# Problem 1

1. True. For $S$ to generate $V$, $span(S)=V$; then by definition $\forall\vec{x}\in V$, $\vec{x}\in span(S)$ so by definition of span they each can be written as a linear combination of vectors in $S$.
2. False. At least one vector in $S$ must be linear combination of other vectors in $S$, not all.
3. True. By definition of linear independence.
4. False. For example, the vector space $P(F)$, the set of polynomials over a field $F$, has an infinite dimensional basis.
5. False. The basis is simply $\varnothing$.
6. False. Many bases can exist. It is easy to see that for $\mathbb{R}^2$, the standard basis $\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\}$ and a similar but distinct basis $\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\}$ exist.
7. False. $\dim(P_n(\mathbb{R}))=n+1$, since there are $n$ powers of $X$ and a constant value.

# Problem 2

Suppose $\mathcal{B}$ is a basis for $V$. Then by definition, $\mathcal{B}$ is linearly independent and spans $V$. Then (1) is immediately satisfied. Assume by contradiction that $\exists\vec{x}\in\mathcal{B}$ such that $\mathcal{B}\setminus\{\vec{x}\}$ spans $V$. Since $\vec{x}\in\mathcal{B}$ and $\mathcal{B}$ is a basis for $V$, then $\vec{x}\in V$. But $\mathcal{B}\setminus\{\vec{x}\}$ spans $V$, i.e. $span(\mathcal{B}\setminus\{\vec{x}\})=V$, hence $\vec{x}\in span(\mathcal{B}\setminus\{\vec{x}\})$. But by the Theorem on Equivalent Definitions for Linear Dependence, if $\exists\vec{x}\in\mathcal{B}$ such that $\vec{x}\in span(\mathcal{B}\setminus\{\vec{x}\})$, then $\mathcal{B}$ is linearly dependent, contradiction; hence $\forall\vec{x}\in\mathcal{B}$, $\mathcal{B}\setminus\{\vec{x}\}$ does not span $V$.

Suppose $\mathcal{B}$ spans $V$ and $\forall\vec{x}\in\mathcal{B}$, $\mathcal{B}\setminus\{\vec{x}\}$ does not span $V$. To be a basis, we need $\mathcal{B}$ spans $V$ and $\mathcal{B}$ is linearly independent. The first is satisfied by assumption. Assume by contradiction that $\mathcal{B}$ is linearly dependent, or by Theorem on Equivalent Definitions for Linear Dependence $\exists\vec{x}\in\mathcal{B}$ such that $\vec{x}\in span(\mathcal{B}\setminus\{\vec{x}\})$. But since $\vec{x}\in span(\mathcal{B}\setminus\{\vec{x}\})$, we must have that $span(\mathcal{B}\setminus\{\vec{x}\})=span(\mathcal{B}\setminus\{\vec{x}\}\cup\{\vec{x}\})=span(\mathcal{B})=V$. But $\forall\vec{x}\in\mathcal{B}$, $\mathcal{B}\setminus\{\vec{x}\}$ does not span $V$, contradiction. Hence $\mathcal{B}$ is also linearly independent and thus a basis for $V$.

## Problem 3

The Basis Reduction Theorem guarantees that if a finite set of vectors $G$ spans $V$, then there exists a basis for $V$, $\mathcal{B}$, such that $\mathcal{B} \subseteq G$. In this case, the subset $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_5}\}$ is a basis for $\mathbb{R}^3$.

## Problem 4

We saw that $\dim(P_n) = n+1$. Since $W = \{f \in P_n(\mathbb{R}) \mid f(0) = 0\}$, where $f = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, we have an extra constraint that $f(0) = a_0 = 0$. Then the constant factor is always fixed to $0$, so we only need to worry about the $n$ powers of $x$. Thus $\dim(W) = n$.

## Problem 5

Since $0$ and $1$ are in every field, we can tweak the standard basis for $F^5$ to satisfy the specific constraints of $W_1$ and $W_2$. For $W_1$, we see that the constraint is effectively $a_1 = a_3 + a_4$. Then take $\mathcal{B} = \{(0,1,0,0,0),(1,0,1,0,0),(1,0,0,1,0),(0,0,0,0,1)\}$ as the basis. It is easy to see that the set is linearly independent. Also, for $\vec{x} \in W_1$, $\vec{x} = (a_3 + a_4, a_2, a_3, a_4, a_5)$, and we see that $\vec{x} = a_2(0,1,0,0,0) + a_3(1,0,1,0,0) + a_4(1,0,0,1,0) + a_5(0,0,0,0,1)$. Then $\vec{x}$ can be expressed as a linear combination of vectors in $\mathcal{B}$, hence by definition $\vec{x} \in span(\mathcal{B})$. Since $\vec{x}$ was an arbitrary vector in $W_1$, this is true for all $\vec{x} \in W_1$, hence $span(\mathcal{B}) = W_1$ and $\mathcal{B}$ is a basis by definition.

For $W_2$, our constraint is $a_2 = a_3 = a_4$ and $a_1 = -a_5$, where $-a_5$ denotes the additive inverse of $a_5$. Then we can take our basis to be $\mathcal{B} = \{(1,0,0,0,-1),(0,1,1,1,0)\}$. Again it is easy to see that these vectors are linearly independent. For $\vec{x} \in W_2$, $\vec{x} = (a_1, a_2, a_2, a_2, -a_1)$, hence $\vec{x} = a_1(1,0,0,0,-1) + a_2(0,1,1,1,0)$. Again this expresses $\vec{x}$ as a linear combination of vectors in $\mathcal{B}$, so $\vec{x} \in span(\mathcal{B})$; this is true for arbitrary $\vec{x} \in W_2$, so $W_2 = span(\mathcal{B})$ and $\mathcal{B}$ is a basis.

Then, by definition of dimension, we see that $\dim(W_1) = 4$ and $\dim(W_2) = 2$.

## Problem 6

1. We see that $\dim(V)=3$. Since $|\mathcal{B}|=3$, then we only need that $\mathcal{B}$ is linearly independent or $\mathcal{B}$ spans $V$ for it to be a basis by Theorem 5. We show that $\mathcal{B}$ is linearly independent. Consider $a(1,2,1)+b(2,1,0)+c(1,-2,2)=(0,0,0)$. We see this results in the system of equations $\begin{cases} a+2b=0 \\ 2a+b-2c=0 \\ a+2c=0 \end{cases}$. Solving, we see that $\begin{cases} a+2b=0 \\ 2a+b-2c=0 \\ a+2c=0 \end{cases} \Rightarrow \begin{cases} 2b-2c=0 \\ 2a+b-2c=0 \\ a+2c=0 \end{cases} \Rightarrow \begin{cases} b=c \\ 2a+3c=0 \\ a+2c=0 \end{cases} \Rightarrow \begin{cases} b=c \\ -c=0 \\ a+2c=0 \end{cases} \Rightarrow \begin{cases} b=0 \\ c=0 \\ a=0 \end{cases}$. Hence the only solution is the trivial linear combination, and so $\mathcal{B}$ is linearly independent and thus a basis.

2. Once again, $\dim(Mat(2,\mathbb{R}))=2\cdot 2=4$, and $|\mathcal{B}|=4$, so we only need to show $\mathcal{B}$ is linearly independent. However, we see that $\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$ can be expressed as a linear combination of the other three vectors: $\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}=a\begin{pmatrix} 3 & -3 \\ 0 & 2 \end{pmatrix}+b\begin{pmatrix} 0 & 0 \\ -5 & 2 \end{pmatrix}+c\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$
$\begin{cases} 3a-c=-1 \\ -3a+c=1 \\ -5b=2 \\ 2a+2b-c=1 \end{cases} \Rightarrow \begin{cases} 3a-c=-1 \\ -3a+c=1 \\ b=-2/5 \\ 2a-c=9/5 \end{cases} \Rightarrow \begin{cases} 3a-c=-1 \\ a=-14/5 \\ b=-2/5 \\ 2a-c=9/5 \end{cases} \Rightarrow \begin{cases} c=-37/5 \\ a=-14/5 \\ b=-2/5 \end{cases}$. Then $a\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}+b\begin{pmatrix} 3 & -3 \\ 0 & 2 \end{pmatrix}+c\begin{pmatrix} 0 & 0 \\ -5 & 2 \end{pmatrix}+d\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has a solution where not all coefficients are $0$, namely $a=-1$, $b=-14/5$, $c=-2/5$, $d=-37/5$. Then by definition $\mathcal{B}$ is not linearly independent and so is not a basis.

3. We see that $\dim(\mathbb{Q}^3)=3$, but $|\mathcal{B}|=4$. By definition of dimension, if $\mathcal{B}$ is a basis then $\dim(\mathbb{Q}^3)=|\mathcal{B}|$, but $3\neq 4$, hence $\mathcal{B}$ is not a basis.

4. $\dim(P_2(\mathbb{R}))=2+1=3$ and $|\mathcal{B}|=3$, so we only need to show that $\mathcal{B}$ is linearly independent. Consider $a(x^2+2x-1)+b(3x-x+2)+c(x^2+x+1)=0$. Then,

$\begin{cases} a+c=0 \\ 2a-b+c=0 \\ -a+2b+c=0 \end{cases} \Rightarrow \begin{cases} a=-c \\ -2c-b+c=0 \\ c+2b+c=0 \end{cases} \Rightarrow \begin{cases} a=-c \\ b=-c \\ b=-c \end{cases}$. But we see that $a=-1, b=-1, c=1$ is a solution to the system, and not all coefficients are zero; then by definition $\mathcal{B}$ is linearly dependent and thus not a basis.

# Problem 7

1. Let $\mathcal{B}=\{\overrightarrow{x_1},\ldots,\overrightarrow{x_n}\}$ be a basis for $W_1 \cap W_2$. By definition, $\mathcal{B}$ must be linearly independent, and since $\mathcal{B} \subset W_1 \cap W_2 \subset W_1, W_2$, we can apply the basis extension theorem, i.e. there exists bases $\mathcal{B}_1$ for $W_1$ and $\mathcal{B}_2$ for $W_2$ such that $\mathcal{B} \subset \mathcal{B}_1$ and $\mathcal{B} \subset \mathcal{B}_2$. Let $\mathcal{B}_1=\{\overrightarrow{x_1},\ldots,\overrightarrow{x_n},\overrightarrow{y_1},\ldots,\overrightarrow{y_k}\}$ and $\mathcal{B}_2=\{\overrightarrow{x_1},\ldots,\overrightarrow{x_n},\overrightarrow{z_1},\ldots,\overrightarrow{z_m}\}$. We see that $\mathcal{B}_1 \cup \mathcal{B}_2=\{\overrightarrow{x_1},\ldots,\overrightarrow{x_n},\overrightarrow{y_1},\ldots,\overrightarrow{y_k},\overrightarrow{z_1},\ldots,\overrightarrow{z_m}\}$. Also, by definition, if $\vec{x} \in W_1+W_2$, then $\vec{x}=\overrightarrow{w_1}+\overrightarrow{w_2}$, where $\overrightarrow{w_1} \in W_1$ and $\overrightarrow{w_2} \in W_2$. But since $\mathcal{B}_1$ is a basis for $W_1$ and $\mathcal{B}_2$ for $W_2$, we have $\overrightarrow{w_1}=\sum_{i=1}^{n} a_i \overrightarrow{x_i}+\sum_{i=1}^{k} b_i \overrightarrow{y_i}$ and $\overrightarrow{w_2}=\sum_{i=1}^{n} c_i \overrightarrow{x_i}+\sum_{i=1}^{m} d_i \overrightarrow{z_i}$. Then $\vec{x}=\sum_{i=1}^{n}(a_i+c_i)\overrightarrow{x_i}+\sum_{i=1}^{k} b_i \overrightarrow{y_i}+\sum_{i=1}^{m} d_i \overrightarrow{z_i}$, a linear combination of exactly the vectors in $\mathcal{B}_1 \cup \mathcal{B}_2$; hence $W_1+W_2=span(\mathcal{B}_1 \cup \mathcal{B}_2)$. To show that $\mathcal{B}_1 \cup \mathcal{B}_2$ is linearly independent, we consider $\sum_{i=1}^{n} a_i \overrightarrow{x_i}+\sum_{i=1}^{k} b_i \overrightarrow{y_i}+\sum_{i=1}^{m} d_i \overrightarrow{z_i}=\vec{0}$. Then, $\sum_{i=1}^{n} a_i \overrightarrow{x_i}+\sum_{i=1}^{k} b_i \overrightarrow{y_i}=-\sum_{i=1}^{m} d_i \overrightarrow{z_i}$. We see that the left hand side is in $span(\mathcal{B}_1)=W_1$, while the right is in $W_2$. Then, $-\sum_{i=1}^{m} d_i \overrightarrow{z_i} \in W_1 \cap W_2$, or $-\sum_{i=1}^{m} d_i \overrightarrow{z_i}=\sum_{i=1}^{r} e_i \overrightarrow{x_i}$; then $\sum_{i=1}^{r} e_i \overrightarrow{x_i}-\sum_{i=1}^{m} d_i \overrightarrow{z_i}=\vec{0}$. But this is exactly a linear combination of vectors from $\mathcal{B}_2$, so $e_i=d_i=0$. Then $\sum_{i=1}^{n} a_i \overrightarrow{x_i}+\sum_{i=1}^{k} b_i \overrightarrow{y_i}=\vec{0}$; but again this is a linear combination of vectors from $\mathcal{B}_1$, so $a_i=b_i=0$. Then $\mathcal{B}_1 \cup \mathcal{B}_2$ is linearly independent and thus a basis for $W_1+W_2$. Finally, we see that $\dim(V)=|\mathcal{B}_1 \cup \mathcal{B}_2|=|\mathcal{B}_1|+|\mathcal{B}_2|-|\mathcal{B}_1 \cap \mathcal{B}_2|=\dim(W_1)+\dim(W_2)-\dim(W_1 \cap W_2)$.

2. Suppose $V = W_1 \oplus W_2$. Then $W_1 \cap W_2 = \{\vec{0}\}$ by definition. From the above,

$$\dim(V) = \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

$$= \dim(W_1) + \dim(W_2) - \dim(\{\vec{0}\})$$

$$= \dim(W_1) + \dim(W_2) - 0 = \dim(W_1) + \dim(W_2)$$

Suppose $\dim(W_1) + \dim(W_2) = \dim(V) = \dim(W_1 + W_2)$. From the above, we see that for this to hold true we must have $\dim(W_1 \cap W_2) = 0$. The only vector space with a dimension of 0 is $\{\vec{0}\}$; then $W_1 \cap W_2 = \{\vec{0}\}$, which combined with the assumption that $V = W_1 + W_2$ yields $V = W_1 \oplus W_2$ by definition.