Quiz #3

December 2015

Please write your name below. Then complete the exam in the space provided. There are THREE questions. You may refer to one page of notes: standard paper, both sides, any content you wish.

(Name and signature)

1. *Moving average model, part 1 (predicting the future).* Consider the stochastic process

$$x_t = \sigma(w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2}), \tag{1}$$

where $\{w_t\}$ is a sequence of independent normal random variables with means equal to zero and variances equal to one. Assume, as usual, that at date t we know the current and past values of x and w, but not the future values.

(a) At date t, what is the (conditional) distribution of x_{t+1}? (10 points)

(b) What definition of the state z_t is sufficient to describe this distribution? (10 points)

(c) What is $E_t(x_{t+k}) \equiv E(x_{t+k}|z_t)$ for $k = 1, 2, 3, ...$? (10 points)

Solution:

(a) Shifting this forward one period, we have

$$x_{t+1} = \sigma(w_{t+1} + \theta_1 w_t + \theta_2 w_{t-1}).$$

The last two terms are known at t, the first one is not. Therefore we have

$$\begin{aligned} E_t x_{t+1}) &= \sigma(\theta_1 w_t + \theta_2 w_{t-1}) \\ \text{Var}_t(x_{t+1}) &= \sigma^2. \end{aligned}$$

Since w_{t+1} is normal, the conditional distribution of of x_{t+1} is normal with this mean and variance.

(b) If we look at the conditional mean above, we see that it depends on (w_t, w_{t-1}), so we define that as the state.

(c) Shifting equation (1) ahead by k periods and computing the expectation as of date t, we have

$$\begin{aligned} E(x_{t+1}) &= \sigma(\theta_1 w_t + \theta_2 w_{t-1}) \\ E(x_{t+2}) &= \sigma\theta_2 w_t \\ E(x_{t+3}) &= 0. \end{aligned}$$

As we have seen before, the MA(2) has a two-period memory.

2. *Moving average model, part 2 (dividend valuation).* We now use the same MA(2) to model dividends,

$$d_t \;=\; x_t \;=\; \sigma(w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2}),$$

and consider a claim to the dividend sequence $(d_d, d_{t+1}, d_{t+2}, \ldots)$. Suppose the price of this claim satisfies

$$q_t \;=\; d_t + \delta E_t(q_{t+1}),$$

where $E_t(q_{t+1}) = E(q_{t+1}|z_t)$ is the expectation of the price at $t+1$ given the state at t.

(a) Express the price as the discounted value of future dividends. (10 points)

(b) Express the price as a function of (w_t, w_{t-1}, w_{t-2}). (10 points)

(c) Suppose w_t rises by one. What is the effect on d_t? On q_t? Why do they differ? (10 points)

Solution:

(a) Repeated substitution (and a terminal condition) gives us

$$q_t \;=\; d_t + \delta E_t(d_{t+1}) + \delta^2 E_t(d_{t+2}) + \cdots \;=\; \sum_{j=0}^{\infty} \delta^j E_t(d_{t+j}).$$

(b) Since d is MA(2), we only have a few non-zero terms in the sum:

$$\begin{aligned} q_t &= \sigma(w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2}) + \sigma(\theta_1 w_t + \theta_2 w_{t-1}) + \sigma(\theta_2 w_t) \\ &= \sigma(1 + \delta\theta_1 + \delta^2\theta_2) w_t + \sigma(\theta_1 + \delta\theta_2) w_{t-1} + \sigma\delta^2\theta_2 w_{t-2}. \end{aligned}$$

(c) We see the coefficient of w_t above: $\sigma(1 + \delta\theta_1 + \delta^2\theta_2)$. In the expression in parentheses: the number 1 is the direct impact on the current dividend, $\delta\theta_1$ is the discounted impact on the expected dividend at $t+1$, and $\delta^2\theta_2$ is the discounted impact on the expected dividend at $t+2$.

There's a practical point here. When dividends rise, the effect on the price is both the direct effect on the price and the impact on expected future dividends.

3. *Moving average model, part 3 (bond valuation).* We shift gears now and use the MA(2) as a pricing kernel:

$$\log m_t \;=\; \sigma(w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2}).$$

Our experience with similar models leads us to guess that bond prices take the form

$$\log q_t^n \;=\; A_n + B_n w_t + C_n w_{t-1}$$

for some coefficients $\{A_n, B_n, C_n\}$.

(a) What is the price q_t^1 of a one-period bond? The initial forward rate f_t^0? (10 points)

(b) Derive recursions connecting $(A_{n+1}, B_{n+1}, C_{n+1})$ to (A_n, B_n, C_n). (10 points)

(c) What are (A_n, B_n, C_n) for $n = 0, 1, 2, 3$? (10 points)

(d) What are forward rates f_t^n for $n = 0, 1, 2$? (10 points)

(e) Suppose the forward risk premium fp_t^n is defined by

$$f_t^n = E_t(f_{t+n}^0) + fp_t^n.$$

What are the forward risk premiums for $n = 1, 2$? (10 points)

Solution:

(a) This is the usual "mean plus variance over two" expression:

$$\log q_t^1 = \sigma^2/2 + \sigma(\theta_1 w_t + \theta_2 w_{t-1}).$$

The forward rate is the negative: $f_t^0 = -\log q_t^1$.

(b) We attack this in the usual recursive way, one maturity at a time:

$$\log q_t^{n+1} = \log E_t \left(e^{\log m_{t+1} + \log q_{t+1}^n} \right).$$

With our loglinear guess, the exponent is

$$\begin{aligned} \log m_{t+1} + \log q_{t+1}^n &= \sigma(w_{t+1} + \theta_1 w_t + \theta_2 w_{t-1}) + A_n + B_n w_{t+1} + C_n w_t \\ &= A_n + (\sigma + B_n) w_{t+1} + (\sigma\theta_1 + C_n) w_t + (\sigma\theta_2) w_{t-1}. \end{aligned}$$

Mean plus variance over two then gives us

$$\begin{aligned} \log q_t^{n+1} &= A_n + (\sigma + B_n)^2/2 + (\sigma\theta_1 + C_n) w_t + (\sigma\theta_2) w_{t-1} \\ &= A_{n+1} + B_{n+1} w_t + C_{n+1} w_{t-1}. \end{aligned}$$

Lining up terms, we have

$$\begin{aligned} A_{n+1} &= A_n + (\sigma + B_n)^2/2 \\ B_{n+1} &= \sigma\theta_1 + C_n \\ C_{n+1} &= \sigma\theta_2. \end{aligned}$$

(c) The coefficients start with $A_0 = B_0 = C_0 = 0$. The recursions then imply

Maturity n	A_n	B_n	C_n
0	0	0	0
1	$\sigma^2/2$	$\sigma\theta_1$	$\sigma\theta_2$
2	$\sigma^2[1+(1+\theta_1)^2]/2$	$\sigma(\theta_1+\theta_2)$	$\sigma\theta_2$
3	$\sigma^2[1+(1+\theta_1)^2+(1+\theta_1+\theta_2)^2]/2$	$\sigma(\theta_1+\theta_2)$	$\sigma\theta_2$

(d) Recall that the forward rate is

$$\begin{aligned} f_t^n &= \log(q_t^n/q_t^{n+1}) \;=\; \log q_t^n - \log q_t^{n+1} \\ &= (A_n - A_{n+1}) + (B_n - B_{n+1})w_t + (C_n - C_{n+1})w_{t-1} \end{aligned}$$

That gives us

$$\begin{aligned} -f_t^0 &= \sigma^2/2 + \sigma\theta_1 w_t + \sigma\theta_2 w_{t-1} \\ -f_t^1 &= \sigma^2(1+\theta_1)^2/2 + \sigma\theta_2 w_t \\ -f_t^2 &= \sigma^2(1+\theta_1+\theta_2)^2/2. \end{aligned}$$

An MA(2) has a two-period memory, so by the time we get to the two-period-ahead forward rate, there's no effect of the state.

(e) Expected future short rates are

$$\begin{aligned} f_t^0 &= -\sigma^2/2 - \sigma\theta_1 w_t - \sigma\theta_2 w_{t-1} \\ E_t(f_{t+1}^0) &= -\sigma^2/2 - \sigma\theta_2 w_t \\ E_t(f_{t+2}^0) &= -\sigma^2/2. \end{aligned}$$

The term premiums are therefore

$$\begin{aligned} tp_t^1 &= f_t^1 - E_t(f_{t+1}^0) \;=\; \sigma^2[1-(1+\theta_1)^2]/2 \\ tp_t^2 &= f_t^2 - E_t(f_{t+1}^0) \;=\; \sigma^2[1-(1+\theta_1+\theta_2)^2]/2. \end{aligned}$$

Both are constant: they don't depend on the