How to Simulate it in Isabelle: Towards Formal Proof for Secure Multi-Party Computation

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Outline

- Motivation for formal methods in Cryptography.
- Introduce Secure Multi-Party Computation.
- How is security defined in SMPC?
- CryptHOL, the framework we use in Isabelle.
- Basic proof techniques.
- Toy example to demonstrate how formalisation works.

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"Security proof for even simple cryptographic systems are dangerous and ugly beasts. Luckily, they are only rarely seen: they are usually safely kept in the confines of "future full-versions" of papers, or only appear in cartoon-ish form, generically labelled as ... "proof sketch""

Bristol Crypto Group. 2017

Contributions

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Starting from Lindell's tutorial 'How to Simulate it: A tutorial on the simulation proof technique.'

- Demonstrated how the simulation-based proof method can be formalised.
- Defined computational indistinguishability - up to polynomial time distinguishers.

Protocols formalised:

- Secure multiplication protocol.
- Noar Pinkas Oblivious Transfer.
- Protocol that securely computes an AND gate.

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- Recommended as an *"emerging approach that enhances privacy protections"* in the report of the US commission on Evidence-Based Policymaking, 2017.

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- SMPC began to become more widely implemented.
  - First real life deployment was in 2008 at a Danish sugar beat auction.
- Could be considered as a counterpart to Homomorphic encryption.

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***What are we considering*?**

- Two party setting.
- Semi honest adversary model - honest but curious adversaries.
  - Adversaries follow the protocol specification exactly.
  - They attempt to learn additional information by analysing the transcript of messages received during the execution.

Simulation based security: intuition

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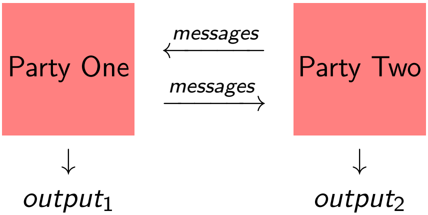
The Real World



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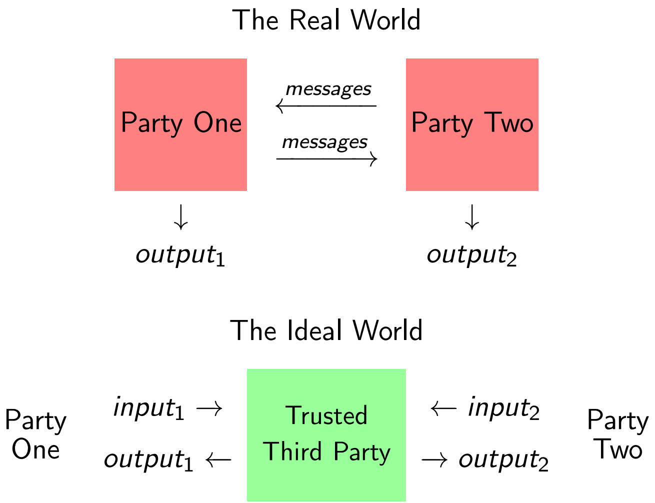

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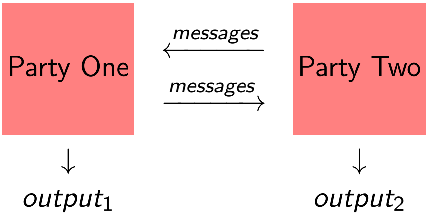

The Ideal World

- Show that the two worlds are *equivalent* or *indistinguishable*.

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The Ideal World

- Construct *simulator*, S_1, which only takes as input the input and output of party one. Outputs the *simulated view of the party*.

Show the two output distributions are *computationally indistinguishable*.

$$\{Real_{View1}(input_1, input_2)\} \stackrel{c}{\equiv} \{S_1(input_1, out_1)\}$$

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- A game defines the goal of an attacker explicitly.
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- Frameworks have been developed to formalise game-based proofs - EasyCrypt, FCF.
- Cryptographers view game-based and simulation-based proofs as distinct.
- We use a game-based framework, CryptHOL, to do simulation-based proofs.

Formalising security in Isabelle: CryptHOL

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- Defines theory on sub probability mass functions (spmfs).
- Can reason about probabilistic programs.
- Designed with game-based proofs in mind.

CryptHOL: some key features

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- Many protocols require uniform sampling from sets.
  - $\mathit{uniform} : \alpha\ \mathit{set} \Rightarrow\ \alpha\ \mathit{spmf}$
  - $\mathit{sample}_{\mathit{uniform}}\ n \equiv \mathit{uniform}\ \{..< n\}$
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- Much of our reasoning comes from the functorial structure $\mathit{map}_{\mathit{spmf}}$.
  - $\mathit{map}_{\mathit{spmf}} : (\alpha \Rightarrow \beta) \Rightarrow \alpha\ \mathit{spmf}\ \Rightarrow \beta\ \mathit{spmf}$
  - $\mathit{map}_{\mathit{spmf}}\ f\ p = \mathit{bind}_{\mathit{spmf}}\ p\ (\lambda x.\ \mathit{return}_{\mathit{spmf}}\ (f\ x))$

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To show equality between the views.

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- This is a contradiction.

Refresher

What have we seen so far?

- Why formal methods are useful to cryptography.
- What SMPC is and how security is defined.
- The basics of CryptHOL.
- The proof methods we use.

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Now we will see a toy example of how we actually formally prove security for an Oblivious Transfer protocol.

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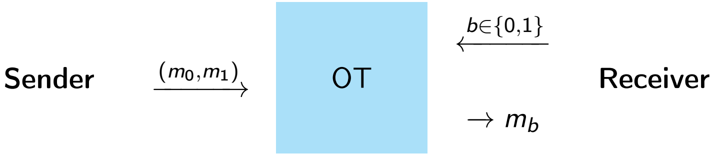

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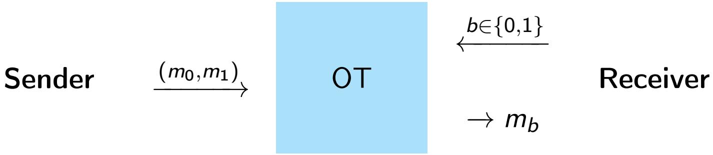

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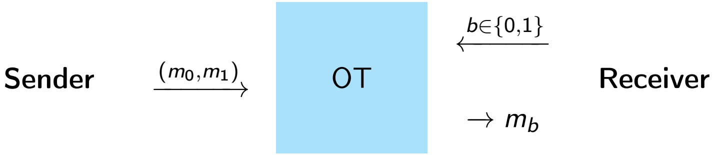

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- The *Receiver* learns nothing of m_{b-1} and the *Sender* does not learn b.

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Trusted Initialiser

$$r_0, r_1, d \xleftarrow{\$} \{0,1\}$$

$$\begin{array}{ccc}
\mathbf{P_1} & & \mathbf{P_2} \\
m_0, m_1 \in \{0,1\} & & b \in \{0,1\} \\
r_0, r_1 & & d, r_d \\
e & \xleftarrow{\quad\quad \mathrm{e} \quad\quad} & e = b \oplus d \\
f_0 = m_0 \oplus r_e & & \\
f_1 = m_1 \oplus r_{1-e} & \xrightarrow{\quad\quad f_0, f_1 \quad\quad} & f_0, f_1 \\
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\begin{aligned}
&R_2\ (m_0, m_1)\ b = do\ \{ \\
&\qquad r_0, r_1, d \leftarrow coin_{spmf}; \\
&\qquad let\ e = b \oplus d; \\
&\qquad let\ r_e = (if\ \ e\ then\ r_1\ else\ r_0); \\
&\qquad let\ r_{1-e} = (if\ \ e\ then\ r_0\ else\ r_1); \\
&\qquad return_{spmf}(m_0 \oplus r_e, m_1 \oplus r_{1-e})\}
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and apply it twice to show:

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This implies security for party two, namely:

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Theorem The Bit Oblivious Transfer protocol is information theoretic secure in the semi honest adversary model.

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- These methods are the main ways in which SMPC is realised.

Thank you