COMPLEX NETWORKS, CSYS303

HW02 WRITE-UP

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0.1 Problem 1

Tokunaga's law is statistical but we can consider a rigid version. Take $T_1 = 2$ and $R_T = 2$, and then draw an example network of order $\Omega = 4$ with these parameters.

Tokunaga's laws give us the following two properties:

Property 1: $T_{u,v} = T_{u-v}$

that the number of side streams of order v that enter streams of order u is equal to the total number of side streams you would expect to enter any stream of order equal to the difference of u and v, or that the network is scale independent in that the number of streams entering higher order streams as tributaries, is independent of the size of the order;

Property 2: $T_{u,v} = T_1(R_T)^{u-v-1}$

and that the number of side streams grows exponentially with the difference in stream orders.

Given the parameters, we can use the second law to calculate the number of incoming streams of varying order, into stream segments of order Ω, $\Omega - 1$, $\Omega - 2$, ... etc. We should see the relationship established in **Property 1** from the average values that emerge from the calculation. **Property 1** also give us a nice way of summarizing the information embedded in our example network:

$$T_3 = 2^1 2^2 = 8$$

$$T_2 = 2^1 2^1 = 4$$

$$T_1 = 2^1 2^0 = 2$$

Given the values above, we can construct our example below:

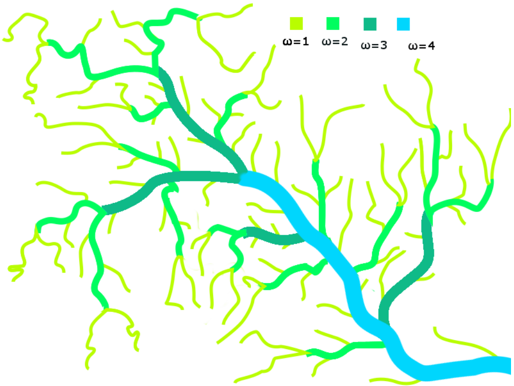

Figure 1: Example of a river network branching structure. The parameters for this particular network are $T_1 = 2$, $R_T = 2$ and order $\Omega = 4$

0.2 Problem 2

Show $R_s = R_\ell$. In other words, show that Horton's law of stream segments matches that of main stream lengths, and do this by showing they imply each other.

Recall that stream segment lengths sum to give main stream segment lengths:

$$\bar{\ell}_\omega \approx \sum_{i=1}^{\omega} \bar{s}_i \tag{1}$$

From Horton, we get the following relationships:

$$\frac{\bar{\ell}_{\omega+1}}{\bar{\ell}_\omega} = R_\ell \implies \bar{\ell}_{\omega+1} = R_\ell \bar{\ell}_\omega \tag{2}$$

$$\frac{\bar{s}_{\omega+1}}{\bar{s}_\omega} = R_s \implies \bar{s}_{\omega+1} = R_s \bar{s}_\omega \tag{3}$$

We want to show equivalence in **1**, through **2** and **3**, in order to prove that $R_s = R_\ell$. First using property **3**, we can expand the sum on the RHS:

$$\sum_{i=1}^{\omega} \bar{s}_i \implies \bar{s}_1 + R_s\bar{s}_1 + R_s\bar{s}_2 + R_s\bar{s}_3 + \ldots + R_s\bar{s}_{\omega-1}$$

$$= \bar{s}_1 + R_s\bar{s}_1 + R_s^2\bar{s}_1 + R_s^3\bar{s}_1 + \ldots + R_s^{\omega-1}\bar{s}_1$$

$$= \bar{s}_1(1 + R_s + R_s^2 + R_s^3 \ldots + R_s^{\omega-1})$$

$$\implies \bar{\ell}_\omega = \bar{s}_1 \sum_{i=0}^{\omega-1} R_s^i = \bar{s}_1 \sum_{i=1}^{\omega} R_s^{i-1} \tag{4}$$

What we have is the sum of a geometric series. We can use that fact to rewrite the summation in the form $\Sigma_{k=1}^n ar^{k-1} = \frac{a(r^n-1)}{r-1}$ for $r > 1$, or $\Sigma_{k=1}^n ar^{k-1} = \frac{a(1-r^n)}{1-r}$ for $r < 1$. By definition, we know that $R_s > 1$, so from **(4)** we may say that

$$\bar{\ell}_\omega = \bar{s}_1 \frac{R_s^\omega - 1}{R_s - 1}$$

$$\implies \bar{\ell}_\omega \propto R_s^\omega, \text{for large } \omega \tag{5}$$

Assuming **(5)**, we can extend our argument to say that

$$\frac{\bar{\ell}_{\omega+1}}{\bar{\ell}_\omega} \propto \frac{R_s^{\omega+1}}{R_s^\omega} \tag{6}$$

Hence, **(5)** can be rewritten to be

$$\bar{\ell}_{\omega+1} \approx R_s \ell_\omega \tag{7}$$

Finally, recall that Horton's law of stream lengths, **(2)**, that the ratio of two streams of sequential order is equal to R_ℓ. Hence, from **(7)** we get

$$\frac{\bar{\ell}_{\omega+1}}{\bar{\ell}_{\omega}} = R_s$$

$$\implies R_\ell = R_s \therefore$$

Now, we want to show that the relationship holds when starting with **(3)**. First, notice that by adding the next highest order stream segment length to the sum on the RHS of **(1)** we get

$$\bar{\ell}_{\omega+1} = \bar{s}_{\omega+1} + \sum_{i=1}^{\omega} \bar{s}_i \tag{8}$$

From our assumption of **(1)**, **(8)** implies the following:

$$\bar{\ell}_{\omega+1} - \bar{\ell}_{\omega} = \bar{s}_{\omega+1}$$

$$\implies R_\ell^{\omega}\ell_1 - R_\ell^{\omega-1}\ell_1 = \bar{s}_{\omega+1}$$

$$\implies R_\ell^{\omega}\left(1 - \frac{1}{R_\ell}\right)\ell_1 = \bar{s}_{\omega+1}$$

$$\implies R_\ell^{\omega} \propto \bar{s}_{\omega+1} \tag{9}$$

As before, we can extend **(9)** to show that

$$\frac{R_\ell^{\omega}}{R_\ell^{\omega-1}} \propto \frac{\bar{s}_{\omega+1}}{\bar{s}_{\omega}} \tag{10}$$

and from **(10)**, **(9)** can be rewritten as

$$R_\ell \bar{s}_\omega = \bar{s}_{\omega+1} \tag{11}$$

Finally, recall that Horton's law of stream segment length, **(3)**, the ratio of stream segments of sequential order is equal to R_s. Hence, from **(11)** we find

$$R_\ell = \frac{\bar{s}_{\omega+1}}{\bar{s}_\omega} \tag{12}$$

$$\implies R_\ell = R_s \therefore \tag{13}$$

We have shown equality using two of Horton's laws for stream length and stream segment length, and can therefore conclude **(13)**.

0.3 Problem 3

Tokunaga's law implies Horton's laws → In lectures, we established the following:

$$n_\omega = 2n_{\omega+1} + \sum_{\omega'=\omega+1}^{\Omega} T_{\omega'-\omega} n_{\omega'} \tag{14}$$

From here, derive Horton's law for stream numbers: $\frac{n_\omega}{n_{\omega+1}}$, where $R_{n>1}$ and is independent of ω, and find R_n in terms of Tokunaga's two parameters T_1 and R_T.

From **Problem 1**, we may use the two properties established by Tokunaga; that the number of streams entering higher order streams as tributaries, is independent of the size of the order and the number of side streams grows exponentially with the difference in stream orders, and substitute

$$T_{\omega'-\omega} = T_1 R_T^{\omega'-\omega-1} \tag{15}$$

$$\implies n_\omega = 2n_{\omega+1} + \sum_{\omega'=\omega+1}^{\Omega} T_1 R_T^{\omega'-\omega-1} n_{\omega'} \tag{16}$$

Given **(16)**, then the following also holds:

$$n_{\omega+1} \;=\; 2n_{\omega+2} \;+\; \sum_{\omega'=\omega+2}^{\Omega} T_1 R_T^{\omega'-\omega-2} n_{\omega'} \tag{17}$$

Notice that, besides the starting index of the geometric series, multiplying **(17)** by the ratio R_T ensures that the product within the sum of **(16)** and **(17)** will be the same, since

$$\begin{aligned}
&R_T \sum_{\omega'=\omega+2}^{\Omega} T_1 R_T^{\omega'-\omega-2} n_{\omega'} \\
&= R_T\Big(T_1 R_T^0 n_{\omega+2} \;+\; T_1 R_T^1 n_{\omega+3} \;+\; T_1 R_T^2 n_{\omega+4} \;+\; \ldots \;+\; T_1 R_T^{\Omega-\omega-2} n_\Omega\Big) \\
&= \Big(T_1 R_T^1 n_{\omega+2} \;+\; T_1 R_T^2 n_{\omega+3} \;+\; T_1 R_T^3 n_{\omega+4} \;+\; \ldots \;+\; T_1 R_T^{\Omega-\omega-1} n_\Omega\Big) \\
&= \sum_{\omega'=\omega+2}^{\Omega} T_1 R_T^{\omega'-\omega-1} n_{\omega'}
\end{aligned}$$

By taking the difference **(17)**R_T − **(16)** therefore, we get

$$R_T n_{\omega+1} \;-\; n_\omega \;=\; 2R_t n_{\omega+2} \;+\; \sum_{\omega'=\omega+2}^{\Omega} T_1 R_T^{\omega'-\omega-1} n_{\omega'} \;-\; 2n_{\omega+1} \;-\; \sum_{\omega'=\omega+1}^{\Omega} T_1 R_T^{\omega'-\omega-1} n_{\omega'}$$

$$\implies\; R_T n_{\omega+1} \;-\; n_\omega \;=\; 2R_T n_{\omega+2} \;+\; \cancel{\sum_{\omega'=\omega+2}^{\Omega} T_1 R_T^{\omega'-\omega-1}} n_{\omega'} \;-\; 2n_{\omega+1} \;-\; T_1 n_{\omega+1} \;-\; \cancel{\sum_{\omega'=\omega+2}^{\Omega} T_1 R_T^{\omega'-\omega-1}} n_{\omega'}$$

$$\implies\; -n_\omega \;=\; -R_T n_{\omega+1} \;+\; 2R_T n_{\omega+2} \;-\; 2n_{\omega+1} \;-\; T_1 n_{\omega+1}$$

$$\implies\; \frac{n_\omega}{n_{\omega+1}} \;=\; \frac{\cancel{n_{\omega+1}}(2 \;+\; R_T \;+\; T_1)}{\cancel{n_{\omega+1}}} \;-\; \frac{2R_T n_{\omega+2}}{n_{\omega+1}}$$

$$\implies\; \frac{n_\omega}{n_{\omega+1}} \;=\; (2 \;+\; R_T \;+\; T_1) \;-\; \frac{2R_T n_{\omega+2}}{n_{\omega+1}} \tag{18}$$

We now utilize Horton's law of stream numbers, which says that ratio of the number of streams of sequential order is independent of order ω, i.e:

$$\frac{n_\omega}{n_{\omega+1}} = R_n \tag{19}$$

to extend **(18)**:

$$R_n = (2 + R_T + T_1) - \frac{2R_T}{R_N} \tag{20}$$

We would like to write R_n in terms of T_1 and R_T by separating terms on the LHS and RHS respectively. To do this, we will multiply each side of the expression by R_n to get

$$R_n^2 = (2 + R_T + T_1)R_n - 2R_T$$

$$0 = R_n^2 - (2 + R_T + T_1)R_n + 2R_T \tag{21}$$

Notice that we can now solve for the roots of R_n in terms of the parameters R_T and T_1 using the quadratic formula.

$$R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2} \therefore \tag{22}$$

0.4 Problem 4

Show $R_n = R_a$ by using Tokunaga's law to find the average area of an order ω basin, $\bar{a}_\omega$, in terms of the average area of basins of order 1 to $\omega - 1$.

Connect $\bar{a}_\omega$ to the average areas of basins of lower orders as follows:

$$\bar{a}_\omega = 2\bar{a}_{\omega-1} + \sum_{\omega'=1}^{\omega-1} T_{\omega,\omega'}\bar{a}_{\omega'} + 2\delta\bar{s}_\omega \tag{23}$$

The first term on the right hand side corresponds to the two 'generating' streams of order $\omega - 1$. The second term (the sum) accounts for side streams entering the sole order ω stream segment in the basin. And the last term gives the contribution of 'overland flow,' i.e., flow that does not arrive in the main stream segment through a stream. The length scale δ is the typical distance from stream to ridge.

As we did in the last problem, we may use Tokunaga's law to rewrite **(23)** as follows:

$$\bar{a}_{\omega} \;=\; 2\bar{a}_{\omega-1} \;+\; \sum_{\omega'=1}^{\omega-1} T_1 R_T^{\omega-\omega'-1} \bar{a}_{\omega'} \;+\; 2\delta \bar{s}_{\omega} \tag{24}$$

In addition, notice that by incrementing the index by one, we get an equivalent expression:

$$\bar{a}_{\omega+1} \;=\; 2\bar{a}_{\omega} \;+\; \sum_{\omega'=1}^{\omega} T_1 R_T^{\omega-\omega'} \bar{a}_{\omega'} \;+\; 2\delta \bar{s}_{\omega+1} \tag{25}$$

We will use similar tactics to the ones used in **Problem 3** to reduce the expression in terms of the parameters T_1 and R_T. Notice that by multiplying **(23)** by R_T, we re-index of the geometric series (the sum term) such that it is equivalent to the sum term in **(24)**.

$$R_T \sum_{\omega'=1} T_1 R_T^{\omega\,-\,\omega'-1} \bar{a}_{\omega'}$$

$$=\; R_T\Big(T_1 R_T^{\omega\,-\,2}\bar{a}_1 \;+\; T_1 R_T^{\omega\,-\,3}\bar{a}_2 \;+\; T_1 R_T^{\omega\,-\,4}\bar{a}_3 \;+\; \ldots \;+\; T_1 R_T^{0}\bar{a}_{\omega-1}\Big)$$

$$=\; T_1 R_T^{\omega\,-\,1}\bar{a}_1 \;+\; T_1 R_T^{\omega\,-\,2}\bar{a}_2 \;+\; T_1 R_T^{\omega\,-\,3}\bar{a}_3 \;+\; \ldots \;+\; T_1 R_T^{1}\bar{a}_{\omega-1}$$

$$\sum_{\omega'=1}^{\omega-1} T_1 R_T^{\omega\,-\,\omega'} \bar{a}_{\omega'} \therefore$$

We then calculate the difference R_T**(24) - (25)** to get

$$\bar{a}_{\omega} R_T \;-\; \bar{a}_{\omega+1} \;\ldots$$

$$=\; 2\bar{a}_{\omega-1}R_T \;+\; \sum_{\omega'=1}^{\omega-1} T_1 R_T^{\omega\,-\,\omega'} \bar{a}_{\omega'} \;+\; 2\delta\bar{s}_{\omega}R_T \;-\; 2\bar{a}_{\omega} \;-\; \sum_{\omega'=1}^{\omega} T_1 R_T^{\omega\,-\,\omega'} \bar{a}_{\omega'} \;-\; 2\delta\bar{s}_{\omega+1}$$

$$\implies\; \bar{a}_{\omega}R_T \;-\; \bar{a}_{\omega+1} \;=\; 2\bar{a}_{\omega-1}R_T - 2\bar{a}_{\omega} \;-\; T_1\bar{a}_{\omega} \;+\; 2\delta\Big(\bar{s}_{\omega}R_T \;-\; \bar{s}_{\omega+1}\Big)$$

Dividing by $\bar{a}_\omega$, we get

$$\frac{\bar{a}_\omega R_T - \bar{a}_{\omega+1}}{\bar{a}_\omega} = \frac{2\bar{a}_{\omega-1}R_T}{\bar{a}_\omega} - \frac{2\bar{a}_\omega}{\bar{a}_\omega} - \frac{T_1\bar{a}_\omega}{\bar{a}_\omega} + \frac{2\delta\Big(\bar{s}_\omega R_T - \bar{s}_{\omega+1}\Big)}{\bar{a}_\omega}$$

$$\frac{\bar{a}_\omega R_T - \bar{a}_{\omega+1}}{\bar{a}_\omega} = \frac{2\bar{a}_{\omega-1}R_T}{\bar{a}_\omega} - \frac{2\bar{a}_\omega}{\bar{a}_\omega} - \frac{T_1\bar{a}_\omega}{\bar{a}_\omega} + 2\delta\frac{\bar{s}_\omega R_T}{\bar{a}_\omega} - 2\delta\frac{\bar{s}_{\omega+1}}{\bar{a}_\omega} \tag{26}$$

We typically find that $\bar{a}_\omega \gg \bar{s}_\omega R_T$ for large ω because $R_T \approx 2$. Hence, for the purpose of argument, we may claim that

$$\lim_{\omega\to\infty} 2\delta\frac{\bar{s}_\omega R_T}{\bar{a}_\omega} \to 0 \qquad \text{and} \qquad \lim_{\omega\to\infty} 2\delta\frac{\bar{s}_{\omega+1}}{\bar{a}_\omega} \to 0, \qquad \text{for large } \omega \tag{27}$$

With this claim, we first remind ourselves that Horton's law of stream basin area says there is a constant ratio among stream basins of sequential order $\frac{\bar{a}_{\omega+1}}{\bar{a}_\omega} = R_a$, and then extend **(26)** by saying

$$R_T - R_a \approx \frac{2R_T}{R_a} - 2 - T_1$$

$$\implies 0 = R_a + \frac{2R_T}{R_a} - R_T - T1 - 2$$

$$\implies 0 = R_a^2 - R_a\Big(R_T + T_1 + 2\Big) + 2R_T \tag{28}$$

Similarly to **Problem 3**, we solve the quadratic formula for R_a to arrive at

$$R_a = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1) - 8R_T}}{2} \tag{29}$$

We have now found **(22)** and **(29)** to be equivalent. Hence, we can conclude $R_a = R_n \therefore$