Lab Report #3: Consumption, Risk, & Portfolio Choice

Revised: September 23, 2015

Due at the start of class. You may speak to others, but whatever you hand in should be your own work. Please include your Matlab code.

1. *Entropy.* Later in the course, we define the entropy of a positive random variable x by

$$H(x) \;=\; \log E(x) - E(\log x).$$

(We'll also explain why we call this entropy.)

(a) Use Jensen's inequality to show that $H(x) > 0$ as long as x is random.

(b) Suppose $x = e^y$ with $y \sim \mathcal{N}(\kappa_1, \kappa_2)$. What is the entropy of x?

2. *Constrained optimization.* Consider the problem: choose x and y to maximize

$$f(x, y) \;=\; \log(x-1) + \log(y-2)$$

subject to $2x + y \leq 7$.

(a) What is the Lagrangian associated with this problem?

(b) What are the first-order conditions?

(c) What values of x and y solve the problem? What is the Lagrange multiplier?

3. *Assets, returns, and Arrow securities.* Consider a two-period event tree. At date 0, we purchase one unit of asset (or security, the words are interchangeable) j for price q^j. At date 1, we get dividend $d^j(z)$, which depends on the state z. Let us say, specifically, that there are two assets and two states, with dividends

Asset	State 1	State 2
1 ("bond")	2	2
2 ("equity")	2	3

The prices of the assets are $q^1 = 3/2$ (bond) and $q^e = 2$ (equity).

(a) Does it matter that the bond pays two in each state? How does it differ from a bond that pays one in each state? Or one hundred?

(b) What are the (gross) returns on these assets? The expected returns?

(c) An Arrow security pays one in a specific state, nothing in other states. Here we have two states, hence two Arrow securities. Their dividends are

Security	State 1	State 2
Arrow 1	1	0
Arrow 2	0	1

What quantities of the two assets (bond, equity) reproduce the dividend of the second Arrow security? What must the prices be of the two Arrow securities?

(d) Assets can be thought of as collections of Arrow securities. If we know the prices of Arrow securities, we can find the prices of other assets by adding up the values of their state-specific dividends. Use the prices of the Arrow securities computed above to find the prices of the two assets.

4. *Portfolio choice.* An investor must decide how to allocate his saving between a riskfree bond and equity. We approximate the world with two states, each of which occurs with probability 1/2. The returns by state are

Security	State 1	State 2
1 ("bond")	1.1	1.1
2 ("equity")	0.9	1.5

That is, one unit invested at date 0 yields the returns listed in the table at date 1.

The investor's problem is to choose current consumption c_0 and the fraction of saving a to invest in equity to solve

$$\begin{aligned} \max_{c_0, a} \quad & u(c_0) + \beta \sum_z p(z) u[c_1(z)] \\ s.t. \quad & c_1(z) \;=\; (y_0 - c_0)[(1-a)r^1 + a r^e(z)]. \end{aligned}$$

If $a > 1$, the agent has a levered position, borrowing to fund investments in equity greater than saving. As usual, $u(c) = c^{1-\alpha}/(1-\alpha)$. Where the problem calls for numbers, we'll use $\beta = 0.9$ and $\alpha = 2$.

(a) What is saving here?

(b) What are the upper and lower bounds on a consistent with positive consumption $c_1(z)$ in all states z? Lower bound?

(c) What are the first-order conditions for c_0 and a? *Comment: I recommend substituting the expression for $c_1(z)$ into the utility function.*

(d) Use Matlab to solve the first-order condition for a numerically. What value of a maximizes utility? Make sure it implies positive consumption in all states. *Comment: I did this by varying a manually until its first-order condition was satisfied. You could also compute the first-order condition for a grid of values for a, and choose the one that comes closest to satisfying the first-order condition.*