1 Theoretical Kinematics

1.1 Plücker Coordinates

Julius Plücker (June 16, 1801 – May 22, 1868) introduced the method of referencing lines as a set of coordinates, called *Plücker coordinates* [1], which maps a line in projective three-dimensional space to a point in projective five-dimensional space (k-dimensional subspaces of n-dimensional space). This creates a new geometry where lines in three-dimensional space are the points of the new geometry[2].

To define Plücker coordinates, we first define the meet and join operations of two linear subspaces. The *meet* of two subspaces is defined to be their maximum common intersection and the *join* of two subspaces is defined to be the minimum space enclosing both subspaces. For example, in general position, a line meets a plane at a point and a line joins a point into a plane.

1.1.1 Notation

Homogeneous coordinates make calculations possible in projective space. The notation used will be:

$(V:0)$	–	A homogeneous vector, with $V = (V_x, V_y, V_z)$.
$(P:w)$	–	A homogeneous point, with $P = \left(\frac{P_x}{w}, \frac{P_y}{w}, \frac{P_z}{w}\right)$
$[N:n]$	–	A plane satisfing $N \cdot P + n = N_x P_x + N_y P_y + N_z P_z + nw = 0$ for any point P on the plane with $N = (N_x, N_y, N_z)$ being the normal to the plane, n the moment of normal.
$\{U:V\}$	–	A Plücker coordinate for a line, with U, V three-element vectors.
$U \cdot V$	–	The dot product of vectors U and V.
$U \times V$	–	The cross product of vectors U and V.

1.1.2 Derivation

Let us consider projective three-dimensional space with homogeneous coordinates. We will present a derivation for line coordinates based on [2] and [3] . First the standard determinant definition of Plücker coordinates, then a similar mapping based on geometry.

Let L be a line and let P and Q be two distinct points on L and let $[E:e]$ and $[F:f]$ be two distinct planes that contain L. Thus L is the join of P and Q and the meet of E and F. Assume the homogeneous coordinates for points P and Q are $(p_0 : p_1 : p_2 : p_3)$ and $(q_0 : q_1 : q_2 : q_3)$ respectively (i.e. $P = \left(\frac{p_0}{p_3}, \frac{p_1}{p_3}, \frac{p_2}{p_3}\right)$, $Q = \left(\frac{q_0}{q_3}, \frac{q_1}{q_3}, \frac{q_2}{q_3}\right)$). Since points P and Q both lie on planes $[E:e]$ and $[F:f]$, we have following equations:

$$E \cdot P + e = 0, \quad E \cdot Q + e = 0 \quad \text{and} \quad F \cdot P + f = 0, \quad F \cdot Q + f = 0$$

Let λ_{ij} denote $p_i q_j - p_j q_i$, using either plane we can equate elements and eliminate to get:

$$\begin{aligned}
0E_x + \lambda_{10}E_y + \lambda_{20}E_z + \lambda_{30}e &= 0 \\
\lambda_{01}E_x + 0E_y + \lambda_{21}E_z + \lambda_{31}e &= 0 \\
\lambda_{02}E_x + \lambda_{12}E_y + 0E_z + \lambda_{32}e &= 0 \\
\lambda_{03}E_x + \lambda_{13}E_y + \lambda_{23}E_z + 0e &= 0
\end{aligned}$$

From the definition of λ_{ij}, we know that $\lambda_{ij} = -\lambda_{ij}$ and $\lambda_{ii} = 0$. Therefore, only six different values describe the line L through P and Q. The Plücker coordinates from different pairs of points on a line only differ by a non-zero constant factor, being homogeneous coordinates of projective five-dimensional space. If we choose $\{\lambda_{01} : \lambda_{02} : \lambda_{03} : \lambda_{32} : \lambda_{13} : \lambda_{21}\}$ as the coordinates the system of homogeneous (linear) equations, having a non-trivial solution, gives the following constraint:

$$\lambda_{01}\lambda_{32} + \lambda_{02}\lambda_{13} + \lambda_{03}\lambda_{21} = 0$$

A similar mapping of a line L in projective three-dimensional space to a point in projective five-dimensional space can be achieved if we define the displacement vector from point P to point Q on the line ($U = Q - P$) as the first three of Plücker coordinates and the cross product of the position vectors as the other three ($V = Q \times P$). We now have

$$L = \{U : V\} = \{Q - P : Q \times P\}$$

with $U.V = 0$ providing an equivalent constraint to that of the linear equations.

1.1.3 Relevant Plücker Coordinate Relationships

Here are a list of Plücker coordinate relations[3] relevant to the analysis presented below:

$L = \{Q - P : Q \times P\}$, for A and B distinct points on L, and line is directed $P \to Q$
$L = \{U : U \times P\}$, for U the direction of L and P a point on L
$L = \{pQ - qP : Q \times P\}$, for $(P : p)$ and $(Q : p)$ distinct homogeneous points on L
$L = \{S \times T : tS - sT\}$, for $(S : s)$ and $(T : t)$ distinct planes containing L

1.2 Inverse Kinematics of the Novint Falcon Controller

1.2.1 Geometric Analysis

Figure 1 shows an overview of the geometry with two triangular platforms. The nine key points, A_i,, B_i, C_i,are visible. The centre of the smaller EE (end effector) is located at position P displaced by (P_x, P_y, P_z) from the origin O at the centre of FF (fixed frame). Note the design constraints. c is the side length of EE, a the side of FF, r_c the distance C_iB_i and r_a the distance A_iB_i. A further constraint is imposed by the circular trajectory of B_i at radius r_a from the centre A_i (the dotted lines). A sphere, radius r_c, centred on C_i gives the locus of B_i at radius r_a from centre A_i. The plane of this arc cuts the sphere in a small circle in the same plane. The intersection of the arcs and circles yield B_i, with the solution external to FF and EE being valid. The desired actuated R-joint angles are measured from an edge or line view of FF to A_iB_i as θ_i. The leg subscript i is omitted in all following equations. It is obvious that a joint angle θ must be computed separately for each leg.

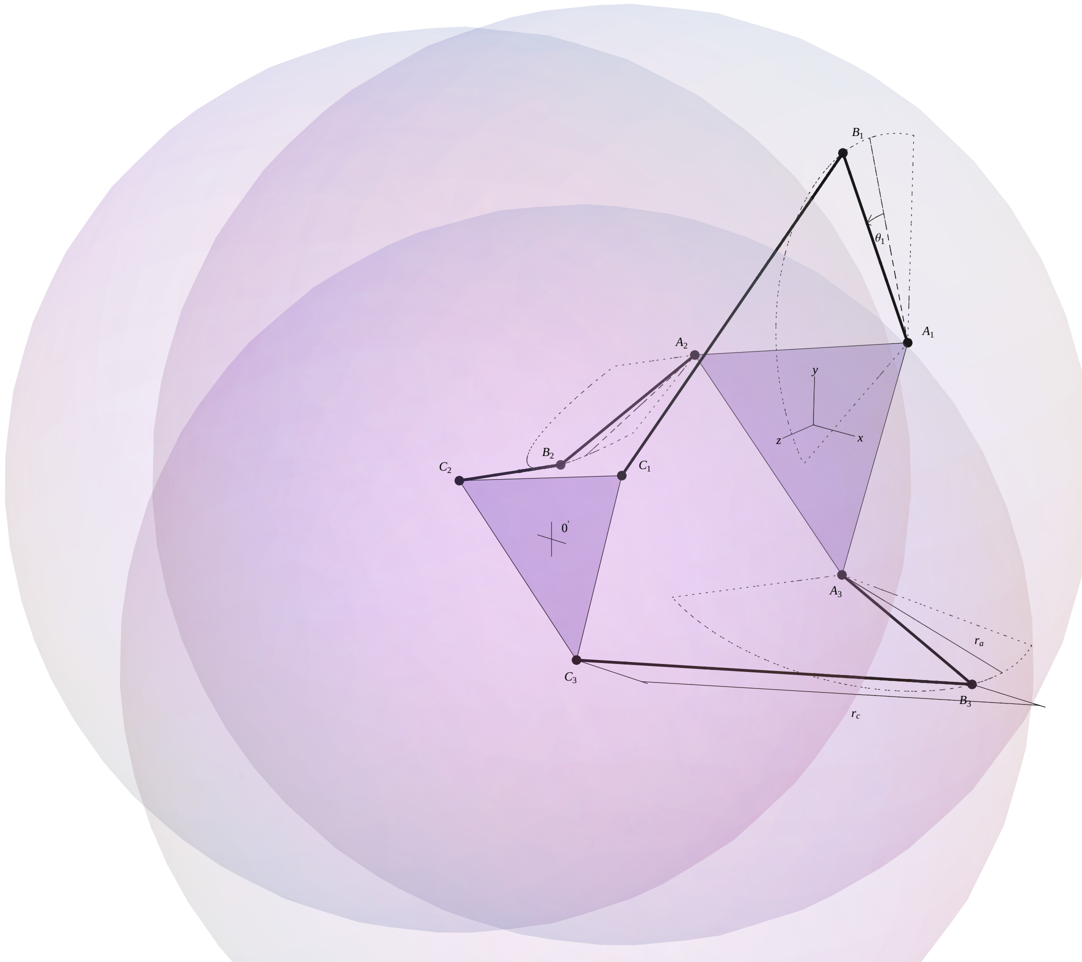

Figure 1: Inverse kinematics overview

1.2.2 Inverse Kinematic Computation

In this discussion, a line L will be intersected not with the sphere described above, centred on ankle C, but with the algebraically simpler one on $A = (A_x, A_y, A_z)$. The homogeneous coordinates of a point $(B : w)$ on it are given by

$$(A_x w - B_x)^2 + (A_y w - B_y)^2 + (A_z w - B_z)^2 - r_a^2 w^2 = 0 \tag{1}$$

where $A_z = 0$ in the frame chosen. We define L as the line through the two possible solutions of B. It is obtained by interecting the plane of a thigh circle on A with the plane of a circle produced by the interection of the sphere given by Eq. 1 and the one of radius r_c centred on C. The homogeneous coordinates $[T:t]$ of the three vertical thigh planes are:

$$
\begin{aligned}
[T_1:t_1] &= \left[\cos\left(\frac{\pi}{12}\right):\sin\left(\frac{\pi}{12}\right):0:-\frac{a}{2}\right] &&= \left[\frac{1+\sqrt{3}}{2\sqrt{2}}:\frac{-1+\sqrt{3}}{2\sqrt{2}}:0:-\frac{a}{2}\right]\\
[T_2:t_2] &= \left[\cos\left(\frac{3\pi}{4}\right):\sin\left(\frac{3\pi}{4}\right):0:-\frac{a}{2}\right] &&= \left[-\frac{1}{\sqrt{2}}:\frac{1}{\sqrt{2}}:0:-\frac{a}{2}\right]\\
[T_3:t_3] &= \left[\cos\left(\frac{17\pi}{12}\right):\sin\left(\frac{17\pi}{12}\right):0:-\frac{a}{2}\right] &&= \left[\frac{1-\sqrt{3}}{2\sqrt{2}}:\frac{-1-\sqrt{3}}{2\sqrt{2}}:0:-\frac{a}{2}\right]
\end{aligned}
$$

Coordinates of the plane of the circle of intersection between the spheres centred on A and C (Figure 2) are the coefficients of the linear equation which is the difference between the two sphere equations. Its plane coordinates are $[S:s]$. Explicitly, a thigh and shin sphere intersection circle plane has coordinates

$$[S:s] = \left[(C_x - A_x):(C_y - A_y):C_z:\left(r_c^2 - r_a^2 + A_x^2 - C_x^2 + A_y^2 - C_y^2 - C_z^2\right)/2\right]$$

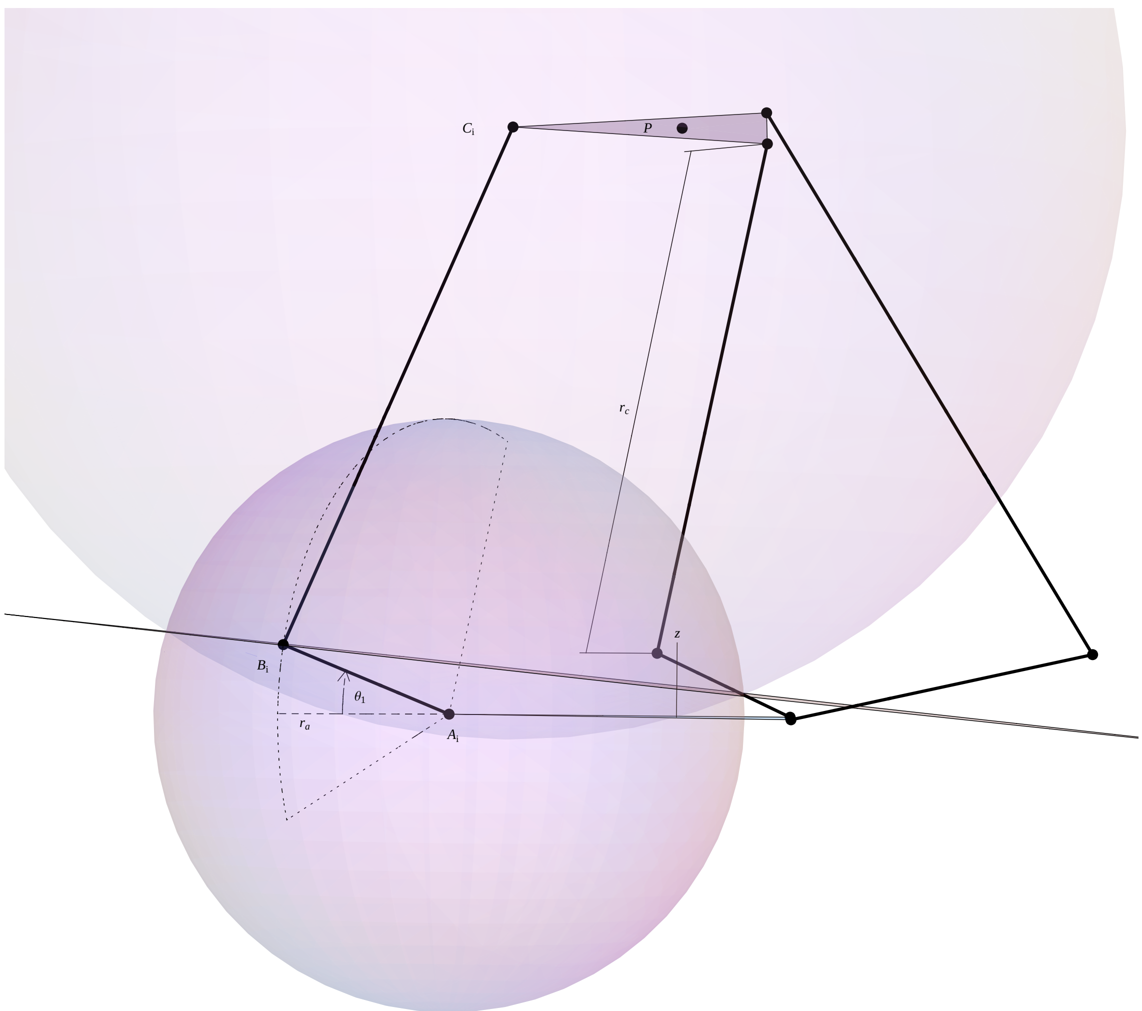

Figure 2: Intersection of the C_i and A_i spheres

The next step is to calculate the Plücker coordinates of the line that is the meet of these two planes:

$$L = \{S \times T : tS - sT\} = \{U : V\}$$

Using the relationship of points on a plane for any point B on the line, we have $T \cdot B + t = 0$ and $S \cdot B + s = 0$. We can combine these equalities to create a set of linear equations:

$$\begin{aligned}
0B_x + (S_xT_y - S_yT_x)\,B_y + (S_xT_z - S_zT_x)\,B_z + (S_xt - sT_x)\,w &= 0 \\
(S_yT_x - S_xT_y)\,B_x + 0B_y + (S_yT_z - S_zT_y)\,B_z + (S_yt - sT_y)\,w &= 0 \\
(S_xT_y - S_yT_x)\,B_x + (S_xT_y - S_yT_x)\,B_y + 0B_z + (S_xt - sT_x)\,w &= 0 \\
(sT_x - S_xt)\,B_x + (sT_y - S_yt)\,B_y + (sT_z - S_zt)\,B_z + 0w &= 0
\end{aligned} \tag{2}$$

The coefficients are the Plücker coordinates of the line $L = \{U : V\} = \{U_x : U_y : U_z : V_x : V_y : V_z\}$. The first two lines of Eq. 2 can be rearranged to give

$$\begin{aligned}
B_x &= \frac{U_x}{U_z} B_z + \frac{V_y}{U_z} w \\
B_y &= \frac{U_y}{U_z} B_z - \frac{V_x}{U_z} w
\end{aligned}$$

and substtituted into Eq. 1 to produce a quadratic in B_z, the z components for the intersections of the line L and the sphere of r_a centred on A:

$$\begin{aligned}
0 = \left[U_x^2 + U_y^2 + U_z^2\right] B_z^2 - 2w\left[(A_xU_z - V_y)\,U_x + (A_yU_z + V_x)\,U_y\right] B_z + \\
w^2\left[(A_xU_z - V_y)^2 + (A_yU_z + V_x)^2 + A_zU_z^2 - r_a^2U_z^2\right]
\end{aligned} \tag{3}$$

This is all that is required to find $\theta = \sin^{-1}(B_z/r_a)$. Each solution can be tested to obtain the correct angle, with angle in the range $\theta\left\{\frac{7\pi}{18}, \frac{11\pi}{18}\right\}$ requiring special attention to determine if $\theta = \sin^{-1}(B_z/r_a)$ or $\theta = \pi - \sin^{-1}(B_z/r_a)$.

One method is to use each value of B_z and the relations above to calculate the possible knee locations(B_x, B_y, B_z); the vectors $A \to B$ and $A \to C$, along with the normal of the thigh plane $[T : t]$ provide the following test for validity:

$$T \cdot ((B - A) \times (C - A)) > 0 \quad \text{,for valid } B_z \tag{4}$$

References

[1] J. Plücker, *Neue Geometrie des Raumes.* Leipzig, 1868.

[2] D. M. Mount and F. T. Pu, "Binary Space Parititions in Plücker Space," *Algorithm Engineering and Experimentation, (ALENEX'99)*, 1999.

[3] K. Shoemake, "Plücker Coordinate Tutorial," *Ray Tracing News*, 1998.