Modern Algorithmic Game Theory

Martin Schmid
Department of Applied Mathematics
Charles University in Prague

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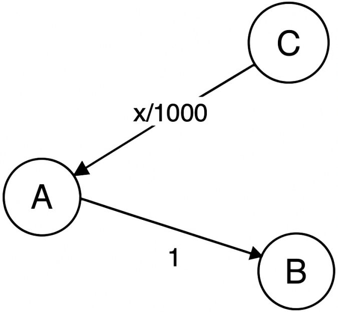
Recap

- In the last two weeks, we started talking about normal-form games, introduced two solution concepts and their properties, and finally described Support Enumeration, an algorithm that finds all possible Nash equilibria in a given game.
- The problematic part about this algorithm is that its runtime is exponential in the number of actions and solving larger games quickly becomes intractable.
- Today, we will first focus on a couple of multi-agent dynamics in non-zero-sum games
- We will then come back to zero-sum games and introduce the first self-play style algorithm – *Fictitious Play*

Unexpected Dynamics in Game Theory

Congestion Games and The Price of Anarchy

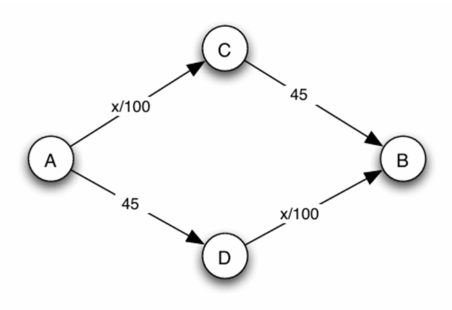


Congestion Games and The Price of Anarchy

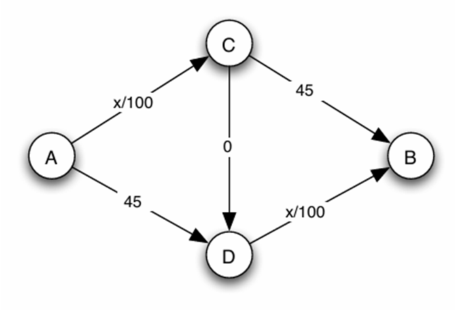
- First, let us compute the minimum possible delay
- Suppose x drivers go to road 2 and $1000 - x$ go to road 1
- Then, the total delay is
$$\frac{x^2}{1000} + (1000 - x)$$
- This expression is minimized when $x \approx 500$, that is, 500 drivers go to road 2 and the other 500 to road 1
- The total delay is then
$$500 \times \frac{1}{2} \; + \; 500 \times 1 \; \approx \; 750 \text{ minutes}$$

Braess' Paradox

- 4,000 agents



(a) Original roads/actions

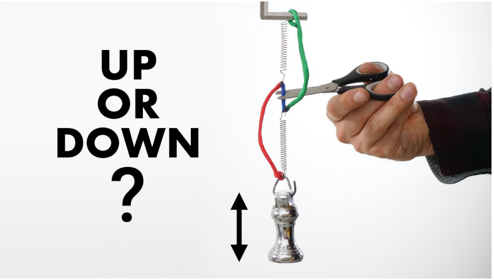


(b) Adding a new road/action

Braess' Paradox

- This actually sometimes happens!
- Could this happen in a single-agent game?
- Are agents rational?

Braess' Paradox in Springs



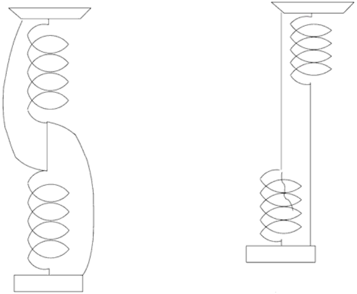


Figure 2: Strings and springs. Severing a taut string lifts a heavy weight.

Tragedy of the Commons

- Shared resource; getting more increases your reward, but puts strain on the resource

Tragedy of the Commons

	extract	preserve
extract	(50, 50)	(80, 20)
preserve	(20, 80)	(100, 100)

Table: Commonize Costs - Privatize Profits Game

Evaluating the Quality of Strategy Profiles

Motivation

How can we evaluate the quality of an agent, i.e. how well it plays a game?

- In single-agent environments we can easily detevaluateermine how good an agent is based on the score it achieves (e.g. in Atari)
- However, the agent's performance in multi-agent environments directly depends on the quality of its opponents
- ELO-based comparisons are problematic due to intransitivity
- We will use measures that tell us how "close" to an optimal policy we are in terms of performance rather than distance (e.g. KL divergence)

Metrics

We define the following metrics:

- the incentive of player i to deviate

$$\delta_i(\pi) = \max_{\pi'_i} u_i(\pi'_i, \pi_{-i}) - u_i(\pi) = u_i(b(\pi_{-i}), \pi_{-i}) - u_i(\pi)$$

- the total incentive to deviate

$$NashConv(\pi) = \sum_{i \in \mathcal{N}} \delta_i(\pi)$$

- the average incentive to deviate

$$Exploitability(\pi) = \frac{NashConv(\pi)}{|\mathcal{N}|}$$

Metrics in Zero-Sum Games

- In zero-sum games, the definition of $NashConv(\pi)$ simplifies to

$$NashConv(\pi) = \sum_{i \in \mathcal{N}} u_i(b(\pi_{-i}, \pi_{-i}),$$

- In two-player zero-sum games ($u = u_1 = -u_2$), $NashConv(\pi)$ can be written as

$$NashConv(\pi) = \max_{\pi'_i} u(\pi'_i, \pi_{-i}) - \min_{\pi'_{-i}} u(\pi_i, \pi'_{-i})$$

ϵ-Nash Equilibrium

Definition: ϵ-Nash Equilibrium

Strategy profile (π_i, π_{-i}) is an ϵ-**Nash equilibrium** if none of the players can improve by more than ϵ by unilaterally deviating from their policy. Mathematically,

$$\forall i \in \mathcal{N}, \forall \pi_i' \in \Pi_i : u_i(\pi_i, \pi_{-i}) \geq u_i(\pi_i', \pi_{-i}) - \epsilon$$

- Iterative algorithms may not be able to find an exact Nash equilibrium due to finite computation time and/or numerical instabilities
- Therefore, we need to use a looser definition of an optimal strategy profile
- If a strategy profile π is an ϵ-Nash equilibrium, it holds that $\max_{i \in \mathcal{N}} \delta_i(\pi) \leq \epsilon$
- Consequently, a strategy profile is a Nash equilibrium $\iff$ *Exploitability*$(\pi) = 0$

Fictitious Play

Fictitious Play

- An iterative algorithm where players repeatedly play against each other and keep track of the **empirical distribution** over their opponent's previously played actions
- Both players **simultaneously** compute a **pure** best response to maximize their expected payoff against their opponent's observed **average** strategy
- The sequence of **average** strategies produced by the algorithm converges in certain classes of games to Nash equilibria; a property called **average-iterate** convergence
- The actual sequence of best-response strategies **does not** converge in general
- Mathematically, a single iteration of the algorithm can be expressed as

$$\bar{\pi}_i^{t+1} \in \left(1 - \frac{1}{t+1}\right)\bar{\pi}_i^t + \frac{1}{t+1} b(\bar{\pi}_{-i}^t),$$

where $\bar{\pi}_i^t = \frac{1}{t}\sum_{k=1}^{t} \pi_i^k$ is player i's average strategy and $\bar{\pi}_{-i}^t$ is defined analogously.

Fictitious Play

- Fictitious Play is a belief-based learning process in repeated games
- Each player assumes that the opponent's play is stationary and equal to the empirical distribution of their past actions
- At every round, player i plays a pure best response to that belief
- It is therefore **myopic**: players maximize their current expected payoff given their current belief, without anticipating future consequences or learning the opponent's update rule
- It is not deterministic if multiple best responses exist; the theory usually assumes any of them can be chosen.

Fictitious Play Example

Let's consider the Matching Pennies game and simulate a couple of steps of Fictitious play. Let n_i^t be the running count of the number of times player i played each action.

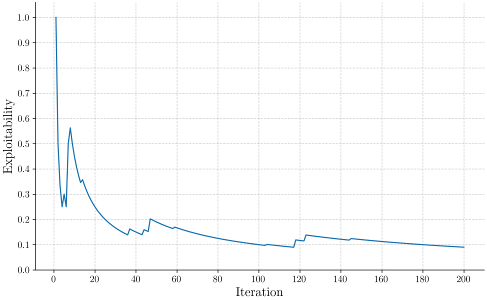
	heads	tails
heads	(1, -1)	(-1, 1)
tails	(-1, 1)	(1, -1)

Table: Matching Pennies

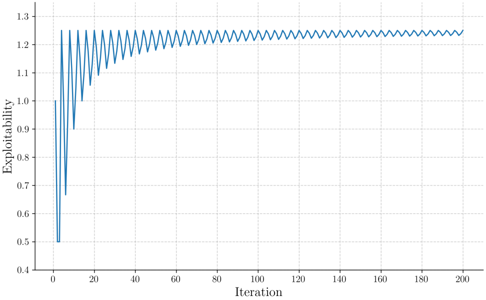
Time	n_1^t	n_2^t	Play
0	(0, 0)	(0, 0)	(h, h)
1	(1, 0)	(1, 0)	(h, t)
2	(2, 0)	(1, 1)	(h, t)
3	(3, 0)	(1, 2)	(t, t)
4	(3, 1)	(1, 3)	(t, t)
5	(3, 2)	(1, 4)	(t, t)
6	(3, 3)	(1, 5)	(t, h)
7	(3, 4)	(2, 5)	(t, h)
8	...	...	...

Average vs. Current Strategy Convergence

- When we best-respond to opponent's previously played action, instead of their average strategy, Fictitious play may not converge to a Nash equilibrium



(a) Best response against the average strategy



(b) Best response against the current strategy

Convergence of Fictitious Play to Pure Strategies

- Let us now study the asymptotic behavior of the sequence of strategy profiles $\{\pi^t\}$ produced by FP, i.e. the convergence properties of the sequence $\{\pi^t\}$ as $t \to \infty$
- We say the sequence $\{\pi^t\}$ converges to π^*, if there exists T, s.t. $\forall t \geq T: \pi^t = \pi^*$
- The following theorem formalizes the property that if the sequence $\{\pi^t\}$ converges, then it has to converge to a Nash equilibrium of the game

Theorem

If the sequence $\{\pi^t\}$ converges to π^*, then π^* is a pure strategy Nash equilibrium. Moreover, suppose that for some t, $\pi^t = \pi^*$, where π^* is a **strict** Nash equilibrium. Then $\pi^{t'} = \pi^*$ for all $t' \geq t$.

Convergence of Fictitious Play to Mixed Strategy

- The sequence $\{\pi^t\}$ converges to a mixed strategy profile π^* in **the time-average sense**, if for each player $i \in \mathcal{N}$ and for all actions $a_i \in \mathcal{A}_i$, we have:

$$\lim_{T \to \infty} \frac{\sum_t \mathbb{1}(\pi_i^t = a_i)}{T} = \pi^*(a_i)$$

Theorem

If the sequence $\{\pi^t\}$ converges to π^* in the time-average sense, then π^* is a mixed strategy Nash equilibrium.

Convergence of Fictitious Play

Games in which Fictitious Play converges are said to have **the fictitious-play property**. The algorithm has been proven to converge for the following classes of games:

- two-player zero-sum games
- two-player non-zero-sum game, where each player has at most two strategies
- games solvable by Iterated removal of strictly dominated strategies
- identical interest games; games where all players have the same payoff function

Convergence of Fictitious Play

On the other hand, in games such as the **Shapley game**, Fictitious Play can cycle indefinitely and fail to converge, depending on the initial conditions.

	x	y	z
a	(2, 1)	(0, 0)	(1, 2)
b	(1, 2)	(2, 1)	(0, 0)
c	(0, 0)	(1, 2)	(2, 1)

Table: The Shapley Game

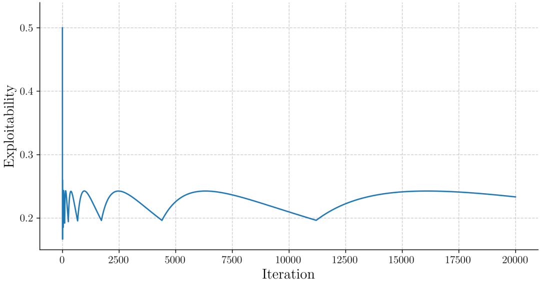


Figure: Convergence of FP when starting with (a, x)

Week 3 Homework

You can find more detailed descriptions of homework tasks in the GitHub repository.

1. Strategy profile evaluation ($\delta_i(\pi)$, $NashConv(\pi)$, $Exploitability(\pi)$)
2. Fictitious play
3. Exploitability convergence plots