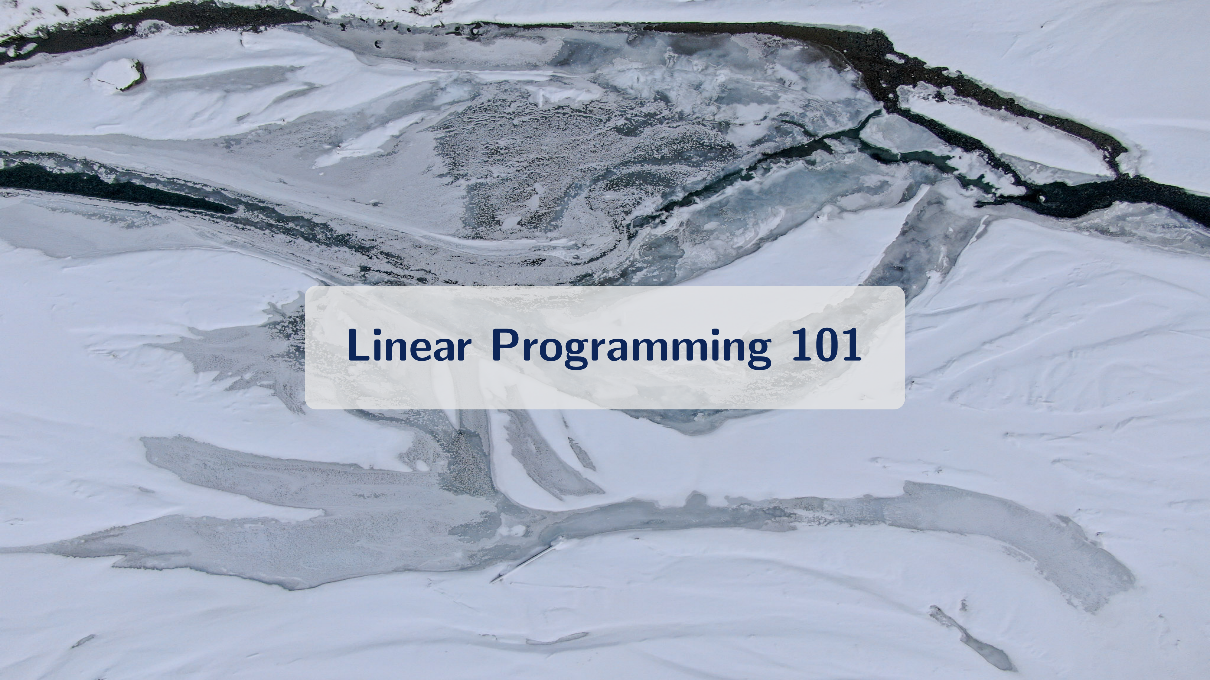
Modern Algorithmic Game Theory

Martin Schmid
Department of Applied Mathematics
Charles University in Prague

October 15, 2025



Linear Programming 101

Very Brief Intro to Linear Programming

- Linear programming deals with optimization of **linear objective functions** ($c^\top x$) that are subject to **linear inequality** ($Ax \leq b$) and **equality** constraints ($Ax = b$)
- Every linear optimization problem can be written in the canonical form

$$\begin{aligned} \max c^\top x \\ Ax \leq b \\ x \geq 0 \end{aligned}$$

- The given set of linear (in)equalities describes a **polyhedron**
- We say that a solution is **feasible** if it satifies the given (in)equalities
- Algorithms like the **Simplex method** or the family of **Interior point methods**, such as the **Ellipsoid algorithm** are commonly used to solve LPs

Primal and Dual Linear Programs

- Consider the two linear programs below

$$\begin{aligned} \max c^\top x \\ Ax \leq b \\ x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \min b^\top y \\ A^\top y \geq c \\ y \geq 0 \end{aligned}$$

- We call them the **primal** and the **dual** linear programs
- We can **always** convert between these two linear programs
- Furthermore, there exists an important theorem linking the two LPs together!

Weak and Strong Duality

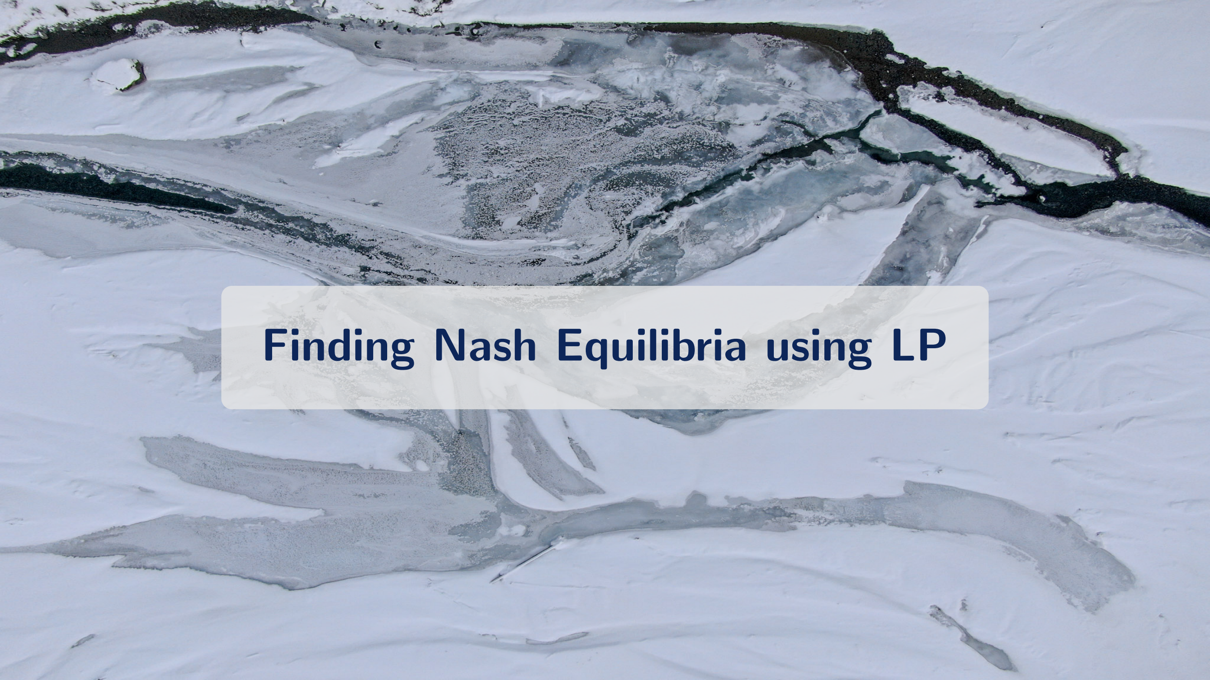
Consider a primal linear program P and its corresponding dual linear program D. The two following theorem link P and D together.

Theorem: Weak Duality

If x is a feasible solution to the primal P and y is a feasible solution to the dual D, then $c^\top x \leq b^\top y$.

Theorem: Strong Duality

If P and D are both feasible, then there exist optimal solutions x^* and y^*, such that $c^\top x^* = b^\top y^*$.

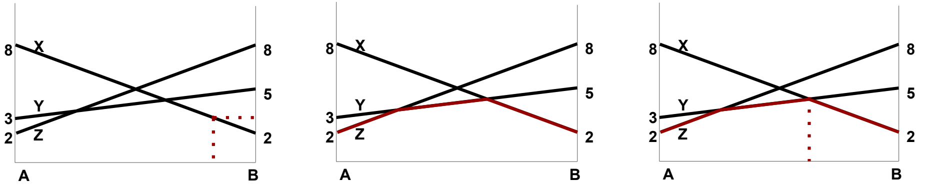


Finding Nash Equilibria using LP

Best Response Value Function

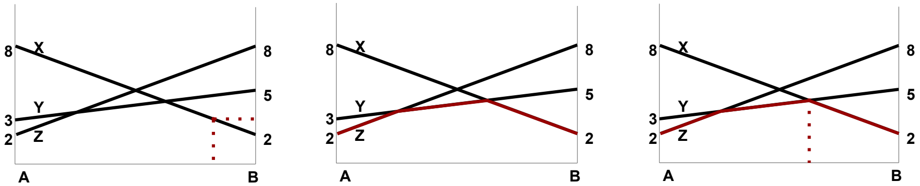
Let's revisit the concept of the best response value function of a 2×3 zero-sum game that we introduced in one of the previous lectures.

- The first figure shows the expected utilities of the row player against the three actions of the column player as a function of the row player's strategy
- The second figure highlights the best response value function when playing against the worst-case adversarial opponent
- The third figure shows the optimal strategy of the row player against the worst-case adversarial opponent



Best Response Value Function

- We can see that the best response value function is **concave** and **piece-wise linear**
- It can be shown that the best response value function is always going to be concave and piece-wise linear as it is a point-wise minimum of an affine function $\min_{\pi_{-i}} \pi_i^\top A \pi_{-i}$, for a fixed π_i
- This is a **key observation** that allows us to formulate finding a strategy maximizing the best response value function as a linear program!



Linear Programming and Nash Equilibria

- In the following section, we will only consider two-player zero-sum games
- Recall that Nash equilibria and Maximin strategies coincide in zero-sum games
- We will derive a pair of linear programs that find Maximin strategies for the two players and thus a Nash equilibrium in zero-sum games
- It also gives us a constructive and efficient way of finding Nash equilibria in two-player zero-sum games in **polynomial time**!

Linear Programming and Nash Equilibria

Let's write an LP that finds a Nash equilibrium in a given two-player zero-sum game.

The row player's point of view:

- Given any strategy π_i that I play in a Nash equilibrium, the column player plays a best response against me – $\min_{\pi_{-i}} \pi_i^\top A\pi_{-i}$
- However, I want to maximize my (maximin) value – $\max_{\pi_i} \min_{\pi_{-i}} \pi_i^\top A\pi_{-i}$

The column player's point of view

- Given any strategy π_{-i} that I play in a Nash equilibrium, the column player plays a best response against me – $\max_{\pi_i} \pi_i^\top A\pi_{-i}$
- However, I want to maximize my (minimax) value $\iff$ minimize the opponent's value – $\min_{\pi_{-i}} \max_{\pi_i} \pi_i^\top A\pi_{-i}$

LP Construction I

- Recall the definition of the value of a Maximin strategy from the previous slide

$$\max_{\pi_i \in \Pi_i} \min_{\pi_{-i} \in \Pi_{-i}} \pi_i^\top A \pi_{-i} \tag{1}$$

- Since the equation is bilinear, we need to decouple the two variables π_i, π_{-i}
- Given a row player's strategy π_i, a best-responding opponent will simply choose the action minimizing its expected payoff vector $\pi_i^\top A$

$$\min_{\pi_{-i} \in \Pi_{-i}} \pi_i^\top A \pi_{-i}$$

- Therefore, the best the row player can do, is to maximize the minimum value in $\pi_i^\top A$, which when combined with Equation (1) leads to the following LP

$$\max_{\pi_i, u} u$$

$$\pi_i^\top A \geq u$$

LP Construction II

- On the previous slide, we derived an LP that finds a row player's Maximin strategy

$$\max_{\pi_i, u} u$$

$$\pi_i^\top A \geq u$$

- By following the symmetrical line of reasoning for the column player, we arrive at the following LP

$$\min_{\pi_{-i}, v} v$$

$$A\pi_{-i} \leq v$$

- Thus, by solving the two LPs, we get a pair of Maximin strategies, which in zero-sum games form a Nash equilibrium!

Minimax Theorem

Theorem

For any two-player zero-sum game, the following holds

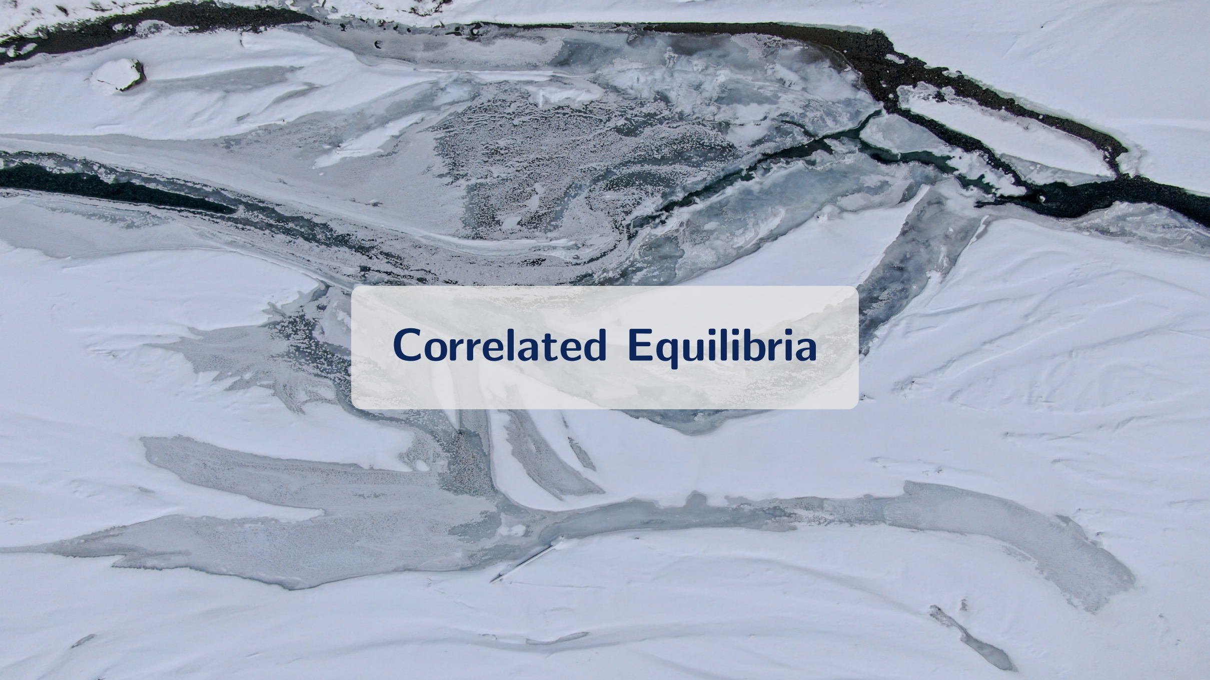
$$\max_{\pi_i} \min_{\pi_{-i}} u_i(\pi_i, \pi_{-i}) = \min_{\pi_{-i}} \max_{\pi_i} u_i(\pi_i, \pi_{-i})$$

- We have constructed an LP that solves $\max_{\pi_i} \min_{\pi_{-i}} \pi_i^\top A \pi_{-i}$
- We have also constructed an LP that solves $\min_{\pi_{-i}} \max_{\pi_i} \pi_i^\top A \pi_{-i}$
- It can be shown that these two LPs are in fact **duals** of each other
- Since both of these programs are feasible, the strong duality theorem guarantees that the optimal values u^* and v^* are equal

Linear Programming and Zero-Sum Games

There is an interesting one-to-one connection between linear programming and two-player zero-sum games.

- Given any two-player zero-sum normal-form game, we can construct an LP that finds the optimal solution
- Given any linear program, we can construct a game, where the optimal strategies in that game correspond to the optimal solution to the linear program



Correlated Equilibria

Relaxing Nash Equilibria

- Consider the Game of Chicken again:

	stop	go
stop	(0, 0)	(-1, 1)
go	(1, -1)	(-10, -10)

- The game has two pure- and one mixed-strategy Nash equilibria:
  - $\pi_1 = (1, 0)$, $\pi_2 = (0, 1)$ with expected utilities $u_1 = -1$, $u_2 = 1$
  - $\pi_1 = (0, 1)$, $\pi_2 = (1, 0)$ with expected utilities $u_1 = 1$, $u_2 = -1$
  - $\pi_1 = (0.9, 0.1)$, $\pi_2 = (0.9, 0.1)$ with expected utilities $u_1 = u_2 = -0.1$

Relaxing Nash Equilibria

- These equilibria describe product distributions over strategy profiles:
  - $\pi_1 = (1,0), \;\; \pi_2 = (0,1)$:
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
  - $\pi_1 = (0,1), \;\; \pi_2 = (1,0)$:
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
  - $\pi_1 = (0.9, 0.1), \;\; \pi_2 = (0.9, 0.1)$:
$$\begin{pmatrix} 0.81 & 0.09 \\ 0.09 & 0.01 \end{pmatrix}$$

Relaxing Nash Equilibria

- What about the following distribution over strategy profiles?

$$\begin{pmatrix} 0.0 & 0.5 \\ 0.5 & 0.0 \end{pmatrix}$$

- Such a profile leads to the expected utilities $u_1 = u_2 = 0$ for both players
- Can we obtain it as a product of players' strategies?
- We can add an external coordinator – a traffic light in this case

Correlated Equilibrium

- In a Nash equilibrium, players choose their strategies independently
- In a correlated equilibrium a coordinator chooses strategies for all players
- Chosen strategies have to be stable; in other words, it has to be in each player's interest to follow the coordinator's advice
- Consider the Game of Chicken once more:
  - The coordinator is free to choose only one player to play go with any probability
  - The other player will play `stop` and receive utility of -1
  - However, they know that ignoring the coordinator and playing go will result in utility of -10

Correlated Equilibrium

Definition: Correlated Equilibrium

A correlated equilibrium is a probability distribution over strategy profiles $a \in \mathcal{A}$. Such a distribution is a correlated equilibrium if for all players $i \in \mathcal{N}$ and all strategies $a_i, a_i' \in \mathcal{A}_i$, it holds

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} p(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in \mathcal{A}_{-i}} p(a_i, a_{-i}) u_i(a_i', a_{-i}),$$

where $p(a) = p(a_i, a_{-i})$ is the probability of strategy profile a.

- Intuitively, the definition says that if player i receives a suggested strategy $a_i \in \mathcal{A}_i$, they cannot increase their expected utility by switching to a different strategy $a_i' \in \mathcal{A}_i$

Relation to Nash Equilibria in Normal-Form Games

Examples

Does a correlated equilibrium always represent a Nash equilibrium?

No, consider the following distribution.

$$\begin{pmatrix} 0.0 & 0.5 \\ 0.5 & 0.0 \end{pmatrix}$$

There is no product of a pair independent distributions that would result in such a distribution.

Relation to Nash Equilibria in Normal-Form Games

Examples

Are Nash equilibria a subset of correlated equilibria?

Yes, Nash equilibria are a special case of correlated equilibria where the distribution over strategy profiles is the product of independent distributions for each player.

Relation to Nash Equilibria in Normal-Form Games

Examples

Is the set of all correlated equilibria convex?

Yes, the set of all correlated equilibria is convex as any convex combination of two probability distribution results in another probability distribution satisfying the same linear inequalities.

Finding Correlated Equilibria

- The definition of a correlated equilibrium describes a set of linear inequalities
- For all pairs of strategies $a_i, a_i' \in \mathcal{A}_i$, we have

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} p(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in \mathcal{A}_{-i}} p(a_i, a_{-i}) u_i(a_i', a_{-i})$$

- The only non-constant part in the inequalities is $p(a_i, a_{-i})$ – these are our variables
- This set of inequalities combined with the conditions that $p(a)$ is a valid probability distribution gives us an LP that finds a correlated equilibrium in a normal-form game
- Optimizing the objective function even lets us find a correlated equilibrium with a specific property, e.g. one maximizing the sum of players' utilities
- In contrast to the LP for finding NEs, we no longer require the game to be zero-sum!

Week 4 Homework

You can find more detailed descriptions of homework tasks in the GitHub repository.

1. Nash equilibria and linear programming in zero-sum games
2. Correlated equilibria and linear programming in general- and zero-sum games