MODELLING COMPLEX SYSTEMS

NONLINEAR DYNAMICS

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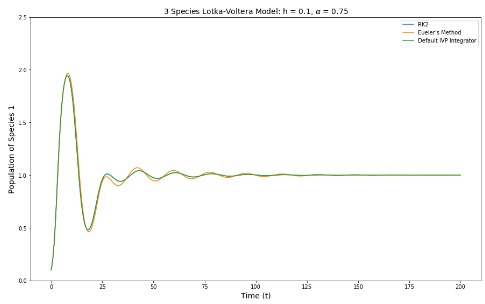
0.1 Problem 2

Numerically Solve the 3-Species Lotka- Volterra System using both Heun's and Euler's methods.

Figure 1 displays how well Euler and Heun's methods were able to approximate the population of species 1 over time given a time step of h = 0.1. We will do a rigorous analysis of error later in the report. Visually, for small h Heun's method performs fairly well. When α is particularly small,there is less complexity in the dynamics of the system and is thus easier to predict. Even with a small step size, Euler's method remains accurate for a very limited amount of time.



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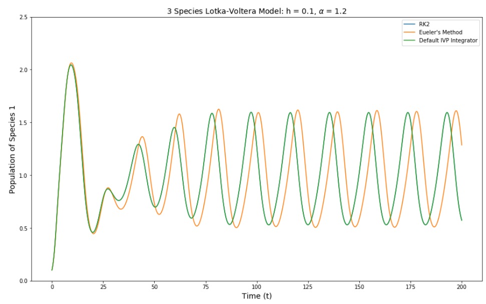

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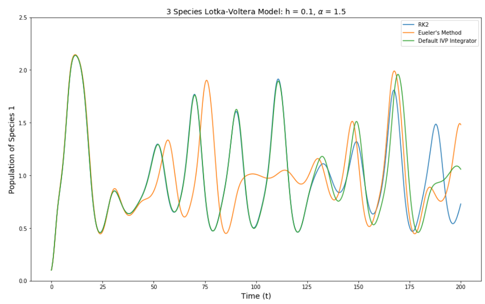

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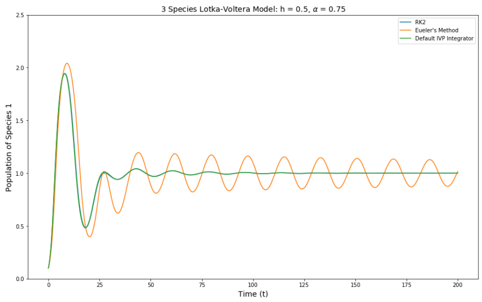
Figure 1: Three plots show how the different models approximate the evolving population of a species with varying values of α. All models approximate the population with h = 0.1 time steps

Below, h is increased to 0.5. The integrators will require less computational effort to perform the approximation to a given time t. However, we are sacrificing the accuracy of the approximation as we will see in the error analysis section. We now notice that Euler's method fails to predict the simplest dynamics, when the system converges to a singular fixed point as displayed in Figure 2a.



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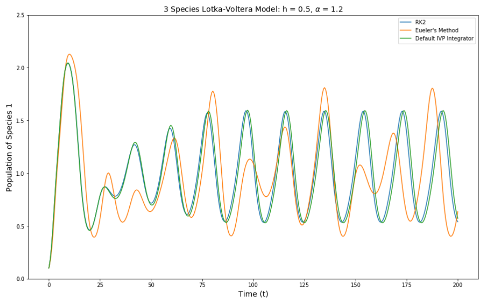

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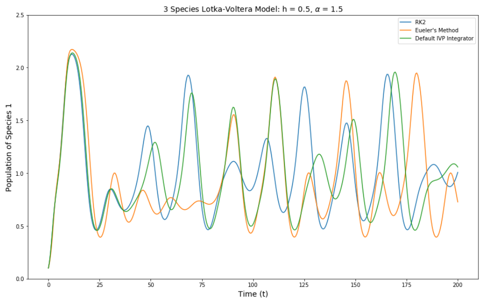

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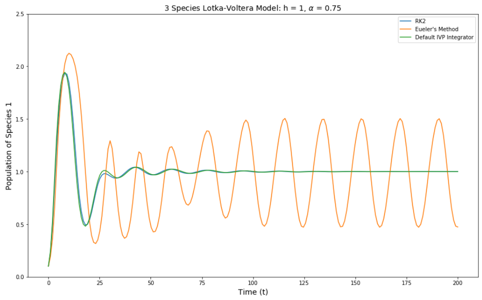
Figure 2: Three plots show how the different models approximate the evolving population of a species with varying values of α. All models approximate the population with h = 0.5 time steps

When h is increased to 1, our approximations really take a turn for the worst. A step size that large leads to gross over approximations when the system undergoes rapid change. Say for example if the fish population triples overnight, the sharks will have a feeding day and in turn the killer whales. The aftermath of this will be a cascading dive in population level from the three species. The integrates use the slope at t_i to predict the population at time t_{i+1}. If there is a quick change in slope, the integrator will hop right over, and propagate massive error through the approximation. In image 3a and 3b, Euler's method explodes and approximations are pushed to (-∞, ∞). Even Huen's method is fairly inaccurate for large α.



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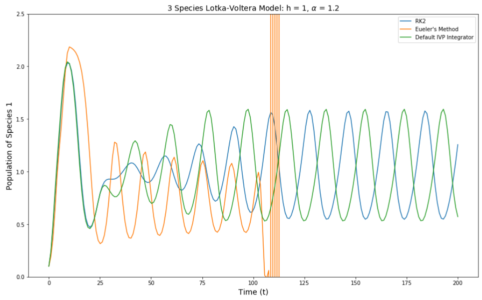

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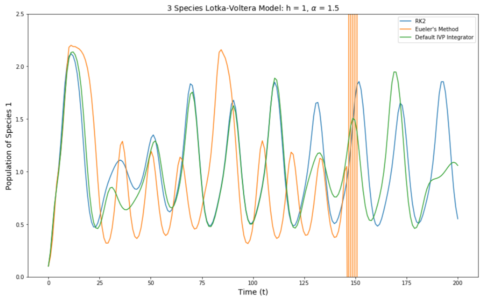

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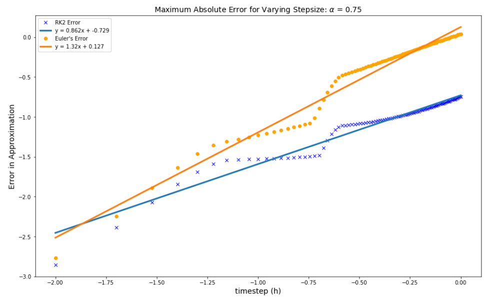
Figure 3: Three plots show how the different models approximate the evolving population of a species with varying values of α. All models approximate the population with h = 1 time steps. It is highly inadvisable to use this step size.

0.2 Problem 3

Error Analysis of Numerical Integration Methods

To test the accuracy of each numerical integrator at different step sizes h, the maximum absolute error was take for 100 h, such that .01$\leq h \leq$1. Given this distribution of maximum absolute errors we can analyze how error behaves as a function of step size for different values of α. Figure 4 illustrates our findings.



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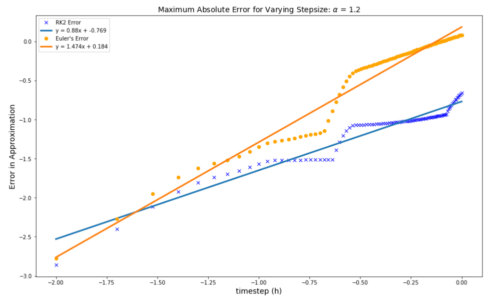

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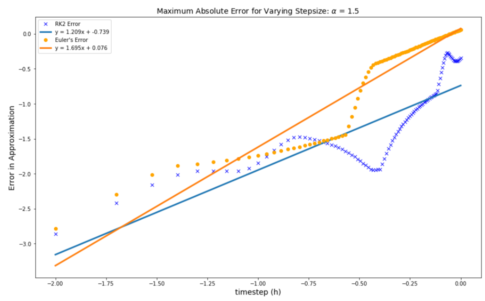

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Figure 4: Three plots show maximum absolute error as a function of step size across three different values of α on a loglog scale with lines of best fit.

As expected, Huen's Method outperforms Euler's methods accross the board. There is a consistent interesting break of scale in error for all step sizes. Around $10^{-.5}$, there is a near instantaneous jump in error. For Euler's method, this jump is nearly a full order of magnitude. Intuitively, as tunable α increases, the jump grows. For $\alpha \geq 1.5$, the dynamics of the system are chaotic. This introduces highly sensitive dependence on initial conditions. If the integrator is off only the slightest amount, its future approximation is likely to be very inaccurate. This intuition is confirmed by a rising positive slope in line of best fit as α increases.

0.3 Problem 4

Show analytically that the global precision of Heun's method is in h^2 (where h is its step size).

$$\frac{dx}{dt} = f(t, x(t)),\ x(t_0) = x_0 \tag{IVP}$$

For Euler's method, we have that x evalauted at step any step t is

$$\begin{aligned} x_e(t_n) &= x_n, \\ x_e(t_{n+1}) &= x_e(t_n) + h\, f(x_e(t_n)) = x_n + h\frac{dx}{dt}, \text{ where } h = t_{n+1} - t_n \end{aligned}$$

and we can calculate the local error of our estimation for x at any time-step t_n by finding the difference between the true value of x evaluated at the next time-step t_{n+1} and our estimated value of x evaluated at the next time-step t_{n+1}. We will assume that the function(s) we are evaluating can be approximated using a Taylor expansion

$$x(t) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{(n)} (t_{n+1} - t_n)$$

$$x(t) = x_0 + \frac{dx}{dt}h + \frac{1}{2!}\frac{d^2x}{dt^2}h^2 + \frac{1}{3!}\frac{d^3x}{dt^3} + \ldots$$

where

$$x(t_{n+1}) - x_e(t_{n+1}) = x_0 + \frac{dx}{dt}h + \frac{1}{2!}\frac{d^2x}{dt^2}h^2 - (x_0 + h\frac{dx}{dt}) = \frac{1}{2}\frac{d^2x}{dt^2}h^2$$

But of course, the expansion is actually an infinite sum, so we say that there is some remainder term that we are not accounting for in the approximation. Hence, while we have only taken the difference between the actual value $x(t)$ and our approximation using Euler's method to be $\frac{1}{2}\frac{d^2x}{dt^2}h^2$, we have neglected to include the remaining portion of the sum of terms that represent the full Taylor expansion. Since our step size (h) is very small, we simplify the remaining error term by writing $\mathcal{O}$ of the next largest value for h. We have just found this to be h^2, so our local error of our approximation using Euler's method is $\mathcal{O}(h^2)$.

If we use Euler's method to approximate any point in our function, then we are approximating the function at every point x over some interval, let's call it, T. Thus, the total number of approximations we make is the length of the interval $|T|$ divided by the distance of a single step h. The global error of the approximation would be the sum of all error at each time-step. This is approximately equal to the local error evaluated at all points in T divided by the size of our step h. Hence, the global error of Euler's method is $\frac{h^2}{h} = h$.

Under similar assumptions, we can use the same methods to calculate the global error of Heun's method. Heun's method modifies Euler's method by assuming that we can make a "better" approximation of the function at some value t_{n+1} by taking the average value of the slopes at $x(t_n)$ and $x(t_n + h)$. This looks like

$$
\begin{aligned}
x_h(t_n) &= x_n, \\
x_h(t_{n+1}) &= x(t_n) + \frac{h}{2}\Big(f(x_h(t_n)) + f(x_h(t_{n+1})) \Big)
\end{aligned}
$$

For simplicity, we will take x evaluated at our initial time $t_0 = 0$ to be x_0, and evaluate x at the next time-step, h. If we expand Heun's method, we get

$$
\begin{aligned}
x_h(t_0) &= x_0, \\
x_h(h) &= x_0 + \frac{h}{2}\left(\frac{dx}{dt}\Big|_{t_0} + f\Big(x_h(t_0) + hf(x(t_0)) \Big) \right)
\end{aligned}
$$

$$
= x_0 + \frac{h}{2}\left(\frac{dx}{dt}\Big|_{t_0} + f\Big(x_0 + hf(x_0) \Big) \right)
$$

$$
= x_0 + \frac{h}{2}\left(\frac{dx}{dt}\Big|_{t_0} + f\Big(x_0 + hf(x_0) \Big) \right)
$$

$$
= x_0 + \frac{h}{2}\left(\frac{dx}{dt}\Big|_{t_0} + \frac{dx}{dt}\Big|_{t_0} + h\frac{d^2x}{dt^2}\Big|_{t_0} \right)
$$

$$
= x_0 + h\frac{dx}{dt}\Big|_{t_0} + \frac{h^2}{2}\frac{d^2x}{dt^2}\Big|_{t_0}
$$

Again, we can subtract this from the true value of the function at t_0 to find the local error:

$$x(h) - x_h(h) = x_0 + h\frac{dx}{dt}\mid_{t_0} + h^2\frac{1}{2!}\frac{d^2x}{dt^2}\mid_{t_0} + h^3\frac{1}{3!}\frac{d^3x}{dt^3}\mid_{t_0} - (x_0 + h\frac{dx}{dt}\mid_{t_0} + h^2\frac{1}{2!}\frac{d^2x}{dt^2}\mid_{t_0})$$

$$= \frac{1}{3!}\frac{d^3x}{dt^3}h^3$$

So the local error for Heun's method is $\mathcal{O}(h^3)$. We can then calculate the global error to be $\frac{h^3}{h} = h^2$.

0.4 Problem 5

The 2-species Lotka-Volterra system is very similar to the SIS epidemic model studied in class. Why did we not see cycles in that system? What about SIR or SIRS systems?Should we expect chaos in classic epidemic models?

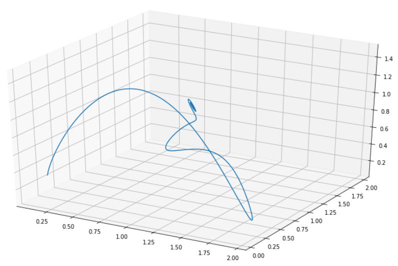
SIS is a state transition model in which the interactions between *susceptible* and *infected* units of a population are observed. The system models how units can be in one state at a given discrete time, and transition to the other state at another discrete point in the future. When observing these transitions, we often consider a closed system in which the total population does not change over time. Hence, if we let S and I represent the sub-populations of *susceptible* and *infected* units respectively, and let N represent the total population of the system, then $S + I = N$. Since N is the only dependent variable in the system (on S and I), the system has only one degree of freedom. Hence, we do not expect any cycles in the system.

If we consider one of the other epidemic models, say *SIR*, our system changes slightly. Now, there are three states; *susceptible*, *infected*, and *recovered*, where *susceptible* units can become *infected* and *infected* units can become *recovered*, where they remain. Now, the dynamics are such that as time goes on, the total number of *recovered* units will approach the overall size of the population ($S + I + R = N$, and $R = N$ for large time, t). Hence, we have two degrees of freedom initially, and then one as time increases (and depending on the recovery rate of the system). So, we do not expect chaos to arise out of this system, as we always know the precise outcome (i.e: the population tends towards a fixed point).

Finally, in the *SIRS* system, units that are *susceptible* can become *infected*, which in turn can *recover*, but also become *infected* again. So now, the total population of the system is $S + I + R = N$, and two degrees of freedom govern the dynamics of the population.

In the Lotka-Voltera model, at least three species are required for chaos to take place in the population state space. However, if we are using a discrete model, this is not true. The Flake book makes a good analogy for why 3 dimensions are necessary in continuous space. It is impossible to draw an infinite continuous line on a 2D plane without lifting off the plane. A line of infinite length on a 2D surface will intersect itself at every point on the plane. However, if we add a dimension, we can draw a line in 3D space that comes arbitrarily close to intersecting itself, infinitely often. The system that generates such a line is chaotic.

We see in Figure 5a, the system contains a single fixed point attractor that all initial conditions within the basin of attraction will approach. In 5b, $\alpha = 1.2$ which yields periodic behavior, this is also visible in a time series of one species population, evident in Figure 1b. When α is cranked to 1.5, chaos emerges from the 3 Species system. Figure 5c shows the chaotic attractor present in the 3 species population dynamics.



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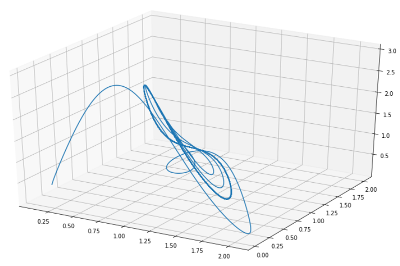

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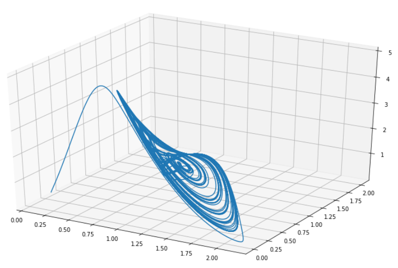

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Figure 5: From top left, top right, bottom left, we see the population dynamics of species one when $\alpha = .75$,1.2 and 1.5 respectively.

0.5 Problem 6

How many fixed points exist in the n-species Lotka-Volterra system? How many of those, if any, contain a co-existence of all species (i.e. with no extinctions)?

Despite popular belief and much deliberation, there are certainly 2 fixed points that exist in the n-species Lotka-Volterra system. The first fixed point, $X_n = 0$ will result when every species is dead, i.e. when there are extinctions. This necessitates that all X_i from i to n = 0. We will prove that the only other fixed point is a vector of size n with elements $x_i = 1$. After showing that $X_n = 0$ is a fixed point of the system we will rewrite our system as:

$$\begin{bmatrix} \sum_{j=1}^{n} A_{i,j}(1-X_j) \\ \sum_{j=1}^{n-1} A_{i,j}(1-X_j) \\ . \\ . \\ \sum_{j=1}^{1} A_{i,j}(1-X_j) \end{bmatrix} = \begin{bmatrix} 0 \\ . \\ . \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A_n - \sum_{j=1}^{n} A_{i,j}(X_j) \\ . \\ . \\ A_1 - \sum_{j=1}^{1} A_{i,j}(X_j) \end{bmatrix} = \begin{bmatrix} 0 \\ . \\ . \\ 0 \end{bmatrix}$$

$$r_i = \sum A_{i,j} \tag{1}$$

$$r_i - AX = 0 \tag{2}$$

$$\vec{X}_i = A^T A = \vec{1} \tag{3}$$

Thus, all $X_i = 1$ is a fixed point with no extinctions. There are two values for which the system is fixed, when all X from i to n are 1 and when all X from i to n are 0.