New Goodness-of-Fit Tests for the Kumaraswamy Distribution: A Simulation Study

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Abstract:

The two-parameter distribution introduced by Kumaraswamy (1980) is a very flexible alternative to the Beta distribution with the same (0,1) support. Originally proposed in the field of hydrology, it has subsequently received a good deal of positive attention in both the theoretical and applied statistics literatures. Interestingly, the problem of testing formally for the appropriateness of the Kumaraswamy distribution appears to have received little or no attention to date. To fill this gap, in this paper we apply the "biased transformation" methodology proposed by Raschke (2009) to several standard goodness-of-fit tests based on the empirical distribution function. A simulation study reveals that these (modified) tests perform well in the context of the Kumaraswamy distribution, in terms of both low size distortion, and respectable power. In particular, the "biased transformation" Anderson-Darling test dominates the other tests that are considered.

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1. Introduction

The two-parameter distribution introduced by Kumaraswamy (1980) is a very flexible alternative to the Beta distribution with the same (0,1) support. Originally proposed for the analysis of hydrological data, it has subsequently received a good deal of attention in both the theoretical and applied statistics literature. For example, Sundar and Subbiah (1989), Seifi *et al*. (2000), Ponnambalam *et al*. (2001), Ganji *et al*. (2006), and Courard-Hauri (2007 provide applications in various fields, and theoretical extensions are implemented by Cordeiro and Castro (2010), Bayer and Pumi (2017), and Cordeiro *et al*. (2018), among others.

The distribution function for a random variable, X, that follows the Kumaraswamy distribution is

$$F(x) = 1 - (1 - x^a)^b \quad a, b > 0 ; \qquad 0 < x < 1 \qquad , \tag{1}$$

which can be inverted to give the quantile function,

$$Q(y) \equiv F^{-1}(y) = \left[1 - (1 - y)^{1/b}\right]^{1/a} \quad ; \qquad 0 < y < 1 \tag{2}$$

The corresponding density function is:

$$f(x) = abx^{a-1}(1 - x^a)^{b-1} \qquad , \tag{3}$$

where 'a' and 'b' are both shape parameters. Some examples of the forms that this density can take are illustrated in Figure 1. In particular, in common with the beta density $f(x)$ is unimodal if $a > 1$ and $b > 1$; uniantimodal if $a < 1$ and b <1; increasing (decreasing) in x if $a > 1$ and $b \leq 1$ ($a \leq 1$ and b >1; and constant if $a = b = 1$. Nadarajah (2008) notes that the Kumaraswamy distribution is in fact a special case of a generalized beta distribution proposed by McDonald (1984).

The r'th central moment of the Kumaraswamy distribution exists if $r > -a$, and is given by

$$E(X^r) = bB\left(1 + \frac{r}{a}, b\right), \tag{4}$$

where B (. , .) is the complete beta function; and from (2), the median of the distribution is

$$m_d = \left[1 - 0.5^{1/b}\right]^{1/a} . \tag{5}$$

See Jones (2009), Garg (2008), and Mitnik (2013) for a detailed discussion of the additional properties of the Kumaraswamy distribution.

These properties, compared with those of the beta distribution, are considered by many to give the Kumaraswamy distribution a competitive edge. For example, compared with the formula for the cumulative distribution function of the beta distribution, the invertible closed-form expression in (1) is seen by some as being advantageous in the context of computer-intensive simulation analysis, and modelling based on quantiles. The latter consideration is of particular interest in the context of regression analysis. Beta regression, based on the closed-form mean of that distribution, is well-established (e.g., Ferrari and Cribari-Neto, 2004), but robust regression based on the median is impractical. In contrast, the median of the Kumaraswamy distribution has the simple form given in (5), and so robust regression based on this distribution is straightforward. See Mitnik and Baek (2013) and Hamedi-Shahraki *et al*. (2021), for example.

Interestingly, the problem of testing formally for the appropriateness of the Kumaraswamy distribution appears to have received little or no attention in the literature. Goodness-of-fit tests based on the empirical distribution function (EDF) are obvious candidates, but their properties are unexplored for this distribution. Raschke (2009) observed that such tests were unavailable for the beta distribution, and he proposed a "biased transformation" that he then applied to the test of Anderson and Darling (1952, 1954) to fill this gap. He also used this approach to construct an EDF test for the gamma distribution. Subsequently, Raschke (2011) provided extensive simulation results that favoured the use of the "bias-transformed" Anderson-Darling test over various other tests based on the EDF, such as those of Kuiper (1962), Watson (1961), the Cramér-von Mises test (Cramér,1928; von Mises, 1928) and the Kolmogorov-Smirnov test (Kolmogorov, 1933; Smirnov, 1948).

In this paper we apply Raschke's methodology to the problem of constructing EDF goodness-of-fit tests for the Kumaraswamy distribution, and we compare the performances of several such standard tests in terms of both size and power. We find that Raschke's method performs well in this context, with the Kolmogorov-Smirnov and Cramér-von Mises tests exhibiting the least size distortion, and the Anderson-Darling test being a clear choice in terms of power against a wide range of alternatives.

In the next section we introduce the "biased transformation" testing strategy suggested by Raschke and describe the five well-known EDF tests that we consider in this paper. Section 3 provides the results of a simulation experiment that evaluates the sizes and powers of the tests; and an empirical application is included in section 4. Some concluding remarks are presented in section 5.

2. Rasccke's "biased transformation" testing

In very simple terms, the procedure proposed by Raschke involves the use of a transformation that converts the problem of testing the null hypothesis that the data follow the Kumaraswamy distribution into one of testing the null hypothesis of normality. The latter, of course, is readily performed using standard EDF tests. More specifically, the steps involved are as follows (Raschke, 2011, p.80):

(i) Assuming that the data, *X*, follow the Kumaraswamy distribution, estimate the shape parameters, *a* and *b*, using maximum likelihood (ML) estimation. See Lemonte (2011, pp. 1972-1973) and Jones (2009, pp.76-77) for details of the ML estimator for this distribution.

(ii) Generate sample of *Y*, where $Y = \Phi^{-1}(F(X))$, where Φ is the distribution function for the standard Normal distribution, and $F(.)$ is given in (1).

(iii) Obtain the ML estimates of the parameters of the normal distribution for *Y*.

(iv) Apply an EDF test for normality to the *Y* data.

(v) For a chosen significance level, α, the null hypothesis, reject H_0: "*X* is Kumaraswamy" if H_0': "Y is Normal" is rejected.

We consider five standard EDF tests for normality at step (iv), with the *n* values of the *Y* data in ascending order. See Stephens (1986) for more details. The first two of these tests are based on the two quantities $D^+ = \max_{(i)} [i/n - F(Y_i)]$, $D^- = \max_{(i)} [F(Y_i) - (i-1)/n]$, and $D = \max [D^+, D^-]$. The Kolmogorov-Smirnov test statistic is $D^* = D(\sqrt{n} - 0.01 + 0.85/\sqrt{n})$; and Kuiper's test statistic is $V^* = V(\sqrt{n} + 0.05 + 0.82/\sqrt{n})$, where $V = (D^+ + D^-)$. In each case, H_0' is rejected if the test statistic exceeds the appropriate critical value.

Further, defining $W^2 = \sum_{i=1}^{n}[F(Y_i) - (2i-1)/(2n)]^2$, the Cramér-von Mises test statistic is given by $W^{2*} = W^2(1.0 + 0.5/n))$. Similarly, if $U^2 = W^2 - n\{\sum_{i=1}^{n}[F(Y_i]/n) - 0.5\}^2$, the Watson's test statistic is defined as $U^{2*} = U^2(1.0 + 0.5/n)$. Finally, the Anderson-Darling test statistic is defined as $A^{2*} = A^2(1.0\_0.75/n/ + 2.25/n^2)$, where $A^2 = -n - \sum_{i=1}^{n}\{(2i-1)(ln[F(Y_i)] + \ln[1 - F(Y_{n+1-i})])\}/n$. Again, for these last three tests, the null hypothesis is rejected if the test statistic exceeds the appropriate critical value. In the next section we consider nominal significance levels of α = 5% and α =10%. The relevant critical values for the five tests are obtained from Table 4.7 of Stephens (1986, p.123), and appear in the last row of Table 1 in the next section.

3. A simulation study

Using Raschke's "biased transformation" each of the five EDF tests for the Kumaraswamy null hypothesis has been evaluated in a simulation experiment, using the R (R Core Team, 2024). In all parts of the Monte Carlo study, 10,000 Monte Carlo replications were used. The 'univariateML' package (Moss and Nagler, 2022) was used for obtaining the ML estimates of the Kumaraswamy distribution in step (i), and the 'GoFKernel' package (Pavia, 2022) was used to invert the distribution in step (ii), in the last section. Random numbers for the truncated log-normal and triangular distribution were generated using the 'EnvStats' package (Millard and Kowarik, 2023); while those for the Kumaraswamy distribution itself were generated using the 'VGAM' package (Yee, 2023). The 'trapezoid' package (Hetzel, 2022), and the 'truncnorm' package (Mersmann *et al*., 2023) were used to generate random variates from the trapezoidal and truncated normal distributions respectively; and the R base 'stats' package was used for the beta variates. Finally, random variates from the truncated gamma distribution were generated using the 'cascsim' package (Bear *et al*., 2022); and those for the truncated Weibull distribution were obtained using the 'ReIns' package (Reynkens, 2023). The R code that was used for the simulation experiment is available for downloading from https://github.com/DaveGiles1949/r-code .

In the first part of the experiment we investigate the true "size" of each of the five EDF tests for various sample sizes (n) and a selection of values of the parameters (a and b) of the null distribution. As noted above, the tests are applied using nominal significance levels of both 5% and 10%, and we are concerned here with the extent of any "size distortion" that may arise.

The results obtained with six representative (a, b) pairs, and sample sizes ranging from n = 10 to n = 1,000 are shown in Table 1. The corresponding Kumaraswamy densities appear in Figure 1. The simulated sizes of all of the tests are very close to the nominal significance levels in all cases. This result is very encouraging, and provides initial support for adopting the "biased transformation" EDF testing strategy for the Kumaraswamy distribution.

Of the five tests considered, the Kolmogorov-Smirnov test performs best, in terms if least absolute difference between the nominal and simulated sizes, in 16 of the 36 cases at the 5% nominal level and 10 of the 36 cases at the 10% nominal level in Table 1. In the latter case it is out-performed by the Cramér-von Mises test, which dominates for 14 of the 36 cases that are considered. Further, there is a general tendency for simulated sizes of all of the tests to exceed the nominal significance levels when $n \leq 25$, while the converse is true (in general) when $n \geq 50$. An exception is when both of the distribution's parameters equal

0.5, do the density is uniantimodal. These size distortions are generally small, but their direction has implications for the results relating to the powers of the tests.

The second part of the Monte Carlo experiment investigates the powers of the five tests against a range of alternative hypotheses. The latter all involve distributions on the (0 , 1) interval, with some distributions truncated accordingly. It should be noted that the simulated powers that are reported are "raw powers", and are not "size-adjusted". That is, the various critical values that are used are those reported at the end of Table 1. In practical applications, this is how a researcher would proceed.

The results of this part of the study are reported in Table 2. The same set of samples sizes (n) is used as in Table 1. A wide range of parameter values was considered for each of the alternative distributions, and a representative selection of the results that were obtained are reported here.

One immediate result that emerges is that, with only two exceptions, all of the tests are "unbiased" in all of the settings considered. That is, the power of the test exceeds the nominal significance level. The only exceptions that were encountered are when the alternative distributions is truncated log-normal, with both parameters equal to 0.5, and with a sample size of n = 10. This is a very encouraging result. A test that is "biased" has the unfortunate property that it rejects the null hypothesis less frequently when it is false than when it is true. Moreover, as the various tests are "consistent", their powers increase as the sample size increases, for any given case.

The results in Table 2 also provide overwhelming support for the Anderson-Darling test in terms power. Interestingly, this result is totally consistent with the conclusion reached by Rashcke (2011) for the same "biased transformation" EDF tests in the context of the beta distribution. This may reflect that fact that the latter distribution and the Kumaraswamy distributions have densities that are capable of following very similar shapes, depending on the values of the associated parameters. Moreover, Stephens (1986) recommends the Anderson-Darling test over other EDF tests in general.

The Anderson-Darling test has the highest power among all five tests, in all cases, except for very small samples when the alternative distribution is Trapezoidal, with parameters $m_1 = 5/8$, $m_2 = 7/8$; $n_1 = n_3 = 2$; and for the truncated Weibull alternative with n = 10. Of the other tests under study, the Cramér-von Mises test ranks second in terms of power, followed by Watson's test and the Kolmogorov-Smirnov test. We find that Kuiper's test is the least powerful, in general.

As was noted in section 1, the density for the Kumaraswamy distribution can take shapes very similar to those of the beta density, as the values of the two shape parameters vary in each case. The densities for the alternative beta distributions that are considered in the power analysis are depicted in Figure 2, and may be compared with the Kumaraswamy densities in Figure 1. This similarity suggests that there may be instances where the proposed EDF tests have relatively low power. If the data are generated by a beta distribution whose characteristics can be mimicked extremely closely by a Kumaraswamy distribution with the same, or similar, shape parameters, the tests may fail to reject the latter distribution. An obvious case in point is when the values of both of these shape parameters are 0.5, and the densities of both distributions are uniantimodal, though not identical. As can be seen in Figure 1, the density for the Kumaraswamy distribution is slightly asymmetric in this case, while its beta distribution counterpart is symmetric. The relatively low power of all of the EDF tests, even for n = 1,000, in this case can be seen in Table 2.

In view of these observations, we have considered a wide range of different values for the shape parameters associated with the beta distributions that are considered as alternative hypotheses in the power analysis of the EDF tests. A representative selection of the results appears in Table 2. There, we see that although the various tests have modest power when the data are generated by Beta (2, 4), Beta (4,2), and Beta (3,3) distributions, they perform extremely well against several other beta alternatives.

4. Empirical Applications

To illustrate the effectiveness of the "biased transformation" Anderson-Darling test, we present two applications with actual (economic) data. The R code and associated data files can be downloaded from https://github.com/DaveGiles1949/r-code .

The first application uses data for the size of the so-called "hidden economy", or "underground economy" for 158 countries in 2017. These data measure the size of the hidden economy (HE) relative to the value of Gross Domestic Product (GDP) in each country, and are reported by Medina and Schneider (2019). These ratios range from 0.0543 for Switzerland, to 0.5578 for Bolivia, with a mean of 0.2741 and a standard deviation of 0.1120.

When a Kumaraswamy distribution is fitted to the data, the estimates of the two shape parameters are 2.6065 and 20.7094. See Figures 3(a) and 3(c). The value for the Anderson-Darling statistic is 0.5344, which is less than the 10% critical value of 0.631, and so we would not reject the hypothesis that the data follow a Kumaraswamy distribution. If a beta distribution is fitted to the data, the estimates of the two shape parameters are 3.8801 and 10.3253. See Figures 3(a) and 3(b). The corresponding Anderson-Darling statistic (using the “biased transformation” and the beta distribution) is 1.0577. This exceeds the 5% critical value of 0.752, leading us to reject the hypothesis that the data follow a beta distribution. These two test results support each other.

The second application uses a sample data for the Gini indices for income inequality in 69 countries in 2017, as reported by the World Bank (2024). The Gini index ranges in value from 0 (perfect equality) to 1 (perfect inequality). In our sample the smallest value is 0.2320 (for the Slovak Republic) and the largest value is 0.5330 (for Brazil). The sample mean and standard deviation are 0.3522 and 0.0701 respectively. When a Kumaraswamy distribution is fitted to the data the estimates of the two scale parameters are 5.3065 and 165.9645. See Figures 4(a) and 4(c). The Anderson-Darling statistic is 0.8635, which exceeds the 5% critical value, suggesting a rejection of the hypothesis that the data are Kumaraswamy-distributed. Fitting a beta distribution to the data yields estimates of 16.6524 and 30.6073 for the shape parameters. The corresponding Anderson-Darling statistic is 0.3668, suggesting that the hypothesis that the data are beta-distributed cannot be rejected.

5. Conclusions

The Kumuraswamy distribution is an alternative to the beta distribution that has been applied in statistical studies in a wide range of disciplines. Its theoretical properties are well-established, but the literature is lacking a discussion of formal goodness-of-fit tests for this distribution. In this paper we have applied the “biased transformation” methodology suggested by Raaschke (2009) to various standard tests based on the empirical distribution function, and investigated their performance for the Kumaraswamy distribution.

The results of our simulation experiment that focuses on both the size and power of these tests can be summarized as follows. The “biased transformation” EDF goodness-of-fit testing strategy performs well for the Kumaraswamy distribution, against a wide range of possible alternatives, though it needs to treated with caution against certain beta distribution alternatives. In all cases, the Anderson-Darling test emerges clearly as the most powerful test of those considered, and is recommended for practitioners.

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Table 1: Simulated sizes of the EDF tests *,**

n	K-S	Kuiper	C-M	Watson	A-D	K-S	Kuiper	C-M	Watson	A-D
				***a* = 3.0, *b* = 1.0**						
			***α* = 5%**					***α* = 10%**		
10	0.0599	0.0596	0.0575	0.0581	0.0651	0.1195	0.1208	0.1124	0.1188	0.1271
25	0.0507	0.0475	0.0491	0.0485	0.0506	0.1042	0.0984	0.1008	0.1027	0.1024
50	0.0520	0.0490	0.0474	0.0463	0.0481	0.1037	0.0990	0.0954	0.0972	0.0986
100	0.0501	0.0481	0.0502	0.0493	0.0496	0.1013	0.0938	0.0951	0.0967	0.0967
				***a* = 3, *b* = 2**						
			***α* = 5%**					***α* = 10%**		
10	0.0589	0.0588	0.0574	0.0566	0.0646	0.1183	0.1214	0.1134	0.1188	0.1270
25	0.0500	0.0471	0.0482	0.0489	0.0497	0.1046	0.0982	0.1000	0.1022	0.1043
50	0.0521	0.0479	0.0471	0.0464	0.0484	0.1029	0.0990	0.0948	0.0983	0.0988
100	0.0492	0.0476	0.0497	0.0491	0.0494	0.1007	0.0932	0.0949	0.0976	0.0959
				***a* = 3, *b* = 4**						
			***α* = 5%**					***α* = 10%**		
10	0.0585	0.0597	0.0561	0.0550	0.0638	0.1193	0.1210	0.1133	0.1189	0.1277
25	0.0492	0.0464	0.0482	0.0486	0.0487	0.1053	0.0978	0.0990	0.1015	0.1037
50	0.0505	0.0475	0.0459	0.0451	0.0472	0.1007	0.0981	0.0941	0.0977	0.0982
100	0.0497	0.0467	0.0488	0.0480	0.0486	0.1002	0.0925	0.0932	0.0967	0.0949
				***a* = 3, *b* = 8**						
			***α* = 5%**					***α* = 10%**		
10	0.0577	0.0593	0.0559	0.0551	0.0628	0.1174	0.1209	0.1130	0.1178	0.1255
25	0.0479	0.0460	0.0474	0.0481	0.0485	0.1040	0.0972	0.0978	0.1013	0.1033
50	0.0498	0.0470	0.0447	0.0434	0.0463	0.1002	0.0969	0.0924	0.0976	0.0972
100	0.0494	0.0462	0.0481	0.0475	0.0475	0.0986	0.0926	0.0922	0.0956	0.0927
				***a* = 2, *b* = 2.5**						
			***α* = 5%**					***α* = 10%**		
10	0.0598	0.0590	0.0570	0.0560	0.0642	0.1184	0.1214	0.1133	0.1192	0.1267
25	0.0500	0.0470	0.0486	0.0486	0.0496	0.1048	0.0981	0.0991	0.1019	0.1034
50	0.0515	0.0479	0.0460	0.0460	0.0479	0.1021	0.0988	0.0951	0.0986	0.0985
100	0.0489	0.0468	0.0496	0.0494	0.0492	0.1010	0.0929	0.0946	0.0974	0.0957

Table 1 (continued): Simulated sizes of the EDF tests *,**

n	**K-S**	**Kuiper**	**C-M**	**Watson**	**A-D**	**K-S**	**Kuiper**	**C-M**	**Watson**	**A-D**
					***a* = 2.0, *b* = 5.0**					
			***α* = 5%**					***α* = 10%**		
10	0.0586	0.0597	0.0557	0.0550	0.0630	0.1184	0.1208	0.1130	0.1184	0.1269
25	0.0491	0.0463	0.0478	0.0486	0.0489	0.1055	0.0977	0.0983	0.1015	0.1032
50	0.0503	0.0472	0.0457	0.0445	0.0468	0.1003	0.0973	0.0931	0.0976	0.0976
100	0.0496	0.0464	0.0488	0.0474	0.0485	0.1000	0.0923	0.0930	0.0959	0.0941
					***a* = 2.0, *b* = 20.0**					
			***α* = 5%**					***α* = 10%**		
10	0.0576	0.0593	0.0550	0.0550	0.0620	0.1170	0.1205	0.1106	0.1172	0.1248
25	0.0473	0.0455	0.0474	0.0481	0.0476	0.1034	0.0969	0.0978	0.1018	0.1022
50	0.0492	0.0464	0.0442	0.0430	0.0456	0.0999	0.0963	0.0911	0.0974	0.0964
100	0.0485	0.0463	0.0469	0.0466	0.0459	0.0978	0.0923	0.0913	0.0943	0.0923
					***a* = 1.0, *b* = 3.0**					
			***α* = 5%**					***α* = 10%**		
10	0.0589	0.0592	0.0560	0.0553	0.0642	0.1189	0.1209	0.1131	0.1192	0.1273
25	0.0497	0.0464	0.0486	0.0486	0.0493	0.1051	0.0980	0.0993	0.1021	0.1037
50	0.0512	0.0477	0.0460	0.0455	0.0474	0.1014	0.0986	0.0948	0.0981	0.0985
100	0.0495	0.0470	0.0493	0.0487	0.0487	0.1002	0.0926	0.0940	0.0968	0.0955
					***a* = 0.5, *b* = 0.5**					
			***α* = 5%**					***α* = 10%**		
10	0.0625	0.0603	0.0574	0.0582	0.0679	0.1260	0.1199	0.1179	0.1203	0.1346
25	0.0592	0.0527	0.0619	0.0590	0.0643	0.1154	0.1071	0.1158	0.1168	0.1235
50	0.0650	0.0571	0.0689	0.0624	0.0742	0.1267	0.1165	0.1285	0.1238	0.1366
100	0.0864	0.0730	0.0940	0.0819	0.1017	0.1563	0.1356	0.1605	0.1509	0.1706
Crit.	0.895	1.489	0.126	0.117	0.752	0.819	1.386	0.104	0.096	0.631

* Crit. = Upper-tail critical values when the normal distribution's parameters are both estimated. Source: Stephens (1986, p.123), Table 4.7.

** K-S = Kolmogorov and Smirnov; C-M = Cramér and von Mises; A-D = Anderson and Darling.

Table 2: Simulated powers of the EDF tests *

n	K-S	Kuiper	C-M	Watson	A-D	K-S	Kuiper	C-M	Watson	A-D
					Triangular (mode = 1/4)					
			α = 5%					**α = 10%**		
10	0.0784	0.0752	0.0765	0.0754	0.0878	0.1439	0.1409	0.1426	0.1463	0.1604
25	0.0978	0.0868	0.1097	0.0987	0.1175	0.1766	0.1550	0.1857	0.1785	0.1988
50	0.1422	0.1218	0.1707	0.1498	0.1806	0.2397	0.2061	0.2597	0.2422	0.2742
100	0.2470	0.2121	0.3051	0.2639	0.3160	0.3740	0.3182	0.4157	0.3773	0.4333
250	0.5491	0.4885	0.6490	0.5831	0.6723	0.6762	0.6224	0.7564	0.7030	0.7712
500	0.8529	0.8178	0.9246	0.8865	0.9347	0.9196	0.8940	0.9573	0.9351	0.9637
1000	0.9929	0.9909	0.9978	0.9961	0.9986	0.9975	0.9963	0.9992	0.9986	0.9994
					Triangular (mode = 7/8)					
			α = 5%					**α = 10%**		
10	0.0676	0.0711	0.0660	0.0644	0.0755	0.1314	0.1330	0.1283	0.1353	0.1446
25	0.0736	0.0680	0.0822	0.0768	0.0866	0.1422	0.1294	0.1455	0.1462	0.1538
50	0.0969	0.0884	0.1144	0.1052	0.1233	0.1797	0.1626	0.1966	0.1854	0.2124
100	0.1575	0.1336	0.1928	0.1659	0.2191	0.2561	0.2294	0.2964	0.2703	0.3264
250	0.3588	0.3192	0.4530	0.3883	0.5122	0.5053	0.4532	0.5853	0.5231	0.6366
500	0.6568	0.6198	0.7771	0.6994	0.8386	0.7845	0.7446	0.8622	0.8127	0.9001
1000	0.9364	0.9310	0.9789	0.9603	0.9890	0.9730	0.9671	0.9904	0.9815	0.9954
					Truncated Log-Normal (meanlog = 0, sdlog = 1)					
			α = 5%					**α = 10%**		
10	0.0774	0.0758	0.0762	0.0738	0.0883	0.1458	0.1409	0.1420	0.1452	0.1614
25	0.1011	0.0862	0.1147	0.1003	0.1260	0.1836	0.1560	0.1918	0.1796	0.2139
50	0.1536	0.1212	0.1836	0.1548	0.2073	0.2533	0.2037	0.2842	0.2528	0.3148
100	0.2778	0.2193	0.3431	0.2785	0.3907	0.4153	0.3310	0.4672	0.4035	0.5182
250	0.6143	0.5376	0.7436	0.6318	0.8054	0.7399	0.6646	0.8319	0.7519	0.8795
500	0.9105	0.8815	0.9713	0.9290	0.9867	0.9600	0.9397	0.9870	0.9657	0.9936
1000	0.9981	0.9975	1.0000	0.9992	1.0000	0.9996	0.9994	1.0000	0.9999	1.0000

Table 2 (continued): Simulated powers of the EDF tests *

n	K-S	Kuiper	C-M	Watson	A-D	K-S	Kuiper	C-M	Watson	A-D
			Truncated Log-Normal (meanlog= 0.5, sdlog = 0.5)							
			***α* = 5%**					***α* = 10%**		
10	0.0398	0.0400	0.0375	0.0382	0.0432	0.0786	0.0757	0.0751	0.0765	0.0841
25	0.0653	0.0589	0.0672	0.0635	0.0718	0.1303	0.1167	0.1313	0.1272	0.1382
50	0.0798	0.0671	0.0868	0.0760	0.0956	0.1503	0.1290	0.1539	0.1459	0.1673
100	0.1164	0.0924	0.1349	0.1106	0.1474	0.1997	0.1679	0.2168	0.1946	0.2374
250	0.2427	0.1834	0.2966	0.2365	0.3371	0.3668	0.2937	0.4120	0.3540	0.4568
500	0.4474	0.3538	0.5422	0.4401	0.6156	0.5861	0.4880	0.6661	0.5742	0.7337
1000	0.7557	0.6784	0.8670	0.7678	0.9163	0.8594	0.7980	0.9255	0.8623	0.9582
			Truncated Normal (mean = 0.5, sd = 0.1)							
			***α* = 5%**					***α* = 10%**		
10	0.0704	0.0648	0.0686	0.0629	0.0799	0.1348	0.1253	0.1260	0.1287	0.1476
25	0.0839	0.0770	0.0952	0.0856	0.1076	0.1540	0.1368	0.1644	0.1517	0.1798
50	0.1188	0.0986	0.1395	0.1166	0.1607	0.2031	0.1698	0.2209	0.1991	0.2457
100	0.1928	0.1506	0.2309	0.1868	0.2682	0.3029	0.2497	0.3393	0.2945	0.3834
250	0.4263	0.3493	0.5179	0.4206	0.5840	0.5625	0.4782	0.6409	0.5542	0.6987
500	0.7151	0.6509	0.8345	0.7321	0.8822	0.8227	0.7661	0.8988	0.8336	0.9310
1000	0.9516	0.9408	0.9891	0.9675	0.9957	0.9813	0.9723	0.9954	0.9857	0.9984
			Truncated Normal (mean = 0.8, sd = 0.8)							
			***α* = 5%**					***α* = 10%**		
10	0.0577	0.0586	0.0547	0.0559	0.0647	0.1170	0.1166	0.1137	0.1163	0.1306
25	0.0519	0.0500	0.0529	0.0517	0.0540	0.1040	0.0981	0.1045	0.1080	0.1096
50	0.0544	0.0507	0.0584	0.0559	0.0573	0.1081	0.1003	0.1044	0.1063	0.1088
100	0.0558	0.0544	0.0558	0.0552	0.0574	0.1193	0.1075	0.1115	0.1126	0.1154
250	0.0681	0.0657	0.0720	0.0692	0.0765	0.1293	0.1244	0.1284	0.1241	0.1349
500	0.0873	0.0867	0.0999	0.0904	0.1099	0.1648	0.1578	0.1696	0.1628	0.1790
1000	0.1404	0.1300	0.1566	0.1410	0.1717	0.2318	0.2196	0.2476	0.2374	0.2695

Table 2 (continued): Simulated powers of EDF tests *

n	K-S	Kuiper	C-M	Watson	A-D	K-S	Kuiper	C-M	Watson	A-D
	Trapezoidal (m_1 = 1/8, m_2 = 3/8; n_1 = n_3 = 2)									
	α = 5%					**α = 10%**				
10	0.0636	0.0652	0.0606	0.0605	0.0691	0.1250	0.1249	0.1193	0.1254	0.1372
25	0.0632	0.0570	0.0651	0.0620	0.0730	0.1282	0.1169	0.1293	0.1261	0.1400
50	0.0771	0.0652	0.0830	0.0703	0.0968	0.1473	0.1275	0.1537	0.1417	0.1704
100	0.1124	0.0877	0.1292	0.1054	0.1531	0.1934	0.1616	0.2146	0.1889	0.2477
250	0.2306	0.1772	0.2885	0.2196	0.3537	0.3521	0.2873	0.4103	0.3433	0.4821
500	0.4214	0.3613	0.5405	0.4248	0.6512	0.5705	0.4937	0.6686	0.5650	0.7640
1000	0.7258	0.7011	0.8737	0.7625	0.9371	0.8409	0.8148	0.9322	0.8639	0.9694
	Trapezoidal (m_1 = 5/8, m_2 = 7/8; n_1 = n_3 = 2)									
	α = 5%					**α = 10%**				
10	0.0587	0.0618	0.0570	0.0575	0.0678	0.1221	0.1241	0.1172	0.1232	0.1331
25	0.0553	0.0518	0.0570	0.0572	0.0603	0.1136	0.1040	0.1121	0.1151	0.1176
50	0.0628	0.0577	0.0622	0.0580	0.0681	0.1185	0.1102	0.1231	0.1201	0.1309
100	0.0726	0.0637	0.0822	0.0740	0.0936	0.1439	0.1222	0.1477	0.1390	0.1635
250	0.1270	0.1021	0.1473	0.1203	0.1781	0.2181	0.1827	0.2343	0.2051	0.2822
500	0.2183	0.1782	0.2735	0.2157	0.3452	0.3429	0.2856	0.3934	0.3329	0.4788
1000	0.3973	0.3560	0.5320	0.418	0.6536	0.5543	0.5058	0.6674	0.5676	0.7692
	Trapezoidal (m_1 = 5/8, m_2 = 7/8; n_1 = n_3 = 2)									
	α = 5%					**α = 10%**				
10	0.0626	0.0732	0.0637	0.0687	0.0719	0.1271	0.1389	0.1291	0.1395	0.1439
25	0.0711	0.0862	0.0781	0.0858	0.0809	0.1402	0.1560	0.1517	0.1663	0.1576
50	0.0973	0.1285	0.1224	0.1313	0.1274	0.1860	0.2172	0.2143	0.2302	0.2213
100	0.1543	0.2168	0.2118	0.2317	0.2272	0.2707	0.3330	0.3306	0.3574	0.3496
250	0.3816	0.5086	0.5225	0.5503	0.5607	0.5477	0.6386	0.6588	0.6863	0.6912
500	0.7178	0.8402	0.8635	0.8791	0.8885	0.8447	0.9095	0.9238	0.9322	0.9389
1000	0.9719	0.9914	0.9938	0.9953	0.9969	0.9905	0.9971	0.9983	0.9989	0.9991

Table 2 (continued): Simulated powers of the EDF tests *

n	K-S	Kuiper	C-M	Watson	A-D	K-S	Kuiper	C-M	Watson	A-D
	Truncated Gamma (2, 3)									
	α = 5%					**α = 10%**				
10	0.2211	0.2728	0.2364	0.2498	0.3155	0.3266	0.3958	0.3626	0.3759	0.4511
25	0.5018	0.6293	0.5807	0.5813	0.7294	0.6489	0.7385	0.7025	0.7058	0.8269
50	0.8745	0.9442	0.9062	0.9023	0.9748	0.9391	0.9708	0.9516	0.9507	0.9890
100	0.9985	0.9998	0.9989	0.9983	1.0000	0.9997	0.9999	0.9996	0.9996	1.0000
250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Truncated Gamma (2, 6)									
	α = 5%									
	α = 10%									
10	0.1423	0.1321	0.1550	0.1423	0.1834	0.2234	0.2082	0.2346	0.2222	0.2713
25	0.2556	0.2388	0.3222	0.2818	0.3717	0.3662	0.3426	0.4142	0.3788	0.4710
50	0.4407	0.4211	0.5428	0.4775	0.6116	0.5646	0.5251	0.6368	0.5872	0.7005
100	0.7104	0.6954	0.8124	0.7516	0.8649	0.8072	0.7799	0.8703	0.8268	0.9083
250	0.9740	0.9743	0.9910	0.9830	0.9958	0.9878	0.9860	0.9956	0.9906	0.9977
500	0.9998	0.9997	1.0000	0.9997	1.0000	1.0000	0.9999	1.0000	0.9999	1.0000
1000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Truncated Weibull (shape = 2, scale = 1))									
	α = 5%					**α = 10%**				
10	0.0569	0.0553	0.0508	0.0524	0.0540	0.1138	0.1120	0.1052	0.1099	0.1086
25	0.0587	0.0537	0.0608	0.0588	0.0630	0.1192	0.1084	0.1156	0.1148	0.1266
50	0.0643	0.0580	0.0696	0.0644	0.0745	0.1243	0.1168	0.1275	0.1243	0.1345
100	0.0852	0.0702	0.0935	0.0807	0.1004	0.1508	0.1306	0.1563	0.1455	0.1701
250	0.1450	0.1108	0.1607	0.1345	0.1739	0.2354	0.1943	0.2535	0.2244	0.2744
500	0.2478	0.1900	0.2874	0.2381	0.3143	0.3634	0.2943	0.3938	0.3456	0.4239
1000	0.4308	0.3457	0.5167	0.4260	0.5630	0.5717	0.4757	0.6333	0.5563	0.6767

Table 2 (continued): Simulated powers of the EDF tests *

n	K-S	Kuiper	C-M	Watson	A-D	K-S	Kuiper	C-M	Watson	A-D
					Beta (3, 3)					
			α = 5%					**α = 10%**		
10	0.0545	0.0573	0.0539	0.0528	0.0606	0.1128	0.1115	0.1065	0.1105	0.1215
25	0.0506	0.0484	0.0534	0.0513	0.0542	0.1047	0.0982	0.1036	0.1054	0.1088
50	0.0527	0.0501	0.0565	0.0542	0.0569	0.1114	0.0975	0.1075	0.1073	0.1149
100	0.0577	0.0532	0.0600	0.0586	0.0629	0.1215	0.1112	0.1186	0.1160	0.1232
250	0.0790	0.0641	0.0821	0.0706	0.0897	0.1492	0.1266	0.1492	0.1369	0.1571
500	0.1151	0.0863	0.1148	0.0964	0.1302	0.1939	0.1603	0.1947	0.1714	0.2145
1000	0.1847	0.1339	0.2065	0.1586	0.2340	0.2919	0.2257	0.3078	0.2603	0.3401
					Beta (20, 20)					
			α = 5%					**α = 10%**		
10	0.0710	0.0670	0.0689	0.0664	0.0790	0.1344	0.1288	0.1296	0.1325	0.1493
25	0.0892	0.0765	0.0974	0.0858	0.1100	0.1608	0.1397	0.1738	0.1610	0.1941
50	0.1340	0.1048	0.1629	0.1347	0.1808	0.2313	0.1871	0.2501	0.2235	0.2810
100	0.2395	0.1784	0.2922	0.2287	0.3373	0.3682	0.2842	0.4157	0.3494	0.4655
250	0.5380	0.4486	0.6598	0.5398	0.7355	0.6814	0.5858	0.7703	0.6677	0.8338
500	0.8580	0.8040	0.9427	0.8691	0.9692	0.9295	0.8896	0.9719	0.9291	0.9875
1000	0.9925	0.9904	0.9995	0.9953	0.9999	0.9981	0.9968	0.9997	0.9987	0.9999
					Beta (4, 2)					
			α = 5%					**α = 10%**		
10	0.0562	0.0588	0.0562	0.0578	0.0630	0.1155	0.1153	0.1102	0.1138	0.1229
25	0.0509	0.0532	0.0507	0.0503	0.0522	0.1003	0.1019	0.1009	0.1046	0.1072
50	0.0542	0.0516	0.0555	0.0544	0.0574	0.1080	0.0982	0.1050	0.1071	0.1100
100	0.0549	0.0540	0.0562	0.0545	0.0559	0.1079	0.1038	0.1074	0.1076	0.1076
250	0.0672	0.0613	0.0641	0.0579	0.0649	0.1237	0.1134	0.1186	0.1156	0.1232
500	0.0740	0.0647	0.0761	0.0674	0.0795	0.1414	0.1245	0.1346	0.1283	0.1433
1000	0.1048	0.0787	0.1085	0.0905	0.1167	0.1866	0.1489	0.1808	0.1635	0.1956

Table 2 (continued): Simulated powers of the EDF tests *

n	K-S	Kuiper	C-M	Watson	A-D	K-S	Kuiper	C-M	Watson	A-D
					Beta (2, 4)					
			α = 5%					α = 10%		
10	0.0567	0.0584	0.0533	0.0557	0.0601	0.1118	0.1127	0.1075	0.1131	0.1194
25	0.0521	0.0544	0.0553	0.0539	0.0575	0.1053	0.1032	0.1066	0.1069	0.1120
50	0.0571	0.0539	0.0579	0.0534	0.0587	0.1108	0.1036	0.1121	0.1117	0.1180
100	0.0599	0.0563	0.0631	0.0599	0.0674	0.1168	0.1057	0.1164	0.1116	0.1238
250	0.0799	0.0671	0.0782	0.0689	0.0840	0.1455	0.1262	0.1436	0.1335	0.1569
500	0.1066	0.0835	0.1156	0.0973	0.1254	0.1896	0.1540	0.1897	0.1686	0.2060
1000	0.1720	0.1248	0.1855	0.1498	0.2118	0.2744	0.2068	0.2879	0.2426	0.3124
					Beta (3, 20)					
			α = 5%					α = 10%		
10	0.0580	0.0585	0.0562	0.0550	0.0651	0.1180	0.1170	0.1125	0.1147	0.1297
25	0.0662	0.0581	0.0688	0.0640	0.0738	0.1270	0.1182	0.1287	0.1243	0.1413
50	0.0833	0.0678	0.0879	0.0781	0.0953	0.1516	0.1291	0.1574	0.1452	0.1677
100	0.1182	0.0932	0.1374	0.1117	0.1504	0.2104	0.1643	0.2211	0.1936	0.2384
250	0.2426	0.1747	0.2923	0.2226	0.3365	0.3674	0.2825	0.4119	0.3431	0.4621
500	0.4480	0.3450	0.5458	0.4331	0.6145	0.5923	0.4819	0.6718	0.5656	0.7343
1000	0.7586	0.6753	0.8664	0.7577	0.9139	0.8600	0.7928	0.9258	0.8546	0.9548
					Beta (0.5, 0.5)					
			α = 5%					α = 10%		
10	0.0547	0.0557	0.0482	0.0491	0.0577	0.1154	0.1121	0.1053	0.1129	0.1224
25	0.0500	0.0506	0.0529	0.0531	0.0542	0.1067	0.0997	0.1073	0.1120	0.1131
50	0.0500	0.0501	0.0532	0.0505	0.0541	0.1048	0.1042	0.1052	0.1102	0.1070
100	0.0538	0.0533	0.0553	0.0542	0.0585	0.1064	0.1056	0.1070	0.1069	0.1086
250	0.0663	0.0615	0.0675	0.0631	0.0702	0.1263	0.1165	0.1235	0.1210	0.1259
500	0.0834	0.0726	0.0853	0.0772	0.0898	0.1551	0.1363	0.1530	0.1457	0.1600
1000	0.1192	0.0993	0.1275	0.1137	0.1363	0.1983	0.1750	0.2071	0.1895	0.2232

* For the truncated log-normal distribution, "meanlog" = mean of the distribution of the non-truncated random variable on the log scale.; "sdlog" = standard deviation of the distribution of the non-truncated random variable on the log scale.

For the trapezoidal distribution, m_1 = mode 1, m_2 = mode 2; n_1 = growth parameter, n_3 = decay parameter.

Figure 1: Kumaraswamy densities

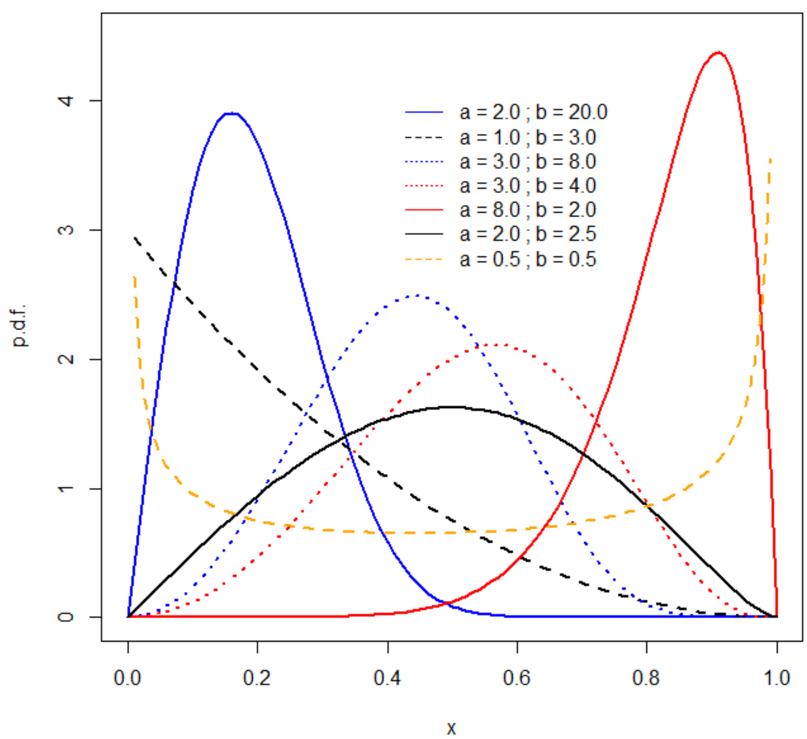


Figure 2: Beta densities

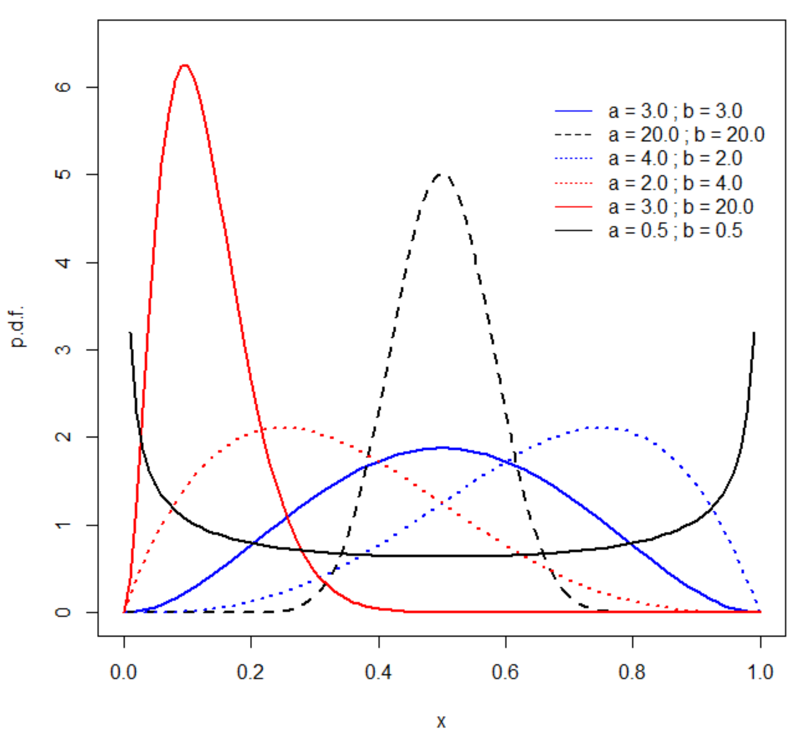
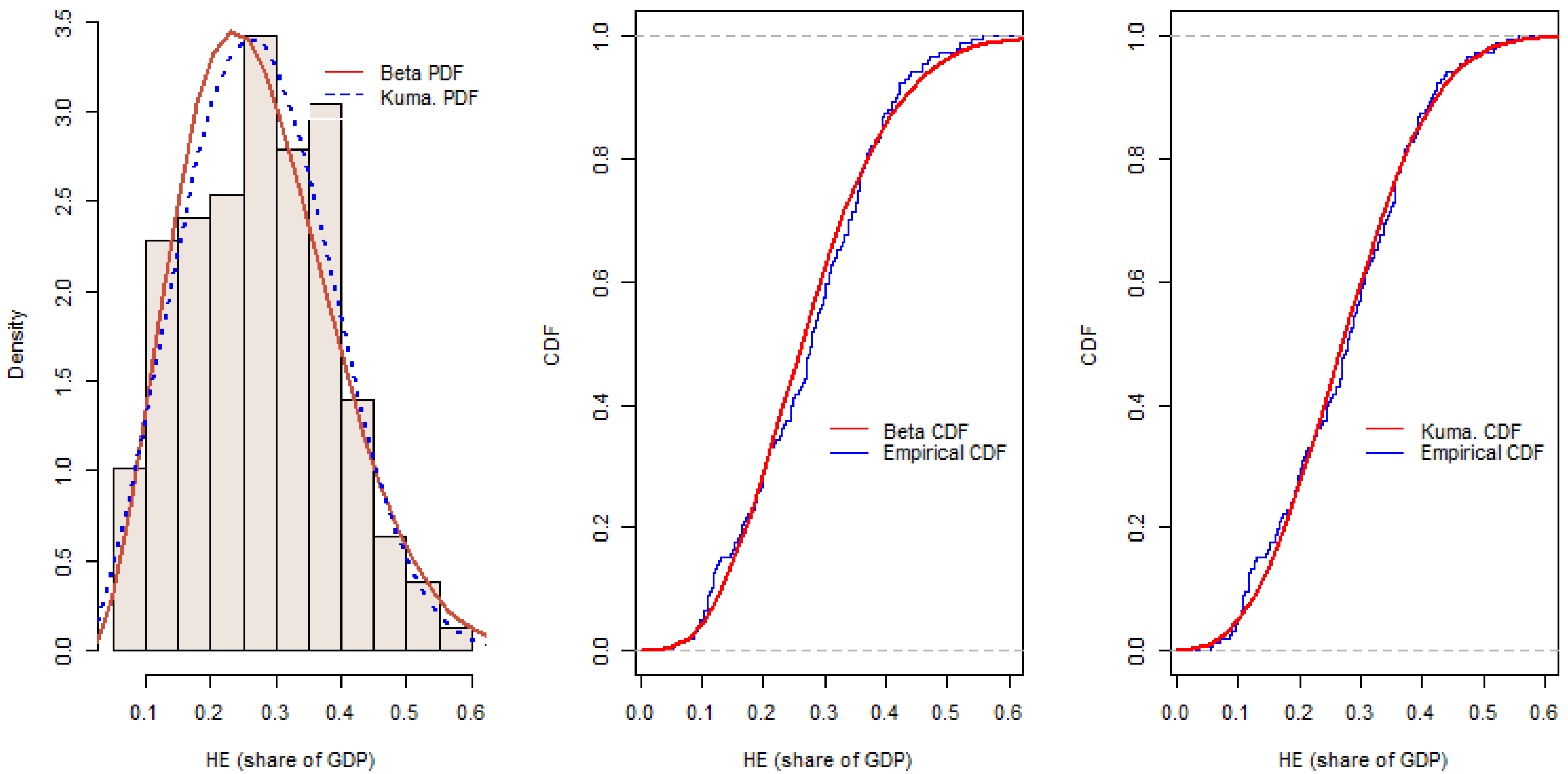


Figure 3 (a): Hidden Economy Densities **Figure 3 (b): Distribution Functions** **Figure 3 (c): Distribution Functions**



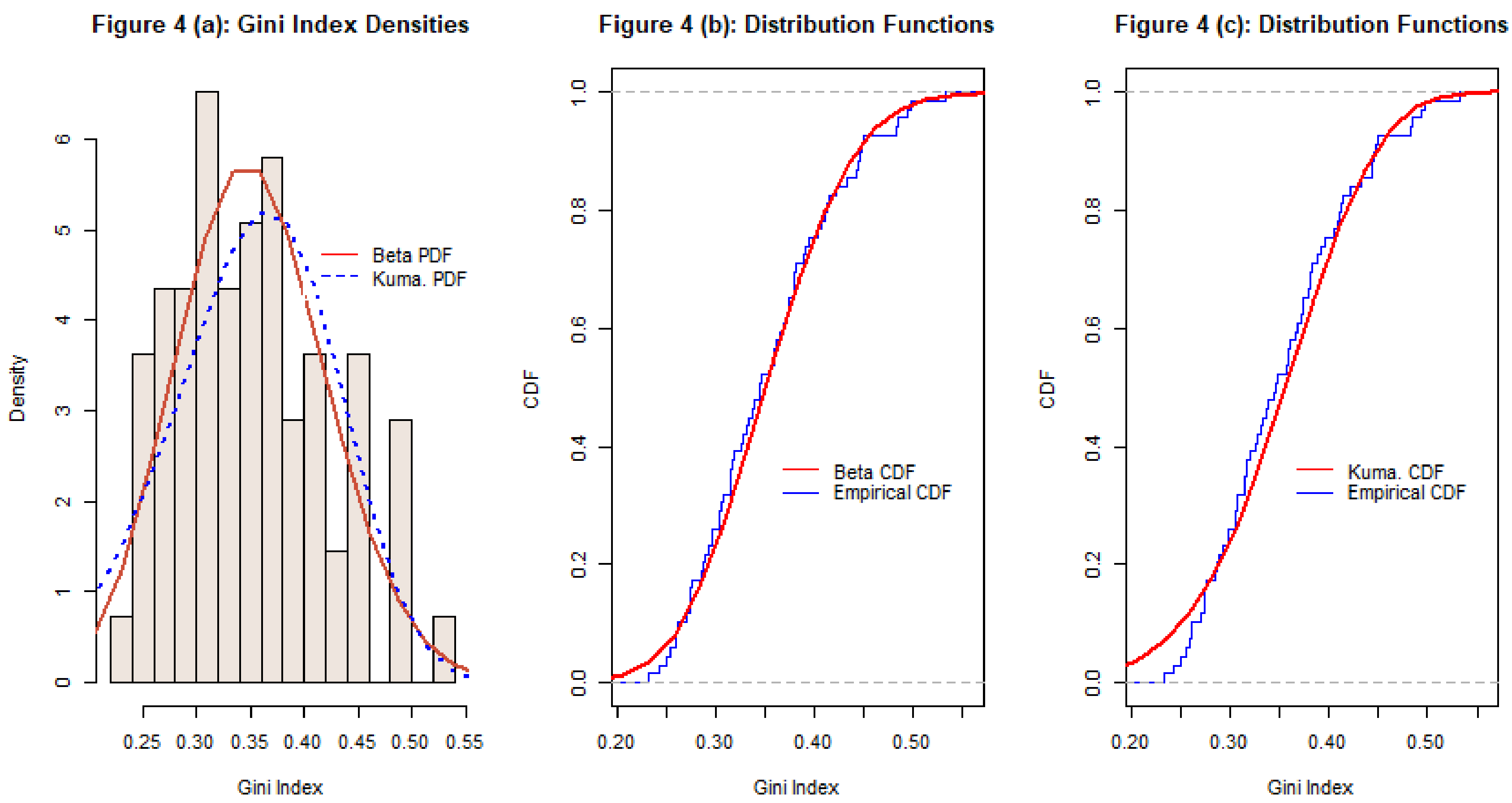


Figure 4 (a): Gini Index Densities
Density
Gini Index
Beta PDF
Kuma. PDF
0.25 0.30 0.35 0.40 0.45 0.50 0.55
0 1 2 3 4 5 6
Figure 4 (b): Distribution Functions
CDF
Gini Index
Beta CDF
Empirical CDF
0.20 0.30 0.40 0.50
0.0 0.2 0.4 0.6 0.8 1.0
Figure 4 (c): Distribution Functions
CDF
Gini Index
Kuma. CDF
Empirical CDF
0.20 0.30 0.40 0.50
0.0 0.2 0.4 0.6 0.8 1.0

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