Re-interpreting the "Least Squares Ratio" Regression Estimator

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Abstract:

The Ratio Least Squares (RLS) regression estimator is obtained by minimizing the sum of squared "relative" residuals, rather than minimizing the sum of squared differences between the actual and "fitted" values of the dependent variable. We show that this estimator is simply the weighted least squares estimator for the case where the error variance is proportional to the square of the dependent variable. This equivalence explains the robustness of the RLS estimator to sample outliers. It also implies that much of the literature associated with the RLS estimator lacks novelty.

Key Words: Regression analysis; least squares ratio estimator; generalized least squares

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1. Introduction

There are numerous alternative estimators for the coefficient vector of a linear regression model. The properties of these estimators, and the resulting forecasts from the model, depend on what conditions are satisfied by the model's specification. For example, the Ordinary Least Squares (OLS) is "best linear unbiased" and consistent if the regressors are uncorrelated with the errors, and if the errors themselves have a zero mean, are serially independent, and are homoskedastic. On the hand, if the regressors are random and correlated with the errors, then the consistent Instrumental Variables (IV) estimator may be chosen in favour of the (inconsistent) OLS estimator. A Quantile Regression (QR) estimator may be preferred if the sample includes significant outliers.

It is noteworthy that many regression coefficient estimators are special cases of their alternatives. For example, the OLS estimator is algebraically identical to the IV estimator if the instrument matrix is identical to the regressor matrix. The Generalized Least Squares (GLS) estimator is identical to the Weighted Least Squares (WLS) estimator if the covariance matrix of the error vector is diagonal, and equal to the OLS estimator if this matrix is scalar. When a new estimator is proposed, and is claimed to be "novel", it is important to ask if this estimator is actually related to one or more existing estimators. Among other things, if such a relationship exists, this may provide us with some results about the estimator's properties, based on the existing literature.

Akbilgic and Akinci (2009) proposed an alternative to the OLS estimator of the regression coefficient vector that is based on the ratio of the actual and fitted values of the model's dependent variable, rather than the difference between these values. Let the regression model be:

$$y = X\beta + \varepsilon \quad ; E(\varepsilon) = 0; \; V(\varepsilon) = \sigma^2 I_n \tag{1}$$

where X is $(n \times k)$, β is $(k \times 1)$, y and ε are $(n \times 1)$, and the columns of X are non-random. Denote the predicted value of the y vector by $\hat{y}$. Then, while the OLS estimator of β is obtained by minimizing $\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$, the Least Squares Ratio (LSR) estimator is obtained by minimizing $\sum_{i=1}^{n}(\frac{y_i - \hat{y}_i}{y_i})^2$. Consequently, while the OLS estimator is $b = (X'X)^{-1}X'y$, the LSR estimator takes the form:

$$\hat{\beta}_{LSR} = \left[\left(\frac{X}{y}\right)'\left(\frac{X}{y}\right)\right]^{-1}\left(\frac{X}{y^2}\right)'y \tag{2}$$

In (2), the matrix $(\frac{X}{y})$ is defined by dividing the values X_{ij} by y_i , for $i = 1, 2, \ldots, n$, and for $j = 1, 2, \ldots, k$; and the matrix $\left(\frac{X}{y^2}\right)$ is obtained by dividing the values X_{ij} by y_i^2, again for $i = 1, 2, \ldots, n$, and for $j = 1, 2, \ldots, k$.

Based on a simulation experiment, Akbilgic and Akinci (2009) found that in the absence of outliers in the sample, the OLS estimator is somewhat superior to the LSR estimator, in terms of mean squared error. However, when outliers are present the LSR estimator was "consistently superior" to OLS in this respect. Similar results were reported by Güneri and Göktaş (2017).

2. WLS interpretation

The algebraic expression for the LSR estimator is, in fact, just a particular WLS estimator. To see this, consider the following alternative formulation of the underlying model:

$$y = X\beta + \varepsilon \quad ; \; E(\varepsilon) = 0; \; V(\varepsilon) = \sigma^2 diag.(y_i^2) \tag{3}$$

where the $(n \times n)$ error covariance matrix takes the form, $\sigma^2 diag.(1/w_i)$, and $w_i = (\frac{1}{y_i^2})$. If we define the $(n \times n)$ matrix, $W = diag.(w_i)$, then the WLS estimator of the coefficient vector is:

$$\hat{\beta}_{WLS} = (X'WX)^{-1}X'Wy. \tag{4}$$

$$= \left[\left(W^{\frac{1}{2}}X\right)'\left(W^{\frac{1}{2}}X\right)\right]^{-1}(X'W)y$$

$$= \left[\left(\frac{X}{y}\right)'\left(\frac{X}{y}\right)\right]^{-1}\left(\frac{X}{y^2}\right)'y$$

$$= \hat{\beta}_{LSR} \quad . \tag{5}$$

So, the LSR estimator for the standard regression model in (1) is simply the WLS (and GLS) estimator for the case where the model's error term is serially independent, but heteroskedastic with the variance of each element of the error term proportional to the square of the corresponding value of the dependent variable. So, although the error covariance matrix is observable, it is also random, and this has some implications that are discussed in the next section**.**

3. Discussion

First, it is interesting to note that the LSR estimator is essentially the same as the "Least Squares Percentage Regression" estimator proposed by Tofallis (2008). Further, given the equivalence of the LSR estimator and this particular WLS estimator, the results in the literature relating to the robustness of LSR to outliers in the sample data are hardly surprising. Outliers in the data increase the variance of the dependent variable (and hence the variance of the errors) in proportion to some function of the values of the y_i data. This is taken into account explicitly when this WLS estimator is applied, while it is ignored when the OLS estimator is used. In this respect the findings of Akbilgic and Akinci (2009) and Güneri and Göktaş (2017) are just as would have been expected had the equivalence of the LSR and WLS estimators been understood.

As the LSR estimator is already a WLS estimator designed to handle a specific form of heteroskedasticity, it is somewhat ironic that various authors (including Bhatti *et al*. (2023), Satyanarayana and Ismail (2023), and Zafar and Aslam (2023)) have discussed "extensions" of the LSR estimator to allow heteroskedastic errors. Essentially, these studies simply consider regression models with errors that have a more complex heteroskedastic structure than they actually discuss. In the same vein, the estimator discussed by Satyanarayana *et al.* (2025), merely allows for autorcorrelation, and that proposed by Yazici (2016) deals with M-estimation, each in the context of a model with heteroskedastic errors. Similarly, the ridge and Liu estimators discussed by Jadhav and Kashid (2018) and Giresunlu *et al*. (2024) are just standard variants of these estimators for a (particular) heteroskedastic model.

The equivalence of LSR and WLS implies that other algebraic results follow immediately. One example is the goodness-of-fit of the model under LSR estimation. Define $SSR = \sum_{i=1}^{n} w_i(y_i - \hat{y}_i)^2$ and $SST = \sum_{i=1}^{n} w_i(y_i - \bar{y}_W)^2$, where $\bar{y}_W = (\sum_{i=1}^{n} w_i y_i)/(\sum_{i=1}^{n} w_i)$ and $\hat{y}_i$is the i'th. element of $\hat{y} = X\hat{\beta}_{LSR}$. Then, using the result that $w_i = (1/y_i^2)$, for i = 1, 2, …, n, the coefficient of determination under LSR estimation is:

$$
\begin{aligned}
R^2 &= 1 - SSR/SST \\
&= 1 - \sum_{i=1}^{n} \left[\frac{(y_i - \hat{y}_i)}{y_i}\right]^2 / \sum_{i=1}^{n} [(y_i - \bar{y}_W)/y_i]^2 \qquad , \qquad (6)
\end{aligned}
$$

where $\bar{y}_W = \sum_{i=1}^{n}(1/y_i) / \sum_{i=1}^{n}(1/y_i^2)$.

Interestingly, the theoretical literature associated with the LSR estimator focuses entirely on point estimation and point forecasts. While the various simulation studies discussed in that literature provide evidence on the mean squared error and mean absolute error of the coefficient estimator, there is no discussion of the formula for the covariance matrix of $\hat{\beta}_{LSR}$, or for the standard errors. This is understandable, as this estimator is a non-linear function of the random *y* vector. Usually, the covariance matrix for the WLS estimator would take the form, $V(\hat{\beta}_{WLS}) = \sigma^2(X'WX)^{-1}$. However, this is not the case here as the derivation of this formula requires that *W* is *non-random*.

Although an analytical expression for the standard errors of the coefficient estimates is not readily obtainable, the non-parametric bootstrap can be used to construct standard errors that are valid in finite samples. To simplify the notation, define $\hat{\beta} = \hat{\beta}_{LSR} = \hat{\beta}_{WLS}$, and let $\hat{\beta}_j$ be the *j*'th. element of this vector (*j* = 1, 2, …, *k*). If there are *B* bootstrap samples (of size *n*), and $\hat{\beta}_j^{(p)}$ is the estimate of β_j based on the *p*'th. bootstrap sample, then the bootstrap standard error of $\hat{\beta}_j$ is obtained as:

$$s.e.(\hat{\beta}_j) \approx \sqrt{\frac{1}{(B-1)}\left(\sum_{m=1}^{B} \hat{\beta}_j^{(m)} - \frac{1}{B}\sum_{r=1}^{B} \beta_j^{(r)}\right)^2} \ . \qquad (7)$$

In summary, the so-called Least Squares Ratio regression estimator is simply a specific example of the Weighted Least squares estimator. Consequently, the associated literature lacks much of the novelty that is claimed, and the finding that the LSR estimator is robust to sample outliers is a natural consequence of the form of its WLS interpretation.

Disclosure of interest

The author reports there are no competing interests to declare.

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