Modern Algorithmic Game Theory

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Recap

Last week, we started talking about simultaneous decision making. We covered:

- normal-form games
- general- and zero-sum games
- pure and mixed strategies
- expected utilities and best responses
- iterated removal of dominated strategies

We have also started talking about solution concepts and formally introduced **Maximin strategies** and **Nash equilibria**.

Maximin Strategy

Maximin

Definition: Maximin Strategy

A Maximin strategy of player i is a strategy that guarantees the highest possible expected utility against the worst-case opponent. We define it as:

$$\arg\max_{\pi_i \in \Pi_i} \min_{\pi_{-i} \in \Pi_{-i}} u_i(\pi_i, \pi_{-i}) = \arg\max_{\pi_i \in \Pi_i} BRV_i(\pi_i)$$

- We assume everyone else is *"out there to get us"*
- A maximin policy maximizes our expected utility assuming the worst-case scenario
- We also use $v_i = \max_{\pi_i} \min_{\pi_{-i}} u_i(\pi_i, \pi_{-i})$ to denote the Maximin value of player i

Maximin in Pure Strategies

	`cooperate`	`defect`
`cooperate`	(-1, -1)	(-3, 0)
`defect`	(0, -3)	(-2, -2)

Table: Prisoner's Dilemma

Let's reason from the row player's perspective:

- the worst-case payoff when playing `cooperate` is -3 and when playing `defect` is -2
- therefore, the strategy maximizing the worst-case payoff is `defect`

Maximin in Pure Strategies

	rock	paper	scissors
rock	0	-1	1
paper	1	0	-1
scissors	-1	1	0

Table: Rock Paper Scissors

Let's reason from the row player's perspective:

- the worst-case payoff is always -1, no matter what action we choose
- however, we can do better than this when we consider mixed strategies
- the uniform strategy gives the expected utility of 0, but how can we find it?

Optimizing Against Best Response

- In two-player zero-sum games, it holds that

$$\arg\max_{\pi_i \in \Pi_i} \min_{\pi_{-i} \in \Pi_{-i}} u_i(\pi_i, \pi_{-i}) = \arg\max_{\pi_i \in \Pi_i} u_i(\pi_i, b_{-i}(\pi_i))$$

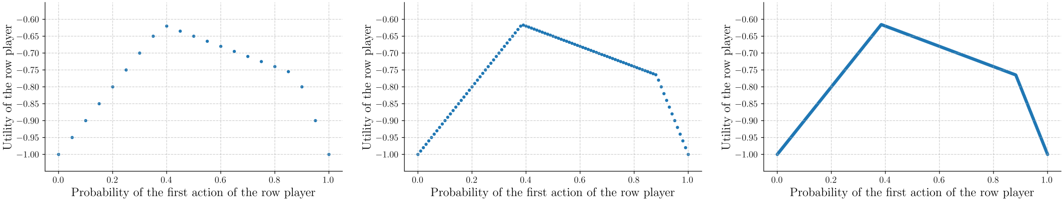
- We are optimizing against a best-responding opponent
- Let's consider the following $2 \times N$ zero-sum game

	A	B	C
X	-1	0	-0.8
1-X	1	-1	-0.5

- The worst-case payoff for the row player is again -1 no matter what action they play
- Let's visualize the best response value function $f(\pi_i) = u_i(\pi_i, b_{-i}(\pi_i))$

Best Response Value Function

- We can see that placing the probability of around 0.385 on the first action yields a much better expected utility of around -0.6!
- The piece-wise linear structure of the best response value function can be leveraged to create a closed-form linear program that finds the Maximin strategy (and its value) that maximizes the function



(a) Step size 0.05 (b) Step size 0.01 (c) Step size 0.001


Nash Equilibrium

Nash Equilibrium

Definition: Nash Equilibrium

Strategy profile (π_i, π_{-i}) is a Nash equilibrium if none of the players can benefit from unilaterally deviating from their policy. Mathematically,

$$\forall i \in \mathcal{N}, \forall \pi_i' \in \Pi_i : u_i(\pi_i, \pi_{-i}) \geq u_i(\pi_i', \pi_{-i})$$

- Represents a stable solution where players cannot individually improve their utility
- We can also look at this as each player playing a best response against other players
- It is easy to verify – all strategies must be best responses

Nash Equilibrium in Pure Strategies

	cooperate	defect
cooperate	(-1, -1)	(-3, 0)
defect	(0, -3)	(-2, -2)

Table: Prisoner's Dilemma

- Only a single Nash equilibrium, both players play `defect`, even though playing `cooperate` would lead to a higher expected utility for both players

	stop	go
stop	(0, 0)	(-1, 1)
go	(1, -1)	(-10, -10)

Table: The Game of Chicken

- Two pure Nash equilibria – (`stop`, `go`) and (`go`, `stop`)

Nash Equilibrium in Pure Strategies

	rock	paper	scissors
rock	0	-1	1
paper	1	0	-1
scissors	-1	1	0

Table: Rock Paper Scissors

- We can naively try finding a pure strategy Nash equilibrium by enumerating all pure strategy profiles and verifying each strategy is a best response
- However, pure strategy Nash equilibrium might not exist!
- Consider Rock Paper Scissors, for any pure action of one player, their opponent has profitable deviation $\rightarrow$ no pure strategy Nash equilibrium

Support Enumeration

We now present a brute-force algorithm that can find all Nash equilibria, both pure and mixed, in any normal-form game by enumerating and checking all pairs of supports.

- Recall the **best response support condition** that states that all actions in the support of a best response strategy are also best responses and so have the same expected utilities
- Therefore, finding NEs essentially boils down to finding the right pairs of supports
- For a given pair of supports, we can compute the probabilities of individual actions in the supports by solving a system of linear equations[1]
- After solving the system of equations, we still need to check that actions outside of the supports do not lead to higher expected utilities; if they did, the given pair of supports and the found strategies would not form a NE in the original game

[1] For games with three or more players, the system of equations becomes non-linear

Support Enumeration

- For some support and strategy vector π_i (where actions outside of the support have probability 0) consider elements of the following payoff vector $\pi_i A_i$
- All the elements in the payoff vector $\pi_i A_i$ must correspond to the best-response value $BRV(\pi_i)$ from the perspective of the **opponent**
- In other words, the player needs to mix actions in their support so that the values of actions in the opponent's support are best-responding (and thus all the same)
- Now, consider the following game with supports $\{\mathtt{X}, \mathtt{Y}\}$ and $\{\mathtt{A}, \mathtt{B}\}$

	A	B	C
X	(0, 0)	(0, 1)	(-10, -10)
Y	(1, 0)	(-10, -10)	(-10, -10)
Z	(-10, -10)	(-10, -10)	(-10, -10)

Support Enumeration

	A	B	C
X	(0, 0)	(0, 1)	(-10, -10)
Y	(1, 0)	(-10, -10)	(-10, -10)
Z	(-10, -10)	(-10, -10)	(-10, -10)

- The row player needs to mix actions in their support $\{\mathtt{X}, \mathtt{Y}\}$ so that the values of the column player for actions in their support $\{\mathtt{A}, \mathtt{B}\}$ are all the same

$$0\pi_1(\mathtt{X}) + 0\pi_1(\mathtt{Y}) = v_2$$
$$1\pi_1(\mathtt{X}) - 10\pi_1(\mathtt{Y}) = v_2$$

- By the same reasoning for the column player, we get

$$0\pi_2(\mathtt{A}) + 0\pi_2(\mathtt{B}) = v_1$$
$$1\pi_2(\mathtt{A}) - 10\pi_2(\mathtt{B}) = v_1$$

- Solution: $\pi_1(\mathtt{X}) = 0.909091 = \pi_2(\mathtt{A})$ and $\pi_1(\mathtt{Y}) = 0.090909 = \pi_2(\mathtt{B})$

Support Enumeration

	A	B	C
X	(0, 0)	(0, 1)	(-10, -10)
Y	(1, 0)	(-10, -10)	(-10, -10)
Z	(-10, -10)	(-10, -10)	(-10, -10)

- Repeating the same steps as on the previous slides but this time for supports $\{\mathtt{X}, \mathtt{Z}\}$ and $\{\mathtt{A}, \mathtt{B}\}$, we get:

$$0\pi_1(\mathtt{X}) - 10\pi_1(\mathtt{Y}) = v_2$$
$$1\pi_1(\mathtt{X}) - 10\pi_1(\mathtt{Y}) = v_2$$

- ... and for the column player:

$$0\pi_2(\mathtt{A}) - 0\pi_2(\mathtt{B}) = v_1$$
$$-10\pi_2(\mathtt{A}) - 10\pi_2(\mathtt{B}) = v_1$$

- There's no solution and therefore no NE candidate for the given supports

General-Sum Games

Maximin Strategies vs Nash Equilibria

- Are the two solution concepts the same?
- Why would we want to play them?

Non-Rational Opponents, Deviating from NEs

Let's elaborate on the properties of Nash equilibria.

- Suppose we play against a non-rational opponent.
- A non-rational player does not necessarily try to maximize their utility, they can play arbitrary strategies
- Consider a Nash equilibrium (π_1, π_2) and say we decide to play π_1, what are our guarantees?
- Even though π_2 maximizes our opponent's utility, they can make mistakes and select different (non-equilibristic) strategy π_2'
- Choosing a different strategy than π_2 cannot lead to a better expected utility for the opponent
- However, it can lead to a much worse outcome for us, it can be the case that $u_1(\pi_1, \pi_2) \gg u_1(\pi_1, \pi_2')$ and so opponent's mistakes can hurt us

Deviating from Nash Equilibria

- Consider Prisoner's Dilemma and assume we play our equilibrium strategy (`defect`)
- A rational opponent would also play `defect` leading to our reward of -2, but a non-rational opponent can also choose `cooperate` which leads to our reward of 0

- Now consider The Game of Chicken and a NE strategy profile (`go`, `stop`)
- Say, we play `go` but our non-rational opponent makes a mistake and also plays `go`, leading to our reward of -10 instead of 1

	`cooperate`	`defect`
`cooperate`	(-1, -1)	(-3, 0)
`defect`	(0, -3)	(-2, -2)

Table: Prisoner's Dilemma

	`stop`	`go`
`stop`	(0, 0)	(-1, 1)
`go`	(1, -1)	(-10, -10)

Table: The Game of Chicken

Rational Players, Multiple Equilibria Problem

- Suppose there are two optimal strategy profiles in the game (π_1, π_2^a) and (π_1, π_2^b)
- The opponent is indifferent between their two strategies, they do not care which strategy they choose (given our strategy π_1), since $u_2(\pi_1, \pi_2^a) = u_2(\pi_1, \pi_2^b)$
- However, we might care! We can imagine a situation where $u_1(\pi_1, \pi_2^a) \neq u_1(\pi_1, \pi_2^b)$
- Even though both players play optimally, different optimal strategies can lead to different utilities!
- Consider the following game with two NEs - (go, `plan a`) and (go, `plan b`)
- In either of the two equilibria, the column player's expected utility is the same; however, the row player's utility is considerably different

	`plan a`	`plan b`
`go`	(10, 5)	(1, 5)
`wait`	(0, 0)	(0, 0)

Zero-Sum Games

Maximin and Minimax Values

We will now investigate the important relationship between the maximin values v_i and v_{-i} of players i and $-i$, respectively. We already know that

$$\begin{aligned} v_i &= \max_{\pi_i} \min_{\pi_{-i}} u_i(\pi_i, \pi_{-i}) \\ v_{-i} &= \max_{\pi_{-i}} \min_{\pi_i} u_{-i}(\pi_i, \pi_{-i}) \end{aligned}$$

However, we can also show that

$$\begin{aligned} v_{-i} &= \max_{\pi_{-i}} \min_{\pi_i} -u_i(\pi_i, \pi_{-i}) \\ &= \max_{\pi_{-i}} - \max_{\pi_i} u_i(\pi_i, \pi_{-i}) \\ &= - \min_{\pi_{-i}} \max_{\pi_i} u_i(\pi_i, \pi_{-i}) \end{aligned}$$

Minimax Theorem

Theorem

For any two-player zero-sum game, the following holds

$$\max_{\pi_i} \min_{\pi_{-i}} u_i(\pi_i, \pi_{-i}) = \min_{\pi_{-i}} \max_{\pi_i} u_i(\pi_i, \pi_{-i})$$

- The Minimax theorem states a critical result – the maximin and the minimax values are in balance $v_i = -v_{-i}$.
- We refer to this unique value as the **game value** and denote it by GV_i
- The theorem was proven by John Von Neumann in 1928 and has dramatic consequences for two-player zero-sum games
- Von Neumann himself later wrote *"As far as I can see, there could be no theory of games ... without that theorem ... I thought there was nothing worth publishing until the Minimax theorem was proved."*

Nash Equilibrium $\iff$ Maximin Strategies

Theorem: Nash equilibrium $\implies$ Maximin strategies

For any two-player zero-sum game and for any pair of mixed strategies π_1, π_2, it holds that if (π_1, π_2) forms a Nash equilibrium then $\pi_1 \land \pi_2$ are both Maximin strategies.

Theorem: Maximin strategies $\implies$ Nash equilibrium

For any two-player zero-sum game and for any pair of mixed strategies π_1, π_2, it holds that if $\pi_1 \land \pi_2$ are both Maximin strategies then (π_1, π_2) forms a Nash equilibrium.

Key Motivating Property

Theorem

If we follow an optimal policy when playing from both positions, the expected utility against any opponent is greater than or equal to zero:

$$(\pi_i, \pi_{-i}) \in Nash \; : \; u_i(\pi_i, \pi'_{-i}) + u_{-i}(\pi'_i, \pi_{-i}) \geq 0 \; \forall \pi'_i, \pi'_{-i}$$

Summary of the Solutions Concepts

The following table summarizes the existence guarantees, and the complexity of verifying and computing the two solution concepts in general- and zero-sum two-player games.

We consider Maximin strategy to be a strategy that maximizes a player's worst-case payoff. However, in zero-sum games we consider it to be a strategy that attains the unique game value GV_i, which in general requires mixed Maximin strategies.

<table>
<tr><th rowspan="2"></th><th colspan="2">General-Sum</th><th colspan="2">Zero-Sum</th></tr>
<tr><th>Maximin</th><th>Nash</th><th>Maximin</th><th>Nash</th></tr>
<tr><td>Pure</td><td>$\checkmark$</td><td>$\times$</td><td>$\times$</td><td>$\times$</td></tr>
<tr><td>Mixed</td><td>$\checkmark$</td><td>$\checkmark$</td><td colspan="2">$\checkmark$</td></tr>
<tr><td>Verification</td><td>Easy</td><td>Easy</td><td colspan="2">Easy</td></tr>
<tr><td>Computation</td><td>Easy</td><td>Hard</td><td colspan="2">Easy</td></tr>
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Week 2 Homework

You can find more detailed descriptions of homework tasks in the GitHub repository.

1. Best response value function
2. Support enumeration
3. Prove that Nash equilibrium $\iff$ Maximin strategies in zero-sum games