PETROPHYSICS, VOL. 42, NO. 2 (MARCH-APRIL 2001), P. 118–136; 5 FIGURES, 1 TABLE

Calculating Water Saturation in Electrically Anisotropic Media

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ABSTRACT

The influence of resistivity anisotropy upon field electrical measurements was described at least as early as 1920. Electrical logging instrument responses in vertical boreholes drilled through horizontal layers were, fortuitously, not affected by anisotropy, not because the formations were isotropic, but because commercially available logging instruments responded to the same tensor component of resistivity that is typically measured in core analysis, and the other tensor components of anisotropy remained unsampled and could be ignored in formation evaluation. Log responses significantly influenced by anisotropy were rare enough that there was not much concern with, or appreciation of, this problem. Consequently Archie (1942) formulated his "law" relating water saturation to resistivity in terms of the resistivity of an isotropic medium. Although his resulting formulation is a relatively complicated power law, it nevertheless has remained the industry-standard paradigm for quantitative formation evaluation for (so far) 59 years. With the advent of highly deviated and horizontal drilling, reservoir anisotropy can no longer be ignored in interpretation theory.

Since it has been customary to compute water saturations from resistivity observations obtained from logging instruments based on a scalar equation, the question naturally arises as to how anisotropy will influence the computation of water saturation. The usual saturation equations can be cast into a form that contains both tensor factors, such as the conductivity, and scalar factors such as porosity and water saturation. The scalar water-saturation factor has solutions in terms of the tensor factors. However, by diagonalizing the conductivity tensor its components uncouple and the familiar scalar Archie equation can be applied separately to the tensor components.

INTRODUCTION

Archie formulated his "law" relating water saturation to resistivity in terms of the resistivity of an isotropic medium in 1942. Although his formulation is a relatively complicated power law, it nevertheless remains the industry-standard paradigm. The existence of resistivity anisotropy in rocks was recognized at least two decades prior to Archie's work, but anisotropy has been ignored in resistivity interpretation practice until recently. When wells were typically drilled vertically, only the horizontal component of the resistivity tensor influenced logging instrument responses; consequently the vertical component of resistivity could be ignored in interpretation without any consequence.

With the advent of large numbers of deviated—especially horizontal—wells being logged, the influence of anisotropy can be ignored only at the price of incorrect interpretations (see Figure 1). In 1997 a method for generalizing Archie's relation for water saturation calculation in the presence of macroscopic anisotropy (i.e., at the log scale) was described by Klein, Martin, and Allen. In this article we introduce a method for obtaining water saturation in microscopically (i.e., at the pore scale) anisotropic media.

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In an isotropic medium, none of the quantities in Archie's law depend upon a direction. Thus not only are the porosity φ, brine conductivity σ_w, and water saturation S_w, bulk properties of the medium, but so also are the bulk conductivity σ_t and the Archie exponents m and n. However, the values of the latter three parameters depend upon pore geometry. When pore geometry varies with direction, then the bulk conductivity and every property derived from bulk conductivity should also vary with direction. Thus σ_t, m, and n should be direction dependent in rocks with direction-dependent pore geometry. The utility of Archie's equation is that it can be solved for water saturation. Water saturation is direction-independent. If a medium is anisotropic special core analysis may still be performed on core plugs to establish Archie's parameters. However, there are no single σ_t, m, and n values characteristic of the sample; the measured values of the parameters depend upon the orientation (direction of measurement) of the anisotropic sample.

This leads to the questions that prompted our research:

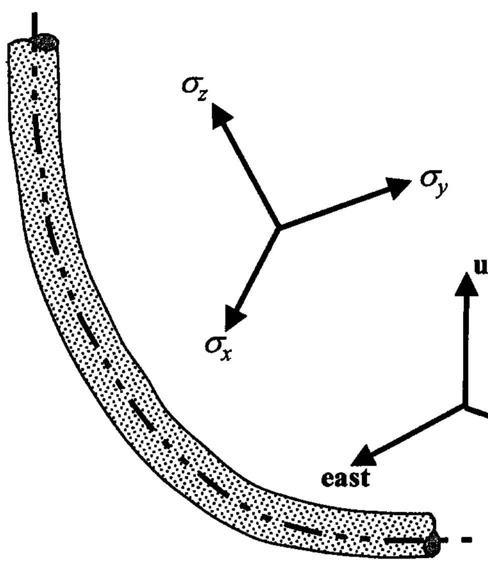

FIG. 1 A borehole penetrating an anisotropic formation. The conductivity tensor is constant; i.e., the same at all points although it differs in three orthogonal directions, but it has no special orientation with respect to map coordinates. In the situation pictured, the angle between the hole axis and the principal axes of the tensor changes continually with depth. Thus a conventional log would respond with a continuously changing apparent resistivity in the medium with direction-dependent, but otherwise constant, conductivity. When the hole axis is parallel to the conductivity tensor axis σ_z, a logging instrument would respond to σ_h provided that $\sigma_x = \sigma_y = \sigma_h$. But in general, the logging instrument response will be to a complicated combination of all of σ_x and σ_y and σ_z. Core samples from the hole illustrated here could be used to obtain the cubical samples discussed in the text.

Can we find a direction-dependent Archie theory that can be applied to arbitrary relative orientations of a rock and a measuring apparatus? Can the results of conductivity observations or experiments be predicted given a full list of parameters for the theory? Finally, can water saturation be predicted given complete conductivity (or resistivity) observations in situ and a complete set of laboratory-derived values for the Archie parameters?

We will show that it is possible to generalize the usual methods used to estimate water saturation from conductivity (or resistivity) in isotropic media so that water saturation is computable from measurements taken on a randomly oriented (with respect to a measurement system) sample in an anisotropic medium.

However, we will also show that application of the generalization is unnecessary if the principal axes of the conductivity tensor in a sample are aligned favorably with the measuring apparatus. We show further that even in cases where physical alignment of the medium and measuring apparatus did not occur, so long as the conductivity data space is fully sampled (meaning that all the components of the conductivity tensor are sampled) the data can always be projected into a coordinate system aligned with the conductivity tensor principal axes. In such cases the tensor formulation will uncouple to yield either two or three versions of the traditional isotropic (or scalar) formula, one for each independent axis of the conductivity tensor. It is certainly simpler to derive the water saturation directly from, say, the formula for the horizontal component of conductivity (as has always been done) than to use a formula that mixes all the components. It is, nevertheless, of theoretical interest to see how the well-known scalar special cases that have been used in log analysis since the 1940s can be obtained from a fully general theoretical understanding of the connection between water saturation and porosity (scalars), the pore-geometric factors m and n, and formation conductivity (tensors).

As in Archie's (1942) original paper, our purpose is not to discuss how the conductivity that we will use can be obtained in situ in the formation using logging instruments. Primarily we are concerned with quantities that could be measured in a laboratory. On the other hand, to be useful in a practical sense our theory must also apply in some fashion to well log data. We have, therefore, not resisted our temptation to remark on the implications of our theory for past, present, and future instrumentation where such remarks would be of general interest.

Our new result is the introduction of a formalism that allows the extension of the usual resistivity-water saturation relationships to anisotropic media by an almost trivial method. Without context, the significance of our generalization of Archie's equation and its solution for water saturation might not be clear. Accordingly we attempt to supply enough context to illuminate the connections between rock samples in the laboratory, rocks in situ, laboratory-derived measurement of electrical properties, log-derived estimates of electrical properties, and our result. With these connections in sight we hope to make our result as accessible to, say, a newly-minted field engineer in a service company as it would be to a long-time practitioner of theoretical formation evaluation.

It seems appropriate to begin our article with a background narrative on the subject of conductivity anisotropy, beginning with the history of its recognition in surface geoelectrical methods, and its subsequent appearance as a concern in interpretation theory in the logging literature, followed by a discussion of the reasons for the recent renewal of interest in this problem. Second, we discuss how the components of the conductivity would be obtained in the laboratory. Third, the significance of the porosity-formation resistivity factor and water saturation-resistivity index functions in terms of conducting phase geometry is explained. Finally we follow with the mathematical formulation of the problem of determination of water saturation in anisotropic media. We offer solutions to both the generalization of Archie's law, and an alternative formulation based on electrical efficiency theory.

Our intended audience is the entire formation evaluation community. The equations in the latter part of our article may cause qualms in some members of the community. Before embarking upon our topic, we take the unusual step in the next section of sharing the mathematical format that we use with the reader in an attempt to raise the confidence level of readers who would usually be put off by equations.

A Cure for Math Anxiety

The audience targeted by this article spans a spectrum of mathematical expertise. The analytical level of our exposition is intended to be intermediate. Thus, for those with a high level of mathematical training, the result will follow in a single, trivial step from the statement of the problem.

For those at the opposite end of the mathematical training spectrum, the mere notation in which the problem is stated will pose a barrier needing to be breached before our result can be appreciated. In the center are many with all the necessary prerequisites who have not had a prior occasion to put together the elements of their mathematical knowledge in the pattern needed to pose and answer the question which is discussed herein. It is our aim to provide a heuristic presentation that should be accessible to any interested formation evaluationist regardless of his level of mathematical training.

For those familiar with the solutions of simultaneous equations in terms of matrices of coefficients and vectors of knowns and unknowns, and with series expansions of the transcendental functions, our presentation is self-contained. Others—if they are willing to accept series expansions on faith—need not be left behind, for simultaneous equations are the subject of elementary algebra. At most we will introduce a new notation for a familiar problem. In algebraic terms we will deal with no more than three equations in three unknowns—a quite manageable system.

We will introduce and frequently use the term "tensor." Those who have looked into the book on tensor analysis, and subsequently put it aside after struggling through a maze of curvilinear coordinates, metrics, subscripts, superscripts, contravariant and covariant components, contractions, etc., take heart. Our tensors are friendly. In fact as we use the word it will refer only to, at most, simple 3×3 matrices in Cartesian coordinates. The linear algebra operations required on these tensors are limited to the multiplication of a 3×3 matrix into a 3×1 column vector, and multiplication of a 3×3 matrix into another 3×3 matrix. Further, all derivations are relegated to Appendix A.

The mathematical statements in the article's text follow a pattern: we first write down the usual (i.e., isotropic) version of a saturation equation, followed by writing down the same equation three times—with different parameters for each coordinate direction—resulting in a 3×3 system of linear equations, and finally writing this system of equations in "matrix-vector," or "tensor" notation. The mathematically anxious should not find themselves trying to "follow" derivations in the text because all the mathematical statements outside the appendices are at most definitions of notations, or in the case of the series expansions of transcendental matrix functions, operational definitions which show a series of steps on the right side of an equation that will yield the specific matrix indicated by the notation on the left side. (The operations appearing on the right are defined in Appendix A.) The only exceptions to this are the solutions to the tensor equations. However, these solutions are formally exactly the same as for the familiar scalar equations in isotropic media. The only difference is the replacement of symbols for certain scalar factors in the isotropic formulation with the corresponding symbols for tensors in the anisotropic formulation. Thus for the formation evaluationist who can solve Archie's equation in terms of logarithms and exponentials (or who is familiar with the elementary algebraic technique of "completing the square") our solutions are a straightforward extension of his present knowledge.

BACKGROUND

Anisotropy and Instrument Responses

Interest in the influence of anisotropy upon geoelectrical measurements, whether made with electrodes arrayed on the surface of the earth or, later, deployed in a borehole, has a considerable history (Schlumberger, 1920). In surface electrical methods there are obvious preferred directions—e.g., horizontal and vertical; strike and dip—that suggest the possibility of anisotropic conductivity. In surface prospecting, conductivity parallel to strike is expected to be greater than conductivity perpendicular to strike. Early in the use of electrical prospecting it was discovered from theoretical considerations that in profiling perpendicular to the strike direction across *vertically dipping* anisotropic beds, the expected anisotropy does not influence the array response; i.e., the response remains that of conductivity parallel to strike. This lack of response has long been known as the "paradox of anisotropy" (Schlumberger et al., 1934, p. 172–173).

The paradox refers to a strange-seeming fact. Consider the linear electrode array used in surface resistivity profiling oriented with its axis parallel to the bedding plane normals of a vertically dipping medium. Currents emitted at the source electrode penetrate the formation; there are both axial and radial (or transverse[1]) components of current, where by "axial" we mean the component of current flowing parallel to the electrode array axis. At the potential electrodes the current is flowing overwhelmingly in the direction of the *axial* component of conductivity, yet the apparent conductivity response of the array is to the *transverse* components of tensor conductivity. In other words the array response calculated using the usual formula assuming an isotropic medium produces the value of the conductivity perpendicular to the direction of current flow. Using the usually employed linear electrode array, such surveys are "blind" to the component of tensor conductivity parallel to the array axis.

While the postulated geometry of vertically dipping beds may be rare in surface outcrop, imagine rotating the profile line of the survey into a vertical orientation and the (originally vertical) bedding planes into horizontal planes, and one has an image of a vertically drilled wellbore penetrating horizontal layers. Thus, while rare in surface prospecting, the paradox of anisotropy applied to most early wellbore electrical surveys (i.e., those in vertical wells penetrating horizontal beds). Variants of surface arrays were tested and employed in boreholes throughout the 1920s (Schlumberger et al., 1934)—these arrays were all blind to the vertical component of conductivity. Moreover, when axial dipole induction instruments were introduced in the 1950s it was determined, again theoretically, that induction instruments (although less paradoxically) also respond primarily to the horizontal component of conductivity.

It had been determined in the laboratory that shale possessed anisotropic conductivity, and that the shale coefficient of anisotropy ranges from one to three (Schlumberger et al., 1934; Keller and Frischknecht, 1966). Anisotropy was usually invoked to explain anomalous instrument responses recorded on logs in shales (Kunz and Moran, 1958). The early focus on shale anisotropy probably contributed to the widespread notion that the bulk reservoir rocks were, in contrast, isotropic.

In the historical period when the early results were obtained, most wells were drilled vertically. Most reservoir bedding planes lie within a few degrees or at most a few tens of degrees from horizontal. Considering the paradox of anisotropy, logging tools respond primarily to the horizontal component of conductivity. Special core analysis used to obtain Archie's saturation exponent n and porosity exponent m employs core plugs selected so that these parameters were generally obtained parallel to bedding planes, i.e., at least approximately horizontally. Thus, the Archie parameters and logging instrument responses in vertical wells are systematically (if unintentionally) chosen consistently. In favorable circumstances (e.g., thick isotropic or the simplest anisotropic beds), which frequently occur in vertical wells, Archie's law using log-derived resistivity is a satisfactory predictor of reservoir water saturation.

Above we mentioned "the expected anisotropy" of the earth, and it is important to be more precise regarding what is meant by this phrase. Although in general a medium might possess different values of the same physical property in each of three mutually orthogonal directions, when referring to the stratified sedimentary rocks considered as targets in oil exploration, it is reasonable to expect that the two components of tensor conductivity parallel to the bedding planes will be more similar to one another than to the component perpendicular to the bedding planes. If the two components in the bedding plane are exactly equal, then the differential equations for the electromagnetic fields de-

[1]The terminology can be confusing. In the theory of mechanical waves, particle displacement in a "longitudinal" wave is parallel to the direction of propagation; for a "transverse" wave particle displacement is perpendicular to the direction of propagation. Similarly, in the earth gridding system of longitude and latitude, the "longitude" lines are those that connect the poles. If the earth's sphere were distorted into a very prolate spheroid the lines of longitude would be nearly parallel to the axis of the spheroid. Thus, the natural generalization (in English) would seem to require "longitudinal" to refer to a direction parallel to the linear array axis (in profiling the array "propagates" over the terrain colinearly with the array axis). "Transverse" would be the direction perpendicular to the array axis. However, in the early literature the term "longitudinal" is used in the same sense that we now use "'transverse," while the term "transversal" is used in the same sense that we are calling "axial", and would call "longitudinal" if we liked that term. However, Conrad Schlumberger (1920; p. 40) coined the terms used in the early literature by analogy with a capacitor made of alternating sheets of paper and tin. There is conductivity along the tin sheets (in the longitudinal direction) and no conductivity perpendicular to the sheets (transversal direction). So *longitudinal* meant (for Schlumberger) "in the directional of the bedding planes" and *transversal* meant "perpendicular to the bedding planes.")

couple and are relatively easily solved and instrument responses can be most easily predicted and interpreted. If the three components of tensor conductivity are all different, the differential equations remain coupled and intractable as far as analytic methods are concerned. Consequently, in almost all the literature on the response of logging instruments to anisotropic media, only the simplest special case with the horizontal components of conductivity (called σ_h) equal and a distinct vertical component (called σ_v) is treated. A medium possessed of a conductivity anisotropy of this type is variously termed transversely isotropic (having identical properties in two orthogonal directions and a different value for the property in the third orthogonal direction), uniaxial, or birefringent. (This vocabulary is a natural generalization from crystal optics where the direction-dependent propagation of electromagnetic waves through material media was first studied.) Although the uniaxial medium is an idealization, in many cases the underlying assumption is probably satisfied closely enough for useful results. Importantly, the resulting mathematical problem is tractable. However, we should not forget (especially when confronted with enigmatic log responses), that in nature the mathematically inconvenient general case can, and surely does, happen—perhaps often. Selected published data (Sawyer et al., 1971; Zhao et al., 1994) illustrating anisotropy are reprinted in Appendix B.

Study of logging tool responses in uniaxial media led to the analysis of instrument responses when the axis of the logging instrument is oriented arbitrarily with respect to the axes of the conductivity tensor. The apparent conductivity response of the usual wireline conductivity instruments to a transversely isotropic medium is

$$\sigma_a = \sqrt{\sigma_h^2 \cos^2\alpha + \sigma_h \sigma_v \sin^2\alpha}$$

where α is the angle between the instrument axis and the (nominally) vertical axis of the conductivity tensor (Moran and Gianzero, 1979). That is, a conventional instrument's sole output, σ_a, is a complicated mixture of σ_v, σ_h, and α. If $\alpha = 0$, $\sigma_a = \sigma_h$; if $\alpha = 90°$, $\sigma_a = \sqrt{\sigma_v \sigma_h}$.

Although we have not done so here, these log responses are often expressed in terms of a so-called coefficient of anisotropy, λ, defined as $\lambda = \sqrt{\sigma_h / \sigma_v}$. This number arises as a scale factor in the separation of variables solution to Maxwell's equations in transversely isotropic media. We avoid this term since it is a frequent source of confusion because it expresses the ratio of conductivities not directly, but in terms of the square root of that ratio, a smaller number. Thus a coefficient of anisotropy of 10 corresponds to a conductivity ratio of 100! Also under-appreciated is the fact that the coefficient of anisotropy is meaningful only in the case of transverse isotropy. Worthington (1981) models fully anisotropic media using a transversely isotropic model by the expedient of defining an equivalent horizontal conductivity using $\sigma_h \approx \sqrt{\sigma_x \sigma_y}$ and $\sigma_v = \sigma_z$. This is probably an acceptable approximation when $\sigma_x \approx \sigma_y$. However, in a general discussion, where three very different conductivity components are considered there is no concept to correspond to a single "coefficient" of anisotropy.

As far as logging instrument responses are concerned, in infinite isotropic media only axial fields exist at the measure transducers (axial electric field for galvanic, and axial magnetic field for induction, instruments), and the instrument output completely characterizes the data space. But in the presence of anisotropy, there will usually exist field components transverse to the instrument axis that the conventional galvanic or axial dipole induction instruments, with their absence of transversely mounted transducers and their consequent inability to respond to transverse components of the electromagnetic field, cannot detect. The consequence is that the data space is incompletely sampled, and that the formation conductivity cannot be properly characterized from the incomplete data set recorded by these instruments.

One result of this study is to quantify how much data is missing in such cases. It turns out that the conventional logging instruments discussed above sample only one out of six independent components of the conductivity tensor, or 1/6 of the data space. In isotropic formations induction data are six-fold redundant and the conventional instrument response fully characterizes the medium. But obviously, if formations are not isotropic, a lot of conductivity information either is not separately resolved, or goes entirely unsampled and unrecorded.

The method described in this article cannot be applied to conventionally-obtained logging data; however, where forward (or inverse) modeling can be used to infer the direction-dependent conductivity components from logs, and the findings of our study are used, resistivity-saturation interpretations will benefit. Recently fielded instruments could in principle resolve all the components of the conductivity tensor (Kriegshäuser et al., 2000) although they do not do so at present.

Renewal of Interest in Reservoir Anisotropy

Until recently it was believed by most practicing formation evaluationists that anisotropy was mostly restricted to shale. It was thus believed to be a relatively complicated phenomenon that occurred in relatively uninteresting rocks. So the study of anisotropy as a log response was mostly ignored or left to specialists. The growing use of horizontal drilling has led to the recent realization that hydrocarbon bearing reservoirs not only can be, but probably *usually* are anisotropic in their conductivity and other properties. It is now clear that the range of anisotropy in water-saturated reservoir rocks is similar to shales (Klein, Martin, Allen, 1997; KMA). In the presence of hydrocarbons the conductivity anisotropy ratio can be quite large (e.g., >10). This leads to the question of how we shall best interpret conductivity data derived from logs recorded in horizontal wells in terms of water saturations in the reservoir, or even whether this can be done.

Anisotropy can occur at different scales and from different causes. Anisotropy due to pores having different shapes and connectivities in different directions is termed microanisotropy. We define macroanisotropy as a bulk property of a set of locally isotropic, alternating conductive-resistive thin beds. To fit the definition of macroanisotropy the beds should be individually unresolvable on a well log so that only an average of their effects can be observed. This is scale-dependent, relying on the transmitter-receiver spacing of particular logging instruments to define whether the formation is "perceived" by the logging instrument as isotropic or anisotropic. For example a sequence of quasiperiodic resistive and conductive laminated beds, each internally isotropic, with the (quasi) period less than half the transmitter-receiver spacing, will give rise to anisotropic conductivity in the bulk medium.

Anisotropy of this type was under discussion as early as 1933 (Schlumberger et al., 1934). The bulk conductivity of such rocks can be expressed as an average weighted by the relative volume of the conductive and resistive components. If the conductive component has a volume V_{sh} and a conductivity σ_{sh}, while the resistive component has a volume $1 - V_{sh}$ and a conductivity σ_{sd} then the horizontal component of macroscopic conductivity σ_h can be computed using

$$\sigma_h = V_{sh}\,\sigma_{sh} + \left(1 - V_{sh}\right)\sigma_{sd} = \frac{1}{R_h} \tag{1}$$

and the vertical component of macroscopic conductivity σ_v can be computed using

$$\frac{1}{\sigma_v} = \frac{V_{sh}}{\sigma_{sh}} + \frac{\left(1 - V_{sh}\right)}{\sigma_{sd}} = R_v\,. \tag{2}$$

R_h and R_v are the corresponding resistivities. Each bed in a unit can have different Archie parameters and porosity. KMA developed a bulk saturation equation for the case of rocks of this type. Their model is derived from the packaging of thin locally isotropic layers into alternating units. However, the success of the KMA method for practical interpretation depends on the layers being thick enough and isotropic enough to permit the determination of their individual Archie parameters by core analysis or other means. When microscale anisotropy occurs due either to fine-grained lamellae or other pore-geometric traits, the KMA method cannot be applied; a more general method is needed.

A good case can be made that the occurrence of macroscale anisotropy of the type analyzed by KMA should be common. The frequency of occurrence of microscale anisotropy is a harder conjecture to make, but both of the laboratory data sets that are available in the literature employ samples that possess anisotropy on the pore scale (see Appendix B). Since the relative frequency of occurrence of the two modes of anisotropy is not known, it may be of practical as well as academic interest to have a method to analyze microscale anisotropy.

Although the choice of interpretation technology would depend upon the anisotropy mode, logging instrument responses would be indifferent to whether the mode of anisotropy were of the thin lamellae type or the pore-geometric type, given equally thick intervals. But how *would* logging instruments respond? It turns out that using the response of the present ensemble of commercially available conductivity logging instruments, the data available are insufficient to resolve the tensor components of the conductivity; the newest generation of logging instrument, capable of sampling the full induction data space (i.e., instruments with three orthogonal transmitters exciting three orthogonal receivers), is required for that.

In this article we shall leave aside the question of interpreting logging tool responses and concentrate on how a sample of anisotropic formation can be characterized in the laboratory, and how a saturation relationship can be developed.

CHARACTERIZING ANISOTROPY IN THE LABORATORY

Formation samples are usually available in the form of cylindrical plugs or whole core. The cylindrical shape results from the samples having been obtained using rotating coring bits; however, the shape is also convenient for insertion into conventional cylindrical pressure vessels. Flow, whether of fluids or electrical current, is constrained to be parallel to the cylinder, or longitudinal, axis of this kind of apparatus. Properties normal to the longitudinal axis are difficult to measure (Givens and Kennedy, 1992a, 1992b; Kennedy and Givens, 1992). To measure the orthogonal tensor components of a formation sample, it should be prepared with cubical symmetry. Cut as a cube the sample may be tested in a simple jig by changing the sample orientation for each new experiment (Sawyer et al., 1971; Sprunt et al., 1990; Zhao, 1994), or with more versatile laboratory apparatus the sample need not be dismounted between experiments.

Consider a homogeneous rock sample cut into cubical form and suppose the edges having lengths Δx, Δy, and Δz, are parallel to the sample's principal axes of conductivity (Figure 2). Imagine that a potential difference is established between the two faces of the cube parallel to the y–z plane. This pair of opposing faces are maintained as equipotential surfaces by the apparatus, establishing an electric field **E**. If current is permitted to flow under these conditions a uniform current density **J** parallel to the x axis is established in the sample. The conductivity is then defined by the electromagnetic field theory constitutive relationship (known as Ohm's law in circuit theory) $\mathbf{J} = \hat{\sigma}\mathbf{E}$ where the vector quantities, as usual, are denoted by boldface type and the caret denotes a tensor. The conductivity tensor $\hat{\sigma}$ has a representation as a matrix, viz.

$$\hat{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}, \qquad (3)$$

where the individual elements of this matrix are the subject of the next paragraph. The computation of $\hat{\sigma}\mathbf{E}$, while not difficult, is obviously a more complicated operation than the more familiar $\sigma\mathbf{E}$ where σ is a scalar. The operation $\hat{\sigma}\mathbf{E}$ is defined in Appendix A.

We shall now elaborate the meaning of the matrix elements in this form of Ohm's law. Figure 2 illustrates the "thought experiment" that we will now use to define a meaning for the matrix elements of $\hat{\sigma}$. Designating the equipotentials imposed on the cube faces as V_x^+ and V_x^-, etc., under the conditions imposed in our experiment the three components of $\mathbf{E} = (E_x, E_y, E_z)$ are given by $E_x = (V_x^+ - V_x^-)/\Delta x$, $E_y = (V_y^+ - V_y^-)/\Delta y$, and $E_z = (V_z^+ - V_z^-)/\Delta z$. Now by hypothesis E_y and E_z are zero since $V_y^+ = V_y^-$, etc.; i.e., under the conditions of the thought experiment shown in Figure 2 the apparatus does not induce potential differences in the y and z directions in the specially cut sample. The current density in the sample is given by $\mathbf{J} = I_x/(\Delta y \Delta z)\mathbf{i}$ with $\mathbf{i}$ being a unit vector in the x direction. So $J_x = \sigma E_x$ suggests that $\sigma = J_x/E_x = (I_x/\Delta y\ \Delta z)/(\Delta V_x/\Delta_x) = (I_x/\Delta V_x)(\Delta x/\Delta y \Delta z)$; by convention we label this component σ_{xx} with the first subscript referring to the direction of ΔV and the second subscript to direction in which the resulting current is sampled. Repeating this procedure in the y and z directions gives σ_{yy} and σ_{zz}.

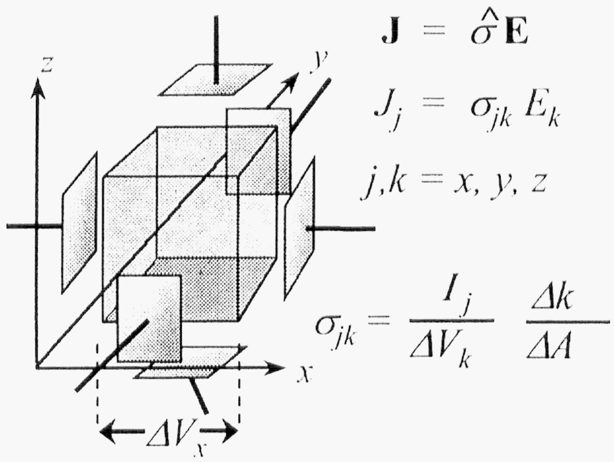

FIG. 2 A cubically cut core sample illustrating schematically a method for determining its conductivity in various directions. The square electrode would be cut the same size as the cube faces in an actual apparatus. Potential differences on the cube faces would be sampled using point electrodes (not shown) connected to a high-impedance voltmeter. If a potential is applied in the x direction and the sample is isotropic, then the equipotentials developed on the surfaces of the cube will be parallel to the y–z plane. In that case σ_{xy} and σ_{xz} will be zero.

We wish to explicitly point to the equipotential surfaces established in, and intersecting the faces of, this cube as all being parallel to the y–z plane. Thus, with the applied potential difference in the x direction, a voltmeter sampling potentials at corresponding points on, say, the top and bottom faces of the cube, registers zero—no potential difference. This indicates that in the experiment we have described to this point $\sigma_{xy} = \sigma_{xz} = 0$, meaning that a potential difference in the x direction does not induce currents to flow in the y or z directions. Put succinctly $\sigma_{ij} = 0$ if $i \neq j$ for this experiment.

We now consider an experiment identical in every respect except that the sample happens to have been cut with its edges oblique to the principal axes of the rock's conductivity tensor (Figure 3). In this case the voltmeter measuring the potential difference between pairs of corresponding points on the opposing cube faces in planes parallel to the y–z plane will register a nonzero potential difference; the equipotentials in this case are not parallel to the applied potential difference. Thus current will flow in the y and z directions (if a circuit is completed external to the cube) regardless that the applied potential difference is in the x direction. If this current is suitably integrated over the cube faces we can compute $\sigma_{xy} = (I_y/\Delta V_x)(\Delta x/\Delta x \Delta z)$, etc. Put simply, the off-diagonal elements of the conductivity tensor give the ratio of a current in one direction to a potential difference in one of the orthogonal directions. This establishes the meaning of the off-diagonal elements of the conductivity tensor, and how to measure them. In addition to σ_{xy}, there are five other off-diagonal elements: σ_{xz}, σ_{yz}, σ_{yx}, σ_{zx}, σ_{zy}. However, the tensor is symmetric so that $\sigma_{xy} = \sigma_{yx}$, etc. Thus there are up to six numerically different elements of $\hat{\sigma}$. It is interesting to note that the signs of the off-diagonal tensor elements can be negative. This merely indicates that the transverse induced currents could circulate in either direction, depending on the tensor orientation with respect to the apparatus. But the axial current always circulates in the direction of the applied field. Stated mathematically, the tensor is positive definite—meaning all the diagonal terms are always greater than zero.

A vertically oriented, conventional galvanic or axial dipole induction logging instrument samples only the horizontal components of $\hat{\sigma}$—i.e., $\sigma_a = \sigma_h = \sigma_{xx} = \sigma_{yy}$ ($\approx \sqrt{\sigma_x \sigma_y}$ if

$\sigma_x \neq \sigma_y$) when the horizontal conductivity component of the medium is transversely isotropic. This signal gives complete information only in an axisymmetric conductivity distribution comprising thick (i.e., thicker than coil or electrode spacing), isotropic beds. Otherwise, the apparent conductivity observed using by a logging instrument gives only partial information regarding $\hat{\sigma}$.

WATER SATURATION RELATIONSHIPS

Archie's Formulation

Archie characterized formation resistivity R_t as

$$R_t = R_w F I \tag{4}$$

where R_w is formation water resistivity, F is formation resistivity factor defined as $F = R_0/R_w$, and I is resistivity index defined as $I = R_t/R_0$, where R_0 is the resistivity of the sample when it is water saturated. Archie's equation can be used for water saturation determination if a connection can be established between F and the formation porosity φ, and between I and formation water saturation S_w. Two experiments are required to make these connections.

To make the connection between F and φ one would ideally wish to observe the change in F, following the diagenetic history of a sample as its porosity evolves, beginning at the time of deposition until the time of coring. This would insure that changes in pore geometry are as representative of the diagenetic process as possible in the presence of diagenetically changing porosity. If this experiment were feasible the result would be a continuous $F - \varphi$ relationship valid for that core, and any regions of the reservoir that start from a statistically similar state and that follow a similar diagenetic history. However, this would not mean that regions that begin in a similar state and are influenced by the same diagenetic processes would necessarily have the same porosity. At the time of sampling the porosity of any given sample would depend upon the time of deposition and the rate at which the process of diagenesis progressed locally. A reasonable assumption can be made that the observed variation in porosity and formation resistivity factor in a reservoir reflects various starting points in a reservoir's initial state and the points in its diagenetic history that are sampled by the core analysis. In other words, nature is performing in a statistical sense the experiment that is infeasible to perform in the laboratory. Thus a representative sample of the reservoir comprises core samples spanning the observed range of porosity. Unfortunately, there can be no guarantee that the rocks represented in the core plugs in fact do represent samples that were initially statistically similar and are experiencing the same diagenetic process in every detail. Different facies in the same reservoir not only originate, but also diagenetically evolve, differently. Randomly sampled cores will reflect these differences.

In the experiment that is actually done, core samples are prepared by saturating them with a standard brine. The resistivity of the core samples is determined, allowing F to be computed for each sample. Finally F is plotted against φ. Since the assumption that each core sample began at the same statistical state and is following the same diagenetic history is an approximation made for convenience, the change in F with porosity is at best satisfied only in a statistical sense. Archie discovered that by plotting the logarithm of each sample's F against the logarithm of its φ that the resulting points fell in a pattern suggestive of a linear relationship in log-log space. Archie fitted a straight line to these data.[2] The slope of the line we now call the porosity exponent m. However, there is usually significant scatter about the line in plots of such data. Assuming the rock is isotropic and discounting experimental error, such scatter most probably indicates subtle differences in the deposition and diagenesis of each sample. (Of course the scatter could also be due in part to unaccounted for anisotropy.) Obviously such a line should pass through $F = 1$ at

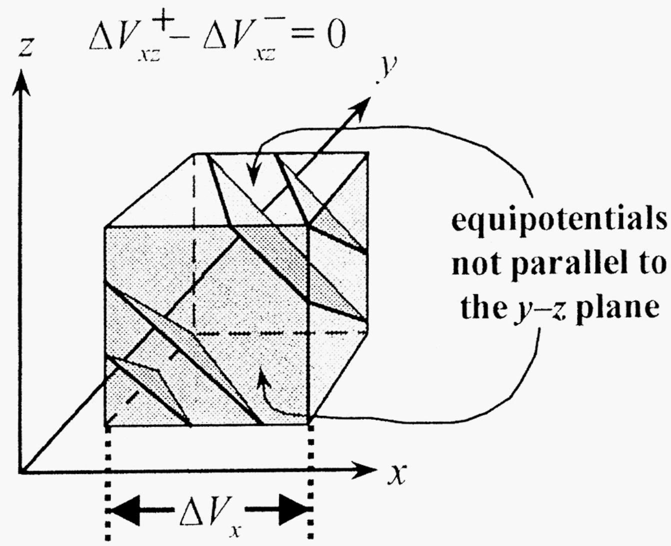

FIG. 3 A cubical core sample with its edges cut obliquely to the principal axes of the conductivity tensor. The six square electrodes shown in Figure 2 are present for this experiment, but are not shown in the interest of clarity. A potential applied in the x direction will induce equipotentials that are not parallel to the y–z plane. Thus if a circuit is completed between corresponding points on the top and bottom of the cube, a current will flow in the z direction.

[2]Archie does not mention how he fitted a line through his data. Archie's line passes through the origin (i.e., $F = 1$ and $\varphi = 100\%$) suggesting that he did not use least squares linear regression.

$\varphi = 100\%$; however, the concept of porosity breaks down, at least for rocks with intergranular pore space, at φ values greater than about 45%. Observers after Archie (e.g., Winsauer, 1952), probably because they studied ensembles of diagenetically unrelated rocks from different locations, felt that as an empirical rule the $F - \varphi$ function would be more useful if an intercept other than (1.0, 100.0%) were permitted. Thus we arrive at the form of Archie's rule that is commonly in use

$$\log F = \log a - m \log \varphi \tag{5}$$

or, $F = a\,\varphi^{-m}$. The negative sign permits m to be used as an intrinsically positive quantity regardless that the slope of the regression line is negative. Performing the same experiment in anisotropic rocks would result in the discovery of a direction-dependent F, and consequently a direction-dependent m and a.

The corresponding experiment for the determination of resistivity index measures R_t of a core sample as S_w is varied from 1.0 to some lower limit as close as possible to 0.0. While conceptually similar, the determination of the $I - S_w$ function in this experiment differs in practice from $F - \varphi$ determination. To obtain $F - \varphi$ it is necessary to use a number of different samples, relying on nature to provide the variability in φ required to define the function, and making the necessary assumption that the functional relationship between the variation of pore geometry and changing porosity is the same for each sample used in the experiment.

However, using a single core prepared with a standard brine, water saturation can be systematically lowered and R_t/R_0 recorded as a function of S_w. Since the sample does not change, the variability of pore geometry is not a factor. But in this case the conducting phase geometry changes as the conducting phase is displaced from the core. However, the conducting phase geometry changes continuously and reproducibly during the experiment. It is observed that in this experiment the data points all plot very close to a line determined by linear regression of the data. Archie analyzed this function on a log-log graph and characterized it as

$$\log I = -n \log S_w \tag{6}$$

or, $I = S_w^{-n}$ as it is usually written. Performing the same experiment in anisotropic rocks would result in the discovery of a direction-dependent I, and consequently a direction-dependent n.

Thus, when direction-dependence is accounted for, Archie's relation is summarized as

$$\begin{aligned} R_{tx} &= R_w F_x I_x = R_w\,\varphi^{-m_x} S_w^{-n_x}, \\ R_{ty} &= R_w F_y I_y = R_w\,\varphi^{-m_y} S_w^{-n_y}, \\ R_{tz} &= R_w F_z I_z = R_w\,\varphi^{-m_z} S_w^{-n_z}, \end{aligned} \tag{7}$$

if $a = 1$ for each direction. These equations hold for measurements obtained in three special orthogonal directions parallel to the directions of the principal components of the resistivity (or conductivity) tensor. If R_t is obtained parallel to one of these special directions, then the corresponding Archie relation would be a satisfactory estimator of water saturation; i.e., although the ms and ns are different in each case, the calculated S_ws would be the same.

However, even when the ms and ns in (7) are known, if the conductivities (or resistivities) are not measured parallel to the principal axes of the conductivity tensor, then none of these formulas supply the correct water saturation.

Electrical Efficiency Formulation

It is possible to generalize any scalar water-saturation resistivity relation into its direction-dependent counterpart. There are many modifications of Archie's theory of water saturation, and there is at least one alternative theory. The latter case will be used to illustrate how any scalar saturation theory can be generalized.

Herrick and Kennedy (1993, 1994) have shown that the problem of water saturation determination can be formulated in terms of pore-geometric factors. The geometric factor E is numerically equal to the ratio of the actual conductance of the rock to its theoretical maximum conductance if the conductive brine phase were configured in its most conductive shape (i.e., a tube). The advantage of forming this ratio is that the brine volume is the same in both the numerator and denominator. Thus brine volume (i.e., φS_w) "divides out" and the ratio depends only upon the effects of the conducting phase geometry. The ratio is equivalent to the efficiency of the rock as a conductor, so Herrick and Kennedy refer to their formulation as electrical efficiency theory. E is determined using the same experimental data that is used to determine m and n. Thus nothing new from the laboratory is required to employ electrical efficiency theory; it is merely a different (but informative) way to organize resistivity-water saturation data.

In terms of electrical efficiency theory, the conductivity of an isotropic water-wet rock saturated with continuous, interpenetrating brine and hydrocarbon phases is given by

$$\sigma_t = \sigma_w \varphi S_w E \tag{8}$$

where E is the pore geometric factor, or electrical efficiency. In this formulation, the volumetric factors φ and S_w are separated from the pore-geometric factor E. Just as F is an empirically determined function of φ and I is an empirically determined function of S_w, so E is empirically related to both φ and S_w. Herrick and Kennedy show that in an isotropic rock the function can be described by a linear relation

$$E = a_t \varphi S_w + b_t \tag{9}$$

where a_t and b_t are empirically determined constants. Substitution of (9) into (8) gives an equation quadratic in φS_w

$$\sigma_t = \sigma_w [a_t(\varphi S_w)^2 + b_t \varphi S_w], \qquad (10)$$

with, as in Archie's m and n, two adjustable parameters a_t and b_t that are empirically determined from the same data used to determine m and n. In anisotropic rocks a_t and b_t and therefore E are direction-dependent. Thus there follow three relations

$$\begin{aligned} \sigma_{tx} &= \sigma_w \left[a_{tx} (\varphi S_w)^2 + b_{tx} \varphi S_w \right], \\ \sigma_{ty} &= \sigma_w \left[a_{ty} (\varphi S_w)^2 + b_{ty} \varphi S_w \right], \\ \sigma_{tz} &= \sigma_w \left[a_{tz} (\varphi S_w)^2 + b_{tz} \varphi S_w \right]. \end{aligned} \qquad (11)$$

Any of these equations can be solved for S_w if σ_t is measured parallel to a principal axis of the conductivity tensor; but, as with Archie's formulation when the measurement is not aligned with the conductivity tensor axes, none of these equations is appropriate.

Obviously, a procedure for calculating the water saturation of an anisotropic medium using conductivity (or resistivity) measurements at arbitrary orientations with respect to the tensor axes is needed.

GENERALIZING THE SCALAR WATER SATURATION RELATIONS

Archie's theory

Consider again a rock sample fashioned into a cube with the edges of the cube cut parallel to the principal axes of the conductivity tensor (the direction of the axes could be estimated by inspection of the rock fabric; Figure 3). Performing the usual special core analysis measurements in turn, using the three pairs of opposite faces in succession as sites of current and fluid injection and removal, Archie parameters in the direction of each principal axis of the conductivity can be obtained. σ_w, φ, and S_w are inherently direction-independent scalars, but m, n, and σ_t will in general depend upon direction. In an isotropic rock the Archie relationship in terms of conductivity is $\sigma_t = \sigma_w \varphi^m S_w^n$ with all the terms in the formula taken as scalars. In anisotropic media the corresponding three Archie equations (i.e., equations (7) in terms of conductivity) are

$$\begin{aligned} \sigma_{tx} &= \sigma_w \varphi^{m_x} S_w^{n_x}, \\ \sigma_{ty} &= \sigma_w \varphi^{m_y} S_w^{n_y}, \\ \sigma_{tz} &= \sigma_w \varphi^{m_z} S_w^{n_z}, \end{aligned} \qquad (12)$$

with the parameters m, n, and σ_t being direction-dependent as indicated by the subscripts. The question now is: Can a tensor equation that can be used to determine water saturation regardless of the orientation of the sample be formulated from these expressions?

Tensors are represented mathematically using a variety of notations, but for the purposes of this discussion a tensor can be conveniently represented by a 3 × 3 matrix. When referred to coordinates other than the principal axes (the usual case), the tensor will be fully populated (i.e., no non-zero elements) and symmetric, as illustrated in the discussion of equation (3) on laboratory-measurements of anisotropy. But in its principal-axis coordinate system the off-diagonal elements of the tensor will be zero.

In the Archie formulation it is easy to express the left hand sides of (12) in terms of a conductivity tensor, viz.,

$$\hat{\sigma}_t = \begin{bmatrix} \sigma_{tx} & 0 & 0 \\ 0 & \sigma_{ty} & 0 \\ 0 & 0 & \sigma_{tz} \end{bmatrix}, \qquad (13)$$

this is readily recognized as a special case of (3); i.e., the same conductivity tensor contained in Ohm's law. However, it might be unclear how to best express the right hand sides of (12) in matrix form since the direction-dependent components are in the exponents. Apparently, both m and n require representation as 3 × 3 matrices: $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$. This suggests that we should consider a relationship of the form $\hat{\sigma}_t = \sigma_w \varphi^{\hat{\mathbf{m}}} S_w^{\hat{\mathbf{n}}}$; however, the right side contains the uncommon mathematical concept of a scalar raised to a matrix power. Evidently, both $\varphi^{\hat{\mathbf{m}}}$ and $S_w^{\hat{\mathbf{n}}}$ are also 3 × 3 matrices. We now pursue how meaning is assigned to these objects.

A formal system that generates equations (12) is

$$\begin{bmatrix} \sigma_{tx} & 0 & 0 \\ 0 & \sigma_{ty} & 0 \\ 0 & 0 & \sigma_{tz} \end{bmatrix} = \sigma_w \begin{bmatrix} \varphi^{m_x} & 0 & 0 \\ 0 & \varphi^{m_y} & 0 \\ 0 & 0 & \varphi^{m_z} \end{bmatrix} \begin{bmatrix} S_w^{n_x} & 0 & 0 \\ 0 & S_w^{n_y} & 0 \\ 0 & 0 & S_w^{n_z} \end{bmatrix}. \qquad (14)$$

Note that (14) is simply a different notation for equation (12), obtained by the same method that is used when any system of simultaneous equations is represented in matrix-vector notation (see e.g., Strang, 1976). The quantity in the square brackets on the left is the conductivity tensor; i.e., equation (13). Since the left side is a tensor, so must be the right. σ_w is intrinsically a scalar. Since the equation must hold when $S_w = 1.0$, $\varphi^{\hat{\mathbf{m}}}$ alone must be a tensor. However, since the equation must also hold when $S_w \neq 1.0$ then $S_w^{\hat{\mathbf{n}}}$ must also be a tensor. The product of two 3 × 3 matrices is itself a 3 × 3 matrix, and the product of two tensors is itself a tensor. Thus, the interpretation of $\varphi^{\hat{\mathbf{m}}} S_w^{\hat{\mathbf{n}}}$ as a tensor is self-consistent. Inspection of equation (14) shows that we have assumed that

$$\varphi^{\hat{\mathbf{m}}} \equiv \begin{bmatrix} \varphi^{m_x} & 0 & 0 \\ 0 & \varphi^{m_y} & 0 \\ 0 & 0 & \varphi^{m_z} \end{bmatrix}. \tag{15}$$

This is not a general definition and gives no rule for constructing $\varphi^{\hat{\mathbf{m}}}$ if $\hat{\mathbf{m}}$ is not diagonal. A more general definition is given by

$$\varphi^{\hat{\mathbf{m}}} \equiv \hat{\mathbf{I}} - \hat{\mathbf{m}}(1-\varphi) + \hat{\mathbf{m}}(\hat{\mathbf{m}} - \hat{\mathbf{I}})\frac{(1-\varphi)^2}{2!} - \hat{\mathbf{m}}(\hat{\mathbf{m}} - \hat{\mathbf{I}})(\hat{\mathbf{m}} - 2\hat{\mathbf{I}})\frac{(1-\varphi)^3}{3!} + \ldots \tag{16}$$

where $\hat{\mathbf{m}}$ is the fully populated 3 × 3 matrix of porosity exponents and $\hat{\mathbf{I}}$ is the 3 × 3 identity matrix (see Appendix A for an explanation of $\hat{\mathbf{I}}$). Note that each operation on the right consists only of the differences, products, and sums, of 3 × 3 matrices—all simple, well-defined operations. We show in Appendix A that the series converges to the required matrix. (Another, closed form but non-elementary, method is briefly discussed in Appendix C.) If an explicit solution for S_w is the requirement, then the saturation exponents must be separated from the water saturation itself, requiring that S_w appear as an isolated scalar factor. If the operation of taking a logarithm could be applied to factors such as $S_w^{\hat{\mathbf{n}}}$ then $\ln S_w^{\hat{\mathbf{n}}} = \hat{\mathbf{n}} \ln S_w$ might serve the purpose. This will require introduction of the logarithm and exponentiation of a matrix. That is,

$$\hat{\sigma}_t = \sigma_w \varphi^{\hat{\mathbf{m}}} S_w^{\hat{\mathbf{n}}} \tag{17}$$

would lead to

$$\ln \hat{\sigma}_t = \hat{\mathbf{I}} \ln \sigma_w + \hat{\mathbf{m}} \ln \varphi + \hat{\mathbf{n}} \ln S_w \tag{18}$$

where the matrix on the left side is the logarithm of the matrix $\hat{\sigma}_t$ and $\hat{\mathbf{I}}$ is the identity matrix. Solving formally for water saturation leads to

$$\hat{\mathbf{I}} S_w = \exp\{\hat{\mathbf{n}}^{-1}(\ln \hat{\sigma}_t - \hat{\mathbf{I}} \ln \sigma_w - \hat{\mathbf{m}} \ln \varphi)\} \tag{19}$$

where $\hat{\mathbf{n}}^{-1}$ is the inverse of the matrix $\hat{\mathbf{n}}$. (This solution is formally identical to the scalar formula, but its scalar counterpart is never written in this form because obviously $\exp\{n^{-1}(\ln \sigma_t - \ln \sigma_w - \ln \varphi)\} = (\sigma_t / \sigma_w \varphi^m)^{1/n}$, and it is this right side that is always presented as the solution to Archie's saturation equation.) Equation (19) is a formal solution that can be employed once the elements of the matrix $\ln \hat{\sigma}_t$ are obtained and a matrix exponentiation operation

$$\exp(\hat{\mathbf{M}}) = \hat{\mathbf{I}} + \hat{\mathbf{M}} + \frac{1}{2}\hat{\mathbf{M}}^2 + \frac{1}{3!}\hat{\mathbf{M}}^3 + \ldots \tag{20}$$

is defined. The matrix powers appearing in (20) are defined as $\hat{\mathbf{M}}^2 = \hat{\mathbf{M}}\hat{\mathbf{M}}$, etc. (Moler and Van Loan, 1978). Similarly, $\ln \hat{\sigma}_t$ might be represented by a series such as

$$\ln \hat{\sigma}_t = (\hat{\sigma}_t - \hat{\mathbf{I}}) - \frac{1}{2}(\hat{\sigma}_t - \hat{\mathbf{I}})^2 + \frac{1}{3}(\sigma_t - \hat{\mathbf{I}})^3 - \ldots \tag{21}$$

if the principal values of $\hat{\sigma}_t$ are all < 2. An alternative expansion would need to be employed if the principal values are > 2. However, series for the logarithm function are slowly convergent and otherwise poorly behaved. More tractable methods are discussed by Lu (1998a). The method discussed in Appendix C can also be used.

The operations presented above (and as extended in Appendix A) prove that $\sigma_t = \sigma_w \varphi^{\hat{\mathbf{m}}} S_w^{\hat{\mathbf{n}}}$ is a mathematically meaningful generalization of Archie's equation for anisotropic media, and that its solution for S_w is in principle computable.

However, we argue in our conclusions that explicit computation of the matric functions is unnecessary.

Electrical Efficiency Theory

As mentioned in equation (8) et seq., the electrical efficiency theory for isotropic media relates the conductivity of a rock to the geometry of its conducting phase by $\sigma_t = \sigma_w \varphi S_w E$ where E is a scalar pore-geometric factor. In terms of the electrical efficiency theory, the experiment described above for an anisotropic rock sample is summarized as

$$\begin{aligned} \sigma_{tx} &= \sigma_w \varphi S_w E_x, \\ \sigma_{ty} &= \sigma_w \varphi S_w E_y, \\ \sigma_{tz} &= \sigma_w \varphi S_w E_z, \end{aligned} \tag{22}$$

or in tensor notation[3] $\hat{\sigma}_t = \sigma_w \varphi S_w \hat{\mathbf{E}}$.

We showed in (8) and (9) that electrical efficiency is related to the amount of the conductive phase (φS_w) in a rock and that for many rocks the relationship is linear, leading to a quadratic equation for S_w, and in anisotropic media leading to the system of equations given by (11). Arranged in the matrix notation equations (11) are

[3]In equation (22) E_x, E_y, and E_z are the components of the electrical efficiency tensor, *not* electric field components.

$$\begin{bmatrix} \sigma_{tx} & 0 & 0 \\ 0 & \sigma_{ty} & 0 \\ 0 & 0 & \sigma_{tz} \end{bmatrix} = \sigma_w \left\{ \begin{bmatrix} a_{tx} & 0 & 0 \\ 0 & a_{ty} & 0 \\ 0 & 0 & a_{tz} \end{bmatrix} (\varphi S_w)^2 + \begin{bmatrix} b_{tx} & 0 & 0 \\ 0 & b_{ty} & 0 \\ 0 & 0 & b_{tz} \end{bmatrix} (\varphi S_w) \right\}. \quad (23)$$

This system suggests a generalization to tensor notation

$$\hat{\sigma}_t = \sigma_w \{ \hat{\mathbf{a}}_t (\varphi S_w)^2 + \hat{\mathbf{b}}_t \varphi S_w \} \quad (24)$$

where the 3×3 matrices are represented by the symbols $\hat{\sigma}_t$, $\hat{\mathbf{a}}_t$, and $\hat{\mathbf{b}}_t$. This can be solved for S_w by the usual technique for quadratic equations of completing the square with care given to the special nature of matrix multiplication. If we begin by dividing the equation through by the scalar factor σ_w the result is

$$\hat{\mathbf{I}} S_w = \frac{1}{2\varphi} \hat{\mathbf{a}}_t^{-1} \left\{ -\hat{\mathbf{b}}_t \pm \left[\hat{\mathbf{b}}_t^2 + \frac{4}{\sigma_w} \hat{\mathbf{a}}_t \hat{\sigma}_t \right]^{\frac{1}{2}} \right\}. \quad (25)$$

Note that the square root of a matrix is required in (25). The matrices involved all contain only positive quantities, so apparently the positive root should be chosen. In this case all the matrix operations are familiar except the square root. A 3×3 matrix has eight square roots, but only one is positive definite and that is the one needed for this case. A method for computing square roots is given by Lu (1998b).

A Final Remark on Generalization

The formal solutions developed above were based on generalizations suggested by diagonal matrices (i.e., their off-diagonal elements were all zero). However, the results also hold for cases where the matrices are fully populated. This can be proved by considering the effect of certain rotations of the coordinate system called similarity transformations. (Details of similarity transformations are further discussed in Appendix A.)

The result is denoted

$$\hat{\sigma}_t = \hat{\mathbf{R}} \hat{\sigma}_D \hat{\mathbf{R}}^T \quad (26)$$

where $\hat{\sigma}_D$ is the diagonal matrix with the principal components (i.e., the direction-dependent conductivities) on the main diagonal with zeros elsewhere as in (13), (14), and (23). $\hat{\mathbf{R}}$ is a matrix describing the "rotation" of the measuring system with respect to the conductivity principal axes, and $\hat{\sigma}_t$ is the fully populated conductivity matrix as in (3), (17), and (24). The factoring of $\hat{\sigma}_t$ into the three factors $\hat{\mathbf{R}}$, $\hat{\sigma}_D$, and $\hat{\mathbf{R}}^T$ can be readily accomplished using common computer 'eigen'-package libraries (e.g., Press et al., 1994).

It is important to notice that $\hat{\sigma}_t$ would be the output of both our laboratory thought experiment and any formation conductivity determination made either in situ by present logging instruments in conjunction with modeling of the instrument's response, or made by the new instruments that sample the full induction data space. But a similarity transform can be used to diagonalize $\hat{\sigma}_t$. Thus multiplication of (26) from the left by $\hat{\mathbf{R}}^T$ and from the right by $\hat{\mathbf{R}}$ results in

$$\hat{\mathbf{R}}^T \hat{\mathbf{R}} \hat{\sigma}_D \hat{\mathbf{R}}^T \hat{\mathbf{R}} = \hat{\mathbf{I}} \hat{\sigma}_D \hat{\mathbf{I}} = \hat{\sigma}_D = \hat{\mathbf{R}}^T \hat{\sigma}_t \hat{\mathbf{R}}. \quad (27)$$

Equation (27) illustrates that if $\hat{\sigma}_t$ is known, it is a simple operation to obtain $\hat{\sigma}_D$. Since the principal axes conductivities can be read directly from the diagonal of $\hat{\sigma}_D$ the horizontal components can be selected directly for use in Archie equations using the horizontally determined m and n values; i.e., the usual procedure. The tensor solution for water saturation is not needed in practice.

CONCLUSIONS

Formation conductivity is a tensor. This fact follows physically from the systematic variation of pore geometry with direction, and implies that the pore geometric terms in Archie's equation and electrical efficiency theory are also direction-dependent. This leads to a more complicated formulation of water saturation-conductivity relationships than have been considered heretofore, regardless of whether we choose to generalize Archie's law to

$$\hat{\sigma}_t = \sigma_w \varphi^{\hat{\mathbf{m}}} S_w^{\hat{\mathbf{n}}}, \quad (17)$$

or to reformulate the problem in terms of a generalized electrical efficiency theory or any of the many water saturation-resistivity relations in common use. We have shown that although Archie's equation can be extended to analyze the formation water saturation in this case, the manipulations are tedious due to the matrix components on the right hand side of Archie's formulation residing in the porosity and saturation exponents. The power-law nature of Archie's theory inevitably leads to the introduction of matric logarithms and exponentiation in its solutions, the result being

$$\hat{\mathbf{I}} S_w = \exp\{ \hat{\mathbf{n}}^{-1} (\ln \sigma_t - \hat{\mathbf{I}} \ln \sigma_w - \hat{\mathbf{m}} \ln \varphi) \}. \quad (19)$$

Similarly, electrical efficiency theory leads to a formal solution containing a matrix square root (i.e., (25)). However, once $\hat{\sigma}_t$ is known, explicit solution (e.g., (19)) of the general equations is not required. Using any convenient method (Press et al., 1992) both the principal components of the general conductivity tensor and their orientations can be found. Once the individual principal components of $\hat{\sigma}_t$ are determined, S_w can be computed using, say, the horizontal component equation, exactly as has always been done in the past.

The tensor method is developed in the context of laboratory measurements taken, for convenience, on cubical samples. However, the method is general. Until recently there was no commercially available logging instrument capable of obtaining the data required to infer $\hat{\sigma}_t$. We have shown that when such instruments become commonplace, water saturation calculations can be performed most simply by first diagonalizing the conductivity tensor, then applying the usual scalar calculation of water saturation.

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APPENDIX A
A SHORT TENSOR TUTORIAL

In terms of isotropic conductivity, the kind encountered in physics 101, Ohm's law in electrodynamics is presented as $\mathbf{J} = \sigma\mathbf{E}$, which is a relationship that declares that the current density **J** and electric field strength **E** are proportional to each other, and moreover **J** and **E** are parallel regardless of their direction in space. This is simple and in many cases true. When this form of Ohm's law holds the material is termed isotropic. However, in general it is observed that **J** and **E** are not parallel in spite of **J** flowing in response to **E**. In other words, **J** is not only scaled by the conductivity property of the medium, but also rotated by it. This type of medium is termed anisotropic.

In anisotropic media Ohm's law is written $\mathbf{J} = \hat{\sigma}\mathbf{E}$. The mathematical operation indicated by $\hat{\sigma}\mathbf{E}$ may be unfamiliar to some readers, but it is simple to define. Written out explicitly in terms of matrix and vector components Ohm's law is

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yz} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{A.1}$$

$$= \begin{bmatrix} \sigma_{xx}E_x + \sigma_{xy}E_y + \sigma_{xz}E_z \\ \sigma_{yx}E_x + \sigma_{yy}E_y + \sigma_{yz}E_z \\ \sigma_{zx}E_x + \sigma_{zy}E_y + \sigma_{zz}E_z \end{bmatrix}. \tag{A.2}$$

The lower right term shows how the elements of $\hat{\sigma}$ are combined with the elements of **E** in a sum-of-products. For specificity, consider the first row of $\hat{\sigma}$. The first component in this row σ_{xx} is multiplied by E_x, the first component of the vector **E**; the second component in the first row of $\hat{\sigma}$, σ_{xy}, is multiplied by the second component of **E**, E_y; the third component in the first row of $\hat{\sigma}$, σ_{xz} is multiplied by the third component of **E**, E_z; the three products are then summed. The result is the first component of the current density vector **J**, J_x. The process proceeds in exactly the same way for the second and third rows of $\hat{\sigma}$. Note that the result of multiplying the 3×3 matrix $\hat{\sigma}$ into the 3×1 column vector **E** is another 3×1 column vector **J**. If the brackets are removed from (A.2) then the three simultaneous equations represented by $\mathbf{J} = \hat{\sigma}\mathbf{E}$ are displayed in the usual algebraic style. Note that each component of **J** depends on all three components of **E**. In other words, **J** and **E** are not parallel.

The multiplication of two matrices follows the same rules, except that the matrix on the right is considered to be a collection of (in this case) three column vectors, and the rule is applied in turn to each column. The result in this case is nine components arrayed in a 3×3 product matrix. With matrix multiplication the order of the factors is important; in general for the matrices $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$, $\hat{\mathbf{A}}\hat{\mathbf{B}} \neq \hat{\mathbf{B}}\hat{\mathbf{A}}$. Operationally this means only that each term in an equation to be multiplied by a matrix must be multiplied from the same side, whether left or right. The distributive and associative properties familiar from ordinary algebra and arithmetic hold for matrices, only the commutative property does not.

An anisotropic medium has three special mutually orthogonal directions in which **J** and **E** components are always parallel. These directions are called the principal axes of the medium, or the principal axes of the conductivity tensor. For a vector component directed parallel to the principal axes Ohm's law still holds in its simple form; e.g., $J_x = \sigma_x E_x$ for the x component, and similarly for the others. That is, the components of **J** and **E** in the principal axes direction are scaled and unrotated exactly as if the medium where isotropic. However, an arbitrarily oriented electric field vector will have vector components in every component direction. What, then, is the effect on the current density? Suppose that the electric field is confined to the x–y plane; that is suppose $\mathbf{E} = E_x\mathbf{i} + E_y\mathbf{j}$. It follows that the components of **J** will be given by

$$\mathbf{J} = J_x\mathbf{i} + J_y\mathbf{j} = \sigma_x E_x\mathbf{i} + \sigma_y E_y\mathbf{j}\,. \tag{A.3}$$

Now, since $\sigma_x \neq \sigma_y$ the components of **E** are scaled by different factors and **J** will not be parallel to **E** (Figure 4). Hence **J** is rotated with respect to **E**. It is this rotation of the

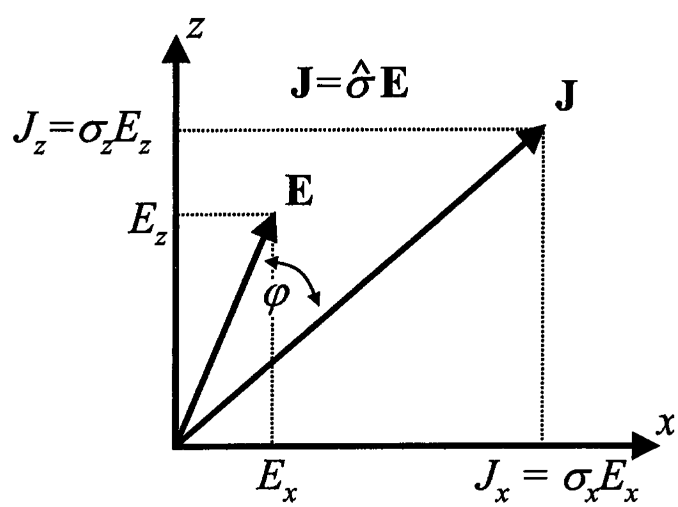

FIG. 4 Illustration of the action of the tensor $\hat{\sigma}$. The components of **J** are proportional to the components of **E**, but the constants of proportionality are different. This has the effect of rotating **J** with respect to the applied electric field **E**.

induced quantity with respect to the inducing quantity that distinguishes anisotropic media from isotropic media.

From a mathematical point of view, the conductivity tensor is a linear vector function that transforms **E** into **J**. The action of the transformation is a separate scaling of the components of **J** with respect to those of **E** that in general results in the resultant vector of the transformation being not parallel to the inducing vector; that is, the linear transformation induces a *rotation* as well as a scaling. But the rotational aspect arises not from a twisting, but from the differential scaling of the conductivity components.

Let us imagine successively imposing electric fields E_x, E_y, and E_z in a medium with differing conductivity in the x, y, and z directions. The resulting current densities are described by

$$J_x = \sigma_x E_x\,, \quad J_y = \sigma_y E_y\,, \quad J_z = \sigma_z E_x\,, \tag{A.4}$$

or in matrix form

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}, \tag{A.5}$$

or in vector form $\mathbf{J}_P = \hat{\sigma}_D \mathbf{E}_P$, where the D subscript indicates the diagonal matrix of conductivity principal components and the P subscript indicates the components of **J** and **E** in the principal axis coordinate system.

The same field in a coordinate system linked to the principal axes coordinates by an arbitrary rotation is

$$\hat{\mathbf{R}}\,\mathbf{J}_P = \hat{\mathbf{R}}\,\hat{\sigma}_D\,\mathbf{E}_P \tag{A.6}$$

where

$$\hat{\mathbf{R}} = \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{bmatrix} \tag{A.7}$$

is a rotation matrix expressing the rotation of the z axis *through an angle* α to a new position in space, and the subsequent rotation of the x axis around the new z axis direction through an angle β (Figure 5). These two rotations are sufficient to align one orthogonal frame with any other that has the same point as origin. A rotation matrix has the property that

$$\hat{\mathbf{I}} = \hat{\mathbf{R}}\,\hat{\mathbf{R}}^{\mathrm{T}} = \hat{\mathbf{R}}^{\mathrm{T}}\hat{\mathbf{R}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{A.8}$$

that is, the matrix is orthonormal—its inverse equals its transpose and their product is the identity matrix. The matrix equation (A.6) can be thus transformed as

$$\hat{\mathbf{R}}\,\mathbf{J}_P = \hat{\mathbf{R}}\,\hat{\sigma}_D\,\hat{\mathbf{I}}\,\mathbf{E}_P = \hat{\mathbf{R}}\,\hat{\sigma}_D\,\hat{\mathbf{R}}^{\mathrm{T}}\hat{\mathbf{R}}\,\mathbf{E}_P\,. \tag{A.9}$$

Grouping the terms with brackets

$$\left[\hat{\mathbf{R}}\mathbf{J}_P\right] = \left[\hat{\mathbf{R}}\,\hat{\sigma}_D\,\hat{\mathbf{R}}^{\mathrm{T}}\right]\left[\hat{\mathbf{R}}\mathbf{E}_P\right] \tag{A.10}$$

and comparing with $\mathrm{s}\,\mathbf{J}_P = \hat{\sigma}_D\,\mathbf{E}_P$ we see that by defining $\mathbf{J} = \hat{\mathbf{R}}\,\mathbf{J}_P$, $\mathbf{E} = \hat{\mathbf{R}}\,\mathbf{E}_P$, and $\hat{\sigma}_t = \hat{\mathbf{R}}\,\hat{\sigma}_D\,\hat{\mathbf{R}}^{\mathrm{T}}$ that Ohm's law is recovered for the arbitrarily oriented coordinate system; viz., $\mathbf{J} = \hat{\sigma}_t\,\mathbf{E}$ where

$$\hat{\sigma}_t = \hat{\mathbf{R}}\,\hat{\sigma}_D\,\hat{\mathbf{R}}^{\mathrm{T}}\,. \tag{A.11}$$

Note that $\hat{\mathbf{R}}\,\hat{\sigma}_D\,\hat{\mathbf{R}}^{\mathrm{T}}$ is a linear transformation that incorporates both scaling and rotation. This type of transformation is termed a similarity transform, and although in general all the components change their values, certain properties of the transformed matrix are the same as in the original matrix. That is, certain properties are invariant under a similarity transform. It is these invariants that permit the discussion of a matrix, which has infinitely many explicit representations depending on α and β, as a single entity—the tensor. Examples of invariants are the sum of the diagonal elements, or trace, of the matrix and the eigenvalues of the matrix. To translate these mathematical statements into their physics interpretation, depending on how the sample in our thought experiment is cut with respect to the principal axes, it has infinitely many possible representations. But in every representation the principal components (or eigenvalues) are the same. For $\hat{\sigma}_t$ the eigenvalues are the conductivities in the direction of the principal axes; i.e., σ_x, σ_y, σ_z.

The vector equations for Archie and efficiency given in principal-axis coordinates can in principle be expressed in arbitrarily oriented coordinates; thus if $\sigma_t = \hat{\mathbf{R}}\,\hat{\sigma}_D\,\hat{\mathbf{R}}^{\mathrm{T}}$ is a tensor with a representation as a fully populated 3×3 matrix, so also are $\varphi^{\hat{\mathrm{m}}}$ and $S_w^{\hat{\mathrm{n}}}$. Thus the fully populated tensor $\varphi^{\hat{\mathrm{m}}}$ can be obtained from the matrices on the right side of equation (16) using $\varphi^{\hat{\mathrm{m}}} = \hat{\mathbf{R}}\varphi^{\hat{\mathrm{m}}_D}\hat{\mathbf{R}}^{\mathrm{T}}$, and similarly for $S_w^{\hat{\mathrm{n}}}$.

However, it is most convenient to do saturation computations using the customary horizontal components of conductivity. Interpretation thus would not involve solution of $\hat{\sigma}_t = \sigma_w\,\varphi^{\hat{\mathrm{m}}} S_w^{\hat{\mathrm{n}}}$ or $\hat{\sigma}_t = \sigma_\omega(\hat{\mathbf{a}}_t(\varphi S_w)^2 + \hat{\mathbf{b}}_t\varphi S_w)$ using matric function operations. The simpler course is to diagonalize the conductivity matrix, essentially reversing the process illustrated in equation (A.11). Multiplying (A.11) from the left by $\hat{\mathbf{R}}^{\mathrm{T}}$ and from the right by $\hat{\mathbf{R}}$ and omitting matrix products equal to $\hat{\mathbf{I}}$ results in

$$\hat{\sigma}_D = \hat{\mathbf{R}}^{\mathrm{T}}\,\hat{\sigma}_t\,\hat{\mathbf{R}}\,. \tag{A.12}$$

The columns of $\hat{\mathbf{R}}^{\mathrm{T}}$ are the eigenvectors of $\hat{\sigma}_t$ while the diagonal of $\hat{\sigma}_D$ contains the eigenvalues. Thus any convenient method of factoring $\hat{\sigma}_t$ into its eigenvalues and vectors can be used to obtain the conductivities directly (Davis, 1973; Press et al, 1992). The result is $\hat{\sigma}_D$. In transversely isotropic media $\sigma_h = \sigma_x = \sigma_y = 1/R_h$ can be read off the matrix diagonal for use in the traditional Archie equation or the electrical efficiency model. The matrix $\hat{\mathbf{R}}$ gives the orientation of the principal axes of the conductivity tensor with respect to the measuring system; i.e., the angles α and β.

Finally, the expansion of $\varphi^{\hat{\mathbf{m}}}$ in equation (16) requires proof. For the diagonal matrix illustrated in the text it is enough to cite the expansion given in Dwight (1961) as formula number 3 on page one. This formula is just the scalar version of the expansion given. It holds trivially for the diagonal matrix since in the expansion of the diagonal matrix the elements do not couple to one another and the action of the equation is just to apply the scalar formula to each diagonal element independently of the others. The remaining question is whether the expansion holds in general for a fully populated $\hat{\mathbf{m}}$ matrix. This amounts to asking whether $\hat{\mathbf{R}}\varphi^{\hat{\mathbf{m}}_D}\hat{\mathbf{R}}^{\mathrm{T}} = \varphi^{\hat{\mathbf{R}}\hat{\mathbf{m}}_D\mathbf{R}^{\mathrm{T}}} = \varphi^{\hat{\mathbf{m}}}$. This relation is in fact true as can be seen by multiplying the relationship for the diagonal form (i.e., equation (15)) from the left by $\hat{\mathbf{R}}$ and from the right by $\hat{\mathbf{R}}^{\mathrm{T}}$. The result is

$$\begin{aligned}\hat{\mathbf{R}}\varphi^{\hat{\mathbf{m}}_D}\hat{\mathbf{R}}^{\mathrm{T}} &= \hat{\mathbf{R}}\hat{\mathbf{I}}\hat{\mathbf{R}}^{\mathrm{T}} \\ &-\hat{\mathbf{R}}\hat{\mathbf{m}}_D(1-\varphi)\hat{\mathbf{R}}^{\mathrm{T}} \\ &+\hat{\mathbf{R}}\hat{\mathbf{m}}_D(\hat{\mathbf{m}}_D-\hat{\mathbf{I}})\frac{(1-\varphi)^2}{2!}\hat{\mathbf{R}}^{\mathrm{T}} \\ &-\hat{\mathbf{R}}\hat{\mathbf{m}}_D(\hat{\mathbf{m}}_D-\hat{\mathbf{I}})(\hat{\mathbf{m}}_D-2\hat{\mathbf{I}})\frac{(1-\varphi)^3}{3!}\hat{\mathbf{R}}^{\mathrm{T}}+\ldots\end{aligned} \tag{A.13}$$

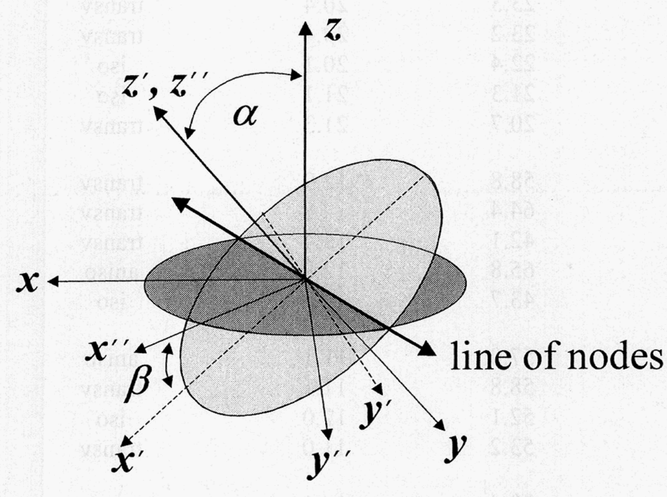

FIG. 5 Defining geometry for α and β. Consider the laboratory (or measurement system) frame to be denoted by (x,y,z). The principal axes of the conductivity tensor are denoted by (x', y', z'). The rotation that maps the laboratory coordinate onto to the principal axes coordinates is first a rotation of the z axis and the z' axis. This moves the x and y axes to x'' and y''. From that point, a rotation of β around the z' axis will bring these axes into alignment with the x' and y' axes.

Now, $\hat{\mathbf{R}}\hat{\mathbf{I}}\hat{\mathbf{R}}^{\mathrm{T}} = \hat{\mathbf{I}}$ since the center factor can be factored into $\hat{\mathbf{R}}^{\mathrm{T}}\hat{\mathbf{R}}$ which then gives $\hat{\mathbf{I}}\hat{\mathbf{I}} = \hat{\mathbf{I}}$ for the term. Next notice that $\hat{\mathbf{R}}^{\mathrm{T}}$ will commute with any (and all) of the scalar factors on the right side of each term in the expansion; i.e., $(1-\varphi)^3/3!\,\hat{\mathbf{R}}^{\mathrm{T}} = \hat{\mathbf{R}}^{\mathrm{T}}(1-\varphi)^3/3!$. The third term in the expansion typifies how each term will reduce. For example

$$\begin{aligned}\hat{\mathbf{R}}\hat{\mathbf{m}}_D(\hat{\mathbf{m}}_D-\hat{\mathbf{I}})\frac{(1-\varphi)^2}{2!}\hat{\mathbf{R}}^{\mathrm{T}} & \\ &= \hat{\mathbf{R}}\hat{\mathbf{m}}_D(\hat{\mathbf{m}}_D-\hat{\mathbf{I}})\hat{\mathbf{R}}^{\mathrm{T}}\frac{(1-\varphi)^2}{2!} \\ &= \hat{\mathbf{R}}\hat{\mathbf{m}}_D\hat{\mathbf{R}}^{\mathrm{T}}\hat{\mathbf{R}}(\hat{\mathbf{m}}_D-\hat{\mathbf{I}})\hat{\mathbf{R}}^{\mathrm{T}}\frac{(1-\varphi)^2}{2!} \\ &= \hat{\mathbf{R}}\hat{\mathbf{m}}_D\hat{\mathbf{R}}^{\mathrm{T}}(\hat{\mathbf{R}}\hat{\mathbf{m}}_D\mathbf{R}^{\mathrm{T}}-\hat{\mathbf{R}}\hat{\mathbf{I}}\hat{\mathbf{R}}^{\mathrm{T}})\frac{(1-\varphi)^2}{2!} \\ &= \hat{\mathbf{m}}(\hat{\mathbf{m}}-\hat{\mathbf{I}})\frac{(1-\varphi)^2}{2!}.\end{aligned} \tag{A.14}$$

This term is formally the same as the term in the equation for the diagonal matrix, but now $\hat{\mathbf{m}}$ represents a fully populated matrix. Obviously, each term on the right side of the expansion can be similarly reduced, thus proving that the expansion given for $\varphi^{\hat{\mathbf{m}}}$ is generally valid, Q.E.D.

APPENDIX B
MEASURED ANISOTROPY

Although Schlumberger et al. (1934) discuss experiments to ascertain the conductivity anisotropy of transversely isotropic rock samples, they content themselves with providing a range of anisotropy coefficients $1.0 \le \lambda \le 2.5$, mentioning that the $\lambda \approx 1$ cases hold for unconsolidated sands. They published no tables of data. Studies to collect data that would support the existence of anisotropy in reservoir rocks at the pore scale have only rarely been published. We know of only two such studies that would apply to our work. Evidence for pore-scale anisotropy was found in both of these studies. Below we have tabulated the data from the two sources, viz. Sawyer et al. (1971) and Zhao et al. (1994). These data support a number of conclusions regarding pore-geometric sources for anisotropy. The apparatus in the two experiments summarized below accepted cubical samples of various core-plug scale sizes (see Table 1 caption), but unlike the apparatus used for our thought experiment, made no provision for sampling potentials on the cube faces orthogonal to applied potentials. Thus a direct determination of all nine tensor components of conductivity was not possible with the apparatus used in the actual experiments, and the direction of the principal axes could not be known with certainty. However, when they occur, different conductivity values in each direction sampled can only be interpreted as evidence of fully general pore scale anisotropy. In the Sawyer et al. study, addi-

TABLE 1 The data in this table has been copied from Sawyer et al. (1971) and Zhao et al. (1994). Both groups of researchers employed small cubical samples (2.5cm and ≤ 7.0cm respectively) for their measurements of sample resistivity. Sawyer et al. give their results in terms of formation resistivity factor, while Zhao et al. give results in terms of unnormalized resistivity. Although we have not done so here, the data could be directly compared using $R_{xx} = F_{xx} R_w$ and the brine parameters (salt concentration and temperature) given by Sawyer et al. The samples were apparently prepared with one pair of faces of the cubes cut parallel to visible horizontal features in the samples. However, while this method might be successful in correctly orienting the vertical principal axis of the resistivity tensor, it would probably not be possible using this method to distinguish the horizontal principal axes. We have therefore labeled the tensor components using double subscripts since single subscripts are used only to indicate the principal components of the resistivity tensor. No attempt to measure the off-diagonal tensor components was made in the experiments summarized in the table below. It is very clear from this data that the resistivity of reservoir rocks can be anisotropic. The rightmost column records our qualitative classification of the samples into one of the three groups: isotropic; transversely isotropic; fully anisotropic. The classifications were made by inspection of the components. Roughly equal component values were taken as equal for purposes of this classification. The data in the table were obtained using water saturated rocks. If similar measurements were taken in the presence of hydrocarbon saturation, the resistivity components would be expected to be even more anisotropic than is observed.

Formation	Sample No.	Formation Factor Vertical F_{zz}	Horizontal minimum F_{xx}	Horizontal maximum F_{yy}	Porosity	Isotropy
Clinton	C1	28.6	25.8	27.0	15.8	iso
"	C2	26.4	24.9	25.3	15.7	iso
"	C3	19.7	18.8	19.2	17.6	iso
"	C4	20.0	19.2	19.3	17.4	iso
"	C5	36.5	33.9	34.2	12.5	transv
"	C6	38.0	35.1	35.3	12.1	transv
"	C7	38.2	34.9	35.1	12.3	transv
"	C8	46.5	41.8	44.1	10.9	aniso
"	C9	46.5	42.6	43.3	11.4	transv
Big Injun	B1	25.9	22.7	23.3	20.4	transv
"	B2	26.2	22.9	23.2	20.4	transv
"	B3	25.0	22.3	22.4	20.1	iso
"	B4	23.1	20.6	21.3	21.1	iso
"	B5	23.1	20.4	20.7	21.3	transv
Bradford Third.	B1	70.4	58.4	58.8	12.9	transv
"	B2	54.4	54.6	64.4	12.8	transv
"	B3	34.1	35.8	42.1	15.4	transv
"	B4	51.0	56.5	65.8	12.8	aniso
"	B5	40.7	41.6	43.7	15.2	iso
Venango Second	V1	61.6	49.2	57.0	11.1	aniso
"	V2	61.8	50.5	58.8	11.1	transv
"	V3	53.3	49.4	52.1	12.0	iso
"	V4	63.7	51.6	53.2	11.0	transv
Gordon	G1	41.1	32.3	39.4	16.1	transv
"	G2	50.3	35.6	42.2	15.3	aniso
"	G3	96.2	73.2	80.3	9.4	aniso
"	G4	77.7	59.9	65.7	10.4	aniso
"	G5	79.2	62.9	71.1	10.4	aniso

(*Continued*)

TABLE 1 Continued.

Formation	Sample No.	Formation Factor Vertical R_{zz}	Horizontal minimum R_{xx}	Horizontal maximum R_{yy}	Porosity	Isotropy
Qian IV	Q1	8.69	7.54	8.09	6.9	aniso
"	Q2	0.81	0.64	0.69	18.9	transv
"	Q3	7.84	2.56	2.75	13.3	transv
"	Q4	7.18	4.75	5.21	6.1	aniso
"	Q5	5.60	4.28	4.55	5.9	transv
"	Q6	3.32	2.35	2.63	12.8	aniso
"	Q7	6.58	4.53	4.83	4.2	transv
"	Q8	7.33	5.47	5.34	4.0	aniso
"	Q9	24.27	14.06	22.22	2.3	transv
"	Q10	3.85	2.49	2.59	12.7	aniso

tional data were collected on adjacent horizontal plugs cut with a 60° angle between the plug axes. Like the data included in the table below, this data also confirms the existence of significant anisotropy in the horizontal plane. We also observe that it is possible in the case of both the cubical samples in the table, and the core plugs just mentioned, for generally anisotropic rocks to appear transversely isotropic or even fully isotropic by a fortuitous choice of sample axes. In these cases absence of evidence for anisotropy cannot be interpreted as evidence of absence unless an additional measurement is obtained in a third direction in the horizontal plane. Thus in the table below it is appropriate to note that at *least* 2/3 of the samples are anisotropic.

The three-axis data that are listed in the table are very suggestive regarding a possible lack of efficacy of present industry-standard methods of core analysis, resistivity logging, and log interpretation in the presence of anisotropic reservoir rocks.

APPENDIX C
A NOTE ON MATRIC FUNCTIONS

In the article we used infinite series representations to define and demonstrate the existence of the various matric functions that appear in the generalized Archie equation. If it were necessary to actually *compute* these functions, there are representations that require only a finite number of terms, and that can be expressed in powers of the matrix no higher than two.

These expansions, however, require that the eigenvalues of the matrix in the function be known. For our 3 × 3 matrices the eigenvalues are easily found, but knowledge of the eigenvalues is not required for the infinite series based definitions. Moreover, while the infinite series based definitions make an obvious connection to the concepts of elementary series expansions, we think that the finite series function definitions do not add particularly to the intuitive clarity of the concept of matric functions. They are, however, interesting and we give a synopsis of the methodology below.

A formula for computing a function F of a matrix $\hat{\mathbf{M}}$ is given by (Michal, 1947)

$$F(\hat{\mathbf{M}}) = \sum_{k=1}^{n} F(\lambda_k)\, Z_k(\hat{\mathbf{M}}) \tag{C.1}$$

where $F(\lambda_k)$ is the scalar function corresponding to the desired matric function, λ_k are the eigenvalues of $\hat{\mathbf{M}}$, and

$$Z_k(\hat{\mathbf{M}}) = \frac{\prod_{k\neq j}(\hat{\mathbf{M}} - \lambda_j \hat{\mathbf{I}})}{\prod_{k\neq j}(\lambda_k - \lambda_j)} \tag{C.2}$$

where $n = 3$ for 3 × 3 matrices. The matric functions arising in the generalized Archie equation are $F(\hat{\mathbf{M}}) = \exp(\hat{\mathbf{M}})$, $F(\hat{\mathbf{M}}) = \ln(\hat{\mathbf{M}})$, and $F(\hat{\mathbf{M}}) = \varphi^{\hat{\mathbf{M}}}$.

As an example of the formula's use consider the expansion of $\varphi^{\hat{\mathbf{m}}}$ where $\hat{\mathbf{m}}$ is the porosity exponent matrix given by

$$\hat{\mathbf{m}} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} \\ m_{yx} & m_{yy} & m_{yz} \\ m_{zx} & m_{zy} & m_{zz} \end{bmatrix} \tag{C.3}$$

having the eigenvalues m_x, m_y, and m_z. Then

$$\varphi^{\hat{\mathbf{m}}} = \frac{1}{(m_x - m_y)(m_x - m_z)(m_y - m_z)} \cdot$$
$$\{(m_x - m_y)[\hat{\mathbf{m}}^2 - (m_y + m_z)\hat{\mathbf{m}} + m_y m_z \hat{\mathbf{I}}]\varphi^{m_x}$$
$$+ (m_x - m_z)[\hat{\mathbf{m}}^2 - (m_x + m_z)\hat{\mathbf{m}} + m_x m_z \hat{\mathbf{I}}]\varphi^{m_y}$$
$$+ (m_y - m_z)[\hat{\mathbf{m}}^2 - (m_x + m_y)\hat{\mathbf{m}} + m_x m_y \hat{\mathbf{I}}]\varphi^{m_z}\}. \tag{C.4}$$

This holds for $m_x \neq m_y \neq m_z$.

ABOUT THE AUTHORS

David Kennedy is currently a research associate at ExxonMobil Upstream Research Company. Prior to that there were 13 years at Mobil, both in research and operation services. His sojourn in the oil patch began in 1973 when Schlumberger was one of the few companies with the foresight to recruit newly-minted Georgia Tech physics graduates. After a five year stint with Schlumberger in the field in California and Alaska, and a brief flirtation with ARCO in Dallas, a full blown romance with graduate studies developed. David acquired MS degrees in physics and earth sciences from the University of Texas at Dallas, but then jilted UTD for an infatuation with U. C. Berkeley, spending four fruitful years there but leaving without a PhD. David has also worked briefly for Sohio and for Lockheed Missiles and Space Company. All this activity served an interest in acquiring, understanding, and interpreting borehole resistivity measurements. Since 1988 David has championed the use of resistivity modeling as a routine element of log interpretation. Lately, David has had the pleasure of serving the industry as editor of the well logging professional journal, *Petrophysics*. David lives in Houston with his lovely wife and two darling children.

David C. Herrick, Ph.D. has recently retired and is now teaching, consulting and fishing (not necessarily in that order). Dave was trained in chemistry and geochemistry at Indiana University and Penn State. He has conducted research, training and technical service during his twenty-five years in the petroleum industry for Conoco, Amoco and Mobil in the areas of geochemistry, petrology and petrophysics. His research interests included resistivity interpretation as a function of pore geometry and mineralogy, nuclear magnetic resonance laboratory studies and log interpretation, and capillary properties of reservoir rocks. His publications include new and fundamental work on interpretation methods for resistivity data. Dave is now giving training courses in the fundamentals of resistivity interpretation.

Tingting Yao received a BS from University of Petroleum in 1993, majoring in petroleum geology and a PhD in geostatistics from Stanford University in 1998. Her PhD research focuses on automatic covariance modeling to characterize the spatial continuity of earth science phenomena and conditional spectral simulation through phase identification to honor the spatially sampled conditioning data. She joined Mobil Technology Company in July 1998 and transferred to ExxonMobil Upstream Research Company in February 2000. She has been working on improved methods for more accurate and efficient geologic modeling and reservoir characterization through integration of various data.