Competitive Equilibrium and Pareto Optimality

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1 Competitive Equilibrium

Household Problem:

$$\max_{C,N} \quad u(C) - \nu(N)$$
$$\text{s.t. } C = WN + \Pi$$

To make things simple, I will use the log form for the utility function:

$$u(C) = \ln C \quad \text{and} \quad \nu(N) = b\frac{1}{1+\varphi}N^{1+\varphi}$$

After plugging the constraint into the utility function, and solving, we get the first order condition (with respect to N):

$$\begin{aligned} \frac{W}{C} - bN^{\varphi} &= 0 \\ \frac{W}{C} &= bN^{\varphi} \\ W &= bN^{\varphi}C \end{aligned} \tag{1}$$

Firm's Problem:

$$\max_N \quad \Pi = Y - WN$$

Again, following the lecture notes:

$$Y = Af(N) = AN^{1-\alpha}$$

Plugging it in and solving for the FOC:

$$\begin{aligned} Af'(N) &= W \\ (1-\alpha)AN^{-\alpha} &= W \end{aligned} \tag{2}$$

Solving for CE:

A competitive equilibrium is an allocation $[C^*, Y^*, N^*]$ and a relative price W^*, such that:

1. The Household chooses C and N to maximize its objective
2. The Firm chooses N to maximize its objective
3. Markets for Goods and Labor clear
4. Firm distributes dividends

In practice: Combine equations (1) and (2) to get

$$\boxed{W = bN^{\varphi}C = (1-\alpha)AN^{-\alpha}} \tag{3}$$

Now, use the fact that markets clear and $C = Y = AN^{1-\alpha}$

$$\begin{aligned} bAN^{1-\alpha+\varphi} &= (1-\alpha)AN^{-\alpha} \\ N^{1+\varphi} &= \frac{(1-\alpha)}{b} \\ N^* &= \left[\frac{(1-\alpha)}{b}\right]^{\frac{1}{1+\varphi}} \end{aligned}$$

Using this, we can easily solve for C^*, Y^* and W^*.

$$Y^* = A\left[\frac{(1-\alpha)}{b}\right]^{\frac{1-\alpha}{1+\varphi}} = C^*$$

$$W^* = (1-\alpha)A\left[\frac{(1-\alpha)}{b}\right]^{\frac{-\alpha}{1+\varphi}}$$

2 Pareto Optimality

The competitive equilibrium allocation is the outcome to the market participants' selfish behavior. One question we could ask is: "how does this outcome compare to others?"

Now, instead of letting the market participants (households and firms) make their own decisions, let us institute a benevolent social planner who chooses consumption and labor *for our agent* in order to maximize his utility.

Planner's Problem:

$$\begin{gathered} \max_{C,N} \quad u(C) - \nu(N) \\ \text{s.t. } C = Af(N) \end{gathered}$$

Set up the Lagrangian:

$$u(C) - \nu(N) + \lambda[AN^{1-\alpha} - C]$$

Recall the specific forms:

$$\max_{C,N} \quad u(C) = \ln C \quad \text{and} \quad \nu(N) = b\frac{1}{1+\varphi}N^{1+\varphi}$$

Solve for the first order conditions:

$$\begin{aligned} FOC(C) : \frac{1}{C} + \lambda(-1) &= 0 \\ \lambda &= \frac{1}{C} \end{aligned}$$

$$\begin{aligned} FOC(N) : -bN^{\varphi} + \lambda[A(1-\alpha)N^{-\alpha}] &= 0 \\ bN^{\varphi} &= \lambda[A(1-\alpha)N^{-\alpha}] \end{aligned}$$

Plug λ in to get:

$$bN^{\varphi} = \frac{1}{C}[A(1-\alpha)N^{-\alpha}]$$

$$\boxed{bN^{\varphi}C = (1-\alpha)AN^{-\alpha}} \tag{4}$$

Now, compare equations (3) and (4); they are the same. Combined with the fact that $C = Af(N) = AN^{1-\alpha}$, we can see that the implied Pareto optimal allocations of ($\hat{C}$, $\hat{Y}$, $\hat{N}$) are the same as the CE ones.

$\Rightarrow$ The CE allocation is also socially optimal: this is the **Welfare Theorem**.

3 Distortions: Income Tax with Endogenous Labor Choice

When does the Welfare Theorem fail? In other words, when are the Competitive Equilibrium (C^*, Y^*, N^*) and the Pareto Allocations ($\hat{C}$, $\hat{Y}$, $\hat{N}$) *not equal*? In order to give an example for this, we will introduce a Government. It will collect income taxes from everyone, and then redistribute all that money in equal amounts to everyone, in a lump-sum fashion.

3.1 Pareto Problem

Spoiler: Nothing changes. It's the same Pareto problem as in the case without taxes.

$$\max_{C,N} \quad u(C) - \nu(N)$$
$$\text{s.t. } C = Af(N)$$

Set up the Lagrangian:

$$u(C) - \nu(N) + \lambda[AN^{1-\alpha} - C]$$

Solve for the first order conditions:

$$\begin{aligned} FOC(C): \frac{1}{C} + \lambda(-1) &= 0 \\ \lambda &= \frac{1}{C} \end{aligned}$$

$$\begin{aligned} FOC(N): -bN^{\varphi} + \lambda[A(1-\alpha)N^{-\alpha}] &= 0 \\ bN^{\varphi} &= \lambda[A(1-\alpha)N^{-\alpha}] \end{aligned}$$

Plug λ in to get:

$$bN^{\varphi} = \frac{1}{C}[A(1-\alpha)N^{-\alpha}]$$

$$\boxed{bN^{\varphi}C = (1-\alpha)AN^{-\alpha}} \tag{5}$$

This is the same condition as in the economy without taxes; the same allocation holds.

3.2 Competitive Equilibrium

Household Problem:

$$\max_{C,N} \quad u(C) - \nu(N)$$
$$\text{s.t. } C = (1-\tau)WN + \Pi + G$$

Note the introduction of the $(1-\tau)$ term into the budget constraint. This is a tax rate on labor income. G is the lump-sum redistribution transfer.

I will continue using the log form for the utility function:

$$u(C) = \ln C \quad \text{and} \quad \nu(N) = b\frac{1}{1+\varphi}N^{1+\varphi}$$

Solving:

$$\begin{aligned} \frac{W(1-\tau)}{C} - bN^{\varphi} &= 0 \\ \frac{W(1-\tau)}{C} &= bN^{\varphi} \\ W &= \frac{bN^{\varphi}C}{(1-\tau)} \end{aligned} \tag{6}$$

Firm's Problem:

It remains unchanged.

$$\max_N \quad \Pi = AN^{1-\alpha} - WN$$

Solving for the FOC:

$$(1-\alpha)AN^{-\alpha} = W \tag{7}$$

Government:

Balances its budget:

$$G = \tau W N$$

Solving:

A competitive equilibrium is an allocation $[C^*,\, Y^*,\, N^*]$, a relative price W^* and a tax rate τ, such that:

1. The Household chooses C and N to maximize its objective
2. The Firm chooses N to maximize its objective
3. The government balances its budget
4. Markets for Goods and Labor clear
5. Firm distributes dividends

Doing the same thing as before, let us combine (6) and (7) to get:

$$\boxed{\left[\frac{1}{1-\tau}\right] bN^{\varphi}C = (1-\alpha)AN^{-\alpha}} \tag{8}$$

Solving it,

$$N^* = \left[\frac{(1-\alpha)(1-\tau)}{b}\right]^{\frac{1}{1+\varphi}} \tag{9}$$

and

$$Y^* = A\left[\frac{(1-\alpha)(1-\tau)}{b}\right]^{\frac{1-\alpha}{1+\varphi}} = C^*$$

$$W^* = (1-\alpha)A\left[\frac{(1-\alpha)(1-\tau)}{b}\right]^{\frac{-\alpha}{1+\varphi}} = (1-\alpha)A\left[\frac{b}{(1-\alpha)(1-\tau)}\right]^{\frac{\alpha}{1+\varphi}}$$

Note the implications: C^*, Y^* and N^* are all lower than in the economy without tax. However, W^* is higher. Intuitively, the tax introduces a wedge between the agent's MRS and the firm's MRT. Without it, the equilibrium is defined at the point where they are equal; with it, the equilibrium is stuck at a point where $MRS < MRT$.