverse of the right side of equation (15)) is decreasing in S, the substitution effect of a rise in R raises saving, while the income effect decreases it. We will assume the substitution effect dominates, as holds with standard preference specifications. The implication is that the saving supply curve will be upward sloping in (R, S) space.

An Explicit Solution

Our model is simple enough to be able to explicitly solve for consumption, as follows:

From rearranging the first order condition for consumption and saving.

$$C_2^i = (\beta R)^{\frac{1}{\sigma}} C_1^i \tag{16}$$

Note that the gross rate of consumption C_2/C_1 depends positively on the gross real rate of interest R. Intutiively, by making saving more attractive a rise in R tilts the houshold to raise consumption tomorrow relative to consumption today. The parameter σ is known as the intertemporal elasticity of substitution since it governs the response of consumption growth to interest rates. By combing this condition with the household's intertemporal budget constraint, we obtain

$$C_1(1 + \frac{1}{R}(\beta R)^{\frac{1}{\sigma}}) = \Pi_1 + \frac{\Pi_2}{R}$$

which reduces to

$$C_1 = \frac{1}{1 + \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma} - 1}} (\Pi_1 + \frac{\Pi_2}{R}) \tag{17}$$

Let us define the households wealth $V_j^i \equiv \Pi_1 + \frac{\Pi_2}{R}$ as the present value of its dividend income. Then we can write:

$$C_1^i = \frac{1}{1 + \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma} - 1}} V^i \tag{18}$$

Borrowing Constraints

Consumption depends on wealth as opposed to income. The timing of income receipts does not matter, only the present value. Note that this results depends on the assumption of perfect capital markets. If the household cannot borrow, then it must be the case that $C_1^i \leq \Pi_1$ This constraint is likely to be be binding if Π_1 is low relative to Π_2.

Infinite Horizon

One can show that if the household has an infinitie horizon:

$$C_1^i = (1 - \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1}) V^i$$

with

$$V^i = \sum_{k=0}^{\infty} (R^{-1})^k \Pi_k$$

If the period length is a year, then a reasonable value of β is 0.96. If σ is close to unity or C_1^i is close to C_2^i,then $(1 - \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}-1}) \approx 1 - \beta$. In this case, a dollar increase in household wealth leads to a four percent increase in consumption in a given year This is known as a wealth effect on consumption. The presence of this wealth effect is why there has been concern about the impact of the decline on the stock market on household consumption. Note that thus far we have abstracted from "human wealth", which is the discounted value of labor income.

Taxes

Finally, we can say something about the impact of tax cuts on consumption spending. Let T_1 and T_2 be taxes the individual pays in periods 1 and 2, respectively:

$$C_1^i = \Pi_1^i - S^i - T_1 \tag{19}$$

$$C_2^i = \Pi_2^i + RS^i - T_2 \tag{20}$$

Now consumption depends on the individual's wealth, net the discounted stream of taxes:

$$C_1^i = \frac{1}{1+\beta^{\frac{1}{\sigma}}R^{\frac{1}{\sigma}-1}}[(\Pi_1 + \frac{\Pi_2}{R}) - (T_1 + \frac{T_2}{R})]$$

What matters is the present value of tax cuts, as opposed to the timing. Permanent tax cuts will have a bigger effect than temporary ones. On the other hand, if the household is credit constrained $C_1^i \leq \Pi_1 - T_1$ with the constraint binding, then a temporary tax cut can have a big effect on the household's consumption.

2.0.2 The Firm Decision Problem

The firm maximizes the discounted stream of profits returned to the household. Given that there is no uncertainty, the firm discounts future profits at the rate $1/R$. Accordingly, the firm chooses $Y_1, Y_c, , I$ and K_2 to solve

$$\max \Pi_1^j + \frac{\Pi_2^j}{R} \tag{21}$$

subject to:

$$\Pi_1^j = A_1(K_1^j)^\alpha \tag{22}$$

$$\Pi_2^j = A_2(K_2^j)^\alpha + (1-\delta)K_2^j - RI^j \tag{23}$$

taking as given R. Here we are assuming that the firm finances investment by borrowing at the rate R. Further,

$$K_2^j = (1-\delta)K_1^j + I^j - \frac{c}{2}(\frac{I^j}{K_1^j})^2 K_1^j$$

.

Again, we can convert the problem into an unconstrained maximization problem by plugging the constraints (22) and (23) into (21). For convenience we know drop the j superscript. The firm then chooses I to solve

$$\max A_1 K_1^\alpha + \frac{1}{R}\{A_2[(1-\delta)K_1 + I^j - \frac{c}{2}(\frac{I}{K_1})^2 K_1]^\alpha + (1-\delta)[(1-\delta)K_1 + I - \frac{c}{2}(\frac{I}{K_1})^2 K_1] - I/R\}$$

The first order necessary conditions for investment is given by:

$$-1 + \frac{1}{R}[A_2 \alpha K_2^{\alpha-1} + 1 - \delta][1 - c\frac{I}{k}] = 0 \tag{24}$$

Equivalently

$$\frac{1}{1 - c\frac{I}{K}} = \frac{[A_2 \alpha K_2^{\alpha-1} + 1 - \delta]}{R} \tag{25}$$

$$\frac{c}{c - \frac{I}{K}} = \frac{1}{c}\frac{[A_2 \alpha K_2^{\alpha-1} + 1 - \delta]}{R} \tag{26}$$

$$= \frac{1}{c}Q \tag{27}$$

where $\frac{[A_1 \alpha K_2^{\alpha-1} + 1 - \delta]}{R}$ is the ratio of the marginal value of a unit of capital to the replacement cost and is known and Tobin's Q ratio. Investment depends positively on Tobin's Q. The sensitive investment is to Q depends on adjustmet costs as measured by c.

Note that we can obtain an investment demand curve by rerranging

$$R = [A_2 \alpha K_2^{\alpha-1} + 1 - \delta][1 - c\frac{I}{K}]$$

Taking into account that K_2 is increasing in I yields an inverse relation between K and R.

Financial Policy

We have been assuming thus far that the firm pays out all its revenue in the first period and then borrows to investment. Under perfect capital markets, it does not matter whether the firm finances investment by retained earnings or borrowing. This is because under perfect capital markets, the firm can borrow and lend freely at the market interest rate R. That is, the opportunity cost to the firm of using retained earnings to finance investment is the same as the cost of borrowing.

As an example, let B be the amount the firm borrows in period 1 and suppose that now this amount may be less than I. The difference then, $I - B$, is the amount the firm finances with reatined earnings. This may be thought of as "firm saving." . Then we may write:

$$\Pi_1^j = A_1(K_1^j)^\alpha + B^j - I^j \tag{28}$$

$$\Pi_2^j = A_2(K_2^j)^\alpha + (1-\delta)K_2^j - RB^{jj} \tag{29}$$

Note however that the present value of dividends is independent of the firm's financial policy:

$$\Pi_1^j + \Pi_2^j/R = A_1(K_1^j)^\alpha - I^j + [A_2(K_2^j)^\alpha + (1-\delta)K_2^j]/R$$

Since the firm must pay off its debt (no Ponzi/Madoff schemes allowed!), the present value of its proceeds from its borrowing is zero and the decision problem is thus exactly as before. The firm's choice of investment is thus exactly as before. What borrowing does is alllow the firm to pay out more to shareholders in dividends in period 1, but this is canceled by having to pay an equal amount less (in present value), in period 2.

The irrelevance of corporate financial policy to the firm's real investment decision is an exactly of the "Miller-Modligliani" theorem. Note that the Miller-Modigliani hypthesis applies only under perfect capital markets.

For example, if the firm cannot borrow, it's investment may be constrained by it's period 1 earnings.

3 Competitive Market Equilibrium

A competitive equilibrium for this economy is an allocation (I, C_1, C_2, Y_1, Y_2, K_2) and a relative price vector (R) such that the household and the firm is each maximizing its respective objective, markets clear, and the economy resource constraints are satisfied.

Capital market:

$$1 = R\frac{\beta(C_2)^{-\sigma}}{(C_1)^{-\sigma}} \tag{30}$$

$$= [A_2\alpha K_2^{\alpha-1} + 1 - \delta][1 - c\frac{I}{K}]\frac{\beta(C_2)^{-\sigma}}{(C_1)^{-\sigma}} \tag{31}$$

Resource Constraints:

$$Y_1 = C_1 + I, \tag{32}$$

$$C_2 = Y_2 + (1-\delta)K_2 \tag{33}$$

Technology Constraints:

$$Y_1 = A_1(K_1)^{\alpha} \tag{34}$$

$$Y_2 = A_2(K_2)^{\alpha} \tag{35}$$

Evolution of Capital

$$K_2 = (1-\delta)K_1 + I - \frac{c}{2}(\frac{I}{K_1})^2 K_1$$

:

The equilibrium price vector is obtained from the respective market clearing condition, i.e.,

$$R = [A_2 \alpha K_2^{\alpha-1} + 1 - \delta][1 - c\frac{I}{K}] = \frac{(C_1)^{-\sigma}}{\beta^{-1}(C_2)^{-\sigma}} \tag{36}$$

4 Adding Employment

4.1 Household Problem

$$\max \sum_{k=1}^{2} \beta^{k-1}[\frac{1}{1-\sigma}(C_k^i)^{1-\sigma} - \frac{1}{1+\varphi}(N_k^i)^{1+\varphi}]$$

s.t

$$C_1^i = W_1^i N_1^i + \Pi_1^i - S^i$$

$$C_2^i = W^i N_2^i + \Pi_2^i + RS^i$$

First Order Necessary Conditions:

(consumption/saving)

$$(C_1^i)^{-\sigma} = R\beta(C_2^i)^{-\sigma}$$

(labor supply)

$$W_k = \frac{(N_k^i)^{\varphi}}{(C_k^i)^{-\sigma}} : \text{ for } k = 1, 2$$

4.2 Firm Problem

$$\max \Pi_1^j + \frac{\Pi_2^j}{R}$$

$$\Pi_1 = Y_1$$

$$\Pi_2 = Y_2 + (1-\delta)K_2 - RI$$

$$Y_k^j = A_k(K_k^j)^{\alpha}(N_k^j)^{1-\alpha} \text{ for } k = 1, 2$$

First Order Necessary Conditions:

(Investment)

$$[A_2\alpha(\frac{K_2}{N_2})^{\alpha-1} + 1 - \delta][1 - c\frac{I}{K}] = R$$

(Labor)

$$(1-\alpha)(\frac{N_k}{K_k})^{-\alpha} = W \text{ for } k = 1, 2$$

4.3 Competitive Equilibrium

A competitive equilibrium for this economy is an allocation ($I,\ N_1, N_2, C_1,\ C_2, Y_1,\ Y_2, K_2$) and a relative price vector (R) such that the household and the firm is each maximizing its respective objective, markets clear, and the economy resource constraints are satisfied.

Capital market:

$$\begin{aligned} 1 &= R\frac{\beta(C_2)^{-\sigma}}{(C_1)^{-\sigma}} && (37) \\ &= [A_2\alpha(\frac{K_2}{N_2})^{\alpha-1} + 1 - \delta][1 - c\frac{I}{K}]\frac{\beta(C_2)^{-\sigma}}{(C_1)^{-\sigma}} && (38) \end{aligned}$$

Labor Market

$$(1-\alpha)(\frac{N_k}{K_k})^{-\alpha} = \frac{(N_k)^{\varphi}}{(C_k)^{-\sigma}} : \text{ for } k = 1, 2$$

Resource Constraints:

$$Y_1 = C_1 + I, \tag{39}$$

$$C_2 = Y_2 + (1-\delta)K_2 \tag{40}$$

Technology Constraints:

$$Y_k = A_k(K_k)^{\alpha}(N_k)^{1-\alpha}; \text{ for k=1,2} \tag{41}$$

Evolution of Capital

$$K_2 = (1-\delta)K_1 + I - \frac{c}{2}(\frac{I}{K_1})^2 K_1$$

:

The equilibrium price vector is obtained from the respective market clearing condition, i.e.,

$$R = [A_2 \alpha K_2^{\alpha-1} + 1 - \delta][1 - c\frac{I}{K}] = \frac{(C_1)^{-\sigma}}{\beta^{-1}(C_2)^{-\sigma}} \tag{42}$$