CS 474/574 Machine Learning
4. Support Vector Machines (SVMs)

Prof. Dr. Forrest Sheng Bao
Dept. of Computer Science
Iowa State University
Ames, IA, USA

September 18, 2020

SVM is a kind of linear classifiers. But a unique type of.

All samples are equal. But some samplers are equaler.

- Let's first see a demo of a linear classifier for linearly separable cases. Pay attention to the prediction outcome.
- Think about the error-based loss function for a classifier: $\sum_i (\hat{y} - y)^2$ where y is the ground truth label and $\hat{(y)}$ is the prediction.
- If $y = +1$ and $\hat{y} = +1.5$, should the error be 0.25 or 0 (because properly classified)?

The perceptron algorithm

- Recall earlier that a sample $(\mathbf{x}_i, y_i)$ is correctly classified if $\mathbf{w}^T\mathbf{x}_i y_i > 0$.
- Let's define a new cost function to be minimized: $J(\mathbf{w}) = \sum\limits_{x_i \in \mathcal{M}} -\mathbf{w}^T\mathbf{x}_i y_i$ where $\mathcal{M}$ is the set of all samples misclassified ($\mathbf{W}^T\mathbf{X}_i y_i < 0$).
- Then, $\nabla J(\mathbf{w}) = \sum\limits_{\mathbf{x}_i \in \mathcal{M}} -\mathbf{X}_i y_i$ (because $\mathbf{w}$ is the coefficients.)
- Only those misclassified matter!
- Batch perceptron algorithm: In each batch, computer $\nabla J(\mathbf{w})$ for all samples misclassified using the same current $\mathbf{w}$ and then update.

Single-sample perceptron algorithm

- Another common type of perceptron algorithm is called single-sample perceptron algorithm.
- Update $\mathbf{w}$ whenever a sample is misclassified.
  1. Initially, $\mathbf{w}$ has arbitrary values. $k = 1$.
  2. In the k-th iteration, use sample $\mathbf{x}_j$ such that $j = k \mod n$ to update the $\mathbf{w}$ by:

$$\mathbf{W}_{k+1} = \begin{cases} \mathbf{W}_k + \rho \mathbf{X}_j y_j & \text{, if } \mathbf{W}_j^T \mathbf{X_j} y_j \leq 0, \text{ (wrong prediction)} \\ \mathbf{W}_k & \text{, if } \mathbf{W}_j^T \mathbf{X_j} y_j > 0 \text{ (correct classification)} \end{cases}$$

     where ρ is a constant called **learning rate**.
  3. The algorithm terminates when all samples are classified correctly.
- Note that $\mathbf{x}_k$ is not necessarily the k-th training sample due to the loop.

An example of single-sample preceptron algorithm

- Feature vectors and labels:
  - $\mathbf{x}'_1 = (0,0)^T$, $y_1 = 1$
  - $\mathbf{x}'_2 = (0,1)^T$, $y_2 = 1$
  - $\mathbf{x}'_3 = (1,0)^T$, $y_3 = -1$
  - $\mathbf{x}'_4 = (1,1)^T$, $y_4 = -1$
- First, let's augment them and multiply with the labels:
  - $\mathbf{x}_1 y_1 = (0,0,1)^T$,
  - $\mathbf{x}_2 y_2 = (0,1,1)^T$,
  - $\mathbf{x}_3 y_3 = (-1,0,-1)^T$
  - $\mathbf{x}_4 y_4 = (-1,-1,-1)^T$

0. Begin our iteration. Let $\mathbf{w}_1 = (0,0,0)^T$ and $\rho = 1$.
1. $\mathbf{W}_1^T \cdot \mathbf{x}_1 y_1 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \leq 0$.

   Need to update $\mathbf{W}$: $\mathbf{W}_2 =$

   $$\mathbf{W}_1 + \rho \cdot \mathbf{x}_1 y_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

2. $\mathbf{W}_2^T \cdot \mathbf{x}_2 y_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1 > 0$. No updated need. But since $\mathbf{w}$ so far does not classify all samples correctly, we need to keep going. Just let $\mathbf{w}_3 = \mathbf{w}_2$.

An example of preceptron algorithm (cond.)

Continue in perceptron.ipynb

14. In the end, we have $\mathbf{W}_{14} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$, let's verify how well it works

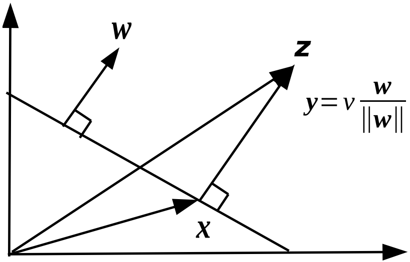
- $$\begin{cases} \mathbf{w}_{14} \cdot \mathbf{x}_1 y_1 & = 1 > 0 \\ \mathbf{w}_{14} \cdot \mathbf{x}_2 y_2 & = 1 > 0 \\ \mathbf{w}_{14} \cdot \mathbf{x}_3 y_3 & = 1 > 0 \\ \mathbf{w}_{14} \cdot \mathbf{x}_4 y_4 & = 1 > 0 \end{cases}$$

- Mission accomplished!
- Note that the perceptron algorithm will not *converge* unless the data is linearly separable.
- What is $\mathbf{w}$ exactly? A linear composition of all training samples!
- Do all samples contribute to $\mathbf{w}$? Not really!

Getting ready for SVMs

- Earlier our discussion used the augmented definition of linear binary classifier: the feature vector $\mathbf{x} = (x_1, \ldots, x_n, 1)^T$ and the weight vector $\mathbf{w} = (w_1, \ldots, w_n, w_b)^T$. The hyperplane is an equation $\mathbf{w}^T\mathbf{x} = 0$. If $\mathbf{w}^T\mathbf{x} > 0$, then the sample belongs to one class. If $\mathbf{w}^T\mathbf{x} < 0$, the other class.
- Let's go back to the un-augmented version. Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ and $\mathbf{w} = [w_1, w_2, \ldots, w_n]^T$. If $\mathbf{w}^T\mathbf{x} + w_b > 0$ then $\mathbf{x} \in C_1$. If $\mathbf{w}^T\mathbf{x} + w_b < 0$ then $\mathbf{x} \in C_2$. The equation $\mathbf{w}^T\mathbf{x} + w_b = 0$ is the hyperplane, where $\mathbf{w}$ only determines the direction of the hyperplane. To build a classifier is to search for the values for $w_1, \ldots, w_n$ and w_b, the bias/threshold.
- For convenience, we denote $g(\mathbf{x}) = \mathbf{w}^T\mathbf{x}$.
- We have proved that $\mathbf{w}$, augmented or not, is perpendicular to the hyperlane.

What is the distance from a sample $\mathbf{z}$ to the hyperplane?



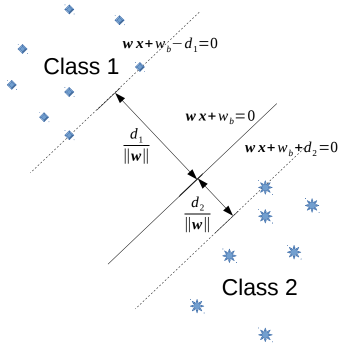
1. Let the point on the hyperplane closest to $\mathbf{z}$ be $\mathbf{x}$. Define $\mathbf{y} = \mathbf{x} - \mathbf{z}$.
2. Because both $\mathbf{y}$ and $\mathbf{w}$ are perpendicular to the hyperplane, we can rewrite $\mathbf{y} = v\frac{\mathbf{w}}{||\mathbf{w}||}$, where v is the Euclidean distance from $\mathbf{z}$ to $\mathbf{x}$ (what we are trying to get) and $\frac{\mathbf{w}}{||\mathbf{w}||}$ is the unit vector pointing at the direction of $\mathbf{w}$.
3. Therefore, $\mathbf{z} = \mathbf{x} + v\frac{\mathbf{w}}{||\mathbf{w}||}$.
4. The prediction for $\mathbf{z}$ is then (subsituting into linear classifier equation):

$$
\begin{aligned}
& \mathbf{w}^T\mathbf{z} + w_b \\
= \; & \mathbf{w}^T\left(\mathbf{x} + v\frac{\mathbf{w}}{||\mathbf{w}||}\right) + w_b \\
= \; & \mathbf{w}^T\mathbf{x} + v\frac{\mathbf{w}^T\mathbf{w}}{||\mathbf{w}||} + w_b = \underbrace{\mathbf{w}^T\mathbf{x} + w_b}_{=0,\text{by definition}} + v\frac{\mathbf{w}^T\mathbf{w}}{||\mathbf{w}||} \\
= \; & v\frac{\mathbf{w}^T\mathbf{w}}{||\mathbf{w}||} = v\frac{||\mathbf{w}||^2}{||\mathbf{w}||} = v||\mathbf{w}||.
\end{aligned}
$$

5. Finally, $v = \overbrace{\mathbf{w}^T\mathbf{z} + w_b}^{prediction}/||\mathbf{w}||$.
6. Thus, if a sample z's distance to a hyperplane $\mathbf{w}^T\mathbf{x} + w_b = 0$ is $d/||\mathbf{w}||$, then $\mathbf{w}^T\mathbf{z} + w_b = d$.

▶ HW: Prove that the distance from the origin to the hyperlane is $\frac{-w_b}{||\mathbf{w}||}$.

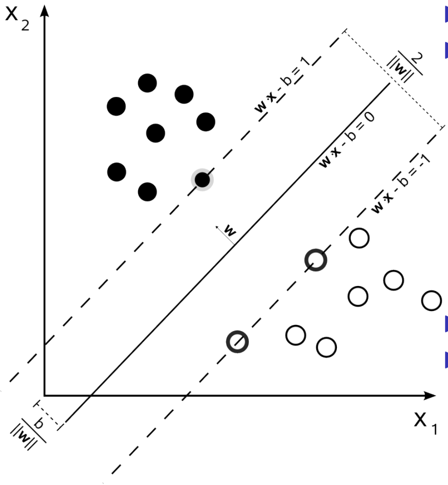
Hard margin linear SVM



- Assume that the minimum distance from any point in Class C_1 and C_2 to the hyperplane are $d_1/||\mathbf{w}||$ and $d_2/||\mathbf{w}||$, respectively, where $d_1, d_2 > 0$.
- Then we have $\mathbf{w}^T\mathbf{x} + w_b - d_1 \geq 0, \forall x \in C_1$, and $\mathbf{w}^T\mathbf{x} + w_b + d_2 \geq 0, \forall x \in C_2$.
- To make the classifier more discriminant, we want to maximize the distance between the two classes, known as the **margin**, i.e. $\max\left(\frac{d_1}{||\mathbf{w}||} + \frac{d_2}{||\mathbf{w}||}\right)$.
- An SVM classifier is also called a *Maximum Margin Classifier*.
- Assuming the two classes are linearly separable, our problem becomes:

$$\begin{cases} \max & \frac{d_1}{||\mathbf{w}||} + \frac{d_2}{||\mathbf{w}||} \\ s.t. & \mathbf{w}^T\mathbf{x} + w_b - d_1 \geq 0, \forall x \in C_1 \\ & \mathbf{w}^T\mathbf{x} + w_b + d_2 \geq 0, \forall x \in C_2 \end{cases}$$

Hard margin linear SVM (cond.)



- We prefer $d_1 = d_2$: both classes are equal.
- Since d_1 and d_2 are constants, we can let them be 1. Let the label $y_k \in \{+1, -1\}$ for sample $\mathbf{x}_k$, we can get a different form:

$$\begin{cases} \max & \frac{2}{||\mathbf{w}||} \\ s.t. & y_k(\mathbf{w}^T\mathbf{x}_k + w_b) \geq 1, \forall \mathbf{x}_k \in C_1 \cup C_2. \end{cases}$$

- Maximizing $\frac{2}{||\mathbf{w}||}$ is equivalent to minimizing $\frac{||\mathbf{w}||}{2}$.
- Finally, we transform it into a quadratic programming problem (**the primal form of SVMs**):

$$\begin{cases} \min & \frac{1}{2}||\mathbf{w}||^2 = \frac{1}{2}\mathbf{w}^T\mathbf{w} \\ s.t. & y_k(\mathbf{w}^T\mathbf{x}_k + w_b) \geq 1, \forall \mathbf{x}_k. \end{cases}$$

Recap: the Karush-Kuhn-Tucker conditions

- Given a nonlinear optimization problem

$$\begin{cases} \min & f(\mathbf{x}) \\ s.t. & h_k(\mathbf{x}) \geq 0, \forall k \in [1..K], \end{cases}$$

where $\mathbf{x}$ is a vector, and $h_k(\cdot)$ is linear, its Lagrange multiplier (or Lagrangian) is:

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{k=1}^{K} \lambda_k h_k(\mathbf{x})$$

- The necessary condition that the problem above has a solution is KKT condition:

$$\begin{cases} \frac{\partial L}{\partial \mathbf{x}} = \mathbf{0}, & \\ \lambda_k \geq 0, & \forall k \in [1..K] \\ \lambda_k h_k(\mathbf{x}) = 0, & \forall k \in [1..K] \end{cases}$$

Properties of hard margin linear SVM

The KKT condition to the SVM problem is

$$\begin{cases} A : \frac{\partial L}{\partial w} = \mathbf{0}, & \\ B : \frac{\partial L}{\partial w_b} = 0, & \\ C : \lambda_k \geq 0, & \forall k \in [1..K] \\ D : \lambda_k[y_k(\mathbf{w}^T \mathbf{x_k} + w_b) - 1] = 0, & \forall k \in [1..K] \end{cases}$$

Properties of hard margin linear SVM

The KKT condition to the SVM problem is

$$\begin{cases} A : \frac{\partial L}{\partial w} = \mathbf{0}, & \\ B : \frac{\partial L}{\partial w_b} = 0, & \\ C : \lambda_k \geq 0, & \forall k \in [1..K] \\ D : \lambda_k [y_k(\mathbf{w}^T \mathbf{x_k} + w_b) - 1] = 0, & \forall k \in [1..K] \end{cases}$$

From Eqs. A and B,

$$\frac{\partial L}{\partial \mathbf{w}} = \mathbf{w} - \sum_{k=1}^{K} \lambda_k y_k \mathbf{x_k} \Rightarrow \mathbf{w} = \sum_{k=1}^{K} \lambda_k y_k \mathbf{x_k}$$

$$\frac{\partial L}{\partial w_b} = \sum_{k=1}^{K} \lambda_k y_k = 0$$

Properties of hard margin linear SVM

The KKT condition to the SVM problem is

$$\begin{cases} A : \frac{\partial L}{\partial w} = \mathbf{0}, & \\ B : \frac{\partial L}{\partial w_b} = 0, & \\ C : \lambda_k \geq 0, & \forall k \in [1..K] \\ D : \lambda_k [y_k(\mathbf{w}^T \mathbf{x_k} + w_b) - 1] = 0, & \forall k \in [1..K] \end{cases}$$

From Eqs. A and B,

$$\frac{\partial L}{\partial \mathbf{w}} = \mathbf{w} - \sum_{k=1}^{K} \lambda_k y_k \mathbf{x_k} \Rightarrow \mathbf{w} = \sum_{k=1}^{K} \lambda_k y_k \mathbf{x_k}$$

$$\frac{\partial L}{\partial w_b} = \sum_{k=1}^{K} \lambda_k y_k = 0$$

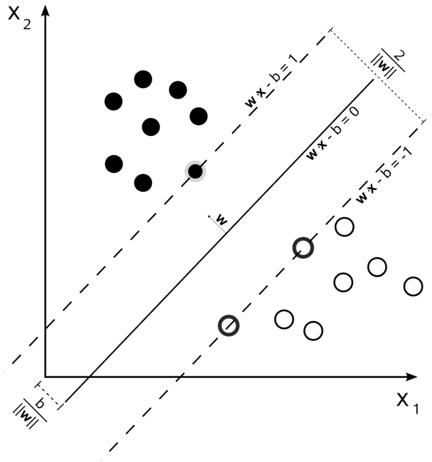
Because λ_k is either positive or 0, the solution of the SVM problem is only associated with samples whose $\lambda_k \neq 0$. Denote them as $N_s = \{\mathbf{x}_k | \lambda_k \neq 0, k \in [1..K]\}$.

Properties of hard margin linear SVM (cont.)

- Therefore, Eq. A can be rewritten into

$$\mathbf{w} = \sum_{\mathbf{x}_k \in N_s} \lambda_k y_k \mathbf{x_k}$$

- The samples $\mathbf{x}_k \in N_s$ collectively determine the $\mathbf{w}$, and thus called **support vectors**, supporting the solution.



The dual form of an SVM

1. Given a nonlinear optimization problem in the **primal** form

$$\begin{cases} \min & f(\mathbf{x}) \\ s.t. & h_k(\mathbf{x}) \geq 0, \forall k \in [1..K], \end{cases}$$

The dual form of an SVM

1. Given a nonlinear optimization problem in the **primal** form

$$\begin{cases} \min & f(\mathbf{x}) \\ s.t. & h_k(\mathbf{x}) \geq 0, \forall k \in [1..K], \end{cases}$$

2. its **dual form** is

$$\begin{cases} \max & L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{k=1}^{K} \lambda_k h_k(\mathbf{x}) \\ s.t. & \lambda_k \geq 0, \forall k \in [1..K], \\ & \nabla L = \mathbf{0} \end{cases}$$

The dual form of an SVM

1. Given a nonlinear optimization problem in the **primal** form

$$\begin{cases} \min & f(\mathbf{x}) \\ s.t. & h_k(\mathbf{x}) \geq 0, \forall k \in [1..K], \end{cases}$$

2. its **dual form** is

$$\begin{cases} \max & L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{k=1}^{K} \lambda_k h_k(\mathbf{x}) \\ s.t. & \lambda_k \geq 0, \forall k \in [1..K], \\ & \nabla L = \mathbf{0} \end{cases}$$

3. Thus for a **primal** SVM problem

$$\begin{cases} \min & \frac{1}{2}||\mathbf{w}||^2 = \frac{1}{2}\mathbf{w}^T\mathbf{w} \\ s.t. & y_k(\mathbf{w}^T\mathbf{x}_k + w_b) \geq 1, \forall \mathbf{x}_k. \end{cases}$$

The dual form of an SVM

1. Given a nonlinear optimization problem in the **primal** form

$$\begin{cases} \min & f(\mathbf{x}) \\ s.t. & h_k(\mathbf{x}) \geq 0, \forall k \in [1..K], \end{cases}$$

2. its **dual form** is

$$\begin{cases} \max & L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{k=1}^{K} \lambda_k h_k(\mathbf{x}) \\ s.t. & \lambda_k \geq 0, \forall k \in [1..K], \\ & \nabla L = \mathbf{0} \end{cases}$$

3. Thus for a **primal** SVM problem

$$\begin{cases} \min & \frac{1}{2}||\mathbf{w}||^2 = \frac{1}{2}\mathbf{w}^T\mathbf{w} \\ s.t. & y_k(\mathbf{w}^T\mathbf{x}_k + w_b) \geq 1, \forall \mathbf{x}_k. \end{cases}$$

4. its **dual form** is

$$\begin{cases} \max & \frac{1}{2}||\mathbf{w}||^2 - \sum\limits_{k=1}^{K} \lambda_k(y_k(\mathbf{w}^T\mathbf{x_k} + w_b) - 1) \\ s.t. & \lambda_k \geq 0, \forall k \in [1..K], \\ & \mathbf{w} = \sum\limits_{k=1}^{K} \lambda_k y_k x_k \;\; (from \;\; \frac{\partial L}{\partial \mathbf{w}} = 0), \\ & \sum\limits_{k=1}^{K} \lambda_k y_k = 0 \;\; (from \;\; \frac{\partial L}{\partial w_b} = 0) \end{cases}$$

The dual form of an SVM (cond.)

$$\begin{cases} \max & \frac{1}{2}||\mathbf{w}||^2 - \sum_{k=1}^{K} \lambda_k(y_k(\mathbf{w}^T\mathbf{x_k} + w_b) - 1) \\ s.t. & \lambda_k \geq 0, \forall k \in [1..K], \\ & \mathbf{w} = \sum_{k=1}^{K} \lambda_k y_k x_k \quad (from \quad \frac{\partial L}{\partial \mathbf{w}} = 0), \\ & \sum_{k=1}^{K} \lambda_k y_k = 0 \quad (from \quad \frac{\partial L}{\partial w_b} = 0) \end{cases}$$

Subsituting $\mathbf{w}$ with $\sum_{k=1}^{K} \lambda_k y_k x_k$, the objective function becomes:

$$L = -\frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K} \lambda_i\lambda_j y_i y_j \mathbf{x}_i^T\mathbf{x}_j + \sum_{k=1}^{K}\lambda_k$$

Thus, the new **dual form** is:

$$\begin{cases} \max & -\frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K} \lambda_i\lambda_j y_i y_j \mathbf{x}_i^T\mathbf{x}_j + \sum_{k=1}^{K}\lambda_k \\ s.t. & \lambda_k \geq 0, \forall k \in [1..K], \\ & \sum_{k=1}^{K} \lambda_k y_k = 0 \end{cases}$$

- The number of unknowns to solve drops from n features to K samples.
- Instead of finding $\mathbf{w}$, find K λ_k's. (Is an SVM really non-parametric?)
- The new SVM: $g(\mathbf{x}) = \mathbf{w}^T\mathbf{x} = \sum_{k=1}^{K} \lambda_k y_k(\mathbf{x}^T\mathbf{x_k}) + w_b$.
- To store an SVM model, just store the support vectors $\mathbf{x}_i$'s, their labels y_i's and weights λ_i's, and the bias w_b.

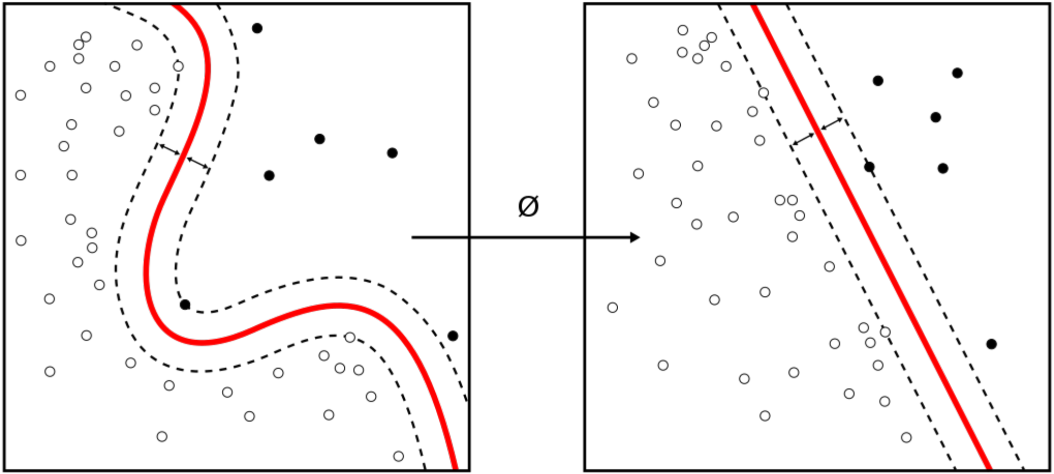
Kernel tricks: achieving non-linearity on SVMs

- In the previous slides, any two samples "interact" with each other thru dot product, e.g., $\mathbf{x_i}^T\mathbf{x}_j$ (in training, between two samples) or $\mathbf{x}^T\mathbf{x_k}$ (in prediction, between a sample to be predicted and a support vector).
- It can be expanded to any operation between two vectors, known as the **kernel function** or **kernel tricks**.
- linear kernel: what we have seen so far in SVMs.
- Gaussian (radial basis function, RBF) kernel:

$$\mathbb{K}(\mathbf{x}, \mathbf{y}) = \exp\left(-\frac{||\mathbf{x} - \mathbf{y}||^2}{\sigma}\right)$$

- There are many kernels other there, but usually linear and Gaussian are good enough.

Transforming a nonlinearly separable problem to a linearly separable one



Source: Wikipedia/SVM.

Generalized Linear Classifier

- Let $f_1(\cdot)$, $f_2(\cdot)$, ..., $f_P(\cdot)$ be P nonlinear functions where $f_p : \mathbb{R}^n \mapsto \mathbb{R}, \forall p \in [1..P]$.
- Then we can define a mapping from a feature vector $\mathbf{x} \in \mathbb{R}^n$ (the **input space**) to a vector in another space $\mathbf{z} = [f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_P(\mathbf{x})]^T \in \mathbb{R}^P$, which is called the **feature space**.
- The problem then becomes finding the value P and the functions $f_p(\cdot)$ such that the two classes are linearly separable.
- Once the space transform is done, we wanna find a weight vector $\mathbf{w} \in \mathbb{R}^P$ such that

$$\begin{cases} \mathbf{w}^T\mathbf{z} + w_b > 0 & \text{if } \mathbf{z} \in C_1 \\ \mathbf{w}^T\mathbf{z} + w_b < 0 & \text{if } \mathbf{z} \in C_2. \end{cases}$$

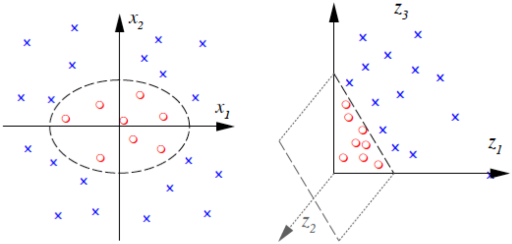
- Essentially, we are building a new hyperplane $g(\mathbf{x}) = 0$ such that $g(\mathbf{x}) = w_b + \sum_{p=1}^{P} w_p f_p(\mathbf{x})$. Instead of computing the weighted sum of elements of feature vector, we compute that of elements of the transformed vector.

Creating features from input features

- For example, $g(\mathbf{x}) = w_b + w_1 x_1 + w_2 x_2 + w_{12} x_1 x_2 + w_{11} x_1^2 + w_{22} x_2^2$
- Here is another example,

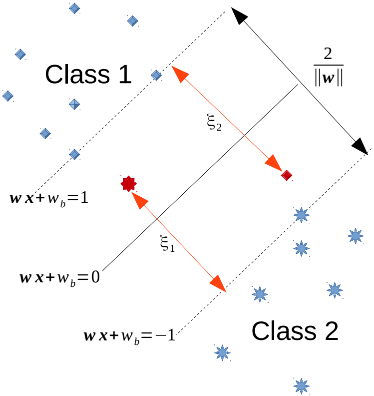
$$\Phi : R^2 \rightarrow R^3$$

$$(x_1, x_2) \mapsto (z_1, z_2, z_3) := (x_1^2, \sqrt{(2)} x_1 x_2, x_2^2)$$



- A good explanation on StackOverflow:
  https://stats.stackexchange.com/questions/46425/what-is-feature-space

Soft margin linear SVM



- We could allow some samples to fall into the margin in exchange for wider margin on the remaining samples.
- Therefore, we have a new optimization problem:

$$\begin{cases} \min & \frac{1}{2}||\mathbf{w}||^2 + C\sum_{k=1}^{K}\xi_k \\ s.t. & y_k(\mathbf{w}^T\mathbf{x}_k + w_b) \geq 1 - \xi_k, \forall \mathbf{x}_k \\ & \xi_k \geq 0. \end{cases}$$

  where C is a constant, and ξ_k is called a **slack** variable defined as $\max(0, 1 - y_i(\mathbf{w}^T\mathbf{x}_k + w_b))$.
- Such SVM is called *soft-margin*.
- The constant C provides a balance between maximizing the margin and minimizing the quality, instead of quantity, of misclassification.
- Next: How to find C and why is slack variable defined so.

Grid search for hyperparameters

- Hyperparameters: Parameters of a model that is not updated in training but set based on experience or arbitrarily.
- Grid search: Create a sequence of values for each hyperparameter and form a grid from them using Cartesian product. Then for each point on the grid, evaluate the performance of the model. Finally, use the one that yields the best performance.
- How to evaluate the performance of a classifer?

Test set

- It would be unfair to evaluate the performance of a classifier using samples seen by the model during training.
- Samples unseen in training and used to evaluate the performance of a model form the **test set**.
- So, from all your data, you split them into two groups **training set** and test set.
- But, is just one test set good?

Cross-validation

- Cross validation (CV): split your data into many pairs of training and test sets. Then evaluate the performance of the classifier on each pair. Usually the test sets do not overlap. And, of course, the training and test sets in each pair do not overlap.
- k-fold CV: Split all data into k folds, equal-size and **non-overlapping**. In each round the CV, use $k-1$ folds for training and the rest one fold for test. Then rotate on the test set. Stop after every fold has been used as test set exactly k times.
- leave-N-out CV (LNOCV): A special case of k-fold CV that only N samples are the test set. When $N = 1$, it becomes leave-one-out CV (LOOCV).

The slack variable and hinge loss

- What is the $\xi = \max(0, 1 - y_i(\mathbf{w}^T\mathbf{x} + w_b))$ when a sample $\mathbf{x}$ is correctly classified?
- It's zero.
- In that case, the constraint is the same as that for hard margin linear SVMs: $y_k(\mathbf{w}^T\mathbf{x} + w_b) \geq 0$.
- The expression $\max(0, 1 - y \cdot \hat{y})$ where $y \in \{+1, -1\}$ is the ground truth label and $\hat{y}$ is prediction for a classifier, is called a **hinge loss**. It's "hinge" because as long as the classification is correct, the loss/error is (capped at) 0.