

Solving an SVM by hand based on KKT conditions

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Although necessary but not sufficient, the KKT conditions are sufficient for optimality if the problem is to minimize a convex function which is the case for an SVM in primal form. Therefore, from KKT conditions alone, we can solve an SVM by hand.

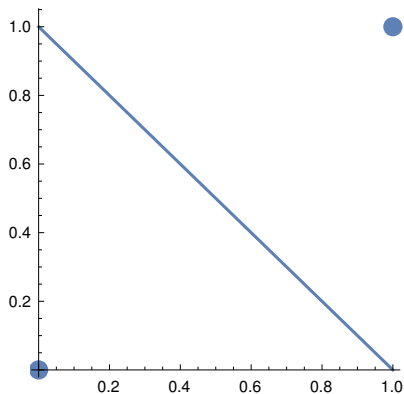
Solving a system of equations in Mathematica

```
In[20]:= Solve[{a1 + a2 == 2, a1 - a2 == 10}, {a1, a2}]  
Out[20]= {{a1 -> 6, a2 -> -4}}
```

Solving a simple SVM of only two samples.

The plot below indicates two two samples, (1,1) of class +1, and (0,0) of class -1. Denote the first one as sample 1 and the second as sample 2. A well-trained SVM would be the one depicted by the blue line.

```
In[32]:= plt1 = ListPlot[{{0, 0}, {1, 1}}, PlotStyle -> {AbsolutePointSize[10]}, AspectRatio -> 1];  
plt2 = ListLinePlot[{{0, 1}, {1, 0}}];  
Show[plt1, plt2]
```



1. First, using the gradient on the bias term that $\sum_{k=1}^K \lambda_k y_k = 0$, we have $(\lambda_1 \ \lambda_2) \begin{pmatrix} +1 \\ -1 \end{pmatrix} = 0$

2. Then, write one equation for each sample in the complementary slackness equation (Equation D in

$$\text{slides): } \lambda_1 \left[+1 \cdot \left(\begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + w_b \right) - 1 \right] = 0$$

$$\lambda_2 \left[-1 \cdot \left(\begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + w_b \right) - 1 \right] = 0$$

3. Finally, the gradient on the non-bias weights: $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \lambda_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \lambda_2 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Expand matrix equations above into a system of linear equations and solve them using the `Solve` function. For typesetting easiness, λ 's are replaced with a 's in the code below.

```
In[22]:= Solve[{ a1 - a2 == 0,
               a1 * (w1 + w2 + wb - 1) == 0,
               a2 * (-1 * wb - 1) == 0,
               w1 == a1,
               w2 == a1
             }, {a1, a2, w1, w2, wb}
]
```

*** Solve: Equations may not give solutions for all "solve" variables.

```
Out[22]= {{a1 -> 0, a2 -> 0}, {a1 -> 0, a2 -> 0, wb -> -1}, {a1 -> 0, a2 -> 0, wb -> 1}, {a1 -> 1, a2 -> 1, wb -> -1}}
```

The first 3 solutions set both a_1 and a_2 to 0. They should be discarded as they remove the constraints from the Lagrange multiplier.

The last solution is what we want, demonstration an equation $x + y - 1 = 0$ which beautifully goes between the two samples at 135 degree. The fact that neither a_1 nor a_2 are zero indicates that both samples are the supporting vectors.

Let's see more examples.

```
In[18]:= Solve[{a1 + a2 - a3 - a4 == 0,
               a1 * (1 * (w1 + w2 + wb) - 1) == 0,
               a2 * (1 * (w1 + 0 + wb) - 1) == 0,
               a3 * (-1 * (0 + 0 + wb) - 1) == 0,
               a4 * (-1 * (0 + w2 + wb) - 1) == 0,
               w1 == a1 + a2,
               w2 == a1 - a4,
               a1 >= 0, a2 >= 0, a3 >= 0, a4 >= 0
             }, {w1, w2, a1, a2, a3, a4, wb}
]
```

*** Solve: Equations may not give solutions for all "solve" variables.

```
Out[18]= {{a1 -> 0, a2 -> 0, a3 -> 0, a4 -> 0}, {a1 -> 1, a2 -> 0, a3 -> 1, a4 -> 0, wb -> -1}}
```

*** Solve: Equations may not give solutions for all "solve" variables.

```
Out[12]= {{a1 -> 0, a2 -> 0}, {a1 -> 0, a2 -> 0, wb -> -1}, {a1 -> 0, a2 -> 0, wb -> 1}, {a1 -> 1, a2 -> 1, wb -> -1}}
```

```
In[19]:= Solve[{a1 - a2 - a3 + a4 == 0,
               a1 * (1 * (w1 + w2 + wb) - 1) == 0,
               a2 * (-1 * (w1 + 0 + wb) - 1) == 0,
               a3 * (-1 * (0 + 0 + wb) - 1) == 0,
               a4 * (1 * (0 + w2 + wb) - 1) == 0,
               w1 == a1 - a2,
               w2 == a1 + a4,
               a1 ≥ 0, a2 ≥ 0, a3 ≥ 0, a4 ≥ 0
               }, {w1, w2, a1, a2, a3, a4, wb}
]
```

*** **Solve:** Equations may not give solutions for all "solve" variables.

```
Out[19]= {{a1 → 0, a2 → 0, a3 → 0, a4 → 0}, {a1 → 1, a2 → 0, a3 → 1, a4 → 0, wb → -1}}
```

*** **Get:** Cannot open CloudObjectLoader`.