

# Conducting Connected Porosity: A Concept for Unifying Resistivity-Porosity Models

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## ABSTRACT

Archie developed his model of resistivity in hydrocarbon-bearing rocks by applying trend analysis to resistivity index – water saturation data, and separately to formation resistivity factor – porosity data. He then combined the separate trends by eliminating a common parameter,  $R_0$ . This produces a serviceable empirical model for the relationship between brine resistivity, porosity, water-saturation, and the bulk resistivity of a rock. However, Archie is silent regarding any underlying physics which might have supported his model. The industry appears to have been incurious as to the physical foundations of the Archie model for 40 years following its introduction. In 1981, Sen published what he called a “Derivation of Archie from First Principles”. However, his derivation is based upon Tetsuya Hanai’s mixing rule for dilute spheres (mineral grains) in a homogenous background (brine), and in the end the porosity exponent Sen derives is  $m = 3/2$ . It seems questionable whether a mixing rule qualifies as a first principle, and the exponent obtained is quite unrocklike. Results of subsequent attempts to derive the Archie model from first principles by a variety of methods have been equally equivocal.

Kennedy (2007) used Archie’s own Nacatoch sandstone data set to derive, by inspection, an alternative formula with a constant, equal to two, exponent and an adjustable parameter playing the role of a percolation threshold. Kennedy and Herrick (2012), starting with three first principles (or self-evident axioms) derived a similar but not identical formula, also depending upon a percolation threshold-like parameter. Further work has shown that, when porosity is recharacterized as “connected porosity participating in conduction” that, not only can the seemingly different formulas from 2007 and 2012 be reconciled, that the Archie model is also brought into agreement with the two models of

Kennedy and Herrick. The new model, although it includes total porosity as a parameter, recognizes that not all of the brine in the pore space participates in conduction, and that the bulk conductivity of the rock depends not on the total porosity but only on the fractional volume of the conducting porosity. When the first principles are adjusted to account for this non-conducting fraction of the pore volume, the normalized conductivity (i.e., reciprocal formation resistivity factor) is proportional to this “conducting connected” porosity and to a geometrical factor whose form is easily correlated to the “connected” porosity. The result is formally identical to the Archie model with a porosity exponent,  $m = 2$ . With this substitution, the formulas of Kennedy and Herrick from 2007 and 2012 take a form identical to the Archie equation. I will claim that this constitutes the first derivation of the Archie model from unequivocal first principles, making the arguments of Kennedy and Herrick for their formulas also applicable to the Archie model, providing the Archie model a sound physical footing for the first time.

## INTRODUCTION

The relationship between the electrical conductivity and porosity of a brine-filled porous medium is of particular interest in formation evaluation in both petroleum engineering and ground water studies. The problem would seem to be simple: in terms of a porous brine-filled cube, the cube has a conductance that depends upon its dimensions and the brine conductivity. When the brine is progressively displaced by a non-conducting mineral, oil or gas phase, (1) the amount of brine is diminished, (2) the cross-sectional area normal to the direction of current flow is diminished, (3) the connectivity of brine remaining is diminished, and (4) tortuosity and constrictivity are increased; thus the conductance of the cube is progressively diminished. In spite of the seeming simplicity of this system, an explanation for the prediction of its properties has remained a challenge to this day. This article will examine three seemingly disparate formulas that relate formation conductivity to its fractional brine volume, fractional brine volume being identical to porosity

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when water saturation is 100 percent. The formulas are Archie's so-called law, and corresponding formulas proposed by Herrick and Kennedy (1994, 2009), Kennedy (2007) and Kennedy and Herrick (2012). When the problem is properly formulated, these formulas will be shown to be identical. This article has significant tutorial content with the goal of making it accessible to a variety of readers.

**Prior Research.** In formation evaluation for petroleum the industry-standard has been the relationship suggested by G. E. Archie in 1941 (published in 1942). In conductivity terms this is

$$\frac{\sigma_t}{\sigma_w} = \varphi^m S_w^n \quad (1)$$

where the usual notational conventions,  $\sigma$  for conductivity,  $\varphi$  for porosity, and  $S_w$  for water saturation, are employed. The superscripts  $m$  and  $n$  are adjustable parameters in Archie's model, usually approximately equal to 2, and the subscripts  $t$  and  $w$  label the bulk rock and pore-filling brine conductivities, respectively. For  $S_w = 1$ , the Archie relationship reduces to

$$\frac{\sigma_0}{\sigma_w} = \varphi^m, \quad (2.a)$$

where the subscript 0 replaces  $t$  to indicate that the rock is completely brine filled. A modification to this formula

$$\frac{\sigma_0}{\sigma_w} = a \varphi^m, \quad (2.b)$$

was first documented by Owen (1952), and this is the form of Archie's model most frequently employed in modern formation evaluation.

However, relations other than Archie's have been suggested, although none has become commonly used in formation evaluation. Without regard to notation for specific variables, note the similarity in form of, for example, Shankland and Waff (1974)

$$\frac{G}{G_0} = \left( \frac{p - p_0}{1 - p_0} \right)^r; \quad (3)$$

Ewing and Hunt (2006) (with a parameter  $a \approx 1$  multiplying the right side)

$$\frac{\sigma(\theta)}{\sigma_b} = a \left( \frac{\theta - \theta_c}{1 - \theta_c} \right)^\mu; \quad (4)$$

and Montaron (2008)

$$\frac{\sigma}{\sigma_w} = \left( \frac{p - p_c}{1 - p_c} \right)^t, \quad (5)$$

where, except for differences in notation, these expressions are all isomorphic to Kennedy (2007); i.e.,

$$\frac{\sigma_0}{\sigma_w} = \left( \frac{\varphi - \varphi_g}{1 - \varphi_g} \right)^2. \quad (6)$$

where  $\varphi_g$  has been called a critical porosity, or percolation threshold; for conductivity in rocks it is more properly called a pseudo-percolation threshold. The similar morphology of Shankland and Waff, Ewing and Hunt, and Montaron is due to their all having provenance in percolation theory, and regardless of notational differences, the quantities in corresponding positions in the formulas have the same meanings as in (6). The Kennedy (2007) formula is derived by inspection of G. E. Archie's Nacatoch sandstone porosity-resistivity data set. Kennedy conjectured that these data satisfy a differential equation (i.e., derivative of normalized conductivity with respect to connected porosity is proportional to connected porosity) subject to boundary conditions that  $\sigma_0 = 0$  at  $\varphi = \varphi_g$  and  $\sigma_0 = \sigma_w$  at  $\varphi = 1$ . The differential equation solved subject to these boundary conditions yields equation (6) above. The convergence of the work of all these authors to similar formulas lends the formulas a certain credibility.

On the other hand, Herrick and Kennedy (2009) imply, and Kennedy and Herrick (2012) explicitly show that, beginning from first principles and without reference to any particular data, that subject to the conditions assumed to apply to conductive rocks and the boundary conditions mentioned above,

$$\frac{\sigma_0}{\sigma_w} = \varphi \left( \frac{\varphi - \varphi_g}{1 - \varphi_g} \right). \quad (7)$$

This formula, being grounded in physical arguments, makes a strong claim as being correct. Moreover, although it is derived without any use of percolation theory percepts, this expression is in agreement with the

pioneering work of Kirkpatrick (1973) in percolation theory.

The left sides of these formulas express bulk rock conductivity normalized by brine conductivity. The right sides all depend upon porosity, but in three different ways. So the question naturally arises, is there a way to reconcile these formulas (i.e., (2), (6) and (7)), or at least, to understand the source of the difference? I shall answer this in the affirmative, and offer to you an explanation below.

Since resistivity in reservoir rocks seems to be so poorly understood, let me begin by offering a thorough tutorial review.

## BACKGROUND

**Continuum Models and Cartoons: A Thermodynamic Analog.** What is a model? The reductionist point of view in physics holds that if reality is viewed on a large enough, or small enough, scale, then it should be simple to understand and explain. The kinetic theory of gases provides an example. At the scale of molecules, the pressure of a gas pushing on the walls of a container can be explained by the change in momentum of a gas molecule as it collides elastically with the container wall. It is simple enough to understand for a single molecule hitting a single wall. The pressure exerted on the wall at any one instant is simply due to the sum of all of the changes in momentum of all of the molecules striking the wall at that instant. The trouble is that the number of molecules in any macroscopic system thwarts any attempt at practical record keeping. On the other hand, at a large enough scale, one can forget about individual molecules and their momenta and consider instead a single number to characterize the system; i.e., its pressure (assuming that the temperature, volume, and number of molecules are all held constant). And there you have both points of view, and both are simple. However, neither model captures the complexity of the scales intermediate between the macroscopic ideal gas law ( $PV = nRT$ ) and the microscopic conservation of momentum for a single molecule and container wall  $d(m_m v_m + m_w v_w)/dt = 0$ . In thermodynamics, these scales are logically and rigorously connected through statistical mechanics. When stated in terms of moles of gas, the quantities in the ideal gas law are considered as continua. In particular, the quantity of gas is *uniformly* distributed throughout the entire volume of

its container without any concept of discrete particles. The unit of the gas constant,  $R$ , is joules per mole per degree kelvin. Even when the right side is expressed in terms of the number of molecules,  $N$ , ( $PV = kNT$ ), the Boltzmann constant having units of joules *per molecule* per degree Kelvin, each molecule is assigned the average energy of all of the molecules in the volume  $V$ , and the gas is still treated as a continuous fluid. So, even when considering the ideal gas on an energy per molecule basis, the molecules are not localized in space but distributed as a continuum throughout the container.

The situation in petrophysical modeling is similar. Rocks are partitioned into solid and fluid phases, the fluids being contained in “pores”, analogous to the “molecules” of the system. The pores interact through pore “throats”, analogous to the interaction of gas molecules by mutual exchanges of momentum during collisions. People make models at this level, often illustrated by cartoons. However, there are millions upon millions of interacting pores. To understand the macroscopic behavior of this system in a tractable model, an analog to the ideal gas equation of state is needed. This analog must somehow capture the overall behavior of the system in terms of a few macroscopically observable quantities, similar to pressure, volume, and temperature in the ideal gas law. For the conductivity property such an analog “equation of state”<sup>2</sup> is  $f = \phi E_0$ , where  $f$  is a normalized conductivity, the ratio of bulk rock conductivity to brine conductivity  $\sigma_0/\sigma_w$ , usually called the formation conductivity factor,  $\phi$  is porosity, and  $E_0$  is a geometrical factor<sup>3</sup> characterizing the effect of the geometry of the pore system exclusive of the pore volume upon the conductivity. Note that all of the parameters vary between zero and one. The physical rock comprises two or three interpenetrating and through-going phases, one conducting, consisting of the brine in the pore space, and a non-conducting rock-forming mineral phase and sometimes a non-conducting hydrocarbon fluid phase. However, since the model is a continuum model, each of these quantities must be thought of as being continuously distributed throughout the volume. For example, Ohm’s law in its field-theoretic form, where a current density  $\mathbf{J} = \sigma \mathbf{E}$ , is induced to flow in a medium having conductivity  $\sigma$  by an applied electric field  $\mathbf{E}$ , applies both at macroscopic and pore scales as long as the media are considered to be homogeneous; it does not apply to individual charge carriers because they are immersed in sea

<sup>2</sup>The ideal gas law expresses relations among pressure and volume, concentration and temperature, in terms of energy, a conserved quantity; the proposed analog is not a relationship among state variables and thus is not an “equation of state”, merely an analog.

<sup>3</sup>The value of  $E_0$  for any rock sample is computed from  $E_0 = f / \phi = \sigma_0 / (\sigma_w \phi)$ . When  $E_0$  from an ensemble of samples is plotted against  $\phi$ , then  $E_0 = a_0 \phi + b_0$ . See Herrick and Kennedy (1993, 1994).

of individual particles, not a continuum. This means that care must be exercised in the joint interpretation of a cartoon-like model of a single (or a few) discreet charge carrying ion(s) and the distributed parameters of a continuum macroscopic model.

Perhaps the most misleading cartoon representation is a drawing of the path of a single migrating ion moving through the pore system. (See Appendix A for a thorough discussion of the several ways this cartoon image is very idealized.) This leads to a mental image of a point particle following a tortuous path through a porous medium, which the single ion (in an ideal medium) most assuredly would do, but this image totally neglects the influence of variable cross sectional area of the path. This cartoon misleads the unwary viewer into thinking that it is *only* tortuosity that contributes to reduction in conductivity, which is not true. Constriction in pore throats plays a comparable, if not greater, role. And, when all such current carriers are considered together to estimate an average current density  $\mathbf{J}$ , that average current density is considered to be distributed throughout the volume, including inside of non-conducting mineral grains.

**Percolation Theory.** According to Wikipedia<sup>4</sup>, “percolation theory describes the behavior of connected clusters in a random graph.” Physicists have studied the conductivity of resistor networks built on cubic lattices as resistors are added (or removed) at random. The formula I use for conductivity in rocks is formally identical to conductivity in resistor<sup>5</sup> networks studied in percolation theory. However, there are differences between conduction in rocks and conduction in resistor networks that must be appreciated. Near the onset of percolation, the conductors in resistor networks and the conducting phase in a rock are configured quite differently. The model that percolation theory employs for the study of the bulk conductance of random networks is resistors connected in a cubic lattice. In the interior lattice, the edges of eight cubes intersect at a point known as a node or site. The edges of four cubes intersect in a line known as a bond. Six bonds radiate from each site: up-down, north-south, east-west. Conductors occupy the bonds and are connected together at the sites. A numerical experiment begins with all of the bonds occupied and the conductance of the network recorded. Then a bond is removed at random, and the fraction of occupied bonds in the network is

recorded. The random removal of bonds and recording of fractional occupation continues until at one step the conductance vanishes. The fraction of bonds occupied at this step is called the percolation threshold, and marks a “phase change” in the network from conducting to non-conducting. For a large enough network, the fractional occupation of bonds when conduction vanishes is always the same, although the resistor configurations are random and would appear to be quite different in detail. This is called the bond percolation model. There are variations on the experiment. Obviously the experiment can be performed in reverse, starting with all of the bonds vacant, and adding conductors at random. In this experiment, the network begins conducting at the percolation threshold which will be at the same fractional occupation as in the random removal experiment. A variation known as the site percolation model begins with the fully occupied bond network, and removes sites at random, recording the fractional occupancy at each step. This removes six bonds at a time. The qualitative behavior of the site experiment is similar to the bond experiment, but the percolation threshold is slightly greater than for bond percolation.

Although this experiment is easily studied and understood, and provides a degree of insight, it is a poor model for conducting networks in rocks. In the random resistor network model, isolated groups of conductors, called clusters, can form as conductors are randomly removed from the network. These are disconnected from the through-going network of conductors, and do not contribute to its conduction, but are still counted in the fraction of occupied bonds. If this model is taken seriously as a surrogate for a rock, the isolated clusters would correspond to isolated porosity. But, this kind of isolated porosity does not develop or exist in sedimentary rocks.

Sandstones begin as fluid-saturated mixtures of mineral grains and brine with all of the fluid in the pore space connected. Diagenesis (compaction and cementation, dissolution and recrystallization) reduces the amount of conducting fluid in the rock, but does not disconnect any of it, at least not until the very final stages of diagenesis. The exception, which is normally volumetrically insignificant, is fluid inclusions which are microscopic bubbles of pore fluid trapped during secondary mineralization such as in quartz cement.

<sup>4</sup> [https://en.wikipedia.org/wiki/Percolation\\_theory](https://en.wikipedia.org/wiki/Percolation_theory)

<sup>5</sup> The passive elements in circuit design are known as inductors, capacitors, and resistors. The name for the resistor, in particular, is a matter of custom. These circuit elements might just as well be called “conductors”, and I will use the terms interchangeably. Conductor is the better terminology for discussing parallel networks because

the total conductance is the simple sum of the conductances connected in parallel. However, the term “resistor” is so customary and entrenched, that by habit, I also use this term freely. No confusion should arise; resistors and conductors refer to one and the same circuit element. (Capacitors were once called “condensers”.)

Whereas the percolation thresholds for resistor networks are in the neighborhood of 0.3, for rocks they are much lower, approaching zero. Moreover, in low-porosity rocks the conducting brine remains a continuous connected phase, interrupted only by the occlusion of pore throats as diagenesis progresses and reduces porosity. In both conductor networks and rocks there are regions in the connected network that do not contribute to conductivity, such as resistors connected to the through-going network at only one node, or dead-end pores. In random resistor networks connectivity increases with an increase in the number of bonds added at random to the network. Likewise, in the rock, mineral grains will cease to be in contact above about 0.45 volume fraction of brine, and at that point all porosity has to be physically “connected”; there is no dead-end pore space and the entire fluid volume will be contributing to the bulk conductance to greater or lesser extents. In summary, the percolation thresholds in rocks and random resistor networks are very different, brine in the rock remains fully connected down to very low porosity values without physically isolated brine ever developing, and toward the high porosity side of the porosity axis, all of the brine in a rock is potentially contributing to conduction at a porosity of 0.45 or perhaps even less. In other words, a rock is not well-modeled by a cubic resistor network. So, although the formulas for conduction are formally identical, the configurations of the conducting phases in the medium at the percolation threshold are quite different.

However, the percolation model is useful as an analog. It seems reasonable that a pore network in a rock can be modeled by an aperiodic network of nodes and bonds, with nodes representing pore bodies and bonds representing pore throats. Thus, the formulas developed using periodic networks, and the names and interpretation of the parameters in the formulas, can be (with some caveats) retained. The values of the “universal” parameters of percolation theory developed on regular lattices, however are not necessarily expected to, and are observed not to, apply. These will play the role of adjustable parameters in rock models formally identical (or isomorphic) to percolation theory formulas.

**Pseudo-Percolation Threshold Model.** Percolation theory examines the connectedness of networks and is widely applied across many disciplines. As used in discussions of the conductivity of rocks, a *percolation threshold* (PT) is the value of porosity at which the through-going network of brine in a rock’s pore system is interrupted by a diagenetic reduction in porosity, at which point the rock is no longer capable of

through-going conduction of electricity at zero frequency. The evolution of a percolation threshold for conductivity in rocks would be difficult to observe in a laboratory experiment, and impossible to observe in nature. However, conductivity-porosity trends established at porosities well above the percolation threshold of a rock can be extrapolated to zero conductivity where the trend will typically intercept the porosity axis at some porosity value different from zero. This intercept is referred to as a *pseudo-percolation threshold* (PPT) to distinguish it from the unobserved and unobservable actual percolation threshold. It is useful as a curve-fitting parameter in that it has a quasi-physical interpretation, with the caveat that it is not to be identified with a real percolation threshold, about which nothing is known for a collection of real rock samples. Defining the normative reduction in conductivity to be the vanishing of conductivity with porosity (a pseudo-percolation threshold of zero), a pseudo-percolation threshold can be positive (representing a conductivity deficit with respect to the norm) or negative (representing a conductivity excess), or zero in special cases. The model based upon the PPT is called, for convenience, pseudo-percolation threshold theory (PPTT).

**Fitting Curves to Data.** When predicting reservoir properties using the correlation of an easily measured property with another, difficult-to-measure, property in order to estimate the difficult to measure property using the easily measured property, the most desirable circumstance is to have data on both properties measured with reliable laboratory techniques on reliable samples such as core plugs from the reservoir in question. Using core plugs, measurements of many properties can be cataloged. In this fortunate circumstance one is free to inspect the various possible correlations among the data and choose the most convenient correlation for use in a trend analysis; that is, the most convenient formula for predicting the dependent variable from the independent variable. If there is a known physical relationship between the variables (e.g., the density-porosity transform) then it makes sense to choose the physical function to describe the trend. This has the advantage of providing a priori physical interpretations for the adjustable parameters of the trend. If the correlation is based purely on a trend observed qualitatively by eye or obtained using a mathematical curve-fitting method there is no particular reason to choose a previously published fitting function merely because it has been published. Doing this amounts to using one data set (i.e., in the publication) to characterize a second data set which may have entirely different properties. Despite this seeming disadvantage, practicing petrophysicists have done this every day

since 1942; i.e., this happens when a formation evaluator adjusts the parameters of G. E. Archie's model to minimize the sum of squared residuals between his observed data and the predictions of the model  $F = a\phi^{-m}$ . It is often presumed that this formula is representative of some physics, but this presumption is unfounded. Although in favorable circumstances Archie's model makes useful predictions of resistivity given porosity, it embodies no explicit physics since Archie's equation was obtained purely by curve fitting with no consideration given to any underlying physics. It is strictly a trend predictor. If data taken on physical samples is available, rather than using Archie's model, one has the option of exploring the universe of fitting functions and should consider doing so in search of the most advantageous fit.

In a sense, the value of Archie's work has been misunderstood. He demonstrated that a correlation could be obtained between porosity and his formation resistivity factor. Rather than emulate his example, people typically apply the correlation he found to other data, not from the same suite of samples he studied. His real lesson to formation evaluators is to look for the best correlation in one's own data and use that correlation. Fortunately for historical log analysis, many rocks have been sufficiently similar to the samples studied by Archie (clean sandstones) that the fitting equation he discovered has proven useful, providing that the fitting parameters  $m$  and  $n$  are determined for the specific rocks under study.

Although it is not evident that Archie used mathematical curve fitting such as the "least squares" method, least squares is most often used to obtain correlations. It is underappreciated that there is nothing special about a least squares fit of data. Drawing a line "by eye" might in fact give a good estimation of the trend in the data. The disadvantage is that no two people are likely to draw exactly the same line. Least squares fitting provides an algorithm that removes this personal bias and provides an objective measure of the magnitude of uncertainty. Everyone inputting identical data to the least squares algorithm will get the same result. This does not mean that the result is the "best" result, except in the "least squares sense". In statistical analysis the least squares method is referred to as the "L2 norm". The L2 norm minimizes the sum of the squared residuals between observations and the fitted function. The L1 norm minimizes the sums of the absolute values between observations and the fitted function. As suggested by the notation, there is an entire family of norms ranging from the L0 to the L- $\infty$  norm. The point here is, if you have data that you want to use to make a predictive model, you have options other than least

squares that may provide predictions superior to least squares.

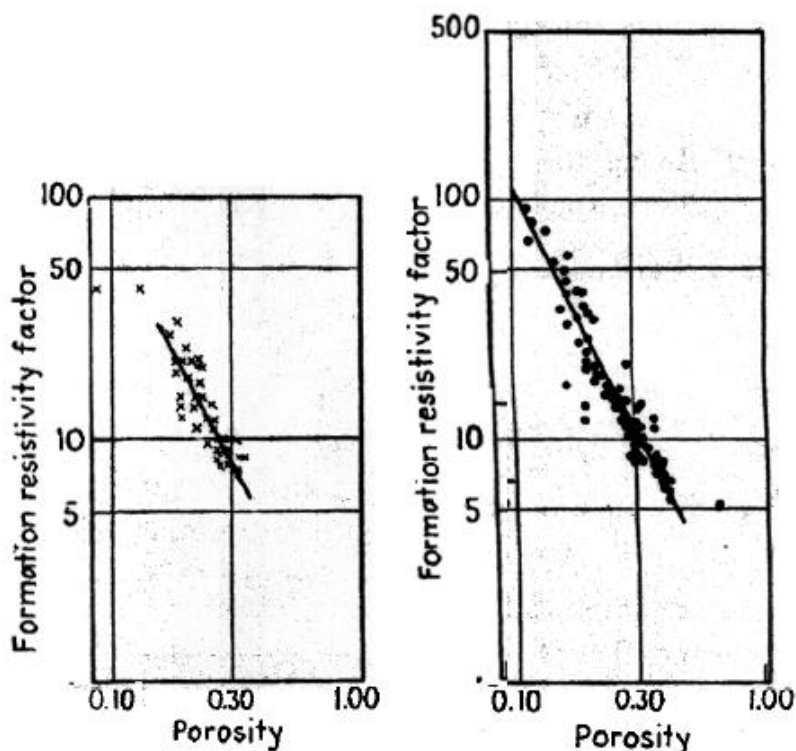
**Resistivity in Brine-Filled Sedimentary Rocks.** This article is about how resistivity and conductivity in rocks change with porosity. It is presumed that the reader is familiar with the assumptions and notations customarily used by petrophysicists in the discussion of this topic. To briefly review, the rocks to be discussed consist of non-conductive mineral grains (e.g., quartz, calcite, dolomite, feldspars, etc.) in intimate contact, and usually (but not always) bound together by non-conductive "cement" comprising silica (quartz), calcite, etc. The void spaces between the grains are fluid-filled, the possible fluids being non-conductive gas and oil and conductive brines. Brine is the only conducting phase in the pore space and, indeed, the entire rock. The brine typically is the wetting phase, covering the surface of every mineral grain exposed to the pore space. Any non-conducting fluid phase fills the central regions of the pore spaces, always leaving a thin aqueous film on the grains. The resistivity of the brine is denoted  $R_w$ , the resistivity of the bulk rock containing the brine is denoted  $R_0$  if completely brine filled, and  $R_t$  if a combination of brine and non-conductive fluids fills the pore space. The ratio  $R_0/R_w = F$  is defined as the formation resistivity factor and is constant for a given rock sample. In this article porosity is denoted as  $\phi$ . The seminal relationship among these quantities (discovered by G. E. Archie in 1941) is  $F = \phi^{-m}$  where  $m$ , an exponent on the porosity, is customarily called by the misnomer "cementation exponent". This article prefers the reciprocal electrical quantities  $\sigma_w$  formation brine conductivity,  $\sigma_0$  bulk brine-filled formation conductivity, and  $\sigma_t$  bulk brine and hydrocarbon filled formation conductivity. The conductivity counterpart to formation resistivity factor  $F = R_0/R_w$  is formation conductivity factor  $f = \sigma_0/\sigma_w = 1/F$ , which is also denoted as "normalized conductivity"  $\sigma_n$ . Thus,  $\sigma_n = \phi^m$  is Archie's law in terms of conductivity, where  $m$  is properly called the "porosity" exponent.

**Trends and Cross Plots.** In this article I will elaborate the relationship between the porosity and the zero-frequency electrical property known as resistivity, and its alter ego, conductivity. By "relationship" is meant a mathematical function where one puts a porosity value into a formula and, after doing the computational task, a value for a resistivity or conductivity is obtained (or vice versa). Such functions are found by fitting various curves to observed data. Unfortunately, these data are often characterized by a tendency to *not* define a trend;

i.e., there is typically a large amount of variation in resistivity or conductivity for a given value of porosity. Consequently, no trend function chosen can accurately predict, let alone reconstruct exactly, any individual point in the data used to construct the trend function. However, this shortcoming is accepted as the best that can be done in a predictive sense. A prediction is made with some, usually unquantified, uncertainty in the result being accepted. From a practical point of view, the choice of the function used to fit the data is immaterial. Parameters can be found for almost any smooth function that will render the difference in residuals (i.e., difference between observations and predictions) between it and any other function negligible. In other words, when the functions are plotted on the same

graph as the data, the locus of the lines can be made to be almost the same. However, when functions are extrapolated outside the range of observations, the situation is very different. Arbitrarily chosen functions will typically rapidly increase, decrease, or oscillate outside the range of observations. When this is the case, although an arbitrary function might make excellent predictions within the domain of the observed data, it cannot be used for extrapolation outside the range of the observations, and there is no physical interpretation that can be given to the parameters of the function.

**Variability in Observations.** Inspection of figure 1 reveals that data points plotted fail to fall on, or even near, a single line. Why is that? In the first place, the



**Figure 1.** Formation resistivity factor versus porosity plots excerpted from Archie's (1942) figures 1 and 2. Archie never published this data in tabular form; it exists in digital form today because it has been painstakingly hand-digitized (Kennedy, 2007). The data in the left figure is from "Gulf Coast Sandstones"; today all will acknowledge that each formation is likely to be unique and mixing samples from different formations does not lead to a trend analysis that can be meaningfully applied to any of the formations contributing data. Therefore, the data in the left plot (which is uncorrelated in  $E_0$ - $\phi$  space) have not been used in this analysis. In the graph on the right, the data are taken from the Nacatoch Sandstone formation near Bellevue Louisiana, northeast of Shreveport. The basis of resistivity-porosity evaluation for the past 75 years has been based upon plotting these data on bi-logarithmic (or log-log) graph paper and writing a formula for the line drawn through the trend. The Nacatoch Sandstone data forms the basis of the numerical examples used in this article because (1) it is in the public domain, and (2) to illustrate that Archie's analysis of these data is not unique in any way; although he designed a certain model, long the industry standard, the model was not thrust upon him by the data but engendered more by the graphical data analysis practices of his era; that is to say plotting data on a variety of graph paper types until a straight line is obtained which is easily fit by eye.

variation is much too large to attribute to experimental error. Experimental uncertainty (called by the misnomer “error”) does exist, but as revealed by water-saturation – resistivity-index plots (made on a single core as its water saturation is diminished by the experimenter from  $S_w = 1$  to as close to zero as is possible, the resistivity-index –  $S_w$  data pairs define a line or curve segment almost perfectly, without a great amount of scatter, demonstrating that resistivity measurements can be made with negligible error. So, if not from experimental error, then whence the source of scatter?

In one view, the observed scatter represents irreducible variability in reservoir properties. But if that is the case, it makes no sense to make predictions based upon some average line fitted to the scattered data; in general, such predictions will fail individually. Only the averages of many individual observations will tend to be correct. A different view of the matter is that a core plug is too small a volume to be representative of large volumes of reservoir. A representative volume (larger than a core plug) would be large enough to contain enough of the rock’s intrinsic heterogeneity so that its average properties would be the same regardless of where it is located in the reservoir. If the trend could be constructed from data measured upon these larger, representative rock volumes, the points defining the trend derived from the larger samples would lie much closer to the trend line, and predictions made from the trend would be valid for large reservoir volumes. Such predictions would, as before, not necessarily be correct for core-plug size volumes, but would become progressively more correct as the sample volumes considered increased to the size of a representative volume.

**Boundary Conditions.** Theoretically, porosity varies between two extremes: zero porosity and 100 percent porosity, or expressed as a fractional volume, porosity is bounded by 0.0 and 1.0. Formation resistivity factor varies between 1 (a friendly number) at 100 percent porosity and infinity (a not-so-friendly number) at some porosity close to or equal to zero porosity. When plotted on log-log graph paper, neither infinite formation resistivity factor nor zero porosity can be represented. The situation is more benign when the formation conductivity factor ( $f = 1/F$ ) is considered. Note that  $f$  is bulk rock conductivity  $\sigma_0$  “normalized” by brine conductivity  $\sigma_w$ . There are many notations in use for this normalized conductivity; for later convenience define  $\sigma_n \equiv \sigma_0/\sigma_w = f = 1/F$ . This ratio varies between zero and one as porosity varies between zero and one. At the right boundary of a graph of normalized conductivity versus porosity, normalized conductivity equals one when porosity equals one. While 100

percent porosity cannot be considered to be a rock, it is the bona fide end member of a mixture of brine and non-conducting mineral grains. The situation on the left boundary is more complicated because (1) conductivity may go to zero (vanish) while there is still some porosity (albeit disconnected) remaining in the rock, or (2) conductivity may not vanish if the rock has some conductive mineral phase as part of its fabric, or (3) conductivity may vanish at zero porosity. The right boundary is always outside the domain of rock sample data and, usually, so is the left boundary. There is a point of view that holds that all interpretation methods should confine themselves to the domain of actual data with extrapolation outside the range of observations viewed as illegitimate, or at least to be avoided. However, if it is desired to design a porosity – conductivity function (for theoretical, as opposed to practical, purposes) that applies across the entire porosity domain, then the boundary conditions have to be honored and become a part of the design of the function. The “right” boundary condition holds at the right edge of the porosity-conductivity graph; the condition being that normalized conductivity equals one when porosity equals one. The “left” boundary condition holds at very low porosity values (not necessarily zero) and is that normalized conductivity “vanishes” at some small value of porosity, known as a (pseudo-) percolation threshold. Once zero normalized conductivity is attained, it is maintained for all values of porosity less than the percolation (and pseudo-percolation) threshold; i.e., conductivity cannot be less than zero.

## Two Methods

In the sections of this article to follow, let me distinguish two methods of learning and knowing. The first is called the Baconian method. The Baconian method, (Francis Bacon 1561-1626) as advocated in Bacon’s *Novum Organum Scientiarum* (*New Instrument of Science*), is where the tenants of the modern scientific method, based upon inductive reasoning from observations made in nature, are first explicitly laid down.

This is contrasted with the Euclidian method, after the author of *Στοιχεῖα* (*Elements*) (Euclid c. 330 B.C.). In *Elements* Euclid establishes plane geometry based upon a collection of five postulates (e.g., postulate 4, all right angles are equal) and five axioms (e.g., axiom 5, the whole is greater than a part), supposedly “self-evident” but not capable of proof, pertaining to points and lines. From these few axioms and postulates, all of the theorems of plane geometry are derived without further reference to the physical world. An example of Euclid’s method closer to our time is classical Newtonian mechanics, which is based upon Newton’s three



(“self-evident” from observations of nature) laws of mechanics, and his postulated gravitational force law.

**Resistivity-Porosity Analysis by the Baconian Method.** The quantitative interpretation of resistivity-porosity data began in 1941 with Archie’s observation of the power law relationship between porosity  $\phi$  and the formation resistivity factor,  $F = R_0/R_w$ . Archie’s data are reproduced in figure 1.

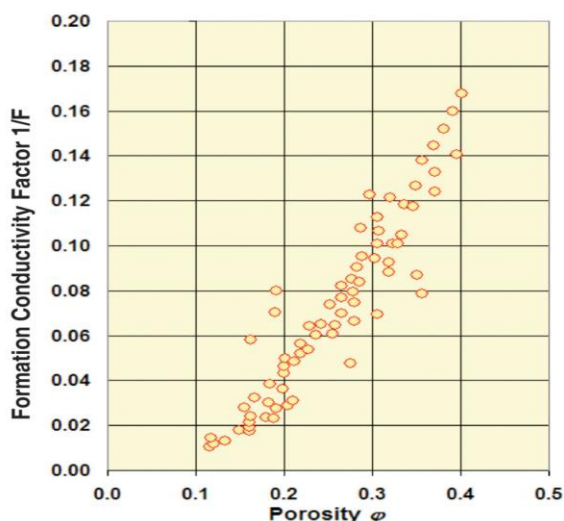
Archie’s trend analysis using log-log graph paper was so useful that the technique immediately became entrenched in the oil industry, well before any attempt was made to understand the physics underlying the function. And, of course, it is still the basis of formation evaluation today because it works so well in so many cases; moreover, many modifications have been invented to make Archie-like formulas work in rocks different from those Archie studied. However, it has never been adequately and unequivocally established that the Archie model as it applies to rocks can be derived from any first principle. Nevertheless, it can be said to be a very successful example of empirical curve fitting.

The Archie model has had limited use in building physical intuitions (indeed, it is the source of much confusion) because of the inverse proportionality of

the logarithm of  $F$  to the logarithm of porosity,  $\phi$ . Further, the situation is not helped by  $F$  tending to infinity at low porosity values. When the Archie relationship is reformulated in terms of conductivity it becomes much easier to understand. Archie rocks are conductive due to the brine that they contain in their pore spaces. The saltier the brine, the more conductive the rock; further, the greater the quantity of brine, the more conductive the rock. As the brine volume (i.e., porosity) tends toward zero, so also the conductivity tends toward zero; likewise, as porosity increases, conductivity increases. These are easily grasped concepts. The question is: *exactly* how does conductivity increase as porosity increases (for a constant salinity brine). It is instructive to look at the Archie Nacatoch Sandstone data for clues.

Similarly to Archie’s definition of the formation resistivity factor, it is convenient to use the formation conductivity divided by brine conductivity ( $\sigma_n = \sigma_0/\sigma_w = 1/F \equiv f$ ) as a parameter. This is called this the normalized conductivity, or formation conductivity factor. Figure 2 shows the Nacatoch Sandstone data plotted as normalized conductivity versus porosity with linearly scaled axes. Consider that the scatter observed in these data is not the result of measurement errors but instead reflects the natural variability in conductivity found in rocks; i.e., rocks in the same formation having the same porosity can have any number of conductivity values scattered about a central trend. In interpreting conductivity (or resistivity) – porosity data sets, one learns to live with the uncertainty in predictions of conductivity made using a function that represents a trend in the conductivity and not a bona fide function of porosity.

The data in figure 2 could be crudely fitted with a straight line; however, the data suggest a trend whose slope increases to the right. In other words, the higher the porosity, the higher the slope, and vice versa. That is, the trend could be well-fitted by a curve. Note that such a curve for these data does not appear to pass through the origin at zero porosity and zero conductivity. It would appear that conductivity goes to zero (i.e., “vanishes”) at some value of porosity greater than zero. In other words, this hypothetical trend suggests that there is brine remaining in the rock at the point where the conductivity vanishes. Conductivity vanishes when the last connected path through the entire pore space is occluded by cement and/or compaction, recrystallization, etc. The name given to the value of porosity at which the through-going connections are interrupted is “percolation threshold”. Already mentioned is that the true percolation threshold of a rock is unobservable. The analogous quantity used in the



**Figure 2.** The trend in the linear plot of formation conductivity factor vs. porosity is arguably less evident than when plotted on log-log paper, but lends itself to a quasi-physical interpretation. The trend posited by Kennedy (2007) is a parabola with vertex somewhere on the porosity axis to the left of the data.

PPTT model is properly called a “pseudo-percolation threshold”, denoted by  $\varphi_g$ . Total porosity  $\varphi$  and connected porosity  $\varphi_c$  are related via the percolation threshold porosity  $\varphi_g$ ; i.e.,

$$\varphi_c = \begin{cases} 0 & 0 < \varphi < \varphi_g \\ \frac{\varphi - \varphi_g}{1 - \varphi_g} & \varphi_c \leq \varphi \leq 1 \end{cases} \quad (8)$$

This function is taken to be zero for  $\varphi < \varphi_g$ . For  $\varphi > \varphi_g$  in a rock this represents the fraction of connected brine, or the connected porosity. The normalization factor  $1 - \varphi_g$  in the denominator is required to insure that the connected porosity is 1 when porosity is 100 percent; i.e., by design, the connected porosity as modeled by (8) approaches 1 as total porosity approaches 1. Not all of the brine in the connected porosity participates in conduction. At, and near, the percolation threshold, much connected brine is situated in locations that do not allow it to participate in electrical conduction; however, as porosity increases, the amount of non-conducting porosity diminishes. This suggests that the pseudo-percolation threshold parameter  $\varphi_g$  ought not be constant, but should depend upon porosity; i.e.,  $\varphi_g = \varphi_g(\varphi)$  which varies from a maximum at the percolation threshold where most of the connected porosity is non-conducting, diminishing to zero toward the right boundary where all of the connected porosity participates in conduction. It is shown in Appendix B that even when  $\varphi_g = \varphi_g(\varphi)$  is accounted for, equation (8) with a constant percolation threshold is correct for a linear diminution of electrically stagnant pore space. Since this is so, we continue to use the symbol  $\varphi_g$  ( $\neq \varphi_g(\varphi) = \text{variable}$ ) for this so-called “critical” porosity.

Our observations of Archie’s Nacatoch sandstone (and other) data lead to a hypothesis regarding how the underlying trend behaves, i.e.,

$$\frac{d(\sigma_0/\sigma_w)}{d\left(\frac{\varphi - \varphi_g}{1 - \varphi_g}\right)} \propto \left(\frac{\varphi - \varphi_g}{1 - \varphi_g}\right) \quad (9.a)$$

or, or more compactly stated

$$\frac{d\sigma_n}{d\varphi_c} \propto \varphi_c. \quad (9.b)$$

Assuming that the derivative is linearly proportional to the connected porosity

$$\frac{d\sigma_n}{d\varphi_c} = \alpha_0 \varphi_c \quad (9.c)$$

This equation is integrated to obtain  $\sigma_n$ ; i.e.

$$\sigma_n = \alpha_0 \int \varphi_c d\varphi_c = \frac{\alpha_0}{2} \varphi_c^2 + \beta_0. \quad (10)$$

The proportionality constant  $\alpha_0$  and integration constant  $\beta_0$  are determined from boundary conditions. In the first place,  $\sigma_n$  must vanish when  $\varphi_c = 0$ , so  $\beta_0$  must be zero. Also,  $\sigma_n = 1$  when total porosity  $\varphi = 1$ , which implies  $\varphi_c = 1$  at the right boundary. Thus

$$1 = \frac{\alpha_0}{2} \quad (11.a)$$

And

$$\alpha_0 = 2 \quad (11.b)$$

and finally

$$\sigma_n = \frac{\sigma_0}{\sigma_w} = \frac{\alpha_0}{2} \varphi_c^2 = \left(\frac{\varphi - \varphi_g}{1 - \varphi_g}\right)^2. \quad (12)$$

A final step will be to fit this function to Archie’s data and find a value of  $\varphi_g$  that minimizes the sum of squared residuals between the trend and the Nacatoch Sandstone data. Presently, we shall evaluate some of the consequences of this function. For now, keep in mind that *no physics* were explicitly invoked to arrive at (12) from the Archie data. Essentially, the spirit of the derivation is similar to Archie’s, purely curve fitting, except that it is done in a space different from Archie’s log-log porosity-resistivity space. The form of the function was arrived at solely by what the actual data suggests to the eye and mind of the interpreter in a linear porosity - linear conductivity space.

**Resistivity-Porosity Analysis by the Euclidian Method.** It is possible to arrive at the same result (i.e., equation (12)) starting from a completely different point of view. If one is interested in basing a theory on self-evident axioms, then in the context of conduction in the porous medium represented by a rock, make the following definitions:

A fluid saturated rock comprises three phases: a non-conducting mineral phase (e.g., quartz or calcite); a

conductive brine fluid phase; optionally, a non-conducting hydrocarbon phase. Electrical conduction takes place only in those volumes of the brine phase that are continuously connected and, due to their location in the pore space, actively participating in conduction; brine isolated in fluid inclusions or electrically stagnant volumes that do not participate in electrical conduction are not part of the conducting brine volume that participates in conduction. Recall that connected porosity is defined to be

$$\varphi_c = \frac{\varphi - \varphi_g}{1 - \varphi_g} \quad (8)$$

such that connected porosity equals zero at  $\varphi = \varphi_g$  and equals one at  $\varphi = 1$ ; these properties are desired in the definition of conducting porosity. With these definitions, the bulk conductivity of the rock should, self-evidently, satisfy three conditions:

- (1) bulk conductivity is proportional to the conductivity of the brine phase
- (2) for constant pore geometry, bulk conductivity is proportional to the fractional volume of the connected brine phase participating in conduction
- (3) bulk conductivity is governed by the geometrical configuration of the connected brine phase and can be represented by a proportionality constant.

These three conditions are the first principles of conduction in rocks.

In the simplest case the rock is totally brine-filled. I define a normalized conductivity parameter as  $\sigma_n = \sigma_0 / \sigma_w \equiv f$  where  $\sigma_0$  is bulk rock conductivity and  $\sigma_w$  brine conductivity and  $f$  is the formation *conductivity* factor. Recall that the normalized conductivity is familiar as  $1/F$ , the reciprocal of the formation resistivity factor in the Archie model. In other words, whereas  $R_0 = F R_w$ ,  $\sigma_0 = f \sigma_w = \sigma_n \sigma_w$ .

In terms of the three principles stated above

$$\sigma_0 \propto \sigma_w \quad (13.a)$$

$$\sigma_0 \propto \varphi_c \quad (13.b)$$

$$\sigma_0 \propto E_c \quad (13.c)$$

where  $\varphi_c$  is the fractional brine volume in the conducting connected pore space and  $E_c$  is a parameter that is used to characterize the geometry of the conducting

phase. Expressing all of the proportionalities in a single relationship gives

$$\sigma_0 = \sigma_w \varphi_c E_c. \quad (14)$$

When written in this way equation (14) is called the geometrical factor theory (GFT) and has been (with variations in notation and interpretation of the parameters) extensively discussed by Herrick and Kennedy (1993, 1994, 2009) and Kennedy and Herrick (2012). It is advantageous to express this relationship in terms of the normalized conductivity, i.e.,

$$\sigma_n = \varphi_c E_c, \quad (15)$$

the advantage being that all the quantities in the relationship now are numbers between 0 and 1.

In principle  $E_c$  can be determined from a geometrical description of the pore geometry in terms of the location of the boundaries between conducting and non-conducting phases, and this has been illustrated for some simple geometries (Herrick and Kennedy, 1993, 1994). In practice such methods are intractable. For real pore systems  $E_c$  is determined from conductivity-porosity data taken on core plugs. For a given plug

$$E_c = \frac{\sigma_n}{\varphi_c} = \frac{\sigma_0}{\sigma_w} \left( \frac{1 - \varphi_g}{\varphi - \varphi_g} \right). \quad (16)$$

This gives a means for computing  $E_c$  for a physical specimen for known  $\sigma_w$ ,  $\sigma_0$  and  $\varphi$  provided that  $\varphi_g$  can be estimated. The quantities  $\sigma_w$ ,  $\sigma_0$  and  $\varphi$  are known from measurements made on the sample.  $\varphi_g$  is estimated by a numerical procedure (described below) using an ensemble of samples. However, viewed as a function, (16) obviously contains the point (1,1) at  $\varphi = 1$ . By its design,  $E_c$  also contains the point (0,0) at  $\varphi = \varphi_g$ ; i.e.,  $\sigma_n \rightarrow 0$  as  $E_c \rightarrow 0$ . Thus, the end points on the  $E_c$  versus  $\varphi_c$  trend are defined. Invoking the principle of parsimony, presume that these end points are joined by a straight line segment; i.e.,  $E_c = a_c \varphi_c + b_c$  with  $a_c = 1$  and  $b_c = 0$ . This choice reflects previous observations in Archie rocks (Herrick and Kennedy; 1993, 1994). It has been often observed that for Archie rocks in a plot of  $E_0$  versus  $\varphi$  that the slope of the trend is always close to 1 and the intercept always close to zero. In these plots  $E_0$  intercepts the porosity axis at a pseudo-percolation threshold. By including a pseudo-percolation threshold in  $\varphi_c$ , I have

essentially moved the origin of the  $E_c$  versus  $\varphi_c$  plot to the pseudo-percolation threshold. The right boundary remains at the brine point (1,1). This does nothing to change the linearity of such trends, and thus it is reasonable to assert that (or assign)  $E_c = \varphi_c$  and adopt  $\varphi_g$  as an adjustable parameter used to honor this assertion in the sense of minimizing residuals in the least squares sense.

To fit a data set to this line  $E_c$  is plotted against connected porosity  $\varphi_c$ , calculated using a sequence of guesses (or successive approximations) for  $\varphi_g$ , stopping with the  $\varphi_g$  value that minimizes some objective function, for example, the sum of squared residuals. The trend constructed using this method satisfies

$$E_c = \varphi_c \quad (17)$$

identically by hypothesis. However, the data fit the hypothesized trend very well. (Herrick and Kennedy have noted the similar relationship  $E_0 \approx \varphi$  in their previous work, 1993 eqn. (20) et seq., 2009 eqn. (10) et seq., and 2012 eqn. (7). That is

$$\sigma_n = \varphi_c E_c = \varphi_c^2 \quad (18)$$

and thus, the GFT from the Euclidian method is

$$\frac{\sigma_0}{\sigma_w} = \left( \frac{\varphi - \varphi_g}{1 - \varphi_g} \right)^2. \quad (19)$$

This is *exactly* the same relationship found by inspection of Archie's Nacatoch Sandstone data (equation (12)). The use of the principle of parsimony is justified a posteriori by the observation that many, if not most, real data follow the postulated trend.

## DISCUSSION

We have thus used two independent methods (Baconian and Euclidian) that lead to the same result. The pseudo-percolation threshold theory method relies only upon inferences made after observing the Nacatoch sandstone dataset. The geometrical factor theory was developed by the Euclidian method of stating self-evident postulates and deriving the result as a theorem. The convergence of these two independent methods provides some support for the validity of the result.

Finally, note that whereas in the Baconian derivation of (12) the method was based upon observations, in the use of Euclidian reasoning no specific data was used.

There are only the three seemingly self-evident statements, the principle of parsimony, and, lastly, comparison of the predictions of the theory to real data. Explicit curve-fitting was not a part of the design process.

Accepting the suggested relationship  $E_c = \varphi_c$  and using the definition  $\sigma_n = \sigma_0/\sigma_w$ , then  $\sigma_n = \varphi_c E_c$  can be stated in a number of ways:

$$\begin{aligned} \frac{\sigma_0}{\sigma_w} &= \varphi_c E_c = \varphi_c^2 = \varphi_c \left( \frac{\varphi - \varphi_g}{1 - \varphi_g} \right) \\ &= \left( \frac{\varphi - \varphi_g}{1 - \varphi_g} \right) E_c = \left( \frac{\varphi - \varphi_g}{1 - \varphi_g} \right)^2. \end{aligned} \quad (20)$$

Written in this way, in this one series of identities are embedded all of: (1) Archie's law,  $\sigma_0/\sigma_w = \varphi_c^2$ ; (2) GFT (per Kennedy and Herrick, 2012),  $\sigma_0/\sigma_w = \varphi_c (\varphi - \varphi_g)/(1 - \varphi_g)$ ; and (3) PPTT (per Kennedy, 2007),  $\sigma_0/\sigma_w = ((\varphi - \varphi_g)/(1 - \varphi_g))^2$ . A great deal of confusion is obviated when the distinction between total porosity  $\varphi$  and the very nearly equal quantity, conducting connected porosity,  $\varphi_c$ , is recognized, the two being related by the porosity at the (pseudo-) percolation threshold  $\varphi_g$  (i.e.,  $\varphi_c = (\varphi - \varphi_g)/(1 - \varphi_g)$ ;  $\varphi_g \approx 0 \Rightarrow \varphi_c \approx \varphi$ ). With this distinction recognized and accounted for in the notation, there is *no difference* among the three! The empirically derived Archie relationship, transformed into its conductivity representation,  $\sigma_0/\sigma_w = \varphi^m$  becomes  $\sigma_0/\sigma_w = \varphi_c^2$ , the formula theoretically derived using the GFT theorem,

$$\frac{\sigma_0}{\sigma_w} = \varphi \left( \frac{\varphi - \varphi_g}{1 - \varphi_g} \right). \quad (21)$$

is corrected by replacing the first porosity factor  $\varphi$  with conducting connected porosity  $\varphi_c$ . With this substitution the empirically-based formula from PPTT is seen to be not just similar to, but identical to, the Archie relationship when conducting connected porosity is accounted for in the Archie relationship.

Note that the PPT for the Nacatoch Sandstone data set is very close to zero ( $\varphi_g = -0.01$ ), which explains why when it is ignored for this data,  $m$  is so very nearly, but not precisely, equal to 2.  $m$  would be exactly 2 if the PPT were taken into account, and for

$\varphi_g \rightarrow 0$ , then  $\varphi_c \rightarrow \varphi$  and  $\sigma_0/\sigma_w = \varphi^2$ . This serves as a derivation of the Archie law from first principles, making obvious the reason for the ubiquitous observation that  $m \neq 2$  but  $m \approx 2$ ; i.e.,  $\varphi_g$  is not accounted for.

**Remarks on Prior Results.** Kennedy and Herrick (2012) recently published the formula, equation (21). This formula is derived based upon

$$\sigma_0 = \sigma_w \varphi E_0 \quad (22)$$

and the definition  $E_0 = \sigma_0/(\sigma_w \varphi)$ . Note that (21) has two roots:  $\varphi = 0$  and  $\varphi = \varphi_g$ . The second root is hidden in (22). Equation (22) was originally derived (Herrick and Kennedy, 1993, 1994) for Archie rocks without regard for the possible existence of a percolation threshold and is in agreement with Archie's model at  $\varphi = 0$ . They observed that for Archie rocks  $E_0$  follows a linear trend with respect to  $\varphi$ ,  $E_0 = a_0\varphi + b_0$ . In those publications Herrick and Kennedy were concerned with determining the adjustable parameters  $a_0$  and  $b_0$  from conductivity-porosity data.

Subsequently it was realized that with the application of the conditions prevailing at the left and right boundaries to determine  $a_0$  and  $b_0$ , then (22) implies (21). With hindsight, I realize that (22) in the Herrick and Kennedy (1993, 1994) papers we neglected the  $\varphi_g \neq 0$  root. The form of (21) does agree with the results published by percolation theory pioneer Scott Kirkpatrick (1973), and so was provisionally accepted. However, the discrepancy between (12) and (21) motivated continued study of the problem. Basically, considering the empirically derived result (i.e., equation (12)) to be correct, and if there were no flaws in the logic leading to (21), then there had to be a flaw in the premises producing the contradiction in results. Examination of principle 2 in the GFT led to its reformulation (i.e., replacing  $\varphi$  with  $\varphi_c$  and  $E_0$  with  $E_c$ ), which leads to consistent, indeed identical, results between the Baconian and Euclidian methods. As to why these results are in disagreement with the Kirkpatrick (1973) formula is left as an open question at this writing, but may have to do with the dissimilarity of rocks and resistor networks.

Kennedy and Herrick (2012) invited the formation evaluation community to challenge their results. Accepting the challenge, I believe to have discovered a way to frame the three postulates that not only produces results consistent with observations, but also succeeds in unifying the prior, apparently inconsistent,

results and brings the Archie model into harmony with the geometrical factor theory.

You will notice that the arguments above are predicated upon the correctness of the function displayed in equation (17). But the analysis leading to equation (17) is premised upon a conjectured trend. A different form for the conjecture would, of course, produce a different result. So, am I guilty of *petitio principii*, (assuming in the premise what I will claim to prove, a form of circular reasoning)? Not being a logician, I am open to a demonstration showing that this is the case, I do not think that it is. The discussion in the paragraph immediately following equation (16), together with equation (17) are "Occam's razor" (or "principle of parsimony") conjectures about observed data; the three first principles are free of connections to any particular data. These appear to be independent assumptions to me.

Be this as it may be, Archie's law imposes zero conductivity and zero porosity to coincide; it is clear from the observation of much data over seven decades that this is a weakness of the model requiring the introduction of an additional adjustable parameter. Inclusion of the so-called tortuosity factor  $a$  does not decouple zero conductivity from zero porosity; rather, it decouples the brine point from a normalized conductivity of unity. This might improve the fitting power of the Archie power law, but it is clearly "unphysical". Introduction of a pseudo-percolation threshold puts the adjustability at the opposite end of the porosity axis, allows conductivity to vanish while porosity remains in the rock, and keeps the normalized conductivity at 100 percent porosity equal to 1, where it belongs. The physical interpretation of this model makes sense at the upper and lower porosity limits, which Archie's law with "tortuosity" factor does not. Thus, whether or not the functional form can be logically deduced, it can certainly be introduced as an ad hoc function that satisfies the boundary conditions, and the resulting formula will be the same.

## CONCLUSION

The original formulation of the geometrical factor theory as

$$\sigma_0 = \sigma_w \varphi E_0 \quad (22)$$

with  $E_0 = a_0 \varphi + b_0$  is always valid when restricted to the porosity range of data acquired on real rocks, with  $a_0$  and  $b_0$  being determined by a trend in the data without reference to boundary conditions.

However, when for theoretical purposes, this formulation is extrapolated to the right and left boundaries of the porosity axis, the vanishing of conductivity at a non-zero value of porosity must be accounted for in the formulation, leading to the concept of connected porosity participating in conduction (i.e., conducting connected porosity,  $\varphi_c$ ) as being the support of the bulk conductivity. In this investigation three disparate formulations of the relationship between the bulk conductivity of an Archie rock, the conductivity of the brine in the pore space, and the amount of pore space, can be unified into a single formula by recognition of the role played by the inclusion or omission of a percolation threshold (more accurately, the pseudo-percolation threshold) in the formulations.

The derivation of Archie's law using the Euclidian method is a serendipitous result of this attempt to reconcile equations (6) and (7). However, this particular derivation begins with very general postulates (i.e., does not presuppose any model other than implied by the postulates) and ends with an Archie equation with the porosity exponent equal to precisely 2, with the variability in the function shifted from the exponent  $m$  to a pseudo-percolation threshold  $\varphi_g$ . This provides an explanation of why the porosity exponent is approximately, but not exactly, 2 in the industry-standard Archie model that does not provide for a percolation threshold parameter. The Archie model always provided a conundrum: it is clear where in the rock the porosity resides, and clear where the mineral, brine, (and hydrocarbon) phases reside; but where do  $m$  (and  $n$ ) reside? In this formulation, the exponent is seen as a pure number unconnected to pore geometry, with the adjustability assigned to the pseudo-percolation threshold parameter, which has a quasi-physical pore geometric interpretation. It is clear that it is an indicator of where an apparent percolation threshold lies on the porosity axis.

Archie's law can now be seen as an empirically derived approximation to a rigorously derived porosity-conductivity relationship. That  $m$  is approximately 2 is seen as a consequence of the Archie relationship having been developed using an  $F$ - $\varphi$  plot rather than an  $F$ - $\varphi_c$  plot. In an  $F$ - $\varphi_c$  plot the slope of the trend would have been 2 exactly, with the adjustable parameter as  $\varphi_g$  rather than  $m$ . I believe that this may be the first derivation of an Archie-like equation with a proper value for the  $m$  exponent to descend directly from physical first principles, not having to resort to effective media approximations, statistical arguments, or other artificial constructs as

fractal pore models (Nigmatullin et al., 1992), continuum percolation theory (Hunt, 2004), etc.

The unification of the three formulas, and the realization that the Archie law, properly formulated in conductivity space in terms of conducting connected porosity can at last be understood in physical terms seemingly brings a conclusion to this two-decade line of research.

Extension of this formalism to the case  $S_w < 1$  is discussed in Kennedy (2007). The reader should remember that these results are derived using Archie's Nacatoch sandstone data set and are strictly applicable only to Archie rocks.

If petrophysics, as an empirical science, has a weakness, it has been the lack of thorough discussion of its percepts over the decades. Therefore, I invite, welcome and encourage – indeed I challenge – my colleagues to produce any collegial discussion and debate that the ideas that I have introduced might engender. If the ideas presented herein are not the final words on this topic, a healthy marketplace of ideas will eventually produce the most robust result.

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## About the Author



David Kennedy, in the time-honored tradition of “oil patch trash”, is a true vagabond. Having attended three universities, and receiving three degrees, David has worked in the “oil patch” since 1973 for two service companies, five oil and gas operators, one geophysical data acquisition company, not to mention

brief stints at both Northrop and Lockheed, and odd jobs around the house. He did stay with Mobil and Exxon for 20 years to prove to his mother that he could hold a job, but clearly that is arguable. Beginning with training at the Schlumberger learning center in 1973, David was always confused by explanations of how rocks conduct electricity, and how induction logs responded to conducting rocks. Although he is still a bit confused, things are now much clearer, and he thinks he has contributed in a small way to the understanding of conductivity anisotropy in reservoir rocks, and how induction instruments respond to it. David has six U.S. patents covering the measurement of resistivity anisotropy in the laboratory and using tri-axial logging instruments. David has about 35 publications as author or co-author, mostly on induction logging instruments and petrophysics of conductive rocks, and maybe has a few papers still left in him. His most personally rewarding jobs (although he liked them all) have been as a Schlumberger field engineer 1973-1977, and his recent employment at Southwestern Energy where he lived the dream of finding oil and gas. David has served the SPWLA as Editor of *Petrophysics*, Vice President of Publications, Vice President of Technology, and President. Although in love with the oil and gas business, David is looking forward to slowing down a bit following the next few years. The slowing down schedule has recently accelerated a bit; first layoff in 43 years at age 69 – not bad, Mom. Any continuing contributions to the industry will be in the field of training. Watch for a new kind of formation evaluation learning from QED Petrophysics LLC.



## APPENDIX A: Cartoons vs. Reality

The cartoon version of a streamline as the path that a charged particle, such as a sodium or chlorine ion, would follow through a medium should be taken with a grain of that same salt before it dissolves in solution. A particle in solution (be it an ion or a dust mote) is bombarded continually from all sides by other particles (principally molecules of the solvent) in the solution. These particles are traveling at hundreds of meters per second between collision sites and careen into neighboring particles at every possible angle of incidence. If attention is focused upon an individual particle, be it an ion in solution, or a solvent molecule, its motion is thoroughly random and lacks any coherent directionality. In statistics its behavior is described by a method called the “drunkard’s walk”. Although the particle of interest at any instant is displaced in a definite direction, it is overwhelmingly likely that it will be replaced by a similar particle displaced in the opposite direction. And so on, for every direction in the medium, and there is no net transport of material.

What happens when the medium is suffused with an electric field? First, there is no effect upon neutral particles other than there might be a degree of induced polarization. However, the polarized neutral particles would feel a torque aligning them with the field, but otherwise their random displacements would be unaffected. If the solvent is water, which has a permanent electric dipole moment, each molecule will experience a torque tending to align it in the direction of the applied field, but other than the alignment, there is no net material transport due to the applied field.

For ions in solution, the picture is different. An ion having charge  $q$  will feel a net force  $\mathbf{F}$  in the direction of the applied field  $\mathbf{E}$ , given by  $\mathbf{F} = q\mathbf{E}$ . The ion will thus be accelerated in the direction of the force, which is in the same direction as the applied field  $\mathbf{E}$ . The ion also feels the force of the thermal collisions which it constantly undergoes, and these forces are typically much larger than the force due to the applied electric field. (This force is given by the rate of change in the particle’s momentum in each collision.) However, the forces due to thermal jostling are random, without a preferred direction, whereas the force due to the applied field is constant, imagined to persist for a long time compared to the time between thermal collisions (about a pico-second). Thus, viewed over a period long compared to the time between thermal collisions, an ion will have a component of motion in the direction of the applied field superposed upon its random thermal motion. When all of the ions in the solution are considered together, the net effect is an electric

current with the charged ions being the current. In this case there *is* net material transport of individual ions.

The situation in the solution is complicated. In the first place, there are (at least) two species of oppositely charged ions. These feel forces in opposite directions and thus move simultaneously in opposite directions. You might think that these currents, being of opposite charge would cancel, but the fact is that a positive charge moving to the left and a negative charge moving to the right both are considered an electric “current” in the *same* direction. The reason for this can be understood in terms of the magnetic field produced by the current. Positive charge moving to the left produces a magnetic field which circulates clockwise when viewed in the direction of the applied electric field. (This is the “right hand rule relationship between current and magnetic field direction”). Negative charge moving to the right *also* produces a magnetic field which circulates clockwise when viewed in the direction of the applied field (This is the “left hand rule relationship between current and magnetic field direction”). Viewed from outside the conducting brine, using a compass to detect the magnetic field, it would not be possible to say whether the charge carriers were positive moving in the direction to the applied field, or negative moving in a direction opposite to the applied field. And so “conventional” electric current is defined as producing a clockwise magnetic field centered on the charge carrier when viewed in the direction of the applied electric field regardless of the sign and direction of transport of the ion.

Although there is charge transport, in equilibrium, for a steady direct current, there is no *net* charge transport. Any charge leaving a volume on the left is replaced by a similar charge entering the volume on the right. The solution remains electrically neutral on the macroscopic scale at all times.

The speeds attained by the charge carriers are minuscule compared to their thermal speeds. For example, the mobility of the sodium ion in aqueous solution at 25° C is  $5.19 \times 10^{-8}$  (m/sec)/(volt/m); for chlorine the mobility is  $7.91 \times 10^{-8}$  (m/sec)/(volt/m). Put in a voltage gradient to compute the “drift” speed. For typical gradients the speeds are on the order of  $10^{-8}$  (m/sec). The thermal speed at the same temperature is about 590 m/sec (This is an average; some of the molecules move faster, some move slower, at any given time. The speed of each molecule changes with each collision. Each molecule experiences on the order of  $2 \times 10^{12}$  collisions per second. So the speed of an individual molecule will rapidly vary from faster than average to slower than average, and back.)



Finally, for various technical reasons, the applied electric field oscillates at frequencies ranging from 1 Hertz up to the gigahertz range for some logging services. This means that whatever displacement in the direction of the applied electric field that an ion attains in one half-cycle of the applied field, is cancelled (on average) by the displacement in the opposite direction during the next half-cycle. So, in a typical pore system energized by a typical logging instrument (or a core sample in a laboratory) there is *no* transport of charge at all. The only thing to be transported is energy.

The picture of the phenomena explicated at the molecular level is quite a bit more detailed and complicated than is captured by a cartoon showing streamlines. Nevertheless, the streamline representation is a useful approximation for the paths that individual currents would take if the currents represented charge carriers that actually circulated through the pore system, end-to-end. They do represent the lines of applied electric field, which are immaterial and not subject to the frenzy of molecular thermal dynamics.

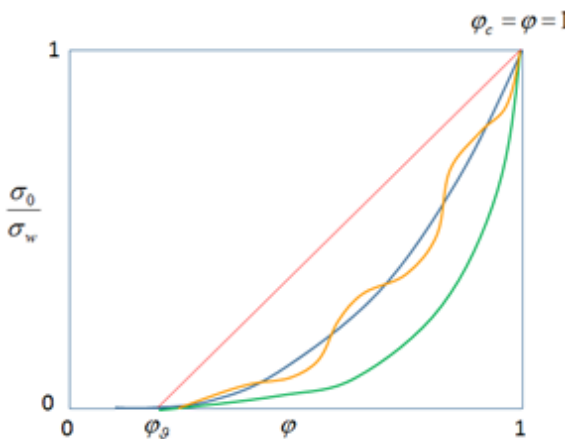
Retuning now to the representation of tortuosity by streamlines, the length of a streamline compared to the end-to-end length of a sample is only one of the controls on the bulk conductivity of a pore system. The role of varying cross section of the conductor along the current path, or streamline, is not well-captured in this kind of cartoon representation.

The net result is the apparent macroscopic current flow which is described by the current pathways or streamlines which follow tortuous paths through the pore system. The current density, the amount of ‘current’ flowing through any miniscule area of pore space, is highest in small constrictions (i.e., the pore throats). By analogy with water flowing through a narrow section of a stream, where the flow rate (cubic meters per second) is the greatest, but is the limiting factor that determines the total amount of water that can flow in the stream, while electric current density is also highest in constrictions, the small size of constrictions limits the total amount of current which can pass through. The smallest constrictions along any “flow tube” (a tubular volume bounded by a closed surface which contains all of the current entering the sample in a small cross-sectional area on one end of the specimen as it continues to the other end) determines the amount of current that flows along the entire path. If all conduction pathways through a rock specimen are considered, then it is easy to conclude that the total cross-sectional area of all of the smallest constrictions along all the paths is an important geometric property which, along with

the average tortuosity of all the conduction paths, limits the total amount of electric current flowing through the rock.

## APPENDIX B

Imagine a thought experiment that varies porosity and observes normalized conductivity. What can be said about the porosity-conductivity function to be expected? Two points are predictable prior to any observation: (1) the “brine point” porosity is 100 percent and normalized conductivity is 1; (2) at the low porosity end of the porosity axis, the conductivity remains zero until the porosity rises above a certain “critical” value. The porosity at the critical value has to be determined by observation for each particular rock, but such a porosity has to exist (although it could be zero porosity). For porosities greater than the critical porosity the normalized conductivity begins to increase (gradually at first) with increases in porosity. As porosity increases, the rate at which normalized conductivity increases also increases: the conductivity increases, and the rate of increase increases. This rules out certain functions that satisfy the boundary conditions (See Figure B1.). For example, a straight line is ruled out because its rate of increase is constant; functions that oscillate are also ruled out. Functions like the blue and green curves are possible candidates.



**Figure B1.** Some trial normalized conductivity versus porosity functions, all satisfying the boundary conditions at minimum and maximum porosity. Not every candidate can meet other constraints upon the function. For example, observation of data rules out the linear function; logical considerations rule out oscillating functions. Functions consistent with observation are smooth curves connecting the boundaries. Polynomials and power laws are consistent with curves of this shape.

Now consider real observations, say the Nacatoch sandstone data. This data has too much scatter to define a trend uniquely, but it does suggest that

$$\frac{d\sigma_n}{d\varphi_c} \propto \varphi_c \quad (9.b)$$

Admittedly, this proposed proportionality is a conjecture. For example, the derivative might also be taken as proportional to a fractional power of porosity, say  $\varphi_c^{p/q}$ ,  $p$  and  $q$  being integers. The data suggest proportionality, but do not prove it. Accepting the suggestion as a working hypothesis, the idea can be checked for consistency with observation. The proportionality hypothesis implies that

$$\sigma_n \propto \varphi_c^2 \quad (B.1)$$

identifying one of the possible functions in Figure B1, and implying that normalized porosity is exactly proportional to conducting connected porosity squared. If the proportionality is expressed in terms of total porosity instead of connected porosity, an exponent almost but not quite equal to 2 (Archie's  $m$ ) is required. So the working hypothesis expressed in equation (9.b) leads directly to an Archie-like law. Archie's method was based upon an ad hoc correlation between logarithms of porosity and formation resistivity factor; this derivation is also based upon an ad hoc supposition of proportionality between the rate of change of normalized conductivity with respect to conducting porosity, and conducting porosity. So it is seen that if conductivity and porosity are constrained to vanish together, then  $\varphi_g \equiv 0$ , and  $m \approx 2$ ; however, if the constraint is relaxed, then  $\varphi_g \neq 0$ , and  $m \equiv 2$ . So, the choice between the models can be based upon observation of whether or not  $\varphi_g \equiv 0$  in every case. Clearly, it is not.

What is the physical interpretation? For randomly populated cubic conductor networks interpretations have been worked out. As conductors are added at random to an initially unpopulated cubic lattice, clusters of disconnected conductors form. These contribute to the fractional probability function (that a bond (or site) is occupied), but not to conductivity. Eventually, a conductor is added that closes a circuit connecting two isolated clusters to produce a through-going cluster, and the network becomes conducting. This threshold is crossed when the probability that a bond (site) is occupied is about<sup>6</sup> 0.2488 (0.3116). However, at this

threshold there are still isolated clusters that do not participate in conduction. As bonds are added at random, the isolated clusters are gradually connected to the through-going network until, at last, the entire network is connected. Even when this occurs, not all of the conductors will participate in conduction, as some will be connected to the network at only one end; others will have the same potential at opposite ends and carry no current.

Conduction in rocks is qualitatively different. The conducting phase comprises a diaphanous network of connected brine filaments. These remain connected to very low porosity values, well below the percolation thresholds in cubic conductor lattices. However, at low porosity values much of the brine phase, although connected to the brine network, is not participating in conduction, as it is situated in dead-end pores, or (more likely) electrically stagnant volumes of the pore space. As porosity increases, the amount of brine participating in conduction also increases, accounting for the observed increase in the rate at which conductivity increases with porosity; volumes of stagnant brine, dead end pores, etc., diminish as a greater fractional volume is occupied by brine. At some point, all of the porosity is connected and increasingly participating in conduction, as stagnant volumes are increasingly reduced. However, as suggested in figure B2, the conductivity at every porosity, is less than it would be if the percolation threshold were zero and conductivity began its rise at zero porosity. The normalized conductivity in the "connected porosity" model does not catch up to the conductivity that starts at zero porosity till porosity (and connected porosity) merge at 100 percent porosity.

The narrative above is based upon connected porosity defined by

$$\varphi_c = \frac{\varphi - \varphi_g}{1 - \varphi_g} = \frac{\varphi}{1 - \varphi_g} - \frac{\varphi_g}{1 - \varphi_g} = \alpha\varphi + \beta. \quad (B.2)$$

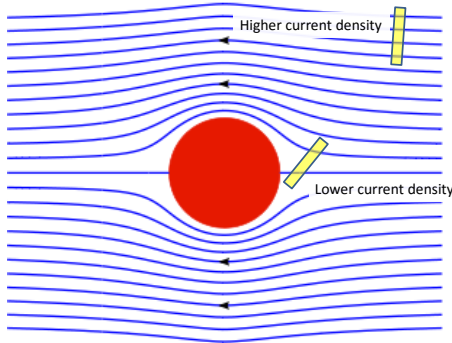
There is potentially a problem with calling this model (i.e.,  $\varphi_c$ ) "connected" porosity. The pseudo-percolation threshold  $\varphi_g$  represents a fixed fractional volume of the pore space not participating in conduction. However, at porosities higher than  $\varphi_g$ , the amount of porosity not participating in conduction is not a constant value given by  $\varphi_g$ , but a fraction of the pore volume that diminishes with increasing porosity, until it finally vanishes altogether at 100 percent porosity (or

<sup>6</sup>[https://en.wikipedia.org/wiki/Percolation\\_threshold](https://en.wikipedia.org/wiki/Percolation_threshold)

maybe at a lesser value of porosity). Thus,  $\varphi_g$  should be a function of the porosity,  $\varphi_g(\varphi)$ . In recognition of this increasingly connected porosity, should the conducting connected porosity be properly defined as

$$\varphi_c = \frac{\varphi - \varphi_g(\varphi)}{1 - \varphi_g(\varphi)} \quad (\text{B.3})$$

where the percolation threshold term is replaced by a function of porosity  $\varphi_g(\varphi)$  that varies from its maximum value at the percolation threshold, call it  $\varphi_{pt}$ , to zero at 100 percent porosity (or maybe attains zero at an even lower porosity)? There is (at present) no way of knowing what such a function would look like; however, various guesses can be tried and the results observed. However, whatever a guess may be, it cannot perturb the function previously constructed for normalized conductivity too much, because the data (i.e., the Nacatoch sandstone) is already fit very well when  $\varphi_g$  is presumed to be a constant. Also, remembering that there is no data from rocks much above  $\varphi = 0.40$ , whatever is happening at higher porosities is going to be in the realm of thought experiments, or at least experiments on materials other than rocks.



**Figure B2.** This figure illustrates that even when mineral grains become fully brine supported, they still exert an influence upon the electrical properties of the medium. Close to the surface of the sphere (representing a surrogate mineral grain), volumes of relatively electrically stagnant brine develop. The current densities in these volumes is much lower than in similar volumes far from the grain. Volumes having less than maximum current density are equivalent the same volumes having a lower conductivity; hence the bulk conductivity of the medium is reduced. Theoretically, the influence of the grains does not vanish until the last grain is removed from the brine; this argument justifies normalizing the connected brine phase by a factor of  $1 - \varphi_g$ .

In percolation theory normalized conductivity is always presented as being proportional to the difference between porosity and porosity at a percolation threshold raised to a power: i.e.,

$$\frac{\sigma_0}{\sigma_w} \propto (\varphi - \varphi_{pt})^q \quad (\text{B.4})$$

where  $\varphi_{pt}$  is a “critical” value of porosity marking a phase change from non-conducting to conducting rock at the percolation threshold.

Now, since the porosity above the percolation threshold increasingly consumes (or subsumes, or connects) originally electrically stagnant porosity as porosity is increased, let us posit that the critical porosity is a function of porosity; i.e.,

$$\varphi_g \equiv \varphi_g(\varphi). \quad (\text{B.5})$$

This function has got to satisfy two boundary conditions: it must equal the critical porosity when porosity itself is equal to the critical porosity; it must vanish when porosity equals one. Let us invent a trial function for this quantity. Let

$$\varphi_g(\varphi) = \left( \frac{1 - \varphi}{1 - \varphi_{pt}} \right) \varphi_{pt} \quad \varphi_{pt} \leq \varphi_g(\varphi) \leq 1 \quad (\text{B.6})$$

be the candidate function. At the left boundary, plugging in  $\varphi = \varphi_{pt}$

$$\varphi_g(\varphi_{pt}) = \left( \frac{1 - \varphi_{pt}}{1 - \varphi_{pt}} \right) \varphi_{pt} = \varphi_{pt}, \quad (\text{B.7})$$

and note that the left boundary condition is satisfied. At the right boundary plugging in  $\varphi = 1$

$$\varphi_g(1) = \left( \frac{1 - 1}{1 - \varphi_{pt}} \right) \varphi_{pt} = 0 \quad (\text{B.8})$$

and the right boundary condition is satisfied. So the trial function satisfies the boundary conditions, and between diminishes linearly to the right. Now plug the trial function into the argument of the original conductivity proportionality (i.e., B.4))

$$\begin{aligned}\varphi - \varphi_g(\varphi) &= \varphi - \left( \frac{1 - \varphi}{1 - \varphi_{pt}} \right) \varphi_{pt} = \frac{(1 - \varphi_{pt})\varphi - (1 - \varphi)\varphi_{pt}}{1 - \varphi_{pt}} \\ &= \frac{\varphi - \varphi\varphi_{pt} - \varphi_{pt} + \varphi\varphi_{pt}}{1 - \varphi_{pt}} = \frac{\varphi - \varphi_{pt}}{1 - \varphi_{pt}}.\end{aligned}\quad (\text{B.9})$$

And so, making the substitution into the conductivity proportionality equation

$$\frac{\sigma_0}{\sigma_w} \propto (\varphi - \varphi_g(\varphi))^q = \left( \frac{\varphi - \varphi_{pt}}{1 - \varphi_{pt}} \right)^q, \quad (\text{B.10})$$

and one discovers the constant critical porosity  $\varphi_{pt}$  playing the same role that was originally assigned to  $\varphi_g$  when it was presumed to be a constant, and  $\varphi_{pt}$  means the same thing, too. So one might just as well stick to the original notation. That is

$$\varphi_{pt} \equiv \varphi_g \quad (\text{B.11})$$

with a recognition that this function accounts for non-conducting porosity diminishing linearly to the right from a maximum at  $\varphi_g$  to zero at  $\varphi = 1$ .

The above considerations show that

$$\frac{\sigma_0}{\sigma_w} \propto \left( \frac{\varphi - \varphi_g}{1 - \varphi_g} \right)^q \quad (\text{B.12})$$

implicitly imposes a linear increase in the connectedness of the porosity as porosity increases and a concomitant decrease in the electrically stagnant volume of the brine.

There be two remaining, related, considerations to address in order to exhaust this topic. The linear reduction in stagnant porosity, although simple, is probably not realistic. It seems more reasonable that stagnant porosity would diminish rapidly with increasing porosity just to the right of the percolation threshold with the rate of diminution decreasing with increasing porosity; in other words, a non-linear decrease seems reasonable to imagine. And, it is also reasonable that, at least to a first approximation, that stagnant porosity might vanish at some porosity less than 100 percent.

To account for this in a model, let

$$\varphi = \varphi_c + \varphi_{stag} \quad (\text{B.13})$$

where  $\varphi_c$  is conducting connected porosity and  $\varphi_{stag}$  is connected, but non-conducting, porosity, and construct a formula for  $\varphi_{stag}$  that satisfies  $\varphi_{stag} = \varphi_{pt}$  at the left boundary and  $\varphi_{stag} = 0$  beginning at some value of porosity  $\varphi_x < 1.0$ . That is

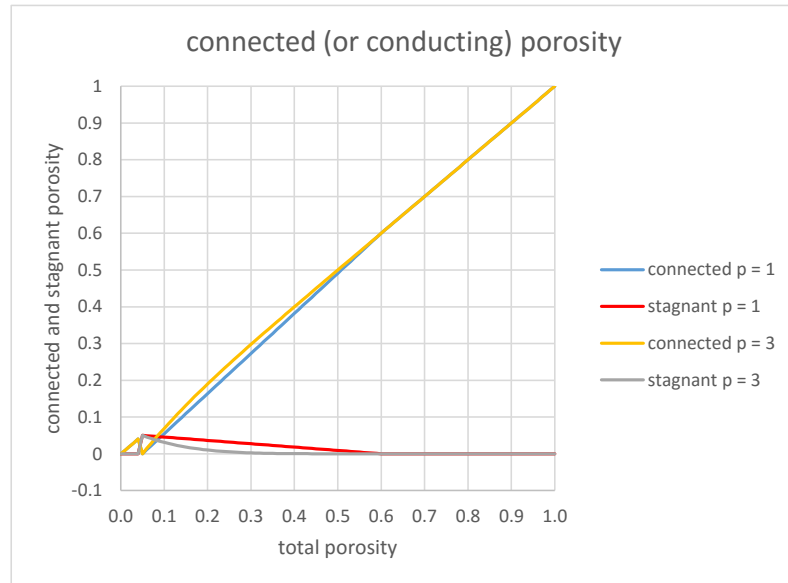
$$\varphi_{stag} = \begin{cases} \left( \frac{\varphi_x - \varphi}{\varphi_x - \varphi_{pt}} \right) \varphi_{pt} & \text{if } \varphi_{pt} \leq \varphi < \varphi_x \\ 0 & \text{if } \varphi_x \leq \varphi \leq 1 \end{cases} \quad (\text{B.14})$$

with  $\varphi_c = \varphi - \varphi_{stag}$  over the same intervals. Note by substitution of  $\varphi_{pt}$  and  $\varphi_x$  for  $\varphi$  that  $\varphi_{stag}$  satisfies the desired conditions. The decrease of this  $\varphi_{stag}$  function is linear. To create a function that satisfies the same boundary conditions, but also decreases more rapidly with increasing porosity, use

$$\varphi_{stag} = \left( \frac{\varphi_x - \varphi}{\varphi_x - \varphi_{pt}} \right)^p \varphi_{pt} \quad \varphi_{pt} \leq \varphi \leq \varphi_x \quad (\text{B.15})$$

for  $p > 1$ . This function not only has the desired behavior as far as rapid diminution is concerned, but also has a derivative evaluating to zero at  $\varphi = \varphi_x$ , so that it merges smoothly into the rightmost segment of the function  $\varphi_{stag} = 0$ ,  $\varphi_x \leq \varphi \leq 1$ . An example featuring  $\varphi_x = 0.60$  for the cases  $p = 1$  and  $p = 3$  is shown in figure B3.

From the arguments above, I have shown: (1) that equation (8) with  $\varphi_g$  taken as a constant critical porosity is a good approximator for connected porosity  $\varphi_c$ ; (2) when it is necessary to take the porosity dependence of  $\varphi_g(\varphi)$  into account, functions that satisfy the necessary boundary conditions and approximate any desired rate of decay are easily constructed.



**Figure B3.** Linear ( $p = 1$ ) and nonlinear ( $p = 3$ ) connected and stagnant porosity for the case where  $\phi_x = 0.60 < 1.0$ . Note the effect is subtle at moderate and higher porosities, and could be safely neglected.