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Shreve (p 4) 66

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N/W ⇒ check values there $h = 1/2$!! easy!!! + corrections!!

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CONSIDERATION ON THE COMPOSITION OF DRAINAGE NETWORKS AND THEIR EVOLUTION

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ABSTRACT

An extended interpretation of the term of the "cyclic net" defined by Scheidegger leads to a better understanding of the laws of drainage composition and the evolution of drainage basins.

The average number ${}_{\kappa}\varepsilon_{\lambda}$ of streams of order λ entering a stream of order κ from the sides provides two parameters: $\varepsilon_1 = {}_{\kappa}\varepsilon_{\kappa-1}$ and $K = ({}_{\kappa}\varepsilon_{\lambda} / {}_{\kappa}\varepsilon_{\kappa-1})^{1/(\kappa-\lambda-1)}$. A model of drainage basins is built on the assumption that each of the parameters is constant for various values of κ and λ in a network. The law of stream numbers of the model is formulated as ${}_{\kappa}\mu_{\lambda} = Q(Q^{\kappa-\lambda-1} - P^{\kappa-\lambda-1})(2+\varepsilon_1 - P)/(Q-P) + P^{\kappa-\lambda-1}(2+\varepsilon_1)$, where ${}_{\kappa}\mu_{\lambda}$ is the average number of streams of order λ in a basin order κ, $P = [2+\varepsilon_1 + K - \sqrt{(2+\varepsilon_1+K)^2 - 8K}]\,/\,2$ and $Q = [2+\varepsilon_1 + K + \sqrt{(2+\varepsilon_1+K)^2 - 8K}]\,/\,2$. On some reasonable assumptions, the law of basin areas and the law of stream lengths are also formalized by using ε_1 and K, viz., $A_{\lambda} = Q^{\lambda-l}A_l$ and $L_{\lambda} = Q^{(\lambda-l)/2}L_l$, where l is the lowest order of streams or basins, A_{λ} is the average area of basins of order λ, A_l is the average area of basins of the lowest order, L_{λ} is the average length of streams of order λ and L_l is the average length of stream of the lowest order. The condition of the "cyclic net" is satisfied basically in the model, because the relation of the streams of order λ to the streams of order $(\lambda+\eta)$ is the same as the relation of the streams of order $(\lambda+1)$ to the streams of order $(\lambda+1+\eta)$.

$Q^{1/2} = R_L$?? true for Shreve ?

$R_N \equiv R_a$

The equation which describes the law of stream numbers gives graphs on the Horton diagram which tend to be concave upward, except the case of $K=0$, and seems to be more adequate to describe the relationship between stream orders and numbers of actual drainage networks than Horton's formula. The average values of ε_1 and K in infinite topologically random channel networks are 1 and 2 respectively for various values of κ and λ. The most probable networks in the set of infinite topologically random channel networks also satisfy $\varepsilon_1 = 1$ and $K=2$.

Horton's R_n is a different thing

The law of allometric growth of drainage basins is formulated by using ε_1 and K as $m(t) = [\delta t + \ln\{(Q-P)/(2+\varepsilon_1-P)\}]\,/\,\ln Q + l$, where $m(t)$ is the order of a basin at time t and δ is constant. This equation holds exactly for basins of infinitely large value of $[m(t) - l]$ and to a fairly good approximation for basins of a comparatively large value of it.

It can be said that the model corresponds to basins in an equilibrium state and encompasses basins of the maximum entropy as a special case. The model seems to be very advantageous not only to investigate the composition of drainage networks but also to

explain their development.

I INTRODUCTION

In a paper famous among students of geomorphology and hydrology, Horton (1945) published an ordering system of streams and two important empirical laws of drainage composition. In his law of stream numbers, the relation between stream order and number of streams of each order in a basin is expressed as an inverse geometric series and, in his law of stream lengths, the relation between stream order and average length of streams of each order as a geometric series (Horton, 1945).

The two laws were originally conceived in terms of Horton's ordering method. But a modification was later proposed by Strahler (1952) in order to avoid the necessity of subjective decisions of parent streams, which is inherent in Horton's system. Further, it was proved by Scheidegger (1968) that, in a basin in which the two laws are satisfied in Horton's system, they are also satisfied in Strahler's system and vice versa. In this paper, Strahler's system is used exclusively by following examples of most recent students.

Horton (1945) also implied that there should be a law analogous to the law of stream lengths for basin areas, and the law of basin areas was later formalized by Schumm (1956).

The first attempt to give a theoretical explanation to these three laws was made by Leopold and Langbein (1962) based on statistical thermodynamics. Their explanation was made by using a model created by a graphical method (Leopold and Langbein, 1962). Such a method has some disadvantages: it requires rather highly simplified basic stream patterns in order to be practical, and it inherently involves Monte Carlo methods which make difficult the control of variables, such as the number of streams of the first order or the order of networks (Shreve, 1966).

A firm analystic approach to investigate topological and metrical characteristics of drainage networks based on the random graph theory was initiated by Shreve (1966, 1967). Shreve's concepts of link magnitude and topologically random channel networks provide the equations which describe the law of drainage composition of the expected state of randomly generated networks and contribute to clarify properties of these networks (Shreve, 1966, 1967, 1969; Tokunaga, 1972a, 1972b, 1974, 1975, 1977; Werner, 1972).

A different type of approaches to explain the laws of drainage composition theoretically has been made by Woldenberg (1966) and Scheidegger (1970). Then the explanation was made on the assumption that a drainage network is the result of a regular and cyclic growth process, in which new parts of the network are created with ever the same bifurcation ratio. This picture was formalized based on the allometric growth theory (Woldenberg, 1966; Scheidegger, 1970).

It can be said that the two types of models, the cyclic model and the randam graph models, have been widely introduced. It should be, however, noted that both the types have also disadvantages. The cyclic model shows mathematical inconsistency in any network except "structurally Hortonian networks" (Tokunaga, 1966; Smart, 1967; Scheidegger, 1968), and the random graph models themselves provide no equation to describe the composition of networks of which topology and metric are affected by non-random force.

yes it does!! but not in your interpretat !!

Furthermore, the two types of models are not compatible with each other, because the expected state of subbasins in infinite topologically random channel networks does not satisfy Horton's law of stream numbers (Shreve, 1969; Tokunaga, 1974).

This paper is written to show that the model proposed by the writer (1966) not only satisfies the condition of the "cyclic net" defined by Scheidegger (1970) in a different way from Horton's law of stream numbers but also encompasses one of the random graph models as a special case (Tokunaga, 1974), that two parameters used in the model combine the law of stream numbers with the law of basin areas, the law of stream lengths, and so on (Tokunaga, 1975), and that the allometric growth of drainage basins is also explained by using the model (Tokunaga, 1979).

cool

This paper summarizes the writer's previous publications (Tokunaga, 1966, 1972a, 1972b, 1974, 1975, 1977, 1979). Some detailed and additional considerations on properties of the model are referred to in those publications.

II CYCLIC SYSTEM

The term of the "cycle of rivers" was used at first by Scheidegger (1968). By his definition, a drainage network is considered to be cyclic, when it satisfies the condition that "each cycle (referring to a particular stream order) is entirely similar to the previous and following cycles" (Sheidegger, 1970). Then we may define a term "cyclic system" as a system which satisfies the above condition. Then Horton's law of stream numbers is considered to be a mathematical expression of such a system. Horton himself, in fact, had essentially an idea of the cyclic system and has attempted a hydrophysical explanation of his laws in terms of a growth process (Horton, 1945).

all based on dubs interp

It is, however, clear that, in any network except "structurally Hortonian networks" (Scheidegger, 1968), Horton's law of stream numbers itself proves to be mathematically inconsistent, because, in a network which exactly satisfies Horton's law of stream numbers, the subbasins do not satisfy it at all, or not with the same bifurcation ratio as the main basin (Tokunaga, 1966; Smart, 1967). This requires us to examine Horton's law of stream numbers itself but by no means enforces us to abandon the concept of the cyclic system. A model proposed by the writer (Tokunaga, 1966) not only satisfies the definition of the cyclic system in a different way from Horton's law of stream numbers but also provides adequate equations to describe the laws of drainage composition (Tokunaga, 1966, 1974, 1975).

The model is built as follows. The average number ${}_\kappa\varepsilon_\lambda$ of streams of order λ entering a stream of order κ from the sides provides parameters ε_1 and K, on the assumption that the following relations are satisfied in a network.

1. $${}_m\varepsilon_{m-1} = {}_{m-1}\varepsilon_{m-2} = \cdots\cdots = {}_\kappa\varepsilon_{\kappa-1} = \cdots\cdots = {}_\lambda\varepsilon_{\lambda-1} = \cdots\cdots = {}_{l+1}\varepsilon_l = \varepsilon_1$$

$${}_m\varepsilon_{m-2} = {}_{m-1}\varepsilon_{m-3} = \cdots\cdots = {}_\kappa\varepsilon_{\kappa-2} = \cdots\cdots = {}_\lambda\varepsilon_{\lambda-2} = \cdots\cdots = {}_{l+2}\varepsilon_l = \varepsilon_2$$

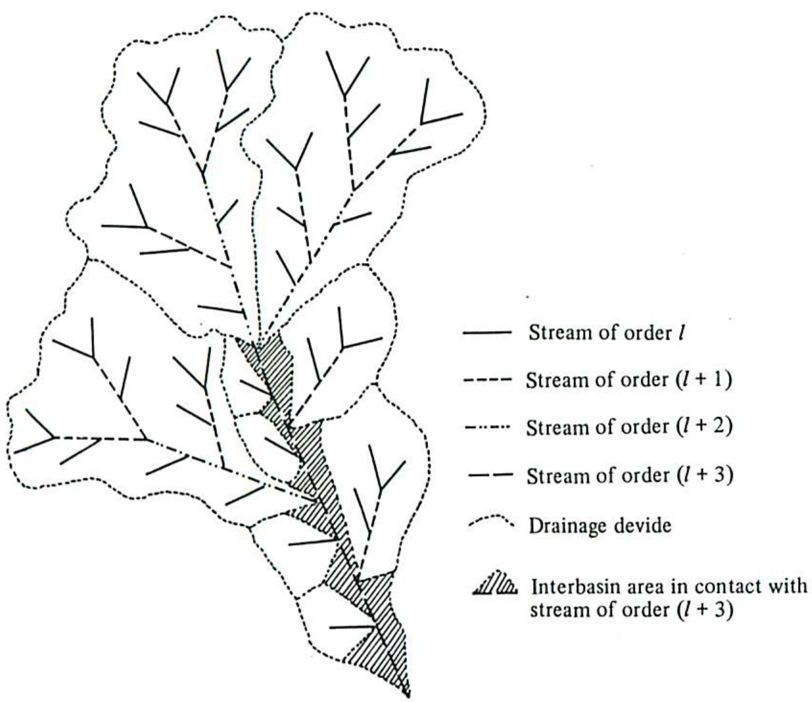
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

. . . .

$$_m\varepsilon_{m-\zeta} = {}_{m-1}\varepsilon_{m-1-\zeta} = \cdots = {}_\kappa\varepsilon_{\kappa-\zeta} = \cdots\cdots = {}_\lambda\varepsilon_{\lambda-\zeta} = \cdots\cdots = {}_{l+\zeta}\varepsilon_l = \varepsilon_\zeta$$

2. $$\frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_3}{\varepsilon_2} = \cdots\cdots\cdots = \frac{\varepsilon_\zeta}{\varepsilon_{\zeta-1}} = K$$

where m is the highest order and l is the lowest order on maps of a given scale. The validity of the assumption in actual drainage basins will be discussed later in this chapter. It will also be shown in Chapter IV that the average or most probable state of infinite topologically random channel networks* satisfies the assumption. A hypothetical network with $\varepsilon_1 = 1$ and $K = 2$ is illustrated in Fig. 1.



* The term of infinite topologically random channel networks is used in this paper to mean a set of topologically distinct channel networks with an infinite number of streams of the lowest order which are equally likely (cf. Chapter IV) as well as in the writer's previous publications (1972b, etc.).

exactly what I did –

The following equations are derived from the assumption (see Fig. 1).

$${}_\kappa\mu_\lambda = 2\,{}_\kappa\mu_{\lambda+1} + \sum_{\eta=\lambda+1}^{\kappa} \varepsilon_1 K^{\eta-\lambda-1}\,{}_\kappa\mu_\eta \qquad \text{(II-1)}$$

$${}_\kappa\mu_{\lambda+1} = 2\,{}_\kappa\mu_{\lambda+2} + \sum_{\eta=\lambda+2}^{\kappa} \varepsilon_1 K^{\eta-\lambda-2}\,{}_\kappa\mu_\eta \qquad \text{(II-2)}$$

where ${}_\kappa\mu_\lambda$ is the average number of streams of order λ in a basin of order κ, ${}_\kappa\mu_\kappa = 1$ and ${}_\kappa\mu_{\kappa-1} = 2 + \varepsilon_1$. Subtracting equation (II-2) x K from equation (II-1) yields the equation

$${}_\kappa\mu_\lambda = (2 + \varepsilon_1 + K)\,{}_\kappa\mu_{\lambda+1} - 2K\,{}_\kappa\mu_{\lambda+2} \qquad \text{(II-3)}$$

This recurrence equation provides the following equation in which ${}_\kappa\mu_\lambda$ is given in the form of sum of series (Tokunaga, 1966, 1974).

$${}_\kappa\mu_\lambda = \frac{Q^{\kappa-\lambda-1} - P^{\kappa-\lambda-1}}{Q - P}\, Q(2 + \varepsilon_1 - P) + P^{\kappa-\lambda-1}(2 + \varepsilon_1) \qquad \text{(II-4)}$$

where $P = [2+\varepsilon_1+K-\sqrt{(2+\varepsilon_1+K)^2-8K}]\,/\,2$ and $Q = [2+\varepsilon_1+K+\sqrt{(2+\varepsilon_1+K)^2-8K}]\,/\,2$. The mathematical procedure to obtain equation (II-4) is shown in Appendix I. Equation (II-4) expresses the law of stream numbers of the model. The model satisfies the definition of the cyclic system, because the relation of the streams of order λ to the streams of order $(\lambda + \eta)$ is the same as the relation of the streams of order $(\lambda + 1)$ to the streams of order $(\lambda + 1 + \eta)$, where $\eta = \pm 1, \pm 2, \ldots$ (see Fig. 1). Equation (II-4) gives graphs on the Horton diagram which tend to be concave upward in the part of the higher orders except the case of $K = 0$. When $K = 0$ and $\varepsilon_1 \neq 0$, we obtain $\varepsilon_2 = \varepsilon_3 = \cdots\cdots = \varepsilon_\eta = 0$. Then equation (II-4) expresses the law of stream numbers of "structurally Hortonian networks" and gives straight lines on the Horton diagram (Tokunaga, 1966, 1972a, 1972b).

Shreve (1966) applied a parabolic curve to each plot of 246 actual networks on the Horton diagram and calculated the value of coefficient of quadratic term to give the best fit to each plot. The sample mean values of coefficient are consistently positive, showing that the curves tend to be slightly concave upward, and these values decrease monotonically with increasing order of networks (Table 1). The values obtained by the similar application to plots given by substituting $\varepsilon_1 = 1$ and $K = 2$ into equation (II-4) are also presented in Table 1. The values show the similar tendency with the above sampled mean values. This implies that equation (II-4) is more adequate than Horton's formula to describe the

Table 1. Average of values of coefficient of quadratic term. (After Tokunaga 1974)

	order of networks or value of $(m-l+1)$				
	3	4	5	6	7
Actual networks (Shreve (1966))	0.160 (152)	0.095 (64)	0.042 (23)	0.013 (5)	0.010 (2)
Network with $\varepsilon_1 = 1$ and $K = 2$	0.084	0.053	0.035	0.025	0.018

The numbers in parentheses indicate the numbers of networks.

relationship of stream numbers to orders of actual drainage networks. The assumption that each of parameters ε_1 and K is constant in a network is examined in four subbasins in the Toyohira River Basin (Tokunaga, 1966). The calculated values of ${}_{\kappa}\varepsilon_{\lambda}$ approximately satisfy the assumption (Table 2).

Table 2. Values of ${}_{\kappa}\varepsilon_{\lambda}$, ε_{η} and K of four subbasins in the Toyohira River Basin*. (After Tokunaga 1966)

Subbasin	${}_{l+1}\varepsilon_l$	${}_{l+2}\varepsilon_{l+1}$	${}_{l+3}\varepsilon_{l+2}$	${}_{l+4}\varepsilon_{l+3}$	Average, ε_1
AA	1.26(214/170)	1.13(44/39)	1.29(9/7)		1.23
AB	1.13(196/174)	1.11(42/38)	2.17(13/6)**	1.00(3/3)**	1.12
BA	1.37(127/93)	1.21(23/19)	1.25(5/4)		1.28
BB	1.20(332/277)	1.05(59/56)	0.64(9/14)**	1.00(3/3)**	1.13
Average	1.24	1.13	1.27	1.00**	1.19
	${}_{l+2}\varepsilon_l$	${}_{l+3}\varepsilon_{l+1}$	${}_{l+4}\varepsilon_{l+2}$		Average, ε_2
AA	2.44 (95/39)	3.57(25/7)			3.01
AB	2.87(109/38)	3.67(22/6)	1.67(5/3)**		3.27
BA	3.37(64/19)	4.25(17/4)			3.81
BB	3.23(181/56)	2.86(40/14)	4.33(13/3)		3.47
Average	2.98	3.59	4.33		3.39
	${}_{l+3}\varepsilon_l$				Average, ε_3
AA	9.00(63/7)				9.00
AB	8.67(52/6)				8.67
BA	13.25(53/4)				13.25
BB	9.29(130/14)				9.29
Average	10.05				10.05

Subbasin	$K(\varepsilon_2/\varepsilon_1)$	$K(\varepsilon_3/\varepsilon_2)$	Average, K
AA	2.45	2.99	2.72
AB	2.92	2.65	2.79
BA	2.98	3.48	3.23
BB	3.07	2.68	2.88
Average	2.86	2.95	2.91

* Miscalculations in the original paper are corrected.
** The values are not used for the calculations of ε_{η} and K.

Let $(\kappa - \lambda) \to \infty$ in equation (II-4), then we obtain

$$ {}_{\kappa}\mu_{\lambda} = Q^{\kappa-\lambda}\,\frac{2+\varepsilon_1 - P}{Q-P} \qquad \text{(II-5)} $$

Here the value of Q which corresponds to the gradient of the asymptote to the plot in the part of the lower order points on the Horton diagram, does not depend on the orders of

excellent, mistakes !!

$Q = R_n$!!

Oops !

networks. Thereby Q was defined anew as a bifurcation ratio (Tokunaga, 1966).

Parameters ε_1 and K give us more information about the topology of a network than Q or Horton's bifurcation ratio, because even a given value of Q or Horton's bifurcation ratio provides many types of drainage patterns, depending on the values of ε_1 and K.

$(\varepsilon_1, k) \Leftrightarrow (\varepsilon_1, k)$

III INTERBASIN AREAS, BASIN AREAS AND STREAM LENGTHS

The law of stream numbers concerns the topology of a drainage network. On the contrary, the laws of basin areas and stream lengths concern the metric. Some students have paid effort to prove empirically that these three laws are not independent of each other (Morisawa, 1962; Smart, 1968). Theoretical considerations on such a problem need some assumptions (Shreve, 1967, 1969; Tokunaga, 1975). Shreve (1967), for example, derived the law of stream lengths from the law of stream numbers on the assumption that all of links have the same length in a topologically random channel network. He also derived the law of basin areas from the laws of stream numbers and stream lengths on the assumption that a basin area is proportional to the sum of the number of exterior links and the number of interior links in it and the drainage density is uniform in the whole basin. The writer does not agree to the latter assumption because the drainage density in the neighbourhood of streams of the higher order seems higher comparing with the density around the divide in an actual drainage basin. This will be discussed by using the writer's model in Chapter IV.

Shreve's equations describe only the expected state of subbasins in an infinite topologically random channel network and therefore the stream length ratio and the basin area ratio are given numerically in them (Shreve, 1969). Such equations are not given in the parameter representations.

It has been shown that the parameters ε_1 and K provide the equations which express the laws of numbers of interbasin areas, basin areas, areas of interbasin areas and stream lengths (Tokunaga, 1975). The equations are given as follows:

$$ N_{\lambda \cdot l} = 2 + \varepsilon_1 + \frac{\varepsilon_1 K (K^{\lambda-l-1} - 1)}{K-1} \qquad \text{(III-1)} $$

$$ A_{\lambda} = Q^{\lambda-l} A_l \qquad \text{(III-2)} $$

$$ \beta_{\lambda \cdot l} = \frac{\left[Q^{\lambda-l} - 2Q^{\lambda-l-1} - \dfrac{\varepsilon_1 (Q^{\lambda-l} - K^{\lambda-l})}{Q-K} \right] A_l}{2 + \varepsilon_1 + \dfrac{\varepsilon_1 K (K^{\lambda-l-1} - 1)}{K-1}} \qquad \text{(III-3)} $$

ouch

$$ L_{\lambda} = L_l Q^{(\lambda-l)/2} \qquad \text{(III-4)} $$

a bunch of ?

where $N_{\lambda \cdot l}$ is the average number of interbasin areas in contact with a stream of order λ, A_{λ} is the average area of basins of order λ, A_l is the average area of basins of the lowest order, $\beta_{\lambda \cdot l}$ is the average area of interbasin areas in contact with a stream of order λ, L_{λ} is the average length of streams of order λ and L_l is the average length of streams of the lowest order, when the lowest order is given by l (see Fig. 1).

The number of streams entering directly a stream of a given order is equal to the number of interbasin areas in contact with it (see Fig. 1). Consequently, we obtain equation (III-1) by summing up numbers of streams of each order entering directly a stream of order λ according to the assumption stated in the previous chapter (Tokunaga, 1975).

The last three equations are derived on some or all of the following assumptions. (1) A basin is divided into infinitesimally small subbasins and interbasin areas according to the assumption in the previous chapter. Then we can define A_j, $N_{\lambda \cdot j}$ and $\beta_{\lambda \cdot j}$ for smaller integral j, to the extent of $j=-\infty$, than l in all the same way as to define A_λ, $N_{\lambda \cdot l}$ and $\beta_{\lambda \cdot l}$ (see Appendix II). (2) The average area of subbasins of order η is larger than the average area $\beta_{\lambda \cdot \eta}$ of interbasin areas to the extent of $\eta=-\infty$. (3) The value of $K/(2+\varepsilon_1)$ is smaller than 1. (4) The following relation exists between basin areas and stream lengths.

$$L_\lambda = C\sqrt{A_\lambda}$$

what about Hack?!

where C is a non-dimensional coefficient concerning the shape of basins.

Equation (III-1) and the assumption in the previous chapter provide the following equation (see Fig. 1):

$$A_\lambda = 2A_{\lambda-1} + \sum_{\eta=l}^{\lambda-1} \varepsilon_1 K^{\lambda-\eta-1} A_\eta + \left[2 + \varepsilon_1 + \frac{\varepsilon_1 K(K^{\lambda-l-1}-1)}{K-1}\right]\beta_{\lambda \cdot l} \quad \text{(III-5)}$$

A shift of index in the above equation gives

$$A_{\lambda-1} = 2A_{\lambda-2} + \sum_{\eta=l}^{\lambda-2} \varepsilon_1 K^{\lambda-\eta-2} A_\eta + \left[2 + \varepsilon_1 + \frac{\varepsilon_1 K(K^{\lambda-l-2}-1)}{K-1}\right]\beta_{\lambda-1 \cdot l} \quad \text{(III-6)}$$

A recurrence equation which expresses the relation between A_λ, $A_{\lambda-1}$ and $A_{\lambda-2}$ is derived from the above two equations on the assumptions (1), (2) and (3), and it provides equation (III-2) on assumption (1). The mathematical procedure is shown in Appendix II.

The total area of interbasin areas in contact with a stream of order λ is obtained by subtracting the total area of subbasins of orders lower than λ which feed streams entering directly a stream of order λ from the area of basin of order λ (see Fig. 1). Therefore, equation (III-3) is derived from equations (II-4), (III-1) and (III-2) (Tokunaga, 1975). The mathematical procedure is shown in Appendix III.

Substituting $L_\lambda = C\sqrt{A_\lambda}$ and $L_l = C\sqrt{A_l}$ into equation (III-2) and eliminating C from the consequential equation yield equation (III-4).

Let us examine the assumptions. In infinite mathematical models, i.e. infinite topologically random channel networks, the unit of basin area is given arbitrarily and only the relative size of basin area has a meaning (Shreve, 1967, 1969, 1974; Tokunaga, 1975). Then it is considered mathematically reasonable to assume a basin of finite size consisting of infinitesimally small subbasins of the lowest order and interbasin areas, instead of an infinite basin consisting of subbasins of the lowest order and interbasin areas of finite size. Therefore, assumption (1) does not raise any problem in the above mentioned models. This assumption is satisfied approximately in an actuall drainage basin of comparatively large

area, because a very small value of A_η / A_λ can be obtained for a large value of $(\lambda-\eta)$ while the value of η is given finite. Assumption (2) is considered reasonable because, if the average area $\beta_{\lambda \cdot \eta}$ of interbasin areas is larger than the average area of subbasins of order η, streams should appear in many of the interbasin areas and such areas can not be any longer interbasin areas. The average state as well as the most probable state of topologically random channel networks satisfies $\varepsilon_1 = 1$ and $K=2$ (shown in the next chapter). This supports assumption (3). This assumption is also confirmed in the Toyohira River Basin (Table 2). Assumption (4) is considerd reasonable from the aspect of dimensional analysis because both sides of the equation have the same dimension. Some students have stated that mainstream length in drainage networks varies statistically in proportion to basin area raised to a power (Hack, 1957; Gray, 1961; Shreve, 1974). Equations (III-2) and (III-4) provide an equation which gives the values of slopes of plots of mainstream length versus basin area on a logarithmic paper for various values of $(\lambda - l)$. Then the value of slope decreases monotonically as $(\lambda - l)$ increases and reaches 1/2 for $(\lambda - l) \to \infty$ (Tokunaga, 1975). The scatter diagram for 461 actual drainage basins shows the same tendency (Shreve, 1974). This is highly encouraging for assumption (4).

IV TOPOLOGICALLY RANDOM CHANNEL NETWORKS

Within the set of all topologically distinct channel networks, the number $N(i, n)$ of ways in which n streams of a given order (here, let it be $(\eta-1)$) can produce i streams of the next higher order is (Shreve, 1966; Werner, 1972)

$$N(i, n) = \binom{n-2}{n-2i} 2^{n-2i} \frac{\binom{2i-1}{i}}{2i-1} \quad \text{(IV-1)}$$

This equation provides the equation which gives the average $E(n)$ of the distribution $N(i, n)$ as follows (Werner, 1972):

$$E(n) = \frac{\sum_{i=1}^{n/2} iN(i, n)}{\sum_{i=1}^{n/2} N(i, n)} = \frac{n}{4}\left(1 + \frac{1}{2n-3}\right) \quad \text{(IV-2)}$$

Equation (IV-1) also provides the equation which gives the average number $E_l(n)$ of streams of $(\eta-1)$ entering a link of order higher than $(\eta-1)$ from the sides in infinite topologically random channel networks as follows (Tokunaga, 1974):

$$E_l(n) = \frac{\sum_{i=1}^{n/2} (n-2i)N(i, n)}{\sum_{i=1}^{n/2} (2i-1)N(i, n)} \quad \text{(IV-3)}$$

Elimination of $\sum_{i=1}^{n/2} iN(i, n)$ and $\sum_{i=1}^{n/2} N(i, n)$ from equations (IV-2) and (IV-3) leads to the following result.

$$E_l(n) = \frac{n^2 - 2n}{n^2 - 3n + 3}, \ \lim_{n\to\infty} E_l(n) = 1 \tag{IV-4}$$

This implies that the average state of infinite topologically random channel networks satisfies $\varepsilon_1 = 1$. The implication is confirmed by proving that the average number of streams of order $(\eta - 1)$ entering a link of order η from the sides is equal to the average number of streams of order $(\eta - 1)$ entering a link of order higher than η in infinite topologically random channel networks (Tokunaga, 1974). The proof is given in Appendix IV. Then we can picture the cycles that a stream of order $(\eta - 1)$ enters a stream of order η from the sides on the average dividing the latter into two links and each of these links receives a stream of order $(\eta - 2)$ on the average, when the streams of order $(\eta - 2)$ join the networks, and so on. This means the average state of infinite topologically random channel networks also satisfies $K = 2$.

The writer has proved by using set-theory that the values of ε_1 and K of the most probable networks in infinite topologically random channel networks are also 1 and 2 respectively (Tokunaga, 1977). The proof and the brief explanation of the mathematical procedure are given in Appendix V. Substituting $\varepsilon_1 = 1$ and $K = 2$ into equations (II-4), (III-1), (III-2), (III-3) and (III-4) yields the equations which express the average or most probable state of compositions of infinite topologically random channel networks. The equations are written as follows:

$${}_{\kappa}\mu_{\lambda} = \frac{2}{3} 4^{\kappa - \lambda} + \frac{1}{3} \tag{IV-5}$$

$$N_{\lambda \cdot l} = 2^{\lambda - l} + 1 \tag{IV-6}$$

$$A_{\lambda} = 4^{\lambda - l} A_l \tag{IV-7}$$

$$\beta_{\lambda \cdot l} = \frac{2^{\lambda - l - 1}}{2^{\lambda - l} + 1} A_l \tag{IV-8}$$

certainly approaches a constant!

$$L_{\lambda} = 2^{\lambda - l} L_l \tag{IV-9}$$

Equations (IV-5), (IV-6), (IV-7), (IV-8) and (IV-9) are respectively special cases of equations (II-4), (III-1), (III-2), (III-3) and (III-4). Equation (IV-5) shows that the average or most probable state of infinite topologically random channel networks does not satisfy Horton's law of stream numbers. Shreve (1969) proved that the expected magnitude of a randomly drawn link of order κ is $(2^{2\kappa - 1} + 1)/3$ in an infinite topologically random channel network. The same equation is obtained by substituting $\lambda = l = 1$ into equation (IV-5). This means that the expected values of ε_1 and K of a randomly drawn subnetwork from an infinite topologically random channel network are 1 and 2 respectively.

The amount of length of a stream of order λ divided by the total area of the interbasin areas in contact with the stream of order λ represents approximately the drainage density of area around the stream of order λ. The average value of such amounts in infinite topologically random channel networks is calculated by using equations (IV-6), (IV-8) and

(IV–9) as follows:

$$\frac{L_{\lambda}}{N_{\lambda \cdot l}\beta_{\lambda \cdot l}} = \frac{2C^2}{L_l} \tag{IV-10}$$

This value reaches twice as high as the average value C^2/L_l of drainage densities of the basin areas of order l which represents approximately the drainage density of area around the divide. Then Shreve's assumption that the drainage density is uniform in the whole basin is not confirmable on the writer's model. If one still wishes to make Shreve's assumption to be compatible with equation (IV-7), he has to restrict the application of equation (IV-9) to the streams of orders equal to and higher than $(l + 1)$ and to assume the average length of streams of order l the same as that of streams of order $(l + 1)$ (Shreve, 1969). Data obtained from the measurement in actual drainage basins, however, do not support it so strongly (Shreve, 1969). Furthermore, it can be said that it is not so reasonable to break a cycle concerning stream lengths to make a cyclic system concerning basin areas. Therefore, it can be presumably concluded that the writer's assumptions are more favourable than Shreve's, to derive the laws of basin areas and stream lengths from the law of stream numbers even in infinite topologically random channel networks. Besides, the writer's assumptions present the additional laws: the laws of numbers of interbasin areas and areas of interbasin areas.

In finite topologically random channel networks, the average values of ε_1 and K of the most probable networks are very close to 1 and 2 respectively, inspite of exhibiting certain systematic deviations, and the deviations decrease monotonically as the magnitude of networks increases, especially in lower order subbasins (Tokunaga, 1972a, 1972b). This implies that the larger the basin, the more sufficiently the assumption stated in chapter II is satisfied in it, especially in its lower order subbasins, even in the case that ε_1 and K take values other than 1 and 2 respectively.

V ALLOMETRIC GROWTH OF DRAINAGE NETWORKS, OR BASINS

The idea to explain the laws of drainage composition in terms of a growth process was formalized at first by Woldenberg (1966). Then his work was done by recognizing Horton's law of stream numbers as a reliable law (Woldenberg, 1966). Scheidegger (1970) investigated the allometric growth of drainage networks based on the cyclic model as well as the random graph model. Then he regarded only the networks which obey Horton's law of stream numbers as cyclic, although he had noted that Horton's law has mathematical inconsistency in a strict sense as stated in Chapter II and is not exactly satisfied even in the random graph model. Such cyclic networks, viz. "structurally Hortonian networks", are very rare as Scheidegger himself mentioned (Sheidegger, 1970). Consequently, he had to conclude that "nature seems to favour the random graph model" (Sheidegger, 1970). Then he considered both the cyclic model and the random graph model to indicate stationary states respectively. Should the problem be solved by adopting the alternative of the two stationary models?

The writer's model also explains the allometric growth of drainage basins which satisfy the definition of the cyclic system encompassing the average or most probable state of infinite topologically random channel networks (Tokunaga, 1979). The mathematical procedure is carried out in similar way to Scheidegger's (1970).

Substitute m into κ, and l into λ in equation (II-5), then the equation becomes as follows:

$$_{m}\mu_{l} = Q^{m-l}\frac{2+\varepsilon_1-P}{Q-P} \quad \text{(V-1)}$$

where m is the order of a basin.

Comparatively large basins are considered to satisfy approximately equations (III-2) and (V-1). The following equation is derived by substituting $\lambda = m$ into equation (III-2) and eliminating Q^{m-l} from the consequential equation and equation (V-1).

$$A_m = \alpha_{m}\mu_{l} \quad \text{(V-2)}$$

where $\alpha = (Q-P)A_l / (2+\varepsilon_1-P)$ and α is regarded as constant in a drainage basin.

Scheidegger used a similar relation to the above as an assumption in his cyclic and random graph models (Scheidegger, 1970). It should be, however, noted that the relation is applied fairly well in a comparatively large basin. The allometric growth of basins which satisfy equation (V-2) is formalized in much the similar way to Scheidegger's method (1970). The number of streams of the lowest order in a basin is assumed to be a continuous function of time t.

Let $_{m}\mu_{l}(t)$ denote the number of streams of the lowest order at time t and substitute $_{m}\mu_{l} = {_{m}\mu_{l}}(t)$ into equation (V-1), then the following equation is obtained.

$$m = \left[\ln \frac{_{m}\mu_{l}(t)(Q-P)}{2+\varepsilon_1-P}\right] / \log Q + l \quad \text{(V-3)}$$

Equation (V-3) gives the order of the basin at time t.

Here let $_{m}\dot{\mu}_{l}(t)$ denote the rate of addition of streams of the lowest order to the basin at time t, then the following equation is derived from equation (V-2) on Woldenberg's assumption that the rate of growth of the basin (capture of streams of the lowest order) is proportional to the size of the basin (Woldenberg, 1966).

$$_{m}\dot{\mu}_{l}(t) = \gamma A_m = \delta_{m}\mu_{l}(t) \quad \text{(V-4)}$$

where γ and δ are constants. Then the consequences of Woldenberg's assumption are as follows:

$$_{m}\mu_{l}(t) = e^{\delta t} \quad \text{(V-5)}$$

Let a certain number of streams of the lowest order be present at the beginning (for $t = t_0$) and let $m(t)$ denote the order of the basin at time t, then the following equation is obtained by substituting equation (V-5) into equation (V-3).

$$m(t) = \left[\delta t + \ln \frac{Q-P}{2+\varepsilon_1-P}\right] / \log Q + l \quad \text{(V-6)}$$

The above equation expresses approximately the allometric growth of large drainage basins

satisfying the definition of the cyclic system mentioned in Chapter II.

Setting $\varepsilon_1 = 1$, $P = 1$ and $Q = 4$ (these correspond to $\varepsilon_1 = 1$ and $K = 2$) and substituting these values into equation (V-6), leads to the equation which expresses approximately the allometric growth of a basin corresponding to the average or most probable state of subbasins in infinite topologically random channel networks (Tokunaga, 1979).

• The values of $_{m}\mu_{l}$ calculated by equation (II-4) and by equation (II-5) are very close to each other for a considerably large value of $(m-l)$. For example, we obtain $_{m}\mu_{l} = 171$ by equation (II-4) and $_{m}\mu_{l} = 170.667$ by equation (II-5) for $(m-l) = 4$, when $\varepsilon_1 = 1$ and $K = 2$. On the contrary, we obtain $(m-l) = 4$ by the former equation and $(m-l) = 4.001$ by the latter for $_{m}\mu_{l} = 171$. These values should also be regarded as to be very close to each other.

The allometric growth of drainage basins can be more precisely expressed by displaying the numerical values calculated by equation (II-4) for various values of ε_1, K and $(\kappa - \lambda)$. It is, however, inferred from the above sampled examination that such a procedure is not necessary to understand the allometric growth of basins.

VI NON-RANDOM FORCE ACTING ON DRAINAGE NETWORKS

The composition of any actual drainage basin is strongly controlled by randomness (Leopold and Langbein, 1962; Shreve, 1966, 1967, 1969, 1974; Liao and Scheidegger, 1968; Werner, 1972; Tokunaga, 1972a, 1972b, 1974, 1975). It is also true that the composition of most of actual drainage basins is affected by non-random force (Werner, 1972; Tokunaga, 1974). Some facts and evidences to prove it can be easily presented. If the composition of a network of which magnitude exceeds 1000 is determined completely at random, it rarely happens that the value of $_{m}\mu_{l} / {_{m}\mu_{l+1}}$ is larger than 4.5 or smaller than 3.1 (Tokunaga, 1977). However, such networks which satisfy the above condition can be easily found out in nature (Morisawa, 1962; Tokunaga, 1966). The averages of values of coefficient for actual networks in Table 1, as regards the samples of which populations are considered to be large enough, reach 1.2 ~ 1.9 times as large as the values obtained by similar application to the plots by equation (IV-5). This means that any of the average values of Q of the samples given by applying equation (II-4) to actual drainage basins is larger than 4.

An equilibrium state of a system is kept on the balance of two opposing forces, viz., randomness and non-random force. Therefore, thermodynamics theory requires that the equations to describe the composition of a network in an equilibrium condition encompasses the equations to describe the composition of a network in the state of the maximum entropy. The cyclic system which grows allometrically is considered to be attained equilibrium (Woldenberg, 1966) and the average or most probable state of topologically random channel networks indicates a network of the maximum entropy. Then equations (II-4), (III-1), (III-2), (III-3) and (III-4) satisfy the above mentioned requirement.

Certain intensity of the non-random force acting uniformly on a network should give the corresponding values to the parameters of these equations. It is needed, at present, to find any methods to identify the non-random force isolating adequately from the randomness (Werner, 1972; Tokunaga, 1975, 1977). More advanced researches on these equations will lead to finding such a method and make it possible to explain regional differences of the values of the parameters.

VII CONCLUSION

The model of drainage basins proposed by the writer encompasses the average or most probable state of infinite topologically random channel networks as a special case as well as satisfies the definition of the cyclic system, viz., the condition of the "cyclic net" defined by Scheidegger (1970). The model also encompasses "structurally Hortonian networks".

The equation which describes the law of stream numbers of the model seems to be more suitable for actual drainage networks than Horton's formula. The parameters used in the equation also provide the laws of numbers of interbasin areas, basin areas, areas of interbasin areas and stream lengths. The allometric growth of drainage basins is also explained fairly well by using the model. Therefore, the model is considered to correspond to drainage basins in an equilibrium state encompassing basins of the maximum entropy as a special case. Further advanced researches on the model will provide a method to identify the non-random force acting on networks in such a state and explain regional differences of the composition of drainage networks.

In addition to the above conclusion, it is considered to be worth noting that the larger the basin, the more favourable the assumption in Chapter II, assumption (1) in Chapter III and equation (V-3) become for the basin, because the reason why the considerable parts of the continents are occupied by so large basins, e.g., the Amazon basin, might be expected in it.

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Appendix I

Equation (II-3) holds for all $\lambda \leqq \kappa-2$. Then, substiting $(\lambda+\eta-1)$ into λ, we obtain

$$_{\kappa}\mu_{\lambda+\eta-1} = (2+\varepsilon_1+K)_{\kappa}\mu_{\lambda+\eta} - 2K_{\kappa}\mu_{\lambda+\eta+1} \qquad \text{(A-I-1)}$$

By using P and Q, the above equation is rewritten as follows:

$$_{\kappa}\mu_{\lambda+\eta-1} - P_{\kappa}\mu_{\lambda+\eta} = Q_{\kappa}\mu_{\lambda+\eta} - QP_{\kappa}\mu_{\lambda+\eta+1} \qquad \text{(A-I-2)}$$

Equation (A-I-2) x $Q^{\eta-1}$ is

$$Q^{\eta-1}{}_{\kappa}\mu_{\lambda+\eta-1} - Q^{\eta-1}P_{\kappa}\mu_{\lambda+\eta} = Q^{\eta}{}_{\kappa}\mu_{\lambda+\eta} - Q^{\eta}P_{\kappa}\mu_{\lambda+\eta+1} \qquad \text{(A-I-3)}$$

The product of equation (A-I-3) for $\eta = 1, 2, \cdots\cdots, \kappa-\lambda-1$ is

$$\prod_{\eta=1}^{\kappa-\lambda-1}(Q^{\eta-1}{}_{\kappa}\mu_{\lambda+\eta-1} - Q^{\eta-1}P_{\kappa}\mu_{\lambda+\eta}) = \prod_{\eta=1}^{\kappa-\lambda-1}(Q^{\eta}{}_{\kappa}\mu_{\lambda+\eta} - Q^{\eta}P_{\kappa}\mu_{\lambda+\eta+1}) \qquad \text{(A-I-4)}$$

where $_{\kappa}\mu_{\kappa} = 1$. This means

$$_{\kappa}\mu_{\lambda} - P_{\kappa}\mu_{\lambda+1} = Q^{\kappa-\lambda-1}(2+\varepsilon_1) - Q^{\kappa-\lambda-1}P \qquad \text{(A-I-5)}$$

This relation holds for all $\lambda \leqq \kappa-1$. Substituting $(\lambda+\eta-1)$ into λ in equation (A-I-5) and multiplying consequential equation by $P^{\eta-1}$, we obtain

$$P^{\eta-1}{}_{\kappa}\mu_{\lambda+\eta-1} - P^{\eta}{}_{\kappa}\mu_{\lambda+\eta} = Q^{\kappa-\lambda-\eta}P^{\eta-1}(2+\varepsilon_1) - Q^{\kappa-\lambda-\eta}P^{\eta} \qquad \text{(A-I-6)}$$

The sum of equation (A-I-6) for $\eta = 1, 2, \cdots\cdots, (\kappa-\lambda-1)$ is

$$_{\kappa}\mu_{\lambda} - P^{\kappa-\lambda-1}{}_{\kappa}\mu_{\kappa-1} = \sum_{\eta=1}^{\kappa-\lambda-1}[Q^{\kappa-\lambda-\eta}P^{\eta-1}(2+\varepsilon_1) - Q^{\kappa-\lambda-\eta}P^{\eta}] \qquad \text{(A-I-7)}$$

Rearranging equation (A-I-7), we obtain equation (II-4).

AppendixII

Let us suppose basins of orders lower than l in the interbasin areas of which average areas are denoted by $\beta_{\lambda\cdot l}$ and $\beta_{\lambda-1\cdot l}$ and interbasin areas among these basins and streams of order λ and order $(\lambda-1)$, according to assumption (1). Then equations (III-5) and (III-6) are rewritten as follows:

$$A_{\lambda} = 2A_{\lambda-1} + \sum_{\eta=l}^{\lambda-1}\varepsilon_1 K^{\lambda-\eta-1}A_{\eta} + \sum_{\eta=j}^{l-1}\varepsilon_1 K^{\lambda-\eta-1}A_{\eta} + \left[2+\varepsilon_1+\frac{\varepsilon_1 K(K^{\lambda-j-1}-1)}{K-1}\right]\beta_{\lambda\cdot j} \qquad \text{(A-II-1)}$$

$$A_{\lambda-1} = 2A_{\lambda-2} + \sum_{\eta=l}^{\lambda-2} \varepsilon_1 K^{\lambda-\eta-2} A_\eta + \sum_{\eta=j}^{l-1} \varepsilon_1 K^{\lambda-\eta-2} A_\eta + \left[2+\varepsilon_1 + \frac{\varepsilon_1 K(K^{\lambda-j-2}-1)}{K-1}\right]\beta_{\lambda-1\cdot j} \qquad \text{(A-II-2)}$$

where j is the lowest order. A basin of a given order contains $(2+\varepsilon_1)$ subbasins of the next lower order on the average. Then

$$A_\lambda > (2+\varepsilon_1)A_{\lambda-1} > \cdots\cdots\cdots > (2+\varepsilon_1)^{\lambda-l}A_l >$$
$$\cdots\cdots\cdots > (2+\varepsilon_1)^{\lambda-j}A_j \qquad \text{(A-II-3)}$$

$$\varepsilon_1 K^{\lambda-j}\left(\frac{1}{2+\varepsilon_1}\right)^{\lambda-j} A_\lambda > \varepsilon_1 K^{\lambda-j} A_j \qquad \text{(A-II-4)}$$

Therefore, when A_λ has a finite value, as $(\lambda-j)\to\infty$, $\varepsilon_1 K^{\lambda-j}A_j \to 0$ in basins in which assumption (3) holds. Assumption (2) means that $\beta_{\lambda\cdot j} < A_j$ and $\beta_{\lambda-1\cdot j} < A_j$. Then $\left[2+\varepsilon_1+\frac{\varepsilon_1 K(K^{\lambda-j-1}-1)}{K-1}\right]\beta_{\lambda\cdot j} \to 0$ and $\left[2+\varepsilon_1+\frac{\varepsilon_1 K(K^{\lambda-j-2}-1)}{K-1}\right]\beta_{\lambda-1\cdot j} \to 0$ as $(\lambda-j)\to\infty$. Then, subtracting (A-II-2) x K from (A-II-1), we obtain the recurrence equation

$$A_\lambda = (2+\varepsilon_1+K)A_{\lambda-1} - 2KA_{\lambda-2} \qquad \text{(A-II-5)}$$

By using P and Q, this equation is rewritten as follows:

$$A_\lambda - PA_{\lambda-1} = QA_{\lambda-1} - QPA_{\lambda-2} \qquad \text{(A-II-6)}$$

This relation holds for all $\lambda \geqq l+2$. Then substituting $(\lambda-\eta+1)$ into λ in equation (A-II-6) leads to

$$A_{\lambda-\eta+1} - PA_{\lambda-\eta} = Q(A_{\lambda-\eta} - PA_{\lambda-\eta-1}) \qquad \text{(A-II-7)}$$

The product of equation (A-II-7) for $\eta = 1, 2, \cdots\cdots, \lambda-l-1$ is

$$\prod_{\eta=1}^{\lambda-l-1}(A_{\lambda-\eta+1} - PA_{\lambda-\eta}) = \prod_{\eta=1}^{\lambda-l-1}[Q(A_{\lambda-\eta} - PA_{\lambda-\eta-1})] \qquad \text{(A-II-8)}$$

This means

$$A_\lambda - PA_{\lambda-1} = Q^{\lambda-l-1}(A_{l+1} - PA_l) \qquad \text{(A-II-9)}$$

This relation holds for all $\lambda \geqq l+1$. Then, substituting $(\lambda-\eta+1)$ into λ in equation (A-II-9) and multiplying the consequential equation by $P^{\eta-1}$, we obtain

$$P^{\eta-1}(A_{\lambda-\eta+1} - PA_{\lambda-\eta}) = Q^{\lambda-\eta-l}P^{\eta-1}(A_{l+1} - PA_l) \qquad \text{(A-II-10)}$$

The sum of equation (A-II-10) for $\eta = 1, 2, \ldots\ldots, \lambda-l$ is

$$A_\lambda - P^{\lambda-l}A_l = Q^{\lambda-l-1}(A_{l+1} - PA_l) \times \sum_{\eta=1}^{\lambda-l}\left(\frac{P}{Q}\right)^{\eta-1} \qquad \text{(A-II-11)}$$

Rearranging the above equation, we obtain

$$A_\lambda = \frac{Q^{\lambda-l} - P^{\lambda-l}}{Q-P}(A_{l+1} - PA_l) + P^{\lambda-l}A_l \qquad \text{(A-II-12)}$$

Then

$$\frac{A_\lambda}{A_{\lambda-1}} = \frac{Q^{\lambda-l}\left[\left\{1-\left(\frac{P}{Q}\right)^{\lambda-l}\right\}\frac{A_{l+1}-PA_l}{Q-P} + \left(\frac{P}{Q}\right)^{\lambda-l}A_l\right]}{Q^{\lambda-l-1}\left[\left\{1-\left(\frac{P}{Q}\right)^{\lambda-l-1}\right\}\frac{A_{l+1}-PA_l}{Q-P} + \left(\frac{P}{Q}\right)^{\lambda-l-1}A_l\right]} \qquad \text{(A-II-13)}$$

Let $(\lambda-l)\to\infty$ in the above equation, then we obtain

$$A_\lambda = QA_{\lambda-1} \qquad \text{(A-II-14)}$$

We may suppose the same relation between A_{l+1} and A_l according to assumption (1). Then we obtain equation (III-2) by substituting $A_{l+1} = QA_l$ into equation (A-II-12).

Appendix III

The area $S(\beta_{\lambda\cdot l})$ occupied by the interbasin areas, of which average area is denoted by $\beta_{\lambda\cdot l}$, is obtained by subtracting the sum total of areas of the subbasins of orders from l to $(\lambda-1)$ from A_λ.

Namely,

$$S(\beta_{\lambda\cdot l}) = A_\lambda - (2+\varepsilon_1)A_{\lambda-1} - \varepsilon_1 K A_{\lambda-2} - \cdots\cdots\cdots - \varepsilon_1 K^{\lambda-l-1}A_l$$
$$= \left[Q^{\lambda-l} - 2Q^{\lambda-l-1} - \frac{\varepsilon_1(Q^{\lambda-l} - K^{\lambda-l})}{Q-K}\right]A_l$$

Dividing $S(\beta_{\lambda\cdot l})$ by $N_{\lambda\cdot l}$ gives equation (III-3).

Appendix IV

When n streams of order $(\eta-1)$ form a network, the average number $E_{e,l}(n)$ of streams of order $(\eta-1)$ entering a link of order η (here a stream of order η) from the sides and the average number $E_{i,l}(n)$ of streams of order $(\eta-1)$ entering a link of order higher than η in the set of all topologically distinct channel networks are given as follows:

$$E_{e,l}(n) = \frac{\sum_{i=1}^{n/2}\frac{(n-2i)i}{2i-1}N(i,n)}{\sum_{i=1}^{n/2} iN(i,n)} \qquad \text{(A-IV-1)}$$

$$E_{i,I}(n) = \frac{\sum_{i=1}^{n/2} \frac{(n-2i)(i-1)}{2i-1} N(i,n)}{\sum_{i=1}^{n/2} (i-1) N(i,n)} \tag{A-IV-2}$$

Put $A(i,n) = [(n-2i)N(i,n)]/(2i-1)$, then the following equation is derived for $E_{i,I}(n)/E_{e,I}(n)$.

$$\frac{E_{i,I}(n)}{E_{e,I}(n)} = \frac{\sum_{i=1}^{n/2} iN(i,n) \sum_{i=1}^{n/2} A(i,n)(i-1)}{\left[\sum_{i=1}^{n/2} iN(i,n) - \sum_{i=1}^{n/2} N(i,n)\right] \sum_{i=1}^{n/2} iA(i,n)}$$

$$= \frac{n(n-1)}{(n-2)(n-3)} \left[1 - \frac{\sum_{i=1}^{n/2} A(i,n)}{\sum_{i=1}^{n/2} iA(i,n)}\right] \tag{A-IV-3}$$

This equation shows that $\lim_{n\to\infty} [E_{i,I}(n)/E_{e,I}(n)] = 1$ if $\lim_{n\to\infty} \left[\sum_{i=1}^{n/2} A(i,n) / \sum_{i=1}^{n/2} iA(i,n)\right] = 0$. The proof is given below. Let us evaluate

$$\frac{A(i,n)}{A(i-1,n)} = \frac{(n-2i+1)(n-2i)}{4(i-1)(2i-1)} \left(\frac{2i-3}{i}\right) \tag{A-IV-4}$$

if $i \geqq 4$, then $(2i-3)/i > 1$, and $[(n-2i+1)(n-2i)]/[4(i-1)(2i-1)]$ decreases as i increases for a given value of n. Here substitute $i = n/8$ into $[(n-2i+1)(n-2i)]/[4(i-1)(2i-1)]$

$$\frac{(\frac{3n}{4}+1)\frac{3n}{4}}{4(\frac{n}{8}-1)(\frac{n}{4}-1)} > \frac{(\frac{3n}{4})^2}{4 \cdot \frac{n}{8} \cdot \frac{n}{4}} = \frac{72}{16} > 1$$

Therefore, $A(i,n)/A(i-1,n) > 1$ for $4 \leqq i \leqq n/8$. Substitute $i = 2, 3$ into equation (A-IV-4), then

$$\frac{A(2,n)}{A(1,n)} = \frac{(n-3)(n-4)}{24}, \quad \frac{A(3,n)}{A(2,n)} = \frac{(n-5)(n-6)}{40} \tag{A-IV-5}$$

These are larger than 1 for $n \geqq 16$. Hence the sequence is monotonically increasing for $i \leqq n/8$ at least, when $n \geq 16$, so

$$A(\frac{n}{16}-1, n) < A(\frac{n}{16}+2, n)$$

$$2A(\frac{n}{16}-2, n) < 2A(\frac{n}{16}+3, n)$$

....

....

$$iA(\frac{n}{16}-i, n) < iA(\frac{n}{16}+i+1, n)$$

....

....

$$(\frac{n}{16}-1)A(1,n) < (\frac{n}{16}-1)A(\frac{n}{8}, n)$$

Therefore,

$$\sum_{i=1}^{n/16-1} (\frac{n}{16}-i)A(i,n) = \sum_{i=1}^{n/16-1} iA(\frac{n}{16}-i, n)$$

$$< \sum_{i=1}^{n/16-1} iA(\frac{n}{16}+i+1, n) = \sum_{i=n/16+2}^{n/8} (i - \frac{n}{16} - 1)A(i,n) \tag{A-IV-6}$$

Here

$$\sum_{i=1}^{n/2} iA(i,n) = \sum_{i=1}^{n/16} iA(i,n) + \sum_{i=n/16+2}^{n/8} (i - \frac{n}{16} - 1)A(i,n) + (\frac{n}{16}+1) \sum_{i=n/16+1}^{n/8} A(i,n) + \sum_{i=n/8+1}^{n/2} iA(i,n) \tag{A-IV-7}$$

From inequality (A-IV-6) and equation (A-IV-7), it follows that

$$\sum_{i=1}^{n/16} iA(i,n) + \sum_{i=1}^{n/16-1} (\frac{n}{16}-i)A(i,n) + (\frac{n}{16}+1) \sum_{i=n/16+1}^{n/8} A(i,n) + \sum_{i=n/8+1}^{n/2} iA(i,n)$$

$$= \frac{n}{16} \sum_{i=1}^{n/16} A(i,n) + (\frac{n}{16}+1) \sum_{i=n/16+1}^{n/8} A(i,n) + \sum_{i=n/8+1}^{n/2} iA(i,n)$$

$$= \frac{n}{16} \sum_{i=1}^{n/2} A(i,n) + \sum_{i=n/16+1}^{n/8} A(i,n) + \sum_{i=n/8+1}^{n/2} (i - \frac{n}{16})A(i,n) < \sum_{i=1}^{n/2} iA(i,n)$$

Therefore,

$$\frac{n}{16} \sum_{i=1}^{n/2} A(i,n) < \sum_{i=1}^{n/2} iA(i,n) \tag{A-IV-8}$$

Here

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n/2} A(i,n)}{\frac{n}{16} \sum_{i=1}^{n/2} A(i,n)} = 0 \tag{A-IV-9}$$

Therefore,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n/2} A(i,n)}{\sum_{i=1}^{n/2} iA(i,n)} = 0 \qquad \text{(A-IV-10)}$$

From equations (IV-4), (A-IV-3) and (A-IV-10), it follow that

$$\lim_{n\to\infty} E_{e,l}(n) = \lim E_{i,l}(n) = 1$$

Appendix V

The number $N(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_\lambda, \cdots\cdots, 1)$ of topologically distinct channel networks of order λ having $n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_\lambda, \cdots\cdots, 1$ streams of order $l, (l+1), \cdots\cdots, \eta, \cdots\cdots, \lambda, \cdots\cdots, \Omega$ respectively is given by the following equation (Shreve, 1966).

$$N(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_\lambda, \cdots\cdots, 1) = \prod_{\omega=l}^{\Omega-1} 2^{n_\omega - 2n_{\omega+1}} \binom{n_\omega - 2}{n_\omega - 2n_{\omega+1}} \qquad \text{(A-V-1)}$$

The equation which expresses the number $N(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_\lambda)$ of topologically distinct channel networks having $n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_\lambda$ streams of order $l, (l+1), \cdots\cdots, \eta, \cdots\cdots, \lambda$, respectively, is derived from equations (A-V-1) and (IV-1).

$$N(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_\lambda) = \left[\prod_{\omega=l}^{\lambda-1} 2^{n_\omega - 2n_{\omega+1}} \binom{n_\omega - 2}{n_\omega - 2n_{\omega+1}}\right] \frac{\binom{2n_\lambda - 1}{n_\lambda}}{2n_\lambda - 1} \qquad \text{(A-V-2)}$$

Let $S(n_l)$ be the set of all topologically distinct channel networks having n_l streams of the lowest order and $S(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_\lambda)$ be the set of all topologically distinct channel networks having $n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_\lambda$ streams of order $l, (l+1), \cdots\cdots, \eta, \cdots\cdots, \lambda$, respectively, then the following two relations are derived.

$$S(n_l) \supset S(n_l, n_{l+1}) \supset \cdots\cdots \supset S(n_l, n_{l+1}, \cdots\cdots, n_\eta) \supset \cdots\cdots \supset S(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_\lambda) \qquad \text{(A-V-3)}$$

$$\begin{aligned} S(n_l) &= \sum_{n_{l+1}=0}^{m_{l+1}} S(n_l, n_{l+1}) = \cdots\cdots \\ &= \sum_{n_{l+1}=0}^{m_{l+1}} \cdots\cdots \sum_{n_\eta=0}^{m_\eta} S(n_l, n_{l+1}, \cdots\cdots, n_\eta) = \cdots\cdots \\ &= \sum_{n_{l+1}=0}^{m_{l+1}} \cdots\cdots \sum_{n_\eta=0}^{m_\eta} \cdots\cdots \sum_{n_\lambda=0}^{m_\lambda} S(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_\lambda) \end{aligned} \qquad \text{(A-V-4)}$$

where $m_{l+1} = n_l/2$ or $(n_{l+1}-1)/2, \cdots\cdots, m_\eta = n_{\eta-1}/2$ or $(n_{\eta-1}-1)/2, \cdots\cdots, m_\lambda = n_{\lambda-1}/2$ or $(n_{\lambda-1}-1)/2$, and $S(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_\lambda) = S(n_l, n_{l+1}, \cdots\cdots, n_{\eta-1}, 1)$, when $n_\eta = 1$. If $n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}$ values are given and these values are sufficiently large, there exists a subset of $S(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda)$ for every n_λ value which satisfies $1 \leqq n_\lambda \leqq n_{\lambda-1}/2$ or $(n_{\lambda-1}-1)/2$. Then we can take out all the topologically distinct channel networks which are given by n_λ value to provide a maximum value of $N(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda)$ from the collection of the subsets of $S(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda)$ and define the set of these networks. Let denote the set by $S_{\lambda\cdot max}(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda)$. The procedure to obtain the value of n_λ of $S_{\lambda\cdot max}(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda)$ is shown below.

If a small variation in n_λ value leaves $N(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda)$ unchanged when $n_l\ n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}$ values are given, then $N(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda)$ has a maximum. Thus n_λ value which satisfies

$$\frac{N(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda)}{N(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda - 1)} = 1 \qquad \text{(A-V-5)}$$

provides the maximum value. From equations (A-V-5) and (A-V-2), it follows that

$$\frac{(n_{\lambda-1} - 2n_\lambda + 2)(n_{\lambda-1} - 2n_\lambda + 1)}{4n_\lambda(n_\lambda - 1)} = 1 \qquad \text{(A-V-6)}$$

For very large values of $n_{\lambda-1}$ and n_λ, this expression is replaced by

$$\frac{(n_{\lambda-1} - 2n_\lambda)^2}{4n_\lambda^2} = 1 \qquad \text{(A-V-7)}$$

By solving equation (A-V-7), we obtain

$$n_\lambda = \frac{n_{\lambda-1}}{4} \qquad \text{(A-V-8)}$$

From this result and the definition of $S_{\lambda\cdot max}(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda)$, it follows that

$$S_{\lambda\cdot max}(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda) = S(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, \tfrac{1}{4}n_{\lambda-1}) \qquad \text{(A-V-9)}$$

This relation holds for $\lambda = l+1 \sim \infty$. Thus the following relations are obtained sequencially by substituting in relation (A-V-9) the consecutive integral values of λ starting with $\lambda = l+1$.

$$\left.\begin{aligned}
&S_{l+1\cdot max}(n_l, n_{l+1}) = S(n_l, \tfrac{1}{4}n_l)\\
&S_{l+2\cdot max}(n_l, \tfrac{1}{4}n_l, n_{l+2}) = S(n_l, \tfrac{1}{4}n_l, (\tfrac{1}{4})^2 n_l)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&S_{\eta\cdot max}(n_l, \tfrac{1}{4}n_l, \cdots\cdots, (\tfrac{1}{4})^{\eta-l-1}n_l, n_\eta)\\
&\quad = S(n_l, \tfrac{1}{4}n_l, \cdots\cdots, (\tfrac{1}{4})^{\eta-l-1}n_l, (\tfrac{1}{4})^{\eta-l}n_l)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&S_{\lambda\cdot max}(n_l, \tfrac{1}{4}n_l, \cdots\cdots, (\tfrac{1}{4})^{\eta-l}n_l, \cdots\cdots, (\tfrac{1}{4})^{\lambda-l-1}n_l, n_\lambda)\\
&\quad = S(n_l, \tfrac{1}{4}n_l, \cdots\cdots, (\tfrac{1}{4})^{\eta-l}n_l, \cdots\cdots, (\tfrac{1}{4})^{\lambda-l-1}n_l, (\tfrac{1}{4})^{\lambda-1}n_l)
\end{aligned}\right\} \quad \text{(A-V-10)}$$

When $n_l, n_{l-1}, \cdots\cdots, n_{\eta-1}$ values are given, let $S_{\eta\cdot max}(n_l, n_{l+1}, \cdots\cdots, n_\eta, \frac{1}{4}n_\eta, \cdots\cdots, (\frac{1}{4})^{\lambda-\eta-1}n_\eta, (\frac{1}{4})^{\lambda-\eta}n_\eta)$ be the set of all topologically distinct channel networks with n_η value which provides a maximum value of $N(n_l, n_{l+1}, \cdots\cdots, n_\eta, \frac{1}{4}n_\eta, \cdots\cdots, (\frac{1}{4})^{\lambda-\eta}n_\eta)$. Then n_η value of $S_{\eta\cdot max}(n_l, n_{l+1}, \cdots\cdots, n_\eta, \frac{1}{4}n_\eta, \cdots\cdots, (\frac{1}{4})^{\lambda-\eta}n_\eta)$ is obtained by solving

$$\frac{N(n_l, n_{l+1}, \cdots\cdots, n_\eta, \frac{1}{4}n_\eta, \cdots\cdots, (\frac{1}{4})^{\lambda-\eta}n_\eta)}{N(n_l, n_{l+1}, \cdots\cdots, n_\eta-1, \frac{1}{4}n_\eta, \cdots\cdots, (\frac{1}{4})^{\lambda-\eta}n_\eta)} = 1 \qquad \text{(A-V-11)}$$

From equations (A-V-11) and (A-V-2), it follows that

$$\frac{(n_\eta-2)\,(n_{\eta-1}-2n_\eta+2)\,(n_{\eta-1}-2n_\eta+1)}{n_\eta(2n_\eta-2)\,(2n_\eta-3)} = 1 \qquad \text{(A-V-12)}$$

For very large values of $n_{\eta-1}$ and n_η, the constant terms in the parentheses may be omitted. Then equation (A-V-12) leads to

$$n_\eta = \frac{n_{\eta-1}}{4} \qquad \text{(A-V-13)}$$

This relation holds for $\eta=l+1\sim\lambda$. Thus the following relations are derived sequentially by decreasing the consecutive integral values η starting with $\eta=\lambda$.

$$\left.\begin{aligned}
&S_{\lambda\cdot max}(n_l, n_{l+1}, \cdots\cdots, n_{\lambda-1}, n_\lambda)\\
&\quad = S(n_l, n_{l+1}, \cdots\cdots, n_{\lambda-1}, \tfrac{1}{4}n_{\lambda-1})\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&S_{\eta\cdot max}(n_l, n_{l+1}, \cdots\cdots, n_\eta, \tfrac{1}{4}n_\eta, \cdots\cdots, (\tfrac{1}{4})^{\lambda-\eta}n_\eta)\\
&\quad = S(n_l, n_{l+1}, \cdots\cdots, n_{\eta-1}, \tfrac{1}{4}n_{\eta-1}, \cdots\cdots, (\tfrac{1}{4})^{\lambda-\eta+1}n_{\eta-1})\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&S_{l+1\cdot max}(n_l, n_{l+1}, \cdots\cdots, (\tfrac{1}{4})^{\lambda-l-1}n_{l+1})\\
&\quad = S(n_l, \tfrac{1}{4}n_l, \cdots\cdots, (\tfrac{1}{4})^{\lambda-l}n_l)
\end{aligned}\right\} \quad \text{(A-V-14)}$$

When n_l takes an infinite value, there exists a subset of $S(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda)$ for every set of $n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda$ values in $S(n_l)$. Let us take out only the subsets of which $n_{\lambda-1}$ is infinite from $S(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda)$ and consider the collection of these subsets. Then equation (A-V-14) means that $S(n_l, \frac{1}{4}n_l, \cdots\cdots, (\frac{1}{4})^{\eta-l}n_l, \cdots\cdots, (\frac{1}{4})^{\lambda-l}n_l)$ is the subset which has the largest population of topologically distinct channel networks in the collection. This implies $S(n_l, \frac{1}{4}n_l, \cdots\cdots, (\frac{1}{4})^{\eta-l}n_l, \cdots\cdots, (\frac{1}{4})^{\lambda-l}n_l)$ has the largest population of topologically distinct channel networks among all subsets of $S(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_{\lambda-1}, n_\lambda)$ with a given infinite value of n_l. The implication is confirmed by proving that there exists no subset of which number of streams of order η is finite and population of topologically distinct channel networks is larger than $N(n_l, \frac{1}{4}n_l, \cdots\cdots, (\frac{1}{4})^{\eta-l}n_l, \cdots\cdots, (\frac{1}{4})^{\lambda-l}n_l)$. Let $n'_\eta, \cdots\cdots, n'_{\zeta-1}$ denote only finite numbers of streams of order $\eta, \cdots\cdots, (\zeta-1)$, respectively to distinguish from numbers $n_\eta, \cdots\cdots, n_{\zeta-1}$ and n'_ζ be 1, then $N(n_l, n_{l+1}, \cdots\cdots, n'_\eta, \cdots\cdots, n'_{\zeta-1}, 1)$ represents the number of topologically distinct channel networks in a subset of $S(n_l, n_{l+1}, \cdots\cdots, n_\eta, \cdots\cdots, n_\lambda)$ of which the number of streams of order η is finite. Put

$$D(n_l) = \frac{N(n_l, n_{l+1}, \cdots\cdots, n'_\eta, \cdots\cdots, n'_{\zeta-1}, 1)}{N(n_l, n_{l+1}, \cdots\cdots, n_{\eta-1}, \frac{1}{4}n_{\eta-1}, \cdots\cdots, (\frac{1}{4})^{\lambda-\eta+1}n_{\eta-1})} \qquad \text{(A-V-15)}$$

and let us evaluate the value of $D(n_l)$. By using equation (A-V-2), $D(n_l)$ is written as follows:

$$D(n_l) = \frac{\binom{n_{\eta-1}-2}{n_{\eta-1}-2n'_\eta} 2^{n_{\eta-1}-2n'_\eta}}{\binom{n_{\eta-1}-2}{\frac{1}{2}n_{\eta-1}} 2^{\frac{1}{2}n_{\eta-1}}}$$

$$\times \frac{A(n'_\eta, n'_{\eta+1}, \cdots\cdots\cdots, n'_\zeta)}{B(\frac{1}{4}n_{\eta-1}, (\frac{1}{4})^2 n_{\eta-1}, \cdots\cdots\cdots, (\frac{1}{4})^{\lambda-\eta+1} n_{\eta-1})}$$

$$= \left[\prod_{i=1}^{n_{\eta-1}/2-2n'_\eta} \left(\frac{n_{\eta-1}-2i-2}{n_{\eta-1}-2n'_\eta-i+1}\right)\right]$$

$$\times \frac{A(n'_\eta, n'_{\eta+1}, \cdots\cdots\cdots, n'_\zeta)}{B(\frac{1}{4}n_{\eta-1}, (\frac{1}{4})^2 n_{\eta-1}, \cdots\cdots\cdots, (\frac{1}{4})^{\lambda-\eta+1} n_{\eta-1})} \qquad \text{(A-V-16)}$$

where

$$A(n'_\eta, n'_{\eta+1}, \cdots\cdots\cdots, n'_{\zeta-1}, 1) = \prod_{\omega=\eta}^{\zeta-1} \binom{n'_\omega - 2}{n'_\omega - 2n'_{\omega+1}} 2^{n'_\omega - 2n'_{\omega+1}}$$

and

$$B(\frac{1}{4}n_{\eta-1}, (\frac{1}{4})^2 n_{\eta-1}, \cdots\cdots\cdots, (\frac{1}{4})^{\lambda-\eta+1} n_{\eta-1})$$

$$= \left[\prod_{\omega=\eta}^{\lambda-1} \binom{(\frac{1}{4})^{\omega-\eta+1} n_{\eta-1} - 2}{\frac{1}{2}(\frac{1}{4})^{\omega-\eta+1} n_{\eta-1}} 2^{\frac{1}{2}(\frac{1}{4})^{\omega-\eta+1} n_{\eta-1}}\right]$$

$$\times \frac{\binom{2(\frac{1}{4})^{\lambda-\eta+1} n_{\eta-1} - 1}{(\frac{1}{4})^{\lambda-\eta+1} n_{\eta-1}}}{2(\frac{1}{4})^{\lambda-\eta+1} n_{\eta-1} - 1}$$

In equation (A-V-16), $\prod_{i=1}^{n_{\eta-1}/2-2n'_\eta} \left(\frac{n_{\eta-1}-2i-2}{n_{\eta-1}-2n'_\eta-i+1}\right) \to 0$ and $A(n'_\eta, n'_{\eta+1}, \cdots\cdots\cdots, n'_{\zeta-1}, 1)/B(\frac{1}{4}n_{\eta-1}, (\frac{1}{4})^2 n_{\eta-1}, \cdots\cdots\cdots, (\frac{1}{4})^{\lambda-\eta+1} n_{\eta-1}) \to 0$, as $(\frac{1}{4})^{\lambda-\eta+1} n_{\eta-1} \to \infty$. Namely $D(n_l) = 0$, when $(\frac{1}{4})^{\lambda-\eta+1} n_{\eta-1} = \infty$. Now, from relations (A-V-14), if follows that

$$N(n_l, \frac{1}{4}n_l, \cdots\cdots\cdots, (\frac{1}{4})^{\eta-l} n_l, \cdots\cdots\cdots, (\frac{1}{4})^{\eta-l} n_l)$$

$$\geq N(n_l, n_{l+1}, \cdots\cdots\cdots, n_{\eta-1}, \frac{1}{4}n_{\eta-1}, \cdots\cdots\cdots, (\frac{1}{4})^{\lambda-\eta+1} n_{\eta-1}) \qquad \text{(A-V-17)}$$

when $(\frac{1}{4})^{\lambda-\eta+1} n_{\eta-1} = \infty$. Therefore,

$$\frac{N(n_l, \cdots\cdots\cdots, n'_\eta, \cdots\cdots\cdots, n'_{\zeta-1}, 1)}{N(n_l, \frac{1}{4}n_l, \cdots\cdots\cdots, (\frac{1}{4})^{\eta-l} n_l, \cdots\cdots\cdots, (\frac{1}{4})^{\lambda-l} n_l)} = 0 \qquad \text{(A-V-18)}$$

when $n_l = \infty$. Here, let $S_{max}(n_l, n_{l+1}, \cdots\cdots\cdots, n_\eta, \cdots\cdots\cdots, n_\lambda)$ be the subset of $S(n_l, n_{l+1}, \cdots\cdots\cdots, n_\eta, \cdots\cdots\cdots, n_\lambda)$ which has the largest population of topologically distinct channel networks for a given value of n_l. When $n_l = \infty$, the following relation is derived from relations (A-V-14) and equation (A-V-18).

$$S_{max}(n_l, n_{l+1}, \cdots\cdots\cdots, n_\eta, \cdots\cdots\cdots, n_\lambda)$$

$$= S(n_l, \frac{1}{4}n_l, \cdots\cdots\cdots, (\frac{1}{4})^{\eta-l} n_l, \cdots\cdots\cdots, (\frac{1}{4})^{\lambda-l} n_l) \qquad \text{(A-V-19)}$$

This means that the bifurcation ratio of the most probable channel networks in infinite topologically random channel networks is 4.

The value ε_1 of the most probable channel networks is obtained by using the above result. Let $n_{e\cdot\eta-1}$ be the total number of streams of order $(\eta-1)$ entering the streams of order η from the sides, then, within the set of all topologically distinct channel networks, the number $N(n_{\eta-1}, n_\eta)$ of ways in which $n_{\eta-1}$ streams of order $(\eta-1)$ can produce n_η streams of order η is given by the following equation.

$$N(n_{\eta-1}, n_\eta) = \sum_{n_{e\cdot\eta-1}=0}^{n_{\eta-1}-2n_\eta} \left[\binom{n_\eta + n_{e\cdot\eta-1} - 1}{n_{e\cdot\eta-1}} 2^{n_{e\cdot\eta-1}}\right.$$

$$\left.\times \binom{n_{\eta-1} - n_{e\cdot\eta-1} - n_\eta - 2}{n_{\eta-1} - n_{e\cdot\eta-1} - 2n_\eta} 2^{n_{\eta-1} - n_{e\cdot\eta-1} - 2n_\eta}\right]$$

$$\times \frac{\binom{2n_\eta - 1}{n_\eta}}{2n_\eta - 1} \qquad \text{(A-V-20)}$$

Here $\binom{n_\eta + n_{e\cdot\eta-1} - 1}{n_{e\cdot\eta-1}} 2^{n_{e\cdot\eta-1}}$ is the number of ways in which $n_{e\cdot\eta-1}$ steams of order $(\eta-1)$ enter n_η streams of order η from the sides, $\binom{n_{\eta-1} - n_{e\cdot\eta-1} - n_\eta - 2}{n_{\eta-1} - n_{e\cdot\eta-1} - 2n_\eta} 2^{n_{\eta-1} - n_{e\cdot\eta-1} - 2n_\eta}$ is the number of ways in which $(n_{\eta-1} - n_{e\cdot\eta-1} - 2n_\eta)$ streams of order $(\eta-1)$ enter $(n_\eta - 1)$ links constituting streams of orders higher than η and $\binom{2n_\eta - 1}{n_\eta}/(2n_\eta - 1)$ is the number of topologically distinct channel networks having n_η sources. When the general term of the sum in equation (A-V-20) has a maximum value, a small variation in the distribution of streams of $(\eta-1)$, i.e., a small decrease in the number of streams of order $(\eta-1)$ entering streams of order η from the sides and the compensating increase in the number of streams of order $(\eta-1)$ entering links of orders higher than η, leave this value unchanged.

$$\frac{\binom{n_\eta+n_{e\cdot\eta-1}-1}{n_{e\cdot\eta-1}} 2^{n_{e\cdot\eta-1}} \binom{n_{\eta-1}-n_{e\cdot\eta-1}-n_\eta-2}{n_{\eta-1}-n_{e\cdot\eta-1}-2n_\eta} 2^{n_{\eta-1}-n_{e\cdot\eta-1}-2n_\eta}}{\binom{n_\eta+n_{e\cdot\eta-1}-2}{n_{e\cdot\eta-1}-1} 2^{n_{e\cdot\eta-1}-1} \binom{n_{\eta-1}-n_{e\cdot\eta-1}-n_\eta-1}{n_{\eta-1}-n_{e\cdot\eta-1}-2n_\eta+1} 2^{n_{\eta-1}-n_{e\cdot\eta-1}-2n_\eta+1}}$$

$$= \frac{(n_\eta+n_{e\cdot\eta-1}-1)\,(n_{\eta-1}-n_{e\cdot\eta-1}-2n_\eta+1)}{n_{e\cdot\eta-1}(n_{\eta-1}-n_{e\cdot\eta-1}-n_\eta-1)} = 1$$

For very large values of $n_{\eta-1}$, this expression may be replaced by

$$\frac{(n_\eta+n_{e\cdot\eta-1})\,(n_{\eta-1}-n_{e\cdot\eta-1}-2n_\eta)}{n_{e\cdot\eta-1}(n_{\eta-1}-n_{e\cdot\eta-1}-n_\eta)} = 1 \qquad \text{(A-V-21)}$$

Substituting equation (A-V-13) into equation (A-V-21) leads to

$$n_{e\cdot\eta-1} = \frac{n_{\eta-1}}{4}$$

Finally we obtain

$$\varepsilon_1 = {}_\eta\varepsilon_{\eta-1} = \frac{n_{e\cdot\eta-1}}{n_\eta} = 1 \qquad \text{(A-V-22)}$$

for the most probable networks in infinite topologically random channel networks.

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