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Review for Quiz #3

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I'll focus as usual on the big picture, which here involves dynamics. For each topic/result/concept, I recommend you construct a numerical example or two to remind yourself how it works.

1 Stochastic processes

The term for the combination of dynamics and randomness is *stochastic process*. It's common to characterize them by their one-period conditional distribution: the distribution over outcomes next period given some description of the current situation, which we refer to as the *state*.

The simplest example is a tree. A state in this case is a node in the tree. The outcomes next period are branches from the node. The conditional distribution consists of the probabilities of these branches. We construct probabilities of longer paths by multiplying the probabilities of individual branches.

Trees are easy to describe but cumbersome to use; the description of a state — the node in a tree — is essentially the complete history of outcomes to that point. Generally we impose more structure than this. We proposed stochastic processes with three properties:

- *Markov.* There's a (hopefully simple) description of the state that tells us all we need to know about the current situation.
- *Stationary.* The distribution one next period's outcomes conditional on this period's state has the same structure at all dates.
- *Stable.* The conditional distribution over states in the distant future settles down to a unique equilibrium distribution.

There are lots of examples of such processes, but we'll use examples that are linear: AR, MA, and ARMA models. An example is an ARMA(1,1):

$$x_{t+1} \quad = \quad \varphi x_t + \sigma(w_{t+1} + \theta w_t),$$

where w_t is a sequence of independent standard normal random variables. This is Markov with state $z_t = (x_t, w_t)$. Conditional on z_t, x_{t+1} is normal with mean $\varphi x_t + \sigma\theta w_t$ and variance σ^2. If $|\varphi < 1$, the distribution of x_{t+k} conditional on z_t settles down: it's normal with mean zero and variance $\sigma^2 + (\varphi + \theta)^2/(1 - \varphi^2)$.

We can describe the properties of a stationary, stable, Markov process with the autocovariance and autocorrelation functions: covariances and correlations, respectively, of a variable x_t and its own future x_{t-k}. We say x is persistent if the autocorrelations $\rho(k)$ are "large." Interest rates are a good example: the first autocorrelation $\rho(1)$ is well above 0.9 for monthly data.

2 Forward-looking models

There's a sense in which time runs backwards in economics: current decisions depend on what we expect to happen in the future. It doesn't really run backwards, because what we expect depends on the past and present, but *expectations* of future outcomes show up in lots of places: in valuing assets (what do we expect dividends to be?), in deciding how much to consume (what will our future income be?), and so on.

The simplest example of this is the linear model

$$y_t \;=\; \lambda E_t(y_{t+1}) + x_t.$$

Here y_t is the variable of interest (the stock price, for example) and x_t is the "fundamental" the drives it (the dividend). We generally solve this forward in time and apply the law of iterated expectations to get

$$y_t \;=\; \sum_{j=0}^{\infty} \lambda^j E_t(x_{t+j}).$$

This works as long as $|\lambda| < 1$ and x is stable. Some models of bubbles add an extra term to the solution: $c_t \lambda^{-t}$ where c_t is a martingale.

3 Bond pricing

We started with definitions: given the prices of zero coupon bonds (the prices at date t of a payoff of one at date $t+n$), we define continuously-compounded yields and forward rates. Prices, yields, and forward rates are different ways of presenting the same information: given one, we can compute the other two.

We value assets as we always do: the no-arbitrage theorem tells us there exists a positive pricing kernel that values cash flows. With bonds, the cash flows are constant, the pricing kernel is all we need. We do this recursively:

$$q_t^{n+1} \;=\; E_t(m_{t+1} q_{t+1}^n)$$

starting with $q_t^0 = 1$ (the price of one today is one). In words: an $(n+1)$-period bond is a claim to an n-period bond in one period.

The approach in finance is to find a pricing kernel that delivers realistic bond prices — in a sense, we reverse engineer it.

What might such a pricing kernel look like? We showed that an iid pricing kernel generated a constant interest rate and a constant flat yield curve. So that can't be it. Evidently we need some dynamics in the pricing kernel.

We focused our attention on loglinear models, which sometimes go by the label "exponential-affine." We can illustrate the idea with the Vasicek model, which we'll express here by

$$\begin{aligned}
\log m_{t+1} &= -\lambda^2/2 - x_t + \lambda w_{t+1} \\
x_{t+1} &= (1-\varphi)\delta + \varphi x_t + \sigma w_{t+1}.
\end{aligned}$$

[You might ask yourself what the dynamics of $\log m$ look like once we've substituted for x.] With this structure, bond prices are loglinear in the state variable x_t. We can express them by

$$\log q_t^n \quad = \quad A_n + B_n x_t$$

for some constants (A_n, B_n). [As practice, you might derive recursions that connect (A_{n+1}, B_{n+1}) to (A_n, B_n). What is the one-period yield?]

Given such a model, we can choose parameters to approximate the salient features of interest rates. We chose four such features and linked them to the four parameters of the model: the mean, variance, and autocorrelation of the short rate; and the mean slope of the yield curve. The Vasicek model is the simplest such example.

There are a couple conceptual issues here worth noting.

- Expectations of the future. This model is forward-looking, but there's a changing in perspective from our earlier work, where the conditional mean was all that mattered. There we had terms like $y_t = E_t(x_{t+1})$ (not literally, we're making a conceptual point). Here we have, in the case of a one-period bond,

  $$\log q_t^1 \quad = \quad \log E_t \left(e^{\log m_{t+1}} \right).$$

  This has a similar form to the cumulant generating function of $\log m_{t+1}$, and will introduce all of its cumulants into the solution, not just its mean.

- Markov structure. As long as the model has a stationary Markov structure, bond prices of all maturities will be functions of the state. Here the function is loglinear, which is convenient but not (you might guess) a general result.

We also touched on two other loglinear models and showed how they can be used to address the observed predictability of forward rates. Predictability of asset returns and related objects is a central issue in finance. Wee establish such patterns with regressions, portfolios sorted on observed variables, and other means. Our emphasis was on models that reproduce known patterns of predictability.