Matrix Forensics

A brief guide to matrix math and its efficient implementation

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github.com/r-barnes/MatrixForensics

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1 | Introduction

Goals: TODO

Contributing: Please contribute on Github at `https://github.com/r-barnes/MatrixForensics` either by opening an issue or making a pull request. If you are not comfortable with this, please send your contribution to `rijard.barnes@gmail.com`.

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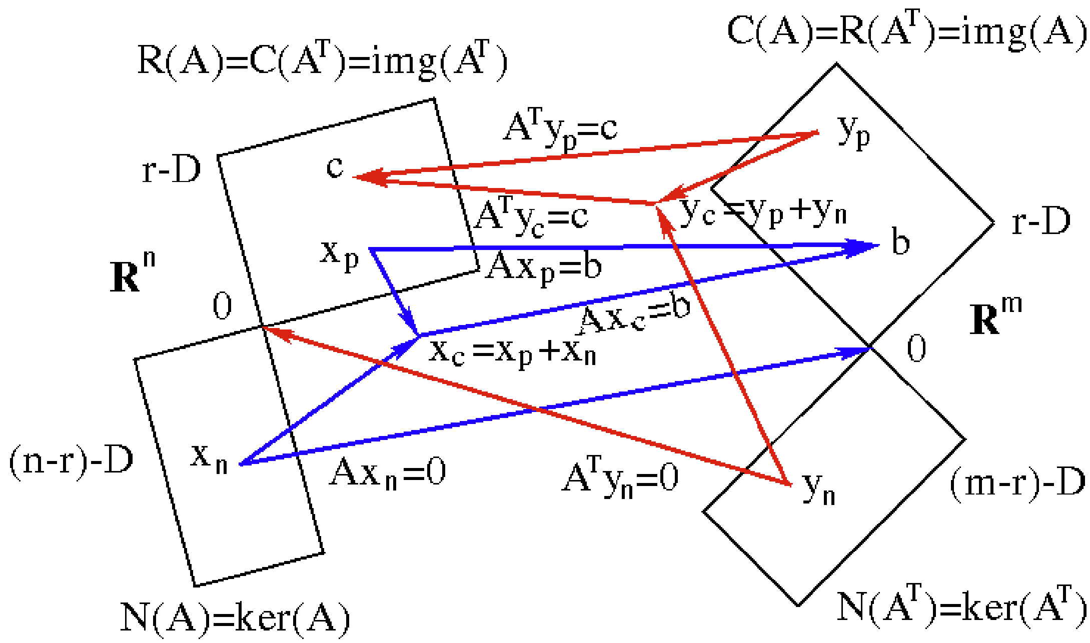
Funding: TODO

2 Nomenclature

$\mathbf{A}$	Matrix.
$\mathbf{a}$	(Column) vector.
a	Scalar.
$\mathbf{A}_{ij}$	Matrix indexed. Returns ith row and jth column.
$\mathbf{A} \circ \mathbf{B}$	Hadamard (element-wise) product of matrices A and B.
$\mathcal{N}(\mathbf{A})$	Nullspace of the matrix $\mathbf{A}$.
$\mathcal{R}(\mathbf{A})$	Range of the matrix $\mathbf{A}$.
$\det(\mathbf{A})$	Determinant of the matrix $\mathbf{A}$.
$\mathrm{eig}(\mathbf{A})$	Eigenvalues of the matrix $\mathbf{A}$.
$\mathbf{A}^H$	Conjugate transpose of the matrix $\mathbf{A}$.
$\mathbf{A}^T$	Transpose of the matrix $\mathbf{A}$.
$\mathbf{A}^+$	Pseudoinverse of the matrix $\mathbf{A}$.
$\mathbf{x} \in \mathbb{R}^n$	The entries of the n-vector $\mathbf{x}$ are all real numbers.
$\mathbf{A} \in \mathbb{R}^{m,n}$	The entries of the matrix $\mathbf{A}$ with m rows and n columns are all real numbers.
$\mathbf{A} \in \mathbb{S}^n$	The matrix $\mathbf{A}$ is symmetric and has n rows and n columns.
$\mathbf{I}_n$	Identity matrix with n rows and n columns.
$\{0\}$	The empty set

3 Basics

3.1 Fundamental Theorem of Linear Algebra



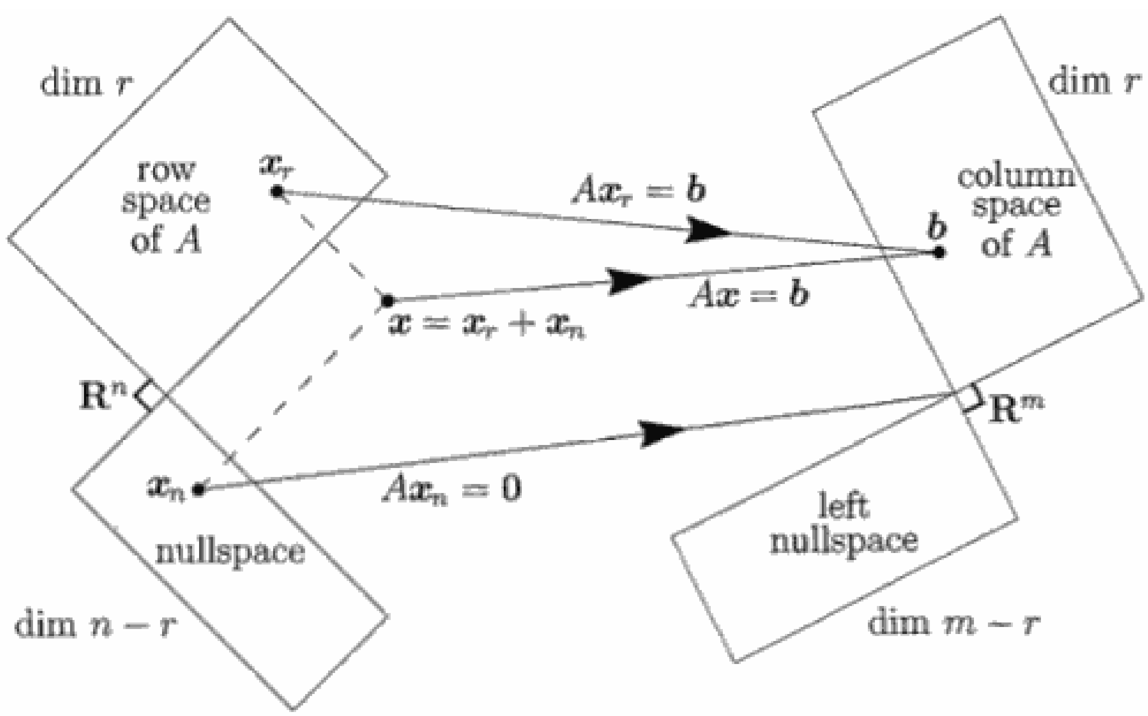
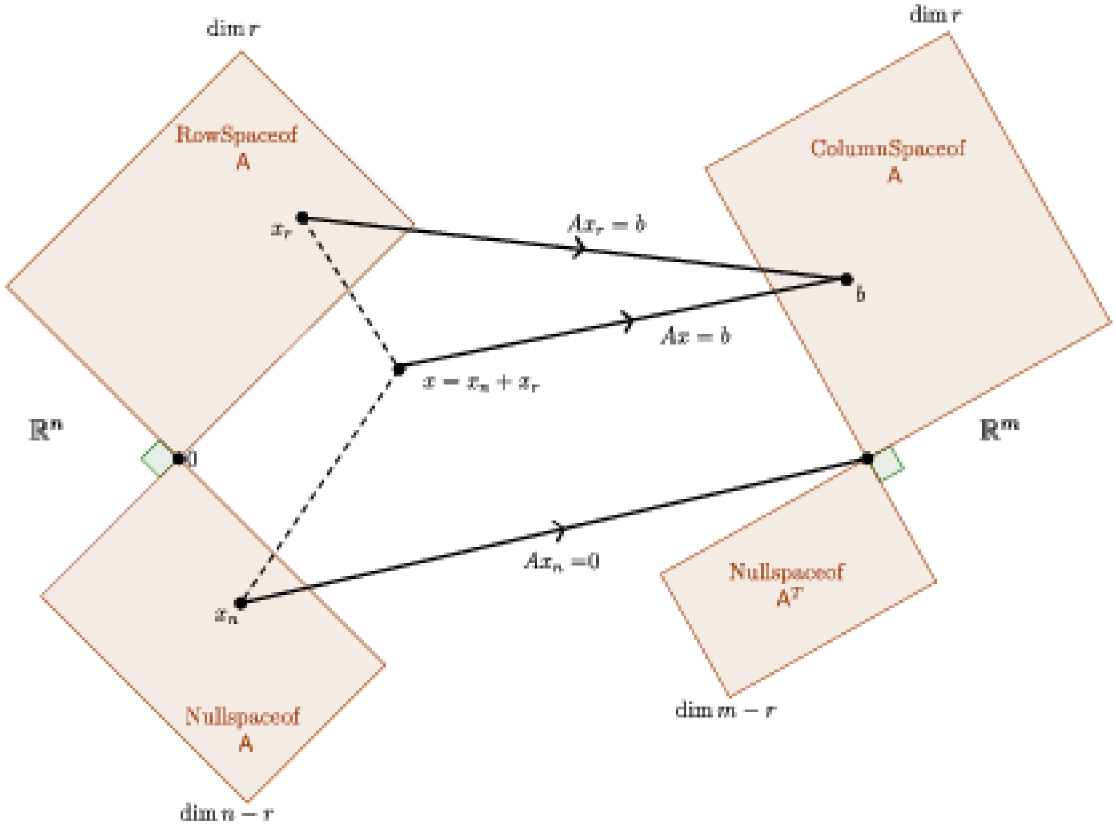
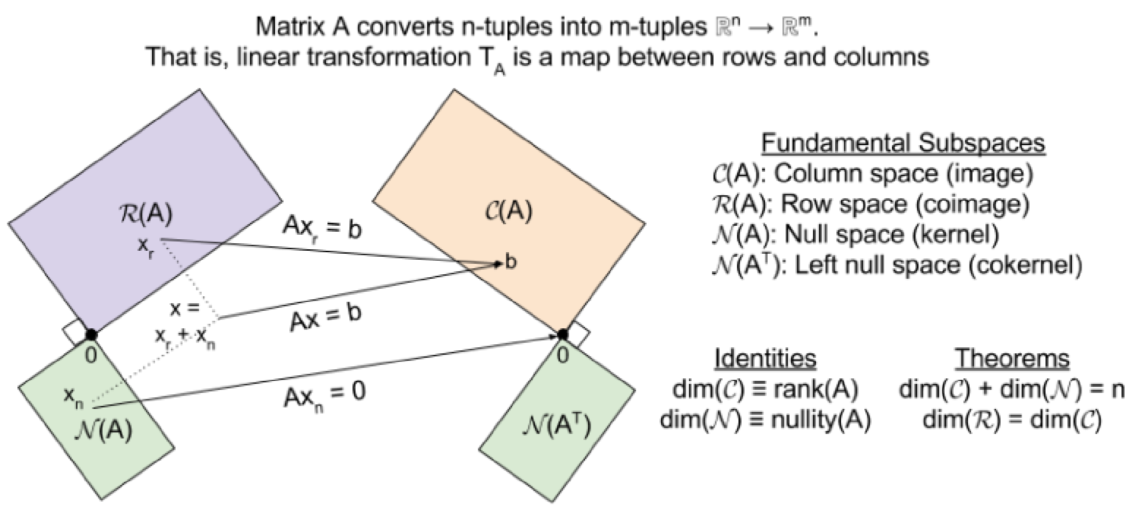


Figure 3.4 The true action $Ax = A(x_{\text{row}} + x_{\text{null}})$ of any m by n matrix.


dim r
RowSpaceof
A
x_r
$Ax_r = b$
dim r
ColumnSpaceof
A
b
$Ax = b$
$x = x_n + x_r$
$\mathbb{R}^n$
$\mathbb{R}^m$
0
$Ax_n = 0$
Nullspaceof
A^T
x_n
dim $m - r$
Nullspaceof
A
dim $n - r$


Matrix A converts n-tuples into m-tuples $\mathbb{R}^n \to \mathbb{R}^m$.
That is, linear transformation T_A is a map between rows and columns
$\mathcal{R}$(A)
x_r
$Ax_r = b$
$\mathcal{C}$(A)
b
x =
$x_r + x_n$
Ax = b
0
0
x_n
$Ax_n = 0$
$\mathcal{N}$(A)
$\mathcal{N}(A^T)$
Fundamental Subspaces
$\mathcal{C}$(A): Column space (image)
$\mathcal{R}$(A): Row space (coimage)
$\mathcal{N}$(A): Null space (kernel)
$\mathcal{N}(A^T)$: Left null space (cokernel)
Identities
dim($\mathcal{C}$) ≡ rank(A)
dim($\mathcal{N}$) ≡ nullity(A)
Theorems
dim($\mathcal{C}$) + dim($\mathcal{N}$) = n
dim($\mathcal{R}$) = dim($\mathcal{C}$)

3.2 Matrix Properties

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC} \qquad \text{(left distributivity)} \tag{1}$$
$$(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA} \qquad \text{(right distributivity)} \tag{2}$$
$$\mathbf{AB} \neq \mathbf{BA} \qquad \text{(in general)} \tag{3}$$
$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \qquad \text{(associativity)} \tag{4}$$

3.3 Matrix Multiplication

$$(\mathbf{AB})_{kl} = \sum_m \mathbf{A}_{km}\mathbf{B}_{ml} \quad \mathbf{A} \in \mathbb{R}^{k,m}, \mathbf{B} \in \mathbb{R}^{m,l} \tag{5}$$

3.4 Time Complexities

Operation	**Input**	**Output**	**Algorithm**	**Time**
Matmult	$A, B \in n \times n$	$n \times n$	Schoolbook	$O(n^3)$
			Strassen [1]	$O(n^{2.807})$
			Best	$O(n^\omega)$
Matmult	$A \in n \times m, B \in m \times p$	$n \times p$	Schoolbook	$O(nmp)$
Inversion	$A \in n \times n$	$n \times n$	Gauss–Jordan elimination	$O(n^3)$
			Strassen [1]	$O(n^{2.807})$
			Best	$O(n^\omega)$
SVD	$A \in m \times n$	$m \times m, m \times n, n \times n$ $m \times r, r \times r, n \times r$		$O(mn^2)$ $(m \geq n)$
Determinant	$A \in n \times n$	Scalar	Laplace expansion	$O(n!)$
			Division-free [2]	$O(n!)$
			LU decomposition	$O(n^3)$
			Integer preserving [3]	$O(n^3)$
Back substitution	A triangular	n solutions	Back substitution	$O(n^2)$

A comment on ω

The lower bound on matmult time complexity is $O(n^\omega)$, where ω is an unknown constant bounded by $2 \leq \omega \leq 2.373$. Algorithms achieving lower values of ω tend to be less efficient in practice for all but the largest matrices. Of the algorithm with times of less than $O(n^3)$, only the Strassen algorithm has seen serious attempts at optimized implementation. Most matmult implementations use highly optimized variants of the standard $O(n^3)$ algorithm. At this point, memory and bus speeds dominate the performance of implementations, so simple Big-O notation cannot be used to reliably compare matmult performances.

Name	**Year**	ω
Standard	-	3
Strassen [1]	1969	2.807
Pan [4]	1978	2.796
Bini et al. [5]	1979	2.78
Schönhage [6]	1981	2.548
Schönhage [6]	1981	2.522
Romani [7]	1982	2.517
Coppersmith and Winograd [8]	1982	2.496
Strassen [9]	1986	2.479
Coppersmith and Winograd [10]	1990	2.376
Williams [11]	2012	2.37294
Le Gall [12]	2014	2.3728639
Williams [11]	2012	2.3727

4 Derivatives

4.1 Useful Rules for Derivatives

For general $\mathbf{A}$ and $\mathbf{X}$ (no special structure):

$$\partial \mathbf{A} = 0 \quad \text{where } \mathbf{A} \text{ is a constant} \tag{6}$$

$$\partial(c\mathbf{X}) = c\partial\mathbf{X} \tag{7}$$

$$\partial(\mathbf{X} + \mathbf{Y}) = \partial\mathbf{X} + \partial\mathbf{Y} \tag{8}$$

$$\partial(\operatorname{tr}(\mathbf{X})) = \operatorname{tr}(\partial(\mathbf{X})) \tag{9}$$

$$\partial(\mathbf{X}\mathbf{Y}) = (\partial\mathbf{X})\mathbf{Y} + \mathbf{X}(\partial\mathbf{Y}) \tag{10}$$

$$\partial(\mathbf{X} \circ \mathbf{Y}) = (\partial\mathbf{X}) \circ \mathbf{Y} + \mathbf{X} \circ (\partial\mathbf{Y}) \tag{11}$$

$$\partial(\mathbf{X}^{-1}) = -\mathbf{X}^{-1}(\partial\mathbf{X})\mathbf{X}^{-1} \tag{12}$$

$$\partial(\det(\mathbf{X})) = \operatorname{tr}(\operatorname{adj}(\mathbf{X})\partial\mathbf{X}) \tag{13}$$

$$\partial(\det(\mathbf{X})) = \det(\mathbf{X})\operatorname{tr}(\mathbf{X}^{-1}\partial\mathbf{X}) \tag{14}$$

$$\partial(\ln(\det(\mathbf{X}))) = \operatorname{tr}(\mathbf{X}^{-1}\partial\mathbf{X}) \tag{15}$$

$$\partial(\mathbf{X}^T) = (\partial\mathbf{X})^T \tag{16}$$

$$\partial(\mathbf{X}^H) = (\partial\mathbf{X})^H \tag{17}$$

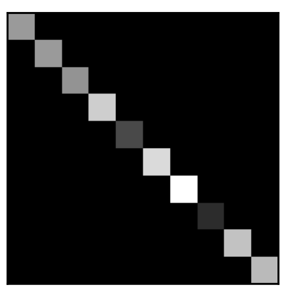
5 | Matrix Rogue Gallery

5.1 Non-Singular vs. Singular Matrices

For $\mathbf{A} \in \mathbb{R}^{n,n}$ (initially drawn from [13, p. 574]):

Non-Singular	**Singular**
$\mathbf{A}$ is invertible	$\mathbf{A}$ is not invertible
The columns are independent	The columns are dependent
The rows are independent	The rows are dependent
$\det(\mathbf{A}) \neq 0$	$\det(\mathbf{A}) = 0$
$\mathbf{Ax} = 0$ has one solution: $\mathbf{x} = 0$	$\mathbf{Ax} = 0$ has infinitely many solutions
$\mathbf{Ax} = \mathbf{b}$ has one solution: $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$	$\mathbf{Ax} = \mathbf{b}$ has either no or infinitely many solutions
$\mathbf{A}$ has n nonzero pivots	$\mathbf{A}$ has $r < n$ pivots
$\mathbf{A}$ has full rank $r = n$	$\mathbf{A}$ has rank $r < n$
The reduced row echelon form is $\mathbf{R} = \mathbf{I}$	$\mathbf{R}$ has at least one zero row
The column space is all of $\mathbb{R}^n$	The column space has dimension $r < n$
The row space is all of $\mathbb{R}^n$	The row space has dimension $r < n$
All eigenvalues are nonzero	Zero is an eigenvalue of $\mathbf{A}$
$\mathbf{A}^T\mathbf{A}$ is symmetric positive definite	$\mathbf{A}^T\mathbf{A}$ is only semidefinite
$\mathbf{A}$ has n positive singular values	$\mathbf{A}$ has $r < n$ singular values

5.2 Diagonal Matrix



$$A = \operatorname{diag}(a_1, \ldots, a_n) = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \tag{18}$$

Square matrix. Entries above diagonal are equal to entries below diagonal.

Number of "free entries": $\frac{n(n+1)}{2}$.

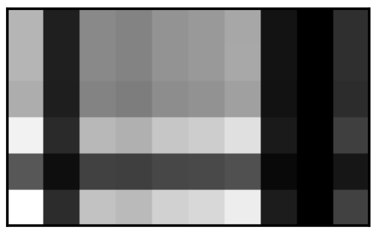
Special Properties

$$\mathrm{eig}(A) = a_1, \ldots, a_n \tag{19}$$

$$\det(A) = \prod_i a_i \tag{20}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{a_1} & & \\ & \ddots & \\ & & \frac{1}{a_n} \end{bmatrix} \tag{21}$$

5.3 Dyads



$\mathbf{A} \in \mathbb{R}^{m,n}$ is a dyad if it can be written as

$$\mathbf{A} = \mathbf{u}\mathbf{v}^T \quad \mathbf{u} \in \mathbb{R}^m, \mathbf{v} \in \mathbb{R}^n \tag{22}$$

Special Properties

- The columns of $\mathbf{A}$ are copies of $\mathbf{u}$ scaled by the values of $\mathbf{v}$.
- The rows of $\mathbf{A}$ are copies of $\mathbf{u}^T$ scaled by the values of $\mathbf{v}$.
- If $\mathbf{A}$ is a dyad, it acts on a vector $\mathbf{x}$ as $\mathbf{A}\mathbf{x} = (\mathbf{u}\mathbf{v}^T)\mathbf{x} = (\mathbf{v}^T\mathbf{u})\mathbf{x}$.
- $\mathbf{A}\mathbf{x} = c\mathbf{u}$ ($\mathbf{A}$ scales $\mathbf{x}$ and points it along $\mathbf{u}$).
- $\mathbf{A}_{ij} = \mathbf{u}_i\mathbf{v}_j$.
- If $\mathbf{u}, \mathbf{v} \neq 0$, then $\mathrm{rank}(\mathbf{A}) = 1$.
- If $m = n$, $\mathbf{A}$ has one eigenvalue $\lambda = \mathbf{v}^T\mathbf{u}$ and eigenvector $\mathbf{u}$.
- A dyad can always be written in a normalized form $c\tilde{\mathbf{u}}\tilde{\mathbf{v}}^T$.

5.4 Hermitian Matrix

$\mathbf{H} \in \mathbb{C}^{m,n}$ is Hermitian iff

$$\mathbf{H} = \mathbf{H}^H \tag{23}$$

where $\mathbf{H}^H$ is the conjugate transpose of $\mathbf{H}$.

For $\mathbf{H} \in \mathbb{R}^{m,n}$, Hermitian and symmetric matrices are equivalent.

Special Properties

$$\mathbf{H}_{ii} \in \mathbb{R} \tag{24}$$
$$\mathbf{H}\mathbf{H}^H = \mathbf{H}^H\mathbf{H} \tag{25}$$
$$\mathbf{x}^H\mathbf{H}\mathbf{x} \in \mathbb{R} \ \ \forall \mathbf{x} \in \mathbb{C} \tag{26}$$
$$\mathbf{H}_1 + \mathbf{H}_2 = \text{Hermitian} \tag{27}$$
$$\mathbf{H}^{-1} = \text{Hermitian} \tag{28}$$
$$\mathbf{A} + \mathbf{A}^H = \text{Hermitian} \tag{29}$$
$$\mathbf{A} - \mathbf{A}^H = \text{Skew-Hermitian} \tag{30}$$
$$\mathbf{A}\mathbf{B} = \text{Hermitian iff } \mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A} \tag{31}$$
$$\det(\mathbf{H}) \in \mathbb{R} \tag{32}$$
$$\text{eig}(\mathbf{H}) \in \mathbb{R} \tag{33}$$

5.5 Idempotent Matrix

A matrix $\mathbf{A}$ is idempotent iff

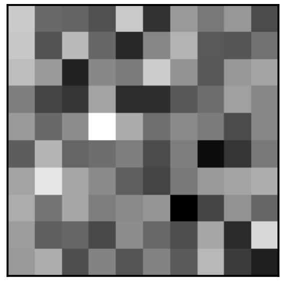
$$\mathbf{A}\mathbf{A} = \mathbf{A} \tag{34}$$

Special Properties

$$\mathbf{A}^n = A \ \ \forall n \tag{35}$$
$$\mathbf{I} - \mathbf{A} \ \text{ is idempotent} \tag{36}$$
$$\mathbf{A}^H \ \text{ is idempotent} \tag{37}$$
$$\mathbf{I} - \mathbf{A}^H \ \text{ is idempotent} \tag{38}$$
$$\text{rank}(\mathbf{A}) = \text{tr}(\mathbf{A}) \tag{39}$$
$$\mathbf{A}(I - \mathbf{A}) = 0 \tag{40}$$
$$\mathbf{A}^+ = \mathbf{A} \tag{41}$$
$$f(s\mathbf{I} + t\mathbf{A}) = (\mathbf{I} - \mathbf{A})f(s) + \mathbf{A}f(s+t) \tag{42}$$
$$\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A} \implies \mathbf{A}\mathbf{B} \text{ is idempotent} \tag{43}$$
$$\text{eig}(\mathbf{A})_i \in \{0, 1\} \tag{44}$$
$$\mathbf{A} \text{ is always diagonalizable} \tag{45}$$

$\mathbf{A} - \mathbf{I}$ may not be idempotent.

5.6 Orthogonal Matrix



(Not much visible structure)

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{46}$$

A matrix $\mathbf{U}$ is orthogonal iff:

$$\mathbf{U}^T\mathbf{U} = \mathbf{U}\mathbf{U}^T = I \tag{47}$$

Square matrix. The columns form an orthonormal basis of $\mathbb{R}^n$.

Special Properties

- The eigenvalues of $\mathbf{U}$ are placed on the unit circle.
- The eigenvectors of $\mathbf{U}$ are unitary (have length one).
- $\mathbf{U}^{-1}$ is orthogonal.

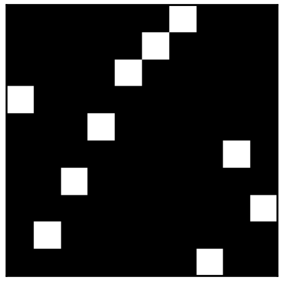
$$\mathbf{U}^T = \mathbf{U}^{-1} \tag{48}$$
$$\mathbf{U}^{-T} = \mathbf{U} \tag{49}$$
$$\mathbf{U}^T\mathbf{U} = \mathbf{I} \tag{50}$$
$$\mathbf{U}\mathbf{U}^T = \mathbf{I} \tag{51}$$
$$\det(\mathbf{U}) = \pm 1 \tag{52}$$

Orthogonal matrices preserve the lengths and angles of the vectors they operator on. The converse is true: any matrix which preserves lengths and angles is orthogonal.

$$\|\mathbf{U}\mathbf{x}\|_2^2 = (\mathbf{U}\mathbf{x})^T(\mathbf{U}\mathbf{x}) = \mathbf{x}^T\mathbf{U}^T\mathbf{U}\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|_2^2 \quad \forall \mathbf{x} \tag{53}$$

$$\|\mathbf{U}\mathbf{A}\mathbf{V}\|_F = \|\mathbf{A}\|_F \quad \forall \mathbf{A}, \mathbf{U}, \mathbf{V} \text{ with } U, V orthogonal \tag{54}$$

5.7 Permutation Matrix



TODO

5.8 Positive Definite

$\mathbf{A} \in \mathbb{S}^n$ is positive definite (denoted $\mathbf{A} \succ 0$) if any of the following are true:

- $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0, \forall \mathbf{x} \in \mathbb{R}^n$.
- $\text{eig}(\mathbf{A}) > 0$

Special Properties

- If $\mathbf{A}$ is PD and invertible, $\mathbf{A}^{-1}$ is also PD.
- If $\mathbf{A}$ is PD and $c \in \mathbb{R}$ then $c\mathbf{A}$ is PD.
- The diagonal entries $\mathbf{A}_{ii}$ are real and non-negative, so $\text{tr}(\mathbf{A}) \geq 0$.
- $\det(\mathbf{A}) > 0$
- For $\mathbf{A} \in \mathbb{R}^{m,n}$, $\mathbf{A}^T \mathbf{A} \succ 0 \iff \mathbf{A}$ is full-column rank ($\text{rank}(\mathbf{A}) = n$)
- For $\mathbf{A} \in \mathbb{R}^{m,n}$, $\mathbf{A} \mathbf{A}^T \succ 0 \iff \mathbf{A}$ is full-row rank ($\text{rank}(\mathbf{A}) = m$)
- $\mathbf{P} \succ 0$ defines a full-dimensional, bounded ellipsoid centered at the origin and defined by the set $\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^n : x^T \mathbf{P}^{-1} x \leq 1\}$. The eigenvalues λ_i and eigenvectors u_i of $\mathbf{P}$ define the orientation and shape of the ellipsoid. u_i are the semi-axes while the lengths of the semi-axes are given by $\sqrt{\lambda_i}$. Using the Cholesky decomposition, $\mathbf{P}^{-1} = \mathbf{A}^T \mathbf{A}$, an equivalent definition of the ellipsoid is $\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{A}\mathbf{x}\|_2 \leq 1\}$.

5.9 Positive Semi-Definite

$\mathbf{A}$ is positive semi-definite (denoted $\mathbf{A} \succeq 0$) if any of the following are true:

- $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0, \forall \mathbf{x} \in \mathbb{R}^n$.
- $\text{eig}(\mathbf{A}) \geq 0$

Special Properties

- For $\mathbf{A} \in \mathbb{R}^{m,n}$, $\mathbf{A}^T\mathbf{A} \succeq 0$
- For $\mathbf{A} \in \mathbb{R}^{m,n}$, $\mathbf{A}\mathbf{A}^T \succeq 0$
- The positive semi-definite matrices $\mathbb{S}_+^n$ form a convex cone. For any two PSD matrices $\mathbf{A}, \mathbf{B} \in \mathbb{S}_+^n$ and some $\alpha \in [0, 1]$:

$$\mathbf{x}^T(\alpha\mathbf{A} + (1-\alpha)\mathbf{B})\mathbf{x} = \alpha\mathbf{x}^T\mathbf{A}\mathbf{x} + (1-\alpha)\mathbf{x}^T\mathbf{B}\mathbf{x} \geq 0 \quad \forall\mathbf{x} \tag{55}$$

$$\alpha\mathbf{A} + (1-\alpha)\mathbf{B} \in \mathbb{S}_+^n \tag{56}$$

- For $\mathbf{A} \in \mathbb{S}_+^n$ and $\alpha \geq 0$, $\alpha\mathbf{A} \succeq 0$, so $\mathbb{S}_+^n$ is a cone.

5.10 Projection Matrix

A square matrix $\mathbf{P}$ is a projection matrix that projects onto a vector space $\mathcal{S}$ iff

$$\mathbf{P} \text{ is idempotent} \tag{57}$$

$$\mathbf{P}\mathbf{x} \in \mathcal{S} \quad \forall\mathbf{x} \tag{58}$$

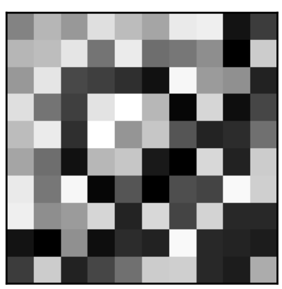
$$\mathbf{P}\mathbf{z} = \mathbf{z} \quad \forall\mathbf{z} \in \mathcal{S} \tag{59}$$

5.11 Singular Matrix

A square matrix that is not invertible.

$\mathbf{A} \in \mathbb{R}^{n,n}$ is singular iff $\det \mathbf{A} = 0$ iff $\mathcal{N}(A) \neq \{0\}$.

5.12 Symmetric Matrix



$\mathbf{A} \in \mathbb{S}^n$ is a symmetric matrix if $\mathbf{A} = \mathbf{A}^T$ (entries above diagonal are equal to entries below diagonal).

$$\begin{bmatrix} a & b & c & d & e & f \\ b & g & l & m & o & p \\ c & l & h & n & q & r \\ d & m & n & i & s & t \\ e & o & q & s & j & u \\ f & p & r & t & u & k \end{bmatrix} \tag{60}$$

Special Properties

$$\mathbf{A} = \mathbf{A}^T \tag{61}$$
$$\mathrm{eig}(A) \in \mathbb{R}^n \tag{62}$$

Number of "free entries": $\frac{n(n+1)}{2}$.

If $\mathbf{A}$ is real, it can be decomposed into $\mathbf{A} = \mathbf{Q}^T\mathbf{D}\mathbf{Q}$ where $\mathbf{Q}$ is a real orthogonal matrix (the columns of which are eigenvectors of $\mathbf{A}$) and $\mathbf{D}$ is real and diagonal containing the eigenvalues of $\mathbf{A}$.

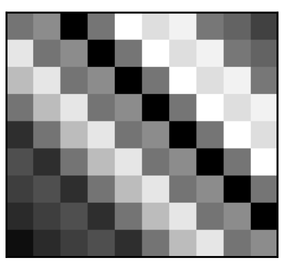
For a real, symmetric matrix with non-negative eignevalues, the eigenvalues and singular values coincide.

5.13 Skew-Hermitian

A matrix $\mathbf{H} \in \mathbb{C}^{m,n}$ is Skew-Hermitian iff

$$\mathbf{H} = -\mathbf{H}^H \tag{63}$$

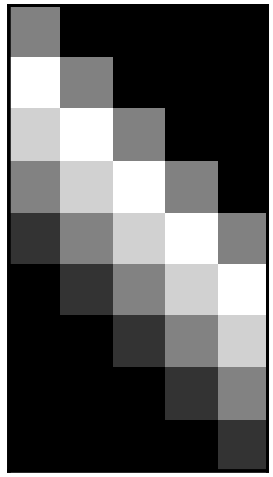
5.14 Toeplitz Matrix, General Form



Constant values on descending diagonals.

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{bmatrix} \tag{64}$$

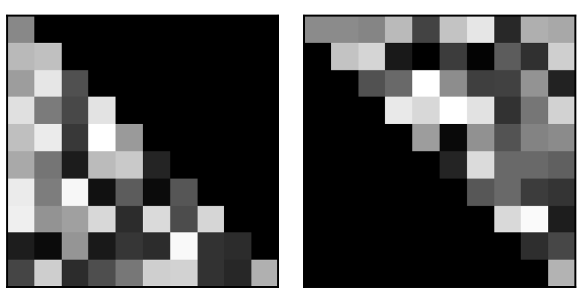
5.15 Toeplitz Matrix, Discrete Convolution



Constant values on main and subdiagonals.

$$\begin{bmatrix} h_m & 0 & 0 & \dots & 0 & 0 \\ \vdots & h_m & 0 & \dots & 0 & 0 \\ h_1 & \vdots & h_m & \dots & 0 & 0 \\ 0 & h_1 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & h_1 & \ddots & h_m & 0 \\ 0 & 0 & 0 & \ddots & \vdots & h_m \\ 0 & 0 & 0 & \dots & h_1 & \vdots \\ 0 & 0 & 0 & \dots & 0 & h_1 \end{bmatrix} \tag{65}$$

5.16 Triangular Matrix



$$\begin{bmatrix} a & b & c & d & e & f \\ & g & h & i & j & k \\ & & l & m & n & o \\ & & & p & q & r \\ & & & & s & t \\ & & & & & u \end{bmatrix} \quad \begin{bmatrix} a & & & & & \\ b & g & & & & \\ c & h & l & & & \\ d & i & m & p & & \\ e & j & n & q & s & \\ f & k & o & r & t & u \end{bmatrix} \tag{66}$$

Square matrices in which all elements either above or below the main diagonal are zero. An upper (left) and a lower (right) triangular matrix are shown above.

For an upper triangular matrix $A_{ij} = 0$ whenever $i > j$; for a lower triangular matrix $A_{ij} = 0$ whenever $i < j$.

Special Properties

$$\mathrm{eig}(A) = \mathrm{diag}(A) \tag{67}$$

$$\det(A) = \prod_i \mathrm{diag}(A)_i \tag{68}$$

The product of two upper (lower) triangular matrices is still upper (lower) triangular.

The inverse of a nonsingular upper (lower) triangular matrix is still upper (lower) triangular.

5.17 Vandermonde Matrix

$$V = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \dots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \dots & \alpha_m^{n-1} \end{bmatrix} \tag{69}$$

Alternatively,

$$V_{i,j} = \alpha_i^{j-1} \tag{70}$$

Uses

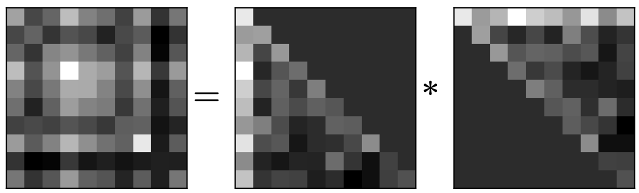
Polynomial interpolation of data.

Special Properties

- $\det(V) = \prod_{1 \le i < j \le n} (x_j - x_i)$

6 Matrix Decompositions

6.1 LLT/UTU: Cholesky Decomposition

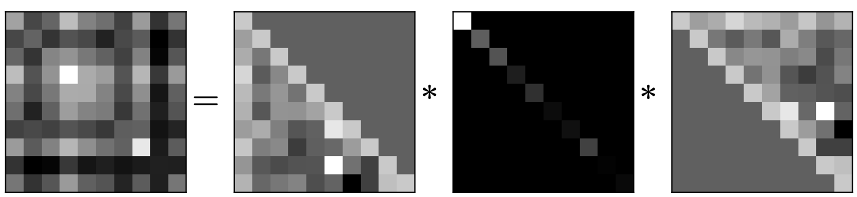


If $\mathbf{A}$ is symmetric, positive definite, square, then

$$\mathbf{A} = \mathbf{U}^T\mathbf{U} = \mathbf{L}\mathbf{L}^T \tag{71}$$

where $\mathbf{U}$ is a unique upper triangular matrix and $\mathbf{L}$ is a unique lower-triangular matrix.

6.2 LDLT



If $\mathbf{A}$ is a non-singular symmetric definite square matrix, then

$$\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T = \mathbf{L}^T\mathbf{D}\mathbf{L} \tag{72}$$

where $\mathbf{L}$ is a unit lower triangular matrix and $\mathbf{D}$ is a diagonal matrix. If $\mathbf{A} \succ 0$, then $\mathbf{D}_{ii} > 0$.

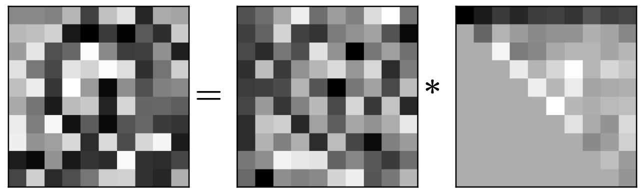
6.3 PCA: Principle Components Analysis

Find normalized directions in data space such that the variance of the projections of the centered data points is maximal. For centered data $\tilde{\mathbf{X}}$, the mean-square variation of data along a vector $\mathbf{x}$ is $\mathbf{x}^T\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\mathbf{x}$.

$$\max_{\mathbf{x}\in\mathbb{R}^n,\ \|\mathbf{x}\|_2=1} \mathbf{x}^T\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\mathbf{x} \tag{73}$$

Taking an SVD of $\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T$ gives $H = \mathbf{U}_r\mathbf{D}^2\mathbf{U}^T$, which is maximized by taking $\mathbf{x} = \mathbf{u}_1$. By repeatedly removing the first principal components and recalculating, all the principal axes can be found.

6.4 QR: Orthogonal-triangular

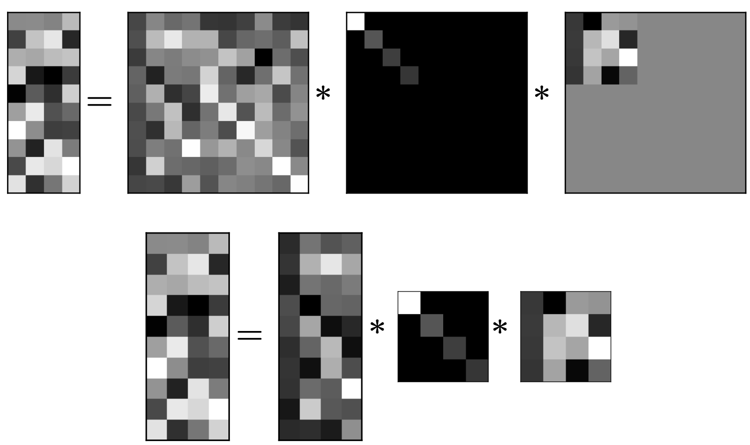


For $\mathbf{A} \in \mathbb{R}^{n,n}$, $\mathbf{A} = \mathbf{QR}$ where $\mathbf{Q}$ is orthogonal and $\mathbf{R}$ is an upper triangular matrix. If $\mathbf{A}$ is non-singular, then $\mathbf{Q}$ and $\mathbf{R}$ are uniquely defined if $\mathrm{diag}(\mathbf{R})$ are imposed to be positive.

Algorithms

Gram-Schmidt.

6.5 SVD: Singular Value Decomposition



Any matrix $\mathbf{A} \in \mathbb{R}^{m,n}$ can be written as

$$\mathbf{A} = \mathbf{UDV}^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T \tag{74}$$

where

$$U = \text{eigenvectors of } \mathbf{AA}^T \qquad \mathbb{R}^{m,m} \tag{75}$$

$$D = \mathrm{diag}(\sigma_i) = \sqrt{\mathrm{diag}(\mathrm{eig}(\mathbf{AA}^T))} \qquad \mathbb{R}^{n,m} \tag{76}$$

$$V = \text{eigenvectors of } \mathbf{A}^T\mathbf{A} \qquad \mathbb{R}^{n,n} \tag{77}$$

Let σ_i be the non-zero singular values for $i = 1, \ldots, r$ where r is the rank of $\mathbf{A}$; $\sigma_1 \geq \ldots \geq \sigma_r$.

We also have that

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i \tag{78}$$

$$\mathbf{A}^T\mathbf{u}_i = \sigma_i \mathbf{v}_i \tag{79}$$

$$\mathbf{U}^T\mathbf{U} = \mathbf{I} \tag{80}$$

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{81}$$

$\mathbf{D}$ can be written in an expanded form:

$$\tilde{\mathbf{D}} = \begin{bmatrix} \mathbf{D} & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} \tag{82}$$

The final $n-r$ columns of $\mathbf{V}$ give an orthonormal basis spanning $\mathcal{N}(\mathbf{A})$. An orthonormal basis spanning the range of $\mathbf{A}$ is given by the first r columns of $\mathbf{U}$.

$$\|\mathbf{A}\|_F^2 = \text{Frobenius norm} = \text{tr}(\mathbf{A}^T\mathbf{A}) = \sum_{i=1}^{r} \sigma_i^2 \tag{83}$$

$$\|\mathbf{A}\|_2^2 = \sigma_1^2 \tag{84}$$

$$\|\mathbf{A}\|_* = \text{nuclear norm} = \sum_{i=1}^{r} \sigma_i \tag{85}$$

The **condition number** κ of an invertible matrix $\mathbf{A} \in \mathbb{R}^{n,n}$ is the ratio of the largest and smallest singular value. Matrices with large condition numbers are closer to being singular and more sensitive to changes.

$$\kappa(\mathbf{A}) = \frac{\sigma_1}{\sigma_n} = \|A\|_2 \cdot \left\|A^{-1}\right\|_2 \tag{86}$$

Low-Rank Approximation

Approximating $\mathbf{A} \in \mathbb{R}^{m,n}$ by a matrix $\mathbf{A}_k$ of rank $k > 0$ can be formulated as the optimization probem

$$\min_{\mathbf{A}_k \in \mathbb{R}^{m,n}} \|\mathbf{A} - \mathbf{A}_k\|_F^2 : \text{rank}\,\mathbf{A}_k = k, 1 \leq k \leq \text{rank}(\mathbf{A}) \tag{87}$$

The optimal solution of this problem is given by

$$\mathbf{A}_k = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T \tag{88}$$

where

$$\frac{\|\mathbf{A}_k\|_F^2}{\|\mathbf{A}\|_F^2} = \frac{\sigma_1^2 + \ldots + \sigma_k^2}{\sigma_1^2 + \ldots + \sigma_r^2} \tag{89}$$

$$1 - \frac{\|\mathbf{A}_k\|_F^2}{\|\mathbf{A}\|_F^2} = \frac{\sigma_{k+1}^2 + \ldots + \sigma_r^2}{\sigma_1^2 + \ldots + \sigma_r^2} \tag{90}$$

is the fraction of the total variance in $\mathbf{A}$ explained by the approximation $\mathbf{A}_k$.

Range and Nullspace

$$\mathcal{N}(\mathbf{A}) = \mathcal{R}(\mathbf{V}_{nr}) \tag{91}$$
$$\mathcal{N}(\mathbf{A})^{\perp} \equiv \mathcal{R}(\mathbf{A}^T) = \mathcal{R}(\mathbf{V}_r) \tag{92}$$
$$\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{U}_r) \tag{93}$$
$$\mathcal{R}(\mathbf{A})^{\perp} \equiv \mathcal{N}(\mathbf{A}^T) = \mathcal{R}(\mathbf{U}_{nr}) \tag{94}$$

where $\mathbf{V}_r$ is the first r columns of V and $V_n r$ are the last $[r+1, n]$ columns; similarly for $\mathbf{U}$.

Projectors

The projection of $\mathbf{x}$ onto $\mathcal{N}(\mathbf{A})$ is $(\mathbf{V}_{nr}\mathbf{V}_{nr}^T)\mathbf{x}$. Since $\mathbf{I}_n = \mathbf{V}_r\mathbf{V}_r^T + \mathbf{V}_{nr}\mathbf{V}_{nr}^T$, $(\mathbf{I}_n - \mathbf{V}_r\mathbf{V}_r^T)\mathbf{x}$ also works. The projection of $\mathbf{x}$ onto $\mathcal{R}(\mathbf{A})$ is $(\mathbf{U}_r\mathbf{U}_r^T)\mathbf{x}$.

If $\mathbf{A} \in \mathbb{R}^{m,n}$ is full row rank ($\mathbf{A}\mathbf{A}^T \succ 0$), then the minimum distance to an affine set $\{x : \mathbf{A}\mathbf{x} = \mathbf{b}\}, \mathbf{b} \in \mathbb{R}^m$ is given by $\mathbf{x}^* = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{b}$.

If $\mathbf{A} \in \mathbb{R}^{m,n}$ is full column rank ($\mathbf{A}^T\mathbf{A} \succ 0$), then the minimum distance to an affine set $\{x : \mathbf{A}\mathbf{x} = \mathbf{b}\}, \mathbf{b} \in \mathbb{R}^m$ is given by $\mathbf{x}^* = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$.

Computational Notes

Since $\sigma \approx 0$, a *numerical rank* can be estimated for the matrix as the largest k such that $\sigma_k > \epsilon\sigma_1$ for $\epsilon \geq 0$.

6.6 Eigenvalue Decomposition for Diagonalizable Matrices

For a square, diagonalizable matrix $\mathbf{A} \in \mathbb{R}^{n,n}$

$$\mathbf{A} = U\Lambda U^{-1} \tag{95}$$

where $U \in \mathbb{C}^{n,n}$ is an invertible matrix whose columns are the eigenvectors of $\mathbf{A}$ and Λ is a diagonal matrix containing the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathbf{A}$ in the diagonal.

The columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$ satisfy

$$\mathbf{A}\mathbf{u}_i = \lambda_i\mathbf{u}_i \quad i = 1, \ldots, n \tag{96}$$

6.7 Eigenvalue (Spectral) Decomposition for Symmetric Matrices

A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n,n}$ can be factored as

$$\mathbf{A} = U\Lambda U^T = \sum_i^n \lambda_i\mathbf{u}_i\mathbf{u}_i^T \tag{97}$$

where $U \in \mathbb{R}^{n,n}$ is an orthogonal matrix whose columns $\mathbf{u}_i$ are the eigenvectors of $\mathbf{A}$ and Λ is a diagonal matrix containing the eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ of $\mathbf{A}$ in the diagonal. These eigenvalues are always real. The eigenvectors can always be chosen to be real and to form an orthonormal basis.

The columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$ satisfy

$$\mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_i \quad i = 1, \ldots, n \tag{98}$$

6.8 Schur Complements

For $\mathbf{A} \in \mathbb{S}^n$, $\mathbf{B} \in \mathbb{S}^n$, $\mathbf{X} \in \mathbb{R}^{n,m}$ with $\mathbf{B} \succ 0$ and the block matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{B} \end{bmatrix} \tag{99}$$

and the Schur complement of $\mathbf{A}$ in $\mathbf{M}$

$$S = \mathbf{A} - \mathbf{X}\mathbf{B}^{-1}\mathbf{X}^T \tag{100}$$

Then

$$\mathbf{M} \succeq 0 \iff S \succeq 0 \tag{101}$$

$$\mathbf{M} \succ 0 \iff S \succ 0 \tag{102}$$

7 | Transpose Properties

$$(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T \tag{103}$$

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \tag{104}$$

$$(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} \tag{105}$$

8 Determinant Properties

Geometrically, if a unit volume is acted on by $\mathbf{A}$, then $|\det(\mathbf{A})|$ indicates the volume after the transformation.

$$\det(I_n) = 1 \tag{106}$$
$$\det(\mathbf{A}^T) = \det(\mathbf{A}) \tag{107}$$
$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})} = \det(\mathbf{A})^{-1} \tag{108}$$
$$\det(AB) = \det(BA) \tag{109}$$
$$\det(AB) = \det(A)\det(B) \quad \mathbf{A}, \mathbf{B} \in \mathbb{R}^{n,n} \tag{110}$$
$$\det(c\mathbf{A}) = c^n \det(\mathbf{A}) \quad \mathbf{A} \in \mathbb{R}^{n,n} \tag{111}$$
$$\det(\mathbf{A}) = \prod \mathrm{eig}(\mathbf{A}) \tag{112}$$

9 Trace Properties

For $\mathbf{A} \in \mathbb{R}^{n,n}$

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} \mathbf{A}_{ii} \tag{113}$$

$$\operatorname{tr}(\mathbf{A} + \mathbf{B}) = \operatorname{tr}(\mathbf{A}) + \operatorname{tr}(\mathbf{B}) \tag{114}$$

$$\operatorname{tr}(c\mathbf{A}) = c \operatorname{tr}(\mathbf{A}) \tag{115}$$

$$\operatorname{tr}(\mathbf{A}) = \operatorname{tr}(\mathbf{A}^T) \tag{116}$$

For $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ of compatible dimensions,

$$\operatorname{tr}(\mathbf{A}^T\mathbf{B}) = \operatorname{tr}(\mathbf{A}\mathbf{B}^T) = \operatorname{tr}(\mathbf{B}^T\mathbf{A}) = \operatorname{tr}(\mathbf{B}\mathbf{A}^T) \tag{117}$$

$$\operatorname{tr}(\mathbf{ABCD}) = \operatorname{tr}(\mathbf{BCDA}) = \operatorname{tr}(\mathbf{CDAB}) = \operatorname{tr}(\mathbf{DABC}) \tag{118}$$

(Invariant under cyclic permutations)

10 | Inverse Properties

The inverse of $\mathbf{A} \in \mathbb{C}^{n,n}$ is denoted $\mathbf{A}^{-1}$ and defined such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n \tag{119}$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix. $\mathbf{A}$ is nonsingular if $\mathbf{A}^{-1}$ exists; otherwise, $\mathbf{A}$ is singular.

If individual inverses exist

$$(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{120}$$

more generally

$$(\mathbf{A}_1\mathbf{A}_2 \dots \mathbf{A}_n)^{-1} = \mathbf{A}_n^{-1} \dots \mathbf{A}_2^{-1}\mathbf{A}_1^{-1} \tag{121}$$

$$(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} \tag{122}$$

11 | Pseudo-Inverse Properties

For $\mathbf{A} \in \mathbb{R}^{m,n}$, a pseudoinverse satisfies:

$$\mathbf{A}\mathbf{A}^{+}\mathbf{A} = \mathbf{A} \tag{123}$$

$$\mathbf{A}^{+}\mathbf{A}\mathbf{A}^{+} = \mathbf{A}^{+} \tag{124}$$

$$(\mathbf{A}\mathbf{A}^{+})^{T} = \mathbf{A}\mathbf{A}^{+} \tag{125}$$

$$(\mathbf{A}^{+}\mathbf{A})^{T} = \mathbf{A}^{+}\mathbf{A} \tag{126}$$

11.1 Moore-Penrose Pseudoinverse

$$\mathbf{A}^{+} = \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^{T} \tag{127}$$

where the foregoing comes from a singular-value decomposition and $\mathbf{D}^{-1} = \mathrm{diag}(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_r})$

Special Properties

- $\mathbf{A}^{+} = \mathbf{A}^{-1}$ if $\mathbf{A} \in \mathbb{R}^{n,n}$ and $\mathbf{A}$ is square and nonsingular.
- $\mathbf{A}^{+} = (\mathbf{A}^{T}\mathbf{A})^{-1}\mathbf{A}^{T}$, if $\mathbf{A} \in \mathbb{R}^{m,n}$ is full column rank ($r = n \leq m$). $\mathbf{A}^{+}$ is a left inverse of $\mathbf{A}$, so $\mathbf{A}^{+}\mathbf{A} = \mathbf{V}_r\mathbf{V}_r^{T} = \mathbf{V}\mathbf{V}^{T} = \mathbf{I}_n$.
- $\mathbf{A}^{+} = \mathbf{A}^{T}(\mathbf{A}\mathbf{A}^{T})^{-1}$, if $\mathbf{A} \in \mathbb{R}^{m,n}$ is full row rank ($r = m \leq n$). $\mathbf{A}^{+}$ is a right inverse of $\mathbf{A}$, so $\mathbf{A}\mathbf{A}^{+} = \mathbf{U}_r\mathbf{U}_r^{T} = \mathbf{U}\mathbf{U}^{T} = \mathbf{I}_m$.

12 Hadamard Identities

$$(\mathbf{A} \circ \mathbf{B})_{ij} = A_{ij}B_{ij} \; \forall \; i,j \tag{128}$$

$$\mathbf{A} \circ \mathbf{B} = \mathbf{B} \circ \mathbf{A} \tag{129}$$ [14]

$$\mathbf{A} \circ (\mathbf{B} \circ \mathbf{C}) = (\mathbf{A} \circ \mathbf{B}) \circ \mathbf{C} \tag{130}$$

$$\mathbf{A} \circ (\mathbf{B} + \mathbf{C}) = \mathbf{A} \circ \mathbf{B} + \mathbf{A} \circ \mathbf{C} \tag{131}$$ [14]

$$a(\mathbf{A} \circ \mathbf{B}) = (a\mathbf{A}) \circ \mathbf{B} = \mathbf{A} \circ (a\mathbf{B}) \tag{132}$$ [14]

$$(\mathbf{A}^T \circ \mathbf{B}^T) = (\mathbf{A} \circ \mathbf{B})^T \tag{133}$$

$$(\mathbf{A}^T \circ \mathbf{B}^T) = (\mathbf{A} \circ \mathbf{B})^T \tag{134}$$

$$(\mathbf{x}^T\mathbf{A}\mathbf{x}) = \sum_{i,j} \big((\mathbf{x}\mathbf{x}^T) \circ \mathbf{A}\big) \tag{135}$$

$$\mathbf{x}^T(\mathbf{A} \circ \mathbf{B})\mathbf{y} = \mathrm{tr}((\mathrm{diag}(\mathbf{x})\mathbf{A})^T\mathbf{B}\,\mathrm{diag}(\mathbf{y})) \quad \mathbf{A}, \mathbf{B} \in \mathbb{R}^{m,n} \tag{136}$$ [15]

$$\mathrm{tr}(\mathbf{A}^T\mathbf{B}) = \mathbf{1}^T(\mathbf{A} \circ \mathbf{B})\mathbf{1} \tag{137}$$

13 | Eigenvalue Properties

$\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbf{A} \in \mathbb{R}^{n,n}$ and $u \in \mathbb{C}^n$ is a corresponding eigenvector if $\mathbf{Au} = \lambda\mathbf{u}$ and $\mathbf{u} \neq 0$. Equivalantly, $(\lambda\mathbf{I}_n - \mathbf{A})\mathbf{u} = 0$ and $\mathbf{u} \neq 0$. Eigenvalues satisfy the equation $\det(\lambda\mathbf{I}_n - \mathbf{A}) = 0$.

Any matrix $\mathbf{A} \in \mathbb{R}^{n,n}$ has n eigenvalues, though some may be repeated. λ_1 is the largest eigenvalue and λ_n the smallest.

$$\text{eig}(\mathbf{A}\mathbf{A}^T) = \text{eig}(\mathbf{A}^T\mathbf{A}) \tag{138}$$

(Note that the number of entries in $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$ may differ significantly leading to different compute times.)

$$\text{eig}(\mathbf{A}^T\mathbf{A}) \geq 0 \tag{139}$$

Computation

TODO: eigsh, small eigen value extraction, top-k

14 Norms

14.1 Matrices

Matrix norms satisfy some properties:

$$f(\mathbf{A}) \geq 0 \tag{140}$$
$$f(\mathbf{A}) = 0 \iff \mathbf{A} = 0 \tag{141}$$
$$f(c\mathbf{A}) = |c| f(\mathbf{A}) \tag{142}$$
$$f(\mathbf{A}+\mathbf{B}) \leq f(\mathbf{A}) + f(\mathbf{B}) \tag{143}$$

Many popular matrix norms also satisfy "sub-multiplicativity": $f(\mathbf{AB}) \leq f(\mathbf{A}) f(\mathbf{B})$.

14.1.1 Frobenius norm

$$\begin{aligned} \|\mathbf{A}\|_F &= \sqrt{\operatorname{tr} \mathbf{A}\mathbf{A}^H} && (144) \\ &= \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{A}_{ij}|^2} && (145) \\ &= \sqrt{\sum_{i=1}^{m} \operatorname{eig}(A^H A)_i} && (146) \end{aligned}$$

Special Properties

$$\|\mathbf{Ax}\|_2 \leq \|\mathbf{A}\|_F \|\mathbf{x}\|_2 \quad \mathbf{x} \in \mathbb{R}^n \tag{147}$$

$$\|\mathbf{AB}\|_F \leq \|\mathbf{A}\|_F \|\mathbf{B}\|_F \tag{148}$$

14.1.2 Operator Norms

For $p = 1, 2, \infty$ or other values, an operator norm indicates the maximum input-output gain of the matrix.

$$\|\mathbf{A}\|_p = \max_{\|\mathbf{u}\|_p = 1} \|\mathbf{Au}\|_p \tag{149}$$

$$\begin{aligned} \|\mathbf{A}\|_1 &= \max_{\|\mathbf{u}\|_1 = 1} \|\mathbf{Au}\|_1 && (150) \\ &= \max_{j=1,\ldots,n} \sum_{i=1}^{m} |\mathbf{A}_{ij}| && (151) \\ &= \text{Largest absolute column sum} && (152) \end{aligned}$$

$$\|\mathbf{A}\|_\infty = \max_{\|\mathbf{u}\|_\infty=1} \|\mathbf{Au}\|_\infty \tag{153}$$

$$= \max_{j=1,\ldots,m} \sum_{i=1}^{n} |\mathbf{A}_{ij}| \tag{154}$$

$$= \text{Largest absolute row sum} \tag{155}$$

$$\|\mathbf{A}\|_2 = \text{“spectral norm”} \tag{156}$$

$$= \max_{\|\mathbf{u}\|_2=1} \|\mathbf{Au}\|_2 \tag{157}$$

$$= \sqrt{\max(\text{eig}(\mathbf{A}^T\mathbf{A}))} \tag{158}$$

$$= \text{Square root of largest eigenvalue of } \mathbf{A}^T\mathbf{A} \tag{159}$$

Special Properties

$$\|\mathbf{Au}\|_p \leq \|\mathbf{A}\|_p \|\mathbf{u}\|_p \tag{160}$$

$$\|\mathbf{AB}\|_p \leq \|\mathbf{A}\|_p \|\mathbf{B}\|_p \tag{161}$$

$$\tag{162}$$

14.1.3 Spectral Radius

Not a proper norm.

$$\rho(\mathbf{A}) = \text{spectral radius}(\mathbf{A}) = \max_{i=1,\ldots,n} |\,\text{eig}(\mathbf{A})_i| \tag{163}$$

Special Properties

$$\rho(\mathbf{A}) \leq \|\mathbf{A}\|_p \tag{164}$$

$$\rho(\mathbf{A}) \leq \min(\ \|\mathbf{A}\|_1, \|\mathbf{A}\|_\infty) \tag{165}$$

$$\tag{166}$$

14.2 Vectors

$$\|\mathbf{x}\|_1 = \sum_i |\mathbf{x}_i| \qquad \text{(L1-norm)} \tag{167}$$

$$\|\mathbf{x}\|_p = (\sum_i |\mathbf{x}_i|^p)^{1/p} \qquad \text{(P-norm)} \tag{168}$$

$$\|\mathbf{x}\|_\infty = \max_i |\mathbf{x}_i| \qquad \text{(L}\infty\text{-norm, L-infinity norm)} \tag{169}$$

15 | Bounds

15.1 Matrix Gain

$$\lambda_{\min}(\mathbf{A}^T\mathbf{A}) \leq \frac{\|\mathbf{A}\mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2} \leq \lambda_{\max}(\mathbf{A}^T\mathbf{A}) \tag{170}$$

$$\max_{\mathbf{x}\neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}(\mathbf{A}^T\mathbf{A})} \implies \mathbf{x} = u_1 \tag{171}$$

$$\min_{\mathbf{x}\neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \sqrt{\lambda_{\min}(\mathbf{A}^T\mathbf{A})} \implies \mathbf{x} = u_n \tag{172}$$

15.2 Norms

For $\mathbf{x} \in \mathbb{R}^n$

$$\frac{1}{\sqrt{n}}\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{\operatorname{card}(\mathbf{x})}\|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_2 \leq n\|\mathbf{x}\|_\infty \tag{173}$$

For any $0 < p < q$ we have that $\|\mathbf{x}\|_q \leq \|\mathbf{x}\|_p$.

15.3 Rayleigh quotients

The Rayleigh quotient of $\mathbf{A} \in \mathbb{S}^n$ is given by

$$\frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \quad \mathbf{x} \neq 0 \tag{174}$$

$$\lambda_{\min}(\mathbf{A}) \leq \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \leq \lambda_{\max}(\mathbf{A}) \quad \mathbf{x} \neq 0 \tag{175}$$

$$\lambda_{\max}(A) = \max_{\mathbf{x}:\|\mathbf{x}\|_2=1} \mathbf{x}^T\mathbf{A}\mathbf{x} = u_1 \tag{176}$$

$$\lambda_{\min}(A) = \min_{\mathbf{x}:\|\mathbf{x}\|_2=1} \mathbf{x}^T\mathbf{A}\mathbf{x} = u_n \tag{177}$$

where u_1 and u_n are the eigenvectors associated with $\lambda_{\max}$ and $\lambda_{\min}$, respectively.

16 Linear Equations

The linear equation $\mathbf{Ax} = \mathbf{y}$ with $\mathbf{A} \in \mathbb{R}^{m,n}$ admits a solution iff $\text{rank}([\mathbf{Ay}]) = \text{rank}(\mathbf{A})$. If this is satisfied, the set of all solutions is an affine set $\mathcal{S} = \{\mathbf{x} = \bar{\mathbf{x}} + z : z \in \mathcal{N}(\mathbf{A})\}$ where $\bar{\mathbf{x}}$ is any vector such that $\mathbf{A}\bar{\mathbf{x}} = \mathbf{y}$. The solution is unique if $\mathcal{N}(\mathbf{A}) = \{0\}$.

$\mathbf{Ax} = \mathbf{y}$ is *overdetermined* if it is tall/skinny ($m > n$); that is, if there are more equations than unknowns. If $\text{rank}(\mathbf{A}) = n$ then $\dim \mathcal{N}(\mathbf{A}) = 0$, so there is either no solution or one solution. Overdetermined systems often have no solution ($\mathbf{y} \notin \mathcal{R}(\mathbf{A})$), so an approximate solution is necessary. See section 16.1.

$\mathbf{Ax} = \mathbf{y}$ is *underdetermined* if it is short/wide ($n > m$); that is, if has more unknowns than equations. If $\text{rank}(\mathbf{A}) = m$ then $\mathcal{R}(\mathbf{A}) = \mathbb{R}^m$, so $\dim \mathcal{N}(\mathbf{A}) = n - m > 0$, so the set of solutions is infinite. Therefore, finding a single solution that optimizes some quantity is of interest.

$\mathbf{Ax} = \mathbf{y}$ is *square* if $n = m$. If $\mathbf{A}$ is invertible, then the equations have the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$. See section 16.2.

16.1 Least-Squares

For an overdetermined system we wish to find:

$$\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{y}\|_2^2 \tag{178}$$

Since $\mathbf{Ax} \in \mathcal{R}(\mathbf{A})$, we need a point $\tilde{\mathbf{y}} = \mathbf{Ax}^* \in \mathcal{R}(\mathbf{A})$ closest to $\mathbf{y}$. This point lies in the nullspace of $\mathbf{A}^T$, so we have $\mathbf{A}^T(\mathbf{y} - \mathbf{Ax}^*) = 0$. There is always a solution to this problem and, if $\text{rank}(\mathbf{A}) = n$, it is unique [16, p. 161]

$$\mathbf{x}^* = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y} \tag{179}$$

16.2 Minimum Norm Solutions

For undertermined systems in which $\mathbf{A} \in \mathbb{R}^{m,n}$ with $m < n$. We wish to find

$$\min_{\mathbf{x}:\mathbf{Ax}=\mathbf{y}} \|\mathbf{x}\|_2 \tag{180}$$

The solution $\mathbf{x}^*$ must be orthogonal to $\mathcal{N}(\mathbf{A})$, so $\mathbf{x}^* \in \mathcal{R}(\mathbf{A}^T)$, so $\mathbf{x}^* = \mathbf{A}^T c$ for some c. Substituting into $\mathbf{Ax} = \mathbf{y}$ gives $\mathbf{AA}^T c = \mathbf{y}$, therefore [16, p. 162]:

$$\mathbf{x}^* = \mathbf{A}^T(\mathbf{AA}^T)^{-1}\mathbf{y} \tag{181}$$

17 Updates

17.1 Removing a row from $\mathbf{A}^T\mathbf{A}$ ($\mathbf{A}^T\mathbf{A} \to \mathbf{A}_{\setminus i}^T\mathbf{A}_{\setminus i}$)

Plain English: Matrix times its transpose after eliminating row i from the matrix

Inputs: $\mathbf{A} \in \mathbb{R}^{k,m}, \mathbf{u} \in \mathbb{R}^m, \mathbf{v} \in \mathbb{R}^n$ and i, the row to remove from $\mathbf{A}$

Reduces to: $\mathbf{C} \in \mathbb{R}^{k,l}$

Algorithm:

Recall that

$$(\mathbf{AB})_{kl} = \sum_m \mathbf{A}_{km}\mathbf{B}_{ml} \quad \mathbf{A} \in \mathbb{R}^{k,m}, \mathbf{B} \in \mathbb{R}^{m,l} \tag{182}$$

then we have that

$$(\mathbf{A}^T\mathbf{A})_{kl} = \sum_m \mathbf{A}_{mk}\mathbf{A}_{ml} = \sum_{m \neq i} \mathbf{A}_{mk}\mathbf{A}_{ml} + \mathbf{A}_{jk}\mathbf{A}_{jl} = \sum_{m \neq i} \mathbf{A}_{mk}\mathbf{A}_{ml} + (\mathbf{A}_{k*})_j(\mathbf{A}_{l*})_j \tag{183}$$

where $(\mathbf{A}_k*)_j$ is the jth element of the kth column of $\mathbf{A}$.

Thus,

$$\mathbf{A}_{\setminus i}^T\mathbf{A}_{\setminus i} = \mathbf{A}^T\mathbf{A} - \mathbf{A}_{*j}\mathbf{A}_{*j}^T \tag{184}$$

17.2 $\mathbf{1}_r^T\mathbf{A}\mathbf{1}_c$

Plain English: The sum of the elements of the matrix.

Reduces to: Scalar

Notation: For $\mathbf{A} \in \mathbb{R}^{r \times c}$, $\mathbf{1}_r$ is in $\mathbb{R}^{r \times 1}$ and $\mathbf{1}_c$ is in $\mathbb{R}^{c \times 1}$.

Algorithm: Traverse all the element of the matrix in order keeping track of the sum. For applications where accuracy is important and the matrices have a large dynamic range, Kahan summation or a similar technique should be used.

Update Algorithm: If an entry changes, subtract its old value from the sum and add its new value to the sum.

17.3 $\mathbf{x}^T\mathbf{A}\mathbf{x}$

Plain English: TODO

Reduces to: Scalar

Notation: $\mathbf{A}$ must be in $\mathbb{R}^{i \times i}$. $\mathbf{x}$ is in $\mathbb{R}^{i \times 1}$.

Algorithm: TODO

Update Algorithm: We make use of the identity $(\mathbf{x}^T\mathbf{A}\mathbf{x}) = \sum_{i,j}\left((\mathbf{x}\mathbf{x}^T)\circ\mathbf{A}\right)$. If an entry $\mathbf{A}_{i,j}$ in the matrix changes subtract its old value $\mathbf{x}_i\mathbf{x}_j\mathbf{A}_{ij}$ and add the new value $\mathbf{x}_i\mathbf{x}_j\mathbf{A}'_{ij}$. If an entry $\mathbf{x}_i$ changes TODO.

18 Optimization

18.1 Standard Forms

Least Squares

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2 \tag{185}$$

LASSO

$$\min_{\mathbf{b}\in\mathbb{R}^n} \left(\frac{1}{N}\|\mathbf{y}-\mathbf{X}\mathbf{b}\|_2^2+\lambda\|\mathbf{b}\|_1\right) \tag{186}$$

LP: Linear program

$$\min_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x} \tag{187}$$
$$\text{s.t. } \mathbf{A}_{\text{eq}}\mathbf{x}=\mathbf{b}_{\text{eq}} \tag{188}$$
$$\mathbf{A}\mathbf{x}\leq\mathbf{b} \tag{189}$$

Linear Fractional Program

$$\underset{\mathbf{x}}{\text{maximize}} \quad \frac{\mathbf{c}^T\mathbf{x}+a}{\mathbf{d}^T\mathbf{x}+b} \tag{190a}$$
$$\text{subject to} \quad \mathbf{A}\mathbf{x}\leq\mathbf{b} \tag{190b}$$

Additional constraints must ensure $\mathbf{d}^T\mathbf{x}+b$ has the same sign throughout the entire feasible region.

$$\max_{\mathbf{x}} \ \frac{\mathbf{c}^T\mathbf{x}+a}{\mathbf{d}^T\mathbf{x}+b} \tag{191}$$
$$\text{s.t.} \tag{192}$$

QCQP: Quadratic Constrainted Quadratic Programs

$$\min_{\mathbf{x}} \ \mathbf{x}^T\mathbf{H}_0\mathbf{x}+2\mathbf{c}_0^T\mathbf{x}+\mathbf{d}_0 \tag{193}$$
$$\text{s.t. } \mathbf{x}^T\mathbf{H}_i\mathbf{x}+2\mathbf{c}_i^T\mathbf{x}+\mathbf{d}_i\leq 0, \quad i\in\mathcal{I} \tag{194}$$
$$\mathbf{x}^T\mathbf{H}_j\mathbf{x}+2\mathbf{c}_j^T\mathbf{x}+\mathbf{d}_j=0, \quad j\in\mathcal{E} \tag{195}$$

Note that, in general, QCQPs are NP-Hard.

QP: Quadratic Program

$$\min_{\mathbf{x}} \ \frac{1}{2}\mathbf{x}^T\mathbf{H}_0\mathbf{x}+\mathbf{c}_0^T\mathbf{x} \tag{196}$$
$$\text{s.t. } \mathbf{A}_{\text{eq}}\mathbf{x}=\mathbf{b}_{\text{eq}} \tag{197}$$
$$\mathbf{A}\mathbf{x}\leq\mathbf{b} \tag{198}$$

SOCP: Second Order Cone Program (Standard Form)

$$\min_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x} \tag{199}$$

$$\text{s.t. } \|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i\|_2 \leq \mathbf{c}_i^T\mathbf{x} + \mathbf{d}_i, \quad i = 1, \ldots, m \tag{200}$$

SOCP: Second Order Cone Program (Conic Standard Form)

$$\min_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x} \tag{201}$$

$$\text{s.t. } (\mathbf{A}_i\mathbf{x} + \mathbf{b}_i, \mathbf{c}_i^T\mathbf{x} + \mathbf{d}_i) \in \mathcal{K}_{m_i} \quad i = 1, \ldots, m \tag{202}$$

18.2 Transformations

18.2.1 Linear-Fractional to Linear

We transform a Linear-Fractional Program

$$\underset{\mathbf{x}}{\text{maximize}} \quad \frac{\mathbf{c}^T\mathbf{x} + a}{\mathbf{d}^T\mathbf{x} + b} \tag{203a}$$

$$\text{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{203b}$$

where $\mathbf{d}^T\mathbf{x} + b$ has the same sign throughout the entire feasible region to a linear program using the Charnes–Cooper transformation [17] by defining

$$\mathbf{y} = \frac{1}{\mathbf{d}^T\mathbf{x} + b} \cdot \mathbf{x} \tag{204}$$

$$t = \frac{1}{\mathbf{d}^T\mathbf{x} + b} \tag{205}$$

to form the equivalent program

$$\underset{\mathbf{y}, t}{\text{maximize}} \quad \mathbf{c}^T\mathbf{y} + at \tag{206a}$$

$$\text{subject to} \quad \mathbf{A}\mathbf{y} \leq \mathbf{b}t, \tag{206b}$$

$$\mathbf{d}^T\mathbf{y} + bt = 1, \tag{206c}$$

$$t \geq 0 \tag{206d}$$

We then have $\mathbf{x}^* = \frac{1}{t}\mathbf{y}$.

19 | Algorithms

19.1 Gram-Schmidt

TODO

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