TABLE 8.9
Taxonomy of the solvability of nonlinear optimization problems.

Type of Problem	*Level of Difficulty to Solve on a Computer*
Linear Optimization Problem	Very Easy
Convex Minimization Problem	Relatively Easy
Concave Maximization Problem	Relatively Easy
General Nonlinear Problem	Difficult

Case 2: The objective is to maximize a concave function, and all constraints are linear constraints. In this case, because all linear functions are both convex and concave, then the problem satisfies all four criteria to be a concave maximization problem and so is relatively easy to solve.

Case 3: The problem is a linear optimization model, i.e., the objective function is a linear function, and all constraints are linear constraints. In this case, if the objective is minimization, the problem is a convex minimization problem, because all functions involved are both convex and concave functions. If, instead, the objective is maximization, the problem is a concave maximization problem, because all functions involved are both convex and concave functions. Either way, the problem is easy to solve.

Finally, it turns out that all three of the examples of this chapter (namely, Example 8.1, Example 8.2, and Example 8.3) are either convex minimization problems or concave maximization problems, and so all three examples are easy to solve on a computer.

8.7 CASE MODULES

ENDURANCE INVESTORS

Endurance Private Client Fund (EPCF)

Endurance Investors is a private asset management firm started five years ago by nine founding partners: four graduates of MIT's Sloan School of Management and five graduates of Harvard Business School. Endurance's early clients were predominantly institutional investors such as universities, employee pension funds, and endowments of charitable organizations. As the firm grew, the partners developed different funds to meet the demands of an expanding client base. Two years ago, Endurance added three different types of fixed-income securities funds. Last quarter, in order to meet the needs of a small but growing list of wealthy private clients, Endurance introduced a blue-chip fund called the Endurance Private Client Fund (EPCF).

The strategy for the Endurance Private Client Fund (EPCF) is to give private investors a pure play in a portfolio of the largest leading American companies with a manageable level of risk. EPCF is a portfolio consisting of stock investments in five specially selected blue chip companies (Boeing, Exxon, General Motors, McDonald's, and Procter & Gamble) plus a market index fund, the Standard & Poor's 500 Index fund. The fraction of the portfolio invested in each of these six assets was selected last quarter based on a portfolio optimization model that maximizes the annual expected return of the portfolio for a given allowable level of investment risk.

Asset Portfolios, Expected Return, and Standard Deviation

Given n potential assets (which in the case of EPCF are the $n = 6$ assets of stock in Boeing, Exxon, General Motors, McDonald's, Procter & Gamble, and the Standard & Poor's 500 Index fund), an asset **portfolio** is created by assigning a fractional amount of each investment dollar to invest in each of the n assets. For the Endurance Private Client Fund (EPCF), for example, a portfolio might consist of investing the fractional amounts in each of the six assets according to Table 8.10 below. For the portfolio described in Table 8.10, 10% of every investment dollar is invested in Boeing, 20% of every investment dollar is invested in Exxon, etc. Notice that the portfolio fractions, which are also referred to as the **portfolio weights,** sum to one.

Let us denote the portfolio weight for asset i by the variable X_i for $i = 1, \ldots, n$. Then these weights must satisfy:

$$\sum_{i=1}^{n} X_i = 1.0,$$

that is, the portfolio fractions sum to one. Also, in most typical applications, the portfolio fractions must be nonnegative, that is,

$$X_1 \geq 0, X_2 \geq 0, \ldots, X_n \geq 0.$$

In a typical portfolio setting, the portfolio manager aims to maximize the annual return of the portfolio, while keeping the risk to a minimum. The "return" of the portfolio is the expected annual return of the portfolio. In order to measure the expected return of the portfolio, we need to have data on the expected annual rate of return on each asset in the portfolio. Table 8.11 shows Endurance's newly revised estimates of the annual expected return (in % per year) and the standard deviation of the annual return (in %

TABLE 8.10
Example of a portfolio composition of the assets of the Endurance Private Client Fund.

Asset	*Boeing*	*Exxon*	*General Motors*	*McDonald's*	*Procter & Gamble*	*S&P500*
Asset Ticker	BA	XON	GM	MCD	PG	SP
Fraction of Portfolio	0.10	0.20	0.25	0.05	0.15	0.25

TABLE 8.11
Newly revised estimates of annual expected return and standard deviation of assets.

Asset	*Ticker*	*Expected Return (% per year)*	*Standard Deviation of Return (% per year)*
Boeing	BA	12.69604	19.05455
Exxon	XON	9.92170	12.03149
General Motors	GM	11.80725	24.79470
McDonald's	MCD	13.54907	21.69084
Procter &Gamble	PG	13.45906	21.80891
S&P 500	SP	13.04295	11.71033

per year) of the six assets for the coming year. The numbers in Table 8.11 have been developed by using a combination of relevant market data bases, basic statistical analysis, professional judgment, and market intuition by the partners at Endurance. Consider the data for Boeing in Table 8.11. Let B denote the annual rate of return on stock invested in Boeing in the coming year. Then B is a random variable, and the expected value of B is 12.69604% per year. The standard deviation of B is 19.05455% per year.

Suppose that the EPCF portfolio is comprised of the fractional weights shown in Table 8.10. Then the expected annual return of the portfolio can be computed as follows:

$$(0.10)(12.69604) + (0.20)(9.92170) + (0.25)(11.80725)$$
$$+ (0.05)(13.54907) + (0.15)(13.45906) + (0.25)(13.04295) = 12.1628\%.$$

This computation is valid because of the rule: "The expected value of the weighted sum of random variables is the weighted sum of the expected values of the random variables." More generally, if the expected annual return on asset i is μ_i and the portfolio weight on asset i is X_i for $i = 1, \ldots, n$, then the formula for the expected annual return μ of the portfolio is given by:

$$\mu = \sum_{i=1}^{n} \mu_i X_i.$$

As stated earlier, a portfolio manager aims to maximize the expected annual return of the portfolio, while keeping the risk to a minimum. The "risk" of the portfolio is usually measured as the standard deviation of the portfolio. In order to compute the standard deviation of the portfolio, we need to have data on the standard deviation of the annual return of each asset in the portfolio, as well as data on the correlation between the rates of return of the different assets comprising the portfolio. For the Endurance Private Client Fund, the newly revised estimates of the standard deviations of the annual returns of each asset are shown in the fourth column of Table 8.11. For example, the standard deviation of the annual return on stock invested in Boeing is 19.05455%. Table 8.12 contains the newly revised estimates of the correlations between the rates of return of the different assets. Like the numbers in Table 8.11, the numbers in Table 8.12 have also been developed by using a combination of relevant market data bases, basic statistical analysis, professional judgment, and market intuition by the partners at Endurance.

To illustrate the meaning of Table 8.12, let B denote the annual rate of return on stock invested in Boeing in the coming year and let G denote the annual rate of return on stock invested in General Motors in the coming year. Then B and G are both random variables. The quantity CORR(B, G) is the correlation of the two random variables B and G. According to Table 8.12, Endurance's newly revised estimate of CORR(B, G) is CORR$(B, G) = 0.21902$.

In general, if the correlation matrix of the rate of return of asset i and asset j is given by CORR(i, j) for $i = 1, \ldots, n$ and $j = 1, \ldots, n$, and if the standard deviation

TABLE 8.12
Newly revised estimates of the correlations between annual returns of assets.

	BA	*XON*	*GM*	*MCD*	*PG*	*SP*
BA	1.00000	0.20559	0.21902	0.43523	0.25849	0.49609
XON	0.20559	1.00000	0.11522	0.30249	0.21095	0.56073
GM	0.21902	0.11522	1.00000	0.32526	−0.17682	0.36528
MCD	0.43523	0.30249	0.32526	1.00000	0.14953	0.59082
PG	0.25849	0.21095	−0.17682	0.14953	1.00000	0.55053
SP	0.49609	0.56073	0.36528	0.59082	0.55053	1.00000

of the rate of return of asset i is given by σ_i for $i = 1, \ldots, n$, and if the portfolio weights are given by X_i for $i = 1, \ldots, n$, then the formula for the variance of the portfolio σ^2 is given by:

$$\sigma^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j \text{CORR}(i,j) \cdot X_i \cdot X_j.$$

Therefore, the standard deviation σ of the portfolio is:

$$\sigma = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j \text{CORR}(i,j) \cdot X_i \cdot X_j}.$$

To illustrate the preceding formulas for the variance and the standard deviation of a portfolio of assets, suppose that the EPCF portfolio is comprised of the fractional weights shown in Table 8.10. Then the variance of the portfolio is computed by summing up all of the numbers in the "correlation table" (Table 8.12) for each of the pairs of assets multiplied by their standard deviations and by their corresponding portfolio weights. For the portfolio weights given in Table 8.10, the variance of the portfolio is given by summing up the 36 terms (one term for each entry of the correlation table of Table 8.12) as follows:

$$\begin{aligned}
\sigma^2 &= (1.00)(19.05455)(19.05455)(0.10)(0.10) \\
&\quad + (0.20559)(12.03149)(19.05455)(0.10)(0.20) \\
&\quad + \cdots \\
&\quad + (0.55053)(21.80891)(11.71033)(0.25)(0.15) \\
&\quad + (1.00)(11.71033)(11.71033)(0.25)(0.25) \\
&= 127.963\%^2.
\end{aligned}$$

Therefore the standard deviation of the portfolio is:

$$\sigma = \sqrt{127.963} = 11.312\%.$$

The Portfolio Optimization Model for the Endurance Private Client Fund

The portfolio optimization model for the Endurance Private Client Fund (EPCF) was created last quarter by Brian Jackson (who had joined Endurance after working for several years for another asset management firm) in order to compute the optimal portfolio weights for the newly created EPCF portfolio. Brian's model is shown below:

$$\begin{aligned}
&\text{maximize} && \mu = \sum_{i=1}^{n}\mu_i \cdot X_i. \\
&\text{subject to:} && \\
&\text{Fractions:} && \sum_{i=1}^{n} X_i = 1 \\
&\text{Stand. dev.:} && \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j \text{CORR}(i,j) \cdot X_i \cdot X_j} \le 13.0 \\
&\text{Max single:} && X_i \le 0.30 \quad \text{for } i = \text{BA, XON, GM, MCD, PG, SP} \\
&\text{Nonnegativity:} && X_i \ge 0 \quad \text{for } i = \text{BA, XON, GM, MCD, PG, SP}.
\end{aligned}$$

The objective function is to maximize the expected return of the portfolio. The first constraint states that the fractions of the EPCF investment dollar invested in each of the six assets must sum to one. The second constraint states that the maximum allowable standard deviation of the portfolio must be less than or equal to the pre-specified target allowable standard deviation. This pre-specified standard deviation was set to TARGET = 13.0% last quarter. The fourth set of constraints disallows investing negative amounts in any of the portfolio assets (that is, no short selling is allowed).

The third set of constraints are called the "max single" constraints. These constraints were added to ensure that no more than 30% of the portfolio's funds was invested in any one of the six assets. Brian added these constraints in order to force the diversification of the portfolio. In their collective experience with portfolio management, Brian and the other fund managers have seen benefits from a marketing perspective in offering investors the maximum level of diversity in trying to achieve the mutual fund's objectives.

Guided by the output of the portfolio optimization model, the EPCF fund managers had chosen the portfolio weights for the EPCF portfolio at the start of last quarter. These initial portfolio weights are shown below in the third row of Table 8.13. As a result of changes in stock prices during the quarter, the fractions of the portfolio invested in each of the six assets changed during the quarter. At the end of the quarter, these portfolio fractions are now those shown in the fourth row of Table 8.13.

Brian Jackson's Problem

It is now the start of the new quarter, and Brian Jackson is working on the problem of revising the optimal portfolio weights for the EPCF portfolio based on the revised data estimates on asset performance. Brian has just put together the newly revised data on asset performance: the updated estimates of the expected rates of returns and the updated estimates of the standard deviation of the rates of return for the six assets (Table 8.11), and the updated estimates of the correlations between the rates of the return of the assets (Table 8.12). Brian has placed all of these new data tables in his spreadsheet portfolio optimization model, which is the worksheet ENDURANCE.XLS. However, as Brian has been working on this problem, he has realized that the model is incomplete in two ways that are explained as follows.

The first way that the portfolio model is incomplete has to do with the real costs of transactions related to changing the composition of the portfolio. Even with economies of scale and the use of discount brokers, the cost of buying and selling stocks for the EPCF portfolio is not trivial and is 0.5% of the dollar value of all stock transactions.

To see how transaction costs might affect the value of the EPCF portfolio, suppose for the sake of argument that the Endurance Private Client Fund has exactly K = $1,000,000 in total assets invested only in Boeing and Exxon, as follows: 30% of the EPCF portfolio currently comprises stock in Boeing, and 70% of the EPCF portfolio currently comprises stock in Exxon. Suppose that Endurance is considering changing these values next quarter to new fractions X_1 and X_2 invested in Boeing and Exxon. Then the total amount of the transaction would be:

$$K \cdot (|X_1 - 0.30| + |X_2 - 0.70|)$$

TABLE 8.13
Asset fractions of the EPCF portfolio.

Asset	*Boeing*	*Exxon*	*General Motors*	*McDonald's*	*Procter & Gamble*	*S&P500*
Asset Ticker	BA	XON	GM	MCD	PG	SP
Initial Fraction of Portfolio	0.19	0.18	0.23	0.07	0.13	0.20
Ending Fraction of Portfolio	0.21	0.16	0.21	0.09	0.09	0.24

(where $|x|$ is the absolute value of the quantity x). The transaction costs associated with this transaction would then be

$$0.005 \cdot K \cdot (|X_1 - 0.30| + |X_2 - 0.70|).$$

Without accounting for transaction costs, the expected return of the new portfolio would be:

$$\mu = 12.69604 \cdot X_1 + 9.92170 \cdot X_2,$$

expressed as a percentage, where the expected return numbers are as shown in Table 8.11. In order to account for transaction costs, we need to subtract 0.5% of the total transactions from the above expression. Therefore the expected return of the portfolio (expressed as a percentage), after accounting for transaction costs, is:

$$\mu = 12.69604 \cdot X_1 + 9.92170 \cdot X_2 - 0.5 \cdot (|X_1 - 0.30| + |X_2 - 0.70|).$$

(The reason why we use "0.5" rather than "0.005" in the above expression is because the units of the expected return are expressed as a percentage.)

The second way that the portfolio model is incomplete has to do with limitations on the extent to which the portfolio weights might be allowed to change. For the coming quarter, Brian Jackson obviously wants the optimization model to determine new portfolio weights so as to optimize the expected return of the portfolio, given the new data in Table 8.11 and Table 8.12. However, Brian would rather not change these portfolio weights too dramatically from their current values in the last row of Table 8.13. The reason for this is that he is concerned that a large change in the composition of the EPCF portfolio might be viewed by some of the private clients as an indication that the fund managers do not have a sound investment strategy for the fund. In order to guard against this possibility, Brian has decided to include constraints in the model that will limit the absolute change in any of the portfolio weights to $\pm$ 15% from their current value. For example, the current portfolio weight of Boeing stock is 21% of the portfolio, according to Table 8.13. For next quarter, Brian wants to limit the range of the portfolio weight in Boeing stock to be in the range from $(21 - 15)\%$ to $(21 + 15)\%$, that is, from 6% to 36% of the portfolio.

Assignment:

Part I

The data and the portfolio model for the Endurance Private Client Fund is contained in the spreadsheet ENDURANCE.XLS. This spreadsheet contains the new expected returns, standard deviations, and correlations for the coming quarter. However, other than the new data, the spreadsheet's optimization model itself is the old optimization model from the last quarter. Therefore, it does not contain any transaction costs, and it does not account for the new constraints that the maximum change in any asset's portfolio weight should be $\pm$15% from its current value.

(a) (For this question, ignore the issue of transaction costs and the issue of the maximum change in the portfolio's weights.) One current policy at Endurance Investors is that the standard deviation of the Endurance Private Client Fund (EPCF) should be no greater than 13%. Another policy is that the percentage holding of any one asset in the fund should be no greater than 30% of the fund. Use the Solver to maximize the expected annual return of the fund, subject to these two policy constraints.

(b) What are the optimal portfolio weights for next quarter? What is the expected annual return of the portfolio?

(c) Examine the shadow prices on all of the constraints. What do they tell you about the solution?

(d) Construct the **efficient frontier** of the portfolio. In order to do so, you will need to run the model for a variety of different values of the right-hand-side (RHS) of the standard deviation constraint. Plot the standard deviation as the independent variable (on the horizontal axis) and the maximized expected annual return as the dependent variable (on the vertical axis).

Part II

(e) Modify the model to incorporate the effects of transaction costs (at a cost of 0.5% of the transactions for both purchases and sales) and to incorporate the constraints that the maximum change in any asset's portfolio weight from the previous quarter should be $\pm 15\%$.

(f) What are the optimal portfolio weights for next quarter? What is the expected annual return of the portfolio? How do these numbers differ from your answers in Question (b)?

(g) Examine the shadow prices on all of the constraints. What do they tell you about the solution? Again, how do the shadow prices differ from those in Question (c)?

(h) Create the **efficient frontier** of the portfolio. As before, run the model for a variety of different values of the right-hand-side (RHS) of the standard deviation constraint. Plot the standard deviation as the independent variable (on the horizontal axis) and the maximized expected annual return as the dependent variable (on the vertical axis).

(i) How is the efficient frontier different from your answer to Question (d)?

(j) Based on the model, what portfolio weights would you recommend for the Endurance Private Client Fund?

CAPACITY INVESTMENT, MARKETING, AND PRODUCTION AT ILG, INC.

Mr. Nelson Stein is the Chief Executive Officer of ILG, Inc., an engineering company that he founded almost a decade ago. ILG was an early player in the market for network routers and over the years ILG has acquired a well-deserved reputation for developing innovative networking products and bringing these products to the market very quickly and profitably. However, owing in part to its own success, the market for network routers and related technology is now quite competitive.

ILG has recently developed a new product, tentatively called the "Speed-demon" by the marketing department, that is able to speed up its network routers by a factor of ten or more. Nelson Stein felt that in order to be successful with Speed-demon, ILG would need to take care to coordinate capital planning, manufacturing, and marketing strategy for the new product. He therefore convened a meeting with Jonathan Barr, vice president of manufacturing at ILG; Jenny Thompson, vice president of marketing at ILG; Richard Bradley, chief financial officer at ILG; and Bill Zender, a member of Nelson Stein's staff who is also a recent business school graduate.

In the meeting, Nelson Stein first asked Jonathan Barr about cost estimates for building production capacity for Speed-demon. Jonathan Barr reported that since the technology required to produce Speed-demon is brand new, none of ILG's current production capacity could be used to produce the product. Using cost data from several of ILG's previous new capacity construction projects as the basis for a nonlinear