Review for Quiz #1

(Started: February 10, 2012; Revised: February 20, 2012)

I'll focus on the big picture to give you a sense of what we've done and how it fits together. The key sections are the first one, random variables, and the last one, asset pricing. The sections in between combine practice with random variables with insight into various approaches to asset pricing.

Random variables

Random variables. They're the input to everything we do. Formally, we start with a state z, associated probabilities $p(z)$, and random variables $x(z)$.

Generating functions. We used the moment generating function $h(s) = E(e^{sx})$ and cumulant generating function $k(s) = \log h(s)$ to generate moments and cumulants, resp. The latter are especially useful, since "high-order" cumulants (cumulants κ_j for $j \geq 3$) are zero for the normal distribution. Measures of skewness and excess kurtosis are therefore indications that the distribution is something other than normal.

Common distributions: Bernoulli, Poisson, normal, exponential. The normal and Poisson are commonly used in option pricing.

Risk and risk aversion

We'll use expected utility with a power function most of the time, but the more general treatment sets up the possibility of more general preferences. We may come back to that at the end of the course if we have time. Also important is the role of high-order moments and cumulants, which we'll see again when we look at the equity premium.

Consumption, saving, and portfolio choice

Asset pricing starts with portfolio choice. We worked our way up to a two-period example with dates 0 and 1 and a number of different states z at date 1. Clean solutions are a rarity, but we saw that a theoretical agent holds less of the risky asset when we increase her risk aversion. We also saw the first version of the equation,

$$\sum_z p(z)\{\beta u'[c_1(z)]/u'(c_0)\} r^j(z) \quad = \quad E(mr^j) \quad = \quad 1, \tag{1}$$

for all traded assets j. Here it's a first-order condition for the agent: given returns r^j, choose consumption and portfolios. Later on it will reappear as an asset pricing relation: given m, find the price and return of an asset.

Two-period economies

General equilibrium models are a basic tool of economics. The representative agent version is a good starting point for macroeconomics and finance. It's the predominant model in macro-finance, even as people work on extensions with more complex preferences or multiple agents.

Arrow securities are an important concept here: claims to one unit of the good in a specific state z at date 1. We denote their prices, in units of the date-0 good, by $q(z)$, which we call the *state prices*. The first-order condition of a representative agent implies

$$q(z) \;\;=\;\; p(z)\beta u'[c_1(z)]/u'(c_0), \tag{2}$$

a short step from (1). If we have as many assets as states, we can use them to construct state prices. The possibility of using the mysterious state prices in practical situations is a significant insight.

Asset pricing

We change perspectives here. In the portfolio choice problem, asset prices and returns are given, agents simply decide how much of each asset to buy. In the general equilibrium problem, both prices and quantities are endogenous: they come out of the model. Both paths lead to (1), but the interpretation is different. Here we don't worry about where (1) comes from. We simply say that if we know m, we can compute asset prices and returns.

The first result, which I regard as one of the great accomplishments of modern finance, is that we can price assets with Arrow securities if the economy is "arbitrage-free." Again, apologies for all the q's. If asset j has dividends $d^j(z)$, its price satisfies

$$q^j \;\;=\;\; \sum_z q(z) d^j(z). \tag{3}$$

The theorem says we can always find positive state prices $q(z)$ that satisfy this equation for every traded asset j. We haven't said anything about what the state prices are or where they come from, only that there must be some.

We have two versions of this result that we'll use repeatedly. The first version is based on a *pricing kernel* m, defined implicitly by $q(z) = p(z)m(z)$. The pricing relation (3) becomes

$$q^j \;\;=\;\; \sum_z p(z)m(z)d^j(z) \;\;=\;\; E(md^j).$$

This turns into (1) if we divide by q^j and note that the return is $r^j(z) = d^j(z)/q^j$. We can breathe some life into m by equating it to the marginal rate of substitution of a representative agent, as in (1). The second version is based on *risk-neutral probabilities* p^*, defined implicitly by $q(z) = p(z)m(z) = q^1 p^*(z)$. Here

$$q^1 \;\;=\;\; \sum_z p(z)m(z) \;\;=\;\; E(m)$$

is the price of a one-period riskfree bond. The pricing relation (3) turns into

$$q^j \quad = \quad q^1 \sum_z p^*(z) d^j(z) \quad = \quad q^1 E^*(d^j),$$

where E^* means the expectation computed from the risk-neutral probabilities.

Risk and return

We've barely touched on this, but the pricing kernel version contains an explanation for why some assets have higher expected returns than others. There are two levels to this explanation: the covariance of the return with m and the connection between m and consumption.

To keep things simple, let's scale the units of every asset so that they have the same expected dividend $E(d^j)$. Then any difference in expected return must come from the price: $E(r^j) = E(d^j)/q^j$. The price of an asset follows from

$$q^j \quad = \quad E(md^j) \quad = \quad E(m)E(d^j) + \mathrm{Cov}(m, d^j).$$

The first term is the same for all assets, but the second need not be. Evidently assets whose dividends have the greatest negative covariance with m have the lowest prices, hence the highest expected returns. Put differently: assets are penalized, in terms of price and return, for having high payoffs in states when m is low.

The second level tells us what states those are. It stems from the connection in representative agent models between the pricing kernel and consumption:

$$m(z) \quad = \quad \beta u'[c_1(z)]/u'(c_0).$$

Here we see that states with high $c_1(z)$ have low $m(z)$. Why? Because marginal utility is decreasing. It says, in words, that payoffs when consumption is high are less valuable than payoffs when consumption is low. That's a good rationale for high average returns on assets like equity, whose payoffs are highest in good times, when consumption is high and the value of an additional unit of consumption is low.