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6.02 - Miscellaneous notes related to the ALOHA protocol
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Determining the maximum value of p
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U = N * p * (1-p)^(N-1)

dU/dp:

    Let f(p) = N * p
        f'(p) = N

    Let g(p) = (1-p)^(N-1)
        g'(p) = (N-1) * -1 * (1-p)^(N-2)
              = (1-N) * (1-p)^(N-2)

    dU/dp = f(p)g'(p) + f'(p)g(p)
          = N * p * (1-N) * (1-p)^(N-2) + N * (1-p)^(N-1)
          = (1-p)^(N-2) * [ N * p * (1-N) + N * (1-p) ]
          = (1-p)^(N-2) * [ Np - N^2p + N - Np ]
          = (1-p)^(N-2) * (N - N^2p)
          = N * (1-p)^(N-2) * (1 - N*p)

dU/dp = 0 when p = 1 or p = 1/N.

    d2U/dp:

    Let f(p) = (1-p)^(N-2)
        f'(p) = (N-2) * (1-p)^(N-3) * -1
              = (2-N) * (1-p)^(N-3)

    Let g(p) = N * (1 - Np)
             = N - N^2p
        g'(p) = -N^2

    d2U/dp = f(p)g'(p) + f'(p)g(p)
           = (1-p)^(N-2) * -N^2 + (2-N)*(1-p)^(N-3)*(N-N^2p)
           = (1-p)^(N-3) * [(1-p)*-N^2 + (2-N)*(N-N^2p) ]
           = (1-p)^(N-3) * [pN^2 - N^2 + 2N - 2N^2p -N^2 + N^3p]
           = (1-p)^(N-3) * [2N - pN^2 - 2N^2 + N^3p]
           = (1-p)^(N-3) * N * [2 - pN - 2N + N^2p]
           = (1-p)^(N-3) * N * [2(1-N) + pN(N-1)]
           = (1-p)^(N-3) * N * [2(1-N) - pN(1-N)]
           = (1-p)^(N-3) * N * (1-N) * (2 - pN)

At p=1/N, d2U/dp is < 0 ==> p=1/N is max.

    d2U/dp @ p=1/N = (1 - 1/N) ^ (N-3) * N * (1-N) * (2 - 1)
                     |_____|   |_|   |___|    |_____|
                            > 0           >0    <0        >0

At p=1, d2U/dp = 0, but it's easy to see that this is a min.

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Determining the maximum utilization
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When p = 1/N, U = (1 - 1/N)^(N-1)

Let g(N) = ln ( (1-1/N)^(N-1) ).  Then the limit of g(N) is ln(lim of
f(N) )

1. lim g(N) = -1

    Pf. We get to use some Taylor series:

    g(N) = (N-1) ln (1-1/N)
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         = (N-1) ((-1/N)/1 - (-1/N)^2/2 + (-1/N)^3/3 - (-1/N)^4/4 + ...)
         = (N-1)(-1/N - 1/2N^2 - 1/3N^3 - 1/4N^4 - ...)
         = -(N-1)/N - (N-1)/2N^2 -(N-1)/3N^3 - (N-1)/4N^4 - ...
         = (1-N)/N + (1-N)/2N^2 + (1-N)/3N^3 + (1-N)/4N^4 + ...
         = 1/N - 1 + 1/2N^2 - 1/2N + 1/3N^2 - 1/3N^2 + 1/4N^4 - 1/4N^3 + ...
         = -1 + (1/N - 1/2N) + (1/2N^2 - 1/3N^2) + (1/3N^3 - 1/4N^3) + (1/4N^4 - 1/5N^4)
+ ...
         = -1 + 1/N + 1/6N^2 + 1/12N^3 + 1/20N^4 + ..
```

    As N -> inf, this approaches -1.

2. lim(f(N)) = 1/e, since lim(g(N)) = -1 and lim(g(N)) = ln(lim(f(N)))
   (i.e., -1 = ln(lim(f(N)))).


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Unslotted ALOHA
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Question: what happens in ALOHA when packets are bigger than one slot?
Why do we even need slots?

Consider a protocol known as Purely Unslotted ALOHA. In this
protocol, each node can send any time it wants (literally any time;
there are no slots).

Will this protocol have better or worse utilization than Slotted
ALOHA? To analyze the utilization, we'll use a slight different
protocol:

Almost Unslotted ALOHA: Packet sizes are T slots long, with T very
large. I.e., time slots are very small in relation to the packet
size. If we let the length of our time slot approach zero (and thus
our packet size approach infinitely many time slots), this protocol is
equivalent to Purely Unslotted ALOHA.

To analyze utilization in Almost Unslotted ALOHA, we need to decide
how packets can collide. When packets are only 1 timeslot long (as in
Slotted ALOHA), this is an easy question: Packet X and Packet Y can
only collide if they're sent in exactly the same time slot.

But suppose the packets are 5 time slots long. Let's say Packet X
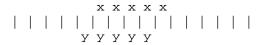takes up these time slots:

```
          x x x x x
| | | | | | | | | | | | | | | | | |
```

If packet Y starts here, it will collide with X:

```
          x x x x x
| | | | | | | | | | | | | | | | | |
   y y y y y
```

Similarly, if it starts here:

```
          x x x x x
| | | | | | | | | | | | | | | | | |
        y y y y y
```

And even here:

```
          x x x x x
| | | | | | | | | | | | | | | | | |
              y y y y y
```

There are 2*5-1 possible timeslots that Y can start in and collide
with X: Any of the slots occupied by X -- of which there are 5, and
any of the (5-1) slots preceding X.

In general, if the packet length is T, there are 2T-1 different
starting timeslots for packet Y to collide with packet X.

So, should utilization increase or decrease with this protocol,
compared to Slotted ALOHA? It should decrease; there are more chances
for packets to collide now.

By how much will it decrease? Let's figure it out. If I send, what
is the probability that no one else will send in the surrounding 2T-1
timeslots?

    $((1-p)^{(N-1)})^{(2T-1)} = (1-p)^{(2T-1)(N-1)}$

    Note: this assumes that each node sends independently in each
    timeslot with probability p. Even though there's no sensing, we
    still wouldn't expect self interference, but this formula allows
    it.

Utilization = throughput / maximum rate.
           = $Np(1-p)^{(2T-1)(N-1)}$ / (1/T)

Why 1/T? The maximum rate is the fraction of a packet in any one
timeslot.

           = $T*N*p(1-p)^{(2T-1)(N-1)}$

If we do algebra and take some derivatives, we get U = T/(2T-1)*e.
With N large, this is roughly 1/2e, or half the utilization of Slotted
ALOHA.