Improved Maximum Likelihood Estimation for the Weibull Distribution Under Length-Biased Sampling

David E. Giles

Department of Economics, University of Victoria

May 2021

Supplementary Material

This supplement presents the results that are needed to define the matrix, A, defined in Equation (22) of the paper. Some of these results can be derived directly, while others were obtained using Maple (Maplesoft, 2020). In what follows, for any real z, $\psi(z) = d ln\Gamma(z)/dz$ is the usual digamma function, $\psi'(z) = d\psi(z)/dz$ is the trigamma function, and $\psi''(z) = d\psi'(z)/dz$ is the psigamma function.

Recall the following expressions in equations (7) - (9) of the paper:

$$\mu_k{}' = E[X^k] = \lambda^k(k+1)/k \tag{S.1}$$

$$E_1 = E[X^k \ln(X)] = [\lambda^k ln(\lambda)(k+1)/k] + \lambda^k\left[(k+1)\psi\left(\frac{1}{k}\right) + k(k+2)\right]/k^2 \tag{S.2}$$

$$\begin{aligned} E_2 &= E[X^k(\ln(X))^2] \\ &= [\lambda^k/k]\left\{(k+1)\big(ln(\lambda)\big)^2 + 2ln(\lambda)\left[(k+1)\psi\left(\frac{1}{k}\right) + k(k+2)\right]/k \right. \\ &\left. + \left[(k+1)\left(\psi\left(\frac{1}{k}\right)\right)^2 + 2k(k+2)\psi\left(\frac{1}{k}\right) + (k+1)\psi'\left(\frac{1}{k}\right) + 2k^2\right]/k^2\right\} \end{aligned} \tag{S.3}$$

Following the same steps that are used to derive E_1 and E_2 in Appendix 1 of the paper, and using Maple to show that

$$\int_0^\infty t^{1+\frac{1}{k}}\big(ln(t)\big)^3 e^{-t}dt = \Gamma\left(\frac{1}{k}\right)\left\{(k+1)\left(\psi\left(\frac{1}{k}\right)\right)^3 + 3k(k+2)\left(\psi\left(\frac{1}{k}\right)\right)^2 + 3\psi\left(\frac{1}{k}\right)\left[2k^2 + (k+1)\psi'\left(\frac{1}{k}\right)\right] + 3k(k+2)\psi'\left(\frac{1}{k}\right) + (k+1)\psi''\left(\frac{1}{k}\right)\right\}/k^2 \quad ,$$

we can obtain:

$$E_3 = E[X^k(\ln(X))^3] = \int_0^\infty \frac{x^k\big(ln(x)\big)^3\left(\frac{x}{\lambda}\right)^k\left(\frac{k}{\lambda}\right)exp\big(-(x/\lambda)^k\big)}{\Gamma[1+1/k]}dx = A_1 + A_2 + A_3 + A_4 \quad , \tag{S.4}$$

where:

$$A_1 = \lambda^k(k+1)(ln(\lambda))^3/k$$

$$A_2 = [3\lambda^k(ln(\lambda))^2/k^2]\left[(k+1)\psi\left(\frac{1}{k}\right) + k(k+2)\right]$$

$$A_3 = [3\lambda^k ln(\lambda)/k^3]\left[(k+1)(\psi\left(\frac{1}{k}\right))^2 + 2k(k+2)\psi\left(\frac{1}{k}\right) + (k+1)\psi'\left(\frac{1}{k}\right) + 2k^2\right]$$

$$A_4 = [\lambda^k/k^4]\left\{(k+1)\left(\psi\left(\frac{1}{k}\right)\right)^3 + 3k(k+2)\left(\psi\left(\frac{1}{k}\right)\right)^2 + 3\psi\left(\frac{1}{k}\right)\left[2k^2 + (k+1)\psi'\left(\frac{1}{k}\right)\right]\right.$$
$$\left. + 3k(k+2)\psi'\left(\frac{1}{k}\right) + (k+1)\psi''\left(\frac{1}{k}\right)\right\}$$

The partial derivatives of E_1 and E_2 can be shown to be:

$$E_1^{(1)} = \frac{\partial E_1}{\partial \lambda} = \lambda^{k-1}\{(k+1)[ln(\lambda) + (1+\psi(1/k))/k] + (k+2)\} \tag{S.5}$$

$$E_1^{(2)} = \frac{\partial E_1}{\partial k} = \left(\frac{(ln(\lambda))^2(k+1)}{k}\right) + \left(\lambda^k ln(\lambda)/k^2\right)\left[(k+1)\psi\left(\frac{1}{k}\right) + k(k+2) - 1\right]$$
$$+(\lambda^k/k^2)\left[\psi\left(\frac{1}{k}\right)(1-2k^2-2k) - \frac{(k+1)\psi'\left(\frac{1}{k}\right)}{k^3} - 2k^2\right] \tag{S.6}$$

$$E_2^{(1)} = \frac{\partial E_2}{\partial \lambda} = \lambda^{k-1}\left[\left\{(k+1)\big(ln(\lambda)\big)^2 + 2ln(\lambda)\left[(k+1)\psi\left(\tfrac{1}{k}\right) + k(k+2)\right]/k + \left[(k+1)\left(\psi\left(\tfrac{1}{k}\right)\right)^2 + \right.\right.\right.$$
$$\left.\left.\left. 2k(k+2)\psi\left(\tfrac{1}{k}\right) + (k+1)\psi'\left(\tfrac{1}{k}\right) + 2k^2\right]/k^2\right\} + \left\{\frac{2(k+1)ln(\lambda)}{\lambda} + 2\left[(k+1)\psi\left(\tfrac{1}{k}\right) + k(k+2)\right]/k\right\}/k\right]$$
$$\tag{S.7}$$

$$E_2^{(2)} = \frac{\partial E_2}{\partial k} = \left\{(k+1)\big(ln(\lambda)\big)^2 + 2ln(\lambda)\left[(k+1)\psi\left(\tfrac{1}{k}\right) + k(k+2)\right]/k + \left[(k+1)\left(\psi\left(\tfrac{1}{k}\right)\right)^2 + 2k(k+\right.\right.$$
$$\left.\left. 2)\psi\left(\tfrac{1}{k}\right) + (k+1)\psi'\left(\tfrac{1}{k}\right) + 2k^2\right]/k^2\right\}\left[\left(\lambda^k/k\right)(ln(\lambda) - 1/k)\right] + \left(\lambda^k/k\right)\left\{\big(ln(\lambda)\big)^2 + (2ln(\lambda)/\right.$$
$$k)\left[\psi\left(\tfrac{1}{k}\right) - \frac{(k+1)\psi'\left(\frac{1}{k}\right)}{k^2} + 2(k+1)\right] - \left(\frac{2ln(\lambda)}{k^2}\right)\left[(k+1)\psi'\left(\tfrac{1}{k}\right) + k(k+2)\right] - \left(\frac{2}{k^3}\right)\left[(k+1)\left(\psi\left(\tfrac{1}{k}\right)\right)^2 + \right.$$
$$\left. 2k(k+2)\psi\left(\tfrac{1}{k}\right) + (k+1)\psi'\left(\tfrac{1}{k}\right) + 2k^2\right] + (1/k^2)\left[\left(\psi\left(\tfrac{1}{k}\right)\right)^2 - \left(\frac{2(k+1)}{k^2}\right)\psi\left(\tfrac{1}{k}\right)\psi'\left(\tfrac{1}{k}\right) + 4(k+\right.$$
$$\left.\left. 1)\psi\left(\tfrac{1}{k}\right) + \left(\frac{3k+4}{k}\right)\psi'\left(\tfrac{1}{k}\right) + \left(\frac{k+1}{k^2}\right)\psi''\left(\tfrac{1}{k}\right) + 4k\right]\right\} \tag{S.8}$$

Next, we require the following third-order partial derivatives of the log-likelihood function, which can be obtained by differentiating equations (15) – (17) in the paper:

$$\frac{\partial^3 l}{\partial \lambda^3} = -2n(k+1)/\lambda^3 + k(k+1)(k+2)\lambda^{-(k+2)}\sum_{i=1}^{n} x_i^k \tag{S.9}$$

$$\frac{\partial^3 l}{\partial\lambda^2\partial k} = \left(\frac{n}{\lambda^2}\right) - \lambda^{-(k+2)}\sum_{i=1}^{n} x_i^k\,[k(k+1)ln(\lambda) - (2k+1)] - k(k+1)\lambda^{-(k+2)}\sum_{i=1}^{n} x_i^k ln(x_i) \tag{S.10}$$

$$\frac{\partial^3 l}{\partial\lambda\partial k^2} = ln(\lambda)\lambda^{-k}\sum_{i=1}^{n} x_i^k\left[(ln(\lambda))^2 - 2/\lambda\right] + 2\lambda^{-k}\sum_{i=1}^{n} x_i^k ln(x_i)\,[(1/\lambda) - (ln(\lambda))^2] \tag{S.11}$$

$$\frac{\partial^3 l}{\partial k^3} = [T_1 + T_2 + T_3 + T_4 + T_5 + T_6] \quad , \tag{S.12}$$

where:

$$T_1 = 2n\left[6k\psi\left(1+\frac{1}{k}\right) + 2\psi'(1+1/k)\right]/k^5$$

$$T_2 = n\left[4k\psi'\left(1+\frac{1}{k}\right) + \psi''(1+1/k)\right]/k^6$$

$$T_3 = (ln(\lambda))^2\lambda^{-k}\left[ln(\lambda)\sum_{i=1}^{n} x_i^k - \sum_{i=1}^{n} x_i^k ln(x_i)\right]$$

$$T_4 = 2ln(\lambda)\lambda^{-k}\left[\sum_{i=1}^{n} x_i^k (ln(x_i))^2 - ln(\lambda)\sum_{i=1}^{n} x_i^k ln(x_i)\right]$$

$$T_5 = \lambda^{-k}\left[\sum_{i=1}^{n} x_i^k (ln(x_i))^3 - ln(\lambda)\sum_{i=1}^{n} x_i^k (ln(x_i))^2\right]$$

$$T_6 = 2n/k^3$$

(The other cross-derivatives follow immediately by the symmetry of differentiation.)

Then, taking the expectations of (S.9) – (S.12), using (S.1) – (S.4), we have:

$$\kappa_{111} = E\left[\frac{\partial^3 l}{\partial\lambda^3}\right] = nk(k+1)(k+3)/\lambda^3 \tag{S.13}$$

$$\kappa_{112} = \kappa_{211} = \kappa_{121} = E\left[\frac{\partial^3 l}{\partial\lambda^2\partial k}\right]$$

$$= n\left\{\lambda^{-2} - \lambda^{-2}(k+1)[k(k+1)ln(\lambda) - (2k+1)]/k - k(k+1)\lambda^{-(k+2)}E_1\right\} \tag{S.14}$$

$$\kappa_{221} = \kappa_{212} = \kappa_{122} = E\left[\frac{\partial^3 l}{\partial\lambda\partial k^2}\right]$$

$$= n(k+1)ln(\lambda)\left[(ln(\lambda))^2 - \frac{2}{\lambda}\right]/k + 2n\left[\frac{1}{\lambda} - (ln(\lambda))^2\right]\left\{(k+1)\left[kln(\lambda) + \psi\left(\frac{1}{k}\right)\right]/k^2 + (k+2)/k\right\} \tag{S.15}$$

$$\kappa_{222} = E\left[\frac{\partial^3 l}{\partial k^3}\right] = T_1 + T_2 + T_6 + n\{(ln(\lambda))^2[(k+1)ln(\lambda)/k\ - \lambda^{-k}E_1] + 2ln(\lambda)\lambda^{-k}[E_2 - ln(\lambda)E_1] + \lambda^{-k}[E_3 - ln(\lambda)E_2]\} \tag{S.16}$$

Next, differentiating the expressions in equations (18) – (20) of the paper, we have:

$$\kappa_{11}^{(1)} = \left(\frac{\partial \kappa_{11}}{\partial \lambda}\right) = 2nk(k+1)/\lambda^2 \tag{S.17}$$

$$\kappa_{11}^{(2)} = \left(\frac{\partial \kappa_{11}}{\partial k}\right) = -n(2k+1)/\lambda^2 \tag{S.18}$$

$$\kappa_{22}^{(1)} = \left(\frac{\partial \kappa_{22}}{\partial \lambda}\right) = n\left\{k\lambda^{-(k+1)}(E_2 - 2ln(\lambda)E_1) - \lambda^{-k}\left(E_2^{(1)} - 2ln(\lambda)E_1^{(1)} - 2E_1/\lambda\right) - 2(k+1)ln(\lambda)/(k\lambda)\right\} \tag{S.19}$$

$$\kappa_{22}^{(2)} = \left(\frac{\partial \kappa_{22}}{\partial k}\right) = n\left\{\frac{\psi''\left(1+\frac{1}{k}\right)}{k^6} + 2\frac{\psi'\left(1+\frac{1}{k}\right)}{k^5} + 6\frac{\psi\left(1+\frac{1}{k}\right)}{k^4} + \left(\frac{2}{k^3}\right) + \frac{(ln(\lambda))^2}{k^2} - \lambda^{-k}\left(E_2^{(2)} - 2ln(\lambda)E_1^{(2)}\right) + \lambda^{-k}ln(\lambda)(E_2 - 2ln(\lambda)E_1)\right\} \tag{S.20}$$

$$\kappa_{12}^{(1)} = \kappa_{21}^{(1)} = \left(\frac{\partial \kappa_{12}}{\partial \lambda}\right) = (n/\lambda^2)\left[(k+1)\big(k - (1 - kln(\lambda))\big) + \frac{kE_1^{(1)}}{\lambda^{k-1}} - \frac{k(k+1)E_1}{\lambda^k} + 1)\right] \tag{S.21}$$

$$\kappa_{12}^{(2)} = \kappa_{21}^{(2)} = \left(\frac{\partial \kappa_{12}}{\partial k}\right) = \left(\frac{n}{\lambda}\right)[1 - (2k+1)ln(\lambda)] + n\lambda^{-(k+1)}\left[E_1(1 - kln(\lambda)) + kE_1^{(2)}\right] \tag{S.22}$$

Then, using (S.13) – (S.22) we can construct

$$a_{ij}^{(l)} = \kappa_{ij}^{(l)} - (\kappa_{ijl}/2); \quad i, j, l = 1, 2$$

$$A^{(l)} = \left\{a_{ij}^{(l)}\right\}\ ; \quad i, j, l = 1, 2$$

Finally, by concatenation we obtain the (2×4) matrix:

$$A = [A^{(1)}\ |\ A^{(2)}|].$$

Reference

Maplesoft (2020). Maple Mathematical Software. Maplesoft, Waterloo, ON.