Lab Report #4: Asset Pricing Fundamentals

Revised: October 29, 2013

Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Please include your Matlab code.

Solution: Brief answers follow, but see also the attached Matlab program and spreadsheet; download the pdf, open, click on pushpins:
Warning: If you don't see a pushpin above, my guess is you have a Mac. The pushpin doesn't appear in Preview, but you can use the Adobe Reader or the equivalent.

1. *Pricing kernels and risk-neutral probabilities with geometric risk.* Consider a representative agent economy with a power utility agent facing "geometric risk." Utility is $u(c) = c^{1-\alpha}/(1-\alpha)$ with risk aversion parameter $\alpha > 0$. Log consumption growth is $x = \log g = \log c_1 - \log c_0$ for $x = 0, 1, 2, \ldots$ with probabilities $p(x) = (1-\omega)\omega^x$ for some parameter ω satisfying $0 < \omega e < 1$. We say that x has a *geometric distribution*.
   (a) What is the pricing kernel for this model?
   (b) What are the state prices?
   (c) What are the risk-neutral probabilities $p^*(x)$? How do they differ from the true probabilities $p(x)$?

Solution:

(a) The pricing kernel is

$$m(x) \;=\; \beta e^{-\alpha x}.$$

(b) State prices are

$$Q(x) \;=\; p(x)m(x) \;=\; (1-\omega)\omega^x \beta e^{-\alpha x}.$$

(c) Risk-neutral probabilities are $p^*(x) = p(x)m(x)/q^1$. Here $q^1 = E(m) = (1-\omega)\beta/(1-\omega e^{-\alpha})$. Risk-neutral probabilities are therefore

$$p^*(x) \;=\; \omega^x e^{-\alpha x}(1-\omega e^{-\alpha}).$$

If $\alpha = 0$ (zero risk aversion), then $p^*(x) = p(x)$. Otherwise p^* is lower than p for large x, and the reverse for small x. As usual, "good states" have lower value, which here is reflected in the risk-neutral probability. In (a) it was reflected in the pricing kernel.

2. *Risk and return in US equity portfolios.* Modern economies issue a wide range of assets, whose returns can have wildly different properties. Here we summarize the properties of returns on some common equity portfolios.

We'll start with data input. Ken French is Gene Fama's frequent coauthor and his website is the go-to place for data on equity returns. Here's the link:

http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

Download the files associated with, respectively, the "Fama-French Factors" and "Portfolios Formed on Size" at the links

http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/ftp/F-F_Research_Data_Factors.zip
http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/ftp/Portfolios_Formed_on_ME.zip.

In both cases, you'll find a txt file inside a zip file. Copy the first table in each txt file into a spreadsheet with the dates aligned. In the first file ("factors") you want the first column (the date), the second (the excess return on the market), and the fifth (the short-term or riskfree interest rate RF). In the second file ("size") you want the first column (the date again) and columns three to five (returns on portfolios of small, medium, and large firms).

When you're done, read the data into Matlab. At this point you should have the riskfree rate and excess returns on the market and three size portfolios (small, medium, and large) — five variables all together. The numbers are monthly, July 1926 to (last I looked) August 2013. I believe they are percentages, but you should check.

(a) Compute the mean, standard deviation, skewness, and excess kurtosis for each excess return series. Which portfolio has the highest excess return? Lowest?

(b) The Sharpe ratio for any asset or portfolio is the mean of its excess return over its standard deviation. Which portfolio has the highest Sharpe ratio? Lowest?

(c) Do the same for log excess returns; that is, compute the mean, standard deviation, skewness, and excess kurtosis for each excess return series. (To get this right, you should add the riskfree rate to each excess return, divide by 100, add one, then take logs. Think about why all this is called for.)

(d) Comment on the nature of risk faced by each of these portfolios. If you were considering an investment of your own money, which properties of returns would you emphasize?

Solution: The idea here is to look at some popular equity portfolios and see how their returns differ from broad-based equity indexes. We see (i) mean excess returns in some cases are larger than the equity premium and (ii) skewness and kurtosis are a standard feature. Skewness differs between returns and log returns, as you might expect: the log is a concave function, so it moderates large positive returns in levels. Finally, we remind ourselves that $E(mr) = 1$ does not imply any particular relation between mean and standard deviation of excess returns. In that respect, the model is similar to the CAPM, where the mean is connected to covariance with the market, not the standard deviation or Sharpe ratio or other simple risk measure.

(a) Last year's numbers below (sorry, I didn't have time to redo this).

Properties of monthly excess returns

	Portfolio			
Statistic	market	small	medium	big
Mean	0.6172	0.9738	0.8532	0.5987
Std dev	5.4571	8.5584	6.8520	5.2757
Skewness	0.1685	2.1813	0.9816	0.1860
Excess kurtosis	7.3997	21.8628	11.7830	7.1738
Sharpe ratio	0.1131	0.1138	0.1245	0.1135

The highest mean excess return is small firms at about 1% a month, the lowest is big firms.

(b) Sharpe ratios also reported above: the ratio of the first row to the second. There's not much difference here among the portfolios, but medium-size firms have the highest Sharpe ratio.

(c) Similar table for log excess returns below:

Properties of monthly log excess returns

	Portfolio			
Statistic	market	small	medium	big
Mean	0.4669	0.6329	0.6224	0.4583
Std dev	5.4556	8.1132	6.7205	5.2682
Skewness	−0.5298	0.4937	−0.0850	−0.4779
Excess kurtosis	6.5282	9.6576	7.5655	6.3824

(d) We care about all of these properties. One worth noting: the excess kurtosis in all of them.

3. *Excess kurtosis and the equity premium.* Consider a 3-state distribution in which the state z takes on the values $\{-1, 0, 1\}$ with probabilities $\{\omega, 1-2\omega, \omega\}$. A random variable x is defined by $x(z) = \mu + \delta z$. Statisticians would say that x has a "categorical distribution."

We'll build a representative-agent economy on this distribution by setting log consumption growth equal to x; that is, $\log g(z) = \log[c_1(z)/c_0] = x(z)$. Utility has the usual additive form,

$$u(c_0) + \beta \sum_z p(z) u[c_1(z)],$$

with $u(c) = c^{1-\alpha}/(1-\alpha)$ (power utility).

We're going to vary ω and see how that affects the equity premium. The idea is to explore the role of excess kurtosis, which you'll see is controlled by ω. In all cases, set

$\alpha = 10$ and $\beta = 0.99$.

(a) What are the mean and variance of x?

(b) What are the traditional measures of skewness and excess kurtosis, γ_1 and γ_2? Under what conditions is γ_2 large?

(c) What is the pricing kernel?

(d) Which is more valuable, a claim to one unit of the good in state $z = -1$ or one unit in state $z = +1$? Why?

(e) We observe that annual log consumption growth has a mean in US data of roughly 0.02 (2%) and a standard deviation of 0.035. Given a value of ω, what values of μ and δ reproduce these values?

(f) Suppose $\omega = 1/6$. Why is this value a useful benchmark? What values of (μ, δ) reproduce the mean and variance of log consumption growth?

(g) Suppose equity is a claim to the growth rate of consumption e^x. What is the equity premium with these values?

(h) Suppose $\omega = 1/20$. How do your choices of (μ, δ) adjust to keep the mean and variance of log consumption growth at their observed values? How does the equity premium change? Why?

Solution: The idea here is to go through a concrete example and compute state prices, the pricing kernel, and risk-neutral probabilities, and think about what each one does.

(a) The easiest way to do this is with the cgf. The mgf is

$$h(s) \;=\; \omega e^{\mu-\delta} + (1-2\omega)e^{\mu} + \omega e^{\mu+\delta}$$

and the cgf is $k(s) = \log h(s)$. The first four cumulants are

$$\begin{aligned}\kappa_1 &= \mu \\ \kappa_2 &= 2\delta^2\omega \\ \kappa_3 &= 0 \\ \kappa_4 &= 2\delta^4\omega(1-6\omega).\end{aligned}$$

The first one is the mean, the second one is the variance.

(b) Skewness and excess kurtosis are

$$\begin{aligned}\gamma_1 &= \kappa_3/\kappa_2^{3/2} &= 0 \\ \gamma_2 &= \kappa_4/\kappa_2^2 &= 1/(2\omega) - 3.\end{aligned}$$

The first is clear, because the distribution is symmetric. Evidently γ_2 is large when ω is small. It's zero when $\omega = 1/6$

(c) The pricing kernel is

$$m(z) \;=\; \beta e^{-\alpha x(z)} \;=\; \beta e^{-\alpha(\mu+\delta z)}.$$

(d) The pricing kernel is decreasing in x and z. In other words, m is higher in states where z is lower ("bad states"). Specifically

$$m(-1) \;=\; \beta e^{-\alpha(\mu-\delta)} \;>\; m(1) \;=\; \beta e^{-\alpha(\mu+\delta)}$$

as long as $\delta > 0$.

(e) The state prices are

$$Q(z) \;=\; p(z)m(z) \;=\; p(z)\beta e^{-\alpha(\mu+\delta z)},$$

with $p(z)$ given in the previous question.

(f) The idea here is to choose parameters to reproduce the mean and variance of log consumption growth in US data. The mean is μ, so we set $\mu = 0.02$. The variance is $2\delta^2\omega$. Given a value of $\omega > 0$, we set $2\delta^2\omega = 0.035^2$ or $\delta = 0.035/(2\omega)^{1/2}$.

(g) With $\omega = 1/6$, excess kurtosis is zero, the same value as the normal distribution. To match the variance, we then need $\delta = 0.0606$. μ remains 0.02.

(h) The equity premium is $E(r^e) - r^1 = 0.0143$ or 1.43%. See the Matlab code for the calculation.

(i) If we reduce ω to 1/20, excess kurtosis goes up to 7. The equity premium rises to 0.0160. The impact is modest, but it illustrates the potential role of adding more kurtosis to the distribution.