Consumption, Saving, & Portfolio Choice
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Another building block for thinking about business cycles and asset returns is what we'll call the consumer's problem: how an individual might decide between current consumption and saving for future consumption, and how that saving is allocated across assets. We'll see how this works in a two-period setting, which captures the essential ideas in a relatively simple way.

This will seem obscure the first time you see it, but fundamental result is the equation

$$E(mr) \;\; = \;\; 1.$$

Variation on this equation account for an embarrassingly large part of macroeconomics and finance.

Kuhn-Tucker conditions

The KT conditions are solution method for a common class of optimization problems. In the standard math treatment, optimization means minimization. Economists prefer to maximize, but the ideas are interchangeable: maximizing $f(x)$ is the same as minimizing $-f(x)$.

The canonical constrained optimization problem is to choose a variable x to maximize a function $f(x)$ subject to a constraint $g(x) \geq 0$. Formally, we might write it as

$$\max_{x} \quad f(x)$$
$$\text{subject to} \quad g(x) \geq 0.$$

We find the solution using the so-called Lagrangean function

$$\mathcal{L}(x, \lambda) \;\; = \;\; f(x) + \lambda g(x),$$

where λ is called the *Lagrange multiplier*. The necessary conditions for a maximum are

$$\partial \mathcal{L}/\partial x \;\; = \;\; f'(x) + \lambda g'(x) \;\; = \;\; 0$$
$$\lambda g(x) \;\; = \;\; 0,$$

with $\lambda \geq 0$ and $g(x) \geq 0$. These are necessary conditions, but with the right structure — which includes any problem in this course — they are also sufficient. What we've done here is add an extra variable λ and an extra equation to the usual unconstrained maximization problem. The second line says that if the multiplier is positive, the constraint binds (holds with equality): $g(x) = 0$. But if $g(x) > 0$, the multiplier must be zero: $\lambda = 0$. *If we're sure*

the constraint binds, we can replace the second equation with $\partial\mathcal{L}/\partial\lambda = g(x) = 0$. That will be true for every example we study in this class.

The same idea extends to problems where x and g are vectors. If x is a vector, we differentiate $\mathcal{L}$ with respect to each of them. And if g is a vector, we make λ a vector of the same dimension. You'll see how this works when we put it to work.

Example 1. I stole this one from Sargent, not sure where he got it. Let $f(x) = -(x+1)^2/2$ and $g(x) = x \geq 0$. The Lagrangean is

$$\mathcal{L} = -(x+1)^2/2 + \lambda x.$$

The necessary conditions are

$$-(x+1) + \lambda = 0$$
$$\lambda x = 0.$$

The first equation implies $\lambda = x+1$. If we substitute this into the second, we get $x(x+1) = 0$, which has two solutions. One is $x = -1$, which violates the constraint. The other is $x = 0$, which implies $\lambda = 1$.

Example 2. Consider an agent with income y who is deciding how many apples a and bananas b to consume. Her utility function is

$$U(a,b) = \beta \log a + (1-\beta) \log b.$$

If prices of apples and bananas are q_a and q_b, respectively, her budget constraint is

$$q_a a + q_b b \leq y.$$

The inequality is there to give this the same form as the generic problem, but you can guess that it binds: there's no reason here not to spend all your money.

We find the solution using the same methods. The Lagrangean is

$$\mathcal{L} = \beta \log a + (1-\beta) \log b + \lambda \left[y - (q_a a + q_b b) \right].$$

The necessary conditions are

$$\beta/a = \lambda q_a$$
$$(1-\beta)/b = \lambda q_b$$
$$\lambda \left[y - (q_a a + q_b b) \right] = 0.$$

The solution includes $\lambda = 1/y$, $a = \beta y/q_a$, and $b = (1-\beta)y/q_b$.

Consumption and saving

We approach consumption and saving the same way we approached consumption of apples and bananas. The goods here are consumption now (c_0) and consumption later (c_1). The utility function has a special form we use for dynamic problems,

$$U(c_0, c_1) \;=\; u(c_0) + \beta u(c_1), \tag{1}$$

where β is the discount factor (a number between zero and one) and u is some convenient function. We'll say that u is increasing ($u' > 0$) and concave ($u'' < 0$). Power utility is a common choice: $u(c) = c^{1-\alpha}/(1 - \alpha)$ with $\alpha > 0$. Log utility $u(c) = \log c$ corresponds to $\alpha = 1$.

We're going to think about the problem of choosing c_0 and c_1 now, at date 0. At date 1, we simply execute our plan. If incomes in the two periods are (y_0, y_1), the budget constraint might be expressed

$$c_0 + qc_1 \;\leq\; y_0 + qy_1 \;=\; Y. \tag{2}$$

Here q is the date-0 price of one unit of date-1 consumption in "c_0 units." It may look more familiar if we write $q = 1/(1+r)$, so that q discounts date-1 quantities are the appropriate rate. We'll write $q = 1/r$ instead, simply because we're lazy: it's a lot easier to write r than $1+r$, especially if you have to do it many times. We refer to r defined this way as the gross interest rate.

Now what, you might ask, does this have to do with saving? The choice of first-period consumption also dictates a choice for saving. Since $q = 1/r$, we can rewrite the budget constraint as

$$c_0 + (1/r)c_1 \;\leq\; y_0 + (1/r)y_1.$$

If we define saving as $s = y_0 - c_0$, this becomes

$$c_1 \;\leq\; rs + y_1.$$

That is: tomorrow's consumption is tomorrow's income plus today's saving with interest. We'll ignore saving from now on, but it's there behind the scenes.

We now have the pieces of the consumption-saving problem: the utility function (1) and the budget constraint (2). We attack it with the usual tools. The necessary conditions for utility maximization are

$$u'(c_0) \;=\; \lambda$$
$$\beta u'(c_1) \;=\; \lambda q$$

plus the budget constraint, which we assume binds (it does).

One implication of these conditions is a relation between consumption and the interest rate. If we take the ratio of the two necessary conditions, we have

$$\beta u'(c_1)/u'(c_0) \;=\; q \; [= 1/r].$$

This says that the marginal rate of substitution between current and future consumption equals the relative price. If we denote the mrs by m, we have

$$\beta u'(c_1)/u'(c_0)r \;=\; mr \;=\; 1. \tag{3}$$

We'll see variants of this equation throughout the course.

[Draw budget line and indifference curves, show tangency. Budget line is $c_0 + c_1/r = Y$, so slope in (c_0, c_1) space is $-r$.]

So what do we have here? We have a connection between preferences, consumption, and the interest rate. If we discount the future more (β is smaller), then for a given interest rate we'll choose more consumption now and less later. This is implied because u' is decreasing (recall $u'' < 0$). Makes sense: if we put less weight on the future, we'll allocate less of our income to future consumption. The interest rate works the same way. If we raise r, which makes future consumption cheaper, we'll do more of it. With power utility, we can be even more specific: $u'(c) = c^{-\alpha}$ and

$$\beta(c_1/c_0)^{-\alpha}r \;=\; 1.$$

Here a high interest rate is associated with high consumption growth.

This connection between consumption and the interest rate will hold in all of our models. Here the interest rate is given and we choose consumption. In market settings, we often reverse the causality and say that the high interest rate is a consequence of high future consumption growth. That's true in the data, too: interest rates are high when the economy (including consumption) is growing quickly and low when it's not.

Let's return to our problem and solve it explicitly for the case of log utility. We solve the necessary conditions for consumption:

$$\begin{aligned}
c_0 &= 1/\lambda \\
c_1 &= \beta/(\lambda q).
\end{aligned}$$

If we substitute into the budget constraint, we have

$$c_0 + qc_1 \;=\; \lambda^{-1}(1+\beta) \;=\; Y. \tag{4}$$

That gives us (see the necessary conditions)

$$\begin{aligned}
c_0 &= [1/(1+\beta)]Y \\
c_1 &= [\beta/(1+\beta)]Y/q.
\end{aligned}$$

This will look familiar from the example in the previous section.

You might try the more general power utility case yourself.

Consumption and portfolio choice 1: Arrow securities

We can do the same thing in risky settings. Continue our two-period environment, but let the state z in the second period be random. We treat consumption in each of these states as separate decisions, which we choose in the usual way at date 0.

[Draw event tree.]

Our agent is now choosing current consumption c_0 and future consumption $c_1(z)$ for each state z. Utility is

$$u(c_0) + \beta \sum_z p(z)u[c_1(z)] \quad = \quad u(c_0) + \beta Eu(c_1), \tag{5}$$

where $p(z)$ is the probability that state z occurs (rational expectations). The budget constraint becomes

$$c_0 + \sum_z q(z)c_1(z) \quad \leq \quad y_0 + \sum_z q(z)y_1(z) \quad = \quad Y, \tag{6}$$

where $q(z)$ is today's price of one unit of date-1 consumption in state z.

We refer to $q(z)$ as the *state price*: for $q(z)$ units of date-0 consumption today, you get one unit of date-1 consumption if state z occurs — and nothing otherwise. So it's the price now of one unit of consumption later if z occurs. Another way to think about $q(z)$ is that it's the price of a risky asset. The return on this asset is $r(z) = 1/q(z)$. If you're familiar with linear algebra, you might guess that these so-called *Arrow securities* form a convenient basis for the state space. In words, this means that it's easy to see how you could use this collection of assets to construct any consumption outcome you want.

The necessary conditions for utility maximization in this case are

$$
\begin{aligned}
u'(c_0) &= \lambda \\
\beta p(z)u'[c_1(z)] &= \lambda q(z) \ [= 1/r(z)],
\end{aligned}
$$

with one of the second equation for every value of z. The ratio — plus the definition $r(z) = 1/q(z)$ — gives us

$$\beta p(z)u'[c_1(z)]/u'(c_0)r(z) \quad = \quad 1.$$

It's not all that useful here, because $r(z)$ for these assets is zero in all states but one, but if we sum over z we get

$$\sum_z \beta p(z)u'[c_1(z)]/u'(c_0)r(z) \quad = \quad E\left(\beta u'(c_1)/u'(c_0)r\right) \quad = \quad 1,$$

a stochastic version of (3). It's a somewhat strange object in this context, since $r(z)$ is zero in all states by one, but shows the underlying unity of what we're doing.

Consumption and portfolio choice 2: arbitrary assets

Here's a version with more explicit portfolio choice and an arbitrary set of assets.

We need to be clear about what we mean by an asset. An asset here is a claim to a random dividend. Let us say, to be concrete, that asset j pays $d^j(z)$ units of the consumption good at date 1 in state z. Different dividend streams constitute different assets. To complete the notation, let us say that asset j sells at date 0 for q^j units of the date-0 consumption good and has gross return as date 1 of $r^j(z) = d^j(z)/q^j$. We reserve $j = 1$ for the (one period) riskfree asset, which has constant dividend $d^1(z) = 1$ and price q^1.

The portfolio choice problem here consists of the same preferences, equation (5), and a suitably modified budget constraint. At date 0, the agent's income goes either to consumption or asset purchases:

$$c_0 + \sum_j a^j q^j \;=\; y_0,$$

where a^j (a for asset) is the number of units purchased of asset j. At date 1, the agent then consumes her income plus the payoffs from her asset positions:

$$c_1(z) \;=\; y_1(z) + \sum_j a^j d^j(z).$$

Note that once we choose asset positions a^j, consumption is determined.

Our portfolio choice problem is then

$$\max_{\{a^j\}} \quad u\left(y_0 - \sum_j a^j q^j\right) + \beta E u\left(y_1(z) + \sum_j a^j d^j(z)\right). \tag{7}$$

The first-order conditions are

$$u'(c_0)\, q^j \;=\; E\left[\beta u'(c_1)\, d^j\right], \tag{8}$$

with one such condition for each asset j. If we divide by the left side, we have

$$1 \;=\; E\left[\beta[u'(c_1)/u'(c_0)]r^j\right], \tag{9}$$

our usual condition.

Arbitrage

We skipped some of the technical details above, which is ok, that's part of our plan. But there's one that's worth a closer look: that not all asset prices q^j and dividends d^j are allowed. We need to rule out combinations that allow investors to get something for nothing.

Once we've ruled them out, we have the resulting combinations are *arbitrage free*. We touch on this now because it will show up later.

First some notation. Let q be a column vector of asset prices, with each element corresponding to a different asset. And let D be a matrix whose jth column is the dividend vector d^j for the j asset, which each row corresponding to a specific state. If J is the number of assets and Z the number of states, then D is Z by J.

An arbitrage is a set of portfolio quantities a (one element for each asset) that generates positive dividends $(Da > 0)$ at nonpositive cost $(q^\top a \le 0)$. The first condition means here that all the elements of Da are greater than or equal to zero and at least one is strictly positive. We say (q, D) are arbitrage free if any portfolio with positive dividends has positive cost: $q^\top a > 0$.

Examples. Which of these price-dividend combinations are arbitrage free?

$$(a) \quad q = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(b) \quad q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(c) \quad q = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(d) \quad q = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

You should stare at this and think about what you see. In (a) we have Arrow securities: each one pays one unit in one state. We see that state 2 is more expensive than state 1. In (b) you might notice that the second asset dominates the first: same price, but the second has greater dividends in state 2 (and the same in state 1). This can't work. If you shorted asset 1 and used the proceeds to invest in asset 2, you get something for nothing. More formally, let $a^\top = (-1, 1)$ and see what you get. In (c) we have the same dividends, but a higher price of asset 2, which kills off the arbitrage opportunity. In (d) we have fewer assets than states, but the same logic applies. You won't be able to find an arbitrage.

If we allowed arbitrage opportunities in the portfolio choice problem, the optimal portfolio would be to go infinite in those positions. Markets generally don't let that happen: the prices adjust and the opportunities go away.

Bottom line

This condition will show up all over:

$$E\left[\beta u'(c_1)/u'(c_0)r\right] = E(mr) = 1$$

for returns r on all assets. Versions of this equation are the basis for much of modern macroeconomics and finance.