# Review for Quiz #1

Revised: October 2, 2015

I'll focus on the big picture to give you a sense of what we've done and how it fits together. Everything is leading up to our central equation, $E(mr) = 1$, and its many variants.

As you work through this, I suggest you construct examples that illustrate each concept and result. If you're stuck, start with the Lab Reports.

## Random variables

Random variables. They're the input to everything we do. Formally, we start with a state $z$, associated probabilities $p(z)$, and random variables $x(z)$. Probabilities $p(z)$ are nonnegative and sum or integrate to one.

Generating functions. We used the moment generating function $h(s) = E(e^{sx})$ and cumulant generating function $k(s) = \log h(s)$ to generate moments and cumulants, resp, of a random variable $x$. This looks mysterious the first time, but it's incredibly convenient. One use is in identifying departures from normality. Measures of skewness and excess kurtosis are indications that the distribution is something other than normal.

Common distributions: Bernoulli, Poisson, normal, exponential. All are commonly used in economics and finance.

Connections between random variables. If we have two or more random variables, we need ways to describe the connections among them — what is formally referred to as dependence. If there's no connection, we say they're independent. More formally, two random variables are independent if the pdf factors: $p_{12}(x_1, x_2) = p_1(x_1)p_2(x_2)$. Otherwise, there's no limit to the kinds of connections we might have. We often start with the simplest connection, the covariance or correlation, which is a measure of linear connection.

Sums and mixtures. Both are useful tools for getting distributions that are something other than normal. With sums, we add independent normal and nonnormal components together, with the result that the nonnormality of the latter carries over to the sum. A mixture has a pdf that's a weighted average of pdf's, hence the mgf is the weighted average of mgf's. This leads to nonnormal behavior even if the components are normal. This is so concise as to be mystifying; you should write down examples to remind yourself what it means.

## Risk and risk aversion

We'll use expected utility with a power function most of the time, but the more general treatment sets up the possibility of more complex preferences.

The central feature is risk aversion. In general settings, we identified risk aversion by comparing the certainty equivalent to mean consumption. Put directly: you are risk averse if you prefer a constant consumption level $\mu$ (the certainty equivalent) that is smaller than $\bar{c} = E(c)$ (the mean) of your consumption. We measure the combination of risk and risk aversion with the risk penalty, defined by $rp = \log(\bar{c}/\mu)$.

Also important is the role of high-order moments and cumulants, which we'll see again when we look at risk premiums and options. One example is Samuelson's expansion, in which we express expected utility in terms of derivatives of utility and the moments of consumption. Another example is our expansion for power utility in terms of the cumulants of $\log c$. Again, we see that it's not only the variance that matters: high-order terms also show up. Power utility agents generally like positive skewness and dislike positive excess kurtosis.

## Consumption, saving, and portfolio choice

Asset pricing starts with portfolio choice. We worked our way up to a two-period example with dates 0 and 1 and a number of different states $z$ at date 1. [Draw the appropriate event tree.]

One approach was based on *Arrow securities*: claims to one unit of the good in a specific state $z$ at date 1. We denote the prices of these securities, in units of the date-0 good, by $Q(z)$, which we call the *state prices*. The first-order conditions of a utility-maximizing agent imply

$$Q(z) \quad = \quad p(z) \frac{\beta u'[c_1(z)]}{u'(c_0)}. \tag{1}$$

Here the equation is interpreted as a demand function: given a price $Q(z)$, how much do we want to consume at date 1 in state $z$ — that is, $c_1(z)$? [It's a little more complicated than that, since it includes $c_0$ as well, but that's the basic idea.]

Another approach is to consider an arbitrary collection of assets. At date 0, we buy asset $j$ at price $q^j$ in units of the date 0 good. At date 1, we get dividend $d^j(z)$, which depends, in general, on which state $z$ occurs. The (gross) return is $r^j(z) = d^j(z)/q^j$. It's often useful to decompose an asset into its state-specific components — into Arrow securities. Recall that Arrow security $z$ gives us one unit of output at date 1 in state $z$, zero in all other states. That means that the dividend $d^j(z)$ in a specific state $z$ can be replicated with the same number of Arrow securities. We can replicate the dividend in all states by purchasing the appropriate number of units of each Arrow security. Its price is then the sum of the prices of the Arrow securities that replicate the pattern of dividends. This is just bookkeeping, there's no maximization involved. Suggestion: make up a two-state example and show how it works.

The consumer maximization problem with an arbitrary collection of assets leads (again) to (1). The first-order conditions also imply

$$\sum_z p(z)\{\beta u'[c_1(z)]/u'(c_0)\} r^j(z) \quad = \quad E(m r^j) \quad = \quad 1, \tag{2}$$

for all traded assets $j$. This equation determines consumptions: given returns $r^j(z)$, choose consumptions to make it hold. Later on the same equation will reappear as an asset pricing relation: given $m$, find the price and return of an asset.

Clean solutions to portfolio choice problems are rare, but we saw that a theoretical agent holds less of the risky asset when we increase her risk aversion. In Merton's formula, the share $a$ invested in the risky asset is

$$a \quad = \quad \frac{1}{\alpha} \frac{E(r^e - r^1)}{\mathrm{Var}(r^e)},$$

a function of risk (the variance of the risky asset's return), return (the expected excess return), and risk aversion ($\alpha$). We won't use this again, but it's a a good sign that things sometimes work out so nicely, even if it's a special case.

## Two-period economies

General equilibrium models are a basic tool of economics. The representative agent version is a useful starting point: simple enough to be manageable, flexible enough to generate clear insights. It's the predominant model in macro-finance, even as people work on extensions with more complex preferences or multiple agents.

The ingredients of a general equilibrium model include:

- List of commodities.
- List of agents.
- Preferences and endowments of agents.
- Technologies for transforming some commodities into others.
- Resource constraints limiting consumption to endowments plus net production.

Once we have these ingredients, we can look at a competitive equilibrium: a set of prices and quantities in which agents maximize utility given prices, firms maximize profits given prices, and supply equals demand (the resource constraints are satisfied). The equilibrium is competitive in the sense that agents and firms take prices as given.

We find a competitive equilibrium in reverse: we find an optimal allocation, and infer prices from marginal rates of substitution. It's a useful shortcut in models with a single "representative" agent.

In asset pricing applications, it's convenient to use an exchange economy, in which the single agent simply consumes the endowment. Prices of *Arrow securities* then come from the marginal rate of substitution of the agent:

$$Q(z) \quad = \quad p(z)\beta u'[c_1(z)]/u'(c_0) \quad = \quad p(z)\beta u'[y_1(z)]/u'(y_0).$$

It's the same equation we saw before, but this time causality goes the other way: endowments $y$ generate prices $Q$. We see, for example, that in states where the endowment $y_1(z)$ is high, the price is low. Why? Because $u'$ is decreasing: the more we have, the less an additional unit is worth to us. In that sense, assets that pay off mostly in good times will have less value than assets that pay off mostly in bad times.

## Notation guide

| Object | Definition |
|---|---|
| *Random variables* | |
| $z$ | state |
| $p$ | probability |
| $x$ | random variable |
| $E$ | expectation: $E[f(x)] = \sum_x f(x)p(x)$ or $\int f(x)p(x)dx$ |
| $\mu'_j$ | raw moment: $E(x^j)$ for positive integer $j$ |
| $\mu_j$ | central moment: $E[(x-\mu'_1)^j]$ |
| $h$ | moment generating function: $h(s) = E(e^{sx})$ |
| $k$ | cumulant generating function: $k(s) = \log h(s)$ |
| $\kappa_j$ | cumulant: $j$th derivative of $k$ evaluated at $s = 0$ |
| $\gamma_1$ | skewness: $\kappa_3/(\kappa_2)^{3/2} = \mu_3/(\mu_2)^{3/2}$ |
| $\gamma_2$ | excess kurtosis: $\kappa_3/(\kappa_2)^2 = \mu_4/(\mu_2)^2 - 3$ |
| *Risk and risk aversion* | |
| $\alpha$ | risk aversion parameter |
| $c$ | consumption |
| $U$ | overall utility |
| $u$ | utility in each state: $U = E[u(c)]$ |
| *Asset prices and returns* | |
| $\beta$ | discount factor in utility |
| $a^j$ | position or share invested in asset $j$ |
| $d^j$ | dividend paid by asset $j$ |
| $q_x$ | price of a generic $x$ |
| $q_0$ | price of consumption at date 0 |
| $q_1$ | price of consumption at date 1 |
| $q_1(z)$ | price of consumption at date 1 in state $z$ |
| $q^j$ | price of generic asset $j$ |
| $q^1$ | price of one-period riskfree bond |
| $q^e$ | price of risky claim ("equity") |
| $Q(z) = q_1(z)/q_0$ | price of Arrow security $z$ |
| $r^j = d^j/q^j$ | return on asset $j$ |