Testing for Size-Biased Sampling in Lifetime Data

David E. Giles

University of Victoria, CANADA

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Abstract:

Size-biased sampling is a phenomenon that arises in many disciplines and it is common with “survival”, or “lifetime” data. This type of sampling occurs when the probability of a population element being sampled is proportional to some weight function that depends on the observed value of that element. This alters the assumed (“base”) distribution of the sample data to a “weighted distribution”. In practice, it is often not known if size-biased sampling has been used, so there is a need for formal testing to select between the “base” distribution and the “weighted” distribution. We consider two variants of a particular such test in the context of distributions from the generalized gamma family. A Monte Carlo simulation experiment is used to generate critical values for the tests, and to evaluate their power properties.

Key Words: Size-biased sampling; hypothesis testing; generalized gamma distribution; power

Author contact:

58 Rock Lake Court, RR1, Bancroft, Ontario K0L 1C0, CANADA; dgiles@uvic.ca ; +1-613-202 9501

(*The author is Professor Emeritus, Department of Economics, University of Victoria, CANADA*.)

1. Introduction

Knowledge of the method by which data have been sampled from a population is of crucial importance in any statistical analysis. Many standard inferential procedures are based on the assumption of simple random sampling, and if a different sampling scheme has been adopted this may have significant implications for the validity and properties of inferences that are drawn from the data. One important "non-standard" sampling scheme that has been studied for the past century is the so-called size-biased sampling. For example, see the seminal contributions of Wicksell (1925), Fisher (1934), Rao (1965), and Cox (1969). This form of sampling arises when the probability of a population element being selected for the sample is proportional to some predetermined weight function, where the latter depends on the observed value of the data for that element.

For example, if the probability of a certain sample value being recorded is directly proportional to the magnitude of that value, we have so-called "length-biased" sampling. If the probability is proportional to the square of that magnitude we have "area-biased" sampling, and so on. Size-biased sampling occurs in many fields, including epidemiology, ecology, environmental and resource economics, forestry, genome mapping, marketing, meteorology, and in a range of reliability and survival studies. For example, see Simon (1980), Nowell *et al.* (1988), Nowell and Stanley (1991), Salant (1997), Gove (2000, 2003a, 2003b), Das and Roy (2011a, 2011b), Beghriche and Zeghdoudi (2019), among others.

In any empirical analysis it is important to know if size-biased sampling has occurred, because it affects the formulation of the correct data density function, and hence the likelihood function. If size-biased sampling has been used, but this is either unknown or ignored, the likelihood function is mis-specified and any inferences that are based upon it will be incorrect, *even asymptotically*. This applies to both frequentist and Bayesian inference. To ensure valid inferences the appropriate "weighted density" (or "moment density") is required as the foundation for the likelihood function. A detailed discussion of this issue is provided by Patil and Ord (1976), for example.

One practical problem that arises in this context is that the analyst may not know whether or not size-biased sampling has been used, as it is not part of any pre-specified experimental design. Rather, it is a characteristic of the context of the sampling process. So, in order to ensure valid subsequent inferences, one might consider the application of some test of the hypothesis, H_0: "Unweighted (simple) random sampling has been used", against the *simple* alternative hypothesis, H_1: "A *specific* weighted (size-biased) sampling has been used". Several parametric and nonparametric tests of H_0 against H_1 have been suggested in the literature. For example, see Navarro and del Aguila (2003), Akman *et al.* (2007), and Economou and Tzavelas (2013). In this paper, we focus on the distribution-free test of the above hypotheses proposed by

Akman *et al.* (2007), and extended by Economou and Tzavelas (2013). The latter authors also consider both likelihood ratio and Wald tests of H_0 against the *composite* alternative hypothesis, H_A: "Some *general* size-biased sampling has been used". However, they find the latter tests to be substantially inferior in terms of power to the test against the simple alternative, H_1, and this motivates the present study.

In the next section we outline the statistical framework that is adopted in this paper, including the formal details of two hypothesis tests that are evaluated, and the base distributions that are considered. These distributions are all ones that are employed widely in the analysis of lifetime (duration) data. Section 3 provides the details of an extensive simulation experiment that we have conducted, first to compute critical values for the tests under a wide range of situations, and second to evaluate the powers of the tests. The latter are presented graphically in section 3, and the critical values are tabulated for various sample sizes in the Appendix. Some concluding comments are provided in section 4.

2. Statistical framework

If a random variable, X, has a density function $f(x;\phi)$, for some parameter (vector), ϕ, then the corresponding "weighted" (or "moment") density, to allow for size-biased sampling of order c, is defined as

$$f_c(x;\phi) = f(x;\phi)x^c/m'_c \qquad (1)$$

where $m'_c = E[X^c]$ is the c^{th} (raw) moment of X. This moment is assumed to exist, so clearly (1) is a proper density. We will refer to $f(x;\phi)$ as the "unweighted", or "base" density function of X.

The case $c = 1$ corresponds to "length biased" sampling; $c = 2$ corresponds to "area biased", *etc*. We will be concerned only with length-biased and area-biased sampling in this paper, as these are the most common forms of size-biased sampling that arise in practice. We can re-phrase our testing problem as one involving the simple null hypothesis, H_0: "the relevant density is $f(x;\phi)$"; *versus* the simple alternative hypothesis, H_1: "the relative density is $f_c(x;\phi)$, for a known c". Or, for example, $H_0: c = 0$ *vs*. $H_0: c = 1$, in the case of potential length-biased sampling.

The test statistic that we consider is based on the quantity

$$\prod_{i=1}^{n}[f_c(x_i;\phi)/f(x_i;\phi)] \quad , \qquad (2)$$

a sufficiently large value of which will suggest that size-biased sampling has been used. Using the definition of $f_c(x_i; \phi)$ in (1), an equivalent test procedure is to reject H_0 in favour of size-biased sampling if the statistic:

$$\lambda = [\prod_{i=1}^{n} x_i]^{1/n} / (m_c')^{1/c} \tag{3}$$

is sufficiently large. However, the statistic in (3) will usually be unobservable, because typically m_c' will be a function of one or more of the elements of the parameter vector, ϕ. In this paper we compare the merits of two ways of dealing with this issue. First, following Economou and Tzavelas (2013), we consider the statistic:

$$\hat{\lambda}_c = [\prod_{i=1}^{n} x_i]^{1/n} / (s_c')^{1/c} \quad , \tag{4}$$

where $s_c' = (1/n) \sum_{i=1}^{n} x_i^c$ is the c^{th} (raw) *sample* moment. Alternatively, following the approach of Giles (2025) we can define the statistic:

$$\tilde{\lambda}_c = [\prod_{i=1}^{n} x_i]^{1/n} / (\tilde{m}_c')^{1/c} \quad , \tag{5}$$

where $\tilde{m}_c'$ is the MLE of m_c', obtained by replacing the relevant elements of the parameter vector, ϕ, with maximum likelihood estimates based on the (unweighted) density, $f(x; \phi)$. As is discussed below, in some situations the two test statistics will coincide. However, in general they may differ and the relative performances of the tests are of interest.

Asymptotically (as $n \to \infty$) these two versions of the test statistic will be identical due to the weak consistency of both method of moments and maximum likelihood estimation. Economou and Tzavelas (2013) establish the asymptotic normality of $\hat{\lambda}$, and so asymptotically $\tilde{\lambda}$ will also follow a (different) normal distribution. However, the two test statistics may differ in value (and in distribution) in finite samples, in which case different critical values will apply. These critical values are obtained by simulation, as is described in the next section. They will be labelled $\hat{k}_c(\alpha)$ and $\tilde{k}_c(\alpha)$ respectively, for a particular choice of c, and a significance level of α. Accordingly, the powers of the tests based on (4) and (5) need to be evaluated separately for any given situation. The asymptotic efficiency of a maximum likelihood estimator is never less than that of its method of moments counterpart, but we cannot generalize about the relative performances in finite samples. Accordingly, while one might anticipate that the $\tilde{\lambda}$ test may out-perform the $\hat{\lambda}$ for some range of sample sizes, this cannot be guaranteed. We explore this in the present study.

All of the above discussion relates to a specific base (unweighted) distribution, $f(x;\phi)$. As this base distribution changes, so do the specific forms and distributions of $\hat{\lambda}_c$ and $\hat{\lambda}_c$; the values of the associated critical values; and the powers of the tests. In this paper we consider a range of base distributions that are special cases of the so-called "generalized gamma distribution" (Stacy, 1962), and are commonly used with lifetime (duration) data. The density function for the generalized gamma distribution is:

$$f(x;a,d,p) = \left(\frac{p}{a^d}\right) x^{d-1} e^{-\left(\frac{x}{a}\right)^p} / \Gamma(\frac{d}{p}) \quad ; \quad x > 0 \, ; \quad a, d, p > 0 \tag{6}$$

and its c^{th} raw moment is:

$$E[X^c] = a^c \Gamma((d+c)/p)/\Gamma(d/p) \quad . \tag{7}$$

So, the size-biased (weighted) counterpart to (6) takes the form

$$f_c(x;a,d,p) = \left(\frac{p}{a^{d+c}}\right) x^{d+c-1} e^{-\left(\frac{x}{a}\right)^p} / \Gamma(\frac{d+c}{p}) \quad ; \quad x > 0 \, ; \quad a, d, p, c > 0 \tag{8}$$

We see from (6) and (8) that the generalized gamma distribution is "form-invariant" under weighted sampling. That is, the weighted distribution also belongs to the generalized gamma family. Specifically, (8) is a generalized gamma density with parameters a, $(d + c)$, and p. See Patil and Ord (1976), and Ducey and Gove (2015, p.124) for further discussion of "form-invariance". The particular members of the generalized gamma family that we consider are the exponential, gamma, half-normal, Rayleigh, and Weibull distributions. These correspond to the generalized gamma distribution for the special cases of $(d = p = 1, a = 1/\theta)$, $(d = b, p = 1, a = 1/\theta)$, $(d = 1, p = 2, a = \sqrt{2\sigma^2})$, $(d = p = 2, a = \sqrt{2\sigma^2})$, and $(d = p = k, a = \theta)$. Size-biased variants of each of these distributions have been discussed by numerous authors, including Blumenthal (1967), Scheaffer (1972), Nowell *et al*. (1988), Jabeen and Jan (2015), Gove (2000, 2003a, 2003b), Ducey (2009), Das and Roy (2011a, 2011b), Mir *et al*. (2013), Tzavelas and Panagiotakos, (2013), Ajami and Jahanshahi (2016), Bashir and Rasul (2018a, 2018b), Perveen and Ahmad (2018), and Giles (2021).

The base density functions and the first two raw population moments for each of these five distributions we are considering are summarized in Table 1, together with the associated size-biased densities. From that information it follows that the length-biased and area-biased counterparts to the exponential distribution with a rate parameter θ are gamma distributions with the same rate parameter, and shape parameters $b = 2$ and $b = 3$ respectively. Further, the length-weighted half-normal distribution is a Rayleigh distribution.

From the base density moments shown in that table we also see that $\tilde{\lambda}_c = \hat{\lambda}_c$ when $c = 1$ for the exponential distribution, and when $c = 2$ for each of the half-normal and Rayleigh distributions.

Table 1: Distributions and moments

Distribution	Density functions	$\boldsymbol{m_1'}$ $[\boldsymbol{\tilde{m}_1'}]$	$\boldsymbol{m_2'}$ $[\boldsymbol{\tilde{m}_2'}]$
Exponential	$f(x;\theta) = \theta e^{-\theta x}$	$1/\theta$	$2/\theta^2$
		$[s_1' = \bar{x}]$	$[2(s_1')^2 = 2\bar{x}^2]$
	$f_c(x;\theta) = \theta^{c+1} x^c e^{-\theta x}/\Gamma(c+1)$		
Half-normal	$f(x;\sigma) = (1/\sigma)\sqrt{(2/\pi)}e^{-x^2/(2\sigma^2)}$	$\sigma\sqrt{2/\pi}$	σ^2
		$\left[\sqrt{2s_2'/\pi}\right]$	$[s_2' = 1/n\sum_{i=1}^n x_i^2]$
	$f_c(x;\sigma) = \left(\frac{1}{\sigma}\right)^{c+1} 2^{0.5-c} x^c e^{-\frac{x^2}{2\sigma^2}}/\Gamma(\frac{c+1}{2})$		
Rayleigh	$f(x;\sigma) = (x/\sigma^2)e^{-x^2/(2\sigma^2)}$	$\sigma\sqrt{\pi/2}$	$2\sigma^2$
		$[\tilde{\sigma}\sqrt{\pi/2}]$	$[s_2' = 1/n\sum_{i=1}^n x_i^2]$
	$f_c(x;\sigma) = \left(\frac{1}{\sigma}\right)^{c+2} 2^{-\frac{c}{2}} x^{c+1} e^{-\frac{x^2}{2\sigma^2}}/\Gamma(1+\frac{c}{2})$		
Gamma	$f(x;b,\theta) = x^{b-1}e^{-\theta x}\theta^b/\Gamma(b)$	b/θ	$b(b+1)/\theta^2$
		$[s_1' = \bar{x}]$	$\tilde{b}(\tilde{b}+1)/\tilde{\theta}^2$
	$f_c(x;b,\theta) = x^{b+c-1}e^{-\theta x}\theta^{b+c}/\Gamma(b+c)$		
Weibull	$f(x;k,\sigma) = \left(\frac{k}{\sigma}\right)(x/\sigma)^{k-1}e^{-(x/\sigma)^k}$	$\sigma\Gamma(1+1/k)$	$\sigma^2\Gamma(1+2/k)$
		$[\tilde{\sigma}\Gamma(1+1/\tilde{k})]$	$[\tilde{\sigma}^2\Gamma(1+2/\tilde{k})]$
	$f(x;k,\sigma) = \left(\frac{k}{\sigma}\right)\left(\frac{x}{\sigma}\right)^{k+c-1} e^{-\left(\frac{x}{\sigma}\right)^k}/\Gamma(1+\frac{c}{k})$		

Note: In all cases, $x > 0$; and all of the parameters are positive. $\tilde{m}_1', \tilde{m}_2', \tilde{b}, \tilde{\theta}, \tilde{\sigma}$, and $\tilde{k}$, are maximum likelihood estimators based on the likelihood function constructed from the associated base (unweighted) density.

3. A simulation experiment

All of the computations in this study were undertaken with the R statistical software (R Core Team, 2021). The various random variates were generated using the 'ggamma' package (Saldanha and Suzuki, 2022). In addition, the 'uniroot' command in the base 'stats' package was used to solve the likelihood equations in those cases where the maximum likelihood estimates have to be obtained numerically. The R code can be downloaded from https://github.com/DaveGiles1949/r-code.

3.1 Critical values

We have conducted an extensive Monte Carlo experiment to establish the finite-sample distributions of each of the test statistics for each null distribution, in a variety of situations. The upper percentiles of these distributions provide the critical values needed to apply the tests for size-biased sampling. The exponential, half-normal, and Rayleigh distributions have a single (scale or rate) parameter, and the MLE's of this parameter can be expressed in closed form in each case. For these null distributions, the upper percentiles of the distributions of $\hat{\lambda}_c$ and $\tilde{\lambda}_c$ are based on 100,000 Monte Carlo replications and these values are independent of the value of the scale or rate parameter. The results appear in Appendix Tables A.1 to A.3 for c = 1 and 2, various sample sizes, and significance levels of α = 1%, 5% and 10%.

The gamma and Weibull distributions have both scale and shape parameters. The critical values are again invariant to the value of the former, but they depend on the (unknown) value of the shape parameter. Accordingly, the critical values presented in Appendix Tables A.4 and A.5 allow for a range of values for this parameter. Shape parameter values of unity are omitted as both the gamma and Weibull distributions collapse to an exponential distribution in this case. For the gamma distribution, it is easily shown that the MLEs of the parameters satisfy $\tilde{b} = \tilde{\theta}\bar{x}$, and so we see in from Table 1 that $\tilde{\lambda}_c = \hat{\lambda}_c$ for this distribution when c = 1. However, the calculations of $\tilde{\lambda}_c$ for the gamma distribution when c = 2, and for the Weibull distribution for c = 1 and 2 require MLEs that are obtained by solving the likelihood equations numerically. Given the additional computation that is required, the results in Tables A.4 and A.5 are based on 50,000 Monte Carlo replications.

The accuracy of these simulated critical values can be assessed by comparing our results for the Weibull distribution in Table A.5 with those in Table 1 of Economou and Tzavelas (2013). The latter results for the $\hat{\lambda}_c$ test, which are for α = 5%, match ours very closely for c = 1 and 2 and various values of the shape parameter (k) and sample sizes, even though in that study they are based on just 5,000 Monte Carlo replications. For all of the distributions, the critical values increase with the sample size, *ceteris paribus*. Also, for each distribution the critical values associated with $c = 2$ are always less than their counterparts for $c = 1$, for any sample size, significance level, or shape parameter value (if relevant). In the case of the gamma and Weibull distributions, the critical values increase as the shape parameter increases in value, for any sample size significance level, or value of c.

3.2 Power evaluations

For each base distribution of interest, and various sample sizes, we have used the simulated critical values to determine the powers of the tests by simulating the rejection rates when the data are generated from the corresponding weighted distribution. The latter are also shown in Table 1. These rejection rates are computed as the fraction of 20,000 simulated values of the test statistics for which $\hat{\lambda}_c > \hat{k}_c(\alpha)$ or $\tilde{\lambda}_c > \tilde{k}_c(\alpha)$. Again, $c = 1$ or 2, and significance levels of $\alpha = 1\%$, 5%, and 10% are considered.

The tests under consideration in this study involve a *simple* null hypothesis and a *simple* alternative hypothesis. For this reason, “power curves”, in which the power is plotted against the degree of departure from the null hypothesis, do not apply. However, for any of the distributions under consideration, and for a given significance level and value of c, the powers of the $\hat{\lambda}_c$ and $\tilde{\lambda}_c$ tests as a function of the sample size are still of interest. A selection of such functions is plotted in Figures 1 to 3.

Figure 1a: Powers against H1: c = 1 (α= 5%)

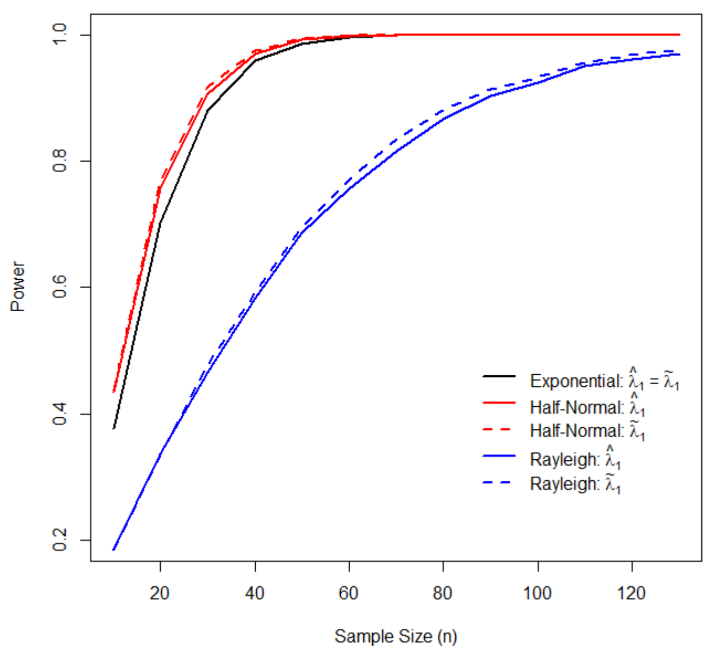


Figure 1b: Powers against H1: c = 2 (α= 5%)

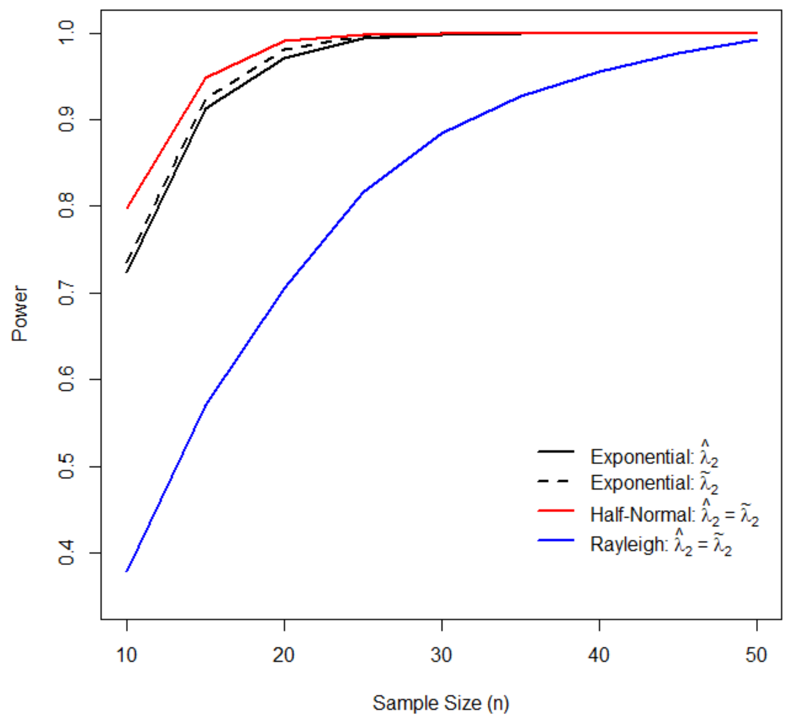
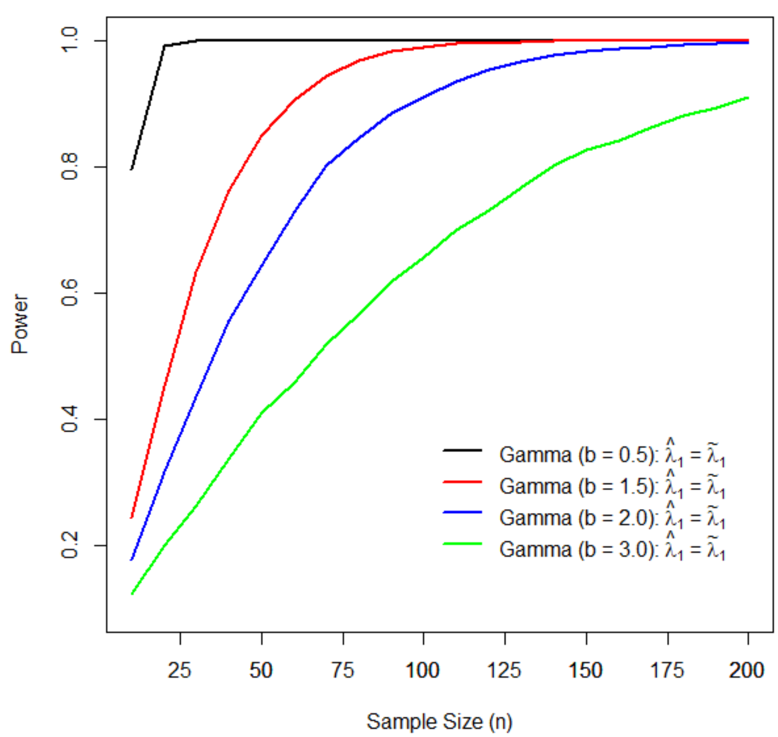
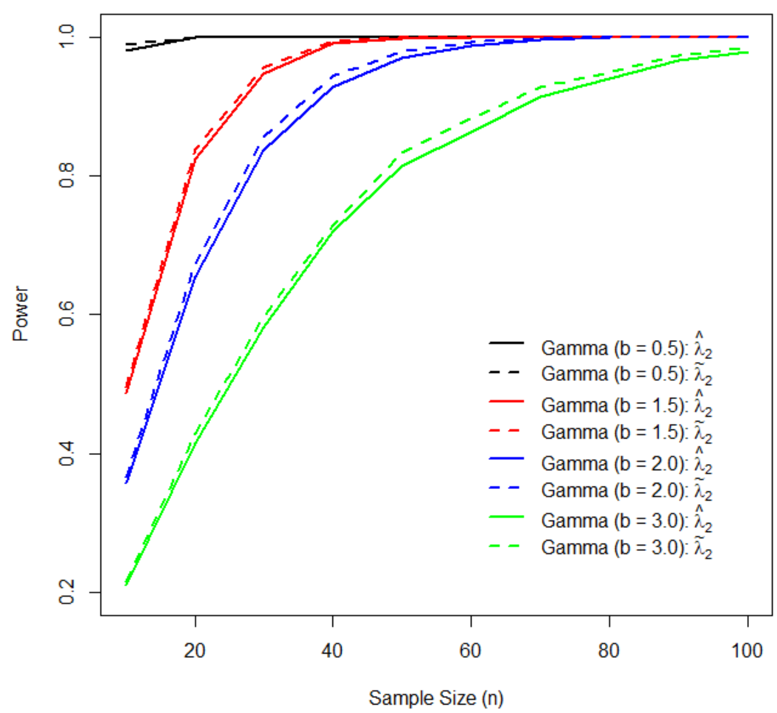


Figure 2a: Powers against H1: c = 1 (α= 5%)



Several basic results emerge from all of these power plots. First, the tests are all "unbiased" in the sense that their power is never less than the significance level. Moreover, the powers are generally substantial in value, even for quite moderate sample sizes. Second, for each distribution and sample size (and for all shape parameter values, if applicable), the powers of all of the tests are greater when $c = 2$ than when $c = 1$. Third, all of the tests are consistent. That is, their power approaches unity as $n \to \infty$. (This has been verified separately for those cases where the figures are limited to modest values of n to ensure meaningful plots.) Further, in both Figures 2 and 3 we see that the powers of the tests decrease (for any n) as the value of the shape parameter increases. All of the above results are consistent with the findings of Economou and Tzavelas (2013) for the $\hat{\lambda}_c$ test in the context of the Weibull distribution.

Figure 2b: Powers against H1: c = 2 (α= 5%)



One of our primary interests lies in the comparison of the relative powers of the $\hat{\lambda}_c$ and $\tilde{\lambda}_c$ tests. In the cases where these test statistics differ for the exponential, half-normal, and Rayleigh distributions, we see in Figures 1a and 1b that there is a slight gain in power by using the $\tilde{\lambda}_c$ test with moderate sample sizes. The two test statistics also differ for the case of area-biased sampling for the gamma distribution. Figure 2b shows that for this case there is also a small gain in power by using the $\tilde{\lambda}_2$ test in small samples. In the case of the Weibull distribution we see in Figures 3a and 3b that the two versions of the tests have essentially the same power, for any n, when testing for either length-biased or area-biased sampling. As noted in section

2, the values of the $\hat{\lambda}_c$ and $\tilde{\lambda}_c$ test statistics converge as $n \to \infty$, and so the associated powers also converge (ultimately to unity).

Figure 3a: Powers against H1: c = 1 (α= 5%)

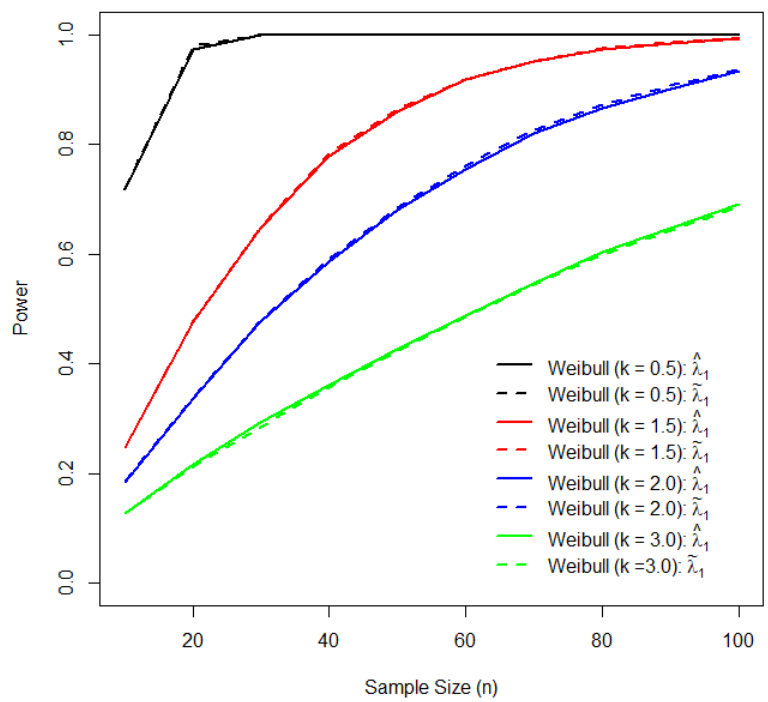
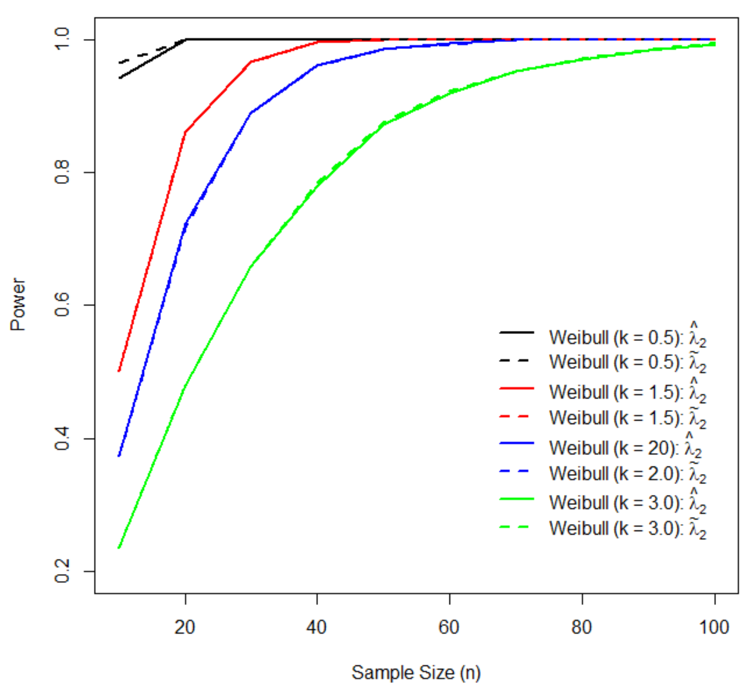


Figure 3b: Powers against H1: c = 2 (α= 5%)



4. Concluding remarks

In this paper we have explored the properties of two variants of a test for the presence of size-biased sampling, where the degree of the potential weighting function is known, and the base distributions are well-known members of the generalized gamma family. These distributions are widely used in the context of lifetime data, where either length-biased or area-biased sampling is common.

Critical values for the test(s) are reported for each of the distributions, and these are then used to evaluate the powers of the tests. The two variants of the test differ with respect to the estimation of the population moment in the denominator of the test statistic. In one case the corresponding sample moment is used; while in the other case the maximum likelihood estimator of the population moment is employed. In some situations the two approaches coincide, and they are asymptotically equivalent for large samples.

In all of the situations consdie4red, the tests are shown to have very good power, even for moderately sized samples, and the variant based on maximum likelihood estimation has a slight advantage over the method of moments version of the test. These results generalize and confirm those reported for just the latter version of the test for the Weibull distribution by Economou and Tzavelas (2013).

These encouraging results suggest that the tests in question should be evaluated for additional base distributions. Further, the present study is limited to the case where the sample data are uncensored. Censored data for the Weibull distribution are considered by Economou and Tzavelas (2013), and the implications of right-censoring should be explored for the other distributions under study for each version of the test.

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Appendix A: Critical values

Table A.1: Critical values - Exponential distribution

Sig. level (α):	**10%**		**5%**		**1%**	
	***c* = 1**	***c* = 2**	***c* = 1**	***c* = 2**	***c* = 1**	***c* = 2**
n						
10	0.7813	0.6556	0.8202	0.7076	0.8825	0.7972
		(0.5524)		(0.5799)		(0.6241)
25	0.6935	0.5460	0.7225	0.5814	0.7738	0.6445
		(0.4904)		(0.5109)		(0.5472)
50	0.6518	0.4970	0.6740	0.5216	0.7134	0.5668
		(0.4609)		(0.4766)		(0.5044)
100	0.6237	0.4651	0.6393	0.4821	0.6692	0.5149
		(0.4410)		(0.4521)		(0.4732)
150	0.6113	0.4511	0.6246	0.4654	0.6488	0.4918
		(0.4322)		(0.4426)		(0.4588)
200	0.6046	0.4435	0.6159	0.4558	0.6367	0.4791
		(0.4276)		(0.4355)		(0.4502)
250	0.5997	0.4382	0.6100	0.4491	0.6288	0.4697
		(0.4241)		(0.4313)		(0.4446)
300	0.5964	0.4345	0.6058	0.4445	0.6230	0.4635
		(0.4217)		(0.4284)		(0.4405)
500	0.5881	0.4253	0.5954	0.4331	0.6091	0.4483
		(0.4158)		(0.4210)		(0.4307)
1,000	0.5801	0.4168	0.5852	0.4222	0.5952	0.4327
		(0.4102)		(0.4138)		(0.4208)
10,000	0.5673	0.4031	0.5689	0.4048	0.5720	0.4080
		(0.4011)		(0.4023)		(0.4045)

Note: For each sample size, the critical values in the first row are $\hat{k}_c(\alpha)$. If $\hat{\lambda}_c$ differs from $\tilde{\lambda}_c$, then the corresponding critical values for the latter, namely $\tilde{k}_c(\alpha)$, appear in parentheses in the second row.

Table A.2: Critical values - Half-normal distribution

Sig. level **(α):**	**10%**		**5%**		**1%**	
n	***c* = 1**	***c* = 2**	***c* = 1**	***c* = 2**	***c* = 1**	***c* = 2**
10	0.8515 (0.9453)	0.7542	0.8815 (0.9996)	0.7975	0.9258 (1.0847)	0.8655
25	0.7818 (0.8319)	0.6637	0.8064 (0.8704)	0.6945	0.8482 (0.9392)	0.7494
50	0.7468 (0.7790)	0.6216	0.7660 (0.8076)	0.6444	0.8004 (0.8601)	0.6863
100	0.7215 (0.7424)	0.5923	0.7356 (0.7631)	0.6089	0.7619 (0.8011)	0.6392
150	0.7104 (0.7273)	0.5803	0.7222 (0.7437)	0.5933	0.7436 (0.7746)	0.6180
200	0.7042 (0.7182)	0.5730	0.7143 (0.7326)	0.5846	0.7332 (0.7596)	0.6061
250	0.6999 (0.7125)	0.5685	0.7094 (0.7254)	0.5788	0.7266 (0.7504)	0.5987
300	0.6967 (0.7081)	0.5650	0.7053 (0.7199)	0.5744	0.7213 (0.7428)	0.5926
500	0.6893 (0.6978)	0.5568	0.6959 (0.7070)	0.5641	0.7083 (0.7239)	0.5776
1,000	0.6817 (0.6877)	0.5487	0.6865 (0.6942)	0.5539	0.6955 (0.7066)	0.5638
10,000	0.6696 (0.6714)	0.5357	0.6711 (0.6735)	0.5374	0.6740 (0.6773)	0.5404

Note: For each sample size, the critical values in the first row are $\hat{k}_c(\alpha)$. If $\hat{\lambda}_c$ differs from $\tilde{\lambda}_c$, then the corresponding critical values for the latter, namely $\tilde{k}_c(\alpha)$, appear in parentheses in the second row.

Table A.3: Critical values - Rayleigh distribution

Sig. level (*α*):	**10%**		**5%**		**1%**	
	***c* = 1**	***c* = 2**	***c* = 1**	***c* = 2**	***c* = 1**	***c* = 2**
n						
10	0.9365 (0.9973)	0.8839	0.9497 (1.0229)	0.9065	0.9692 (1.0613)	0.9406
25	0.9043 (0.9398)	0.8329	0.9158 (0.9598)	0.8506	0.9343 (0.9932)	0.8802
50	0.8869 (0.9107)	0.8071	0.8961 (0.9258)	0.8205	0.9118 (0.9528)	0.8444
100	0.8750 (0.8911)	0.7897	0.8819 (0.9024)	0.7998	0.8942 (0.9230)	0.8180
150	0.8694 (0.8824)	0.7820	0.8753 (0.8918)	0.7903	0.8860 (0.9090)	0.8056
200	0.8662 (0.8773)	0.7775	0.8713 (0.8854)	0.7847	0.8806 (9006)	0.7981
250	0.8640 (0.8738)	0.7744	0.8686 (0.8812)	0.7810	0.8773 (0.8951)	0.7933
300	0.8623 (0.8713)	0.7721	0.8668 (0.8782)	0.7783	0.8747 (0.8908)	0.7895
500	0.8584 (0.8652)	0.7668	0.8618 (0.8705)	0.7714	0.8680 (0.8803)	0.7802
1,000	0.8546 (0.8594)	0.7616	0.8571 (0.8632)	0.7650	0.8618 (0.8705)	0.7715
10,000	0.8484 (0.8499)	0.7532	0.8492 (0.8511)	0.7543	0.8507 (0.8534)	0.7563

Note: For each sample size, the critical values in the first row are $\hat{k}_c(\alpha)$. If $\hat{\lambda}_c$ differs from $\tilde{\lambda}_c$, then the corresponding critical values for the latter, namely $\tilde{k}_c(\alpha)$, appear in parentheses in the second row.

Table A.4: Critical values - Gamma distribution

***Sig. level* (α):**		**10%**		**5%**		**1%**	
		c **= 1**	c **= 2**	c **= 1**	c **= 2**	c **= 1**	c **= 2**
$\boldsymbol{b = 0.5}$	$\boldsymbol{n}$						
	10	0.5695	0.4228 (0.4061)	0.6359	0.4921 (0.4738)	0.7488	0.6166 (0.6028)
	25	0.4402	0.2967 (0.2855)	0.4835	0.3348 (0.3230)	0.5630	0.4103 (0.3971)
	50	0.3857	0.2487 (0.2418)	0.4143	0.2722 (0.2649)	0.4676	0.3178 (0.3093)
	100	0.3509	0.2185 (0.2143)	0.3700	0.2340 (0.2290)	0.4074	0.2646 (0.2587)
	150	0.3366	0.2062 (0.2027)	0.3522	0.2182 (0.2146)	0.3832	0.2434 (0.2386)
	200	0.3285	0.1992 (0.1967)	0.3419	0.2101 (0.2067)	0.3681	0.2312 (0.2273)
	250	0.3230	0.1950 (0.1926)	0.3350	0.2039 (0.2015)	0.2222	0.3587 (0.2193)
	300	0.3190	0.1917 (0.1897)	0.3300	0.2006 (0.1978)	0.3513	0.2172 (0.2143)
	500	0.3100	0.1847 (0.1831)	0.3183	0.1912 (0.1893)	0.3339	0.2035 (0.2011)
	1,000	0.3010	0.1776 (0.1767)	0.3068	0.1820 (0.1808)	0.3174	0.1904 (0.1886)
	10,000	0.2870	0.1668 (0.1665)	0.28888	0.1682 (0.1678)	0.2922	0.1707 (0.1702)
$\boldsymbol{b = 1.5}$	$\boldsymbol{n}$						
	10	0.8552	0.7552 (0.7500)	0.8822	0.7957 (0.7910)	0.9232	0.8588 (0.8573)
	25	0.7924	0.6674 (0.6616)	0.8136	0.6966 (0.6911)	0.8511	0.7487 (0.7450)
	50	0.7608	0.6244 (0.6200)	0.7772	0.6459 (0.6412)	0.8067	0.6858 (0.6800)
	100	0.7401	0.5970 (0.5940)	0.7520	0.6119 (0.6084)	0.7735	0.6397 (0.6359)
	150	0.7304	0.5847 (0.5822)	0.7404	0.5975 (0.5944)	0.7591	0.6219 (0.6180)
	200	0.7250	0.5777 (0.5756)	0.7337	0.5891 (0.5864)	0.7499	0.6096 (0.6065)
	250	0.7213	0.5731 (0.5710)	0.7292	0.5827 (0.5806)	0.7435	0.6012 (0.5979)
	300	0.7188	0.5695 (0.5677)	0.7258	0.5785 (0.5762)	0.7395	0.5960 (0.5933)
	500	0.7123	0.5614 (0.5600)	0.7180	0.5684 (0.5667)	0.7287	0.5815 (0.5794)
	1,000	0.7062	0.5538 (0.5528)	0.7103	0.5589 (0.5576)	0.7180	0.5685 (0.5670)
	10,000	0.6961	0.5413 (0.5410)	0.6973	0.5428 (0.5424)	0.6998	0.5458 (0.5453)

Table A.4 (continued): Critical values - Gamma distribution

Sig. level (α):		10%		5%		1%	
		c = 1	c = 2	c = 1	c = 2	c = 1	c = 2
$b = 2.0$							
	n						
	10	0.8917	0.8100 (0.8064)	0.9124	0.8427 (0.8393)	0.9443	0.8962 (0.8937)
	25	0.8435	0.7368 (0.7328)	0.8600	0.7610 (0.7577)	0.8883	0.8056 (0.8021)
	50	0.8192	0.7018 (0.6983)	0.8319	0.7194 (0.7162)	0.8543	0.0.7516 (0.7485)
	100	0.8022	0.6773 (0.6750)	0.8118	0.6904 (0.6877)	0.8287	0.7140 (0.7113)
	150	0.7947	0.6668 (0.6647)	0.8026	0.6779 (0.6758)	0.8173	0.6989 (0.6958)
	200	0.7902	0.6606 (0.6589)	0.7971	0.6702 (0.6683)	0.8099	0.6876 (0.6856)
	250	0.7872	0.6565 (0.6549)	0.7935	0.6652 (0.6633)	0.8051	0.6808 (0.6787)
	300	0.7852	0.6535 (0.6520)	0.7911	0.6615 (0.6600)	0.8015	0.6763 (0.6740)
	500	0.7800	0.6461 (0.6451)	0.7845	0.6523 (0.6509)	0.7928	0.6635 (0.6617)
	1,000	0.7750	0.6394 (0.6386)	0.7782	0.6439 (0.6428)	0.7842	0.6520 (0.6506)
	10,000	0.7669	0.6281 (0.6279)	0.7679	0.6295 (0.6293)	0.7698	0.6322 (0.6318)
$b = 2.5$							
	n						
	10	0.9138	0.8454 (0.8429)	0.9309	0.8739 (0.8717)	0.9560	0.9167 (0.9159)
	25	0.8744	0.7824 (0.7795)	0.8882	0.8036 (0.8011)	0.9110	0.8406 (0.8382)
	50	0.8543	0.7516 (0.7489)	0.8650	0.7671 (0/7649)	0.8832	0.7959 (0/7941)
	100	0.8404	0.7303 (0.7286)	0.8484	0.7421 (0/7404)	0.8624	0.7632 (0.7612)
	150	0.8341	0.7212 (0.7196)	0.8409	0.7311 (0/7295)	0.8531	0.7492 (0.7475)
	200	0.8307	0.7160 (0.7147)	0.8365	0.7244 (0.7232)	0.8468	0.7401 (0.7384)
	250	0.8282	0.7125 (0.7113)	0.8334	0.7200 (0.7186)	0.8430	0.7343 (0.7327)
	300	0.8264	0.7096 (0.7085)	0.8311	0.7167 (0.7153)	0.8398	0.7296 (0.7279)
	500	0.8222	0.7034 (0.7026)	0.8260	0.7090 (0.7080)	0.8332	0.7193 (0.7181)
	1,000	0.8180	0.6975 (0.6969)	0.8208	0.7014 (0.7007)	0.8258	0.7089 (0.7079)
	10,000	0.8112	0.6874 (0.6873)	0.8121	0.6887 (0.6885)	0.8137	0.6910 (0.6907)

Table A.4 (continued): Critical values - Gamma distribution

Sig. level (α):		10%		5%		1%	
		$c = 1$	$c = 2$	$c = 1$	$c = 2$	$c = 1$	$c = 2$
$b = 3.0$							
	n						
	10	0.9285	0.8699 (0.8679)	0.9427	0.8942 (0.8926)	0.9638	0.9313 (0.9306)
	25	0.8954	0.8148 (0.8125)	0.9069	0.8334 (0.8317)	0.9266	0.8665 (0.8646)
	50	0.8782	0.7874 (0.7857)	0.8871	0.8018 (0.7999)	0.9028	0.8272 (0.8258)
	100	0.8664	0.7690 (0.7674)	0.8731	0.7791 (0.7775)	0.8853	0.7981 (0.7962)
	150	0.8611	0.7608 (0.7596)	0.8667	0.7693 (0.7678)	0.8772	0.7853 (0.7839)
	200	0.8580	0.7557 (0.7548)	0.8630	0.7634 (0.7622)	0.8721	0.7777 (0.7760)
	250	0.8560	0.7527 (0.7518)	0.8605	0.7595 (0.7585)	0.8684	0.7719 (0.7705)
	300	0.8545	0.7504 (0.7495)	0.8586	0.7568 (0.7556)	0.8660	0.7680 (0.7669)
	500	0.8509	0.7449 (0.7443)	0.8542	0.7498 (0.7491)	0.8600	0.7589 (0.7577)
	1,000	0.8473	0.7393 (0.7388)	0.8496	0.7429 (0.7423)	0.8540	0.7495 (0.7489)
	10,000	0.8414	0.7304 (0.7303)	0.8422	0.7316 (0.7314)	0.8436	0.7336 (0.7333)
$b = 3.5$							
	n						
	10	0.9391	0.8870 (0.8857)	0.9513	0.9085 (0.9073)	0.9695	0.9413 (0.9407)
	25	0.9103	0.8387 (0.8369)	0.9202	0.8555 (0.8538)	0.9368	0.8837 (0.8822)
	50	0.8955	0.8147 (0.8131)	0.9032	0.8270 (0.8253)	0.9170	0.9490 (0.8481)
	100	0.8852	0.7978 (0.7965)	0.8909	0.8087 (0.8054)	0.9016	0.8238 (0.8227)
	150	0.8806	0.7902 (0.7894)	0.8855	0.7982 (0.7970)	0.8944	0.8123 (0.8113)
	200	0.8779	0.7859 (0.7851)	0.8822	0.7930 (0.7920)	0.8900	0.8051 (0.8042)
	250	0.8762	0.7832 (0.7824)	0.8800	0.7893 (0.7884)	0.8872	0.8009 (0.8000)
	300	0.8748	0.7810 (0.7803)	0.8784	0.7866 (0.7858)	0.8851	0.7974 (0.7962)
	500	0.8717	0.7761 (0.7756)	0.8745	0.7805 (0.7800)	0.8797	0.7886 (0.7881)
	1,000	0.8685	0.7712 (0.7708)	0.8705	0.7744 (0.7739)	0.8744	0.7801 (0.7798)
	10,000	0.8634	0.7630 (0.7629)	0.8640	0.7640 (0.7639)	0.8653	0.7660 (0.7657)

Table A.4 (continued): Critical values - Gamma distribution

Sig. level **(α):**		**10%**		**5%**		**1%**	
		***c* = 1**	***c* = 2**	***c* = 1**	***c* = 2**	***c* = 1**	***c* = 2**
$\boldsymbol{b = 4.0}$							
	n						
	10	0.9465	0.9002 (0.8990)	0.9574	0.9191 (0.9183)	0.9731	0.9482 (0.9478)
	25	0.9215	0.8568 (0.8556)	0.9303	0.8719 (0.8706)	0.9449	0.8977 (0.8969)
	50	0.9084	0.8348 (0.8338)	0.9152	0.8464 (0.8449)	0.9276	0.8670 (0.8659)
	100	0.8992	0.8200 (0.8191)	0.9043	0.8287 (0.8273)	0.9136	0.8433 (0.8425)
	150	0.8952	0.8132 (0.8125)	0.8996	0.8203 (0.8196)	0.9076	0.8335 (0.8325)
	200	0.8928	0.8095 (0.8088)	0.8967	0.8157 (0.8150)	0.9036	0.8268 (0.8263)
	250	0.8913	0.8069 (0.8063)	0.8947	0.8126 (0.8119)	0.9011	0.8226 (0.8220)
	300	0.8901	0.8049 (0.8044)	0.8933	0.8101 (0.8094)	0.8991	0.8196 (0.8188)
	500	0.8874	0.8005 (0.8000)	0.8898	0.8046 (0.8040)	0.8943	0.8119 (0.8112)
	1,000	0.8846	0.7960 (0.7957)	0.8863	0.7988 (0.7984)	0.8897	0.8040 (0.8036)
	10,000	0.8800	0.7886 (0.7885)	0.8806	0.7895 (0.7894)	0.8817	0.7912 (0.7910)

Note: For each sample size, the critical values in the first row are $\hat{k}_c(\alpha)$. If $\hat{\lambda}_c$ differs from $\tilde{\lambda}_c$, then the corresponding critical values for the latter, namely $\tilde{k}_c(\alpha)$, appear in parentheses in the second row.

Table A.5: Critical values - Weibull distribution

Sig. level (α):		**10%**		**5%**		**1%**	
		$c = 1$	$c = 2$	$c = 1$	$c = 2$	$c = 1$	$c = 2$
$k = 0.5$							
	n						
	10	0.4315	0.2901	0.5038	0.3546	0.6348	0.4895
		(0.4318)	(0.2695)	(0.5047)	(0.3386)	(0.6354)	(0.4801)
	25	0.2976	0.1738	0.3378	0.2049	0.4180	0.2718
		(0.2971)	(0.1565)	(0.3374)	(0.1878)	(0.4191)	(0.2574)
	50	0.2459	0.1315	0.2708	0.1502	0.3223	0.1895
		(0.2454)	(0.1193)	(0.2705)	(0.1372)	(0.3233)	(0.1765)
	100	0.2153	0.1080	0.2317	0.1196	0.2642	0.1431
		(0.2150)	(0.0990)	(0.2315)	(0.1099)	(0.2643)	(0.1329)
	150	0.2029	0.0985	0.2158	0.1076	0.2412	0.1259
		(0.2027)	(0.0913)	(0.2159)	(0.0997)	(0.2416)	(0.1167)
	200	0.1962	0.0934	0.2074	0.1011	0.2282	0.1157
		(0.1961)	(0.0871)	(0.2069)	(0.0940)	(0.2286)	(0.1079)
	250	0.1917	0.0903	0.2016	0.0968	0.2203	0.1100
	(0.1915)	(0.0843)	(0.2015)	(0.0905)	(0.2206)	(0.1028)	
	300	0.1883	0.0878	0.1975	0.0938	0.2146	0.1055
		(0.1882)	(0.0823)	(0.1973)	(0.0879)	(0.2144)	(0.0989)
	500	0.1809	0.0820	0.1876	0.0866	0.2001	0.0954
		(0.1809)	(0.0778)	(0.1874)	(0.0818)	(0.2002)	(0.0896)
	1,000	0.1735	0.0764	0.1781	0.0796	0.1865	0.0854
		(0.1736)	(0.0735)	(0.1781)	(0.0762)	(0.1868)	(0.0813)
	10,000	0.1625	0.0681	0.1639	0.0691	0.1664	0.0709
		(0.1625)	(0.0671)	(0.1639)	(0.0679)	(0.1664)	(0.0694)
$k = 1.5$							
	n						
	10	0.8918	0.8121	0.9136	0.8462	0.9442	0.8971
		(0.8891)	(0.8122)	(0.9110)	(0.8458)	(0.9423)	(0.8956)
	25	0.8405	0.7368	0.8586	0.7621	0.8878	0.8055
		(0.8387)	(0.7374)	(0.8564)	(0.7624)	(0.8858)	(0.8057)
	50	0.8144	0.7009	0.8283	0.7194	0.8533	0.7538
		(0.8130)	(0.7015)	(0.8268)	(0.7200)	(0.8512)	(0.7540)
	100	0.7961	0.6770	0.8064	0.6904	0.8250	0.7149
		(0.7952)	(0.6773)	(0.8053)	(0.6907)	(0.8233)	(0.7152)
	150	0.7882	0.6666	0.7968	0.6777	0.8123	0.6984
		(0.7875)	(0.6670)	(0.7959)	(0.6781)	(0.8111)	(0.6989)
	200	0.7836	0.6607	0.7911	0.6702	0.8048	0.6881
		(0.7829)	(0.6611)	(0.7902)	(0.6706)	(0.8037)	(0.6885)
	250	0.7803	0.6566	0.7872	0.6655	0.7995	0.6813
		(0.7797)	(0.6570)	(0.7864)	(0.6659)	(0.7986)	(0.6819)
	300	0.7780	0.6536	0.7843	0.6618	0.7957	0.6766
		(0.7774)	(0.6539)	(0.7837)	(0.6621)	(0.7949)	(0.6770)
	500	0.7725	0.6468	0.7774	0.6529	0.7865	0.6644
		(0.7721)	(0.6471)	(0.7769)	(0.6531)	(0.7857)	(0.6647)
	1,000	0.7670	0.6398	0.7705	0.6441	0.7771	0.6522
		(0.7667)	(0.6400)	(0.7701)	(0.6443)	(0.7766)	(0.6527)
	10,000	0.7580	0.6287	0.7591	0.6302	0.7613	0.6327
		(0.7579)	(0.6288)	(0.7590)	(0.6302)	(0.7611)	(0.6328)

Table A.5 (continued): Critical values - Weibull distribution

Sig. level **(α):**		**10%**		**5%**		**1%**	
		c = 1	**c = 2**	**c = 1**	**c = 2**	**c = 1**	**c = 2**
$\boldsymbol{k = 2.0}$							
	$\boldsymbol{n}$						
	10	0.9365	0.8842	0.9497	0.9067	0.9680	0.9389
		(0.9343)	(0.8831)	(0.9481)	(0.9053)	(0.9678)	(0.9373)
	25	0.9041	0.8325	0.9156	0.8505	0.9340	0.8801
		(0.9024)	(0.8321)	(0.9142)	(0.8497)	(0.9329)	(0.8791)
	50	0.8869	0.8069	0.8962	0.8206	0.9121	0.8449
		(0.8857)	(0.8068)	(0.8949)	(0.8201)	(0.9110)	(0.8438)
	100	0.8747	0.7894	0.8815	0.7992	0.8938	0.8172
		(0.8738)	(0.7893)	(0.8806)	(0.7990)	(0.8928)	(0.8167)
	150	0.8692	0.7817	0.8751	0.7900	0.8853	0.8053
		(0.8686)	(0.7817)	(0.8742)	(0.7899)	(0.8845)	(0.8048)
	200	0.8661	0.7774	0.8713	0.7846	0.8805	0.7977
		(0.8654)	(0.7774)	(0.8706)	(0.7844)	(0.8796)	(0.7975)
	250	0.8638	0.7744	0.8685	0.7809	0.8768	0.7929
		(0.8633)	(0.7743)	(0.8679)	(0.7808)	(0.8761)	(0.7926)
	300	0.8623	0.7720	0.8666	0.7781	0.8744	0.7893
		(0.817)	(0.7720)	(0.8660)	(0.7781)	(0.8736)	(0.7891)
	500	0.8585	0.7669	0.8618	0.7715	0.8682	0.7801
		(0.8581)	(0.7669)	(0.8614)	(0.7714)	(0.8675)	(0.7800)
	1,000	0.8546	0.7616	0.8571	0.7649	0.8616	0.7712
		(0.8543)	(0.7616)	(0.8567)	(0.7649)	(0.8611)	(0.7711)
	10,000	0.8484	0.7532	0.8492	0.7542	0.8507	0.7563
		(0.8483)	(0.7532)	(0.8491)	(0.7542)	(0.8505)	(0.7563)
$\boldsymbol{k = 2.5}$							
	$\boldsymbol{n}$						
	10	0.9585	0.9222	0.9673	0.9379	0.9794	0.9598
		(0.9573)	(0.9206)	(0.9667)	(0.9363)	(0.9802)	(0.9585)
	25	0.9364	0.8852	0.9444	0.8982	0.9570	0.9193
		(0.9354)	(0.8844)	(0.9437)	(0.8971)	(0.0571)	(0.9180)
	50	0.9244	0.8663	0.9309	0.8764	0.9419	0.8944
		(0.9236)	(0.8657)	(0.9302)	(0.8764)	(0.9416)	(0.8931)
	100	0.9158	0.8532	0.9206	0.8606	0.9292	0.8740
		(0.9152)	(0.8528)	(0.9200)	(0.8600)	(0.9286)	(0.8732)
	150	0.9119	0.8475	0.9160	0.8537	0.9233	0.8648
		(0.9114)	(0.8471)	(0.9156)	(0.8533)	(0.9228)	(0.8644)
	200	0.9096	0.8441	0.9133	0.8495	0.9199	0.8595
		(0.9092)	(0.8438)	(09129)	(0.8492)	(0.9195)	(0.8589)
	250	0.9080	0.8418	0.9113	0.8467	0.9173	0.8557
		(0.9076)	(0.8415)	(0.9110)	(0.8464)	(0.9168)	(0.8553)
	300	0.9068	0.8400	0.9099	0.8447	0.9155	0.8530
		(0.9065)	(0.8398)	(0.9096)	(0.8444)	(0.9151)	(0.8525)
	500	0.9041	0.8361	0.9065	0.8396	0.9111	0.8462
		(0.9039)	(0.8359)	(0.9063)	(0.8394)	(0.9107)	(0.8458)
	1,000	0.9014	0.8320	0.9031	0.8346	0.9064	0.8394
		(0.9012)	(0.8319)	(0.9029)	(0.8344)	(0.9061)	(0.8391)
	10,000	0.8968	0.8255	0.8974	0.8264	0.8985	0.8279
		(0.8967)	(0.8255)	(0.8973)	(0.8263)	(0.8984)	(0.8279)

Table A.5 (continued): Critical values - Weibull distribution

***Sig. level* (α):**		**10%**		**5%**		**1%**	
		***c* = 1**	***c* = 2**	***c* = 1**	***c* = 2**	***c* = 1**	***c* = 2**
$\boldsymbol{k = 3.0}$							
	n						
	10	0.9708	0.9444	0.9771	0.9558	0.9856	0.9717
		(0.9703)	(0.9429)	(0.9772)	(0.9545)	(0.9873)	(0.9707)
	25	0.9549	0.9168	0.9607	0.9266	0.9698	0.9422
		(0.9545)	(0.9158)	(0.9606)	(0.9254)	(0.9708)	(0.9412)
	50	0.9461	0.9024	0.9508	0.9101	0.9588	0.9237
		(0.9458)	(0.9016)	(0.9507)	(0.9093)	(0.9593)	(0.9226)
	100	0.9397	0.8922	0.9432	0.8980	0.9496	0.9083
		(0.9394)	(0.8917)	(0.9431)	(0.8974)	(0.9496)	(0.9074)
	150	0.9368	0.8878	0.9399	0.8926	0.9453	0.9013
		(0.9366)	(0.8874)	(0.9397)	(0.8921)	(0.9452)	(0.9006)
	200	0.9350	0.8852	0.9379	0.8895	0.9427	0.8971
		(0.9349)	(0.8848)	(0.9377)	(0.8890)	(0.9427)	(0.8965)
	250	0.9339	0.8834	0.9363	0.8873	0.9408	0.8942
	(0.9337)	(0.8830)	(0.9362)	(0.8868)	(0.9407)	(0.8937)	
	300	0.9330	0.8820	0.9353	0.8856	0.9395	0.8920
		(0.9328)	(0.8817)	(0.9352)	(0.8853)	(0.9394)	(0.8916)
	500	0.9310	0.8789	0.9328	0.8817	0.9362	0.8869
		(0.9309)	(0.8787)	(0.9327)	(0.8814)	(0.9360)	(0.8864)
	1,000	0.9289	0.8758	0.9302	0.8778	0.9326	0.8815
		(0.9288)	(0.8756)	(0.9301)	(0.8776)	(0.9326)	(0.8812)
	10,000	0.9254	0.8706	0.9259	0.8713	0.9267	0.8725
		(0.9254)	(0.8706)	(0.9258)	(0.8712)	(0.9267)	(0.8724)
$\boldsymbol{k = 3.5}$							
	n						
	10	0.9784	0.9583	0.9831	0.9671	0.9894	0.9790
		(0.9784)	(09571)	(0.9837)	(0.9660)	(0.9916)	(0.9768)
	25	0.9664	0.9371	0.9708	0.9447	0.9776	0.9567
		(0.9664)	(0.9361)	(0.9712)	(0.9438)	(0.9792)	(0.9559)
	50	0.9597	0.9258	0.9633	0.9319	0.9694	0.9425
		(0.9597)	(0.9250)	(0.9636)	(0.9311)	(0.9704)	(0.9416)
	100	0.9548	0.9178	0.9575	0.9223	0.9623	0.9304
		(0.9548)	(0.9172)	(0.9577)	(0.9217)	(0.9628)	(0.9296)
	150	0.9525	0.9142	0.9549	0.9180	0.9590	0.9248
		(0.9525)	(0.9138)	(0.9550)	(09175)	(0.9594)	(0.9242)
	200	0.9512	0.9121	0.9533	0.9156	0.9570	0.9216
		(0.9512)	(0.9117)	(0.9534)	(0.9150)	(0.9574)	(0.9210)
	250	0.9502	0.9106	0.9522	0.9138	0.9556	0.9192
	(0.9503)	(0.9103)	(0.9523)	(0.9133)	(0.9559)	(0.9187)	
	300	0.9496	0.9096	0.9514	0.9124	0.9546	0.9175
		(0.9496)	(0.9093)	(0.9514)	(0.9121)	(0.9547)	(0.9171)
	500	0.9480	0.9071	0.9494	0.9093	0.9520	0.9135
		(0.9480)	(0.9069)	(0.9495)	(0.9090)	(0.9521)	(0.9131)
	1,000	0.9464	0.9046	0.9474	0.9062	0.9493	0.9092
		(0.9464)	(0.9044)	(0.9474)	(0.9059)	(0.9494)	(0.9089)
	10,000	0.9437	0.9004	0.9440	0.9010	0.9447	0.9020
		(0.9437)	(0.9004)	(0.9440)	(0.9009)	(0.9447)	(0.9019)

Table A.5 (continued): Critical values - Weibull distribution

Sig. level **(α):**		**10%**		**5%**		**1%**	
		c **= 1**	c **= 2**	c **= 1**	c **= 2**	c **= 1**	c **= 2**
$k = 4.0$							
	n						
	10	0.9834	0.9680	0.9870	0.9751	0.9919	0.9845
		(0.9838)	(0.9669)	(0.9881)	(0.9743)	(0.9945)	(0.9845)
	25	0.9740	0.9508	0.9774	0.9569	0.9827	0.9664
		(0.9744)	(0.9500)	(0.9783)	(0.9561)	(0.9848)	(0.9659)
	50	0.9687	0.9418	0.9716	0.9467	0.9764	0.9550
		(0.9691)	(0.9411)	(0.9723)	(0.9460)	(0.9777)	(0.9544)
	100	0.9648	0.9352	0.9670	0.9389	0.9708	0.9454
		(0.9651)	(0.9348)	(0.9675)	(0.9384)	(0.9717)	(0.9449)
	150	0.9631	0.9323	0.9653	0.9355	0.9689	0.9409
		(0.9632)	(0.9320)	(0.9653)	(0.9350)	(0.9689)	(0.9405)
	200	0.9620	0.9306	0.9637	0.9335	0.9666	0.9383
		(0.9622)	(0.9303)	(0.9640)	(0.9330)	(0.9672)	(0.9379)
	250	0.9612	0.9294	0.9628	0.9320	0.9655	0.9364
		(0.9614)	(0.9291)	(0.9630)	(0.9316)	(0.9659)	(0.9360)
	300	0.9607	0.9286	0.9621	0.9309	0.9647	0.9351
		(0.9608)	(0.9283)	(0.9624)	(0.9306)	(0.9650)	(0.9347)
	500	0.9595	0.9266	0.9606	0.9284	0.9627	0.9318
		(0.9596)	(0.9263)	(0.9608)	(0.9281)	(0.9629)	(0.9314)
	1,000	0.9582	0.9245	0.9590	0.9258	0.9605	0.9282
		(0.9582)	(0.9243)	(0.9591)	(0.9256)	(0.9607)	(0.9280)
	10,000	0.9560	0.9211	0.9563	0.9215	0.9568	0.9223
		(0.9560)	(0.9210)	(0.9563)	(0.9214)	(0.9569)	(0.9222)

Note: For each sample size, the critical values in the first row are $\hat{k}_c(\alpha)$. If $\hat{\lambda}_c$ differs from $\tilde{\lambda}_c$, then the corresponding critical values for the latter, namely $\tilde{k}_c(\alpha)$, appear in parentheses in the second row.