On the Estimation of the Topp-Leone Distribution

David E. Giles
Department of Economics, University of Victoria
Victoria, B.C., Canada V8W 2Y2

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Abstract

The Topp-Leone distribution is attractive for reliability studies as it has finite support and a bathtub-shaped hazard function. We compare some properties of various estimators of this distribution's shape parameter. These include the method of moments and maximum likelihood estimators; their bootstrap bias-adjusted counterparts; and an analytically bias-adjusted maximum likelihood estimator. The last of these estimators is very simple to apply, and it dominates the other estimators in terms of both relative bias and mean squared error in finite samples.

Keywords: Bias reduction; maximum likelihood; method of moments; unbiased estimation; mean squared error; bathtub hazard; J-shaped distribution; finite support

Mathematics Subject Classifications: 62F10; 62N02; 62N05

Author Contact: David E. Giles, 58 Rock Lake Court, L'Amable, ON, Canada K0L 2L0; email: dgiles@uvic.ca; voice: (613) 332-6833

1. Introduction

In this paper we study the sampling properties of the maximum likelihood (ML) and method of moments (MOM) estimators of the shape parameter of an important distribution that has finite support – the Topp-Leone (T-L) distribution. The selection of distributions whose support is finite is relatively sparse. Obvious examples include the Beta and Uniform distributions, but there are also less well-known examples such as the doubly-truncated Weibull distribution (McEwan and Parresol, 1991), and the distributions of Haupt and Schäbe (1992, 1997), and Schäbe (1994).

Nadarajah and Kotz (2003) "re-discovered" such a distribution, first proposed by Topp and Leone (1955), and this distribution has attracted recent attention – *e.g.*, Ghitany *et al*. (2005), Van Dorp and Kotz (2006), Zhoiu *et al*. (2006), Kotz and Seier (2007), Nadarajah (2009), and Genç (2012). As well as having finite support, the T-L distribution has a uni-modal density with a shape parameter, v. Moreover, the density function is "J-shaped", and the hazard function is "bathtub-shaped", *for all* values of $v \in (0\ ,1)$. The latter unusual characteristic is especially important in reliability applications in a wide range of fields, as is discussed by Reed (2011), for example.

For a given support, the shape parameter of the T-L distribution is easily estimated by ML or MOM. We show that both of these estimators are positively biased in finite samples. The ML estimator can be expressed in closed form, so it is especially simple to compute. It exhibits greater (relative) bias than the method of moments estimator, but smaller relative mean squared error. We derive the analytic bias of the ML estimator to $O(n^{-1})$, and prove that the corresponding bias-corrected estimators based on the different approaches of Cox and Snell (1968) and Firth (1993) are identical for this problem. A simulation experiment illustrates that this bias-corrected ML estimator dominates the MOM estimator in terms of both (relative) bias and mean squared error. In addition, it is superior to bootstrap bias-corrected versions of the MOM and ML estimators, and very simple to construct. Finally, two applications are presented to illustrate our new results with real data.

2. The Topp-Leone distribution

The density for the T-L distribution is:

$$f(x) = (2v/b)\ (x/b)^{(v-1)}\ (1-x/b)\ (2-x/b)^{(v-1)}\ \ ;\ 0 < x < b < \infty\ ;\ \ v > 0. \qquad (1)$$

In contrast to the Beta distribution, for example, the T-L hazard function has a simple closed form, namely:

$$\lambda(x) = (2v/b)y(1-y^2)^{v-1}/[1-(1-y^2)^v] \,, \tag{2}$$

where $y = 1 - (x/b)$.

Following Nadarajah and Kotz (2003, p.317) we will set b = 1, so that

$$f(x) = 2vx^{v-1}(1-x)(2-x)^{v-1} \;;\quad 0 < x < 1 \;;\quad v > 0 \,. \tag{3}$$

The J-shape of this density, and the bathtub shape of the hazard function (*for all* $v \in (0,1)$), are of particular interest, and are illustrated by Nadarajah and Kotz (2003, pp. 312-313). In contrast, if $v > 1$, the density in (3) is still proper, all of its integer-order moments exist, but the density is not J-shaped, and the corresponding hazard function no longer has a bathtub shape. See Kotz and Seier (pp. 6-10) for a discussion of the characteristics of the distribution when $1 < v < 2$, and when $v > 2$.

Under independent sampling, with sample size n, the log-likelihood function is:

$$l = nlog(2) + \; nlog(v) + \sum_{i=1}^{n}[(v-1)log\,(x_i) + log(1-x_i) + (v-1)log(2-x_i)] \quad . \tag{4}$$

Noting that

$$\partial l/\partial v = (n/v) + \sum_{i=1}^{n}[log(x_i\,) + \log(2-x_i\,)] \,, \tag{5}$$

and

$$(\partial^2 l)/(\partial v^2\,) = -n/v^2 \,, \tag{6}$$

it follows trivially that the ML estimator for v can be expressed in closed-form as

$$\hat{v} = -n/\sum_{i=1}^{n} log[x_i(2-x_i)], \tag{7}$$

and so $\hat{v} > 0$. ML estimation of the shape parameter under censored sampling is discussed, for example, by Bavoud (2016b).

Bayoud (2016a, p.74) shows that this ML estimator coincides with the mean of the posterior density (which is gamma in form) in a Bayesian analysis of this problem with a non-informative prior. Accordingly, the ML estimator is also the Bayes estimator of v when the loss function is quadratic.

As the T-L density satisfies the usual regularity conditions, the ML estimator of v is weakly consistent and best asymptotically normal. However, its finite-sample properties are not readily deduced, given that the estimator is a highly non-linear function of the data.

The MOM estimator of v is obtained by solving the equation, $E(X) = \bar{X}$, for v. From Nadarajah and Kotz (2003; p.315), when $b = 1$, $E(X) = 1 - 4^{v}\left(\Gamma(1+v)\right)^{2}/\Gamma(2+2v)$, so the MOM estimator ($\tilde{v}$) is obtained as the solution to the nonlinear equation,

$$\Gamma(2+2v)(\bar{X}-1) + 4^{v}(\Gamma(1+v))^{2} = 0 \,. \tag{8}$$

Although the MOM estimator is also weakly consistent for v, its finite-sample properties have not been explored previously. While the function of v in (8) is non-trivial, straightforward numerical evaluations show that it has a unique (positive) root for all $0 < \bar{X} < 1$, and hence for our situation where $0 < x_i < 1$; $i = 1, 2, \ldots, n$.

By construction, $\hat{v} > 0$ and $\hat{v}_{MOM} > 0$, but it is possible for these estimators to yield estimates that exceed unity, especially for very small n, even if $v \in (0,1)$. This, in itself, is not problematic.

3. Bias-adjusted estimators

One would anticipate the ML and MOM estimators of v may each be biased in finite samples. If so, this could be of concern when the T-L distribution is applied to reliability problems. Here, we investigate this issue, and consider bias-correction strategies for the ML and MOM estimators.

Cox and Snell (1968) provided a framework for estimating the bias, to $O(n^{-1})$ for ML estimators of the parameters of "regular" densities. Then, subtracting the estimated bias from the original ML estimator produces a bias-corrected estimator that is unbiased to $O(n^{-2})$. This type of "corrective" bias adjustment has been applied successfully in many contexts. Recent examples include Cordeiro and Klein (1994), Lemonte *et al*. (2007), Lemonte (2011), Giles *et al*. (2013), Schwartz *et al*. (2013), Xiao and Giles (2014), Schwartz and Giles (2016), and Giles *et al*. (2016).

In general terms, suppose that the parameter vector, θ, is of dimension p. Defining

$$k_{ij} = E(\partial^2 l / \partial\theta_i \partial\theta_j) \quad ; \quad i, j = 1, 2, \ldots, p \tag{9}$$

$$k_{ijl} = E(\partial^3 l / \partial\theta_i \partial\theta_j \partial\theta_l) \quad ; \quad i, j, l = 1, 2, \ldots, p \tag{10}$$

$$k_{ij,l} = E[(\partial^2 l / \partial\theta_i \partial\theta_j)(\partial l / \partial\theta_l)] \quad ; \quad i, j, l = 1, 2, \ldots, p \tag{11}$$

Cox and Snell (1968) showed that the bias of the s^{th} element of the ML estimator of θ ($\hat{\theta}$) is:

$$Bias(\hat{\theta}_s) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p} k^{si}k^{jl}[0.5k_{ijl} + k_{ij,l}] + O(n^{-2})\,; \qquad s = 1, 2, \ldots, p, \tag{12}$$

where k^{ij} is the $(i,j)^{th}$ element of the inverse of the information matrix, $K = \{-k_{ij}\}$.

Cordeiro and Klein (1994) note that this bias expression can be re-written as:

$$Bias(\hat{\theta}_s) = \sum_{i=1}^{p} k^{si} \sum_{j=1}^{p}\sum_{l=1}^{p}[k_{ij}^{(l)} - 0.5k_{ijl}]k^{jl} + O(n^{-2})\,; \qquad s = 1, 2, \ldots, p \tag{13}$$

where

$$k_{ij}^{(l)} = \partial k_{ij} / \partial\theta_l \qquad ; \qquad i, j, l = 1, 2, \ldots, p. \tag{14}$$

The computational advantage of (13) over (12) is that it avoids computing expectations of products (see (11)). Then, defining $a_{ij}^{(l)} = k_{ij}^{(l)} - (k_{ijl}/2)$, for $i, j, l = 1, 2, \ldots, p$; and constructing the matrices:

$$A^{(l)} = \{a_{ij}^{(l)}\}\,; \quad i, j, l = 1, 2, \ldots, p \tag{15}$$

$$A = [A^{(1)} \mid A^{(2)} \mid \ldots\ldots \mid A^{(p)}], \tag{16}$$

Cordeiro and Klein (1994) show that the expression for the $O(n^{-1})$ bias of $\hat{\theta}$ can be re-written as:

$$Bias(\hat{\theta}) = K^{-1}A\,vec(K^{-1}) + O(n^{-2})\,. \tag{17}$$

Here, $K = \{-k_{ij}\}$, is the (expected) Fisher information matrix. Then, a "bias-corrected" ML estimator for θ can then be constructed as:

$$\hat{\theta}_{CS} = \hat{\theta} - \hat{K}^{-1}\hat{A}\,vec(\hat{K}^{-1})\,, \tag{18}$$

where $\hat{K} = (K)|_{\hat{\theta}}$ and $\hat{A} = (A)|_{\hat{\theta}}$.

In our problem, $p = 1$, $K = (n/v^2)$, $(\partial^3 l)/(\partial v^3) = k_{111} = k_{11}^{(1)} = (2n/v^3)$, and $A = a_{11}^{(1)} = (n/v^3)$. So, from (16), $Bias(\hat{v}) = (v/n) + O(n^{-2})$, and the Cox-Snell bias-corrected estimator is

$$\hat{v}_{CS} = \hat{v} - (\hat{v}/n) = (n-1)\hat{v}/n\,, \tag{19}$$

where $\hat{v}$ is defined in (7). This result can be confirmed from Mazucheli *et al*. (2017; p.6), using the R package 'mle.tools'.

Firth (1993) suggested an alternative "preventive" approach to bias correction that involves adjusting the score vector before solving the likelihood equation(s) for the ML estimator. In the present context and notation, Firth's estimator requires that we solve the equation

$$U^* = (\partial l/\partial v) - Avec(K^{-1}) = 0. \tag{20}$$

Using (4), and the expressions for K and A, we immediately obtain the solution:

$$\begin{aligned} \hat{v}_F &= (1-n)/\sum_{i=1}^{n}[\log(x_i) + \log(2 - x_i)] \ . \\ &= (1-n)/\sum_{i=1}^{n}[1 - (x_i - 1)^2] = \hat{v}_{CS} \end{aligned} \tag{21}$$

So, for this particular estimation problem, the Cox-Snell and Firth bias-corrected ML estimators are identical. Moreover, $\hat{v}_{CS}$ is also a particular Bayes estimator. Bayoud (2016a) shows that under a non-informative prior the posterior for v is a gamma distribution with a shape parameter of n, and a scale parameter of $-1/lnT$, where $T = \prod_{i=1}^{n}[x_i(2 - x_i)]$. If $lnT \geq -1$, the mode of this distribution is $(1-n)/lnT = \hat{v}_{CS}$. This is the Bayes estimator of v if we have a "zero-one" loss function.

4. Simulation experiment

The (percentage) biases and mean squared errors of the ML, bias-adjusted ML, and MOM estimators of v have been investigated in a small simulation experiment. This experiment was conducted using the R software environment (R Core Development Team, 2016). The MOM estimator was obtained by solving equation (8) using the *uniroot* function in R. The Monte Carlo simulation involved 50,000 replications, and the T-L random variates were generated *via* the "rtopple" routine in the VGAM package in R (Yee, 2020). The quality of the random variates was checked by comparing the first four empirical moments of a sample of 10,000 simulated values with the corresponding population moments, as given by Nadarajah and Kotz (2003, p.315).

The results appear in Table 1. There, if $\hat{v}_{(i)}$ is the MLE of v obtained from the i^{th} replication of the experiment, then the percentage bias is $\%Bias(\hat{v}) = 100[(1/50000)\sum_{i=1}^{50000}\hat{v}_{(i)} - v]/v$, and $\%MSE(\hat{v}) = 100[(1/50000)\sum_{i=1}^{50000}(\hat{v}_{(i)} - v)^2]/v^2$, *etc*.

It transpires that the percentage biases and MSEs of the ML and bias-corrected ML estimators of v are invariant to the value of v itself. In contrast, the relative bias and relative MSE of the MOM estimator of v depend on the value of that parameter. We see in Table 1 that the MOM estimator exhibits less relative bias, but much greater percentage MSE than does the ML estimator of v. However, the simple Cox-Snell/Firth bias correction is extremely effective. The positive relative bias of the adjusted ML estimator is much less than that of the MOM estimator of v (typically by an order of magnitude); and in addition, bias-correcting the ML estimator also reduces the %MSE of that estimator slightly.

An alternative way to correct for estimator bias is *via* the parametric bootstrap (*e.g*., Efron, 1982). Our simulation study was extended to include (parametric) bootstrap bias-corrected ML and MOM estimators, based on 1,000 bootstrap draws. Again, 50,000 simulation replications were used, and the results are compared with those for the Cox-Snell/Firth correction in Table 2. There, we see that although the bootstrap and analytically bias-corrected ML estimators have similar %MSE's, the latter estimator dominates the former estimator in terms of bias, especially for small n. In addition, both of these "corrected" ML estimators dominate the bootstrap bias-corrected MOM estimator, especially with respect to % MSE.

In summary, the easy-to-apply Cox-Snell/Firth bias-corrected ML estimator is recommended, even for very small sample sizes.

5. Empirical applications

We have conducted two empirical applications to illustrate the consequences of using the various bias-corrected estimators for the Topp-Leone distribution with actual data. The first application involves data from Mazumdar and Gaver (1984) and analyzed previously by Altun and Hamedani (2018). The 23 sample values measure unit capacity factors, using the so-called "SC16" algorithm. The second application uses data provided in Example 2 of Wang (2000), and also analyzed by various authors. These data relate to the life-spans of 18 electronic devices, and their values were divided by 1,000. The sample histograms and the various estimated Topp-Leone densities appear in Figures 1 and 2 respectively.

The estimation results themselves, and goodness-of-fit statistics, are presented in Table 3. Based on the power results described by Al-Zahrani (2012), the Kolmogorov-Smirnov test was chosen for goodness-of-fit. The Topp-Leone model is supported by the data in all cases. In both applications it is clear that there is little difference between the associated ML results, but the MOM estimates are noticeably different.

6. Conclusions

Recently, there has been renewed interest in the Topp-Leone distribution, especially in relation to reliability studies, where its finite support and bathtub-shaped hazard function are very appealing to practitioners. For a fixed support, method of moments estimation of the distribution's shape parameter is straightforward enough, but the maximum likelihood estimator is trivial to compute. Correcting the latter estimator to remove its $O(n^{-1})$ bias analytically is also very straightforward. This analytically bias-corrected estimator is attractive because it has smaller percentage bias and mean squared error than the method of moments estimator in samples of the size likely to be encountered in practice. It also has a natural Bayesian interpretation, and it out-performs the method of moments and maximum likelihood estimators when these are bias-corrected using the parametric bootstrap.

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Fig. 1: SC16 Data

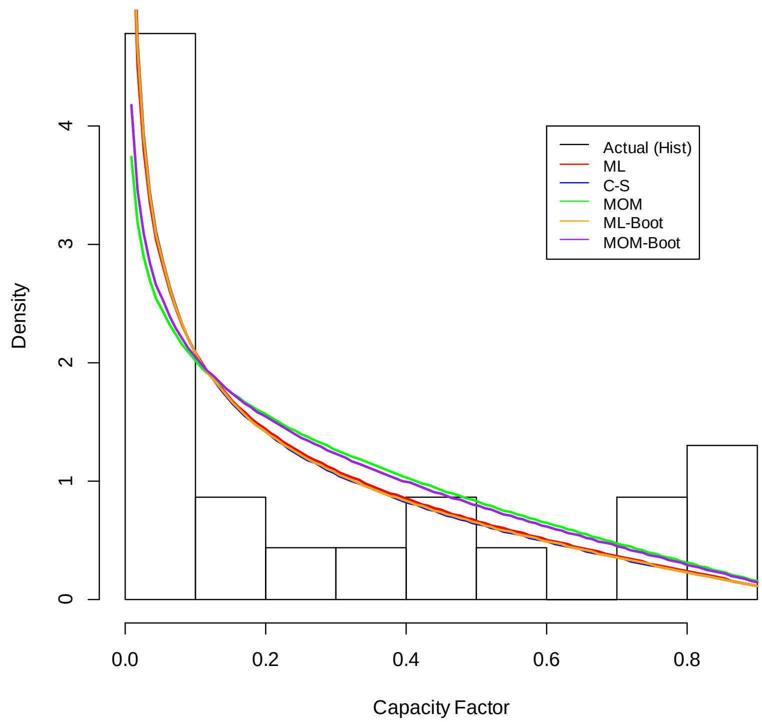

Fig. 2: Electronic Device Data

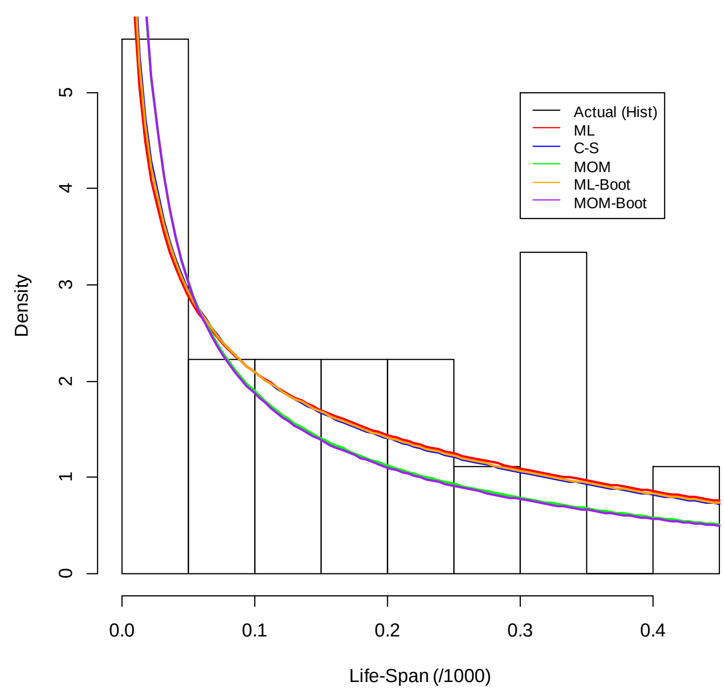

Table 1: Percentage biases [and MSEs] of ML and MOM estimators

n	$\%Bias(\hat{v})$ [$\%MSE(\hat{v})$]	$\%Bias(\hat{v}_{CS})$ [$\%MSE(\hat{v}_{CS})$]	$\%Bias(\tilde{v})$ [$\%MSE(\tilde{v})$]				
			v				
			0.10	**0.25**	**0.50**	**0.75**	**0.90**
10	11.031	-0.072	6.308	6.449	6.773	7.050	7.190
	[16.516]	[12.392]	[84.247]	[41.903]	[28.133]	[23.760]	[22.364]
25	4.049	-0.113	2.110	2.232	2.239	2.493	2.549
	[5.238]	[4.329]	[29.288]	[14.330]	[9.420]	[7.832]	[7.318]
50	2.000	-0.040	0.938	1.029	1.126	1.194	1.226
	[2.206]	[2.081]	[14.303]	[6.917]	[4.499]	[3.718]	[3.465]
75	1.333	-0.018	0.669	0.716	0.774	0.814	0.833
	[1.412]	[1.358]	[9.401]	[4.548]	[2.950]	[2.430]	[2.261]
100	1.001	-0.009	0.538	0.568	0.604	0.631	0.645
	[1.052]	[1.021]	[7.073]	[3.416]	[2.215]	[1.824]	[1.697]
250	0.379	-0.022	0.193	0.191	0.203	0.218	0.222
	[0.409]	[0.404]	[2.817]	[1.357]	[0.877]	[0.721]	[0.670]
500	0.190	-0.011	0.108	0.093	0.094	0.099	0.102
	[0.202]	[0.201]	[1.400]	[0.676]	[0.438]	[0.360]	[0.334]
1000	0.099	-0.001	0.095	0.076	0.068	0.067	0.067
	[0.101]	[0.100]	[0.698]	[0.338]	[0.219]	[0.180]	[0.167]

Table 2: Percentage biases [and MSEs] of bootstrap bias-corrected ML and MOM estimators

n	$\%Bias(\hat{v}_{CS})$ [$\%MSE(\hat{v}_{CS})$]	$\%Bias(\hat{v}_B)$ [$\%MSE(\hat{v}_B)$]	$\%Bias(\tilde{v}_B)$ [$\%MSE(\tilde{v}_B)$] ***v*** **0.10**	**0.25**	**0.5**	**0.75**	**0.90**
10	-0.072	-0.265	-1.224	-1.267	-1.312	-1.330	-1.332
	[12.392]	[12.743]	[67.828]	[32.922]	[21.550]	[17.929]	[16.773]
25	-0.113	0.243	-0.258	-0.146	-0.075	-0.046	-0.036
	[4.329]	[4.385]	[27.719]	[13.481]	[8.780]	[7.247]	[6.748]
50	-0.040	0.080	-0.062	-0.016	0.016	0.024	0.027
	[2.081]	[2.120]	[14.021]	[6.779]	[4.403]	[3.631]	[3.379]
75	-0.018	0.016	-0.002	-0.017	-0.029	-0.031	-0.031
	[1.358]	[1.373]	[9.228]	[4.459]	[2.886]	[2.376]	[2.210]
100	-0.009	0.016	0.108	0.053	0.026	0.019	0.016
	[1.021]	[1.031]	[7.022]	[3.379]	[2.187]	[1.799]	[1.673]
250	-0.022	-0.002	-0.149	-0.098	-0.064	-0.048	-0.042
	[0.404]	[0.401]	[2.788]	[1.340]	[0.865]	[0.711]	[0.661]
500	-0.011	0.015	0.033	0.035	0.032	0.029	0.028
	[0.201]	[0.200]	[1.406]	[0.673]	[0.433]	[0.355]	[0.330]
1000	-0.001	0.017	-0.023	0.004	0.010	0.011	0.012
	[0.100]	[0.100]	[0.694]]	[0.334]	[0.215]	[0.177]	[0.164]

Table 3: Empirical application results*

			SC16 data (*n* = 23)		
Estimates:	$\hat{v}$	$\hat{v}_{CS}$	$\hat{v}_{B}$	$\tilde{v}$	$\tilde{v}_{B}$
	0.594 (0.118)	0.568 (0.113)	0.575 (0.118)	0.780 (0.309)	0.740 (0.309)
K-S:	0.169 [> 0.25]	0.171 [> 0.25]	0.171 [>0.25]	0.260 [> 0.25]	0.241 [>0.25]
			Electronic device data (*n* = 18)		
Estimates:	$\hat{v}$	$\hat{v}_{CS}$	$\hat{v}_{B}$	$\tilde{v}$	$\tilde{v}_{B}$
	0.598 (0.112)	0.565 (0.106)	0.575 (0.112)	0.370 (0.089)	0.360 (0.089)
K-S:	0.224 [> 0.25]	0.211 [> 0.25]	0.215 [>0.25]	0.211 [> 0.25]	0.221 [>0.25]

***** Bootstrapped standard errors, based on 1,000 replications, are in parentheses; "K-S = Kolmogorov-Smirnov test statistic; p-value ranges, based on Table 1 of Al-Zahrani (2012), are in square brackets.

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