The Porosity-Water Saturation-Conductivity Relationship: An Alternative to Archie's Model

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ABSTRACT

The bulk volume fraction of conducting brine β in a reservoir rock is given by the product of porosity and water saturation, $\beta = \varphi S_w$, symmetric in φ and S_w. Bulk conductivity σ_t of a rock varies as this product varies. When $\beta = 1$ bulk conductivity necessarily equals brine conductivity, $\sigma_t = \sigma_w$; conversely, bulk conductivity typically vanishes at some $\beta \neq 0$. A mathematical description of the physical relationship underlying bulk rock conductivity and brine conductivity and the bulk volume fraction of brine must be constrained by these properties.

The classical three-parameter relationship connecting bulk formation conductivity to porosity and water saturation is an ad hoc formula comprising two separate power laws proposed by Archie in 1941 and modified by Winsauer in 1952. The three adjustable parameters a, m, and n, in the usual formulation of these laws are chosen, for particular data sets, by the method of least squares to minimize sums of squared residuals between observed and predicted data. The resulting theory depends upon the term $\varphi^m S_w{}^n$, breaking the symmetry inherent in the bulk volume brine fraction. While the laws have proved useful for predicting water saturations by interpolation within the bounds of observed data sets, they do not usually extrapolate to correct values at the limits of their domain of applicability, suggesting that a pure power law formulation cannot correctly represent the underlying physics.

The classical power laws devolve from an arbitrary selection of one particular class of fitting function from among several possible choices. It is possible to use the same porosity-resistivity data sets used by Archie and Winsauer to define an alternative three-parameter model. The proposed model's adjustable parameters are critical values of porosity, water saturation and conductivity and can be interpreted directly in terms of the percolation thresholds of the medium. The proposed model simultaneously (1) treats the fractional brine volume β as the fundamental variable, (2) both accommodates the boundary condition $\sigma_t = 0$ at $\beta \neq 0$ and satisfies $\sigma_t = \sigma_w$ at $\beta = 1$, and (3) gives a smaller sum of squared residuals using the classical data sets than the classical models. Many additional insights into the conductivity-porosity-water saturation relationship that inhere in this new model are discussed.

The efficacy of the model is illustrated using Archie's and Winsauer's original data by better predicting conductivity from porosity, demonstrated by a smaller sum of squared residuals using those data, and using conductivity-porosity-water saturation data published by Hamada and others in 2002.

INTRODUCTION

In situ estimates of bulk formation resistivity made on petroleum reservoirs by remote sensing instruments located in wellbores have been among the primary observations used in evaluating hydrocarbon volumes for more than a half-century, and resistivity estimates continue to remain an important element of formation evaluation. However, whereas engineering of data acquisition technology has kept pace with developments in new materials and new electronic methods, interpretation technology remains—perhaps arguably—unimproved for more than 60 years.

Empirical relationships among the bulk formation properties that determine the electrical resistivity of reservoir rocks were formulated in the 1940s and 1950s. Originating as ad hoc models, these descriptions have never been understood from a theoretical point of view. This article aims to return to the seminal observations of the electrical behavior of reservoir rocks to investigate whether an entirely different formulation of a formation electrical properties model can lead to a better understanding of the empirical descriptions, new physical insights, and a new interpretation paradigm.

The industry-standard porosity – water saturation – resistivity interpretation method in "clean" clastic rocks and some limestones is based upon the analysis of the data sets used by Archie (1942) and Winsauer and others (1952) [for convenience, hereinafter referred to simply as "Winsauer"]; the former resulting in a model known as Archie's "law", the latter giving rise to the "Humble formula".

The seminal data sets are simply ordered pairs of porosity-resistivity observations arranged as two columns of numbers. The resistivity of the connate water is also available. Archie searched for a simple rule relating the two columns. The search was constrained by the tractability of the representation chosen. There are many possible choices that Archie could have made, but one of the most tractable for the time, considering that in the 1940s every engineer would have had a desk drawer full of semi-log and log-log graph paper, was the plotting of the logarithm of porosity φ against the logarithm of the formation resistivity factor, $F = R_0/R_w$. Especially for Archie's Nacatoch

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sandstone data, a linear relationship between the logarithm of φ and the logarithm of F is strongly suggested. Archie's Nacatoch sandstone data is shown in figure 1. The suggested relationship is a power law expressed as $F = \varphi^{-m}$ with $m \approx 2$. Note that this law predicts that if $\varphi = 1$ then $F = 1$, a correct prediction. Archie's plot has been retroactively interpreted as having physical significance, although Archie[*] himself never made this claim, and there has never been a convincing argument from any physical first principle that suggests a power law in this form is dictated by the laws of physics.

Application of Archie's model to virtually every data set in the 1940s decade was assured, there being no alternative theory. Many instances must have been encountered where a "least squares by inspection" application of Archie's model did not fit the observations very closely. In fact, Archie in his seminal paper offered a second data set titled "gulf coast sandstones" for which the linear trend was less pronounced, albeit still suggested. Winsauer and his coauthors revisited the question of predicting resistivity from porosity and suggested that if the requirement that the least squares line contain the point ($\varphi = 1, F = 1$) were relaxed, the data could be more closely fitted using a power law of the form $F = a\varphi^{-m}$, a not necessarily equal to 1. Archie had noted that $m \approx 2$, and consequently uses a square root in his saturation equation. Based on a (very small) data set assembled from around the world (figure 2), Winsauer

proposed $F = 0.62\ \varphi^{-2.15}$, which is famously known as the "Humble" formula. Because Winsauer's data set was assembled from a variety of widely geographically distributed sandstones, a certain (in hindsight, unwarranted) generality was attributed to the Humble formula, and it was used for a time with wild abandon in the industry. Although the parameter values of the Humble formula certainly do render the sum of the squared residuals for the Humble data set small[*], after a time it was realized that best predictions in a given formation were based upon relations custom-developed for that formation; no generality for the parameter values $a = 0.62$ and $m = 2.15$ could be asserted. However, in the form $F = a\ \varphi^{-m}$ with a and m custom-determined for each data set, the formulation of Winsauer endures as today's industry standard method for representing the porosity-resistivity relation in clean sandstones. Note that if $\varphi = 1$, then $F = a$, which is physically correct only for the special case $a = 1$.

There are problems with the Archie-Winsauer (A-W) formulation of the porosity-resistivity function. If the data sets used to produce the correlations are viewed as having varying porosity due to the process of diagenetic cementation operating upon a rock homogeneous in its porosity at the time of deposition, say by preferentially reducing porosity at a greater rate in finer grained com-
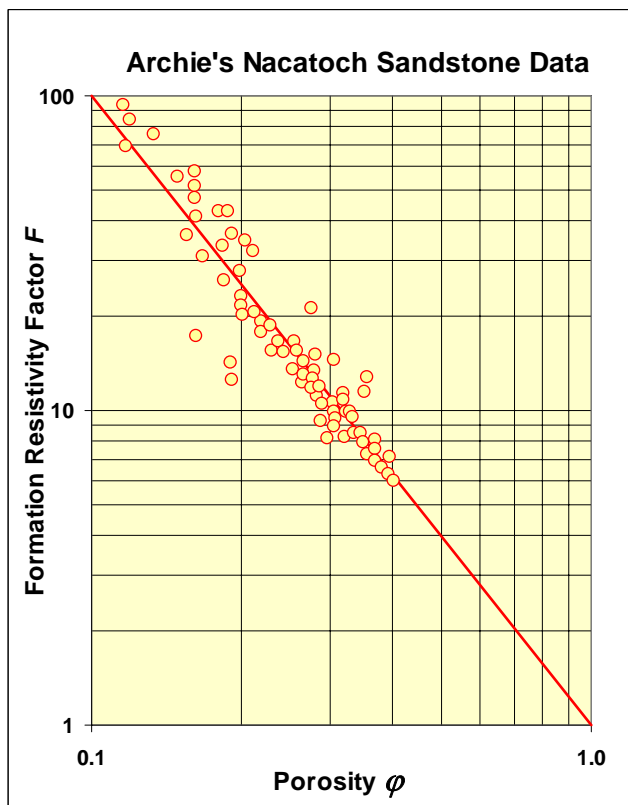

Fig. 2. Winsauer et al. (1952) data set and a line representing the "Humble" formula, $F = 0.62\ \varphi^{-2.15}$.

Fig. 1. Archie's Nacatoch sandstone data.

[*] "It should be remembered that the equations given are not precise and represent only approximate relationships." Archie, 1942
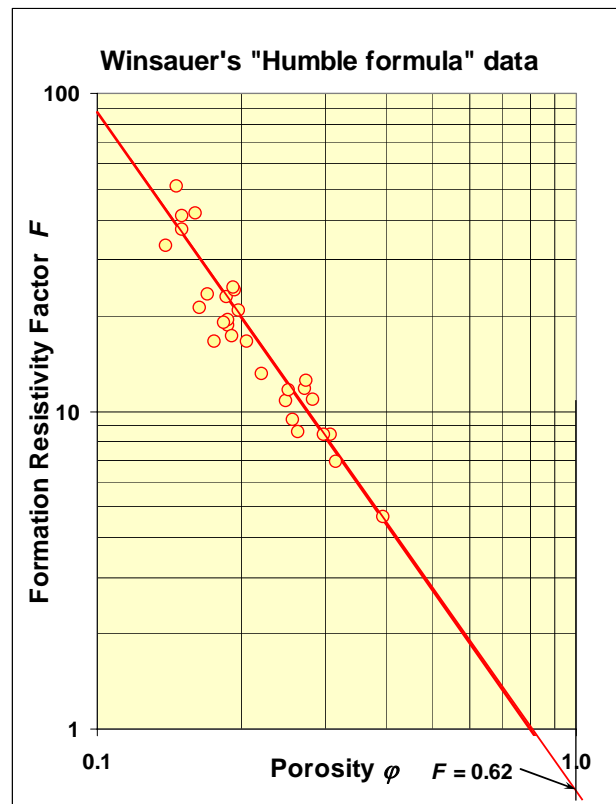
[*] The Humble formula parameters do not minimize the sum of the residuals for Winsauer's data. Perhaps his data was fit "by eye" rather than by mathematical least squares.

ponents, then one expects conductivity to vanish with the porosity still greater than zero. In other words, the rock ceases to conduct electricity when the continuous conduction path through the rock is broken, and the path can be broken while isolated porosity remains in the rock. Conversely, based on extrapolation of the conductivity trend suggested by the observations, some rocks would apparently continue to conduct at zero porosity (e.g., shaly sands).

Archie-Winsauer cannot honor these conditions. Also, clearly if $\varphi = 1$ then $F = 1$, a physical fact that the Winsauer formulation does permit, but does not enforce. Thus, at the boundaries of the porosity domain, the A-W model fails. The failures of fit at the end-points of the porosity domain are summarized in table 1.

A theory to account for conductivity over the whole porosity domain, including the limiting cases, would be more satisfying than a theory that merely fits the observed data, at least if it managed to also fit the observations as well as the A-W model.

RE-EXAMINING THE SEMINAL DATA SETS

The form of the power laws in use for resistivity interpretation was basically imposed by the most readily available method in the 1940s to simplify the representation of relationships; i.e., graph paper that obviated the labor entailed in logarithmic transforms of the data. But there are many other functions that would linearize the relationship among the data. For example, if the square root of $1/F$ is plotted against linear porosity, the data falls on a linear trend. Moreover, there is no imperative to impose linearity if some other relationship is suggested by the data. Does the data conceal any useful trends that would suggest an alternative to present industry standards that might in some way improve upon industry-standard methods? Figure 3 plots the Archie

Nacatoch sandstone data set as formation conductivity normalized by connate water conductivity (i.e., $\sigma_0 / \sigma_w = 1 / F$) on a linear scale, against porosity on a linear scale.

There is a clear suggestion that the parent conductivity distribution underlying the data increases more rapidly than porosity increases. There is also a hint that the underlying function will vanish at a porosity greater than zero. Looking at the data plotted on a scale that includes the 100% porosity end point (figure 4) strengthens the suggestion of curvature.

Porosity-Conductivity Analysis. Making the definition that the "fractional connected brine volume" is the fraction of porosity involved in conduction, represented as $\beta_c = \varphi - \varphi_\vartheta$ where φ_ϑ is the porosity at which the normalized conductivity vanishes, I notice that these data suggest that the rate of change of the normalized conductivity with respect to the fractional connected brine volume might be proportional to the fractional connected brine volume. I tentatively adopt the suggested direct proportionality relationship as a working hypothesis, and check whether the implications of the hypothesis are supported by observation.

These observations are summarized in the mathematical statement

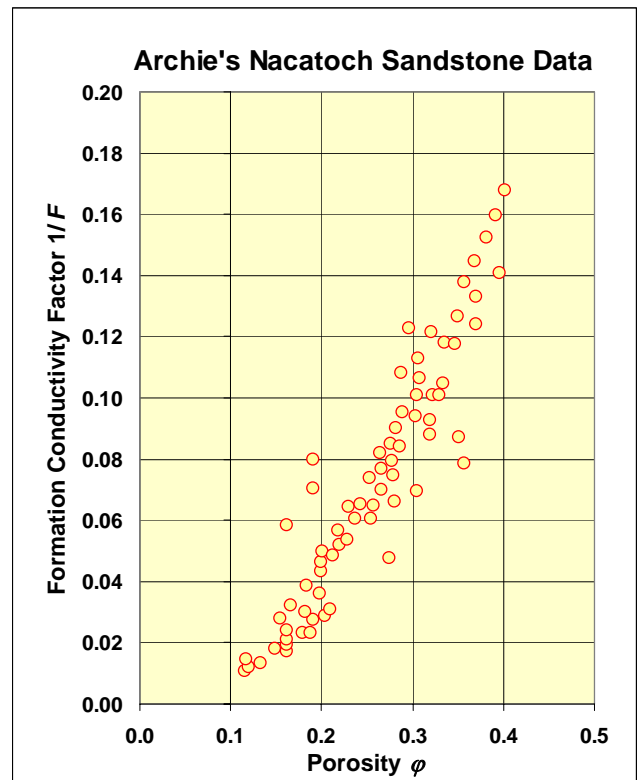|	boundary conditions	
φS_w / σ_t/σ_w	$\varphi S_w = 0$	$\varphi S_w = 1$
observed	$\neq 0$	1
Archie	0	1
Winsauer/ Humble	0	$1/a$
EET	0	$a_t + b_t \neq 1$

Table 1. Observed end-point values and the corresponding values demanded by Archie's and Winsauer's models. Electrical efficiency theory (EET), discussed hereinafter, is an alternative theory that also cannot accommodate the observed boundary conditions. The blue background represents observations; green background indicates agreement with observations; yellow backgrounds indicate disagreement with observations.

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Fig. 3. Archie's data plotted as normalized linear conductivity versus linear porosity.

$$\frac{d(\sigma_0/\sigma_w)}{d(\varphi-\varphi_{\vartheta})} \propto (\varphi-\varphi_{\vartheta}) \qquad (1)$$

where φ_{ϑ} is a "percolation threshold" and where $\varphi-\varphi_{\vartheta}$ is the amount of connected brine volume that supports conduction. Expressing (1) as an equality yields,
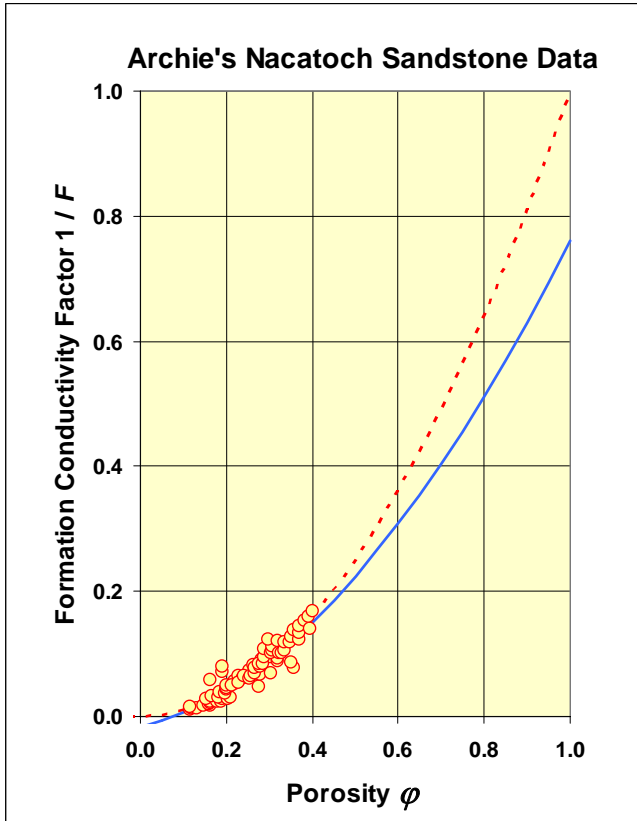
$$\frac{d(\sigma_0/\sigma_w)}{d(\varphi-\varphi_{\vartheta})} = \alpha_0 (\varphi-\varphi_{\vartheta}) \qquad (2)$$

where α_0 is the constant of proportionality. Also note the boundary conditions that $\sigma_0/\sigma_w = 1$ when $\varphi = 1$, and $\sigma_0/\sigma_w = 0$ when $\varphi = \varphi_{\vartheta}$. Equation (2) is immediately integrable, the solution being

$$\frac{\sigma_0}{\sigma_w} = \frac{\alpha_0}{2}(\varphi-\varphi_{\vartheta})^2 + \alpha_1 \qquad (3)$$

with α_1 being a constant of integration. Evaluating the boundary conditions yields $\alpha_1 = 0$ and $\alpha_0 = 2/(1-\varphi_{\vartheta})^2$, whence

$$\frac{\sigma_0}{\sigma_w} = \left(\frac{\varphi-\varphi_{\vartheta}}{1-\varphi_{\vartheta}}\right)^2 . \qquad (4)$$

Fig. 4. Archie's data plotted on a scale to include 100% porosity. The blue curve is an unconstrained quadratic least squares fit to the data in this bi-linear space. Note that this curve intercepts the porosity axis at $\varphi > 0$. The dotted red curve plots $1/F = \varphi^2$, or Archie's law with $m = 2$. Archie's law honors $F = 1$ at $\varphi = 1$, but forces the fit through the origin, an unnatural constraint.

This result, having been derived from Archie's data, should apply only to rocks similar to those studied by Archie and Winsauer, known as "Archie" rocks. Archie rocks are characterized as comprising non-conductive, water-wet, convex, equidimensional mineral grains, with interconnected brine being the only conducting phase in the rock. The domain of the independent variable φ is $0 \le \varphi \le 1$. Rocks occupy only a subinterval of this domain. Archie rocks occupy the "Archie subdomain," roughly $0.10 \le \varphi \le .40$. I also include the brine point $\varphi = 1$ in the Archie subdomain.

φ_{ϑ} is not a true percolation threshold, which is the porosity value at which electrical conduction vanishes, approaching zero from the right on the porosity-normalized conductivity plot. Rather, φ_{ϑ} is the projection to the porosity axis of the trend established by the porosity – normalized conductivity variation in samples representing the reservoir on the Archie-rock subdomain of the porosity domain. In a mature, low-porosity Archie rock, as cementation progresses pore throats begin to close, and the conductivity at some porosity value begins to decrease more rapidly with porosity reduction than on the Archie-rock subdomain, eventually reaching a true percolation threshold, different from the threshold predicted by the porosity–conductivity trend observed on the Archie sub-domain. Consequently, I refer to φ_{ϑ} as a "pseudo" per-colation threshold (PPT). I refer to (4) as a pseudo-percolation threshold theory (PPTT). Experience suggests that core used in analysis of Archie rocks is plugged on the Archie subdomain, with porosity values greater than, say, roughly 0.10, and few core plugs are on the trend that ends at the true percolation threshold. When this con-dition is satisfied the PPT can be used to parametrically characterize the data space in terms of this "pseudo" rock property.

Connections of PPTT to A-W Theory. Having analyzed Archie's data differently than Archie, and having reached a different result, it is interesting to find points at which the two analyses touch. In conductivity notation, Archie's equation for the formation resistivity factor – porosity relationship is $\sigma_0/\sigma_w = a^{-1}\varphi^m$ where Winsauer's a is included in the expression. Archie's model is seen to be a special case of equation (4) where $a = 1$, $m = 2$ and $\varphi_{\vartheta} = 0$. The purpose of m and a as adjustable parameters is to allow sufficient freedom for the Archie-Winsauer function to be fit to observations with a reasonably small residual error. Nevertheless, the A-W function is constrained to pass through $(\varphi, \sigma_0/\sigma_w) = (0,0)$ and is not constrained to contain $(\varphi, \sigma_0/\sigma_w) = (1,1)$. Thus, while A-W[*] may be fit for the purpose of interpolating within the observations, extrapolated to the boundaries it cannot correctly predict the boundary values. Equation (4), on the other hand, has only a single adjustable parameter, but is capable of repre-senting a percolation threshold, $(\varphi, \sigma_0/\sigma_w) = (\varphi_{\vartheta},0)$ and always contains $(\varphi, \sigma_0/\sigma_w) = (1,1)$. In matching the

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[*] Interestingly, Winsauer and his coauthors (1952) suggested a fitting equation with a pseudo-percolation threshold for the Humble data set: $F = (\varphi - 0.06)^{-1.49}$. The idea was not further developed.

boundary conditions (4) is successful, whereas A-W is not. Figure 5 illustrates additional connections between m and φ_ϑ. In particular, low porosity and variation in m are seen to be associated; for porosities greater than about 10 percent and percolation thresholds ranging from $-0.04 \leq \varphi_\vartheta \leq 0.04$, m is almost constant and approximately equal to 2. $m < 2$ is associated with negative percolation thresholds, corresponding to rocks with a conductive matrix component. For φ_ϑ increasingly greater than 0, m is increasingly greater than 2. This suggests that well-cemented rocks have relatively greater percolation thresholds. At low porosity m varies significantly with porosity. While the very low porosity subdomain is not in the range of PPTT normalized conductivity prediction, the need to introduce variable m at low porosity is seen to be attributable to Archie's law lacking a percolation threshold parameter. None of these correspondences contradicts any observations of m or percolation thresholds. However, a percolation threshold has a direct physical interpretation, whereas m is merely the slope of a line[*], suggesting the former model may incorporate more physics than the latter. The criterion for evaluating the relative merit of the PPTT versus the A-W functions vis-a-vis any particular data set will be the degree to which the sums of the squared residuals are minimized.

Connections of PPTT to EET. The normalized conductivity per unit length of a cylindrical brine-filled tube fashioned from insulating material (e.g., quartz) of outer radius R and inner radius r is given by $\sigma_0/\sigma_w = \varphi$ where $\varphi = r^2/R^2$ is the fractional cross-sectional area of brine having conductivity σ_w, and also the porosity of the tube. The cylindrical configu-

Fig. 5. The connection between Archie's m and φ_ϑ, the pseudo-percolation threshold parameter. This "m surface" is a plot of $\sigma_0/\sigma_w = \varphi^m = ((\varphi - \varphi_\vartheta)/(1 - \varphi_\vartheta))^2$ solved for m.

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[*] m is sometimes asserted to be a property of a single core plug. However, since a line requires two points for its definition, either m for a single point does not exist, or since $m = (\log F - \log(1))/(\log \varphi - \log(1)) = \log F / \log \varphi$ then m is the slope of the line connecting the given point to the point $(\varphi, F) = (1,1)$.

ration of the brine phase gives a maximum conductivity for a given volume of brine. Any redistribution of the brine phase into a more complicated geometrical configuration must reduce the conductivity. So, in general, $\sigma_0/\sigma_w = \varphi E_0$ where E_0 is a function of the geometrical configuration of the conducting phase, and where $0 \leq E_0 \leq 1$ (Herrick and Kennedy; 1993, 1994). This will be referred to as the "electrical efficiency theory" (EET).

Although $\sigma_0/\sigma_w = \varphi E_0$ always holds true, E_0 can be determined by analysis only for a few simple cases. For rocks, the form of E_0 is empirically determined using measurements of porosity and conductivity. When EET is applied to Archie rocks and E_0 (= $\sigma_0/(\sigma_w \varphi)$) is plotted against φ, the resulting trend tends to be linear: $E_0(\varphi) = a_0 \varphi + b_0$. From empirical observations, $a_0 \approx 1$ and $b_0 \approx 0$. In summary, for Archie rocks in the Archie subdomain

$$\frac{\sigma_0}{\sigma_w} = \varphi E_0 = a_0 \varphi^2 + b_0 \varphi. \qquad (5.a)$$

Note that $E_0 \approx \varphi$ as a consequence of $a_0 \approx 1$, $b_0 \approx 0$.

PPTT begins directly with observations of σ_0/σ_w vs. φ, leading to equation (4). Expansion of (4) leads to

$$\frac{\sigma_0}{\sigma_w} = \frac{1}{(1-\varphi_\vartheta)^2} \varphi^2 + \frac{-2\varphi_\vartheta}{(1-\varphi_\vartheta)^2} \varphi + \frac{\varphi_\vartheta^2}{(1-\varphi_\vartheta)^2}, \qquad (6.a)$$
$$\equiv a_p \varphi^2 + b_p \varphi + c_p.$$

Note that $a_p + b_p + c_p \equiv 1$ and that for $\varphi_\vartheta \approx 0$, $a_p \gg |b_p| \gg c_p$, and $a_p \approx 1$, $b_p \approx 0$, $c_p \approx 0$. c_p is so small as to be unobservable in practice. Further, from equation (4)

$$\frac{\sigma_0}{\sigma_w} = \varphi \left(\varphi \left(\frac{1 - \varphi_\vartheta/\varphi}{1 - \varphi_\vartheta} \right)^2 \right) \equiv \varphi E_p. \qquad (7) \quad \textbf{RRR}$$

Thus, $E_p \approx \varphi$ explicitly. PPTT and EET applied to Archie rocks lead to the same result. The linearity of E_p vs. φ and the magnitudes of $a_p \approx 1$ and $b_p \approx 0$ are analogous to the corresponding EET quantities. The EET and PPTT approaches are mutually consistent for Archie rocks since both are based on the same observation, namely that the normalized conductivity has a quadratic relationship to water content. The difference is that EET begins with a statement of principle followed by inspection of the E_0 and φ data to observe the form of E_0 vs. φ, whereas PPTT begins with an inspection of σ_0/σ_w vs. φ data, in search of an underlying principle for Archie rocks. However, the convergence of the results strongly suggests that the underlying porosity-normalized conductivity relationship is quadratic, as indicated, and indeed, Archie's relationship with $m = 2$ reduces to a quadratic. Since the form of A-W, EET and PPTT are each based on the same observations, it is not surprising that essentially
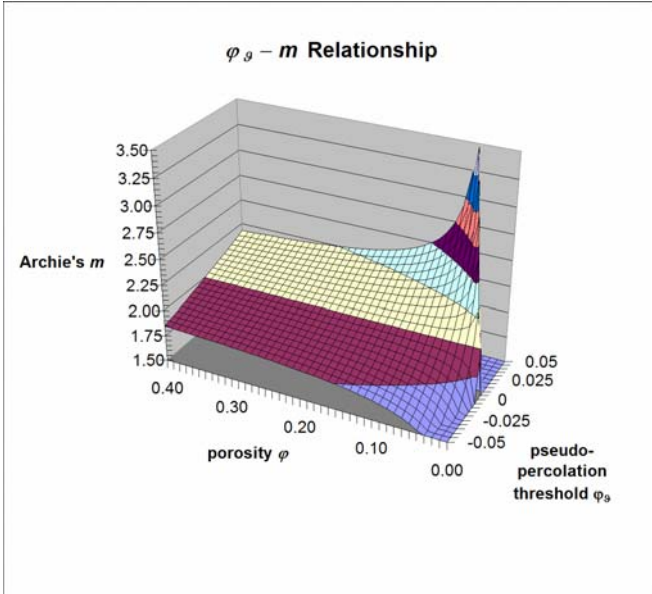
the same limiting case is obtained regardless of the approach employed.

For values of φ typically observed in Archie rocks, the term c_p / φ would be unobservable in the presence of noise. For example, the smallest value of porosity in Archie's Nacatoch sandstone data set is approximatey $\varphi = 0.10$. Using (4), the sum of the squared residuals for this data is minimized when $\varphi_9 = -0.002$. Using this φ_9 to compute the coefficents, then at 0.10 porosity $a_p\varphi^2 = 0.00996$, $b_p\varphi = 0.0003984$ and $c_p/\varphi = 0.0000398$. Thus $E_p \approx a_p\varphi + b_p$, the same relationship stated in EET. The value of c_p / φ is so small compared to a_p and b_p as to be unobservable in practice (0.4 percent of $a_p\varphi^2$). The relative contribution of c_p/φ diminishes at higher porosities. Thus, while EET is more general than PPTT, the properties of the EET parameter $E_0(\varphi)$ for Archie rocks can also be described in terms of a pseudo-percolation threshold.
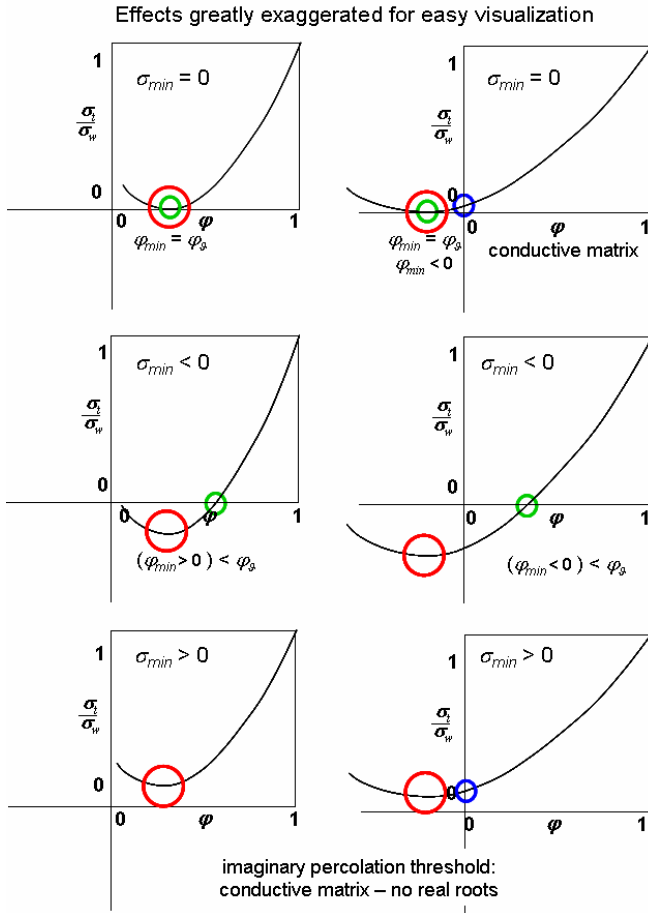
Fig. 6. Generally data will suggest the vertex of the parabola is not tangent to the porosity axis; another degree of freedom in the fitting equation must be introduced to account for this. The red circle locates the vertex of the parabola; the green circle locates the PPT. Note that if the vertex is above the abscissa there is no pseudo-percolation threshold. Rocks with a conductive component, such as clay coating the sand grains, may sometimes exhibit this behavior. The blue circles above the φ axis locate the excess conductivity for a slightly shaly rock. Archie's Nacatoch sandstone has this property.

Although explicitly developed using only Archie rocks, in clay-bearing sandstones the PPTT term c_p/φ becomes non-negligible, and can be considered as representing a parallel conductivity component that is independent of the brine's conductivity and fractional volume. This result was anticipated by the EET, which found for shaly sandstones that $E_0 = a_0\varphi + b_0 + c_0/\varphi$; (Herrick and Kennedy, 1993). According to Herrick and Kennedy, $c_0 = V_{sh}\sigma_{sh} / \sigma_w S_w$, providing a pseudo-percolation threshold-free physical interpretation for the c_0 / φ term. While PPTT does not purport to supply a physical explanation for shaly sandstone conductivity, it can nonetheless be useful for the description of the porosity-normalized conductivity relationship in shaly sandstones.

Additional Degree of Freedom. PPTT as explained to this point is elegant, and consistent with A-W and the behavior of EET. Unfortunately for elegance, in order to allow for best performance as an interpolating tool, the theory as expressed to this point is incomplete. The PPTT function as expressed in (4) is a parabola having its vertex tangent to the $\sigma_0 / \sigma_w = 0$ plane. This function cannot accommodate observations where the trend of the data demands that the fitting function is not tangent to the φ axis. Figure 6 illustrates some possible configurations. In order that the vertex of the parabola can be located above or below the $\sigma_0 / \sigma_w = 0$ axis another degree of freedom in the fitting function is required. With the introduction of this degree of freedom the pseudo-percolation threshold is no longer at the vertex of the parabola, and no longer appears in the conductivity-porosity formula. Equation (4) is modified to include an additional term, and the φ_9 term is replaced by the porosity coordinate of the vertex, which is no longer the pseudo-percolation threshold. The coordinates of the vertex of the parabola are denoted by σ_{min} and φ_{min} respectively, and the data are assumed to satisfy

$$\frac{\dfrac{\sigma_0}{\sigma_w} - \dfrac{\sigma_{min}}{\sigma_w}}{1 - \dfrac{\sigma_{min}}{\sigma_w}} = \left(\frac{\varphi - \varphi_{min}}{1 - \varphi_{min}}\right)^2. \qquad (8)$$

The adjustable parameters are the coordinates of the verticies of the parabola. σ_{min} / σ_w is the minimum normalized conductivity value attained by the function and φ_{min} is the porosity correponding to this value. The left side remains normalized as $\sigma_0 / \sigma_w \to 1$, and reduces to σ_0 / σ_w as $\sigma_{min} \to 0$. Although σ_{min} and φ_{min} do not have the direct physical significance of φ_9, the physical significance of the pseudo-percolation threshold is not lost; the conductivity trend defined by the observations still vanishes at the pseudo-percolation threshold. In that sense φ_9 remains the key "observable" of PPTT; however, the PPT is no longer the curve fitting parameter. The intercept of the conductivity parabola with the porosity axis, (i.e., the PPT, φ_9), is given in terms of the vertex coordinates by

$$\varphi_{\vartheta} = \varphi_{min} + (1 - \varphi_{min})\sqrt{\frac{-\sigma_{min}/\sigma_w}{(1 - \sigma_{min}/\sigma_w)}}, \qquad (9)$$

$$\sigma_{min} < 0,$$

where σ_{min} and φ_{min} are selected to minimize the sum of the squared residuals using (φ, σ_0) observations, and equation (8).

As shown in figure 6, it is possible for the trend implied by the data to have a negative intercept ($\varphi_{\vartheta} < 0$), or even fail completely to intercept the porosity axis. This could happen for a conductive rock matrix. If the corresponding σ_{min} is positive; there is no pseudo-percolation threshold. In such a case φ_{ϑ} from (9) is a complex number with no obvious physical interpretation. However σ_{min} and φ_{min} are still useful for curve fitting.

Equation (8) is analogous to the A-W expression $\sigma_0 / \sigma_w = a^{-1}\varphi^m$. Both equations require two adjustable parameters to best honor the observations. The equations become identical when σ_{min} and φ_{min} are both equal zero, and $m = 2$, $a = 1$. A-W is simpler in appearance, but it cannot accommodate the existence of a pseudo-percolation threshold different from zero except by making m a porosity-dependent variable $m(\varphi)$. A-W gives no prescription for the determination of $m = m(\varphi)$. Further, no a priori apparent physical significance can be assigned to m or a. Finally, for $a \neq 1$, the A-W formula fails to give $F = 1$ at $\varphi = 1$. Although A-W fails to honor the boundary conditions inherent in the conductivity – porosity relationship, the real comparison of efficacy is the minimization of the sum of the squared residuals between the observed data and the predictions of the two models. From the statistical point of view (and assuming the data contains no anomalous outliers), the method with the smallest sum of squared residuals is the better predictor of the relationship between porosity and conductivity.

Comparisons using Archie's data are complicated by his not having published his data except as a postage-stamp-sized figure. Numerical comparisons thus begin with a digitization of this data. Some of the data points are hard to resolve, and different analysts might not only find slightly different coordinates for the data points, but might even disagree as to the number of points. With this caveat in mind, I hand digitized Archie's data. Then using (8), I computed the sums of the squared residuals, using Archie's power law model, to be 0.015234 (using $m = 2$, $a = 1$) and the sums of the squared residuals using PPTT to be 0.015020 (using $\sigma_{min}/\sigma_w = 0.004$, $\varphi_{min} = 0.0075$). The difference is very small, but the small difference in the sums of the squared residuals is in favor of PPTT. Whereas Archie's model demands that conductivity vanish at zero porosity, PPTT indicates a small residual conductivity ($\sigma_0 \approx .004\sigma_w$) at zero porosity. Note that $\sigma_{min} > 0$ implies that the pseudo-percolation threshold is complex; i.e., the curve does not quite intersect the poros-

ity axis. This is consistent with Archie's (1950; p. 947) description of the Nacatoch sandstone as shaly[*].

Archie's data for the Nacatoch sandstone fits $F = \varphi^{-2}$ almost perfectly, although Archie never claims that $m = 2$ for the Nacatoch data set; only that the range of m includes 2. Using my digitization of Archie's data, the sum of the squared residuals is a minimum (0.015220) for $m = 1.995$, hardly statistically distinguishable from $m = 2$ using graphical methods to determine the slope.

The comparable sum of squared residuals for the Winsauer data and model (using $m = 2.15$, $a = 0.62$) is 0.003636 versus 0.003393 using PPTT with $\sigma_{min} / \sigma_w = -0.015$ and $\varphi_{min} = -0.091$, corresponding to $\varphi_{\vartheta} = 0.0416$. Again, this difference, while small, is in favor of the PPTT model. This result is obtained by not including the lowest and highest porosity points in the Winsauer data, rejecting them as outliers. Winsauer himself omitted the lowest porosity point (i.e., his Oil Creek, sample #3) as an outlier in his estimate of the Humble formula parameters. My observation is that the highest porosity point (the Ottawa sand) is also an anomalous outlier. Had it been omitted from Winsauer's data (and using the normalized conductivity residuals in minimization), the Humble formula would have been $F = 0.781 \varphi^{-1.955}$, illustrating the sensitivity of the Humble formula to this one point. Accordingly, when appropriate in my analysis of the Winsauer data, I omit this point as well as omitting Oil Creek sample #3. Since Winsauer listed his data, one has the impression that a numerical regression was actually performed. However, I cannot find a method of regressing the data that yields the Humble formula parameters. Results vary depending upon whether the regression is done using logarithm of porosity and resistivity, linear porosity and resistivity, or linear conductivity and resistivity, etc. Each method requires slightly different values of a and m to minimize the residual for that regression. Since Winsauer does not specify a method, perhaps his method, like Archie's, was graphical. In any event, Winsauer's specification of parameters to two and three significant figures cannot be interpreted as an expression of accuracy or precision in his result. Indeed, although the form of the Humble equation has become the industry-standard, the parameter values are determined anew for each new set of data.

The expansion of (8), followed by comparison to EET, shows that inclusion of σ_{min} / σ_w merely multiplies a_p, b_p, and c_p by a factor of $1 - \sigma_{min} / \sigma_w$ (≈ 1) without changing the behaviors of the coefficients already described. However, the remaining σ_{min} / σ_w (≈ 0) term provides a means to shift the vertex of the parabola slightly off the φ axis if necessary.

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[*] "Friable sandstone, poorly sorted grains, shaly and calcareous; comparatively high porosity for permeability (Nacatoch formation, Bellevue, Louisiana)," Archie 1950.

Although σ_{min} has been introduced as an ad hoc parameter in equation (8), it can be shown that in fact (8) follows from boundary conditions consistent with the observed trend in the data, and thus is less ad hoc in character than it first appears.

CONDUCTIVITY
–WATER SATURATION RELATIONSHIP

The Saturation Exponent. There is no seminal data set comparable to Archie's or Winsauer's for resistivity index – water saturation observations, but data from a recently published article (Hamada and others, 2002) is available to motivate a discussion. The practical significance of conductivity measurements made on reservoir rocks is their use in estimating water saturation given resistivity and porosity from logging instruments. Archie (1942) suggested plotting the logarithm of the water saturation, S_w, against the logarithm of resistivity index, $I = R_t / R_0$. On this plot the data suggest a relationship given by $S_w = I^n$ (always written $I = S_w^{-n}$ in the modern literature) where $n (\approx 2)$ is called the saturation exponent. The data for this plot originates from a single core plug which is assigned a porosity value consistent with its average porosity as determined by core analysis. The value of n assigned to this sample is the slope of a least-squares regression line fitted to the S_w–I data on a bi-logarithmic plot.

Characterization of a reservoir unit requires an n that is representative of the entire unit, and this requires that resistivity index - water saturation analysis be performed upon multiple samples from the same reservoir, both with various samples having the same porosity, and also with samples displaying as much variation in porosity as the reservoir under study provides. The resulting data set yields multiple values of n; one for each core plug in the sample (as illustrated in figure 7). The problem then is to use this data in some fashion to characterize the reservoir. Further, there is no a priori reason to rule out the dependence of saturation exponents upon porosity or upon water saturation. (A water saturation-dependent n is analogous to a porosity-dependent m.) Attempts to honor these dependencies may occasionally be made in formation evaluation, but commonly the data is averaged to yield a single value of n to characterize the entire reservoir. There is no industry-standard method for calculating this average. The arithmetic average $\bar{n} = (1/N)\sum_{i=1}^{N} n_i$ is invariably used; but why should the arithmetic be chosen over, say, the geometric $\bar{n} = \sqrt[N]{\prod_{i=1}^{N} n_i}$ or harmonic $1/\bar{n} = (1/N)\sum_{i=1}^{N} 1/n_i$ averages? In fact, since n is a measure of slope, it can easily be shown that the arithmatic average of multiple n values is not appropriate. For example, consider the extreme case of averaging the slopes of a horizontal line (zero slope) and a vertical line (infinite slope), the result for the arithmetic average being infinite. On the other hand, the average of the angles associated with these

slopes (0° and 90° respectively) is 45°, a more representative result, correponding to a slope of unity. This suggests an average defined as

$$\bar{n} = \tan(\,(1/N)\sum_{i=1}^{N} \tan^{-1}(n_i)\,) \qquad (10)$$

provides an average most properly suited to the data set.

As discussed above, in conventional formation evaluation, the use of saturation exponent data derived from a collection of cores necessarily involves a choice of some method of averaging the data. But this prompts the question: Is it possible to globally analyze this data giving equal (or some statistically appropriate) weight to each data point rather than first analyzing the core plugs individually to extract saturation exponents, then averaging the saturation exponents?

Water Saturation-Conductivity Analysis. First note that the bi-logarithmic water saturation – resistivity index plot can be recast as a plot of water saturation versus "conductivity" index, all the scales being linear. Viewed in this space the conductivity index appears as a non-linear function of water saturation, very similar to the linear porosity–formation conductivity factor plot. The data shown in
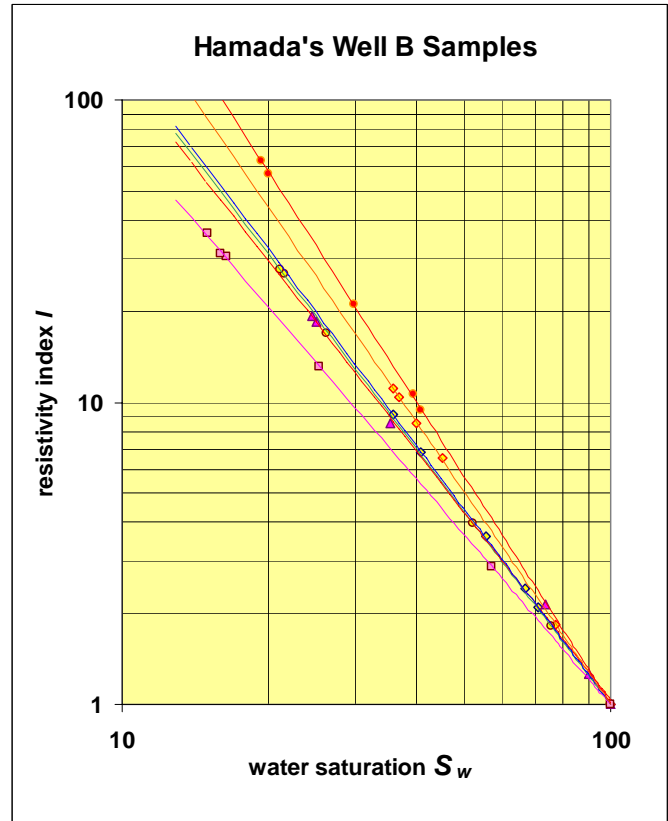
Fig. 7. A water saturation - resistivity index plot using data from Hamada et al. well B. The lines are fitted through the data acquired for a number of core plugs. Power law behavior is strongly suggested, but the suggested trends would curve upward if lower water saturations could be achieved in the desaturation experiment.

figure 7 are replotted in figure 8. Note that the abscissa in figure 8 could be rescaled for each core plug by multiplying S_w by the porosity of the plug φ, scaling the abscissa as fractional brine volume. The result is a plot of conductivity index versus fractional brine volume for each individual core plug. If the ordinates are multiplied by the conductivity formation factor, the result is the normalized conductivity σ_t / σ_w. The resulting data, when plotted above the $\varphi - S_w$ plane (figure 9) is scattered above and below some smooth—but a priori unknown—normalized conductivity surface representing the data's parent distribution in porosity–water saturation space. A representation of this surface can be developed in terms of the data and boundary conditions using a method similar to the development of equations (4) and (8).

Brine Volume-Conductivity Analysis. Inspection of σ_t / σ_w plotted against φS_w with φ held constant (the data for this plot is the same as required for a resistivity index – water saturation plot), with the introduction of a variable ϑ to denote the pseudo-percolation threshold for this case, suggests that the analog to equation (1) for $S_w < 1$ is

$$\frac{d(\sigma_t / \sigma_w)}{d(\varphi S_w - \vartheta)} \propto (\varphi S_w - \vartheta) . \qquad (11)$$

ϑ is a generalization of φ_ϑ to a curve in the φS_w plane, and $\varphi S_w - \vartheta$ is a generalization of the connected brine volume to the case where $S_w < 1$. As with equation (1), the rate of change of normalized conductivity with respect to connected brine volume is proportional to the connected brine volume. In this proportionality it is the fractional volume of the conducting phase, $\beta_c = \varphi S_w - \vartheta$,

rather than porosity alone, that plays the main role in the rate of change of normalized conductivity. Writing (11) with a proportionality constant α_2

$$\frac{d(\sigma_t / \sigma_w)}{d(\varphi S_w - \vartheta)} = \alpha_2 (\varphi S_w - \vartheta) \qquad (12)$$

subject to the boundary condition that $\sigma_t / \sigma_w \to 1$ as $\varphi S_w \to 1$, and $\sigma_t / \sigma_w \to 0$ as $\varphi S_w \to \vartheta$. The general solution is

$$\frac{\sigma_t}{\sigma_w} = \frac{\alpha_2}{2} (\varphi S_w - \vartheta)^2 + \alpha_3 . \qquad (13)$$

The boundary condition $\sigma_t / \sigma_w \to 0$ as $\varphi S_w \to \vartheta$ requires that $\alpha_3 = 0$. The condition that $\sigma_t / \sigma_w \to 1$ when $\varphi S_w \to 1$ requires that $\alpha_2 = 2/(1 - \vartheta_I)^2$, where $\vartheta_I = \vartheta(\varphi S_w = 1)$. Thus

$$\frac{\sigma_t}{\sigma_w} = \left(\frac{\varphi S_w - \vartheta}{1 - \vartheta_I} \right)^2 . \qquad (14.a)$$

This expression must satisfy certain conditions. By analogy with equation (4), the argument in the parenthesis of (14.a) is expected to be the difference between the total fractional brine volume surface σ_t / σ_w, and an isolated fractional brine volume surface β_i, and is expected to reduce to (4) when $S_w = 1$, or to an analogous form in S_w when $\varphi = 1$. The form of the fractional connected brine volume, $\varphi S_w - \vartheta$, suggests that ϑ is more complicated than a simple constant, and the general expectation is that $\vartheta = \vartheta(\varphi, S_w)$.
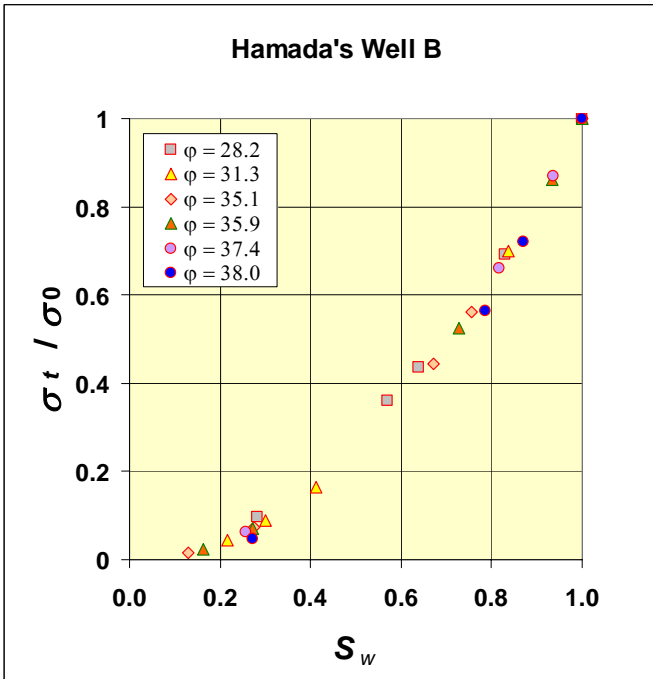
Fig. 8. Linear water-saturation – conductivity-index plots exhibit a resemblance to porosity-conductivity relationships. This is the same data as in figure 7.
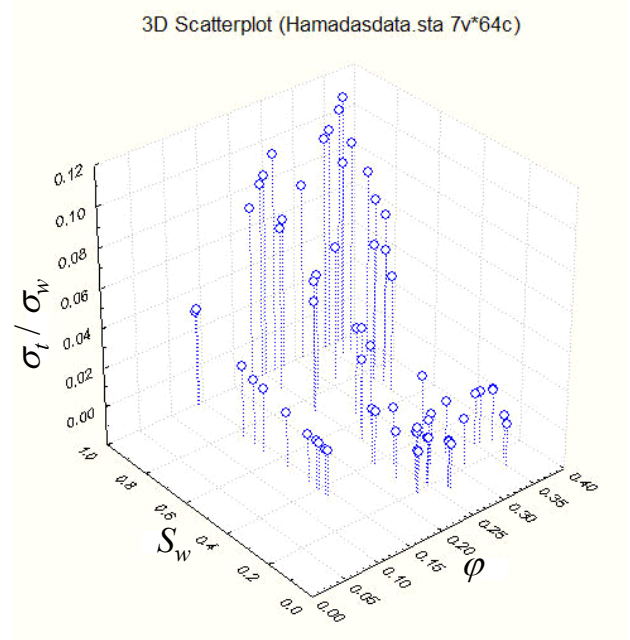
Fig. 9. Hamada's data presented as normalized conductivity above the $\varphi - S_w$ plane. Each core plug generates a row of points having constant φ, variable S_w and σ_t / σ_w.

The Unit Conductivity Cube. Equation (14.a) expresses the normalized conductivity as a function of porosity and water saturation. Noting that both φ and S_w are constrained to the interval between 0 and 1, a convenient representation is to associate φ and S_w with orthogonal axes in a plane with σ_t / σ_w represented as a surface above this plane. Note also that $0 \le \sigma_t / \sigma_w \le 1$, so that the data space is a 3-dimensional unit cube. This cube contains, and renders visualizable, all the data in the porosity – water-saturation – normalized-conductivity experiment, including the boundary values. This is in contrast to the bi-logarithmic scales customarily used, which can display neither zero φ or S_w, nor infinite F or I, nor $\sigma_t / \sigma_w = 0$.

Consider, for example, the Humble formula $\sigma_t / \sigma_w = \varphi^{2.15} S_w^2 / 0.62$ plotted above the $\varphi - S_w$ plane (figure 10). According to the formula, σ_t / σ_w vanishes along the φ and S_w axes, and attains the value of $1.0/0.62 = 1.612$ above the point called the brine pole, $(\varphi, S_w) = (1,1)$. The values at the boundaries required by the Humble formula are not those observed: at $(\varphi, S_w) = (1,1)$, σ_t / σ_w should equal 1.0, and generally σ_t / σ_w does not vanish everywhere precisely on the φ and S_w axes. Conversely, when equation (14.a) is plotted in the unit conductivity cube, for $\varphi S_w = 1$ σ_t / σ_w always is equal to 1.0, and—in general—the normalized conductivity vanishes along a curved contour in the $\varphi - S_w$ plane rather than precisely on the axes. A surface defined by real data will necessarily more closely resemble the pseudo-percolation threshold surface more than the surface defined by the Humble formula. Conductivity measured upon real rocks should van-

ish along a curve on the $\varphi - S_w$ plane and include the brine pole $(\varphi, S_w) = (1,1)$ (figure 11).

Archie rocks will have porosities limited to between about 0.10 and 0.40, Hamada's rocks being an example (figure 9). I have called this subdomain of the porosity-water saturation space the "Archie subdomain" (figure 12). A close look at the Archie subdomain reveals that all the resistivity data acquired on core plugs is accommodated in this plot. For example the data customarily plotted as logarithm of porosity against logarithm of formation resistivity factor for water-filled rocks plots near the red line in the $S_w = 1$ plane as formation conductivity factor (or normalized conductivity) against porosity. The data from the water saturation – resistivity index experiment plots parallel to (the green) lines of constant porosity and variable water saturation. Conductivity vanishes along a contour in the $\varphi - S_w$ plane highlighted by the yellow line. According to (14) normalized conductivity is symmetric in φ and S_w. A white line extending from the origin is the axis of $\varphi - S_w$ symmetry.

The interpretation of $\vartheta(\varphi, S_w)$. In equation (4), which refers to the $S_w = 1$ plane, φ_ϑ is a simple constant; however, the situation is more complicated in the case that $S_w < 1$. A comparison of equations (1) and (11) on the one hand, and (4) and (14.a) on the other, suggests that the argument of the square in (14.a) should be identified with a connected brine volume, β_c. Consider the connected brine volume function in the $S_w = 1$ plane to be extended parallel to the S_w axis. The result is a plane with contours parallel to the S_w axis, inclined upward toward $\varphi = 1$. This plane is described by

$$\beta_{c,\varphi} = \varphi - \frac{\varphi_\vartheta}{1-\varphi_\vartheta}(1-\varphi). \qquad (15)$$
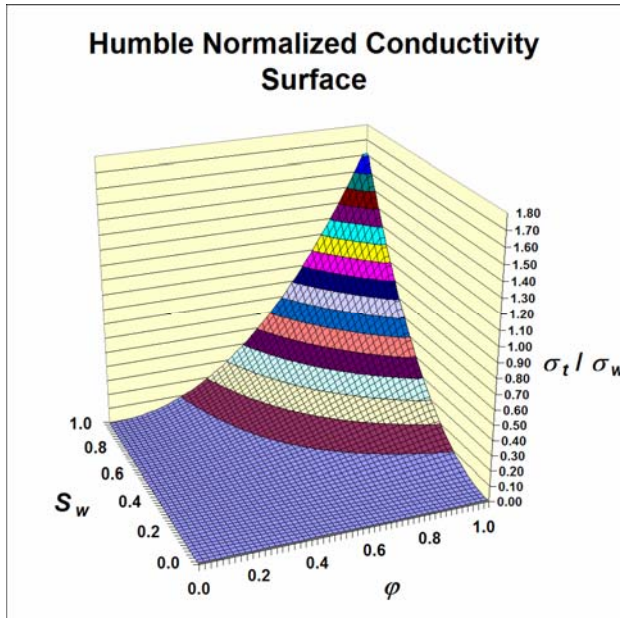
Fig. 10. The surface in the unit conductivity cube defined by the Humble formula. This surface has unphysical properties; conductivity vanishes on the $\varphi = 0$ and $S_w = 0$ axes, and $\sigma_t / \sigma_w \rightarrow 1.0/0.62 = 1.61$ as $\varphi S_w \rightarrow 1$.
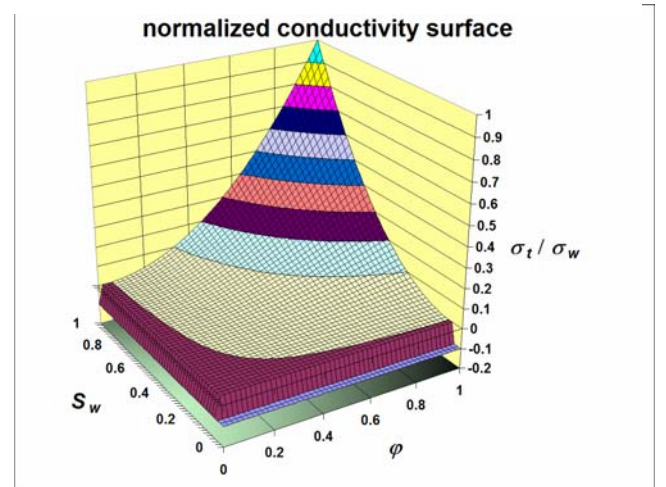
Fig. 11. The normalized conductivity surface in the unit conductivity cube corresponding to the pseudo-percolation threshold theory, equation (14.a). Conductivity vanishes on the boundary contour between the cream-colored surface and the maroon surface, and $\sigma_t / \sigma_w \rightarrow 1$ as $\varphi S_w \rightarrow 1$.

Also consider the analogous connected brine volume function in the $\varphi = 1$ plane extended parallel to the φ axis. This plane is described by

$$\beta_{c,S_w} = S_w - \frac{S_g}{1-S_g}(1-S_w). \qquad (16)$$

The product of these extended planes

$$\beta_c = \left(\varphi - \frac{\varphi_g}{1-\varphi_g}(1-\varphi)\right)\left(S_w - \frac{S_g}{1-S_g}(1-S_w)\right), \qquad (17)$$

expanded and grouped as

$$\beta_c = \varphi S_w -$$
$$\left[\frac{\varphi_g}{(1-\varphi_g)}(1-\varphi)S_w + \frac{S_g}{(1-S_g)}(1-S_w)\varphi \qquad (18)\right.$$
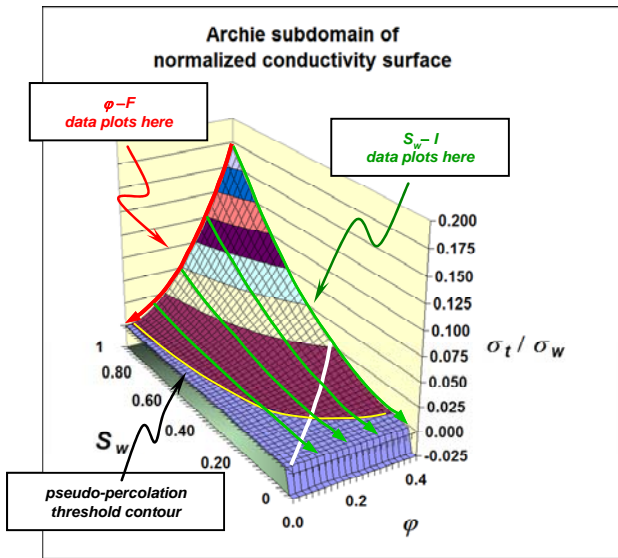$$\left. - \frac{\varphi_g}{(1-\varphi_g)}\frac{S_g}{(1-S_g)}(1-\varphi)(1-S_w)\right]$$

is in the form $\beta - \beta_i$ if the term in the square bracket (i.e., β_i) is interpreted as the isolated fractional brine volume not participating in conduction. Rationalizing this expression gives

$$\beta_c = \frac{\varphi S_w - (\varphi S_g + \varphi_g S_w - \varphi_g S_g)}{1-(\varphi_g + S_g - \varphi_g S_g)}. \qquad (19)$$

Comparing this result to equation (14.a), note that making the definitions

$$\vartheta(\varphi, S_w) \equiv \varphi S_g + \varphi_g S_w - \varphi_g S_g \qquad (20.a)$$

and

Fig. 12. The Archie subdomain in the unit conductivity cube. Porosity-conductivity data plots near the red curve in the $S_w = 1$ plane; water saturation-resistivity index data plots along lines parallel to the green lines. Normalized conductivity vanishes on the yellow contour, the pseudo-percolation threshold. The line of $\varphi - S_w$ symmetry is shown by the white contour.

$$\vartheta_I \equiv \varphi_g + S_g - \varphi_g S_g = \vartheta(1,1) \qquad (20.b)$$

brings the argument of expression (14.a) and (19) into conformance. The factored form of (19) is

$$\beta_c = \frac{(\varphi - \varphi_g)(S_w - S_g)}{(1-\varphi_g)(1-S_g)}. \qquad (21)$$

This clearly reduces to the argument of equation (4) for $S_w \to 1$. Normalized conductivity in terms of connected fractional brine volume $\beta_c = \beta - \beta_i$ can be expressed in several forms. The most convenient form for the display of PPTT adjustable parameters is, in terms of normalized conductivity,

$$\frac{\sigma_t}{\sigma_w} = \left(\frac{\varphi - \varphi_g}{1-\varphi_g}\right)^2\left(\frac{S_w - S_g}{1-S_g}\right)^2. \qquad (14.b)$$

More Electrical Efficiency Theory Connections. For $S_w < 1$ the EET is expressed as (Herrick and Kennedy, 1994)

$$\sigma_t / \sigma_w = E_t \varphi S_w = a_t (\varphi S_w)^2 + b_t \varphi S_w. \qquad (5.b)$$

Expanding (14.b) and making substitutions from (20.a) and (20.b) results in

$$\frac{\sigma_t}{\sigma_w} = \frac{1}{(1-\vartheta_I)^2}(\varphi S_w)^2 + \frac{-2\vartheta}{(1-\vartheta_I)^2}\varphi S_w + \frac{\vartheta^2}{(1-\vartheta_I)^2} \qquad (6.b)$$
$$= a_p (\varphi S_w)^2 + b_p \varphi S_w + c_p$$

It is seen that the relationship of EET as a special case of PPTT remains valid for $S_w < 1$.

Additional Degree of Freedom. As with equation (4), where the surface corresponding to equations (14) meets the $\varphi - S_w$ plane it is everywhere tangent to the plane. In order to accomodate other relationships of the normalized conductivity surface not tangent to the $\varphi - S_w$ plane that data usually require, it is necessary to provide another degree of freedom. This degree of freedom is supplied exactly as in equation (8) to yield

$$\frac{\frac{\sigma_t}{\sigma_w} - \frac{\sigma_{min}}{\sigma_w}}{1-\frac{\sigma_{min}}{\sigma_w}} = \left(\frac{\varphi S_w - \vartheta_{min}}{1-\vartheta_I}\right)^2. \qquad (22.a)$$

Equation (22.a) can also be displayed in factored form as

$$\frac{\frac{\sigma_t}{\sigma_w} - \frac{\sigma_{min}}{\sigma_w}}{1-\frac{\sigma_{min}}{\sigma_w}} = \left(\frac{\varphi - \varphi_{min}}{1-\varphi_{min}}\right)^2\left(\frac{S_w - S_{min}}{1-S_{min}}\right)^2 \qquad (22.b)$$

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from which the definitions $\vartheta_{min} \equiv \varphi S_{min} + \varphi_{min} S_w - \varphi_{min} S_{min}$ and $\vartheta_I \equiv \varphi_{min} + S_{min} - \varphi_{min} S_{min}$ follow. φ_{min} is the value assumed by φ in the $S_w = 1$ plane on the ϑ_{min} contour and S_{min} is the value assumed by S_w in the $\varphi = 1$ plane on the ϑ_{min} contour. $\sigma_t = \sigma_{min}$ everywhere on this contour; i.e., the left side of (22) vanishes. Further, in (22.b) all of the curve-fitting parameters appear not only explicitly, but also with each parameter uncoupled from the others in its own separate factor, making equation (22.b) the most convenient representation for their determination by numerical methods. It is also obvious that in the limit $S_w \rightarrow 1$, equation (22.b) reduces to equation (8) as is required.

φ–S_w Symmetry. The A-W theory, including water saturation, is usually stated as $R_t = a\, \varphi^{-m} S_w^{-n} R_w$, and has the normalized conductivity formulation $\sigma_t / \sigma_w = \varphi^m S_w^n / a$. These formulations seem to give the porosity and water saturation factors unequal effect in influencing conductivity, weighting the factors asymmetrically according to any difference in m and n ($\varphi^m S_w^n \neq S_w^m \varphi^n$). However, the bulk volume fraction of the conducting phase is φS_w, where the factors in this product are formally symmetric ($\varphi S_w = S_w \varphi$). Moreover, although the abscissa of the water saturation – resistivity index plot is conventionally presented scaled in water saturation from 0-to-1, if the labels on the abscissa scale are multiplied by porosity, the axis is converted to a fractional bulk volume of brine axis, the curve shape remaining unchanged. Recalling that Archie's equation is assembled from $F = R_0/R_w = a\varphi^{-m}$ and $I = R_t / R_0 = S_w^{-n}$ by elimination of R_0, forming the product IF, a plot of IF (= R_t/R_w) against φS_w is a graphical means of achieving a similar result. All resistivity data for the formation resistivity factor plot and resistivity index plot can be represented on this single plot (Herrick, 1988). Using these relabeled axes an Archie-like power law would have the form $IF = R_t/R_w = \sigma_w / \sigma_t = (\varphi S_w)^{-n'}$, where $IF \equiv G$ is a generalized resistivity index and n' is the slope of the line that minimizes the sums of the squared residuals. This slope is a generalized saturation exponent, n', based upon the entire data ensemble.

The Archie porosity exponent is recovered from

$$\lim_{S_w \rightarrow 1} n' \rightarrow m$$

where m in this limit has the same value as when it is determined using the conventional $\log\varphi$–$\log F$ plot. However, an explicit porosity exponent, separate from an explicit saturation exponent, is not required in using this method.

Plotting IF against φS_w puts φ and S_w on a formally equal footing using Archie's conventional bi-logarithmic plot. The function $G = (\varphi S_w)^{-n'}$ is symmetric in φ and S_w and reduces to Archie's equation with $m = n$ for all values of n', including $n' = 2$. Moreover, by plotting the generalized resistivity index of an entire ensemble of core plugs against φS_w (instead of merely S_w) on the same plot, and fitting a

single line to the resulting cluster of points, this technique avoids an explicit choice of an averaging method for the saturation exponent. The disadvantage, from the curve-fitting point of view, is that there is only the single adjustable parameter n' (or only two, if Winsauer's a is also used), and this will place a higher limit on the minimum sum of the squared residuals than the use of three adjustable parameters. Like the conventional Archie model, this formulation does not extrapolate correctly to the boundaries.

These considerations further support my assertion that Archie's particular formulation of the formation conductivity function is not inevitable but is merely one among several alternative possible relationships, even in the context of the customary bi-logarithmic plots.

Symmetry in the normalized conductivity function is not unreasonable when normalized conductivity is viewed as a surface over the φ–S_w plane. The gradient in the φ direction (S_w coordinate held constant) is similar to the gradient in the S_w direction (φ coordinate held constant). In both cases the conductivity is diminished by similar reductions in the brine volume fraction, the difference being that in the former case the brine volume is reduced by reduction of the pore volume, the water saturation being constant, while in the latter case the brine volume is reduced by displacing brine with a non-conductive hydrocarbon phase, the porosity being constant. These brine configurations are illustrated in figure 13. Whether the behavior of the normalized conductivity differs in these two orthogonal directions depends upon whether the geometry of the conducting phase in the directions of the respective gradients influences conductivity differently. This is a matter to be decided by observations on rocks, or by experiment. The observations suggest that in many cases similar reductions in fractional brine volume lead to similar reductions in conductivity regardless of the mode of fractional brine volume reduction. This corresponds to the common observation in A-W theory that $m \approx n$, and suggests that bulk conductivity, at least in Archie rocks, is not especially sensitive to the detailed geometry of the conducting phase. This observation suggests that the minimum cross-sectional area of the brine phase normal to current flow controls the bulk conductivity of the medium, since the bulk conductivity is ultimately determined (and limited) by this cross-sectional area. Note that there is no role for tortuosity in this formulation. This supports Herrick's and Kennedy's (1994) demonstration that tortuosity in Archie rocks represents, at most, a negligible perturbation of the primary conduction-limiting mechanism, the pore throat aperture area.

A Saturation Equation. The modeled normalized conductivity - fractional brine volume relationship is

$$\frac{\sigma_t}{\sigma_w} = \frac{\sigma_{min}}{\sigma_w} + \left(1 - \frac{\sigma_{min}}{\sigma_w}\right)\left(\frac{\varphi S_w - \vartheta_{min}}{1 - \vartheta_I}\right)^2 . \quad (23)$$

The usual interest in the conductivity–fractional brine volume relationship in formation evaluation is for the estimation of water saturation from conductivity and porosity as inferred from logging instrument responses. In this form equation (23) is

$$S_w = \frac{1}{\varphi}\left(\vartheta_{min} - (1 - \vartheta_I)\sqrt{\frac{\sigma_t / \sigma_w - \sigma_{min}/\sigma_w}{1 - \sigma_{min}/\sigma_w}}\right) \quad (24)$$

This saturation equation reduces to Archie's saturation equation evaluated with $m = n = 2$ for the case where σ_{min}, φ_{min}, and S_{min} all equal zero.
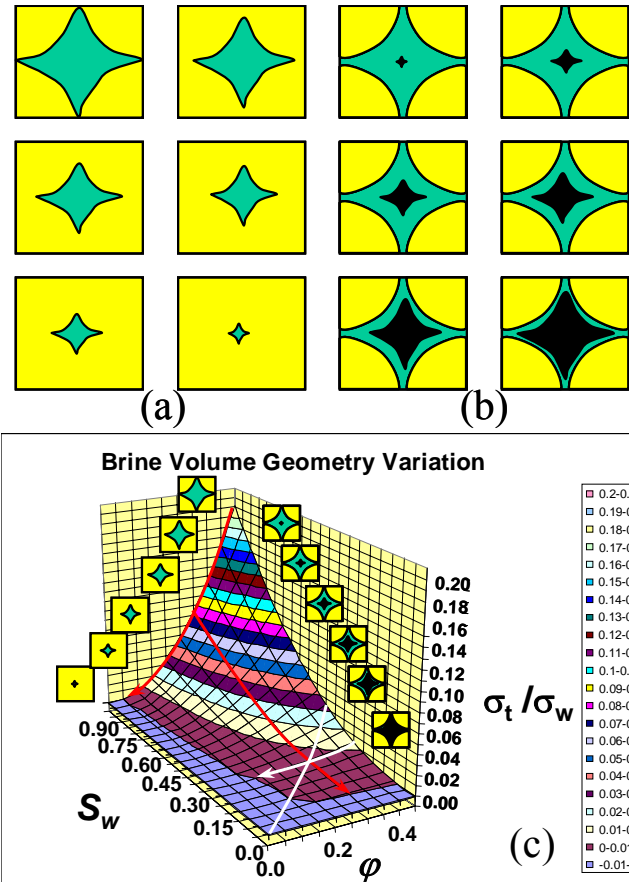

Fig. 13. In real rocks changes in φ produce changes in brine geometry and volume (cartooned in the two left columns above, (a)) that are different from the changes in brine geometry of the same volume produced by changes in S_w (cartooned in the two right columns, (b)). Fractional brine volume changes arising from reduction in porosity (along the line of constant $S_w = 1$) would seemingly result in a significantly different configuration of brine than changes arising from reduction in water saturation (along lines of constant φ). This difference might be expected to induce large differences in how conductivity changes with fractional brine volume according to the mode of brine volume reduction. However, it is observed that frequently conductivity is not particularly sensitive to which mode of brine reduction is operating. In other words, albeit surprising, the normalized conductivity surface ((c) above) seems to be nearly symmetric with respect to an exchange of φ and S_w.

Application to Hamada et al. Data. The use of equation (24) is to estimate water saturation everywhere in the Archie subdomain given a normalized conductivity surface predicted from observations at discrete points in the subdomain. The estimation is affected by finding adjustable parameters that minimize the sums of the squared residuals with respect to observed σ_t / σ_w and the predictions of equation (23); i.e., least squares. Although the PPTT model is nonlinear in its adjustable parameters, σ_t / σ_w is a smooth, well-behaved function of the adjustable parameters. It is a simple matter to use any number of commerically available programs to discover values of the parameters that minimize the sums of the squared residuals between observed data and a PPTT normalized conductivity surface. For example, I used Microsoft Excel's Solver function. Application to Hamada's data finds $\sigma_{min} = 0.059$, $\varphi_{min} = 0.056$, and $S_{min} = 0.232$. Figure 14 is a view of this surface together with the data shown as yellow dots above and below the surface. Figure 15 shows the $\sigma_t / \sigma_w = 0$ plane nearly edge-on and illustrates how the fitted normalized conductivity surface divides the observations. The surface interpolates normalized conductivity given porosity and water saturation. Figure 16 is a similar view of the data with the surface viewed nearly edge-on.

These real data illustrate that often the available data are restricted to a narrow subdomain on the porosity axis. If there is a large amount of scatter in such data, the data may not constrain the adjustable parameters to physically reasonable limits. Under such circumstances, similarly to Winsauer's a parameter (a number which should always be 1.0, but isn't), the parameters of the pseudo-percolation threshold theory may violate the expectation that they be small parameters, close to zero. The more restricted the porosity subdomain, the higher the porosity, and the noisier the data, the greater the possibility will be for physically unmeaningful values of φ_ϑ, and relatively large values of (φ_{min}, S_{min}, σ_{min}) to be observed. However,

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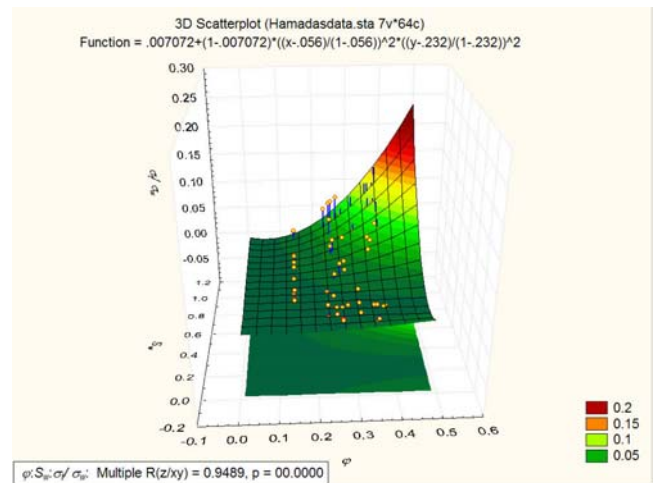
Fig. 14. Hamada's porosity – water-saturation – normalized conductivity data for his well A and well B. Each line of constant-porosity points represents data taken from a single core plug.

the resulting normalized conductivity surface will still be a good interpolator/predictor in the $\varphi - S_w$ subdomain defined by the observations and cannot, by its construction, give silly results at the boundaries. Figures 15 and 16 suggest that robust estimators might be superior to the least squares method for finding adjustable parameters; however, application of robust estimation techniques to Hamada's data is beyond the scope of this study.

INSIGHTS

Having reviewed all of the foregoing points, can anything new be said about the conductivity of reservoir rocks? Introduction of a concept of an ideal Archie rock, characterized by the conductivity relationship

$$\sigma' = (\varphi' S_w')^2 \qquad (25)$$

allows a unified frame where the relations of all the methods can be compared. Contemplation of the adjustable parameters in equation (22.b) suggests that real Archie rocks differ by only small perturbations from this ideal Archie rock. This ideal-rock relationship can be brought into accord with observations if its variables are modified slightly to accommodate the observed deviations from idealness. The modifications can be constructed to honor boundary conditions, observed and inferred, satisfied by the physical conductivity–fractional brine volume relationship. These are

$$\sigma' = \frac{\sigma_t - \sigma_{min}}{\sigma_w - \sigma_{min}} = \frac{\dfrac{\sigma_t}{\sigma_w} - \dfrac{\sigma_{min}}{\sigma_w}}{1 - \dfrac{\sigma_{min}}{\sigma_w}}, \qquad (26.a)$$

$$\varphi' = \frac{\varphi - \varphi_{min}}{1 - \varphi_{min}}, \qquad (26.b)$$

and
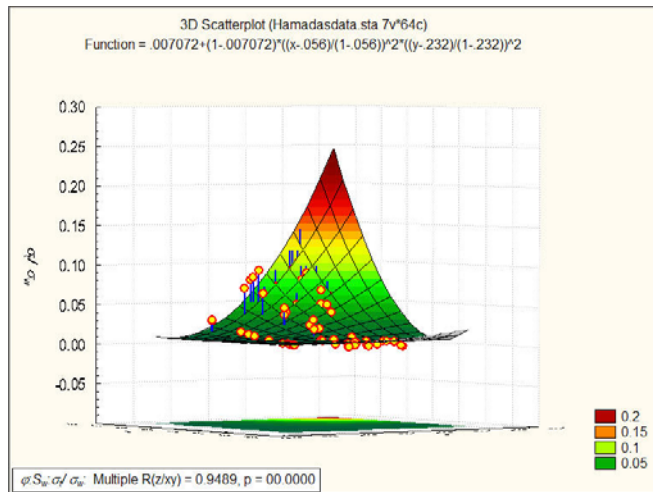
$$S_w' = \frac{S_w - S_{min}}{1 - S_{min}} . \qquad (26.c)$$

These relationships can be viewed as transformations from an ideal Archie space common to all conductivity-porosity-water saturation relationships into the physical space where any particular problem is observed. The correction terms are small perturbations in each of the variables. The adjustable parameter values should correlate with the pore geometry of various rock textures. In the case of $\sigma_{min} = 0$ the percolation threshold parameters become $\varphi_{min} = \varphi_{\vartheta}$ and $S_{min} = S_{\vartheta}$, and lie in the $\sigma_t / \sigma_w = 0$ plane. In this model the brine point in the ideal Archie space and in the unit conductivity cube are always co-located. The two spaces differ by the unit conductivity cube's origin being perturbed just slightly, displacing it from the origin in the ideal Archie space. Viewed from the ideal Archie space, the unit conductivity cube in real porosity–water-saturation–conductivity space behaves like a slightly distorted image of the ideal space, being compressed or streched parallel to its edges into the unit conductivity cube. Substituting (26.a)-to-(26.c) into (25) leads to equation (22.b), an equation perfectly symmetrical in porosity and water saturation, the symmetry being broken slightly in the observations because φ_{min} and S_{min} are determined by different processes and so should be, in general, unequal.

CONCLUSION

This research has been inspired by a hope of discovering whether the Archie-Winsauer power law connecting conductivity to fractional porosity and fractional brine volume could be shown to follow from physical principles, and if so, how so? Full success would have been an enunciation of an unequivocal physical principle from which a porosity–water-saturation–conductivity relationship would follow ineluctably. Be there such a prin-
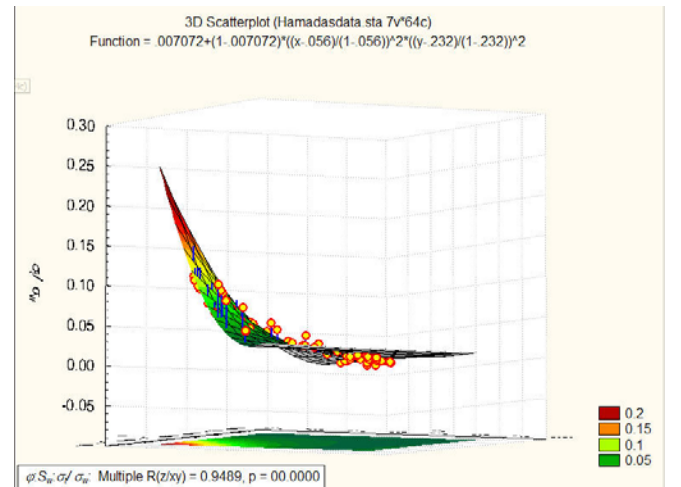

Fig. 15. The normalized conductivity data distributed astride the PPTT normalized conductivity surface that minimizes the sums of the squared residuals about the surface.

Fig. 16. A view showing the PPTT normalized conductivity surface edge-on suggests that a robust fitting scheme might be preferred over standard least squares for some data sets. The effect of robust estimation is to better fit the data by minimizing the influence of outliers.

ciple, it is: the rate of change of normalized conductivity with respect to a change in connected fractional brine volume is proportional to the connected fractional brine volume. This statement is based upon observations of normalized conductivity of Archie rocks on the Archie subdomain and its boundaries on the $\varphi - S_w$ plane. No generalization to other rocks is claimed. Further remarks are offered in the appendix.

At the least, I have shown that the Archie-Winsauer model is valid over, at most, only a portion of the fractional porosity and fractional brine volume range, and therefore cannot qualify as a physical theory itself, nor can it be shown to derive by formal analysis from physical principles. A-W theory merely minimizes the squared residuals between observations and an arbitrarily selected function—logarithm of resistivitiy proportional to logarithm of porosity. The function chosen cannot be used to extrapolate outside the data because it can be shown to have unphysical (and therefore incorrect) behavoir at the boundaries of the $\varphi - S_w$ plane. These findings will not surprise thoughtful formation evaluationists.

I have also shown that for conventional special core analysis data, that it is possible to formulate the porosity – water-saturation – normalized-conductivity relationship problem to honor boundary conditions while simultaneously fitting observations at least as well as the conventional methods. Although the pseudo-percolation threshold is merely quasi-physical, it puts adjustibility into the fitting function where that adjustability can be used to most closely mimic the natural behavoir of conductivity at low porosity and water saturation. Thus, the PPTT model provides insight into why various ad hoc extensions of Archie's law at low porosity and for shaly sands are required. Moreover, while both Archie-Winsauer and pseudo-percolation threshold theory are three-parameter models, the former requires the partioning of data into artificial categories ($S_w = 1$ for determination of a and m, $S_w \leq 1$ for determination of n) followed by the use of ad hoc procedures such as arithmetic averaging of ns; the latter determines all of the fitting parameters simultaneously using all of the data simultaneously, giving all of the data equal influence, seemingly the most "democratic" method to use the data. Further, PPTT treats the fractional brine volume φS_w as an indivisible unit—the conducting phase fractional volume—rather than separating its components and treating them individually; e.g., $\varphi^m S_w^n$. Finally, the pseudo-percolation threshold is at least notionally related to a property of the rock fabric—the PTT, whereas a, m, and n are merely ad hoc model parameters with no a priori physical interpretations implied. PPTT can be used without modification for slightly shaly sands, such as Archie's Nacatoch and Hamada's A and B sands.

Not every feature of the normalized conductivity surface and its limitations has been discussed herein due to space constraints placed upon this article. There is, for example, a large subdomain on the $\varphi - S_w$ plane where the normalized conductivity vanishes because non-conductive hydrocar-

bons form the continuous phase of an emulsion. However, Archie rocks do not belong to this subdomain. There are other caveats as well. Nevertheless, and keeping in mind Archie's cautionary note quoted in the footnote on page 2 above, PPTT remains valid on the Archie subdomain and at the boundaries where normalized conductivity takes on its extreme values: zero and one. I hope to address these and other interesting features of PPTT in a follow-on article in the *Petrophysics* journal.

Although this search for a first principle has not succeeded in finding that principle, it has illustrated that curve- and surface-fitting functions with physically interpretable adjustable parameters can be found. For the industry's seminal data sets the PPTT model interpolates the data at least as efficaciously as the classical A-W model. This provides formation evaluationists with a context for understanding the limitations of classical A-W theory, and a practical alternative to the classical method for water saturation estimation.

Acknowledgements

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Table of Symbols

$\alpha_0, \alpha_2,$	proportionality constants
$\alpha_1, \alpha_3,$	integration constants
φ	(phi) porosity
φ_ϑ	pseudo-percolation threshold
φ'	idealized porosity
φ_{min}	porosity at σ_{min}
ϑ	(theta) pseudo-percolation threshold contour
$\varphi - \varphi_\vartheta$	connected brine volume
S_w	water saturation
S_ϑ	water saturation analog of porosity PPT
S_{w_irr}	"irreducible" water saturation
S_w'	idealized water saturation
S_{min}	water saturation at σ_{min}
β	fractional brine volume $= \varphi S_w = \beta_c + \beta_i$
β_i	isolated fractional brine volume
β_c	connected fractional brine volume
$(\varphi - \varphi_\vartheta)/(1 - \varphi_\vartheta)$	normalized connected brine volume $= \beta_c$
R_0	resistivity if $S_w = 1$
R_t	resistivity if $S_w < 1$
R_w	brine resistivity
V_{sh}	shale volume
σ_0	conductivity if $S_w = 1$
σ_t	conductivity if $S_w < 1$
σ_w	brine conductivity
σ_{min}	minimum conductivity parameter
σ_{sh}	shale conductivity
σ'	idealized conductivity
σ_0/σ_w	normalized conductivity if $S_w = 1$
σ_t/σ_w	normalized conductivity if $S_w < 1$
a	Winsauer's fudge factor
m	Archie's porosity exponent
n	Archie's saturation exponent
n'	generalized saturation exponent
$\bar{n}$	average saturation exponent
F	formation resistivity factor $= R_w / R_0$
f	formation conductivity factor $= 1/F$
I	resistivity index $= R_0 / R_t$
G	generalized resistivity index $= FI$
E_0	EET geometrical factor at $S_w = 1$
E_t	EET geometrical factor at $S_w < 1$
E_p	PPTT analog of E_0 and E_t
a_0, a_t	EET adjustable constants
$b_0, b_t,$	EET adjustable constants
a_p	PPTT analog of a_0 and a_t
b_p	PPTT analog of b_0 and b_t
c_p	new constant predicted by PPTT

APPENDIX

Many physical principles are capable of parsimonious expression. On the other hand, historically many overly pithy principle statements, once generally accepted as reasonable and self evident, have proved to be incorrect; e.g., the earth is flat, the orbits of the planets are circles, heavy objects fall faster than light objects, etc. This latter assertion by Aristotle can be mathematically expressed as

$$v - v_0 \propto \text{mass} \qquad (A.1)$$

where v is speed and v_0 is initial speed. Galileo Galilei, in testing this proposition, found instead that the velocity of falling objects is independent of mass, but depends upon the time spent in falling according to

$$v - v_0 \propto t - t_0 \qquad (A.2)$$

where t is time and t_0 is an initial time. Average speed is the change in displacement, s, during a specified time interval; i.e, $v_{avg} = \Delta s/\Delta t$. Thus Galileo found that

$$\frac{\Delta(s - s_0)}{\Delta(t - t_0)} \propto (t - t_0) \qquad (A.3)$$

or, in words, a change in an increment of displacement $\Delta(s - s_0)$ during a time increment $\Delta(t - t_0)$ is proportional to the time elapsed . This proportionality has the same form as equation (1), and leads to the formula

$$s - s_0 = \frac{g}{2}(t - t_0)^2 \; . \qquad (A.4)$$

where g is a constant acceleration. Equation (A.3) might be construed as a kinematic first principle, but Galileo's kinematical prinicples can be easily derived from Newton's dynamical principles (i.e., the three laws of motion and universal gravitation). Consequently Galileo's kinematical rules are not taught in school as "first" principles; before Newton's work, they might have been. The formulation of (A.3) is based on many observations; likewise the formulation of (1) and (11) is based upon observation. The analogy of (1) and (11) to (A.3) is close. As with Galileo's kinematical principles, surely a more fundamental principle than (1) or (11) must exist. Formation evaluation will remain as much art as science until a principle can be found.

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