Quiz #2

(Spring 2012)

Please write your name below. Then complete the exam in the space provided. There are THREE questions. You may refer to one page of notes: standard paper, both sides, any content you wish.

(Name and signature)

1. (Sharpe ratios) (40 points) We'll look at Sharpe ratios in a two-period representative agent economy. Endowment growth g is Bernoulli:

$$g \;=\; \begin{cases} 1.00 & \text{with probability } 1-\omega \\ 1.10 & \text{with probability } \omega \end{cases}$$

with $\omega = 0.3$. The representative agent has power utility with discount factor $\beta = 0.98$ and risk aversion $\alpha = 5$. Equity is a claim to g.

(a) What is the pricing kernel for this economy? What are the state prices? (10 points)

(b) What are the price and return of a one-period riskfree bond? (10 points)

(c) What is the price of equity? What are the mean and standard deviation of its excess return? What is its Sharpe ratio? (10 points)

(d) What is the maximum Sharpe ratio for this economy? (10 points)

Solution:

(a) The pricing kernel is $m(z) = \beta g(z)^{-\alpha}$. Here we get $m = [0.9800, 0.6085]$. State prices are $q(z) = p(z)m(z)$ or $q = [0.6860, 0.1826]$.

(b) The price of the bond is

$$q^1 \;=\; \sum p(z)m(z) \;=\; 0.8686,$$

which implies $r^1 = 1/q^1 = 1.1513$.

(c) The price of equity is

$$q^1 \;=\; \sum p(z)m(z)g(z) \;=\; 0.8868.$$

The returns are $r^e = [1.1276, 1.2404]$. The mean and standard deviation follow either from a brute-force calculation [$Var(x) = E(x^2) = E(x)^2$] or related formulas for Bernoulli random variables. The mean and standard deviation of the excess return are 0.0102 and 0.0517. The Sharpe ratio is the ratio of the two: $0.0102/0.0517 = 0.1960$.

(d) This is an application of the Hansen-Jagannathan bound. The maximum Sharpe ratio for this pricing kernel is the ratio of the standard deviation of the pricing kernel to its mean. Here we get $0.1702/0.8686 = 0.1960$. Our asset therefore hits the bound. That's something of an accident, it works because of the two-state structure, which means all returns are linear functions of the pricing kernel. Don't worry if that seems obscure to you.

Matlab program:

2. (entropy bound revisited) (30 points) The idea here is to derive the entropy bound from a maximization problem. We'll do this in an arbitrary two-period economy with a finite set of states. Each state z has probability $p(z)$ and pricing kernel $m(z)$. An asset has returns $r(z)$ that satisfy the pricing relation

$$\sum_z p(z)m(z)r(z) \quad = \quad 1. \tag{1}$$

Our mission is to characterize the asset with the highest expected log return,

$$\sum_z p(z) \log r(z).$$

We'll refer to this as the "high-return asset."

(a) What is the entropy of the pricing kernel? Express it in terms of $m(z)$ and $p(z)$. (10 points)

(b) Use Lagrangian methods to find the returns $r(z)$ (one number for each state) that maximize the expected log return while satisfying the pricing relation (1). How is the return on the high-return asset related to the pricing kernel? (10 points)

(c) Show that the high-return asset attains the entropy bound. (10 points)

Solution:

(a) Entropy is defined by

$$H(m) \quad = \quad \log E(m) - E \log m \quad = \quad \log \sum_z p(z)m(z) - \sum_z p(z) \log m(z).$$

(b) The idea is to maximize the expected log return with the pricing relation as a constraint. The Lagrangian is

$$\mathcal{L} \quad = \quad \sum_z p(z) \log r(z) + \lambda \left(1 - \sum_z p(z)m(z)r(z) \right).$$

The first-order condition for $r(z)$ is

$$p(z)/r(z) \quad = \quad \lambda p(z)m(z).$$

You can see here that there's an inverse relation between $r(z)$ and $m(z)$, but we need to eliminate the multiplier λ. If we multiply both sides by $r(z)$ and sum over z, we see that the left side is one and the right side is λ, so we must have $\lambda = 1$. That gives us the maximizing return

$$r(z) \;=\; 1/m(z).$$

We did this in class using Jensen's inequality, but this is more constructive.

(c) The entropy bound says

$$E(\log r - \log r^1) \;\leq\; H(m) \;=\; \log E(m) - E \log m.$$

All we need to do is substitute. We have $r = 1/m$, so $E \log r = -E \log m$. The one-period rate is $r^1 = 1/E(m)$, so $\log r^1 = -\log E(m)$. That gives us

$$E(\log r - \log r^1) \;=\; -E \log m + \log E(m) \;=\; H(m).$$

(The E in front of r^1 is irrelevant here, because r^1 is a constant.)

3. (put and call options) (30 points) We'll look, once again, at the prices of one-year options when the risk-neutral distribution of the underlying is lognormal. If the future value of the underlying is s_{t+1}, then $\log s_{t+1} \sim \mathcal{N}(\kappa_1, \kappa_2)$. We've seen, in this case, that the price of a European put option with strike price k is

$$\begin{aligned} q_t^p &= q_t^1 k N(d) - q_t^1 e^{\kappa_1 + \kappa_2/2} N(d - \kappa_2^{1/2}) \\ d &= (\log k - \kappa_1)/\kappa_2^{1/2}. \end{aligned}$$

(a) What is the price of a call option with the same strike price? (15 points)

(b) What is the no-arbitrage condition for this environment? Use it to derive the BSM formula for call options from your answer to (a). (15 points)

Solution: Axelle says there's a sign error here somewhere, but I haven't had the time to fix it.

(a) The key input here is put-call parity, solved for the call price:

$$q_t^c \;=\; s_t - q_t^1 k + q_t^p$$

That gives us a call price of

$$\begin{aligned} q_t^c &= q_t^1 k[1 - N(d)] + s_t - q_t^1 e^{\kappa_1 + \kappa_2/2} N(d - \kappa_2^{1/2}) \\ &= q_t^1 k N(-d) + s_t - q_t^1 e^{\kappa_1 + \kappa_2/2} N(d - \kappa_2^{1/2}). \end{aligned}$$

The second line follows from the symmetry of the normal distribution: $1 - N(d) = N(-d)$.

(b) The no-arb condition here is $s_t = q_t^1 e^{\kappa_1 + \kappa_2/2}$ or

$$\kappa_1 = \log(s_t/q_t^1) - \kappa_2/2.$$

That allows us to simplify the call price to

$$q_t^c = q_t^1 k N(-d) + s_t N(-d + \kappa_2^{1/2}),$$

with

$$d = \frac{\log(q_t^1 k/s_t) - \kappa_2/2}{\kappa_2^{1/2}}.$$

If we change the sign, we end up with the BSM formula.