Mathy stuff

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We follow Lurie [**HTT**] closely. Let's define sheaves and K-sheaves.

Definition 0.0.1. Let $X \in \mathbf{Top}$ and $\mathcal{C}$ an ∞-category. We define a $\mathcal{C}$-valued sheaf on X to be a presheaf $\mathcal{F} : \mathsf{Open}(X)^{op} \to \mathcal{C}$ such that, for every $U \subseteq X$ and every covering sieve $\mathsf{Open}(X)^{(0)}_{/U} \subseteq \mathsf{Open}(X)_{/U}$,

$$\mathsf{N}(\mathsf{Open}(X)^{(0)}_{/U})^{\triangleright} \hookrightarrow \mathsf{N}(\mathsf{Open}(X)_{/U})^{\triangleright} \to N(\mathsf{Open}(X)) \xrightarrow{\mathcal{F}} \mathcal{C}^{op}$$

We will write $\mathsf{Presh}(X, \mathcal{C})$ for the ∞-category $\mathsf{Fun}(\mathsf{Open}(X)^{op}, \mathcal{C})$ of $\mathcal{C}$-valued presheaves on X. Moreover, we will write $\mathsf{Shv}(X;\mathcal{C})$ for the full subcategory of $\mathsf{Presh}(X;\mathcal{C})$ spanned by the $\mathcal{C}$-valued sheaves on X. Whenever we write $\mathsf{Shv}(X)$ without specifying the target category $\mathcal{C}$, we will always mean sheaves valued in spaces, i.e. $\mathsf{Shv}(X;\mathcal{S})$.

0.0.1 Sheaves on locally compact spaces

In this section we will show that for locally compact Hausdorff spaces there is an equivalence of ∞-categories between $\mathsf{Shv}(X;\mathcal{C})$ and $\mathsf{Shv}_{\mathcal{K}}(X;\mathcal{C})$ where the latter denote so-called $\mathcal{K}$-sheaves and $\mathcal{C}$ is a presentable ∞-category with left exact filtered colimitsd. These are sheaves defined on the collection of compact subsets instead of the opens. Classically it is known that sheaves of sets on such spaces are determined by compact subsets as well as the opens.

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Definition 0.0.2. For a locally compact Hausdorff space X, we write $\mathcal{K}(X)$ for its collection of compact subsets.

Definition 0.0.3. If $K, K' \subseteq X$, we write $K \Subset K'$ if there exists an open subset $U \subseteq X$ between K and K', i.e. $K \subseteq U \subseteq K'$.

Definition 0.0.4. If $K \subseteq X$ is compact, we write $\mathcal{K}_{K\Subset}(X)$ for the set $\{K' \in \mathcal{K}(X) | K \Subset K'\}$.

Definition 0.0.5. A presheaf $\mathcal{F} : \mathsf{N}(\mathcal{K}(X))^{op} \to \mathcal{C}$ is a $\mathcal{K}$-sheaf if it satisfies the following:

1. $\mathcal{F}(\emptyset)$ is final
2. For every pair $K, K' \in \mathcal{K}(X)$, the diagram

$$\begin{tikzcd}
\mathcal{F}(K \cup K') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}(K) \arrow[d] \\
\mathcal{F}(K') \arrow[r] & \mathcal{F}(K \cap K')
\end{tikzcd}$$

   is a pullback in $\mathcal{C}$.
3. For each $K \in \mathcal{K}(X)$, $\mathcal{F}(K)$ is a colimit of $\mathcal{F}|\mathsf{N}(\mathcal{K}_{K\Subset}(X))^{op}$.

Definition 0.0.6. We denote the full subcategory of $\mathsf{Presh}(\mathsf{N}(\mathcal{K}(X));\mathcal{C})$ by $\mathsf{Shv}_{\mathcal{K}}(X;\mathcal{C})$.

Lemma 0.0.7 ([**HTT**]). *Let X locally compact Hausdorff and $\mathcal{C}$ be a presentable ∞-category with left exact filtered colimits. Let $\mathcal{W}$ be an open cover of X and denote by $\mathcal{K}_{\mathcal{W}}(X)$ the set compact subsets of X that are contained in some element of $\mathcal{W}$. Any $\mathcal{K}$-sheaf $\mathcal{F} \in \mathsf{Shv}_K X; \mathcal{C}$ is a right Kan extension of $\mathcal{F}|\mathsf{N}(\mathcal{K}_{\mathcal{W}}(X))^{op}$.*