Quiz #3

April 2013

Please write your name below. Then complete the exam in the space provided. There are TWO questions. You may refer to one page of notes: standard paper, both sides, any content you wish.

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(Name and signature)

1. *Linear models (60 points).* Consider the linear time series model

$$x_t = \varphi x_{t-1} + w_t,$$

with $\{w_t\}$ independent normal random variables with mean zero and variance one. Now consider a second random variable y_t built from x_t and the same disturbance w_t by

$$y_t = x_t + \theta w_t.$$

The question is how this combination behaves.

(a) Is there a state variable for which x_t is Markov? What is the distribution of x_{t+1} conditional on the state at date t? (10 points)

(b) Express x_t as a moving average. What are its coefficients? (5 points)

(c) Is there a state variable for which y_t is Markov? What is the distribution of y_{t+1} conditional on the state at date t? (10 points)

(d) Express y_t as a moving average. What are its coefficients? (5 points)

(e) Under what conditions is y_t stable? That is: under what conditions does the distribution of y_{t+k}, conditional on the state at t, converge as k gets large? (10 points)

(f) What is the equilibrium or stationary distribution of y_t? (10 points)

(g) What is the first autocorrelation of y_t? (10 points)

Solution:

(a) It's Markov with x_t. The distribution of x_{t+1}:

$$x_{t+1} = \varphi x_t + w_{t+1}$$

is conditionally normal with mean φx_t and variance one.

(b) The moving average representation is

$$x_t \;=\; w_t + \varphi w_{t-1} + \varphi^2 w_{t-2} + \cdots .$$

The coefficients are $(1, \varphi, \varphi^2, \ldots)$.

(c) Two answers, both work: the state can be x_t or (more commonly) the vector (y_t, w_t). The distribution of y_{t+1}:

$$y_{t+1} \;=\; \varphi x_t + (1+\theta) w_{t+1} \;=\; \varphi(y_t - \theta w_t) + (1+\theta) w_{t+1}$$

is (conditionally) normal with mean $\varphi x_t = \varphi(y_t - \theta w_t)$ and variance $(1+\theta)^2$).

(d) If we add θw_t to the expression for x_t above, we get

$$y_t \;=\; (1+\theta) w_t + \varphi w_{t-1} + \varphi^2 w_{t-2} + \cdots .$$

(e) It's stable if $|\varphi| < 1$: we need the moving average coefficients to approach zero.

(f) Stationary distribution: x_t is normal with mean zero and variance

$$\mathrm{Var}(x_t) \;=\; (1+\theta)^2 + \varphi^2 + \varphi^4 + \cdots \;=\; (1+\theta)^2 + \varphi^2/(1-\varphi^2).$$

2. *Stochastic volatility and equity pricing (40 points).* The Cox-Ingersoll-Ross model of bond pricing consists of the equations

$$\begin{aligned}
\log m_{t+1} &= -(1+\lambda^2/2) x_t + \lambda x_t^{1/2} w_{t+1} \\
x_{t+1} &= (1-\varphi)\delta + \varphi x_t + \sigma x_t^{1/2} w_{t+1},
\end{aligned}$$

with the usual independent standard normal disturbances w_t. We add to it an equation governing the dividend d_t paid by a one-period equity-like claim:

$$\log d_{t+1} \;=\; \alpha + \beta x_t + \gamma x_t^{1/2} w_{t+1}.$$

Here x_t plays the role of the state: if we know x_t, we know the conditional distributions of $(m_{t+1}, x_{t+1}, d_{t+1})$.

(a) Conditional on x_t, what are the mean and variance of $\log m_{t+1}$? What is its distribution? (10 points)

(b) What is the price q_t^1 of a one-period bond? What is its return $r_{t+1}^1 = 1/q_t^1$? (10 points)

(c) What is the price q_t^e of equity, a claim to the dividend d_{t+1}? What is its return r_{t+1}^e? (10 points)

(d) Conditional on x_t, what is the expected log excess return on equity, $E_t(\log r_{t+1}^e - \log r_{t+1}^1)$? (10 points)

Solution:

(a) Conditional on x_t, $\log m_{t+1}$ is normal with mean and variance

$$\begin{aligned} E_t(\log m_{t+1}) &= -(1+\lambda^2/2)x_t \\ \mathrm{Var}_t(\log m_{t+1}) &= \lambda^2 x_t. \end{aligned}$$

(b) The price uses the "mean plus variance over two" formula:

$$\log q_t^1 = -x_t.$$

The log return is $\log r_{t+1}^1 = x_t$.

(c) The price of equity is

$$\begin{aligned} \log q_t^e &= \log E_t(m_{t+1} d_{t+1}) \\ &= \alpha + [\beta - (1+\lambda^2/2)]x_t + [(\gamma+\lambda)^2/2]x_t \\ &= \alpha + (\beta - 1 + \gamma^2/2 + \gamma\lambda)x_t. \end{aligned}$$

The return is

$$\log r_{t+1}^e = \log d_{t+1} - \log q_t^e = (1 - \gamma^2/2 - \gamma\lambda)x_t + \gamma x_t^{1/2} w_{t+1}.$$

(d) Conditional on x_t, the expected excess return is

$$E_t(\log r_{t+1}^e - \log r_{t+1}^1) = -(\gamma^2/2 + \gamma\lambda)x_t.$$