



**Dispersed:** Friday, October 5, 2018.

**Due:** 11:59 pm, Friday, October 12, 2018.

*Some useful reminders:*

**Deliverator:** Prof. Peter Dodds (peter.dodds@uvm.edu)

**Assistant Deliverator:** David Dewhurst (david.dewhurst@uvm.edu)

**Office:** Farrell Hall, second floor, Trinity Campus

**Office hours:** 10:15 am to 11:30 am, Tuesday and Thursday, and 2:00 pm to 4:00 pm, Wednesday

**Course website:** <http://www.uvm.edu/pdodds/teaching/courses/2018-08UVM-300>

**Bonus course notes:** <http://www.uvm.edu/pdodds/teaching/courses/2018-08UVM-300/docs/dewhurst-pocs-notes.pdf>

---

All parts are worth 3 points unless marked otherwise. Please show all your workings clearly and list the names of others with whom you collaborated.

Please obey the basic life rule: Never use Excel. Or any Microsoft product except maybe Xbox (which sadly will likely not help you here.)

Graduate students are requested to use  $\text{\LaTeX}$  (or related  $\text{\TeX}$  variant).

**email submission:** 1. Please send to david.dewhurst@uvm.edu.

2. PDF only! Please name your file as follows (where the number is to be padded by a 0 if less than 10 and names are all lowercase): CSYS300assignment%02d\$firstname-\$lastname.pdf as in CSYS300assignment06michael-palin.pdf

---

**Please submit your project's current draft** in pdf format via email. Please use this file name format (all lowercase after CSYS):

CSYS300project-\$firstname-\$lastname-YYYY-MM-DD.pdf as in

CSYS300project-lisa-simpson-1989-12-17.pdf where the date is the date of submission (and not, say, your birthdate).

---

1. The 1- $d$  theoretical percolation problem:

Consider an infinite 1- $d$  lattice forest with a tree present at any site with probability  $p$ .

- (a) Find the distribution of forest sizes as a function of  $p$ . Do this by moving along the 1-d world and figuring out the probability that any forest you enter will extend for a total length  $\ell$ .
  - (b) Find  $p_c$ , the critical probability for which a giant component exists.  
Hint: One way to find critical points is to determine when certain average quantities explode. Compute  $\langle l \rangle$  and find  $p$  such that this expression goes boom (if it does).
2. Show analytically that the critical probability for site percolation on a triangular lattice is  $p_c = 1/2$ .  
**Hint**—Real-space renormalization gets it done.:

Direct link: <http://www.youtube.com/watch?v=JlkbU5U7QqU>

3. (3 + 3)

**Coding, it's what's for breakfast:**

- (a) Percolation in two dimensions (2- $d$ ) on a simple square lattice provides a classic, nutritious example of a phase transition.  
Your mission, whether or not you choose to accept it, is to code up and analyse the  $L$  by  $L$  square lattice percolation model for varying  $L$ .  
Take  $L = 20, 50, 100, 200, 500$ , and  $1000$ .  
(Go higher if you feel  $L = 1000$  is for mere mortals.)  
(Go lower if your code explodes.)  
Let's continue with the tree obsession. A site has a tree with probability  $p$ , and a sheep grazing on what's left of a tree with probability  $1 - p$ .

Forests are defined as any connected component of trees bordered by sheep, where connections are possible with a site's four nearest neighbors on a lattice.

Each square lattice is to be considered as a landscape on which forests and sheep co-exist.

Do not bagelize (or doughnutize) the landscape (no periodic boundary conditions—boundaries are boundaries).

(Note: this set up is called site percolation. Bond percolation is the alternate case when all links between neighboring sites exist with probability  $p$ .)

Steps:

- i. For each  $L$ , run  $N_{\text{tests}}=100$  tests for occupation probability  $p$  moving from 0 to 1 in increments of  $10^{-2}$ . (As for  $L$ , you may use a smaller or larger increment depending on how things go.)
  - ii. Determine the fractional size of the largest connected forest for each of the  $N_{\text{tests}}$ , and find the average of these,  $S_{\text{avg}}$ .
  - iii. On a single figure, for each  $L$ , plot the average  $S_{\text{avg}}$  as a function of  $p$ .
- (b) Comment on how  $S_{\text{avg}}(p; N)$  changes as a function of  $L$  and estimate the critical probability  $p_c$  (the percolation threshold).

Helpful reuse of code (intended for black and white image analysis): You can use Matlab's `bwconncomp` to find the sizes of components. Very nice.

#### 4. (3 + 3)

- (a) Using your model from the previous question and your estimate of  $p_c$ , plot the distribution of forest sizes (meaning cluster sizes) for  $p \simeq p_c$  for the largest  $L$  your code and psychological makeup can withstand. (You can average the distribution over separate simulations.)
- Comment on what kind of distribution you find.
- (b) Repeat the above for  $p = p_c/2$  and  $p = p_c + (1 - p_c)/2$ , i.e., well below and well above  $p_c$ .
- Produce plots for both cases, and again, comment on what you find.

#### 5. (3 + 3)

Repeat of the last question from Assignment 4, changing from  $\gamma = 5/2$  to  $\gamma = 3/2$ . Now  $1 < \gamma < 2$  so we should see a very different behavior.

You should be able to reuse everything you set up for Assignment 4.

Here's the question reprinted with  $\gamma$  switched.

For  $\gamma = 3/2$ , generate  $n = 1000$  sets each of  $N = 10, 10^2, 10^3, 10^4, 10^5$ , and  $10^6$  samples, using  $P_k = ck^{-3/2}$  with  $k = 1, 2, 3, \dots$

How do we computationally sample from a discrete probability distribution?

Hint: You can use a continuum approximation to speed things up. In fact, taking the exact continuum version from the first two assignments will work.

- (a) For each value of sample size  $N$ , plot the maximum value of the  $n = 1000$  samples as a function of sample number (which is not the sample size  $N$ ). So you should have  $k_{\max}$  for  $i = 1, 2, \dots, n$  where  $i$  is sample number. These plots should give a sense of the unevenness of the maximum value of  $k$ , a feature of power-law size distributions.
- (b) For each set, find the maximum value. Then find the average maximum value for each  $N$ . Plot  $\langle k_{\max} \rangle$  as a function of  $N$  and calculate the scaling using least squares.

Does your scaling match up with your theoretical estimate?

How to sample from your power law distribution (and kinds of beasts):

We now turn our problem of randomly selecting from this distribution into randomly selecting from the uniform distribution. After playing around a little,  $k = 10^6$  seems like a good upper limit for the number of samples we're talking about.

Using Matlab (or some ghastly alternative), we create a cdf for  $P_k$  for  $k = 1, 2, \dots, 10^6$  and one final entry  $k > 10^6$  (for which the cdf will be 1).

We generate a random number  $x$  and find the value of  $k$  for which the cdf is the first to meet or exceed  $x$ . This gives us our sample  $k$  according to  $P_k$  and we repeat as needed. We would use the exactly normalized  $P_k = \frac{1}{\zeta(3/2)} k^{-3/2}$  where  $\zeta$  is the Riemann zeta function.

Now, we can use a quick and dirty method by approximating  $P_k$  with a continuous function  $P(z) = (\gamma - 1)z^{-\gamma}$  for  $z \geq 1$  (we have used the normalization coefficient found in assignment 1 for  $a = 1$  and  $b = \infty$ ). Writing  $F(z)$  as the cdf for  $P(z)$ , we have  $F(z) = 1 - z^{-(\gamma-1)} = 1 - z^{-1/2}$ . Inverting, we obtain  $z = [1 - F(z)]^{-2}$ . We replace  $F(z)$  with our random number  $x$  and round the value of  $z$  to finally get an estimate of  $k$ .