

Foundational Flaws in Modern Petrophysical Models

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This paper was presented at the SPWLA 57th Annual Logging Symposium held in Reykjavik, Iceland June 25-29, 2016.

ABSTRACT

Fifteen years following the invention of well logging, G. E. Archie invented an empirical method of interpreting the log measurements quantitatively in terms of a relationship between porosity, resistivity, and water saturation. In the decades following Archie's discovery, a relative handful of investigators, among them M. R. J. Wyllie and Walter Rose, William Winsauer and his colleagues, and Monroe Waxman and L. J. M. Smits invested much effort in trying to understand and extend the Archie model by a combination of laboratory and thought experiments, and our modern industry-standard paradigms for resistivity interpretation incorporate the ideas that these pioneers introduced.

However, it can be fairly said that neither these pioneers, nor those who have followed have done much to illuminate the physics of conduction in rocks, and much misinformation has come to be accepted as fact, at least among working formation evaluators.

The major reason for the confusion is that each of the models proposed by the pioneers mentioned above has unrecognized flaws. When their work is taken together in an effort to understand the electrical behavior of rocks the result is confusion.

However, when the flaws in the models due to Wyllie and Rose, Winsauer and colleagues, and Waxman and Smits, are identified, it is possible to construct a conceptual model that avoids the flaws, and that also can be easily understood in physical terms. Our paper points out the flaws in the models of Wyllie and Rose, Winsauer and colleagues, and Waxman and Smits, and models derived therefrom, and shows how the flaws continue to contribute to confusion in understanding and interpretation of formation resistivity relationships. It also illustrates how the flaws

in the first two models can be removed, and that the repaired models are both equivalent to the geometrical factor theory of D. C. Herrick. It is also shown that when properly understood, Waxman and Smits can be reformulated into a logically correct theory, but whose parameters, unfortunately, are not amenable to measurement.

INTRODUCTION

Motivation. Petrophysics is built upon the foundation of a few seminal publications. The very first of these is strictly and unabashedly empirical; that is Archie's model of 1941/1942. Then, following a hiatus of about a decade, the first efforts to understand the bulk conductivity of rocks in terms of the phases (e.g., quartz, brine and hydrocarbons) that comprise them were made in the late 1940s. These efforts are all based upon application of circuit theory, relying upon models conceptually built from discrete circuit elements; namely, resistors in series and parallel. It would be a further four decades before any researchers would ask: Given the constitutive phases of a rock, what should its resistivity depend upon?

Deplorably, in the western university, there is no path to a degree in formation evaluation. As a result, practicing formation evaluators may have been exposed only to the history of their specialties, if they have been exposed to any history of formation evaluation at all. This paper is about flaws in influential seminal publications in resistivity-porosity-water saturation determination. The reader unfamiliar with these papers will not receive the full benefits of our arguments. We therefore encourage our readers to add the publications discussed herein to their personal digital libraries, to peruse them, and to refer to them as they read this article.

In order to discuss flaws in the foundational documents of petrophysics, it is necessary to have a "flawless¹" model for comparison. We present our

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¹Of course we are aware that the model that we are about to present is "flawless" only in the sense that it is self-consistent, without contradiction. Our model is "flawed" by its incompleteness; i.e., it is a parallel conductivity model which ignores any contribution to conduction of its conductive phases due to a series combination of the phases, etc. It is "flawless" only in the sense that, given a two-

model following a review of the development of petrophysics, and the introduction of its “classical” models for parallel conduction.

History. As most readers of this article will know, G. E. Archie first announced a model relating formation resistivity to brine resistivity, porosity, and water saturation in 1941, providing a means for the quantitative interpretation of resistivity logs for the first time since their invention, fourteen years earlier in 1927. Archie’s model does not rely upon (or provide) any physical insight or theory; rather it is strictly an empirical synthesis of two trend analyses based upon observations of resistivity, water saturation, and porosity. The Archie model rests upon two pillars.

The first pillar Archie deduced from data found in four papers published between 1936 and 1938 (see reference list). All of the papers presented resistivity index data plotted against water saturation. These papers focused more upon permeability than water saturation, and the resistivity – water saturation data was not the primary purpose of the papers. Moreover, the data was presented in some papers in terms of conductivity, in other papers in terms of resistivity. There is only a single table of data; all the other data was presented as plots. Visually, from the plots, none of the data suggested any particular regularity. Archie found that when all of the data were converted to resistivity versus water saturation, and all were plotted on a bi-logarithmic grid, that the data were all fit (within observational uncertainty) by the same power law. Thus, Archie deduced from his plot that $I = R_t / R_0 = S_w^{-n}$, where I is resistivity index, R_t the resistivity of the hydrocarbon-bearing formation, R_0 the resistivity of the same, but fully brine-saturated, formation, and S_w is water saturation. The saturation exponent, n , is the slope of the trend line suggested by the bi-logarithmic plot of resistivity index against water saturation. We call this Archie’s “petite” synthesis.

The second pillar is based upon Archie’s own experimental data listing pairs of porosity and brine-saturated rock resistivity. Reservoir samples were saturated with a brine of known resistivity and their bulk resistivity and porosity measured. These data Archie, again, plotted on bi-logarithmic paper and found that the resulting trends appeared to be well-fit by a power law. In a definition similar to resistivity index, Archie defined a formation resistivity factor $F = R_0 / R_w$, where R_w is brine resistivity, and found

from the plots that $R_0 / R_w = \phi^{-m}$ where ϕ is porosity, and m is a porosity exponent, the slope of the trend line suggested by the bi-logarithmic plot of formation resistivity factor against porosity.

Noting that $R_t = R_0 S_w^{-n}$, and that $R_0 = R_w \phi^{-m}$, then $R_t = R_w \phi^{-m} S_w^{-n}$. Some things are worthy of mention at this point: (1) there was no theory used to derive $R_t = R_w \phi^{-m} S_w^{-n}$; this relationship is based solely upon curve fitting and Archie makes no hypothesis as to why this relationship should exist; (2) Archie does not call m a “cementation factor” or “cementation exponent”; these abominable misnomers are attributable to H. Guyod in 1944; (3) there is no a factor introduced in $R_0 / R_w = a \phi^{-m}$ (the a factor first appears in a paper by J. E. Owens, 1952); (4) Archie himself fails to put his relationships together as clearly as we have done above, i.e., $R_t = R_w \phi^{-m} S_w^{-n}$. The six equations that Archie offers in his paper are introduced with such haphazard randomness, that generations of interpreters and textbook writers have been confused by them, to this day.

Archie’s own assessment of his discovery was:

It should be remembered that the equations given are not precise and represent only approximate relationships. It is believed, however, that under favorable conditions their application falls within useful limits of accuracy.

We see by this disclaimer and the cautious presentation of his results, that no fault can be found with the presentation of this seminal discovery, although had he worked a bit more thoroughly on his derivations he might have *saved thousands of students many a headache*. Nonetheless, forming the product of I and F , $IF = R_t / R_w = \phi^{-m} S_w^{-n}$ can rightly be referred to as Archie’s “Grand” synthesis.

Archie was neither a scholar nor scientist, and published only four times in his career. In his third paper he coins the word *petrophysics*, his second enduring contribution to formation evaluation. Archie’s influence is analogous to Columbus’ (who made four voyages to the New World); both made discoveries that have insured their names would be imprinted upon their cultures as pioneers despite their follow-up activities having had little significance or influence. Archie’s 1942 paper is easily the most cited work in the art and science that he named “petrophysics”, just as Columbus’ 1492 voyage may be the most storied cruise in history.

or three-phase parallel conductivity model, in contrast to the classical models critiqued herein, nothing is omitted from our model.

Our title evokes “foundational flaws”, and as we say, Archie’s model, taken as is, has no (logical) flaws. One can imagine an alternative universe in which Archie noticed how much simpler working in geophysical electrical methods would be had he reformulated his data in terms of conductivity, and the entire evolution of our craft might have been more rapid, and less painful and mysterious. Then again, knowing the stubborn conservativeness of our industry, had he done so, his work might have been ignored. Ours is a very practical and useful craft, impinging upon literally trillions of dollars over the lifetime of the industry, and yet, comparatively little effort has been invested in research, and there has been so little discussion of the underlying modeling in terms of fundamental physics that all such papers published since 1942 are easily bound in a single (albeit thick) loose-leaf binder with some room to spare. Contrast this with a topic that will have no foreseeable practical impact: string theory. Thousands of physicists in hundreds of universities have turned this topic inside out, explored every implication that could be thought of (with some implications, no doubt, yet to be thought of), and in general had a vigorous marketplace of ideas for the past 35 or 40 years. In general, string theory has not lived up to early expectations as potentially being a fundamental theory of everything, but it is not for lack of intellectual effort. Contrast this with the development of theoretical petrophysics of low frequency electrical behavior of petroleum-bearing rock formations; all of the seminal papers would easily fit into a loose leaf binder – a very thin one. Our industry has continued to work with models that can easily be shown to have flaws, into which we plug data which we do not fully understand, and, depending upon which company we work for, purport to get accuracy in our results to three or four significant figures although most of the inputs were accurate to only two significant figures. Almost a century into this game, it is perhaps too much to expect that anything will change. But, it certainly will never change unless this deplorable lack of curiosity and argumentativeness is identified and disparaged. As a discipline, petrophysics could use an attitude adjustment.

You will note by the dates that Archie’s discovery was made upon the eve of the United States’ entry into World War II. The war, of course, disrupted everything in the civilized world, including petrophysics. Of course neither the belligerents, nor the occupied countries, were sharing results, if any purely petrophysical research was happening at all. From the war years we have only the tutorial articles by Hubert

Guyod (1944) on log interpretation. Certainly his most impactful words in those articles were:

For the range of sands investigated by Archie, the factor m of equation (12-4) [$F = \phi^{-m}$] was found to vary from 1.3 for unconsolidated sands to 2.0 for cemented formations. The higher the degree of cementation, the larger the factor seems to be, in general. It is therefore logical to call m the “cementation factor.” There is no known method of determining accurately the cementation factor directly.

In fact, m might respond to cement, but only as one of many factors affecting brine geometry. Brine geometry is also affected by grain shape, surface roughness, packing, and other textural parameters. If m had to have a name, “shape factor” would have been a better, and the value-neutral name, “porosity exponent”, the best, choice.

Publications resumed in the late 1940s. Many researchers returned to civilian pursuits after having been involved in military research in the war years.

PARALLEL CONDUCTION MODEL

It will facilitate the discussion to develop an explicit parallel conduction model. Although such a model is explicitly assumed in modeling “shaly sands”, in fact the Archie model can be considered as a parallel conduction model where one of the “conductors” has a conductivity approaching zero; i.e., the mineral grain component. Circuit theory is where such discussions usually begin, with the study of resistors in a circuit. A resistor is a circuit element for which the ratio of voltage across the resistor to current passing through the resistor is a constant, or invariant. The value of this ratio is called the “resistance” of the circuit element. The reciprocal of this ratio is called the conductance. Conductance, then, would be the ratio of current through the conductor to voltage across the conductor. In some sense, conductance is a more natural measure than resistance since for a constant applied potential difference, the current through the conductor varies directly with the intrinsic conductivity of the material comprising the conductor, and also depends intuitively upon the geometry; for a given applied potential difference the larger the cross sectional area, the more current can be carried, and conductance increases; the longer the conductor, the more material to be traversed by the current, the smaller the voltage gradient, and the smaller the amount of current conducted.

The total resistance r_T of two resistors, r_1 and r_2 , connected in parallel is given by a formula from elementary physics for resistors connected in parallel:

$$\frac{1}{r_T} = \frac{1}{r_1} + \frac{1}{r_2} \quad (1)$$

In terms of conductances, this formula simplifies. Electrical engineers use g as a notation for conductance; equation (1) becomes $g_T = g_1 + g_2$. This formula is illustrated in figure 1. (A more thorough discussion of parallel conduction in rocks is provided in Appendix A.) Conductance can be measured directly by putting the conductance element in a circuit, measuring the current through the conductor and the voltage across it; then $g = I/V$. Obviously, the unit of conductance is amperes/volt \equiv Siemen. Alternatively, in terms of material properties and geometry, the conductance should depend upon an intrinsic property of the material reflecting the ease that charged particles in the material move under the influence of an applied electric field. In physics, the notation for this

intrinsic property is σ , called conductivity, with units of Siemens/meter.

Conductivity is a measure of the ease with which charge carriers in a material move under the influence of an electric field. It will depend upon the valence of the charge carriers (i.e., ions or electrons), q , the density of free charge carriers available in a material, n , and the speed v with which they can be induced to move per unit of applied electric field, $v/|E|$. In other words, $\sigma \propto q n \mu$ where $\mu = v/|E|$ is called the mobility². In a brine having charge carriers of both signs, $\sigma = q_+ n_+ \mu_+ + q_- n_- \mu_-$ for each species of dissolved salt; the total conductivity would be the sum of such conductivities. In terms of units, q has units of coulomb/ion, n has units of ion/m³, and μ has units of (m/sec)/(volt/m). Combining these yields coulomb/m³ \times (m/sec)/(volt/m) = ((coulomb/sec) / m²) / (volt/m) = (current density) / (electric field) = $|J|/|E|$. This shows how the microscopic material property relates to the measurable quantities, current density J , and electric field E , or $J = \sigma E$. Thus bulk conductivity can be determined from voltage and current measurements, and the geometry of a sample. It is unnecessary to know the detailed composition of the conducting brine.

When cross sectional area of a given material of conductivity σ is increased, the amount of material available to carry current is increased, and so the conductance increases. For fixed cross-sectional area and applied potential difference, increasing the length of the conductor decreases potential drop per unit length, and thus the current passing through, and decreases the conductance. That is

$$g = \sigma \frac{A}{\ell}, \quad (2)$$

where A is cross-sectional area and ℓ is length of the conductor with conductance g . Now consider two conductors connected in parallel. Let them have physical lengths, ℓ_1 and ℓ_2 , and cross-sectional areas A_1 and A_2 . The conductance of the individual conductors is given by formula (2) with all of the parameters subscripted with a 1 or a 2, respectively. The equivalent conductance σ_0 will be based upon the total

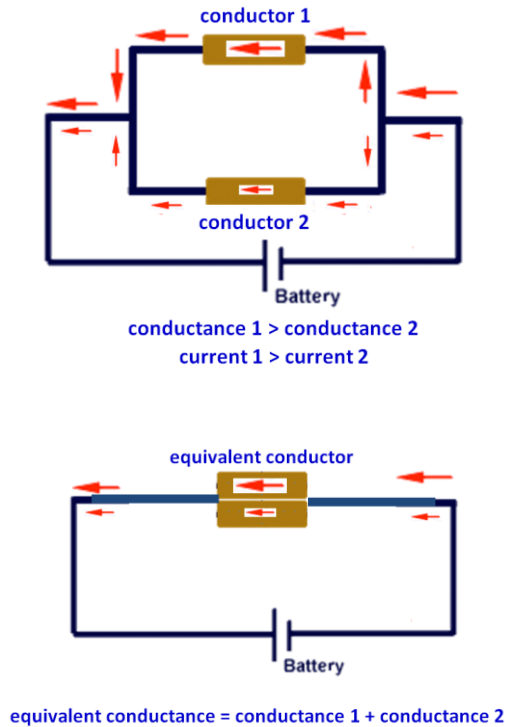


FIG. 1. Two conductors in a parallel circuit. When connected in this way, the conduction is as if the two conductors were brought together into a single conductor, each carrying its own fraction of the current. The equivalent conductance is the sum of the two individual conductances.

²This formula, deduced from logic, in practice holds only for dilute solutions in which the ions from the dissociated salts are well-separated from their nearest-neighbor ions and are uninfluenced by their electric fields. In concentrated solutions, this is not the case and the simple formulas break down; the dependence of conductivity on concentration has to resort to empiricism.

cross-sectional area³ $A_0 = A_1 + A_2$ and the choice of ℓ_0 . Then $g_0 = g_1 + g_2$ is written

$$\sigma_0 \frac{A_0}{\ell_0} = \sigma_1 \frac{A_1}{\ell_1} + \sigma_2 \frac{A_2}{\ell_2} . \quad (3)$$

The equivalent conductivity is then

$$\sigma_0 = \left(\frac{A_1/A_0}{\ell_1/\ell_0} \right) \sigma_1 + \left(\frac{A_2/A_0}{\ell_2/\ell_0} \right) \sigma_2 = x\sigma_1 + y\sigma_2 . \quad (4)$$

The coefficients of the conductivities are obviously “geometric constants” (Waxman and Smits, 1968) or “textural parameters” (Wyllie and Rose, 1950). The formula can be applied to rocks by recognizing the denominators as tortuosities as defined by Winsauer et al. (1952); i.e., $\tau_i = \ell_i / \ell_0$ for i equal 1 or 2. The interpretation of the fractional cross-sectional area depends upon what kind of rock is being modeled. In terms of an Archie rock considered as a parallel conduction model (where the conductivity of the mineral phase is zero), Wyllie and Rose state that “the fundamental definition of a porous medium possessing random pore distribution is that any plane through the medium will always expose a constant fractional void area *proportional* to the porosity” [emphasis added], but they assume that the cross sectional area is $A_1 = A_0\phi$. However, what their words imply is fractional void area $A_1/A_0 = \alpha_1\phi$, where α_1 is a constant of proportionality (i.e., $A_1 = \alpha_1 A_0\phi$). Wyllie and Rose thus implicitly take $\alpha_1 = 1$. This conforms to the practice of petrographers’ estimating porosity by the point-count method, but it is not strictly correct. The effective cross-sectional area experienced by current penetrating a cross-section is less than the observed cross-sectional area because the current flow will be limited by the minimum area constriction in the current path; i.e., the current will be controlled by the minimum area of the pore throats, not the pore bodies. To account for this reduction, a constriction factor, $\alpha \leq 1$ is necessary. The mineral phase will have fractional volume $A_2/A_0 = \alpha_2(1 - \phi)$, but this quantity does not appear in a final expression for bulk conductivity since the conductivity of the mineral phase is zero. Thus, for Archie’s model considered as a parallel conductance model,

$$\begin{aligned} \sigma_0 &= \left(\frac{A_1/A_0}{\ell_1/\ell_0} \right) \sigma_1 + \left(\frac{A_2/A_0}{\ell_2/\ell_0} \right) \sigma_2 \\ &\rightarrow \left(\frac{\alpha_1\phi}{\tau_1} \right) \sigma_1 = \left(\frac{\alpha_1}{\tau_1} \right) \phi \sigma_1 = E_1 \phi \sigma_1 \end{aligned} , \quad (5)$$

where the constriction factor α_1 and tortuosity τ_1 are both “geometric parameters” (Wyllie and Rose, 1950). Since these cannot be independently measured, especially in situ, it is convenient to define a “pore geometric factor” $E_1 = \alpha_1/\tau_1$, as we have done above, labeling the pore geometric factor with the phase subscript. It is also clear from (5) that $\sigma_0/\sigma_1 = E_1\phi \equiv 1/F$; i.e., the formation resistivity factor *combines* the effects of porosity and pore geometry. To examine pore geometry decoupled from porosity it is necessary to normalize F by multiplying it by porosity; $E_1 = 1/(\phi F)$.

Returning to the parallel conductor model with two conductive components, useful in “shaly” sandstone modeling

$$\sigma_0 = \left(\frac{A_1/A_0}{\ell_1/\ell_0} \right) \sigma_1 + \left(\frac{A_2/A_0}{\ell_2/\ell_0} \right) \sigma_2 = x\sigma_1 + y\sigma_2 . \quad (4)$$

Looking ahead in this paper, this formula makes plain how Waxman’s and Smits’ (1968) “appropriate geometric constants” x and y relate to pore geometry and rock texture. Comparing the coefficients in (4) and (5) note that $x = E_1\phi$; this result suggests the generalization $y = E_2\phi$. The fractional pore volume ϕ , constrictivity α , and tortuosity τ for each individual phase has its individual role to play.

We shall make a final observation of the Archie model

$$\sigma_0 = \left(\frac{A_1/A_0}{\ell_1/\ell_0} \right) \sigma_1 = \frac{\alpha_1}{\tau_1} \phi \sigma_1 = E_1 \phi \sigma_1 = \phi^m \sigma_w, \quad (6)$$

where $\sigma_0 = \phi^m \sigma_w$ has been added to the display of equation (5). When there is only a single non-zero conducting phase (i.e., formation brine), in conventional notation (then $E_1 \equiv E_0$) this is written as $\sigma_0 = E_0 \phi \sigma_w = \phi^m \sigma_w$ where E_0 , although not directly observable, can be computed from the three observable quantities, bulk conductivity, brine conductivity, and porosity ($E_0 = \sigma_0 / \phi \sigma_w$). When E_0 is correlated to ϕ (i.e., E_0 vs. ϕ plot), a trend

³Note that $A_0 = A_1 + A_2$ implies that $1 = A_1 / (A_0 + A_2 / A_0)$ and that $A_2 / A_0 = 1 - A_1 / A_0$.

$E_0 = a_0 \varphi + b_0$ (where $a_0 \approx 1$ and $b_0 \approx 0$) is observed in Archie rocks. Thus, the several parameters

$$\frac{\alpha}{\tau} = E_0 = \varphi^{m-1} = a_0 \varphi + b_0 = \frac{1}{\varphi F} \quad (7)$$

are related, and provide several ways to view the pore geometrical factor E_0 . It can be expressed as the ratio of two more fundamental geometrical factors, tortuosity and constriction. But since tortuosity and constriction are not measureable in situ, the correlation of E_0 to porosity is useful. The enigmatic factor φ^{m-1} is somewhat demystified as depending only upon geometry but not directly upon porosity, and φ^m is seen to be the product of porosity φ and the geometrical factor, E_0 ; i.e., $\varphi^m = E_0 \varphi$.

The constriction α and tortuosity τ factors are related to m by

$$m = 1 + \frac{\ln(\alpha/\tau)}{\ln \varphi}. \quad (8)$$

So, to the extent that $m \approx 2$, then $\alpha \approx \tau \varphi$. This leads to some insights. For unconstricted pores, $\alpha = 1$; for fully constricted pores $\alpha = 0$. Constriction can vary over much of this range without much effect upon tortuosity. So, when $\alpha = \varphi \tau$, $m = 2$ is implied; for constant tortuosity, if $\alpha / \tau < \varphi$, then for decreasing α (more constriction), m increases. Conversely, if $\alpha / \tau > \varphi$, then for increasing α (less constriction), m decreases. These findings confirm intuition and are justified in Appendix B.

Model from First Principles

As we have noted, Archie did not begin with a theory, nor did he attempt to impose one, when he discovered his model using trend analysis. However, his model enjoyed enough success that it apparently prevented the pioneers from asking “What *should* the relationship between bulk formation resistivity, brine resistivity, and porosity be?” Even if this question had been asked, it is very difficult to answer in terms of resistivity. Yet, the educational conditioning of the twentieth century petroleum⁴ scientists and engineers was so biased toward thinking in terms of resistivity and circuit theory, it would not be until 1968 that a formation evaluation problem would be framed in

terms of conductivity. Following the formulation of Waxman’s and Smits’ parallel conductivity model in 1968, all theoretical models would be framed in terms of conductivity.

Framed in terms of conductivity, the problem becomes almost trivially simple³. What should the conductivity of a bulk rock sample depend upon? Consider the three following principles:

- (1) Bulk conductivity is proportional to brine conductivity; that is, if brine conductivity is doubled or tripled, etc., bulk conductivity is self-evidently doubled or tripled, etc. Thus, $\sigma_0 \propto \sigma_w$.
- (2) For constant pore geometry, bulk conductivity is proportional to the amount of brine; that is (for constant pore geometry), doubling, or halving, the porosity will self-evidently double, or halve, the bulk conductivity. Thus, $\sigma_0 \propto \varphi$.
- (3) Bulk conductivity depends upon the geometry of the brine. It is self-evident that if the brine in a sample has no end-to-end connectivity, then the bulk conductivity of a sample will be zero; conversely if the brine is arranged with maximum connectivity, in an end-to-end tube, the conductivity will be a maximum (i.e., $\sigma_w \varphi$). Pore geometries in rocks will fall between these limits, and will self-evidently be described by a parametric function that varies between 0 and 1. Thus $\sigma_0 \propto E_0$ where $0 \leq E_0 \leq 1$.

Putting these principles into a formula $\sigma_0 = \sigma_w \varphi E_0$. In conductivity terms, this is what *should* be observed. E_0 is not directly measurable, but may be computed from the three measurable formation properties, σ_0 , σ_w , and φ ; i.e., $E_0 = \sigma_0 / \sigma_w \varphi$. Pore geometry has no intrinsic dependence upon porosity; i.e., there is no simple way to predict E_0 given a description of a pore system. However, there may well be correlation between E_0 and porosity. If such a correlation is observed to exist, it can be exploited by expressing E_0 as a porosity-dependent trend.

Given the model $\sigma_0 = \sigma_w \varphi E_0$ one can remove the effect of brine conductivity on bulk conductivity by dividing (or normalizing) by brine conductivity, the

⁴Scientists in other disciplines were beginning to look for conductivity models in terms of percolation theory, also formulated in terms of conductivity (e.g., Shankland and Waff, 1974).

³These ideas are first enunciated by Herrick and Kennedy (1993,1994); the “three principle” formulation first appeared in Kennedy and Herrick (2012).

result being $\sigma_0 / \sigma_w = \phi E_0$. One can then remove the effect of porosity by dividing (normalizing) by porosity, the result being $\sigma_0 / (\sigma_w \phi) = E_0$. The remaining quantity has to depend only upon pore geometry, since both brine conductivity and porosity are “normalized out” of the expression.

Conversely, beginning with $R_0 = R_w \phi^{-m}$ one can remove the effect of brine resistivity by dividing R_0 by R_w , resulting in $F = R_0 / R_w = \phi^{-m}$. From this expression it is not obvious how to remove the effects of (or normalize for) porosity in order to isolate a pore-geometric factor, and none of the pioneers succeeded in so doing. Through insight gained from the conductivity formulation, the resistivity formulation is normalized for porosity by multiplying by porosity; i.e., $\phi F = \phi R_0 / R_w = \phi^{-(m-1)}$. By our construction, $\phi^{-(m-1)}$ is a geometric factor, independent of porosity; but, how is this obvious by inspection of the expression? It is easy to understand how the pioneers, wedded to a resistivity formulation, made no progress in isolating a pore geometric factor. The factor ϕ^{m-1} hardly appears to be porosity-independent. However, in spite of appearances to the contrary, $E_0 = \phi^{m-1}$ is independent of porosity. (For this to be so, it is necessary that $m = 1 + \ln E_0 / \ln \phi = 1 + \ln(\alpha' \tau) / \ln(\phi)$. This is amplified in Appendix C.)

Having now laid a foundation for parallel conductivity (and resistivity) models, we can compare the details of the pioneers’ formation resistivity theories to the elements of this foundation. The ideal for comparison will be $\sigma_0 = \sigma_w \phi E_0$.

Wyllie & Rose (1950)

According to his obituary⁵, “Jesse Wyllie [1920-2010] is probably best known in his industry as the father of modern well log interpretation.” Unless you worked at the Gulf Oil Corporation, this is probably a little overstated. However, there is no doubt that the Gulf Research facility in Pittsburgh was a hotbed of published petrophysical research from the late 1940s until about 1960, spanning Wyllie’s career in the lab. (Following that he moved into management and had a second career.) Wyllie was a very prolific author and co-author in the late 1940s and 1950s, and he undoubtedly published more articles in more fields relating to petrophysics than any other author. However, his work has had little visible lasting impact,

the exception being his name attached to the “Wyllie Time-Average” equation, an approximation relating compressional slowness (interval transit time, as it was called in Wyllie’s day) and porosity. In 1950, with Walter D. Rose, he published “Some Theoretical Considerations Related to the Quantitative Interpretations of the Physical Characteristics of Reservoir Rock from Electric Log Data”, which will be the subject of this section. Earlier the same year⁶, with H. W. Patnode (1950) he published what may have been the first “shaly sand” paper, “The Presence of Conductive Solids in Reservoir Rocks as a Factor in Electric Log Interpretation”.

Wyllie and Rose (1950) in their section on “Theoretical Considerations”, begin with a reiteration of Archie’s findings, $F = \phi^{-m}$ and $I = S_w^{-n}$. They consider a sample of cross-sectional area A_c , with a total void cross-sectional area of ϕA_c . In their words: “In considering the electrical resistance of a fluid saturated porous medium possessing a random* distribution of pores, we may express the resistance of a single fluid-filled channel having a rather complex shape but of constant cross-sectional area, ϕA_c , ...”. This is illustrated by figure 2. The sample is of length dimension L , but with current path length $L_a > L$. Modernizing the Wyllie and Rose notation, formation brine resistivity is given by R_w , and the bulk sample resistivity is denoted by R_0 . Their procedure is to

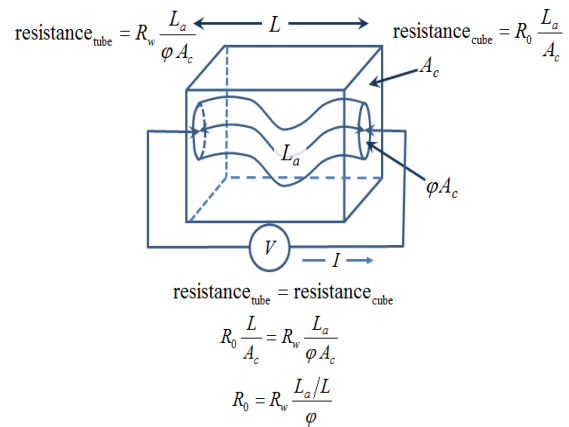


FIG. 2. The model of Wyllie and Rose. The porosity is collected into a sinuous tube of constant cross-sectional area, ϕA_c . In practice, the effective cross-sectional area is less than ϕA_c .

⁵<http://www.aimehq.org/programs/award/bio/m-r-j-wyllie-deceased-2010>

⁶The manuscripts were received at AIME within two days of one another.

equate the resistance of brine in its single tortuous channel of constant cross-sectional area to the resistance of the bulk sample. These resistances are expressed in terms of resistivities, areas, and lengths, viz.,

$$R_0 \frac{L}{A_c} = R_w \frac{L_a}{\phi A_c} \quad (9)$$

The bulk resistivity is then

$$R_0 = R_w \frac{L_a/L}{\phi A_c/A_c} = R_w \frac{\tau}{\phi} \quad (10)$$

Already, at this point, the model has lost its dependence on constrictions in the cross sectional area. The asterisk in the quoted passage refers to a footnote reading: “Note that the fundamental definition of a porous medium possessing random pore distribution is that any plane through the medium will always expose a constant fractional void area *proportional* to the porosity.” [emphasis added] Had this fact been used, the porosity factor in equation (10) would have been $\alpha\phi$, where α is a proportionality constant, $\alpha < 1$, that scales the porosity, viewed as fractional cross-sectional area, to its constricted value. Inclusion of this factor would change (10) into

$$R_0 = R_w \frac{L_a/L}{\alpha\phi A_c/A_c} = R_w \frac{\tau}{\alpha\phi} = R_w \frac{1}{E_0} \frac{1}{\phi} = FR_w \quad (11)$$

Equation (11) is recognized as the resistivity representation of equation (5); i.e., $\sigma_0 = \sigma_w \phi (\alpha/\tau)$. The obvious consequence of Wyllie and Rose putting all of the variability into the tortuosity is that the apparent tortuosity will have to be larger than the actual tortuosity. This anomalously large tortuosity, due to the apparent L_a being forced to include constriction effects in addition to tortuosity, is then made seemingly larger by use of the definition: Tortuosity = $(L_a/L)^2 = T$. Thus, when $L_a/L = 10$, already a large tortuosity, but by the Wyllie and Rose definition, this is $T = 100$. A long-lasting consequence of this definition is that many people, to this day, believe that tortuosities are much larger than they actually are. A graph of porosity versus tortuosity as defined by Wyllie and Rose is reproduced in figure 3.

Thus have we seen that Wyllie and Rose began the first attempt at a theoretical discussion of formation resistivity with a model that omits the most important

control on resistivity, constriction of cross-sectional area normal to the direction of current flow, thus forcing all of the variability in the model into the tortuosity parameter, forcing it to be larger than the actual tortuosity. They then, by their adoption of a definition of tortuosity from the fluid flow literature, compound the effect, generating tortuosity values enormously larger than the actual ratio of current path to sample size. Although, by their definition, the ratio is the square root of the number that they have named tortuosity, people are often not aware of this quirk and thus have an exaggerated idea of how tortuous currents paths in rocks really are.

Figure 4a is the Wyllie and Rose tortuosity-porosity plot with the tortuosity redefined as $\tau = L_a/L$. Using this definition tortuosities are seen to be much smaller numbers. The plot shows lines of iso-formation resistivity factors (iso- F) and iso- m values. The shaded region indicates where Archie rocks would be expected to plot.

Figure 4b plots Archie's Nacatoch sandstone data set, Wyllie's and Rose's data, and the Winsauer et al. data sets. The maximum observed tortuosity is about 4; for high porosity samples, tortuosity approaches about 1.5. The theoretical maximum tortuosity for a simple cubic array of spheres would be $\pi/2 \approx 1.57$.

Figure 5 illustrates how tortuosity varies with F , m , and ϕ . Implicit in this figure is also a variation in constrictivity, α . This was briefly discussed follow-

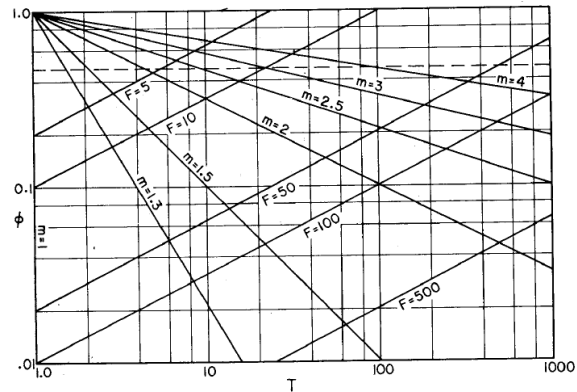


FIG. 1 — POROSITY AS A FUNCTION OF TORTUOSITY FOR CONSTANT VALUES OF FORMATION FACTOR AND CEMENTATION FACTOR

FIG. 3. The porosity versus tortuosity plot of Wyllie and Rose. According to their definition, $T = (L_a/L)^2$. The more common definition is $T = L_a/L$. Note that for $m = 2$, according to this definition of tortuosity, tortuosity at ten percent porosity is 100; at five percent porosity tortuosity is 400! These large numbers, when associated with the word “tortuosity” are very misleading.

ing equation (8) above, and more thoroughly in Appendix B.

How close did Wyllie and Rose come to getting things right? They missed only by omitting a constriction coefficient α on their representation of cross-sectional area, ϕA_c . As shown in equation (11) use of $\alpha \phi A_c$ would have permitted the separation of the resistivity formulation into three factors, brine resistivity, brine fractional volume ($1/\phi$), and a separate geometrical factor, τ/α . Even so, they managed to avoid expressing resistivity as proportional to the inverse square of porosity; see equation (10). They came quite close to the ideal model.

Patnode & Wyllie (1950)

At the risk of belaboring the point, Patnode and Wyllie (1950), in their paper on conductive solids in reservoir rocks, begin their model with the observation of how measured core resistivity is related to brine and clay resistivity with the definitions:

- “... measured resistivity of the core, ρ_{wa} ”
- “... the resistivity of the fluid ..., ρ_c ”
- “... the resistivity of the conductive solids as distributed in the core, ρ_p ...”
- “... resistivity of the fluid as distributed in the core, ρ_w ...”

using a parallel resistivity model, viz

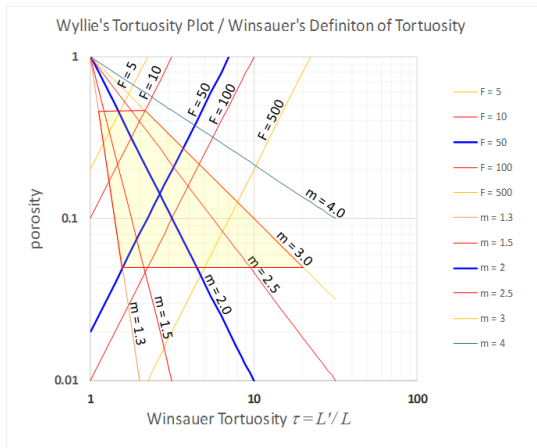


FIG. 4a. The Wyllie & Rose Tortuosity-Porosity plot with the tortuosity axis rescaled to $\tau = L_d/L$. The shaded area in the plot on the left is bounded by $1.3 \leq m \leq 3.0$ and $0.05 \leq \phi \leq 0.45$ which is the region where experimental data is found.

$$\frac{1}{\rho_{wa}} = \frac{1}{\rho_f} + \frac{1}{\rho_w} = \frac{1}{\rho_f} + \frac{1}{F\rho_c} \quad (12)$$

This is a formula that cries out to be expressed in terms of parallel conductivity,

$$\sigma_{wa} = \sigma_f + \sigma_w = \sigma_f + \frac{1}{F} \sigma_c \quad (13)$$

or, using modern subscript notations (with some ad hoc modifications: $\sigma_{0, \text{shaly}}$ refers to the bulk conductivity of the clay bearing rock; $\sigma_{0, \text{clean}}$ refers to the bulk conductivity of the same pore system without a clay component)

$$\sigma_{0, \text{shaly}} = \sigma_f + \sigma_{0, \text{clean}} = \sigma_f + \frac{1}{F} \sigma_w \quad (14)$$

It was an opportunity for the pioneers to shift the paradigm at the very beginning of the discussion of formation resistivity/conductivity theory. But, it was a path not trodden.

Note that the formula (12), as reformulated in (14), presages the Waxman and Smits formulation; it is a parallel conductivity model and the formation brine is weighted by the “geometrical” constant $1/F$, and we have seen that $1/F = \phi E_0$. Patnode and Wyllie do not make this separation of $1/F$ into a fractional volume and a geometric factor. Moreover, Patnode and Wyllie do not separate the factor σ_f into a volumetric component and a geometrical constant appropriate

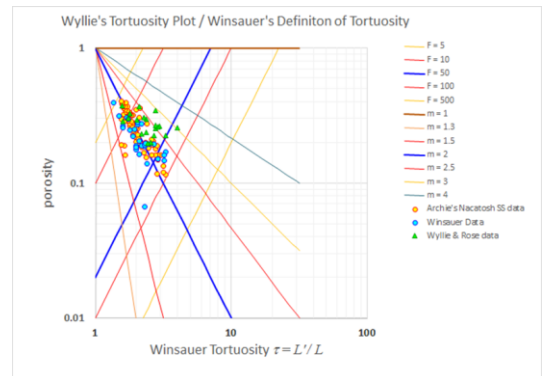


FIG. 4b. This plot shows three data sets: Archie's Nacatoch sandstone dataset; the Wyllie and Rose data; the data from the Winsauer dataset. The largest tortuosity observed in these data is 4. For the largest porosities, tortuosity approximates the theoretical maximum for a cubic lattice of packed spheres, $\pi / 2 = 1.57$.

for the conductive clay phase (e.g., they do not write $\sigma_f = \sigma_{clay} / F_{clay}$). These omissions give some appreciation of the evolution of thought achieved by Waxman and Smits eighteen years later.

Winsauer, Shearin, Masson, Williams (1952)

Winsauer and his coauthors (hereinafter referred to, for brevity, as “Winsauer”) made the second attempt at a theoretical explanation of the resistivity-porosity function. Winsauer had 29 core samples from sandstones widely distributed in geography and time. His stated goal⁷ was to validate a relationship between porosity, tortuosity and formation resistivity factor. To do this, he invented a technique for directly measuring tortuosity. He observed much lower tortuosity values than had been expected (perhaps having fallen under the influence of Wyllie and Rose), and he examined relationships relating various log-derived quantities to observables measurable only in a laboratory, in search of a predictive model. The lasting legacy of Winsauer and his coauthors is the once ubiquitous “Humble formula”. In the intervening six and a half decades, the use of this formula has waned. Rightly so, but it still appears in service company chart books. In the following we examine some of the premises behind the conclusions reached by Winsauer and his coauthors.

In his formulation Winsauer begins with a model

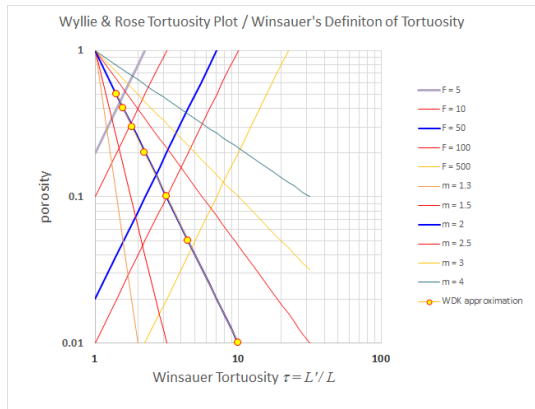


FIG. 5. We have shown that $\phi \approx \alpha / \tau$. If we (arbitrarily, but not irrationally) partition the porosity into $\alpha \approx \sqrt[3]{\phi}$ and $1 / \tau \approx \sqrt[3]{\phi}$, then all points plot near the $m = 2$ line. For constant tortuosity, m increases as α decreases (more constriction); m decreases as α increases (less constriction).

⁷In Winsauer’s words: “It is thus apparent that there is need for a sound, usable relation between the resistivity of a brine-saturated rock and easily determinable physical properties which reflect texture.”

seemingly similar to Wyllie and Rose, but contrary to Wyllie and Rose, mentions several times that cross-sectional area of the brine ought to be a factor considered. Winsauer’s equation (1) is

$$R = \rho \frac{L}{A} \quad (1)$$

This represents the resistance of a cylinder filled with brine of resistivity ρ , having length L and cross sectional area A ; in Winsauer’s notation resistance is denoted by R and resistivity is denoted by ρ . It is a correct equation but it is not what is needed in Winsauer’s subsequent development. Here Winsauer needed the resistance of the same cylinder, but sand-filled with brine in the resulting porosity. This equation would have been

$$R_{\text{rock sample}} = \rho_0 \frac{L}{A} \quad (15)$$

where we have supplied the subscripts for clarity: ρ_0 is sample bulk resistivity. Winsauer’s equation (2) is

$$R' = \rho \frac{L'}{A'} \quad (2)$$

where primes denote the resistivity and dimensions of a tube filled with brine of resistivity ρ . Winsauer then ratios his equations (2) to (1); the result is

$$\frac{R'}{R} = \frac{L'/L}{A'/A} \quad (3)$$

Winsauer claims that this ratio of *resistances* is “by definition, the formation [resistivity] factor, F .” But this is not correct; the formation resistivity factor is a ratio of *resistivities*. Because of the omission of a necessary relation, it is necessary to misinterpret the definition of formation resistivity factor in order to arrive at a correct answer; i.e., Winsauer had to embrace this incorrect notion in order to arrive at his equation (4), which is correct; the resistivity ratio in Winsauer (3) reduces to (4) due to his inappropriate choice of his equation (1). Here we restate Winsauer’s equation (2), labeling the resistance and resistivity with appropriate subscripts. Winsauer (2) then reads:

$$R_{\text{sinuous_tube}} = \rho_w \frac{L'}{A'} \quad (16)$$

What next is needed is a useful relationship among the dimensions of the sinuous tube and the other parameters, all of which are (in principle) easily measured. The condition to impose is that the *resistance* of the sinuous tube and the *resistance* of the rock sample be identical. Using equation (15), omitted by Winsauer, and Winsauer's equation (2) as restated with resistivities and resistances descriptively and properly subscripted, i.e., our equation (16)

$$R_{\text{sinuous tube}} = \rho_w \frac{L'}{A'} = \rho_0 \frac{L}{A} = R_{\text{rock sample}}. \quad (17)$$

Then, forming the ratio of rock sample resistivity to sinuous tube resistivity gives

$$F = \frac{\rho_0}{\rho_w} = \frac{L'/A'}{L/A} = \frac{L'/L}{A'/A} = \frac{\tau}{\psi}. \quad (18)$$

In these relationships formation resistivity factor is *correctly* expressed in terms of a resistivity (not resistance) ratio, and in terms of tortuosity (L'/L) and the ratio of cross-sectional areas (A'/A); i.e., $F = \tau/\psi$.

$$F = \frac{\tau}{\psi}. \quad (4)$$

By Winsauer's omission of one of the needed expressions, he is forced into logical gymnastics in order to arrive at equation (3); in his formulation the ratio of *resistivities* turns out to be 1. But, in fact, it is the ratio of *resistances* that should be 1. However, by adopting the incorrect definition of formation resistivity factor, equation (4) in Winsauer is correct as written. For serious students of the Winsauer paper, reconciling the first three equations with the fourth will be a time-consuming exercise. It is particularly perplexing because the problem is not an error in algebra in going from equations (1) and (2) to equation (3). The problem is that equation (1) is irrelevant, and that one of the needed relationships to get to equation (4) is not present. So, no amount of alternative algebra and checking for errors on the equations in the paper can produce equation (4); one has to figure out that Winsauer has omitted an equation that should be there.

Thus, a fundamental error has been made in Winsauer's formulation of the problem. It appears as if Winsauer and his collaborators, and peer reviewers, failed to really understand the concept of formation resistivity factor, and did not recognize that equation (3), in terms of resistances, is not correct. On the other hand, equation (4) is correct, whether fortuitously

or by design, and our critique going forward is based upon considerations other than the errors in how equation (4) is produced.

In its essentials, the Winsauer model is similar to the Wyllie and Rose model except that a term to represent a reduction in pore cross-sectional area is introduced. The Winsauer model is shown in Figure 6.

Early in his text, Winsauer makes several observations, some accurate and others enigmatic.

For example

“(1) the mean length of path traversed by the current is longer, since the current can no longer travel in straight lines, but must go around the sand grains; and (2) the cross-

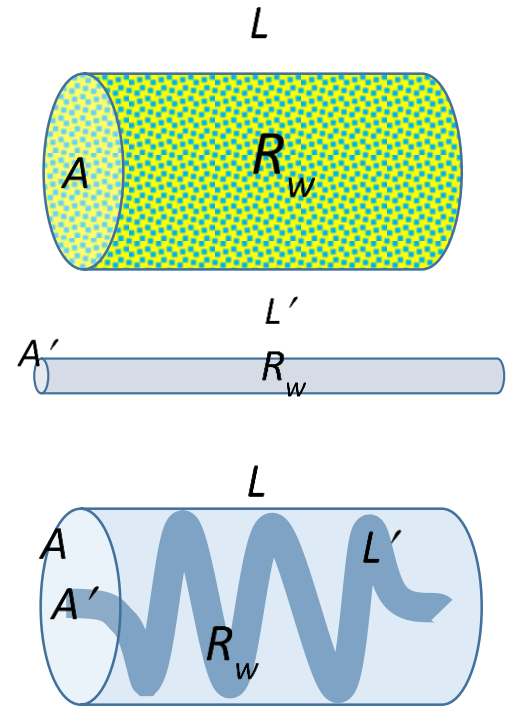


FIG. 6. The equivalent resistivity of a brine-filled porous rock is computed from the dimensions of the porous rock sample, L and A , and the resistance of the sample; i.e., $\rho_0 = R_{\text{sample}} A/L$. The brine in the sample is also imagined to fill a tube with the same resistance, but different dimensions, L' and A' . The ratio $L'/A' = R_{\text{sample}} / \rho_w$ can be computed. A knowledge or assumption about one of the primed dimensions then determines the other. The model imagines the tube to be distorted to fit into the volume of the porous rock sample.

sectional area which is available to current flow is smaller, since sand grains now occupy a large part of the area.”

These observations are standard fare, but note the attention to cross-sectional area, and

“... flow of current from one plane in a sand to a parallel plane can take place by a number of different paths. These diverse

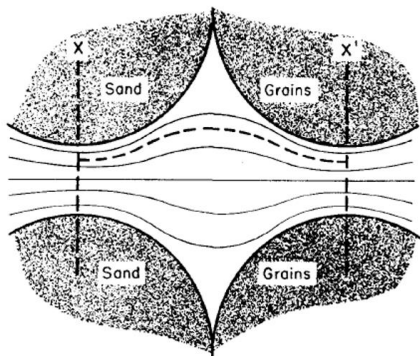


FIG. 7. Winsauer’s et al.’s cartoon of the tortuosity of current streamlines as they traverse model pore throats and pore bodies. This cartoon strongly suggests that a significant increase in constriction, by moving the grains closer to each other along the vertical dashed lines, might have little apparent impact upon the shape or length of the streamlines, and therefore the tortuosity.

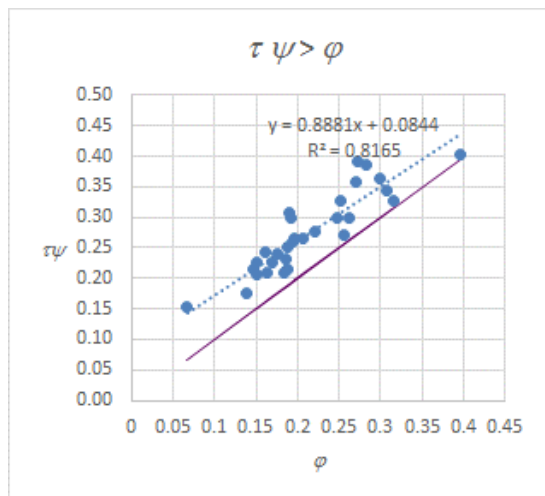


FIG. 8a. According Winsauer’s expectations, the data points should have fallen on, or below, the 45° line. This implies Winsauer’s measured tortuosity values are too high, an alternative mentioned by Winsauer that he was reluctant to accept.

paths, too, can cross and intermesh. ... ”

According to the usual meaning of the words, current paths do not, and cannot, “cross and intermesh”. They must remain sub-parallel throughout the conducting volume.

“Although the tortuosity of a rock is determined primarily by the manner in which the pore channels twist and turn in going around grains, there is also an effect due to the variation of the cross-sectional area of the tortuous paths.”

This may be metaphorical language, but pore channels do not have “twists” if that means a consistent rotation in a single direction. Certainly the paths that current streamlines follow have no “twists”. (In physics jargon, $\nabla \times \mathbf{J} = 0$.) However, we do agree that “cross-sectional area of the tortuous paths” definitely has an effect. Figure 7 reproduces Winsauer’s concept for including both tortuosity and constrictivity in a model.

Based upon the relationship $\tau\psi = (L'/L)(\underline{A}'/A) = L'\underline{A}'/LA = V'/V \equiv \phi$, Winsauer notes that the product $\tau\psi$ should at most equal porosity, but would be expected to be less in clay-free rocks. However, when the observations are plotted (figure 8a), in every case for the Winsauer data set, this expectation is not met. (Winsauer measures F and τ , then estimates ψ from $\psi = \tau/F$.)

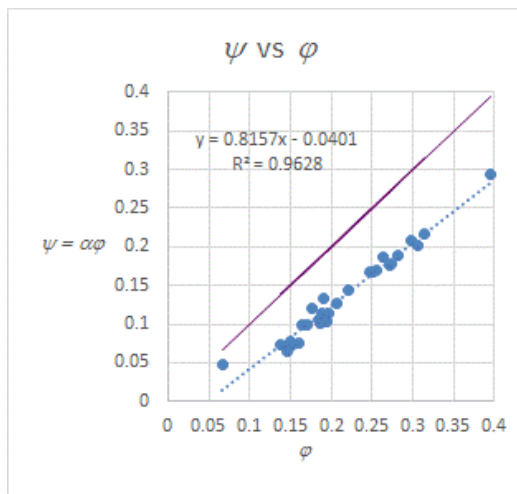


FIG. 8b. The solid line represents $\alpha = 1$. All of the constrictivity values in the Winsauer data set plot on a trend with $\alpha < 1$. For these data $\alpha \approx 0.6$ approximately. Lower tortuosity values imply more constriction and even lower values of α .

In Winsauer's words, "From a consideration of the physical significance of τ and ψ , it might be expected that, unless there is an appreciable amount of pore space which is ineffective in flow of current, the product of τ and ψ should equal the porosity of a sandstone." However, "An examination of the data shows that the product of τ , determined experimentally, and ψ , computed from equation (4), was usually greater than the porosity, and that in no case was it less." Winsauer then speculates upon the source of this surprise, "This behavior might be interpreted as indicating a systematic error in the determination of the tortuosity, but, on the other hand, no source of systematic error is evident." After examining and dismissing several possible explanations, Winsauer concludes with "There is the possibility that constriction and enlargement of the pore channels and intermeshing of the channels should give rise to measured values of τ such that the product $\tau\psi$ exceeds ϕ ."

However, it is also true that $\psi \equiv \alpha \phi < \phi$. With α computed from $\alpha = \tau / F$; remembering that Winsauer has measured both τ and F we have some confidence in this estimate of α . Plotting $\alpha\phi$ against ϕ shows that the expectation is well-met. (See figure 8b.)

This is totally consistent with expectations. This suggests that in fact the tortuosities estimated from Winsauer's experiments are indeed overestimated. This comports with our assertion that tortuosities are generally lower than is commonly believed. A smaller role for tortuosity implies a larger role for constrictivity.

The most impactful result of the Winsauer et al. paper has to be the "Humble" formula for the formation resistivity factor:

$$F = 0.62 \phi^{-2.15}. \quad (19)$$

In the 1950's this became a very popular alternative for Archie's $1/\phi^m \equiv F$ in cases where there was no data available for the determination of F . It came to be called the "Humble formula" after Winsauer's employer. We wish to make clear that Winsauer makes no particular claims for this formula; it is one of two that he offers that fit his data well. It was the consumers of the Winsauer et al. paper that attributed so much significance to this formula. The Humble and Archie formulas are compared in figure 9.

The Humble formula has no intrinsic meaning. First and foremost, it was generated not from 29 samples

taken in the same formation, but from samples taken from 10 states, 3 eras, and 9 epochs. Second, unlike the Archie formula where m can vary to fit various situations, it has *no* adjustable parameters. Further, in the porosity range where most conventional reservoirs of the 1950s were located, there is negligible difference between the predictions made by the Archie formula $F = 1/\phi^2$ and the Humble formula, especially if the porosity exponent were taken as a variable, $F = 1/\phi^m$.

It is hardly as if a formation evaluator could state: "my reservoir is similar to the reservoir used by Winsauer et al. in 1952, and so the Humble formula is appropriate for my rocks and reservoir" because their rocks are not of a single provenance. So why would anyone want to use it except that he has been mentored to use it without question? The use of the formula has been diminishing in the decades following the 1960s.

Although it is no longer identified as such, the Humble formula is still to be found in the latest Schlumberger Chart Books (as SatOH-1 on page 247 of the 2013 Chart Book).

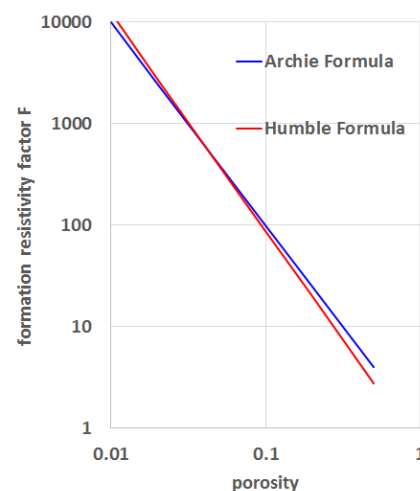


FIG. 9. Comparison of the Humble formula ($m = 2.15$, $a = 0.62$) and the Archie model with $m = 2$, $a = 1$. The predictions made are not much different, and can be made even closer if m is an adjustable parameter $\neq 2$. The Humble formula, being based upon 29 samples from 10 states, 10 epochs, and 3 eras is a trend without intrinsic meaning for *any* reservoir. At the time, at least some people must have thought, because it had been derived from samples widely separated in space and time, it was therefore widely applicable to *any* sandstone. Soberly considered, this is a hard case to make.

How close did Winsauer et al. come to getting things right? Very close. They express formation resistivity factor as the ratio of tortuosity and a fractional area factor, ψ :

$$F = \frac{R_0}{R_w} = \frac{\sigma_w}{\sigma_0} = \frac{\tau}{\psi}, \quad ((\text{Winsauer et al. (4)})). \quad (20)$$

They did not think to express $A' \equiv \alpha \phi A$ and therefore ψ as a product of a constriction coefficient and porosity, and so did not have an equation such as $A'/A \equiv \alpha \phi$ available. If they had made this observation, then from (20)

$$\begin{aligned} \sigma_0 &= \sigma_w \frac{\psi}{\tau} = \sigma_w \frac{A'/A}{L'/L}, \\ &= \sigma_w (\alpha \phi) \left(\frac{1}{\tau} \right) = \sigma_w \phi \left(\frac{\alpha}{\tau} \right) = \sigma_w \phi E_0 \end{aligned}, \quad (21)$$

and they would have written down the ideal model.

Owen (1952)

Owen's work was published in 1952. He was thus aware of Wyllie's and Rose's 1950 publication, and was in correspondence with Winsauer. Owen was first to explicitly use the idea of constriction, and he was first (as far as we know) to express the Archie formula in the form $F = a\phi^{-m}$. His paper had no impact (other than his generalization of the Archie formula) because it was based upon an intricate model pore system. The formulas that resulted worked splendidly for the model pore system, and Owen was able to show how his work might be useful. However, no one was able to express his results in a general way that did not depend upon the details of his model. Although it illustrated some properties of pore systems that had been neglected (the importance of constriction), Owen's ideas did not take hold in the industry.

In the quaint custom of the day, there is a discussion between Wyllie and Owen following the references in the Owen paper. In this discussion we learn (1) that Owen had transmitted a manuscript of his paper to Winsauer about a year before the Winsauer paper was published. Winsauer apparently did not respond to Owen personally, could not cite Owen's paper as it had not yet appeared in print, and did not acknowledge Owen in any way in his paper. Still, one is left to wonder whether Winsauer's attention to cross-sectional area might have been influenced by Owen's communication. Another interesting item

from the Wyllie-Owen exchange is Wyllie's comment that he did not believe that Winsauer et al. had actually measured tortuosity, as they claimed, but merely formation resistivity factor. There are no details in the discussion to support Wyllie's comment, so we are left to guess what he may have meant. Looking at Winsauer's experimental procedure to measure tortuosity, it certainly looks convincing to us.

Of course research continued on the resistivity of reservoir rocks during the 1950s, but unfortunately, nothing new of lasting value was added. Wyllie was trained as an electrochemist. He turned his talents to management in the early 1960s. It would be left to other electrochemists to attempt the next contribution.

Waxman & Smits (1968)

Waxman and Smits may have been the first researchers to formulate a theory of conduction directly in terms of conductance and conductivity. Their model is set out in four parallel conductivity equations. These may be the most confusing set of four equations in the resistivity literature. Here, we attempt to illuminate the work of Waxman and Smits.

Waxman and Smits begin the explanation of their theory with a conductance equation; given the conductance due to the clay component C_c and the conductance due to the electrolyte C_{el} , the *conductance* of a bulk rock sample comprising these components C_{rock} is

$$C_{rock} = C_c + C_{el}, \quad (\text{Waxman \& Smits, eqn.(1)}). \quad (22)$$

In conventional notation (in electrical engineering) this would be written as

$$g_{rock} = g_c + g_{el}, \quad (23)$$

the only difference being the notation for conductance. The *conductivity* of a bulk rock sample C_0 is assumed to be given by, in the parallel conductivity model,

$$C_0 = xC_e + yC_w, \quad (\text{Waxman \& Smits, eqn.(2)}), \quad (24)$$

where C_e is the conductivity of the "clay exchange cations" and C_w is the conductivity of the "equilibrating salt solution". The factors x and y are "appropri-

ate geometrical constants”. In conventional physics notation, this equation would be written as

$$\sigma_0 = x\sigma_e + y\sigma_w \quad (25)$$

the difference being, again, the notation for conductivity. So, whereas Waxman and Smits have used the same notation for conductance and conductivity, C , it is clear from the conventional notations of engineering and physics that these are different quantities. Further, comparing equation (14) [Waxman-Smits (2)] with the parallel conduction model developed in our introduction, that x and y are given by

$$x = \frac{A_e/A_0}{\ell_e/\ell_0} \quad \text{and} \quad y = \frac{A_w/A_0}{\ell_w/\ell_0} \quad (26)$$

where $A_e/A_0 = \alpha_e\phi_e$ and $A_w/A_0 = \alpha_w(\phi - \phi_e)$ are the constricted fractional volumes of the clay exchange cations and brine, respectively, and $\ell_e/\ell_0 = \tau_e$ and $\ell_w/\ell_0 = \tau_w$ are the tortuosities of the exchange cations and brine, respectively. The fractional volume associated with the clay surfaces is denoted as ϕ_e (not to be confused with “effective porosity” which we do not discuss in this article). The parameters α_e and α_w , τ_e and τ_w representing constrictivity and tortuosity exist, but are not separately measurable.

Waxman and Smits

“assume next that the electric current transported by the counterions⁸ associated with the clay travels along the same tortuous path as the current attributed to the ions in the pore water.”

and write

$$x = y = \frac{1}{F^*}, \quad (\text{Waxman \& Smits, eqn. (3)}), \quad (27)$$

⁸Waxman and Smits use the phrase “clay exchange cations” and the word “counterion” interchangeably. Counterion is a generic term used when polarization is an important phenomenon and one’s attention is focused on one of the ion species; the other ion species is the counterion, which could be of either charge, depending upon where one’s attention is focused. Clay molecules are essentially macro (or mega) ions with a net negative charge. By the principle of electroneutrality, the negative charge is balanced by positive charges (i.e., cations) on the clay crystal surface. These positive charges are potentially mobile (i.e., exchangeable), and constitute “clay exchange cations”.

where F^* is the formation resistivity factor of the shaly sand. This is indeed a bold assumption, made necessary in order to proceed to develop the theory, but which would seem to violate what is visually evident in almost any SEM photomicrograph of a clay rich reservoir rock pore system (See figure 10).

To complete their theory, Waxman and Smits put their equations (2) and (3) together to give (in their notation, where $C \equiv \sigma$)

$$C_0 = \frac{1}{F^*}(C_e + C_w), \quad (\text{Waxman \& Smits, eqn. (4)}). \quad (28)$$

We show in Appendix A (eqn. A.8) that

$$C_0 = \frac{\alpha_e}{\tau_e} \phi_e C_e + \frac{\alpha_w}{\tau_w} (\phi - \phi_e) C_w \quad (29)$$

And (28) and (29) are compatible if, and only if,

$$\frac{\alpha_e}{\tau_e} \phi_e = \phi^{m^*} \quad \text{and} \quad \frac{\alpha_w}{\tau_w} (\phi - \phi_e) = \phi^{m^*} \quad (30)$$

where $1/F^* = \phi^{m^*}$. For these equations to be simultaneously satisfied not only would it have to hold that, under the Waxman-Smits assumption, constriction factors and tortuosities be the same for both phases,

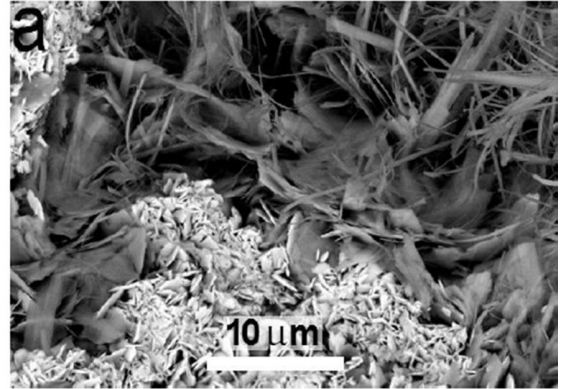


FIG. 10. A scanning electron micrograph of a typical clay-lined pore. When brine-filled and electrically energized, the pore body would be filled with relatively smooth electric current streamlines. On the other hand, the clay surfaces support conduction in a hydrous layer a few molecules thick adjacent to the clay surfaces. Obviously, any counterion current in this layer follows streamlines much more tortuous than streamlines in the pore body and pore throats. (Joanne Welton, 1984, SEM Petrology Atlas).

but that each phase occupy the entire pore space. This would violate the principle of the impenetrability of matter; two objects cannot occupy the same space at the same time. This rock, with x and y equal, would have to be a very special rock, indeed. For other rocks, Waxman and Smits could hold only approximately, depending upon how nearly equal x and y truly are. Thus, (28) cannot be correct.

The fundamental flaw in the Waxman-Smits model is the assumption that $x = y$; as shown above this cannot be correct. Consider equation (28), and ask: if the amount of clay is doubled (or halved) in a pore system, how does this equation change? There are no obvious volume fractions to adjust.

Clavier, Coates, and Dumanior (1977/1984)

Clavier, Coates, and Dumanior, apparently having looked at much data that was not well explained by Waxman's and Smits' model, offered an improved version of the Waxman-Smits model, first in a conference paper in 1977, and later as a much-revised, peer-reviewed, article in 1984. Although they did not frame their objection in terms of logical problems with the Waxman-Smits model, basically their model recognized the necessity of partitioning the pore space into two sub-volumes, one containing the formation brine having a certain salinity and conductivity, and the other containing an anion-depleted, cation-enriched solution close to the clay surfaces; this brine has a conductivity different than the pore-filling formation brine. Thus, they called their model "dual-water". The 1984 version of the paper explicitly frames the problem in terms of fractional volumes. The subvolume of the pore space adjacent to the clay surfaces is expressed in terms of a fractional porosity. The fraction is given in terms of Q_v the concentration of clay counterions per unit pore volume (meq/cc) and the volume occupied by the clay counterions (basically the product of the surface area of the clay crystals and the diameters of two water molecules per unit concentration, v_Q (cc/meq)). Thus the product $v_Q Q_v$ is a unitless fraction of the pore volume $\phi v_Q Q_v$ that we can equate with the quantity ϕ_e from our section on Waxman-Smits. Thus, with $\phi_e = \phi v_Q Q_v$ then the remaining pore volume is $\phi(1 - v_Q Q_v)$ or $(\phi - \phi_e)$ and the Clavier, Coates, and Dumanior (1984) model is written as

$$\begin{aligned} C_0 &= \frac{1}{F_0} (v_Q Q_v C_{cw} + (1 - v_Q Q_v) C_w) \\ &= \phi^m (v_Q Q_v C_{cw} + (1 - v_Q Q_v) C_w) \\ &= \phi^{m-1} (\phi v_Q Q_v C_{cw} + (\phi - \phi v_Q Q_v) C_w) \\ &= \phi^{m-1} (\phi_e C_{cw} + (\phi - \phi_e) C_w) \end{aligned} \quad (31)$$

and it is seen that Clavier, Coates, and Dumanior partitioned the pore space into two complementary sub-volumes. However, the factor ϕ^{m-1} is the geometrical factor, and it remains the *same* for both sub-volumes.

Thus, even as Clavier, Coates, and Dumanior recognize and correct one of the flaws in the Waxman-Smits model, there remains the fundamental flaw that the geometry of the pore bodies and the geometry of the pore surfaces are clearly not the same, this remains unremarked upon by Clavier, Coates, and Dumanior.

Additional information regarding the Waxman-Smits and Dual-Water models can be found in Herrick and Kennedy (2009).

Impasse and a way forward

Is there a way out of this impasse? Maybe. By using an approximation. Herrick and Kennedy have observed that in Archie rocks the pore geometric factor, E_0 , is approximately equal to the porosity ϕ , so that $\phi E_0 = \phi^m$ with $m \approx 2$. If this fact is exploited and assumed to hold for *both* conductive phases, then a conjectured relationship would be

$$\begin{aligned} C_0 &= \frac{\alpha_e}{\tau_e} \phi_e C_{cw} + \frac{\alpha_w}{\tau_w} (\phi - \phi_e) C_w \\ &\approx \phi_e^2 C_{cw} + (\phi - \phi_e)^2 C_w \end{aligned} \quad (32)$$

where $\phi_e = \phi v_Q Q_v$ from the dual-water model.

Concluding Remarks

Following the efforts of Wyllie and Rose, and Winsauer et al., nothing to challenge the Archie model appeared in the petrophysics literature. Indeed, the later shaly sand models were designed to reduce to the Archie model in the limit of zero clay volume. We know of one use of a formula derived based upon percolation theory that appeared in the whole-earth geophysics literature in the mid-1970s (Shankland and Waff, 1974) but seems to have been unknown to

petrophysicists. Attempts to apply the new theory of fractals to petrophysics also met with no success. In 1981 Pabrita Sen made the first attempt to derive Archie's model from what he called "first principles". However, his first principle was the Hanai-Bruggeman effective medium model, and his result is a power law model similar in form to the Archie model, but with $m = 3/2$. Sen's claim seems extravagant to those who do not view effective medium theories as "first principles", and would not claim a successful result unless $m = 2$, at least approximately, was the result. It was not until the late 1980s that David Herrick decided that the way forward was to begin again, ignoring Archie's formula, and ask "How should bulk conductivity of a rock depend upon brine conductivity, fractional pore volume, and pore geometry?" He was able to answer this question with a new model, derived from three self-evident first principles (listed above, page 6) and to which Archie's model reduced in the limit of special pore geometries. However, this model has not aroused much interest in the petrophysics community.

In this article we have tried to call attention to a lack of argumentation that has been prevalent in the theoretical petrophysics literature from its very inception. Perhaps this just reflects the fact that at any one time there were never more than a handful of researchers working on the problem of conductivity in rocks, and these tended, at any one time, to work in a single company. This is a situation that promotes group-think. Obviously our formation evaluation problem is not trivial, but it is not as difficult as many a problem that mankind has put behind itself. Why not this one too? It will take (1) a realization on the part of our young professionals that although people have been doing petrophysics a long time, that does not mean that it is settled science (the daily practice might well be settled, but not the science), (2) a willingness to challenge authority in publication, (3) a taste for debating the logical foundations of models, (4) and a commitment from company managements to support the dialog, not take sides, and provide rewards (or, at least, withhold punishment) to those willing to struggle forward.

As petrophysicists, we sometimes feel a certain envy of geophysicists, whose discipline is more firmly grounded upon physics than our own, and tends to be free of physically undecipherable adjustable parameters having origins in trend analysis. As shown above, it is possible to make some progress in this direction, if only we make the effort to step out of past paradigms, break new ground, and rebuild our discipline with new and firmer foundations.

REFERENCES

- Archie, G. E., 1942, The Electrical Resistivity Log as an Aid in Determining Some Reservoir Characteristics, *Transactions*, AIME, vol. 146.
- Clavier, C, Coates, G., Dumanior, J., 1977, Theoretical and Experimental Bases for the Dual-Water Model for Interpretation of Shaly Sands, *Transactions of the Fall Technical Conference and Exhibition of the SPE*, AIME, SPE 6859.
- Clavier, C, Coates, G., Dumanior, J., 1984, Theoretical and Experimental Bases for the Dual-Water Model for Interpretation of Shaly Sands, *Society of Petroleum Engineers Journal*, SPE 6859.
- Guyod, Hubert, 1952, *Electric Well Logging Fundamentals*, Part 12, Fundamental Data for the Interpretation of Electric Logs, Well Instrument Developing Co., p. 76. (Probable original source is *Oil Weekly*, Vol. 115, No. 38, Oct. 30, 1944.)
- Herrick, D. C. and W. D. Kennedy, 1993, Electrical efficiency: a pore geometric model for the electrical properties of rocks, *Transactions*, SPWLA 34th Annual Logging Symposium June 13-16, paper HH.
- Herrick, D. C. and W. D. Kennedy, 1994, Electrical efficiency-A pore geometric theory for interpreting the electrical properties of reservoir rocks, *Geophysics*, vol.59, No. 6, pp. 918-927.
- Herrick, D. C. and W. D. Kennedy, 2009, On the Quagmire of "Shaly Sand" Saturation Equations, *Transactions SPWLA 50th Annual Logging Symposium*, paper EE.
- Jakosky, J. J., Hopper, R. H., 1937, The Effect of Moisture on the Direct Current Resistivities of Oil Sands and Rocks, *Geophysics*, vol. 1, pp. 33-54.
- Kennedy, W. D., Herrick, D. C., 2012, Conductivity Models for Archie Rocks, *Geophysics*, vol.77, No. 3, pp. WA109-128.
- Leverett, M. C., 1938, Flow of Oil-Water Mixtures through Unconsolidated Sands, *Trans. AIME*, pp. 149-171
- Martin, M., Murray, G. H., Gillingham, W. J, 1938, Determination of the Potential Productivity of

Oil-Bearing Formations by Resistivity Measurements, *Geophysics*, vol. 3, pp. 258-272.

Owen, J. E., 1952, The Resistivity of a Fluid-Filled Porous Body, *Petroleum Transactions*, Vol. 195, AIME, T.P. 3352.

Patnode, H. W., Wyllie, M. R. J., 1950, Presence of Conductive Solids in Reservoir Rocks as a Factor in Electric Log Interpretation, *Journal of Petroleum Technology*, Vol. 1, No. 1, pp. 107-112, AIME T.P. 2797.

Sen, P.N., 1981, The Dielectric and Conductivity Response of Sedimentary Rocks, *Transactions of 55th Annual Fall Technical Conference and Exhibition of the SPE*, Dallas, Texas, Sept. 21-24, SPE 9379.

Shankland, T. S., Waff, H. S., 1974, Conductivity in Fluid-Bearing Rocks, *Journal of Geophysical Research*, Vol. 79, No. 32, pp. 4863-4868.

Waxman, M. H., and Smits L. J. M., 1968, Electrical Conductivities in Oil-Bearing Sands, *SPE Journal*, June, pp. 107-122.

Winsauer W. O., H. M. Shearin, P. H. Masson and M. Williams, 1952, Resistivity of Brine-Saturated Sands in Relation to Pore Geometry, *AAPG Bulletin* 36 (1952): 253-277.

Wyllie, M. R. J., Rose, W. D., 1950, Some Theoretical Considerations Related to the Quantitative Interpretations of the Physical Characteristics of Reservoir Rock from Electric Log Data, *Transactions AIME*, Vol. 189, pp. 105-118, T.P. 2852.

Wyckoff, R. D., Botset, H. G., 1936, The Flow of Gas-Liquid Mixtures Through Unconsolidated Sands, *Physics*, Vol. 7 (September), pp. 325-345.

ACKNOWLEDGEMENT

The authors would like to thank the management of Southwestern Energy for their support and permission to publish this article.

ABOUT THE AUTHORS

David Kennedy has been a student of formation evaluation since 1973 when he entered the industry as a Schlumberger field engineer. It is not an easy discipline to master from theory; as discussed in this paper the theories are rudimentary and, when not explicitly in error, too simple to provide an accurate

description of any pore system. He hopes this article is useful to, and makes life easier for, novice formation evaluators.

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APPENDIX A

Consider two cylinders of different diameter and length each filled with the same conductive brine having conductivity σ_w . (See figure A.1) We will show how to calculate an equivalent conductivity for the pair, both when connected in parallel as individual conductors, and the equivalent conductivity when the same conductors are embedded as porosity in a larger insulating cylinder. Each cylinder has a conductance given by a formula with the form $g = \sigma A/L$. We want to discover the equivalent parallel conduct-

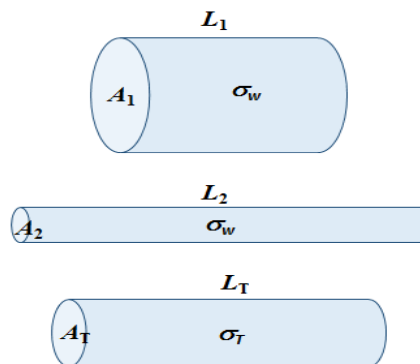


FIG. A.1. Cylinders 1 and 2 made of the same material, but having different dimensions. We require the apparent conductivity of an equivalent cylinder T of the same total cross sectional area and volume.

ance of the cylinders. Let their combined cross-sectional area be given by $A_T = A_1 + A_2$. We need the length L_T of the equivalent cylinder. Its volume will be $A_T L_T = A_1 L_1 + A_2 L_2 = V_T$, so $L_T = V_T / A_T$. These cylinders are illustrated in figure A.1.

The equivalent conductance will be $g_T = g_1 + g_2$ or

$$\sigma_T \frac{L_T}{A_T} = \sigma_w \frac{L_1}{A_1} + \sigma_w \frac{L_2}{A_2}. \quad (\text{A.1})$$

So the parallel equivalent conductivity will be

$$\sigma_T = \sigma_w \left(\frac{L_1/L_T}{A_1/A_T} \right) + \sigma_w \left(\frac{L_2/L_T}{A_2/A_T} \right) = \frac{A_T}{L_T} (g_1 + g_2). \quad (\text{A.2})$$

This is the same as two resistors in parallel, except in this case instead of beginning with known resistance or conductance, we have to calculate the resistance/conductance from resistivity/conductivity and geometry. In terms of geological formations, for example if we consider a model core plug, the terms change, as seen in figure A.2. The situation can still be modeled as two parallel conductors, but the mineral phase conductivity is negligible. However, it has cross-sectional area, A_2 , which is a fraction of the total cross sectional area that will be non-conducting. Hence, the mineral matrix, even though non-conductive, plays a role in the bulk conductivity of the model, and also (of course) in real core samples.

We can estimate porosity in terms of a fractional area of the conducting phase to the entire sample

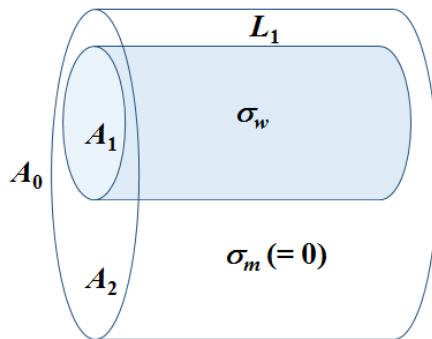


FIG. A.2. Equivalent conductivity in terms of a rock sample. The model has a brine phase of conductivity σ_w and a mineral phase having zero conductivity and cross-sectional area A_2 .

$$\phi = \frac{A_1}{A_0} = \phi_w. \quad (\text{A.3})$$

This porosity can be modeled as being distributed along a tortuous cylinder of constant cross sectional area, as shown in Figure A.3. This is the model considered by Wyllie and Rose (1950).

In reality, the tortuous tube does not have a constant cross-sectional area. An additional parameter is required to account for this feature. Thus, the concept of a “constriction” coefficient is introduced.

Figure A.4 shows how the concept is introduced in steps. Whereas the tubes in figures 1 and 2 are cylinders, in figure 4, the tube at the top is bent, or tortuous. The tube in the center of the set has a linear axis, but has periodic constrictions. The image at the bottom of the figure combines these two effects into a tortuous tube with constrictions. Obviously the cross-sectional area of the conducting tube varies with position along the axis of the tube. Now imagine a core plug with four such tortuous tubes placed in a prism of square cross-section – one in each quadrant. Each tortuous tube is shifted parallel to its axis a random distance so the constrictions are not correlated in any way, but occur at random with respect to the other tubes. Then imagine taking a series of “thin-section” slices moving down the length of the model plug and notice how each tube’s diameter decreases or increases with the successive slices. Figure A.5 illustrates the effects.

Figure A.5a illustrates a thought experiment. This figure diagrammatically depicts a plane normal to the axis of a cylinder, recalling how a thin section would be cut from a core plug. Figure A.5b represents four thin sections cut at random intervals along the prism axis. The notable feature is that porosity of single

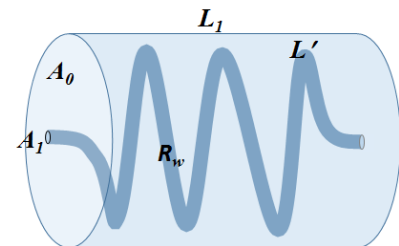


FIG. A.3. Wyllie and Rose (1950) consider a more realistic model with the cylinder of length L having an electric current path length L' , and introducing a tortuosity parameter, $\tau = L'/L$ (a sinuous tube). They formulated the problem in resistivity terms.

thin sections estimated by the “point count” method would underestimate volumetric porosity in every case. This illustrates the meaning of the footnote of Wyllie and Rose quoted in the text on page 8. In figure A.5.c a random slice is reproduced. Consider only the upper left corner. The maximum area of this pore is shown in blue; the area intercepted by the thin section is shown in red. The total area of the core is A_0 . For this thin section the fractional contribution of this pore to the porosity ϕ_1 would be the cross section of the blue circle, A_1 , as a fraction of the total, i.e.,

$$\phi_1 = \frac{A_1}{A_0}. \quad (\text{A.4})$$

However, in the usual case the cross-section of the pore sampled will be constricted; the resulting cross-sectional area, A' , is reduced by the constriction factor at that slice, so we now have

$$\phi_1 = A'/A_0 = \alpha_1 A_1/A_0 \quad (\text{A.5})$$

where α_1 ($0 \leq \alpha_1 \leq 1$) is a constriction factor unique to that slice.

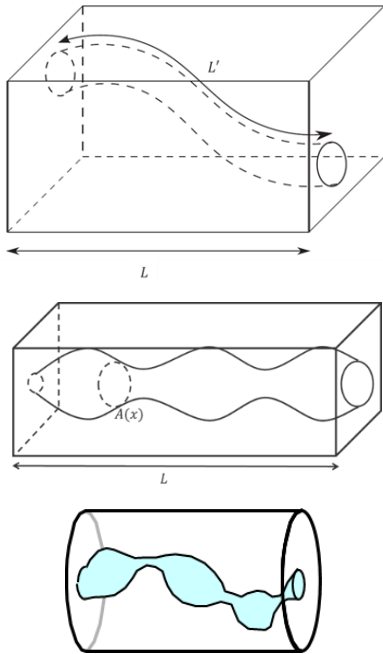
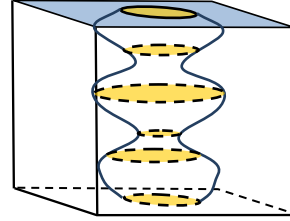


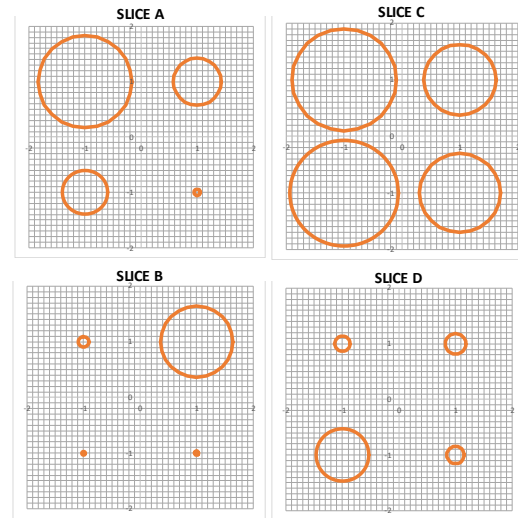
FIG. A.4. A straight cylindrical conducting tube is made tortuous (top). Constrictions are introduced (middle). The bottom diagram is the most realistic model.

Averaging all of the individual slices for all of the tubes in our figure A.5 into a single composite constriction factor α , averaging all the tortuosity param-

(a)



(b)



(c)

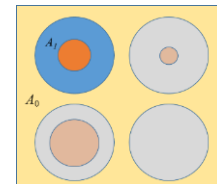


FIG. A.5. a) Examples of thin sections taken at right angles to the axis of three spheres. b) Slices at random intervals along the axis of a core plug of square cross section. Note the “point count” porosity is always a fraction of the actual volumetric porosity. c) The blue (and gray) circles represent the maximum cross-sectional areas along the core axis. The red-shaded areas represent the areas of these tortuous tubes as sampled by the thin section. The conductivity will be diminished by, and controlled by, the smallest areas. Therefore, the effective cross-sectional area of the tortuous tube is not equal to the porosity, but is a diminished fraction of it.

ters for individual tubes into a single composite tortuosity parameter τ , we can define a composite pore geometrical factor E_0 , as

$$E_0 = \frac{A'/A}{L'/L} = \left(\frac{\alpha}{\tau} \right). \quad (\text{A.6})$$

where A and L are the cross-sectional area and length of the sample. Then considering that the conductivity of a brine-filled sample is the product of the conductivity of the brine, the total volume of that brine, and the geometry of the volume that the brine occupies, we have a more accurately defined expression for bulk conductivity, σ_0

$$\sigma_0 = \sigma_w \frac{A'/A}{L'/L} = \sigma_w \phi \left(\frac{\alpha}{\tau} \right) = \sigma_w \phi E_0. \quad (\text{A.7})$$

We now consider our two parallel brine filled cylinders as subvolumes of a larger cylinder. The subvolumes are going to represent dual-water-like quantities, with one of the cylinders representing conduction in the pore bodies and the other presenting conduction along the grain surfaces when clay coated. Both of these phases contribute to porosity and to conduction. Each has a cross sectional area in the “core plug” of A_1 and A_2 respectively. These areas are measured in units of squared length. The core plug itself has cross sectional area A_0 , also measured in units of squared length. The total porosity of the plug will be given by $(A_1 + A_2)/A_0 = \phi$. The model is illustrated in figure A.6.

The effective conductivity will be derived from the sum of the conductances by

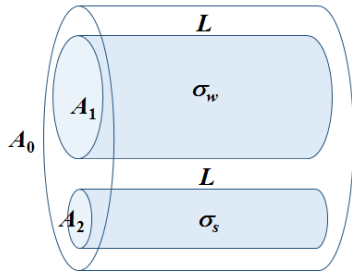


FIG. A.6. Approximating a “dual-water” situation with two effective paths with differing fluid conductivities. Recall the actual tube geometries are to be thought of as seen in the lowermost doodle of figure A.4

$$\sigma_0 \frac{A_0}{L_0} = \sigma_w \frac{A_1}{L_1} + \sigma_s \frac{A_2}{L_2} \quad (\text{A.8})$$

as before. Then

$$\sigma_0 = \left(\frac{A_1/A_0}{L_1/L_0} \right) \sigma_w + \left(\frac{A_2/A_0}{L_2/L_0} \right) \sigma_s; \quad (\text{A.9a})$$

note that this still amounts to a variation on the formula in equation (A.2), viz. $\sigma_0 = L_0/A_0 (g_1 + g_2)$. Let us express the fractional areas in porosity terms. Then

$$\phi = \frac{A_1 + A_2}{A_0} = \frac{A_1}{A_0} + \frac{A_2}{A_0} = \phi_w + \phi_e, \quad (\text{A.10})$$

or, $\phi_w = \phi - \phi_e$, so making substitutions for the ratios of areas, and for the tortuosity τ ratios, L_1/L_0 and L_2/L_0 , then

$$\sigma_0 = \left(\frac{\phi - \phi_e}{\tau_1} \right) \sigma_w + \left(\frac{\phi_e}{\tau_2} \right) \sigma_s. \quad (\text{A.11})$$

Noting that for a rock, the area ratios are not equal to the porosities, but only proportional to them with constants of proportionality α less than 1, and we introduce these parameters into (A.11) to get

$$\sigma_0 = \left(\frac{\alpha_1(\phi - \phi_e)}{\tau_1} \right) \sigma_w + \left(\frac{\alpha_2 \phi_e}{\tau_2} \right) \sigma_s. \quad (\text{A.12})$$

Now, regrouping the terms, we see our standard result emerge

$$\begin{aligned} \sigma_0 &= \left(\frac{\alpha_1}{\tau_1} \right) (\phi - \phi_e) \sigma_w + \left(\frac{\alpha_2}{\tau_2} \right) \phi_e \sigma_s. \quad (\text{A.13}) \\ &= E_w (\phi - \phi_e) \sigma_w + E_s \phi_e \sigma_s \end{aligned}$$

The parallel conductivity model has appeared in terms of two cylinders having differing intrinsic conductivities, different volume fractions of the total porosity, and different geometrical factors, E . Compare to Waxman and Smits (noting that $\sigma_s \rightarrow \sigma_e$)

$$\sigma_0 = E_s \phi_e \sigma_e + E_w (\phi - \phi_e) \sigma_w = x \sigma_e + y \sigma_w, \quad (\text{A.9b})$$

and one can see the structure that should have appeared in the Waxman-Smits x and y coefficients.

APPENDIX B

We have seen that the parameters of the Archie model and the geometrical factor theory are related by

$$m = 1 + \frac{\ln(\alpha/\tau)}{\ln \varphi}. \quad (8)$$

The ratio of constrictivity to tortuosity could be greater than, equal to, or less than porosity. These cases can be correlated with the Archie porosity exponent. For dense rocks, pore throats will be maximally constricted (small α) and tortuosity will be a maximum. The α/τ ratio will have its lowest values. On the tortuosity-porosity plot (figure 5), reducing α corresponds to increasing m .

$$\frac{\alpha}{\tau} < \varphi \Rightarrow \alpha < \varphi\tau \Rightarrow m > 2 \quad (B.1)$$

$$0 \leftarrow \alpha \Rightarrow m > 2 \quad (B.2)$$

For softer rocks, pore throats will be larger compared to pore bodies, and tortuosity will be reduced. In terms of the α/τ ratio, when it is approximately 1, m is approximately 2.

$$\frac{\alpha}{\tau} = \varphi \Rightarrow \alpha = \varphi\tau \Rightarrow m = 2 \quad (B.3)$$

For the case of the softest, least consolidated rocks, pore throats and pore bodies will approach the same size, α will approach 1, tortuosity will approach a minimum value. This will correspond to m values less than 2, also observed on figure 5.

$$\frac{\alpha}{\tau} > \varphi \Rightarrow \alpha > \varphi\tau \Rightarrow m < 2 \quad (B.4)$$

$$\alpha \rightarrow 1 \Rightarrow m < 2 \quad (B.5)$$

APPENDIX C

Recall that $E_0 = \alpha/\tau$. How can φ^{m-1} not depend upon porosity? It depends upon how m is related to porosity and pore geometry. Let $m = 1 + \ln E_0 / \ln \varphi$.

Then $\varphi^{m-1} = \varphi^{1 + \frac{\ln E_0}{\ln \varphi} - 1} = \varphi^{\frac{\ln E_0}{\ln \varphi}}$. Now, taking logarithms, $\ln \varphi^{m-1} = \frac{\ln E_0}{\ln \varphi} \ln \varphi = \ln E_0$, The arguments are equal, thus $\varphi^{m-1} = E_0 \neq E_0(\varphi)$. An alternative

view is that porosity, being a fractional volume, does not depend upon pore geometry. Let

$$\varphi = e^{\frac{\ln E_0}{m-1}} \quad (C.1)$$

Then

$$\ln \varphi = \frac{\ln E_0}{m-1} \ln e \quad (C.2)$$

implies $(m-1) \ln \varphi = \ln E_0$ and $\varphi^{m-1} = E_0 \neq E_0(\varphi)$.