COMPLEX NETWORKS, CSYS303

HW04 WRITE-UP

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Problem 1

Consider a set of rectangular areas with side lengths L_1 and L_2 such that $L_1 \propto A^{\gamma_1}$ and $L_2 \propto A^{\gamma_2}$, where A is area and $\gamma_2 + \gamma_2 = 1$. Assume $\gamma_1 > \gamma_2$ and that $\epsilon = 0$. Now imagine that material from a central source in each of these areas to sinks distributed with density $\rho(A)$, and that these sinks draw the same amount of material per unit time independent of L_1 and L_2.

To find the exact form for how the volume of the most efficient distribution network scales with overall area $A = L_1 L_2$, we can imagine some network where a source is centered at the set of rectangular areas, and delivers material to sinks in the outer quadrants of the space. The minimal distance between any sink and the central source is simply given by the Euclidean norm:

$$\sqrt{x_1^2 + x_2^2}$$

$$= ||\vec{x}||$$

where x_1 represents sink position in the L_1 direction, and x_2 represents sink position in the L_2 direction. Recall that each dimension L_1 and L_2 scales independently, so we need to integrate over the area with respect to the change in both directions to find the minimal volume of the network. Assuming that material from the central source is distributed with some density $\rho(A)$, then the minimal volume of the network is proportional to the following:

$$\mathbf{min} V_{\text{net}} \propto \int_A ||\vec{x}||\rho(A) dx_1 dx_2$$

We turn to the diagram below for a better intuition behind the network scaling. Source material flows from the heart to the blue points (p) distributed about the area. The set of rectangles can be considered the four quadrants of the larger area, and the Euclidean distance from the heart to a sink is the norm of a vector that originates from the source and points to a particular sink p in the network. Assuming allometric scaling of area, we will need to integrate with respect to the changes in both directions of the sinks, dx_1 and dx_2, over all quadrants. This is equivalent to solving the definite integral over the bounds of the first quadrant and multiplying by four.

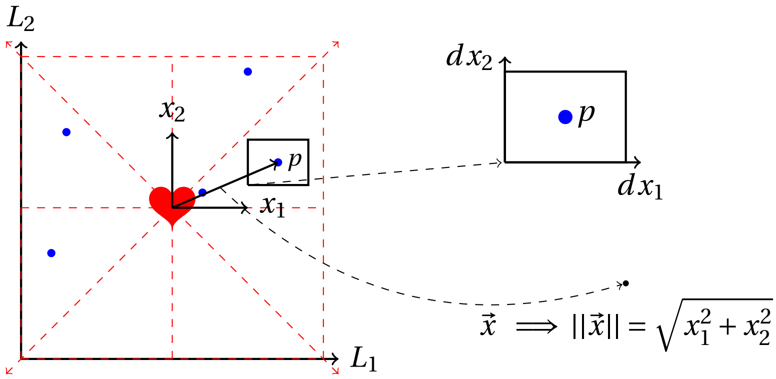


Figure 1: Distribution of material on a network, from a central source (heart) to sinks (p) distributed over an area A. L_1 and L_2 dimensions scale independently as indicated by the arrows, and the box to the right shows the possible change in position of a point p, relative to the scaling of the two dimensions.

We cannot solve directly for $\mathbf{min}\,V_{\text{net}}$ by integrating over the total area of the network:

$$\mathbf{min}\,V_{\text{net}} \propto \int_A ||\vec{x}||\rho(A)\,dx_1\,dx_2$$

$$= \rho(A) \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} dx_1 \int_{-\frac{L_2}{2}}^{\frac{L_2}{2}} dx_2 \sqrt{x_1^2 + x_2^2}$$

However, we will find that in solving the definite integral over the two triangular regions that separate each quadrant of the network (see the dashed red lines in **Fig. 1**), we may solve directly. We will let I_{Δ_1} denote the bottom half of the first quadrant separated by the dashed red line. The lower limits of integration are now 0 as the first triangle intersects the origin of the source. The upper limit of integration over x_1 remains $\frac{L_1}{2}$ while the upper limit of integration over x_2 becomes $\frac{L_2}{L_1}$:

$$I_{\Delta_1} = \rho(A) \int_0^{\frac{L_1}{2}} dx_1 \int_0^{\frac{L_2}{L_1}} dx_2 \sqrt{x_1^2 + x_2^2}$$

We may rewrite the norm of the vector $\vec{x}$ as:

$$\sqrt{x_1^2 + x_2^2} = x_1\sqrt{1 + \left(\frac{x_2}{x_1}\right)^2}$$

to get

$$\rho(A) \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} dx_1 \int_{-\frac{L_2}{2}}^{\frac{L_2}{2}} dx_2\ x_1\sqrt{1 + \left(\frac{x_2}{x_1}\right)^2}$$

We can perform a substitution by letting $u = \frac{x_2}{x_1}$, where $du = \frac{1}{x_1}dx_2$ to get:

$$\rho(A) \int_{0}^{\frac{L_1}{2}} dx_1 \int_{0}^{\frac{L_2}{L_1}} x_1\ du\ x_1\ \sqrt{1 + u^2}$$

$$\rho(A) \int_{0}^{\frac{L_1}{2}} x_1^2 dx_1 \int_{0}^{\frac{L_2}{L_1}} du\ \sqrt{1 + u^2}$$

To simplify the algebra, we will solve for the first half of the integral and set it aside:

$$\rho(A) \int_{0}^{\frac{L_1}{2}} x_1^2 dx_1$$

$$= \rho(A)\left(\frac{x_1^3}{3}\right)\Big|_0^{\frac{L_1}{2}}$$

$$= \rho(A)\left(\frac{L_1^3}{24}\right) \tag{1}$$

Focusing on the second half of the integral, we recognize a relation with the trigonometric identity $\sin^2(\theta) + \cos^2(\theta) = 1$ for any value θ:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\implies \frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$\implies \tan^2(\theta) + 1 = \frac{1}{\cos^2(\theta)}$$

Again, we may substitute $u = \tan(\theta)$, and $du = \sec^2(\theta)\ d\theta = \frac{1}{\cos^2(\theta)}\ d\theta$ to arrive at:

$$\int_0^{\frac{L_2}{L_1}} du\, \sqrt{1 + u^2}$$

$$= \int_0^{\frac{L_2}{L_1}} \frac{1}{\cos^2(\theta)} \sqrt{1 + \tan^2(\theta)}\ d\theta$$

$$= \int_0^{\frac{L_2}{L_1}} \frac{1}{\cos^2(\theta)} \frac{1}{\cos(\theta)}\ d\theta$$

$$= \int_0^{\frac{L_2}{L_1}} \frac{1}{\cos^3(\theta)}\ d\theta$$

We may proceed with integration by parts, which claims $\int w\, dv = wv - \int v\, dw$ for some w, v. Let $w = \frac{1}{\cos(\theta)}$ and $dv = \frac{1}{\cos^2(\theta)} d\theta$. Then, we have that $dw = \tan(\theta)\sec(\theta) d\theta$ and $v = \tan(\theta)$. We can solve as follows:

$$\int w\,dv = wv - \int v\,dw$$

$$= \frac{1}{\cos(\theta)}\ \tan(\theta) - \int \tan(\theta)\ \tan(\theta)\sec(\theta)d\theta$$

$$= \frac{1}{\cos(\theta)}\ \frac{\sin(\theta)}{\cos(\theta)} - \int \tan^2(\theta)\frac{1}{\cos(\theta)}d\theta$$

$$= \frac{1}{\cos(\theta)}\ \frac{\sin(\theta)}{\cos(\theta)} - \int \frac{\sin^2(\theta)}{\cos^2(\theta)}\frac{1}{\cos(\theta)}d\theta$$

Recall that from the pythagorean identity we get:

$$\sin^2(\theta) + \cos^2(\theta) = 1 \implies \sin^2(\theta) = 1 - \cos^2(\theta)$$

We now have the following:

$$\int \frac{1}{\cos^3(\theta)} = \frac{1}{\cos(\theta)}\ \frac{\sin(\theta)}{\cos(\theta)} - \int \frac{1 - \cos(\theta)}{\cos(\theta)}\ \frac{1}{\cos(\theta)}d\theta$$

$$= \frac{\sin(\theta)}{\cos^2(\theta)} + \int \frac{1}{\cos(\theta)}d\theta - \int \frac{1}{\cos^3(\theta)}d\theta$$

The integral of $\frac{1}{\cos(\theta)}$, or $\sec(\theta)$, is a known expression:

$$\int \frac{1}{\cos(\theta)} = \ln\left(\ \tan(\theta) + \frac{1}{\cos(\theta)}\ \right)$$

We will use this fact to arrive at the a solution for $\int \frac{1}{\cos^3(\theta)}d\theta$:

$$\int \frac{1}{\cos^3(\theta)}d\theta = \frac{1}{2}\left[\ \frac{\sin(\theta)}{\cos^2(\theta)} + \ln\left(\ \tan(\theta) + \frac{1}{\cos(\theta)}\ \right)\right]$$

With our original task in mind, we must evaluate the solution over the interval $[0,\ \tan^{-1}(\ \frac{L_2}{L_1}\ )]$.

With the following identities:

$$\tan(\arctan(x)) = x,$$

$$\cos(\arctan(x)) = \frac{1}{\sqrt{(1 + x^2)}},$$

$$\sin(\arctan(x)) = \frac{x}{\sqrt{(1 + x^2)}}$$

we get the following solution when we evaluate over the specific interval:

$$\int_0^{\tan^{-1}(\frac{L_2}{L_1})} \frac{1}{\cos^3(\theta)} d\theta = \frac{1}{2} \left[\frac{\sin(\theta)}{\cos^2(\theta)} + \ln\left(\tan(\theta) + \frac{1}{\cos(\theta)} \right)\right] \Bigg|_0^{\tan^{-1}(\frac{L_2}{L_1})}$$

$$= \frac{1}{2} \left[x\sqrt{1 + x^2} + \ln\left(x + \sqrt{1 + x^2}\right)\right]$$

Here, we let $x = \frac{L_2}{L_1}$ for simplicity. We can reduce this further if we recognize that $\ln\left(x + \sqrt{1 + x^2}\right) = \sinh^{-1}(x)$. Hence, we continue with

$$\int_0^{\tan^{-1}(\frac{L_2}{L_1})} \frac{1}{\cos^3(\theta)} d\theta = \frac{1}{2} \left[x\sqrt{1 + x^2} + \sinh^{-1}(x)\right]$$

The Taylor expansion of the hyperbolic arcsin yields $x - \frac{1}{6}x^3 + \mathcal{O}(x^5)$. However, for simplicity we will truncate the expansion at the first term, x, since $(\frac{L_1}{L_2})^3$ is negligibly small. Hence

$$\frac{1}{2} \left[x\sqrt{1 + x^2} + \sinh^{-1}(x)\right] \approx \frac{1}{2} \left[x\sqrt{1 + x^2} + x\right]$$

Substituting $x = \frac{L_2}{L_1}$, we get

$$\frac{1}{2} \left[x\sqrt{1 + x^2} + x\right] = \frac{1}{2} \left[\frac{L_2}{L_1}\sqrt{1 + \left(\frac{L_2}{L_1}\right)^2} + \frac{L_2}{L_1}\right]$$

Combining our results from **(1)**, we can solve for I_{Δ_1}:

$$\rho(A)\Big(\frac{L_1^3}{24}\Big) \times \frac{1}{2}\left[\frac{L_2}{L_1}\sqrt{1 + \Big(\frac{L_2}{L_1}\Big)^2} + \frac{L_2}{L_1}\right]$$

$$\propto \; \rho(A)\left[L_2 L_1^2 \sqrt{1 + \Big(\frac{L_2}{L_1}\Big)^2} + L_2 L_1^2\right]$$

We need to add the area of the upper triangle of Quadrant 1 to solve. We now have:

$$\rho(A)\left[L_2 L_1^2 \sqrt{1 + \Big(\frac{L_2}{L_1}\Big)^2} + L_2 L_1^2 + L_2 L_1^2 \sqrt{L_2^2 + L_1^2} + L_1 L_2^2\right]$$

We use the following identity to help us group like-terms:

$$L_1\sqrt{1 + \Big(\frac{L_2}{L_1}\Big)^2} = \sqrt{L_1^2 + L_2^2}$$

This gives us the following:

$$\rho(A)\left[L_2 L_1 \sqrt{L_1^2 + L_2^2} + L_2 L_1^2 + L_2 L_1 \sqrt{L_2^2 + L_1^2} + L_1 L_2^2\right]$$

$$= \rho(A)\left[2\, L_2 L_1 \sqrt{L_1^2 + L_2^2} + L_2 L_1^2 + L_1 L_2^2\right]$$

Recall that $L_1 \propto A^{\gamma_1}$ and $L_2 \propto A^{\gamma_2}$ for $\gamma_1 > \gamma_2$, and $\gamma_1 + \gamma_2 = 1$. We can rewrite our expression as follows:

$$\rho(A)\left[2A\sqrt{A^{2\gamma_1} + A^{2\gamma_2}} + A^{1+\gamma_1} + A^{1+\gamma_2}\right]$$

$$= \rho(A)\left[2A\sqrt{A^{2\gamma_1} + A^{2-2\gamma_1}} + A^{1+\gamma_1} + A^{2-\gamma_1}\right]$$

Focusing on the radical term, we will naively claim the following:

$$\sqrt{A^{2\gamma_1} + A^{2-2\gamma_1}}$$

$$\sim \sqrt{A^{2\gamma_1}},\; A^{2\gamma_1} \gg A^{2-2\gamma_1}$$

Hence, we have:

$$= \rho(A)\left[2A\sqrt{A^{2\gamma_1} + A^{2-2\gamma_1}} + A^{1+\gamma_1} + A^{2-\gamma_1}\right] \approx \rho(A)\left[2A\sqrt{A^{2\gamma_1}} + A^{1+\gamma_1} + A^{2-\gamma_1}\right]$$

$$= \rho(A)\left[2A^{1+\gamma_1} + A^{1+\gamma_1} + A^{2-\gamma_1}\right]$$

$$= \rho(A)\left[3A^{1+\gamma_1} + A^{2-\gamma_1}\right]$$

Given the constraints we set on γ_1 and γ_2, we know that γ_1 is bounded such that $\frac{1}{2} < \gamma_1 \leq 1$. We can deduce that the minimal scaling of V_{net} occurs when $\gamma_1 \to \frac{1}{2}$. This implies that

$$V_{\text{net}} \propto \rho(A)\, A^{1+\gamma_1}$$

$$\implies \mathbf{min} V_{\text{net}} \propto A^{\frac{3}{2}} \tag{a}$$

If network volume must remain a constant fraction of overall area, then we have found that

$$cA \propto \rho(A)\, A^{1+\gamma_1}$$

$$\implies \rho(A) \propto cA^{-\gamma_1} \tag{b}$$

Problem 2

Show that for large V and $0 < \epsilon < \frac{1}{2}$, that

$$\mathbf{min} V_{\text{net}} \propto \int_{\Omega_{d,D(V)}} \rho ||\vec{x}||^{1-2\epsilon} d\vec{x} \sim \rho V^{1+\gamma_{\max}(1-2\epsilon)}$$

The scenario can be visualized similarly to the network in **Fig. 1**; an organism has a central source that delivers material over a volume to various sinks (nodes). Here, we have extended the argument to look at d-dimensional shapes in a volume that is $D \geq d$ dimensions.

Recall that we previously defined:

$$L_i = c_i^{-1} V^{\gamma_i}, \text{ where } \gamma_1 + \gamma_2 + \gamma_3 + \ldots + \gamma_d = 1, \tag{2}$$

$$\gamma_1 = \gamma_{\max} \geq \gamma_2 \geq \gamma_3 \geq \ldots \geq \gamma_d, \text{ and} \tag{3}$$

$$c = \prod_i c_i \leq 1, \text{ where } c \text{ is a shape factor.} \tag{4}$$

We will assume that the first k lengths scale in the same way with $\gamma_1 = \ldots \gamma_k = \gamma_{\max}$, and write $||\vec{x}|| = (x_1^2 + x_2^2 + \ldots + x_d^2)^{\frac{1}{2}}$. Hence, $||\vec{x}||^{1-2\epsilon}$ yields

$$||\vec{x}||^{1-2\epsilon} = (x_1^2 + x_2^2 + \ldots + x_d^2)^{\frac{1-2\epsilon}{2}} \tag{5}$$

We can substitute **(5)** into the integral to get

$$\mathbf{min} V_{\text{net}} \propto \int_{\Omega_{d,D(V)}} \rho(x_1^2 + x_2^1 + \ldots + x_d^2)^{\frac{1-2\epsilon}{2}} d\vec{x} \tag{6}$$

We will let $u_i = \frac{x_i}{L_i}$. One can think of this as an indicator of how much in the L_i direction a particular sink $||\vec{x}||$ extends. For each x_i, we get:

$$u_i = \frac{x_i}{L_i} \tag{7}$$

$$\implies x_i = L_i u_i \tag{8}$$

$$\implies dx_i = L_i \, du_i \tag{9}$$

From **(5)** and **(8)** we get the following:

$$||\vec{x}||^{\frac{1-2\epsilon}{2}} = (L_1^2 u_1^2 + L_2^2 u_2^2 + \ldots + L_d^2 u_d^2)^{\frac{1-2\epsilon}{2}} \tag{10}$$

Given $\vec{x}$ is a vector of components and **(8)** and **(9)**, we know the following:

$$\vec{x} = \begin{bmatrix} L_1 u_1 \\ L_2 u_2 \\ \vdots \\ L_d u_d \end{bmatrix}$$

$$\implies d\vec{x} = \prod_i L_i \, du_i = V \tag{11}$$

Hence, from **(10)** and **(11)**, we can re-write **(6)**:

$$\mathbf{min} V_{\text{net}} \propto \int_{\Omega_{d,D(V)}} \rho (L_1^2 u_1^2 + L_2^2 u_2^2 + \ldots + L_d^2 u_d^2)^{\frac{1-2\epsilon}{2}} (L_1 L_2 \ldots L_d) \, du_1 \, du_2 \ldots du_d \tag{12}$$

The terms ρ and $(L_1 L_2 \ldots L_d) = V$ are constant values and may be pulled out of the integrand. Hence, **(12)** becomes:

$$\rho \, (L_1 L_2 \ldots L_d) \int_{\Omega_{d,D(V)}} (L_1^2 u_1^2 + L_2^2 u_2^2 + \ldots + L_d^2 u_d^2)^{\frac{1-2\epsilon}{2}} \, du_1 \, du_2 \ldots du_d$$

$$\rho \, V \int_{\Omega_{d,D(V)}} (L_1^2 u_1^2 + L_2^2 u_2^2 + \ldots + L_d^2 u_d^2)^{\frac{1-2\epsilon}{2}} \, du_1 \, du_2 \ldots du_d \tag{13}$$

Recall the relationship from the assumption made in **(2)**, and that the first k lengths scale in the same way with $\gamma_1 = \ldots \gamma_k = \gamma_{\text{max}}$. Then the following is true:

(13) $\implies$

$$\rho \, V \int_{\Omega_{d,D(V)}} (c_1^{-2} V^{2\gamma_1} u_1^2 + c_2^{-2} V^{2\gamma_2} u_2^2 + \ldots + c_k^{-2} V^{2\gamma_k} u_k^2 + \ldots + c_d^{-2} V^{2\gamma_d} u_d^2)^{\frac{1-2\epsilon}{2}}$$

$\ldots$

$du_1 \, du_2 \ldots du_k \ldots du_d$

$$= \rho\, V \int_{\Omega_{d,D(V)}} V^{\frac{2\gamma_1(1\,-\,2\epsilon)}{2}} (c_1^{-2} u_1^2 \,+\, c_2^{-2} u_2^2 \,+\, \ldots \,+\, c_k^{-2} u_k^2 \,+\, \mathcal{O}(V^{(2\gamma_{k\,+\,1}\,-\,\gamma_k)}))^{\frac{1\,-\,2\epsilon}{2}}$$
$$\ldots\, du_1\, du_2 \,\ldots\, du_k \ldots\, du_d$$

$$= \rho\, V \int_{\Omega_{d,D(V)}} V^{\gamma_1(1\,-\,2\epsilon)} (c_1^{-2} u_1^2 \,+\, c_2^{-2} u_2^2 \,+\, \ldots \,+\, c_k^{-2} u_k^2 \,+\, \mathcal{O}(V^{(2\gamma_{k\,+\,1}\,-\,\gamma_k)}))^{\frac{1\,-\,2\epsilon}{2}}$$
$$\ldots\, du_1\, du_2 \,\ldots\, du_k \ldots\, du_d \tag{14}$$

Notice again that we have the constant term $V^{\gamma_1(1\,-\,2\epsilon)}$ that can pulled in front of the integral in **(14)**. As a result we arrive at a solution:

$$\mathbf{min} V_{\text{net}} \,\propto\, \rho\, V\, V^{\gamma_1(1\,-\,2\epsilon)}$$

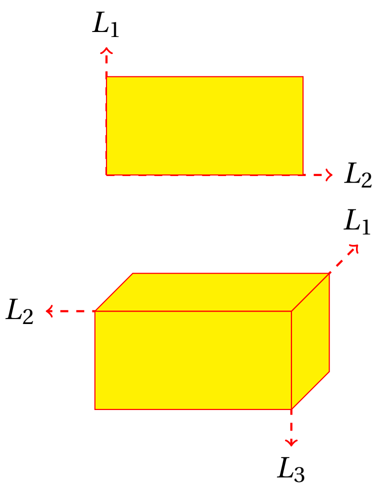
$$\implies \mathbf{min} V_{\text{net}} \,\propto\, \rho\, V^{1\,+\,\gamma_1(1\,-\,2\epsilon)}$$

$$\implies \mathbf{min} V_{\text{net}} \,\propto\, \rho\, V^{1\,+\,\gamma_{\max}(1\,-\,2\epsilon)} \;\therefore \tag{15}$$

Problem 3

(a)

Let S denote the hyper-surface area, and V denote the volume of a hyperrectangle (orthotope). We know that for $V \to \infty$, that $S \propto V^{\beta}$, for some scaling factor β. Extending a rectangle from 2-dimensions, we get the following:



$$S_A = 2\left(L_1 + L_2\right)$$

$$V = L_1 \times L_2$$

$$S_A = 2\left(L_1L_2 + L_1L_3 + L_2L_3\right)$$

$$V = L_1 \times L_2 \times L_3$$

$$\vdots$$

$$S_A \propto \sum_{i=1}^{d} \prod_{j=1, j\neq i}^{d} L_j$$

$$V \propto \prod_{i=1}^{d} L_i$$

Figure 2: For a d-dimensional space, we may construct a hyperrectangle. The exact calculation for volume and surface area of a hyperrectangle is intuitive in 2 and 3 dimensions. As $d \to \infty$, we can see that volume V is proportional to the infinite product over d of the dimensions L_i and surface area S_A is proportional to the infinite sum of products over d of the different combinations of dimensions L_i.

The intuition gained from the figure says that for a hyperrectangle in d-dimensions,

$$V \propto [\ell]^d$$

$$S \propto [\ell]^{d-1}$$

for some unit-less dimension of length, ℓ. From the problem statement, we know that for $V \to \infty$,

$$S \propto V^\beta$$

Using dimensional analysis, we may conclude that

$$[\ell]^{d-1} \propto \left[[\ell]^d\right]^\beta$$

$$\implies \beta \propto \frac{\ln\left([\ell]^{d-1}\right)}{\ln\left([\ell]^d\right)}$$

$$\implies \beta \propto \frac{d-1}{d}\frac{\cancel{\ln([\ell])}}{\cancel{\ln([\ell])}}$$

$$\implies \beta \propto \frac{d-1}{d}$$

$$\implies \beta \propto 1 - \frac{1}{d} \therefore$$

(b)

Based on the findings in (a), we get that $\beta \propto 1 - \frac{1}{d}$. For large d, we find that $\cancelto{0}{1/d}$. Therefore, $\beta \to 1$, for large d. By contrast, when $\cancelto{1}{1/d}$ for $d \to 1$, $\beta \to 0$. Hence, β is bounded.