Some Implications of Preliminary-Test Estimation in the Context of Size-Biased Sampling

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Abstract In this chapter, we consider maximum likelihood estimation of the parameters of certain distributions, when there is the possibility that size-biased sampling has been used, rather than simple random sampling. If a statistical test is used to discriminate between these sampling procedures, and estimation proceeds on the basis of the outcome of that test, then we have a ”preliminary-test estimation” problem that has not previously been investigated. Using three common single-parameter distributions that are members of the Generalized Gamma family and various loss functions, we analyze the relative finite-sample biases and relative risks of such preliminary-test estimators, *via* Monte Carlo simulation. We also address the issue of selecting the optimal critical value for the test in question, based on the ”minimax regret” criterion. We find that a constant critical value is optimal, regardless of the sample size, and that this value is very similar across the three distributions considered.

1 Introduction

Size-biased sampling is a phenomenon that arises in a variety of fields, such as epidemiology, ecology, forestry, reliability studies, survival analysis, marketing, meteorology, genome mapping, and environmental and resource economics. Although it may not be directly observable, size-biased sampling occurs when the probability of a unit being selected into the sample is proportional to a predetermined weight function, with the latter depending on the observed value of the data for that unit. This phenomenon is discussed in the seminal contributions of Wicksell (1925), Fisher (1934), Rao (1965), and Cox (1969). For example, if the probability of an observation being included in a sample is directly proportional to the size of that observation, we have so-called “length-biased” sampling. Size-biased sampling is important because

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it affects the formulation of the appropriate data density, and hence the likelihood function. If this is ignored, the likelihood function is mis-specified and inferences based upon it will be invalid, *even asymptotically*.

If a random variable, X, has a density function $f(x;\theta)$, for some parameter (vector) θ, then the corresponding "weighted" density, to allow for size-biased sampling of order c, is defined as $f_c(x;\theta) = f(x;\theta)x^c/m'_c$, where $m'_r = E[X^r]$ is the rth. moment of X. The case $c = 1$ corresponds to "length-biased" sampling; while $c = 2$ corresponds to "area-biased", *etc.* Size-biased sampling can arise with both discrete and continuous distributions. The empirical literature includes examples based on the discrete binomial, Poisson, and negative binomial distributions, as well as continuous ones such as the exponential, gamma, Weibull, half-normal, Nakagami, and Rayleigh distributions.

An obvious preliminary-testing problem arises here, because in general the researcher will not know whether or not size bias is an issue in the sampling process. Interestingly, it appears that this particular preliminary-testing issue has not been analyzed previously in the literature. Specifically, we might test the hypothesis, H_0: "Simple random sampling was used", against the alternative hypothesis, H_1: "Size-biased sampling was used". If the null hypothesis is not rejected, we would use the maximum likelihood estimate (MLE) of θ, basing the likelihood function on $f(x;\theta)$. On the other hand, if the null hypothesis is rejected, we would use the MLE of θ, with the likelihood function constructed from $f_c(x;\theta) = f(x;\theta)x^c/m'_c$. The (sampling) properties of this preliminary-test estimator will differ from those of the two alternative MLEs, and will depend on the value of θ, the sample size, and the level of significance chosen for the test of H_0.

Several tests of H_0 against H_1 have been proposed in the literature. For example, see Navarro and del Aguila (2003), Akman *et al.* (2007), and Economou and Tzavelas (2013). In this study, we consider preliminary-testing based on the simple distribution-free test introduced by Akman *et al.* (2007), and elaborated upon by Economou and Tzavelas (2013). Using this test, and various distributions from the generalized gamma family, we investigate the relative risk of the preliminary-test estimator (and its two component estimators) under squared-error loss, as well as the relative biases of these estimators. Some of the results are obtained analytically, while others are based on the results of an extensive Monte Carlo simulation experiment.

The next section considers the preliminary-testing problem under study in more detail. The impact of size-biased sampling on the generalized gamma distribution and some of the members of this family are discussed in section 3. The design of the Monte Carlo experiment is described in section 4; and the associated results are presented in section 5. Section 6 provides some concluding remarks.

2 Preliminary-Test Estimation

Preliminary-test estimators arise in a very wide range of situations, and their properties can be complicated. Often, researchers ignore the fact that such estimators have been used, and this can be a cause for serious concern. Such estimators have been studied extensively and deserve due attention.

2.1 The Preliminary-Testing Problem

The preliminary-test estimation problem arises when the choice of estimator is conditioned on the application of some prior statistical test. This conditioning results in a ”preliminary-test” estimator (PTE) whose sampling properties depend upon the choice of test and significance level; the alternative estimators that are used, depending on the test's outcome; the values of all of the parameters in the problem; and the sample size. Motivated by earlier work by Berkson (1942), preliminary-test estimation was first discussed by Bancroft (1944), in the context of estimating a regression parameter after a prior test of significance of one of the model's other parameters. Subsequently, a large literature on this topic (and that of ”preliminary-test testing”) has emerged. This literature has been surveyed by a number of authors, including Bancroft and Han (1977), Judge and Bock (1978), Han *et al.* (1988), Giles and Giles (1993), and Saleh (2014).

2.2 Properties of Preliminary-Test Estimators

In general, preliminary-test estimators have desirable asymptotic (large sample) properties. This is the case in this study because the component estimators ($\tilde{\theta}$ and $\hat{\theta}$) are maximum likelihood estimators. The distributions that are considered satisfy the usual regularity conditions, and so the MLEs are consistent - *if the underlying hypotheses are correct*. The preliminary-test estimator is a weighted combination of these estimators, with random weights whose properties depend on those of the sampling distribution of the test statistic, $\tilde{\lambda}$.

Provided that the test is itself consistent - ensuring that its power approaches one as the sample size grows indefinitely - then the preliminary-test estimator will be consistent. Economou and Tzavelas (2013) establish the asymptotic normality of the sampling distribution of $\tilde{\lambda}$, The consistency of the test is readily confirmed through Monte Carlo simulation, and this is discussed further in section 4.

However, although the preliminary-test estimators under consideration are well-behaved asymptotically, their finite-sample properties require careful attention. First, it should be noted that PTEs are generally inadmissible (*e.g.*, Cohen, 1965) due to the fact that they are a *discontinuous* weighted combination of the component estimators. Typically, they will also be biased (because the random weights generally are not independent of the test statistic's distribution), and the finite-sample risks of the estimators will be of interest. These are the properties on which the present study focuses.

2.3 Preliminary-Testing for Size-Biased Sampling

Now let us consider the preliminary-test estimator of the parameter (vector) θ in the density $f(x;\theta)$, when the *potential level* of size bias is c. As noted in section 1, in this study we investigate preliminary testing using the simple distribution-free test suggested by Akman *et al.* (2007) and extended by Economou and Tzavelas (2013). The test statistic is based on the quantity

$$\prod_{i=1}^{n}[f_c(x_i;\theta)/f(x_i;\theta)], \tag{1}$$

a large value of which will suggest that size-biased sampling has been used.

Using the definition of $f_c(x_i;\theta)$, and noting that $m'_c = E[X^c]$ is typically an unobservable function of θ, an equivalent test procedure is to reject in favour of size-biased sampling if the statistic:

$$\tilde{\lambda} = [\prod_{i=1}^{n} x_i]^{1/n}/(\tilde{m'_c}))^{1/c} \tag{2}$$

is sufficiently large. In (2), $\tilde{m'_c}$ is the MLE of m'_c, in which θ is replaced with, $\tilde{\theta}$, its MLE based on the likelihood function formed using the "base" (or "unweighted") density, $f(x;\theta)$.

Let $\hat{\theta}$ denote the MLE based on the likelihood function formed using the "weighted" density, $f_c(x;\theta)$. Then, the preliminary-test estimator, θ^*, is equal to $\tilde{\theta}$ if $\tilde{\lambda} > k(\alpha)$, and it equals $\hat{\theta}$ if $\tilde{\lambda} \leq k(\alpha)$. Here, k is a critical value that is determined by simulation, and is dependent on the chosen significance level, α. So, we can express the preliminary-test estimator as:

$$\theta^* = \hat{\theta} I_{k(\alpha)} + \tilde{\theta}(1 - I_{k(\alpha)}) = \hat{\theta} + (\hat{\theta} - \tilde{\theta}) I_{k(\alpha)}, \tag{3}$$

where $I_{k(\alpha)}$ is an indicator function taking the value 1 if $\tilde{\lambda} > k(\alpha)$, and zero if $\tilde{\lambda} \leq k(\alpha)$.

We will be concerned with the relative bias and relative risk (both in percentage terms) of this preliminary-test estimator, together with the corresponding measures for $\tilde{\theta}$ and $\hat{\theta}$, under various scenarios. The %MSE provides the relative risk under quadratic loss, multiplied by 100. In this study, the biases and risks of $\tilde{\theta}$ and $\hat{\theta}$ can be obtained analytically for the three of the distributions that we consider. However, for all of the distributions that we explore, the biases and risks of the preliminary-test estimators will be obtained through Monte Carlo simulation, given the complexity of their structure.

We also consider the relative risks under both absolute-error, and LINEX loss functions, again expressed as percentages. Under the former loss function the percentage relative risk equals the mean relative absolute percentage error (%MAE). The LINEX loss function, proposed by Varian (1975) and elaborated upon by Zellner (1986), allows for an asymmetric loss structure. A simple form of this loss function is given by:

$$L(\bar{\theta};\theta) = exp[w(\bar{\theta}-\theta)] - w(\bar{\theta}-\theta) - 1; w \neq 0 \quad (4)$$

where $\bar{\theta}$ is any one of the estimators of θ under consideration, and the parameter 'w' is assigned to determine the degree of asymmetry. If $w > 0$, the over-estimation of θ is (exponentially) more costly than under-estimation; while the converse is true if $w < 0$. For values of 'w' close to zero, the loss function is approximately quadratic. The relative risk (in percentage terms) under LINEX loss is:

$$\%relR_L(\bar{\theta};\theta) = 100E[exp(w(\bar{\theta}-\theta)) - w(\bar{\theta}-\theta) - 1]/\theta \quad (5)$$

3 The Generalized Gamma Distribution and Size-Biased Sampling

Our focus in this paper is on a distribution family that includes many specific distributions that are of practical importance in the presence of possible size-biased sampling. Some of these particular distributions are discussed in detail below.

3.1 Stacy's Distribution

Stacy (1962) proposed a three-parameter generalization of the Gamma probability distribution, which is very flexible in its form and contains a number of well-known distributions as special cases. The probability density function (p.d.f.) for a random variable, Y, that follows this (G-Gamma) distribution is:

$$f(y;a,d,p) = (p/a^d)y^{d-1}exp[-(y/a)^p]/\Gamma(d/p); y > 0; a,d,p > 0 \quad (6)$$

The r'th. raw moment of Y is

$$E[Y^r] = a^r\Gamma((d+r)/p)/\Gamma(d/p). \quad (7)$$

So, under general size-biased sampling, the density of Y becomes:

$$f_c(y;a,d,p) = (p/a^{(d+c)})y^{(d+c)-1}exp[-(y/a)^p]/\Gamma((d+c)/p); y > 0; a,d,p > 0 \quad (8)$$

This is the G-gamma density with parameters a, $(d+c)$, and p; for $c = 1, 2, \ldots$ So, the G-gamma distribution has the property that Patil and Ord (1976) call "form invariance" under weighted sampling. See, also, Ducey and Gove (2015, p.124).

Jabeen and Jan (2015) consider the G-Gamma family as a whole when the sampling is size-biased, but they focus on the associated information measures. Several members of the G-Gamma family of distributions are among those that have been applied in a wide variety of fields in the context of size-biased sampling.

For example, Mir *et al.* (2013) and others discuss the Exponential distribution ($d = p = 1$). The Weibull distribution ($d = p$) is investigated by several authors, including Blumenthal (1967), Scheaffer (1972), Gove (2003a), Das and Roy (2011a), and Giles (2021). Blumenthal (1967) and Schaeffer (1972) also consider the Gamma distribution ($p = 1$) under size-biased sampling; and the Rayleigh distribution ($a = \sqrt{(2\sigma^2)}, d = p = 2$) is discussed by Das and Roy (2011b). The Nakagami distribution ($a = \sqrt{(\Omega/m)}, d = 2m, p = 2$) under length-biased sampling is investigated by Madisir and Ahmed (2018), and Bashir and Rasul (2018) discuss the Half-Normal distribution ($a = \sqrt{(2\sigma^2)}, d = 1, p = 2$) in this context. (Here, m and Ω are the shape and spread parameters of the Nakagami distribution, and σ is the scale parameter for the Half-Normal and Rayleigh distributions.) Three of these distributions are discussed in the next sub-section, and our attention will focus purely on length-biased sampling, rather than more general forms of size-bias.

3.2 Some Single-Parameter Distributions

We consider three single-parameter members of the G-Gamma family for which the MLEs of the parameters can be obtained analytically, and so many of the associated biases and risks can be expressed in closed form. For most members of the G-Gamma family the MLEs must be obtained numerically, which precludes closed form expressions for the associated biases and risks.

Exponential Distribution

To elaborate on some of the general concepts introduced so far, consider the particular case where the random variable, X, follows an Exponential distribution with a rate parameter, β. So, its p.d.f. is:

$$f(x;\beta) = \beta exp(-\beta x) : x > 0; \beta > 0 \tag{9}$$

and $E[X] = (1/\beta)$. This is a special case of the G-Gamma distribution, with parameters $a = 1/\beta$, and $d = p = 1$. It is well-known that the MLE of β based on a sample of n independent observations and (9), is given by $\tilde{\beta} = 1/\bar{x}$, where $\bar{x} = (1/n)\sum_{i=1}^{n} x_i$. Using the result that the sum of n independent Exponential random variables with a common rate parameter β follows a Gamma (n, β) distribution, it follows immediately that $E(\tilde{\beta}) = n\beta/(n-1)$ and $Var(\tilde{\beta}) = n^2\beta^2/[(n-1)(n-2)]$. So, if (9) is the correct density for X, the relative bias and relative mean squared error (MSE) of the MLE, in percentage terms, are:

$$\%relBias(\tilde{\beta}) = 100/(n-1) \tag{10}$$

and

$$\%relMSE(\tilde{\beta}) = 100(n+2)/[(n-1)(n-2)]. \quad (11)$$

Under length-biased sampling, the p.d.f. for X becomes:

$$f_1(x;\beta) = xf(x;\beta)/E[X] = x\beta^2 exp(-\beta x); x > 0; \beta > 0 \quad (12)$$

which is the density for a Gamma (2, β) – distributed random variable with a rate parameter, β.

In this case, the MLE for β is $\hat{\beta} = 2/\bar{x}$, and the sum of the n independent x's follows a Gamma ($2n$, β) distribution. It is easily shown that $E(\hat{\beta}) = 2n\beta/(2n-1)$ and $Var(\hat{\beta}) = 2n^2\beta^2/[(n-1)(2n-1)^2]$. So, if (12) is the correct density for$X$, the relative bias and relative mean squared error (MSE) of the MLE are, in percentage terms:

$$\%relBias(\hat{\beta}) = 100/(2n-1) \quad (13)$$

and

$$\%relMSE(\hat{\beta}) = 100(n+1)/[(n-1)(2n-1)]. \quad (14)$$

Now, consider the situation where (9) is the correct density, but the rate parameter is estimated under the incorrect assumption that length-biased sampling has been used. In this case, it is easily shown that $E(\hat{\beta}) = 2n\beta/(n-1)$ and $Var(\hat{\beta}) = 4n^2\beta^2/[(n-2)(n-1)^2]$, and so:

$$\%relBias(\hat{\beta}) = 100(n+1)/(n-1) \quad (15)$$

and

$$\%relMSE(\hat{\beta}) = 100(n^3+4n^2-3n-2)/[(n-1)^2(n-2)]. \quad (16)$$

Conversely, if length-biased-sampling has been used, so that (12) is appropriate, but this is ignored (or is unknown), we can show that that $E(\tilde{\beta}) = n\beta/(2n-1)$ and $Var(\tilde{\beta}) = n^2\beta^2/[2(n-1)(2n-1)^2]$. It follows that:

$$\%relBias(\tilde{\beta}) = -100(n-1)/(2n-1) \quad (17)$$

and

$$\%relMSE(\tilde{\beta}) = (100(2n^3-5n^2+6n-2))/[2(2n-1)^2(n-1)] \quad (18)$$

Note that the expressions in (10) and (11), as well as (13) to (18) are independent of the true value of the rate parameter. It follows that the relative bias and relative MSE of the preliminary-test estimator of β are also invariant to the true value of that parameter. However, the properties of the last of these estimators cannot easily be evaluated analytically, as the test statistic's distribution is unknown. These properties are explored *via* Monte Carlo simulations later in this paper.

Half-Normal Distribution

Consider the Half-Normal distribution with a scale parameter, σ. Its p.d.f. is:

$$f(x;\sigma) = (\sqrt{2\pi}/\sigma)exp(-x^2/(2\sigma^2)); x > 0; \sigma > 0 \quad (19)$$

In this case, $E[X] = \sigma\sqrt{2/\pi}$, and so the weighted p.d.f. for length-biased sampling is:

$$f_1(x;\sigma) = (x/\sigma^2)exp(-x^2/(2\sigma^2)) \quad (20)$$

which is the G-Gamma density with parameters $a = \sqrt{2\sigma^2}$, $d = p = 2$, or simply the Rayleigh density in (21) below. In this case, $E[X] = \sigma\sqrt{\pi/2}$, and $E[X^2] = 2\sigma^2$.

Based on an independent sample of n values from (19), the MLE of the scale parameter is $\tilde{\sigma} = \sqrt{(1/n)\sum_{i=1}^{n} x_i^2}$. Using the results that for this distribution, $\sum_{i=1}^{n} x_i^2 = \sigma^2\chi^2_{(n)}$, and $E[\chi_{(n)}] = \sqrt{2}\Gamma((n+1)/2)/\Gamma(n/2)$, it follows that the %relBias$(\tilde{\sigma}) = \sqrt{2/n}\Gamma((n+1)/2)/\Gamma(n/2) - 1$. Using the expression for $Var(\chi_{(n)})$ we can then derive $Var(\tilde{\sigma})$ and hence $\%MSE(\tilde{\sigma})$. On the other hand, if H_1 is true and the random sample of data is generated by (20), the MLE of σ is $\hat{\sigma} = \sqrt{(1/(2n))\sum_{i=1}^{n} x_i^2}$. The bias and MSE expressions for $\hat{\sigma}$ when the the data are sampled from (19), and so H_0 is true, are obtained in a similar manner. See the first column of Table 1.

On the other hand, under length-biased sampling (from (20)), it is easily shown that $\sum_{i=1}^{n} x_i^2$ is Gamma$(n, 2\sigma^2)$ - distributed, and its square root follows a Nakagami distribution. These results enable us to obtain the analytical expressions for the %Bias and %MSE for both $\tilde{\sigma}$ and $\hat{\sigma}$ in this case, as shown in Table 1.

Rayleigh Distribution

Finally, we can proceed in a similar manner when X follows a Rayleigh distribution, with a scale parameter σ. Its p.d.f. is:

$$f(x;\sigma) = (x/\sigma^2)exp(-x^2/(2\sigma^2)); x > 0; \sigma > 0 \quad (21)$$

and $E[X] = \sigma\sqrt{\pi/2}$. Based on a random sample of size n, using (21), the MLE of σ is $\tilde{\sigma} = \sqrt{(1/2n)\sum_{i=1}^{n} x_i^2}$.

The density in the case of length-biased sampling is:

$$f_1(x;\sigma) = \sqrt{2}x^2 exp(-x^2/(2\sigma^2))/[\sigma^3\sqrt{\pi}] \quad (22)$$

which is the G-Gamma p.d.f. with parameters $a = \sqrt{2\sigma^2}$, $d = 3$, and $p = 2$. Based on a random sample of size n, using (22), the MLE of σ is $\hat{\sigma} = \sqrt{(1/3n)\sum_{i=1}^{n} x_i^2}$.

Using the result that the sum of n squared Rayleigh-distributed random variables is Gamma-distributed, the %Bias and %MSE of $\tilde{\sigma}$ are easily obtained when the true distribution is (21). We can use the same result to derive the %Bias and %MSE of $\hat{\sigma}$ when (22) is the true distribution. These results are summarized in Table 1. However, the properties of $\tilde{\sigma}$ and $\hat{\sigma}$ when the true distribution is (22) are not easily obtained analytically. Table 2 provides an overview of the distributional relationships discussed above.

Once again, all of the relative bias and relative risk expressions are invariant to the value of the distribution's sole parameter, and the properties of the preliminary-test estimator are evaluated subsequently in a simulation experiment. Both the Half-Normal and Rayleigh distributions are special cases of the so-called Generalized Rayleigh distribution. Xiao and Giles (2014) investigated the bias and MSE of the MLEs of the parameters of the latter distribution under regular sampling, with and without analytical bias corrections.

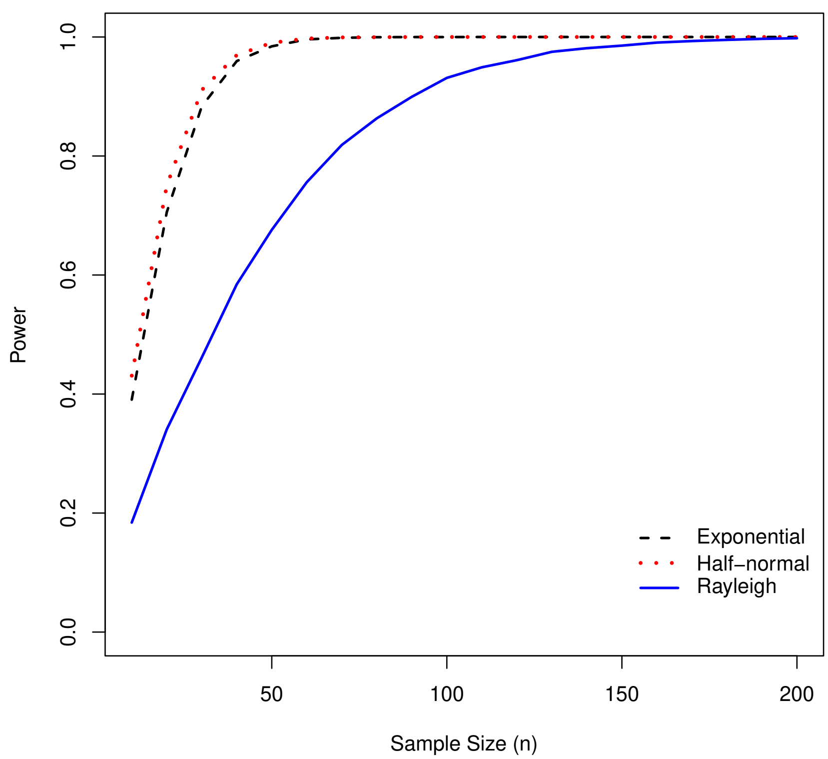

Fig. 1 Powers of the $\tilde{\lambda}$ test when $\alpha = 5\%$.

Table 1 Relative Biases and Relative MSEs of MLEs[a]

MLEs:	$\tilde{\beta}$, $\tilde{\sigma}$	$\hat{\beta}$, $\hat{\sigma}$
***H_0* True**[b]		
Exponential (β)		
Rel. Bias	$1/(n-1)$	$(n+1)/(n-1)$
Rel. MSE	$(n+2)/[(n-1)(n-2)]$	$[n^3+4n^2-3n-2]/[(n-1)^2(n-2)]$
1/2-Normal (σ)		
Rel. Bias	$\sqrt{2/n}[\Gamma((n+1)/2)/\Gamma(n/2)]-1$	$(1/\sqrt{n})[\Gamma((n+1)/2)/\Gamma(n/2)]-1$
Rel. MSE	$2[1-\sqrt{2/n}\Gamma((n+1)/2)/\Gamma(n/2)]$	$3/2-(2/\sqrt{n})\Gamma((n-1)/2)\Gamma(n/2)$
Rayleigh (σ)		
Rel. Bias	$\Gamma(n+1/2)/(\sqrt{n}\Gamma(n))-1$	$\sqrt{2/(3n)}\Gamma(n+1/2)/\Gamma(n)-1$
Rel. MSE	$2[1-\Gamma(n+1/2)/(\sqrt{n}\Gamma(n))]$	$5/3-2\sqrt{2/(3n)}\Gamma(n+1/2)/\Gamma(n)$
***H_1* True**[b]		
Exponential (θ)		
Rel. Bias	$-(n-1)/(2n-1)$	$1/(2n-1)$
Rel. MSE	$[2n^3-5n^2+6n-2]/[2(2n-1)^2(n-1)]$	$(n+1)/[(n-1)(2n-1)]$
1/2-Normal (σ)		
Rel. Bias	$\sqrt{2/n}\Gamma(n+1/2)/\Gamma(n)-1$	$\sqrt{1/n}\Gamma(n+1/2)/\Gamma(n)-1$
Rel. MSE	$3-2\sqrt{2/n}\Gamma(n+1/2)/\Gamma(n)$	$2[1-\sqrt{1/n}\Gamma(n+1/2)/\Gamma(n)]$
Rayleigh[c] (σ)		
Rel. Bias	n.a.	n.a.
Rel. MSE	n.a.	n.a.

[a] Multiply all results by 100 to obtain %relBias and %relMSE
[b] H_0 True: the population density is $f(x)$; H_1 True: the population density is $f_1(x)$
[c] Analytical expressions have not been established in this case

4 A Monte Carlo Simulation Experiment

For each of the distributions discussed in the previous section, we have explored the relative risks and biases of the MLEs of their parameters under unweighted sampling, length-biased sampling, and preliminary testing. Recall that in the case of the Exponential, Rayleigh, and Half-Normal distributions, most of these properties can be computed exactly. In all other situations, the properties of the various estimators are obtained *via* Monte Carlo simulation, with 100,000 replica-

Table 2 Connections Between the Distributions

Base Density ($f(x)$)	$m'_c = E[X^c]$	Size-Biased Density ($f_c(x)$)
Exponential (rate = β)	$\Gamma(c+1)/\beta^c$	Gamma (shape = c + 1, rate = β)
Half-Normal (scale = σ)	$\sqrt{(2^c/\pi)}\sigma^c\Gamma((c+1)/2)$	G-Gamma ($a = \sqrt{2\sigma^2}$, $d = 1 + c$, $p = 2$)
Rayleigh (scale = $\sigma = a/\sqrt{2}$)	$2^{c/2-1}c\sigma^c\Gamma(c/2)$	G-Gamma (a, $d = c + 2$, $p = 2$)

tions in each case. All of the computations are undertaken using the R statistical software (R Core Team, 2021), with random variables being generated using the 'ggamma' package (Saldanha and Suzuki, 2022). The R code can be downloaded from https://github.com/DaveGiles1949/r-code. All of the results are invariant to the scale parameter of each of the distributions being considered, so this value (either σ or $1/\beta$) is set to 1, without loss of generality.

4.1 The Test Statistic

As noted in section 2.2, the critical values for the test of H_0 *vs.* H_1 depend on the distribution(s) in question, the sample size, and possibly the parameter value(s). The critical values are obtained within each part of the simulation study using further Monte Carlo replications. Specifically, for a particular distribution, parameter value, and sample size combination, 100,000 samples of size n are generated from the null distribution. Then, for each sample the expression for the mean of the associated length-biased distribution is evaluated using the MLE(s) of the parameter(s) based on the null distribution. Equation (2) is then applied, thus generating 20,000 values of $\tilde{\lambda}$. For example, in the case of the Exponential distribution, $m'_1 = (1/\beta)$; and replacing β with $\tilde{\beta} = 1/\bar{x}$, yields $\tilde{\lambda} = [\prod_{i=1}^{n} x_i]^{1/n}/\bar{x}$. The construction of $\tilde{\lambda}$ for the alternative hypothesis of length-biased sampling is summarized in Table 3 for each of the distributions under study.

The upper quantiles of the distribution of 100,000 $\tilde{\lambda}$ values provide the desired critical values for the test. A range of sample sizes is considered, and the impact of the choice of significance level for the preliminary tests is explored. The power of the $\tilde{\lambda}$ test is easily simulated for the distributions under consideration. In each case, 20,000 values of the test statistic are generated under the alternative hypothesis, H_1, and the power is defined as the proportion of these values that exceed the associated critical value. These computations are performed for a range of sample sizes, and as shown in Figure 1, the $\tilde{\lambda}$ test is consistent.

Table 3 Construction of the Test Statistic

Base Density ($f(x)$)	m_1'	$\tilde{m_1'}$	$\tilde{\lambda}$
Exponential (rate = β)	$1/\beta$	$\bar{x}$	$[\prod_{i=1}^{n} x_i]^{1/n}/(\bar{x})$
Rayleigh (scale = σ)	$2\sigma\sqrt{2/\pi}$	$2\sqrt{\frac{\sum_i^n x_i^2}{(n\pi)}}$	$[\prod_{i=1}^{n} x_i]^{1/n}/[2\sqrt{\frac{\sum_i^n x_i^2}{(n\pi)}}]$
Half-Normal (scale = σ)	$\sigma\sqrt{\pi/2}$	$\sqrt{\frac{\pi\sum_i^n x_i^2}{(2n)}}$	$[\prod_{i=1}^{n} x_i]^{1/n}/[\sqrt{\frac{\pi\sum_i^n x_i^2}{(2n)}}]$

4.2 Bias and Risk Evaluations

In cases where we do not have an exact closed-form expression for the relative bias or risk of one of the estimators, these quantities are simulated as follows. If $\bar{\theta}_{(i)}$ is the i^{th} simulated value of one of the estimators, $\bar{\theta}$ (= $\tilde{\theta}, \hat{\theta}$, or θ^*) of the parameter in question (θ), then:

$$\%relBias(\bar{\theta};\theta) = 100[\frac{1}{NREP}\sum_{i=1}^{NREP}\bar{\theta}_{(i)} - \theta]/\theta \tag{23}$$

$$\%relMSE(\bar{\theta};\theta) = 100[\frac{1}{NREP}\sum_{i=1}^{NREP}(\bar{\theta}_{(i)} - \theta)^2]/\theta^2 \tag{24}$$

$$\%relMAE(\bar{\theta};\theta) = 100[\frac{1}{NREP}\sum_{i=1}^{NREP}|\bar{\theta}_{(i)} - \theta|]/\theta \tag{25}$$

$$\%relR_L(\bar{\theta};\theta) = 100[\frac{1}{NREP}\sum_{i=1}^{NREP} exp(w(\bar{\theta}_{(i)} - \theta)) - w(\bar{\theta}_{(i)} - \theta) - 1]/\theta \tag{26}$$

Recall that in this study, $NREP = 20,000$. This choice was justified by comparing the exact results provided in section 3.2 with their simulated counterparts.

4.3 Optimal Critical Values

In any preliminary-testing situation, the properties of the preliminary-test estimator depend on the underlying model and its parameter values as well as the sample size and values; and the particular test that is employed to choose between the two

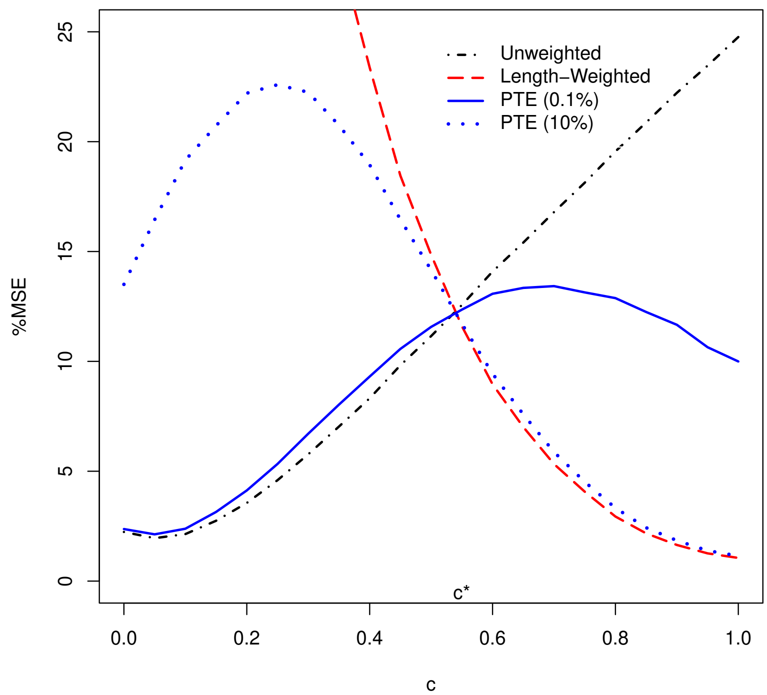

Fig. 2 Exponential distribution: Relative risks of the component and preliminary-test estimators under quadratic loss (%relMSE); $\alpha = 0.1\%$ and $\alpha = 10\%$ for the $\tilde{\lambda}$ test.

”component” estimators. Not only are the properties of the latter test important, but so is the choice of significance level α, (and hence critical value, k). This choice, together with the power of the test, determines the (random) weights that are assigned to the component estimators. In other words, it partially drives the random mixture that forms the preliminary-test estimator. The effect of changing the significance level is illustrated in Figure 2, for the case of the Exponential distribution and quadratic loss, when $n = 50$.

Figure 2 is constructed by allowing for the possibility of non-integer size-bias - that is, $c \in (0, 1)$. As c varies from zero to one in value, the parameter ratio (d/p) for the G-Gamma distribution varies accordingly. In the case of the Exponential distribution, for example, $c \in (0, 1)$ implies $(d/p) \in (1, 2)$. Examples of situations involving fractional biased sampling are discussed by Ducey (2009) and Ducey and Valentine (2009), for example. The intersection of the ”unweighted” and ”length-biased” risk functions depends on the value of n and occurs at $c = c^*$. The risk function for the preliminary-test estimator also passes through this point. This is because the latter estimator is constructed on the basis of null and alternative hypotheses that are ”point” hypotheses. We see that as α increases, the preliminary-test risks ”pivot” monotonically around this intersection point, rising to its left, and falling to its right.

This raises a natural question: ”What choice of significance level is best?” This question has been addressed by a number of authors for various other preliminary testing problems. Several related criteria have been suggested for defining the ”optimal” choice of significance level or critical value. For example, see Sawa and Hiromatsu (1973), Toyoda and Wallace (1976), Brook (1976), Giles *et al.* (1992), and Kibria and Saleh (2006). We will consider the widely adopted criterion of ”mini-max regret”, in the form suggested by Brook (1976).

For each value of c in Figure 2, consider the vertical ”distances” between the risk of the preliminary-test estimator and the *smaller* of the risks of the two component estimators. Here, the latter are the MLEs of β for the Exponential distribution, based on ”unweighted” and ”length-biased” sampling. We will label these distances $d_L(c;\alpha) = [\%relMSE(\beta^*) - \%relMSE(\tilde{\beta})]$, for $c < c^*$, and $d_U(c;\alpha) = [\%relMSE(\beta^*) - \%relMSE(\hat{\beta})]$, for $c \geq c^*$. These are termed the ”regret” of using the preliminary-test estimator instead of the component estimator that has lower (relative) risk for that value of c. These distances show the extent to which preliminary testing worsens the risk of the estimator, relative to the other two options.

The ”mini-max regret” criterion for obtaining an optimal value of α, say α^*, is to select the latter such that both $\underset{(c)}{max}.\{d_L(c;\alpha)\}$ and $\underset{(c)}{max}.\{d_U(c;\alpha)\}$ are minimized. Given the monotonic ”pivoting” of the preliminary-test risk function about the point c^* noted above, this implies finding α^* (and hence k^*) such that $\underset{(c)}{max}.\{d_L(c;\alpha)\} = \underset{(c)}{max}.\{d_U(c;\alpha)\}$. Using k^* as the critical value results in the best possible preliminary-testing strategy from the perspecive of estimation risk.

5 Results

In this section, we present a representative set of results related to the sampling properties of the length-biased sampling preliminary-test estimator for each of the three distributions under study. These properties are compared, graphically, with those of the two component MLEs.

5.1 Relative Biases

Although our primary focus is on the (relative) risks of the various estimators, under different loss functions, the relative biases of the estimators are also of interest. We have examined these in percentage terms, for various sample sizes, and we present the results for the illustrative case where a 5% significance level is chosen for the $\tilde{\lambda}$

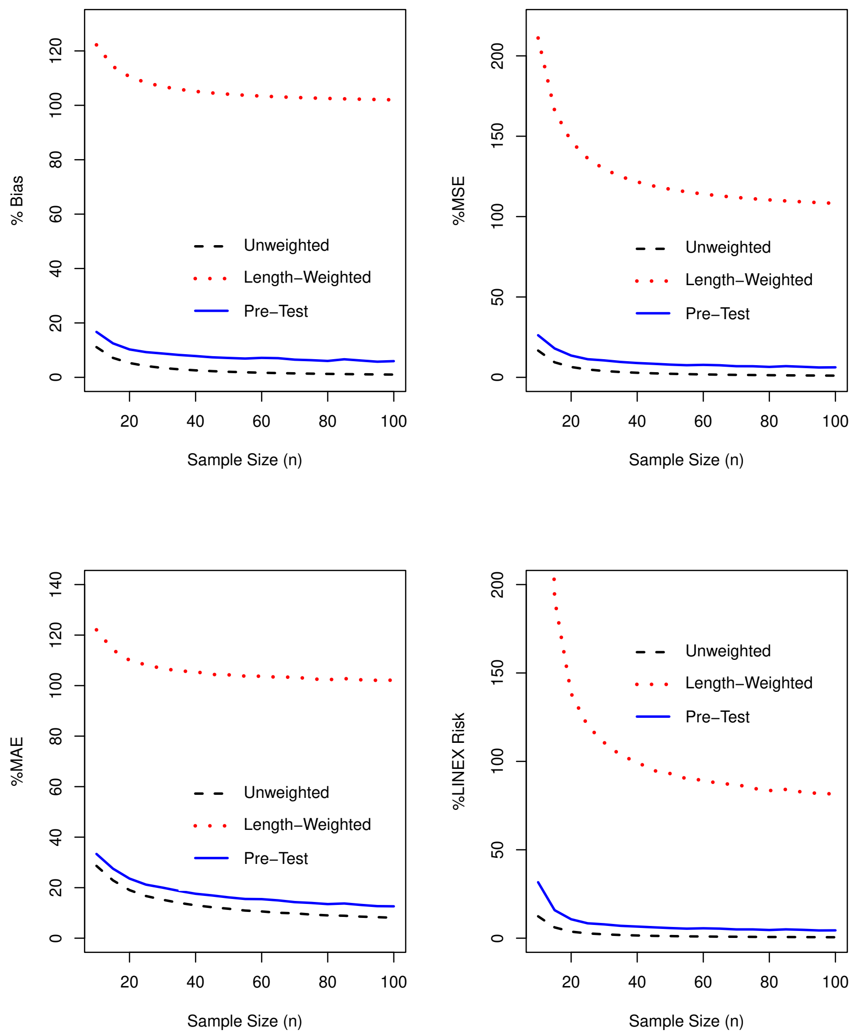

Fig. 3 Exponential distribution: H_0 true. Relative bias and relative risks of the component and preliminary-test estimators, in percentage terms. $\alpha = 5\%$. For the LINEX risk, $w = 2$ and $\beta = 1$.

preliminary test. These results appear in the top-left boxes of Figures 3 to 8, and are ordered as follows.

For the case of the Exponential distribution, the exact relative biases of the component MLEs and the simulated relative bias of the PTE of the rate parameter, β, are shown as a function of the sample size, n for the case where H_0 is true in Figure 3 and for the case where H_1 is true in Figure 4. The corresponding results

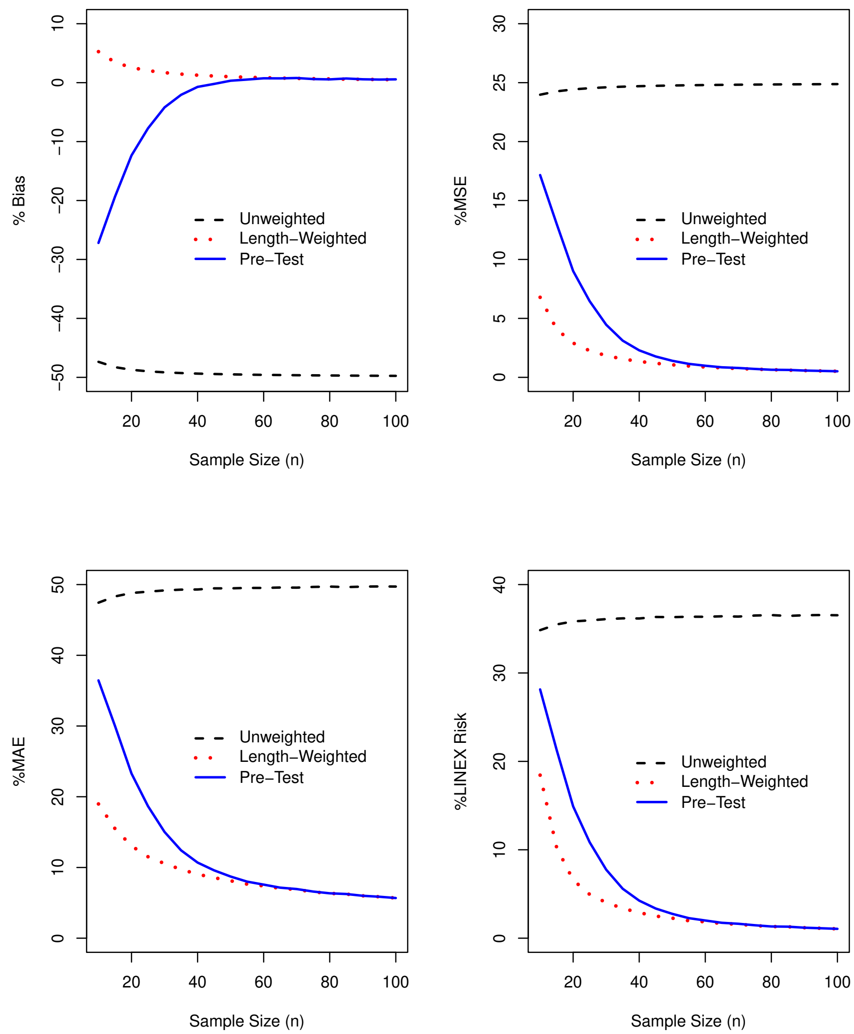

Fig. 4 Exponential distribution: H_1 true. Relative bias and relative risks of the component and preliminary-test estimators, in percentage terms. $\alpha = 5\%$. For the LINEX risk, $w = 2$ and $\beta = 1$.

for the Half-Normal and Rayleigh distributions appear in Figures 5 and 6, as well as Figures 7 and 8. (In the last of these figures, the biases of the component MLEs are also simulated - see the discussion associated with Table 1.)

In all of these figures, we see that the relative bias of the PTE (labelled "Pre-Test" for brevity) always lies between the relative biases of the component MLEs. As expected, when H_0 is true, the bias of the PTE is close to that of the MLE that assumes ”unweighted” sampling. Similarly, when H_1 is true, it is close to the bias of the MLE that assumes ”length-weighted” sampling. The latter gap closes (and the bias approaches zero) as the sample size increases, even modestly. In addition, the absolute biases of the ”component” MLEs themselves can be as large as 50% in value (even when $n = 100$) if the hypothesis under which they are constructed is false. This reflects the inconsistency of these MLEs in such cases.

5.2 Relative Risks

The three (relative) risk functions for the ”unweighted” and ”size-biased” MLEs and for the associated PTEs also appear in Figures 3 to 8, with the same ordering as for the relative biases. The relative LINEX risks of the PTEs depend on the value of the true parameter of each distribution, as well as on the choice of the asymmetry weight (w) for the LINEX loss function. The results for a single choice of each of these variables are illustrated in Figures 2 to 7, but our overall simulation results show that these are representative choices.

We see that the risk function for the preliminary-test estimator lies between the risks of its two ”component estimators”. Interestingly, this implies that the preliminary-test estimator can never be the worst of the three estimators in terms of relative risk. This stands in contrast to many other results in the preliminary-testing literature. It arises because both the null and alternative hypotheses are point hypotheses. This result is discussed further in the next sub-section. The pattern of behaviour of the risk of the PTE relative to the risks of the component MLEs follows that described for the biases in the last sub-section. Again, the inconsistency of the component MLEs when the hypotheses on which they are based is false is clear from the %MSE results in these tables.

5.3 Optimal Critical Values

As found by the various authors referenced in sub-section 4.3, for a given risk criterion and underlying distribution, the mini-max regret approach for determining α^* results in an almost constant optimal critical value, k^*, regardless of the sample size. Moreover, it is almost the same for the %relMSE and %relMAE risks. Of course, this implies that the optimal *significance level* varies greatly with the sample size, and it can be vastly different from the significance levels (such as 5% or 10%) that would commonly be used. This is shown in Table 4, for the three distributions under study and for the different risk functions being considered. This constancy in the critical value across sample sizes is, of course, of considerable help to practitioners. For example, in the case of the Half-Normal distribution, choosing a critical value of approximately 0.8 to 0.85 will optimize the performance of the preliminary-test

Table 4 Optimal Critical Values

	Exponential		Half-Normal		Rayleigh	
n	α^*	k^*	α^*	k^*	α^*	k^*
			%relMSE-Risk			
15	5.0%	0.77	22.1%	0.82	34.5%	0.90
25	3.2%	0.74	13.3%	0.81	27.4%	0.90
50	0.90%	0.72	4.6%	0.81	16.7%	0.90
100	0.07%	0.70	0.62%	0.81	7.7%	0.90
250	0.0002%	0.69	0.006%	0.80	1.02%	0.89
			%relMAE-Risk			
15	8.4%	0.75	20.2%	0.83	33.0%	0.91
25	5.0%	0.72	11.8%	0.82	25.9%	0.90
50	1.3%	0.71	3.9%	0.82	16.0%	0.90
100	0.11%	0.70	0.57%	0.81	7.1%	0.90
250	0.0002%	0.69	0.006%	0.81	1.1%	0.89
			%relLINEX-Risk ($w = 1$)			
15	2.4%	0.80	26.5%	0.80	38.5%	0.89
25	2.0%	0.76	16.2%	0.80	30.2%	0.89
50	0.56%	0.72	5.8%	0.80	18.9%	0.89
100	0.04%	0.71	0.95%	0.80	8.5%	0.89
250	0.00009%	0.70	0.01%	0.80	1.3%	0.89
			%relLINEX-Risk ($w = -1$)			
15	7.7%	0.75	18.5%	0.83	30.7%	0.91
25	4.8%	0.72	10.8%	0.83	24.8%	0.90
50	1.3%	0.71	3.7%	0.82	15.6%	0.90
100	0.14%	0.70	0.51%	0.81	7.0%	0.90
250	0.0002%	0.69	0.003%	0.81	0.89%	0.90
			%relLINEX-Risk ($w = 2$)			
15	1.1%	0.83	31.7%	0.77	42.6%	0.89
25	1.1%	0.77	19.3%	0.79	33.2%	0.89
50	0.35%	0.73	7.3%	0.79	20.3%	0.89
100	0.023%	0.72	1.2%	0.79	9.3%	0.89
250	0.00001%	0.71	0.02%	0.79	1.5%	0.89
			%relLINEX-Risk ($w = -2$)			
15	11.1%	0.73	15.3%	0.85	27.2%	0.92
25	6.7%	0.71	9.1%	0.84	22.3%	0.91
50	2.1%	0.70	3.0%	0.82	13.8%	0.90
100	0.26%	0.69	0.37%	0.82	6.3%	0.89
250	0.0008%	0.68	0.002%	0.82	0.78%	0.90

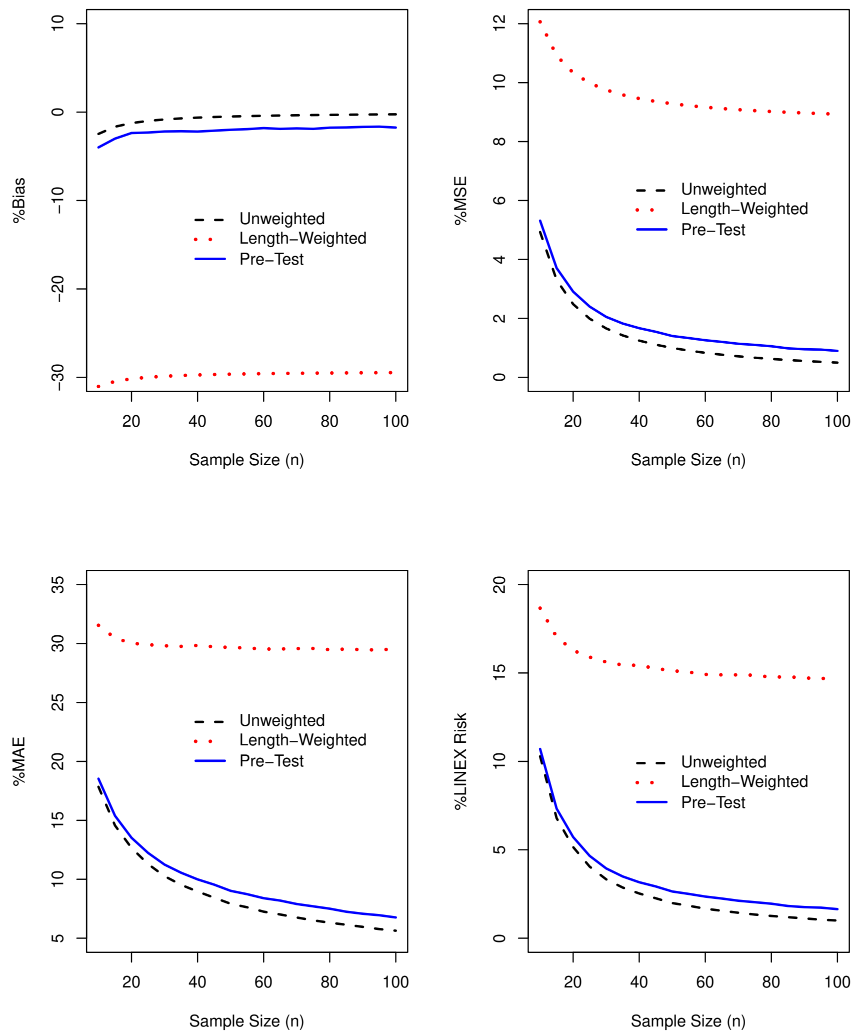

Fig. 5 Half-normal distribution: H_0 true. Relative bias and relative risks of the component and preliminary-test estimators, in percentage terms. $\alpha = 5\%$. For the LINEX risk, $w = 2$ and $\beta = 1$.

estimator for most sample sizes, regardless of the choice of risk function. A choice of $k^* = 0.9$ has the same effect for the Rayleigh distribution.

Figures 9 to 11 illustrate the relative risks of the preliminary-test estimators for each distribution and each loss function, when the choice of critical value for the $\tilde{\lambda}$ preliminary test is optimized in terms of mini-max regret. For comparison, the corresponding risks when $\alpha = 5\%$ are also shown. The results shown in these figures

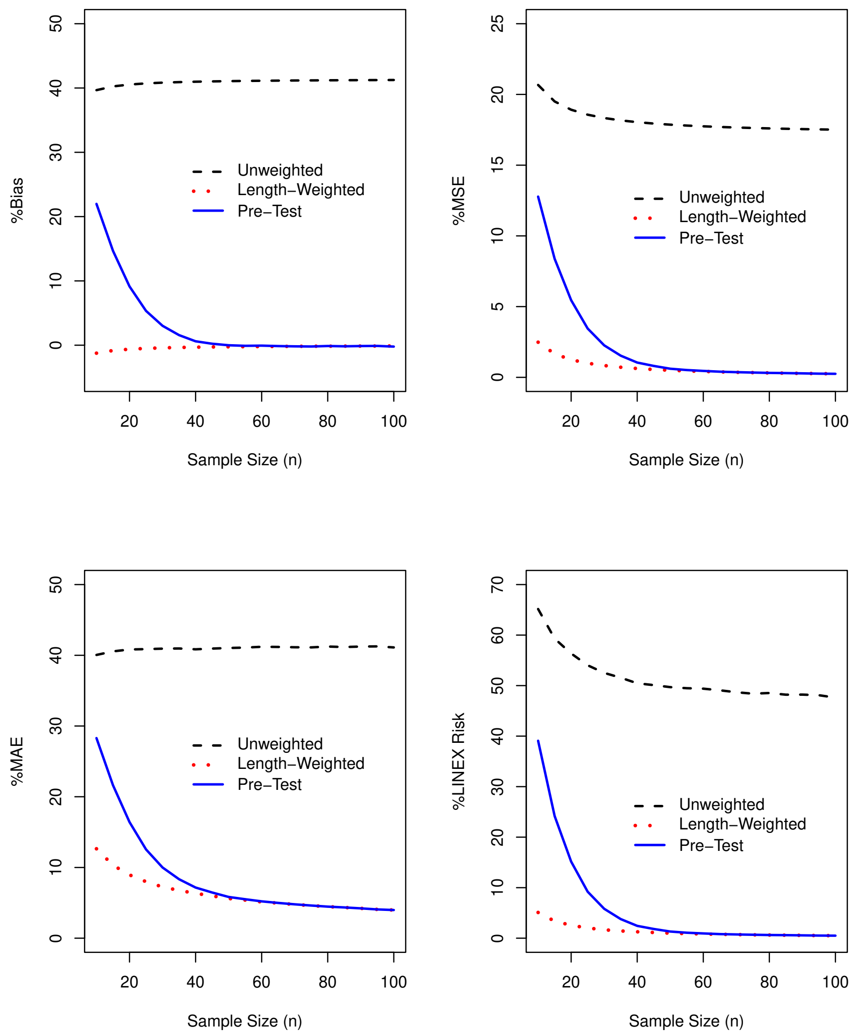

Fig. 6 Half-normal distribution: H_1 true. Relative bias and relative risks of the component and preliminary-test estimators, in percentage terms. $\alpha = 5\%$. For the LINEX risk, $w = 2$ and $\beta = 1$.

are $n = 50$ or $n = 25$; and the LINEX risks are for $w = 2$, and $\beta = 1$ or $\sigma = 1$, for illustrative purposes. These choices are fully representative of other situations that have been explored, and these figures clearly demonstrate the importance of using a (relatively constant) critical value, k^*, as provided in Table 4. Conventional choices of the significance level for the preliminary test can lead to a highly sub-optimal

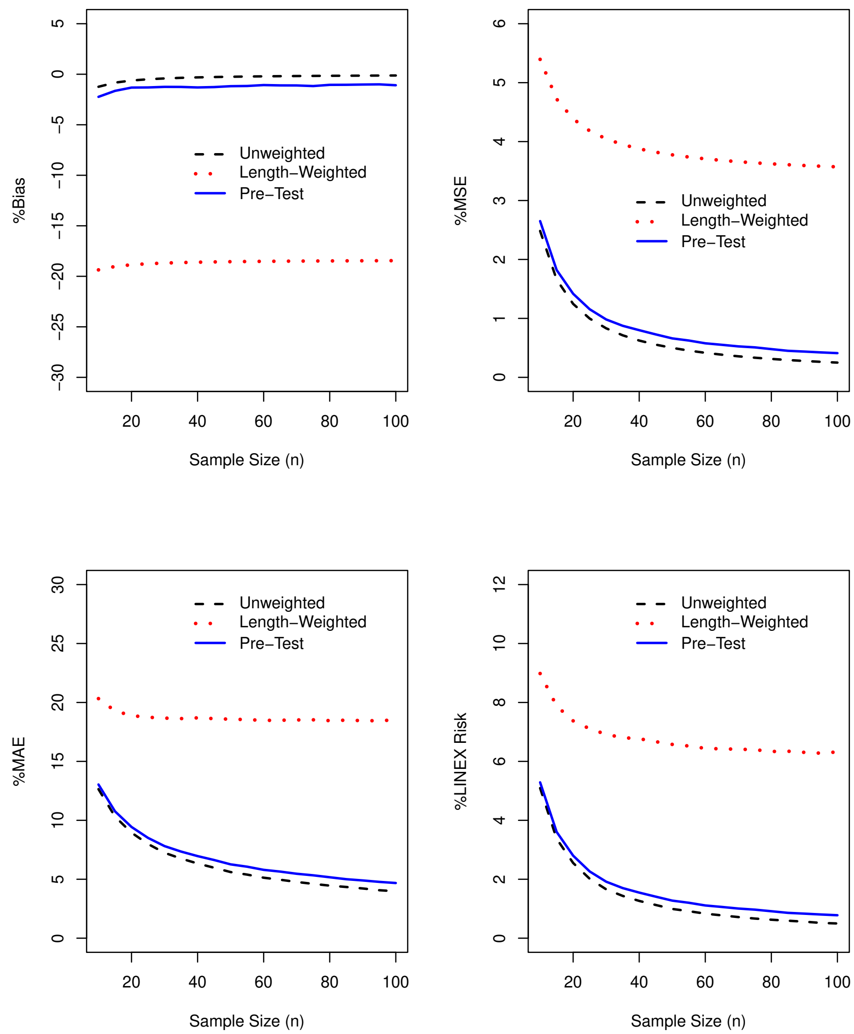

Fig. 7 Rayleigh distribution: H_0 true. Relative bias and relative risks of the component and preliminary-test estimators, in percentage terms. $\alpha = 5\%$. For the LINEX risk, $w = 2$ and $\beta = 1$.

risk for the PTE, regardless of the underlying distribution, the loss function, or the sample size.

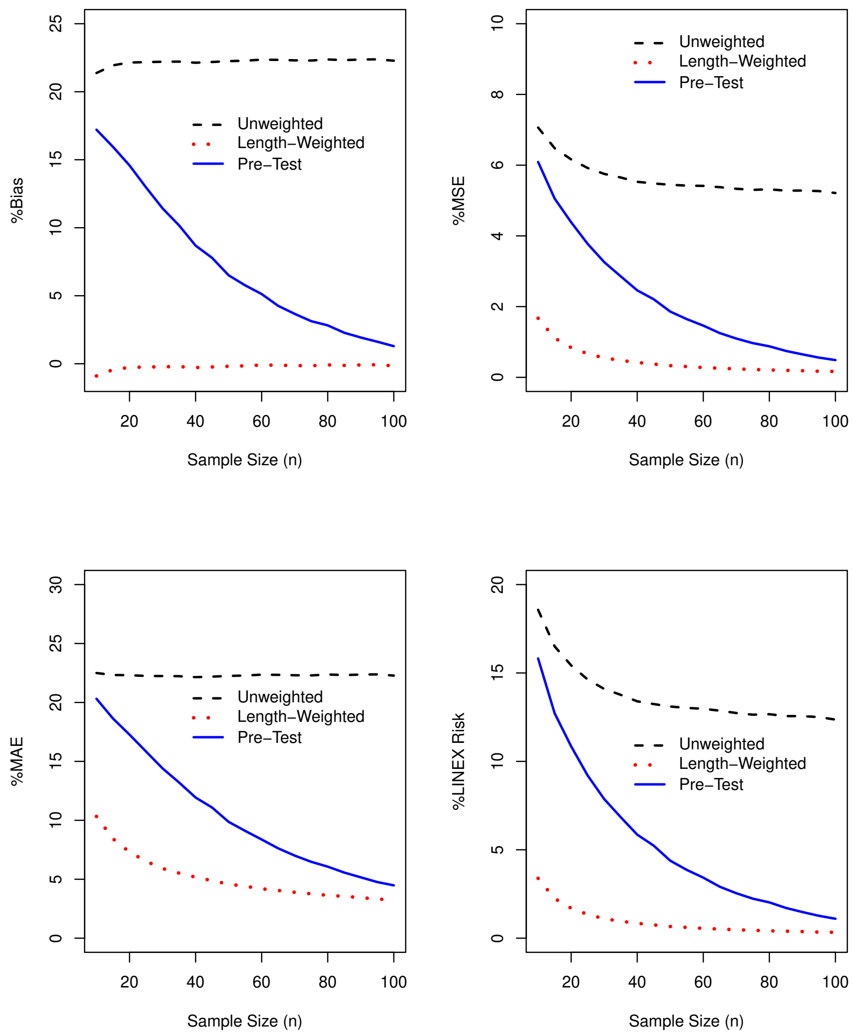

Fig. 8 Rayleigh distribution: H_1 true. Relative bias and relative risks of the component and preliminary-test estimators, in percentage terms. $\alpha = 5\%$. For the LINEX risk, $w = 2$ and $\beta = 1$.

6 Concluding Remarks

Size-biased sampling is a statistical phenomenon that arises in a wide range of disciplines. If it has occurred, but this is ignored, then there can be detrimental consequences for the properties of estimators and tests based on the sampled data.

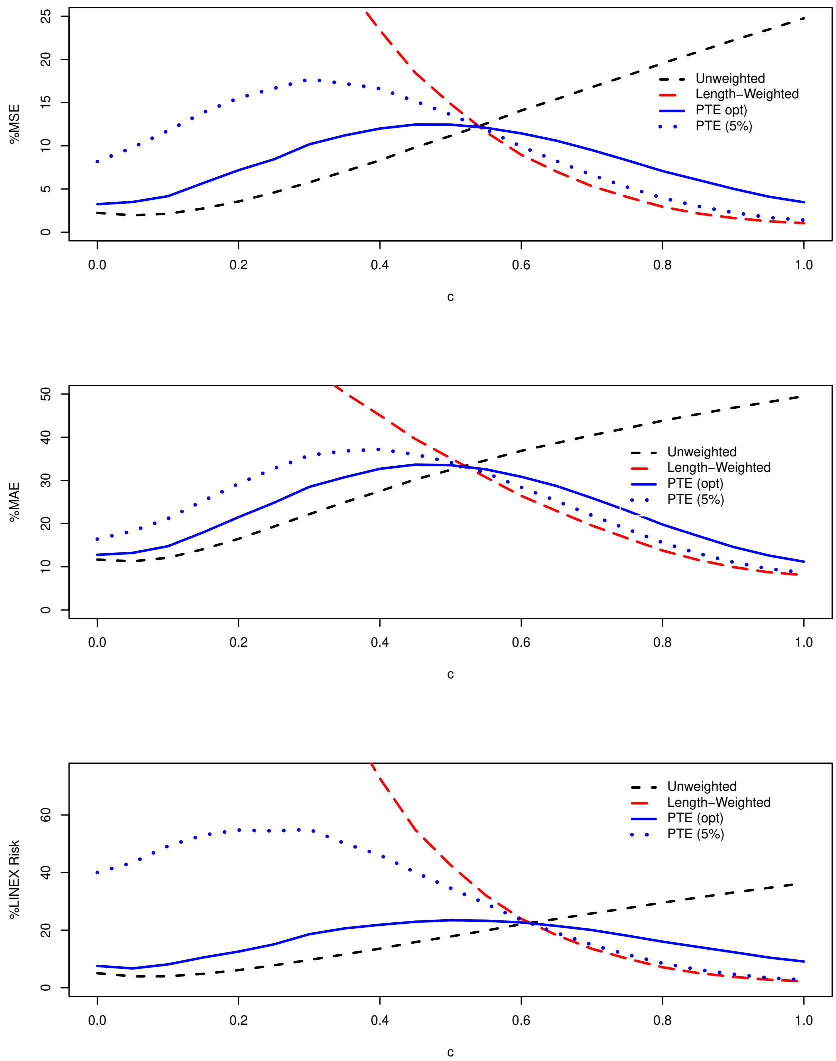

Fig. 9 Exponential distribution: Relative risks of the component and preliminary-test estimators, in percentage terms. $n = 50$; $\alpha = 5\%$ or $\alpha = \alpha^*$ (= opt). For the LINEX risk, $w = 2$ and $\beta = 1$.

One practical issue is that the researcher may not know whether such sampling has been used. If this is the case, then there is a small but important literature that provides tests for the hypothesis that simple random sampling has been used, *versus* the alternative hypothesis that length-biased (or ”weighted”) sampling of a fixed

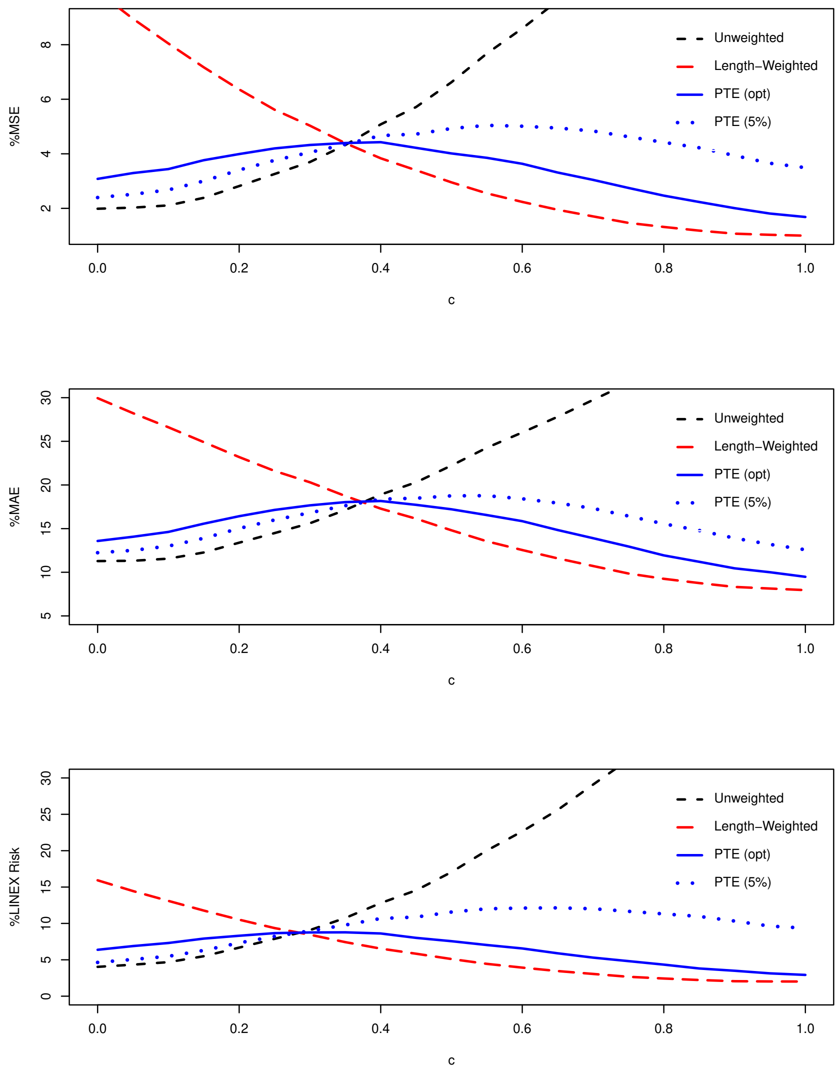

Fig. 10 Half-normal distribution: Relative risks of the component and preliminary-test estimators, in percentage termrs. $n = 25$; $\alpha = 5\%$ or $\alpha = \alpha^*$ (= opt). For the LINEX risk, $w = 2$ and $\sigma = 1$.

order is in place. These tests have asymptotic (large-sample) justification, but there is limited evidence about their performances in small samples.

If a researcher uses such a test to determine which distribution will form the basis of subsequent inferential procedures, then we have an example of ”preliminary testing”. Although the literature on preliminary-test inference (both estimation and

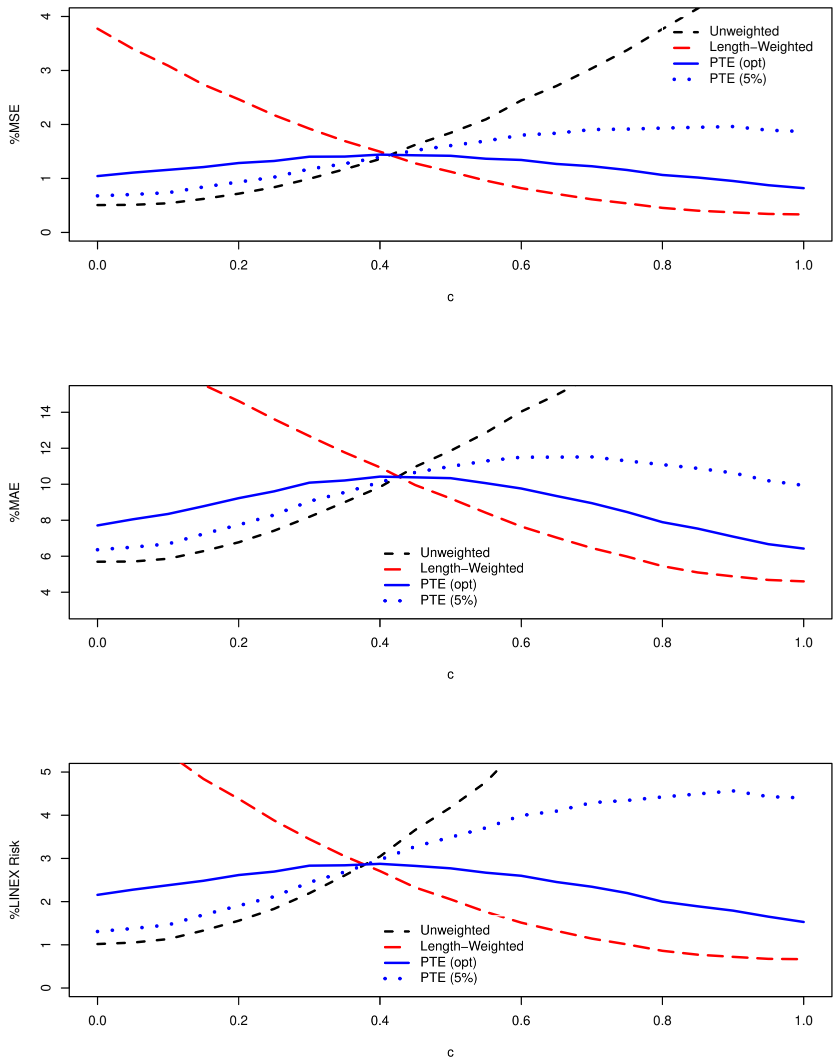

Fig. 11 Rayleigh distribution: Relative risks of the component and preliminary-test estimators, in percentage terms. $n = 50$; $\alpha = 5\%$ or $\alpha = \alpha^*$ (= opt). For the LINEX risk, $w = 2$ and $\sigma = 1$.

testing) is substantial, it seems that to date it has not included the example of testing for size-biased sampling. In this paper, we have attempted to address this issue by exploring the sampling properties of preliminary-test estimators under various loss structures, for three simple distributions that have been used in the biased-sampling literature - the Exponential, Half-Normal, and Rayleigh distributions.

Our results indicate that the relative bias and relative risks of these preliminary-test estimators lie between those of the two ”component” MLEs - those obtained from the ”unweighted” length-biased distributions. Accordingly, if we consider three possible strategies - always assume that H_0 is true, always assume that H_1 is true, or apply a preliminary test using the $\tilde{\lambda}$ statistic - the preliminary-test strategy can never be the best of the three strategies, but it can also never be the worst.

An important finding is that when we seek to optimize the preliminary-testing strategy by choosing the critical value for the $\tilde{\lambda}$ test that satisfies the mini-max regret criterion, then this critical value should be constant across different sample sizes, for a particular distribution. It should not be varied to comply with a conventional significance level such as 5% or 10%. This finding is consistent with the corresponding preliminary-test literature for the linear regression model.

The distributions that have been studied in this paper are specific members of the Generalized Gamma family, and there are several other members of this family that are commonly used in size-biased sampling. These include, for instance, the Weibull, Gamma, and Nakagami distributions. Each of these distributions has more than one parameter, and the associated maximum likelihood estimators cannot be written in closed-form, which complicates the evaluation of the corresponding preliminary-test estimators. However, exploring preliminary testing in the context of these distributions would be an important avenue for further research. Other extensions of the present study could incorporate additional loss functions. Each of those considered in this paper is unbounded, and it would be interesting to undertake a similar investigation based on the ”reflected Normal” loss function (Spiring, 1993; Giles, 2002), and the bounded LINEX (BLINEX) loss function (Wen and Levy, 2001), for example. Moreover, this paper dealt solely with length-biased sampling. Higher-order size-biased sampling, such as area-biased, is important in several fields such as forestry (*e.g.*, Gove, 2003b), and preliminary testing in that context would be worthy of study.

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