A5

October 20, 2022

[6]:
```
import camb
import numpy as np
import matplotlib.pylab as plt
import glob
import time
import corner
```

[7]:
```
glob.glob("./*")
```

[7]:
```
['./obj',
 './A5.ipynb',
 './COM_PowerSpect_CMB-TT-binned_R3.01.txt',
 './planck_chain.txt',
 './COM_PowerSpect_CMB-TT-full_R3.01.txt',
 './planck_chain_tauprior.txt',
 './planck_fit_params.txt']
```

1 Data

The four columns contains multipole, variance, lower & upper errorbars.

We look at multipoles from 2 to 2508. (I guess that for higher multipoles, we are looking at higher angular scales).

[8]:
```
dat = np.loadtxt("COM_PowerSpect_CMB-TT-full_R3.01.txt")
dat.shape
```

[8]:
```
(2507, 4)
```

2 Question a)

2.0.1 Try to understand the script

CAMB model parameters

source: >https://camb.readthedocs.io/en/latest/model.html
>https://camb.readthedocs.io/en/latest/CAMBdemo.html

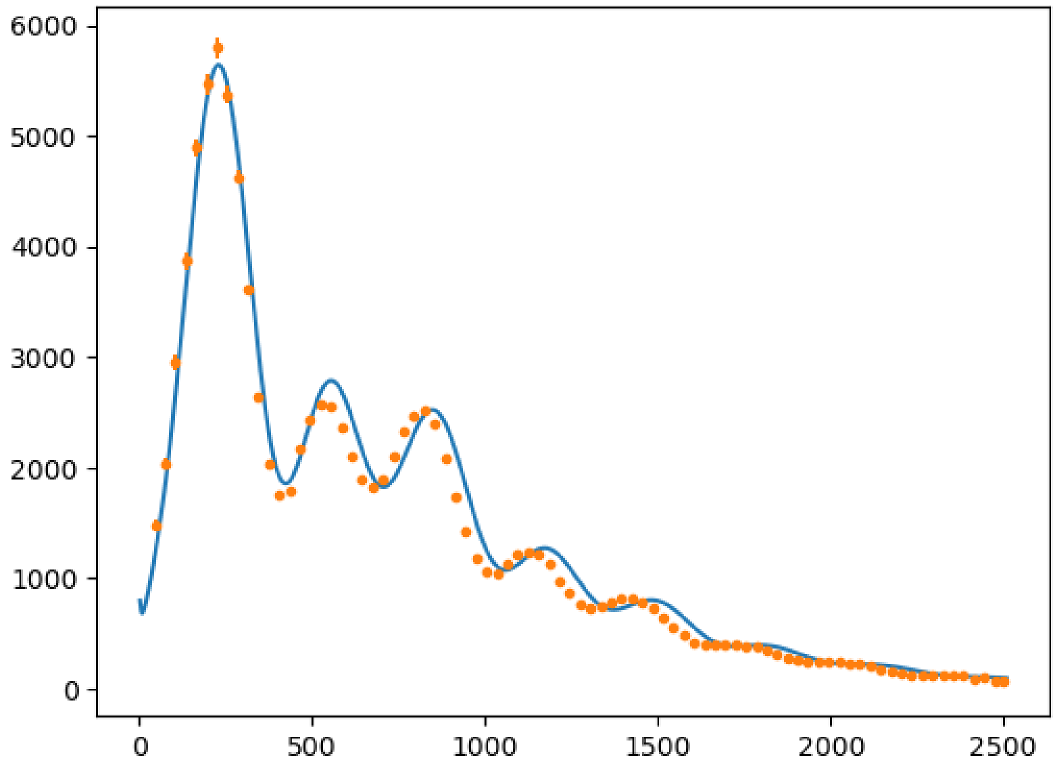
Essentially: * set I.C. for cosmological conditions * set initial power spectrum params * set max multipole we want * calculate specific parameters (including power spectra) * get total power spectra (shaped (lmax,4))

[5]:
```
# parameters for power spectrum model
pars=np.asarray([60,0.02,0.1,0.05,2.00e-9,1.0])
# load data
planck=np.loadtxt('COM_PowerSpect_CMB-TT-full_R3.01.txt',skiprows=1)
# scale
ell=planck[:,0]
# variance
spec=planck[:,1]
# avg. errorbar
errs=0.5*(planck[:,2]+planck[:,3]);
```

[6]:
```
def get_spectrum(pars,lmax=3000):
    # hubble
    H0=pars[0]
    # density of regular baryons
    ombh2=pars[1]
    # Omega_cdm: density of cold dark matter
    omch2=pars[2]
    # Thompson scattering optical depth
    tau=pars[3]
    ## "Amplitude and tilt of init. power spectrum"
        # comoving curvature power at k=pivot_scalar
    As=pars[4]
        # scalar spectral index
    ns=pars[5]

    pars=camb.CAMBparams()

        # mnu: sum of neutrino masses
        # omk: Omega_k curvature parameter
    pars.set_cosmology(H0=H0,ombh2=ombh2,omch2=omch2,mnu=0.06,omk=0,tau=tau)

        # r:  tensor to scalar ratio at pivot
    pars.InitPower.set_params(As=As,ns=ns,r=0)

    # Set parameters to get CMB power spectra accurate to specific a l_lmax.
            # this does not fix the actual output L range,
            # spectra may be calculated above l_max (but may not be accurate␣
↪there).
    pars.set_for_lmax(lmax,lens_potential_accuracy=0)

    # Calculate results for specified parameters and return CAMBdata instance
    results=camb.get_results(pars)
    powers = results.get_cmb_power_spectra(pars,CMB_unit='muK')

    # alternatively ('unlensed_scalar', 'unlensed_total', 'lensed_scalar',␣
↪'tensor', 'lens_potential')
    cmb = powers['total']

    tt = cmb[:,0]     #you could return the full power spectrum here if you␣
↪wanted to do say EE

    # we only have multipls from 2
    return tt[2:]
```

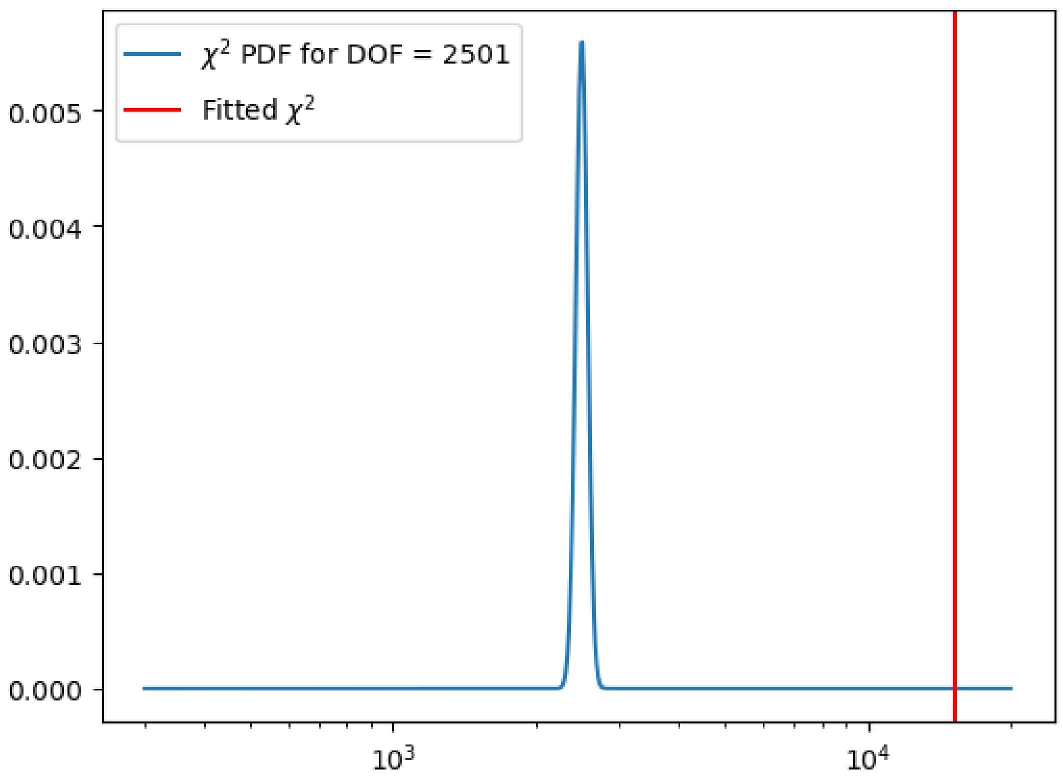
[7]:
```
model=get_spectrum(pars)
model=model[:len(spec)]
resid=spec - model
chisq=np.sum( (resid/errs)**2)
print("chisq is ",chisq," for ",len(resid)-len(pars)," degrees of freedom.")
# read in a binned version of the Planck PS for plotting purposes
planck_binned=np.loadtxt('COM_PowerSpect_CMB-TT-binned_R3.01.txt',skiprows=1)
errs_binned=0.5*(planck_binned[:,2]+planck_binned[:,3]);

plt.clf()
plt.plot(ell,model)
plt.errorbar(planck_binned[:,0],planck_binned[:,1],errs_binned,fmt='.')
plt.show()
```

```
chisq is  15267.937435709791  for  2501  degrees of freedom.
```



```
[8]: from scipy.stats import chi2

     x_chi = np.linspace(300,20000,1000)
     plt.plot(x_chi, chi2.pdf(x_chi, len(resid)-len(pars)),
              label = f"$\chi^2$ PDF for DOF = {len(resid)-len(pars)}")
     plt.axvline(chisq,c='r', label = "Fitted $\chi^2$")
     plt.xscale('log')
     plt.legend()
```

```
[8]: <matplotlib.legend.Legend at 0x13fdb7d90>
```



[9]:
```
chi2.sf(chisq, len(resid)-len(pars))
```

[9]:
```
0.0
```

The likelihood of observing such χ^2 value given our input data is essentially zero from the survival function. ## The fit is **unacceptable**.

[10]:
```
pars = np.array([69, 0.022, 0.12, 0.06, 2.1e-9, 0.95])
model=get_spectrum(pars)
model=model[:len(spec)]
resid=spec - model
chisq2=np.sum( (resid/errs)**2)
dof = len(resid) - len(pars)
```

[11]:
```
print("First set of params:", chisq, '\nwith survival value:' , chi2.sf(chisq,␣
  ↪dof),"\n")
print("Second set of params:", chisq2, '\nwith survival value:' , chi2.
  ↪sf(chisq2, dof))
print(f'\nMEAN: {dof} +/- {np.sqrt(2*dof)}')
print("\nchi^2 value, below which is 90% acceptable: ", chi2.isf(.9,  dof))
```

```
First set of params: 15267.937435709791
```

```
with survival value: 0.0

Second set of params: 3272.2036739044693
with survival value: 1.149548163599582e-23

MEAN: 2501 +/- 70.7248188403477

chi^2 value, below which is 90% acceptable:  2410.8015270308797
```

The new χ^2 value of **3272.2** is lower. However, it is still far from accepting the null hypothesis. Given that the survival probability is `~1e-23`, this new set of `pars` is still **unacceptable.**
Our target is at least below `2501`.

3 Question b): Implementing Numerical Derivatives

Stealing my own code (and adapting to this assignment).

[12]:
```python
def Ndiff(f, x, dx_list):
    diffs = []

    for i in range(len(x)):
        one_idx = np.zeros(len(x))
        one_idx[i] += 1

        dx = dx_list[i]

        x_2m = x.copy()
        x_m  = x.copy()
        x_p  = x.copy()
        x_2p = x.copy()
        x_2m -= 2 *dx * one_idx
        x_m  -= dx * one_idx
        x_p  += dx * one_idx
        x_2p += 2*dx * one_idx

        diffs.append((f(x_2m) + 8 * f(x_p) - 8 * f(x_m) - f(x_2p))/(12 * dx))

    return np.array(diffs)
```

[13]:
```python
def calc_spec(params, lmax = 3000):
    # returns the spectrum as well as the gradient w.r.t. the parameters at␣
 ↪input <p>
    f = lambda params: get_spectrum(params, lmax)
    y = f(params)
    grad = np.zeros([lmax, len(params)])
    grad = Ndiff(f, params, dx).transpose()

    return  y, grad
```

Because the curvature can be not $O(1)$, we have to set the stepsizes in each parameter differently instead of assuming an uniform `dx`.
The differentiation takes up most of the time because we have to take gradient at 2507 points along 6 directions.

[14]:
```
dx = pars *1e-3
```

3.0.1 Levenberg-marquardt

The result comes from more than 30 iterations. The last 20 iterations didn't update the result. We suspect that the NLS fitting has converged.

[15]:
```
def calc_chi(y):
    return np.sum((y - spec)**2/errs**2)

def update(damping, success):
    # if all goes well...
    if success:
        damping *= .3
        if damping <= 0.1:
            damping = 0
    else:
        if damping == 0:
            damping = 1
        else:
            damping *= 2

    return damping
```

[16]:
```
# np.save('obj/fit_p.npy', pars)
# info = [10,1e9,True]
# np.save('obj/info.npy', info)
```

3.0.2 Here we print:

- damping after last iteration
- params after last iteration
- Whether or not $ ^2 $ is less than old value

[17]:
```
steps = 10

p0 = np.load('obj/fit_p.npy')
p = np.array(p0.copy())

invN = np.diag(1/errs**2)

damping, chisq, success = np.load('obj/info.npy')
```

```
[252]: for i in range(steps):
           print("parmas: ".rjust(30), p)
           print("damping: ".rjust(30), damping)

           pred, grad = calc_spec(p)
           pred = pred[:len(spec)]
           grad = np.matrix(grad)[:len(spec),:]

           r = spec - pred
           r = np.matrix(r).transpose()

           lhs = grad.transpose() @ invN @ grad

           curv_mat = np.linalg.inv(lhs)

           lhs += damping * np.diag(np.diag(grad.transpose() @ invN @ grad))

           rhs = grad.transpose() @ invN @ r

           dp = np.linalg.inv(lhs)@(rhs)

           p_new = p.copy()

           for j in range(len(p)):
               p_new[j] = p[j] + dp[j]

           new_chisq = calc_chi(get_spectrum(p_new)[:len(spec)])

           if new_chisq < chisq:
               success = True
               chisq = new_chisq
               p = p_new
               damping = update(damping, success)

           else:
               success = False
               damping = update(damping, success)

           print(np.sqrt(np.diag(curv_mat)))
           print("success: ".rjust(30), success,"\n")


           np.save('obj/fit_p.npy', p)
           np.save('obj/fit_curvmat.npy', curv_mat)
           np.save('obj/info.npy', [damping, chisq, success])
```

```
                      parmas:   [6.78789164e+01 2.23151582e-02 1.18465150e-01
```

```
5.85254433e-02
 2.10767833e-09 9.70995108e-01]
                     damping:  1
[1.17354613e+00 2.30185089e-04 2.63700816e-03 3.64604161e-02
 1.46637861e-10 6.37003260e-03]
                     success:  True

                      parmas:  [6.78777988e+01 2.23142381e-02 1.18464882e-01
5.85338602e-02
 2.10769893e-09 9.71000762e-01]
                     damping:  0.3
[1.17355122e+00 2.30184341e-04 2.63708815e-03 3.64636829e-02
 1.46650114e-10 6.36839407e-03]
                     success:  True

                      parmas:  [6.78751114e+01 2.23123383e-02 1.18465419e-01
5.85587851e-02
 2.10778031e-09 9.71009865e-01]
                     damping:  0
[1.17400299e+00 2.30191737e-04 2.63817276e-03 3.64326459e-02
 1.46522166e-10 6.37087802e-03]
                     success:  False

                      parmas:  [6.78751114e+01 2.23123383e-02 1.18465419e-01
5.85587851e-02
 2.10778031e-09 9.71009865e-01]
                     damping:  1
[1.17400299e+00 2.30191737e-04 2.63817276e-03 3.64326459e-02
 1.46522166e-10 6.37087802e-03]
                     success:  True

                      parmas:  [6.78743861e+01 2.23118535e-02 1.18465672e-01
5.85659363e-02
 2.10780568e-09 9.71011818e-01]
                     damping:  0.3
[1.17392294e+00 2.30309365e-04 2.63673446e-03 3.63658979e-02
 1.46282176e-10 6.37271145e-03]
                     success:  True

                      parmas:  [6.78723275e+01 2.23107209e-02 1.18467290e-01
5.85884626e-02
 2.10789537e-09 9.71013444e-01]
                     damping:  0
[1.17470416e+00 2.30521875e-04 2.63792645e-03 3.63787074e-02
 1.46329231e-10 6.37967495e-03]
                     success:  False

                      parmas:  [6.78723275e+01 2.23107209e-02 1.18467290e-01
```

```
5.85884626e-02
 2.10789537e-09 9.71013444e-01]
                     damping:  1
[1.17470416e+00 2.30521875e-04 2.63792645e-03 3.63787074e-02
 1.46329231e-10 6.37967495e-03]
                     success:  True

                      parmas:  [6.78717244e+01 2.23104279e-02 1.18467848e-01
5.85951216e-02
 2.10792266e-09 9.71013535e-01]
                     damping:  0.3
[1.17480553e+00 2.30657124e-04 2.63788548e-03 3.63840328e-02
 1.46345743e-10 6.38144002e-03]
                     success:  True

                      parmas:  [6.78700276e+01 2.23097893e-02 1.18470190e-01
5.86164516e-02
 2.10801694e-09 9.71011408e-01]
                     damping:  0
[1.17411518e+00 2.30593614e-04 2.63645212e-03 3.63710469e-02
 1.46306508e-10 6.37822981e-03]
                     success:  False

                      parmas:  [6.78700276e+01 2.23097893e-02 1.18470190e-01
5.86164516e-02
 2.10801694e-09 9.71011408e-01]
                     damping:  1
[1.17411518e+00 2.30593614e-04 2.63645212e-03 3.63710469e-02
 1.46306508e-10 6.37822981e-03]
                     success:  True
```

[18]:
```
p = np.load('obj/fit_p.npy')
curv_mat = np.load('obj/fit_curvmat.npy')
```

[20]:
```
p_names = ['Hubble', "Baryon", "CDM", "Optical Depth", "P.S. Amp.", "P.S. Tilt"]
p_err = np.sqrt(np.diag(curv_mat))
for i, n in enumerate(p_names):
    print(f"{n} = ".ljust(20), f"{p[i]:.2E}", "+/-", f"{p_err[i]:.2E}")
```

```
Hubble =             6.79E+01 +/- 1.17E+00
Baryon =             2.23E-02 +/- 2.31E-04
CDM =                1.18E-01 +/- 2.64E-03
Optical Depth =      5.86E-02 +/- 3.64E-02
P.S. Amp. =          2.11E-09 +/- 1.46E-10
P.S. Tilt =          9.71E-01 +/- 6.38E-03
```

```
[320]: fit_string = []

       p_names = ['Hubble', "Baryon", "CDM", "Optical Depth", "P.S. Amp.", "P.S. Tilt"]
       p_err = np.sqrt(np.diag(curv_mat))
       for i, n in enumerate(p_names):
           fit_string.append(f"{n} = ".ljust(20)+f"{p[i]:.2E}"+ " +/- "+ f"{p_err[i]:.
        ↪2E}\n")
```

```
[325]: np.savetxt('planck_fit_params.txt', fit_string, delimiter="\n", fmt="%s")
```

4 Question c) MCMC

We draw our **trial step sizes** from the curvature matrix above based on a Gaussian distribution with $\sigma_{stepsize}$.

It takes about 0.71 seconds to run one step. Most of the time is spent calculating `results` and obtaining the power spectrum within the `CAMB` package.

```
[228]: # pp = pars.copy()
       pp = np.load('obj/chain.npy')[-1]
       steps = 5000
       scaling = .8

       # chisqr = calc_chi(get_spectrum(pp)[:len(spec)])
       chisqr = np.load('obj/chiChain.npy')[-1]

       # # # Initiate new run
       chain = np.zeros([steps, len(pp)])
       chiChain = np.zeros(steps)
```

4.0.1 Generating perturbative steps

Use multivariate with covariance matrix to generate error.
Same as doing np.linalg.cholesky(curv_mat)@ np.random.randn(6)

```
[229]: t1 = time.time()
       for s in np.arange(steps):
           # generate steps
           p_new = pp + np.random.multivariate_normal(mean = np.zeros(len(curv_mat)),␣
        ↪cov = curv_mat) * scaling
           new_chi2 = calc_chi(get_spectrum(p_new)[:len(spec)])
           del_chi = new_chi2 - chisqr

           take = None

           if del_chi >= 0 :
               if np.random.rand() < np.exp(- 0.5 * del_chi):
```

```
            take = True
        else:
            take = False

    else:
        take = True

    if take == True:
        pp = p_new
        chisqr = new_chi2

    chiChain[s] = chisqr
    chain[s, :] = pp

    # print module
    if s%1000==0:
        print(s)
        print(pp)

t2 = time.time()
```

```
0
[6.82603137e+01 2.23918260e-02 1.17444071e-01 4.19451261e-02
 2.03745233e-09 9.69858981e-01]
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
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 Warning: xe at redshift zero is < 1
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 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
1000
[6.75414662e+01 2.23768877e-02 1.20551481e-01 7.12161607e-02
 2.17244809e-09 9.67914584e-01]
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
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 Warning: xe at redshift zero is < 1
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 Warning: xe at redshift zero is < 1
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 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
2000
[6.80365476e+01 2.22508663e-02 1.17705718e-01 1.02822538e-01
 2.29949785e-09 9.70692295e-01]
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
3000
[6.69907982e+01 2.19687705e-02 1.19543128e-01 7.48244503e-02
 2.18737677e-09 9.64205412e-01]
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
 Warning: xe at redshift zero is < 1
 Check input parameters an Reionization_xe
 function in the Reionization module
4000
[6.92454213e+01 2.26094045e-02 1.15577650e-01 7.19070929e-02
 2.15239959e-09 9.74271320e-01]
```
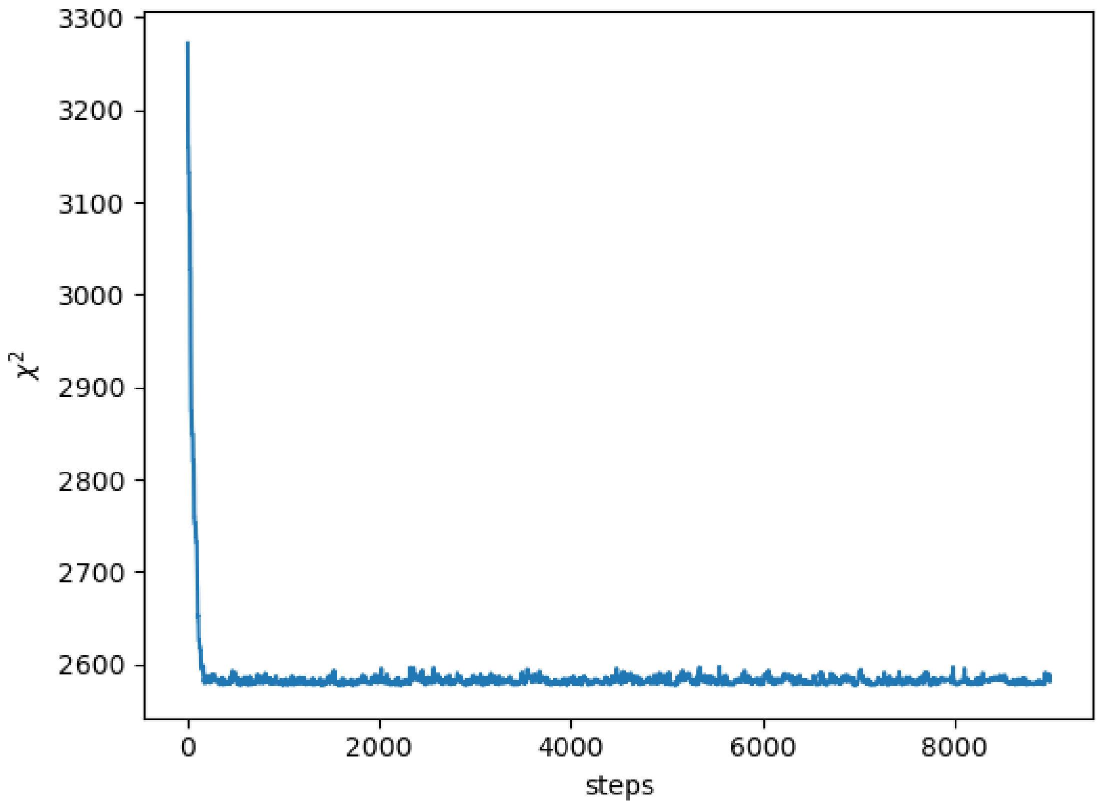
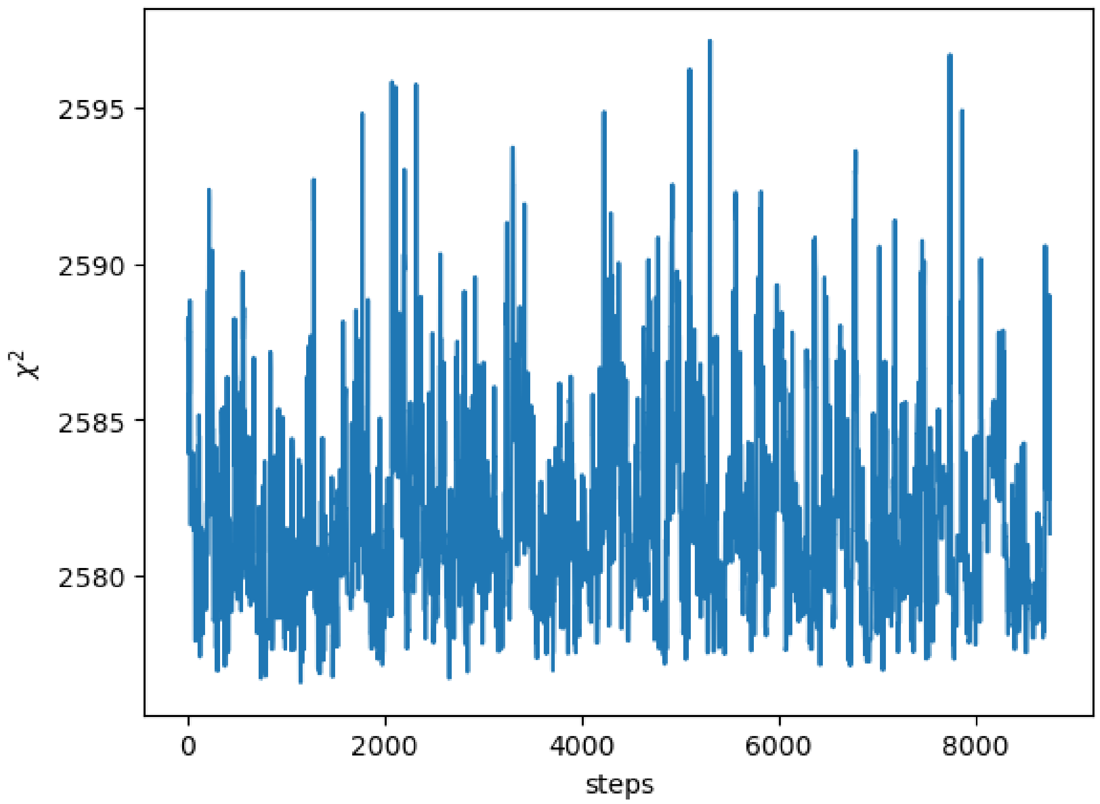

[232]:
```python
# test_chiChain = np.hstack([np.load('obj/chiChain.npy'), chiChain])
# test_chain = np.vstack([np.load('obj/chain.npy'), chain])
```

[309]:
```python
# np.save('obj/chain.npy', chain)
# np.save('obj/chiChain.npy', test_chiChain)
# chain = np.load('obj/chain.npy')[:9000]
# chiChain = np.load('obj/chiChain.npy')
```

[245]:
```python
plt.plot(chiChain)
plt.ylabel("$\chi^2$")
plt.xlabel("steps")
plt.show()
```

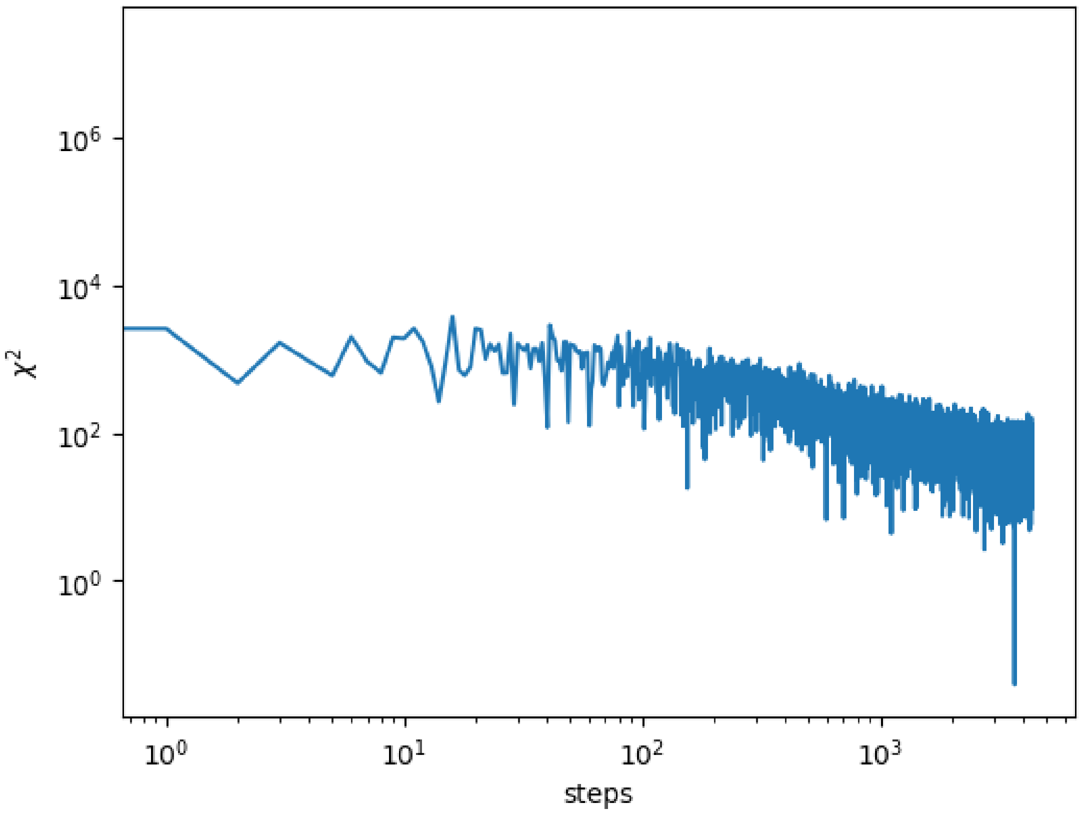


[246]:
```
plt.plot(chiChain[250:])
plt.ylabel("$\chi^2$")
plt.xlabel("steps")
plt.show()
```



```
[247]: plt.loglog(np.abs(np.fft.rfft(chiChain[300:])))
       plt.ylabel("$\chi^2$")
       plt.xlabel("steps")
```

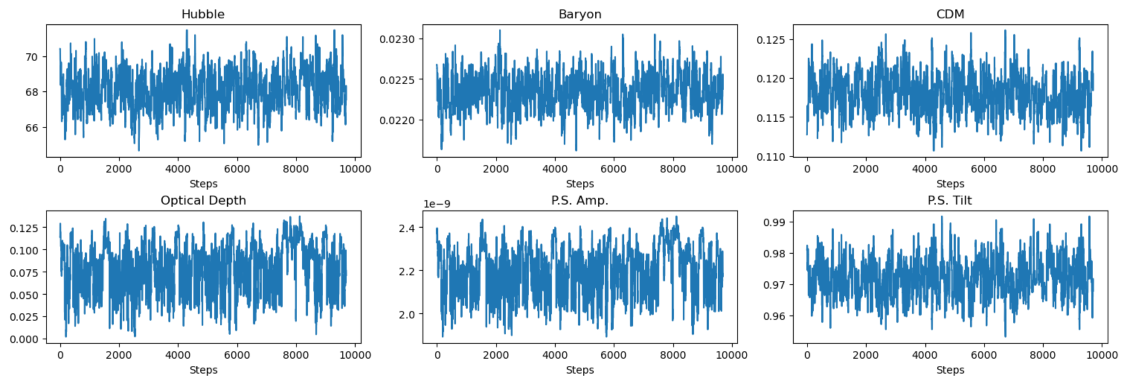
```
[247]: Text(0.5, 0, 'steps')
```



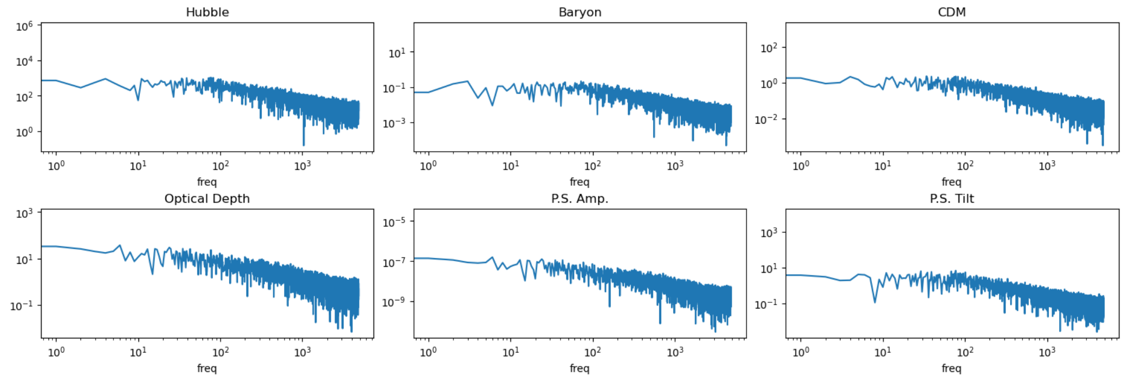
4.0.2 c) Chain Convergence

Looking at FFT of the variables, we see that the spectrum is independent of frequency up to around $\frac{10000}{100} = 100$ steps, below which we see local correlated noise.
The fourth and fifth variables are correlated, and their convergence is weaker than the others (to be dealt with in Question 4)

[248]:
```
fig = plt.figure(figsize = (15,5),constrained_layout = True)
axes = fig.subplots(2,3).flatten()

for i in np.arange(6):
    axes[i].plot(chain[300:, i])
    axes[i].set_xlabel('Steps')
    axes[i].set_title(f"{p_names[i]}")
```



```
[249]: fig = plt.figure(figsize = (15,5),constrained_layout = True)
       axes = fig.subplots(2,3).flatten()

       for i in np.arange(6):
           axes[i].loglog(np.abs(np.fft.rfft(chain[300:, i])))
           axes[i].set_xlabel('freq')
           axes[i].set_title(f"{p_names[i]}")
```



4.1 Reasonable rejection rate

Between $20\% - 25\%$

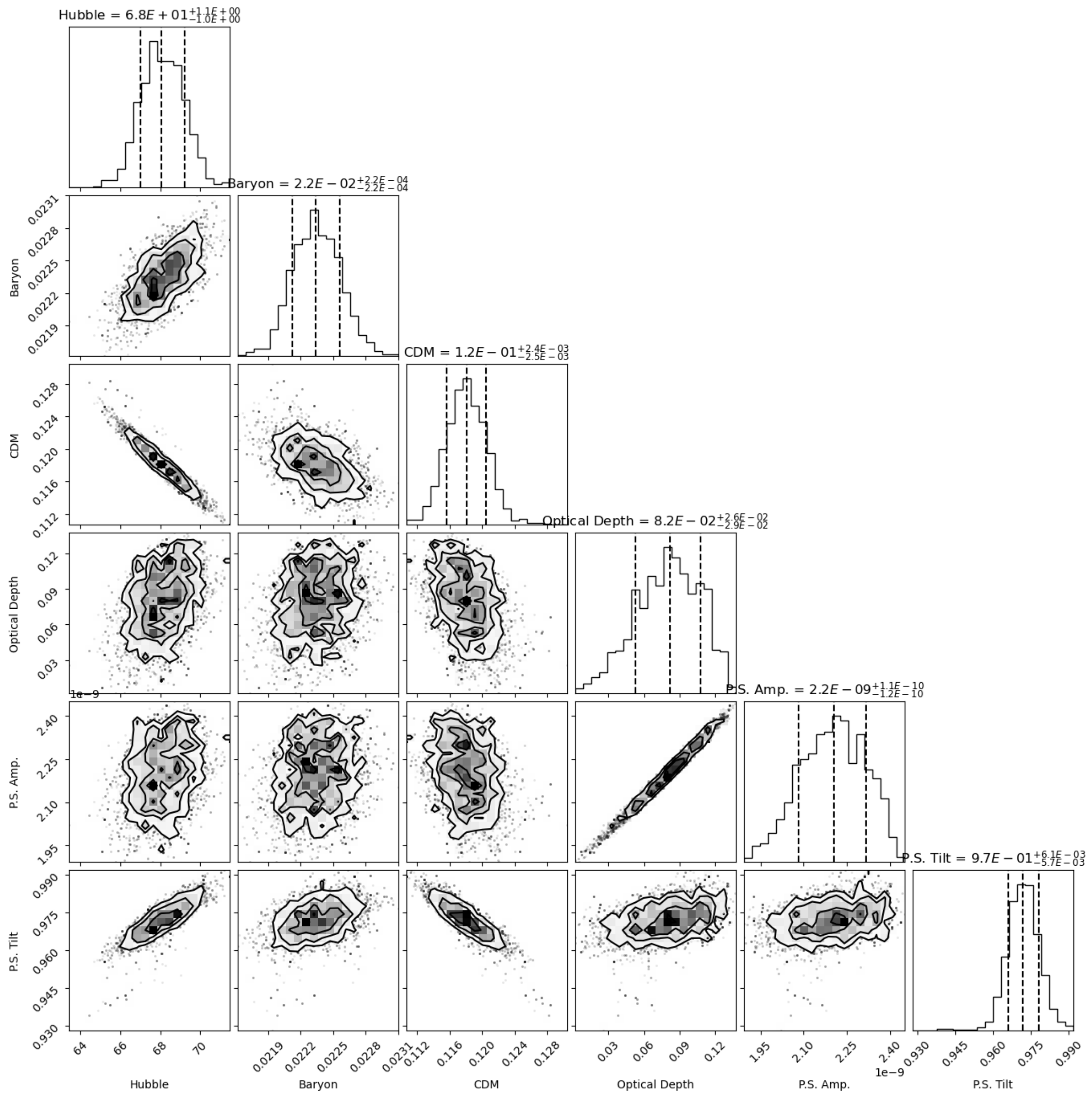
```
[250]: # rejected steps
       acc_rate = (len(chain) - np.sum(np.logical_and(np.diff(chain[:,0])==0,np.
        ↪diff(chain[:,1])==0)))/len(chain)

       print('Acceptance rate is :', acc_rate)
```

```
Acceptance rate is : 0.2486
```

4.2 Corner Plots

[251]:
```
plt.ioff()
corner.corner(chain,
    labels=p_names, quantiles=[0.16, 0.5, 0.84],
    show_titles=True, title_fmt = '.1E')
```

[251]:



4.3 Compute the output variables

[252]:
```
mp = np.mean(chain, axis = 0)
mp_err = np.std(chain,axis = 0)
```

4.4 Dark energy

For a critical universe, $\Omega_b + \Omega_c + \Omega_\Lambda = 1$.

$$h = H_0/100$$

$$\Omega_b = \text{Baryon}/h^2$$

$$\varepsilon(\Omega_b) = \sqrt{\frac{1}{h^4}\sigma^2_{\text{Baryon}} + \frac{4 * \text{Baryon}^2}{h^6}\sigma^2_h}$$
$$\varepsilon(\Omega_c) = \sqrt{\frac{1}{h^4}\sigma^2_{\text{D.M.}} + \frac{4 * \text{D.M.}^2}{h^6}\sigma^2_h}$$

4.4.1 Result: $\Omega_\Lambda = 0.70 + / - 0.02$ (dim.less)

[253]:
```
h = mp[0]/100
eh = mp_err[0]/100
Ob = mp[1]/h**2
eOb = np.sqrt(1/h**4 *mp_err[1]**2 + 4*Ob**2/h**6 * eh**2)
Oc = mp[2]/h**2
eOc = np.sqrt(1/h**4 *mp_err[2]**2 + 4*Oc**2/h**6 * eh**2)

Ol = 1 - Ob - Oc
eOl = np.sqrt(eOc**2 + eOb**2)
```

[254]:
```
print("Dark energy = ", f"{Ol:.2f}", "+/-", f"{eOl:.2f}")
```

```
Dark energy =  0.70 +/- 0.02
```

5 Problem 4

5.1 Tau Prior

Use $\frac{1}{\chi^2}$ as weight to sampling the prvious chain. Calculate covariance with this weight from the previous chain using numpy.cov. Generate new run with this covariance, evaluate new $\chi^2 = \chi^2_{fit} + \chi^2_\tau$

[84]:
```
tau = .0540
e_tau = .0074
```

6 Importance sampling

Weight associated with each step in the MCMC chain is:

$$w = \sqrt{\frac{\sigma^2_\tau}{(\tau_{\text{obs.}} - \mathbb{E}[\tau])^2}}$$

Using this prior information we can calculate the mean and the covariance again.

```
[197]: w = 1/np.sqrt(((chain[:,3] - tau)**2/e_tau**2))
       ip = np.average(chain, axis = 0, weights = w)
       prior_cov = np.cov(chain.T, aweights = w)
       ip_err = np.sqrt(np.diag(prior_cov))

       p_names = ['Hubble', "Baryon", "CDM", "Optical Depth", "P.S. Amp.", "P.S. Tilt"]
       print("IMPORTANCE SAMPLING:")
       for i, n in enumerate(p_names):
           print(f"\t{n} = ".ljust(20), f"{ip[i]:.2E}", "+/-", f"{ip_err[i]:.2E}")

       p_names = ['Hubble', "Baryon", "CDM", "Optical Depth", "P.S. Amp.", "P.S. Tilt"]

       print("\n\nFitting:")
       for i, n in enumerate(p_names):
           print(f"\t{n} = ".ljust(20), f"{p[i]:.2E}", "+/-", f"{p_err[i]:.2E}")
```

```
IMPORTANCE SAMPLING:
        Hubble =            6.88E+01 +/- 1.23E+00
        Baryon =            2.25E-02 +/- 2.26E-04
        CDM =               1.17E-01 +/- 2.79E-03
        Optical Depth =     5.49E-02 +/- 6.24E-03
        P.S. Amp. =         2.08E-09 +/- 2.99E-11
        P.S. Tilt =         9.75E-01 +/- 7.06E-03


Fitting:
        Hubble =            6.79E+01 +/- 1.17E+00
        Baryon =            2.23E-02 +/- 2.31E-04
        CDM =               1.18E-01 +/- 2.64E-03
        Optical Depth =     5.86E-02 +/- 3.64E-02
        P.S. Amp. =         2.11E-09 +/- 1.46E-10
        P.S. Tilt =         9.71E-01 +/- 6.38E-03
```

7 MCMC with Covariance Drawn from Importance Sampling

```
[219]: pp = pars.copy()
       # pp = np.load('obj/tau_chain.npy')[-1]
       steps = 10000
       scaling = .78

       chisqr = calc_chi(get_spectrum(pp)[:len(spec)])
       # chisqr = np.load('obj/tau_chiChain.npy')[-1]

       # Initiate new run
```

```
chain = np.zeros([steps, len(pp)])
chiChain = np.zeros(steps)
```

8 Metropolis Hastings

Note that we can generate Gaussian steps with prior τ information from the new covariance matrix. However, we also need to change the acceptance function by encompassing the prior information. Therefore:

$$\chi^2_{\text{new}} = \chi^2_{\text{fit}} + \chi^2_{\tau}$$

since log-likelihood functions is additive. For joint probability of both the fit and tau, we simply add the chi^2 together.

[220]:
```
for s in np.arange(steps):
    # generate steps
    p_new = pp + np.random.multivariate_normal(mean = np.zeros(len(curv_mat)),␣
 ↪cov = prior_cov) * scaling
    new_chi2 = calc_chi(get_spectrum(p_new)[:len(spec)])

    del_chi = new_chi2 - chisqr + (p_new[3] - tau)**2/e_tau**2

    take = None

    if del_chi >= 0 :
        if np.random.rand() < np.exp(- 0.5 * del_chi):
            take = True
        else:
            take = False

    else:
        take = True

    if take == True:
        pp = p_new
        chisqr = new_chi2

    chiChain[s] = chisqr
    chain[s, :] = pp

    # print module
    if s%500==0:
        print(s)
        print(pp)
```

```
0
[6.9e+01 2.2e-02 1.2e-01 6.0e-02 2.1e-09 9.5e-01]
```

```
500
[6.72294898e+01 2.22047141e-02 1.19827287e-01 7.43409750e-02
 2.19046423e-09 9.62907582e-01]
1000
[6.68712354e+01 2.23129297e-02 1.20719145e-01 6.23728762e-02
 2.13712683e-09 9.62719777e-01]
1500
[6.71217807e+01 2.22019659e-02 1.20322264e-01 6.09498271e-02
 2.12838451e-09 9.65580724e-01]
2000
[6.83007861e+01 2.24226102e-02 1.17234882e-01 5.68430663e-02
 2.09294206e-09 9.74948422e-01]
2500
[6.77218378e+01 2.22476185e-02 1.18313861e-01 5.54497653e-02
 2.09136127e-09 9.70671021e-01]
3000
[6.72179910e+01 2.21518403e-02 1.19206020e-01 6.00799071e-02
 2.12198561e-09 9.67394879e-01]
3500
[6.83811461e+01 2.23792206e-02 1.16963102e-01 5.61688723e-02
 2.09150026e-09 9.73032203e-01]
4000
[6.77595272e+01 2.21778874e-02 1.18727170e-01 4.91780250e-02
 2.06986206e-09 9.71074414e-01]
4500
[6.92855275e+01 2.23872723e-02 1.14856407e-01 4.06339020e-02
 2.01802306e-09 9.75517550e-01]
5000
[6.94902368e+01 2.27082352e-02 1.15363510e-01 5.39702784e-02
 2.07181132e-09 9.78858443e-01]
5500
[6.78263963e+01 2.24635971e-02 1.18676232e-01 7.04712313e-02
 2.16121493e-09 9.71159309e-01]
6000
[6.81180834e+01 2.22753856e-02 1.17531355e-01 5.86548130e-02
 2.09893333e-09 9.72968122e-01]
6500
[6.71505733e+01 2.20977503e-02 1.19677736e-01 4.84059370e-02
 2.07453006e-09 9.65765718e-01]
7000
[6.76990015e+01 2.22927519e-02 1.18676006e-01 5.30710961e-02
 2.08081517e-09 9.73309238e-01]
7500
[6.93895397e+01 2.25291230e-02 1.15218108e-01 5.80909777e-02
 2.08396900e-09 9.81634384e-01]
8000
[6.86466168e+01 2.24492667e-02 1.16718936e-01 6.00208155e-02
 2.10099986e-09 9.77161764e-01]
```

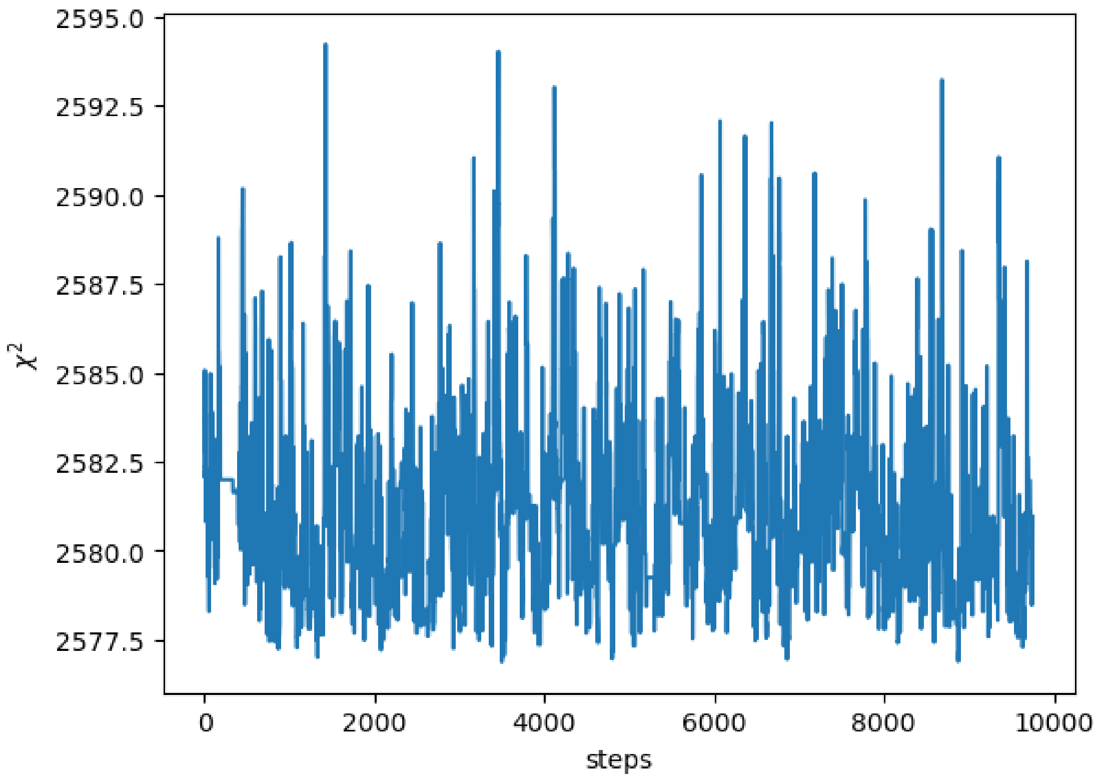
```
8500
[6.72358103e+01 2.21474434e-02 1.20052530e-01 4.70745797e-02
 2.06664834e-09 9.69681253e-01]
9000
[6.91412912e+01 2.23982200e-02 1.15080229e-01 5.41110352e-02
 2.07380287e-09 9.79317250e-01]
9500
[6.87827640e+01 2.24680577e-02 1.16659410e-01 4.76390253e-02
 2.05426554e-09 9.77401727e-01]
```

[225]:
```
# np.save('obj/tau_chain.npy', chain)
# np.save('obj/tau_chiChain.npy', chiChain)
```

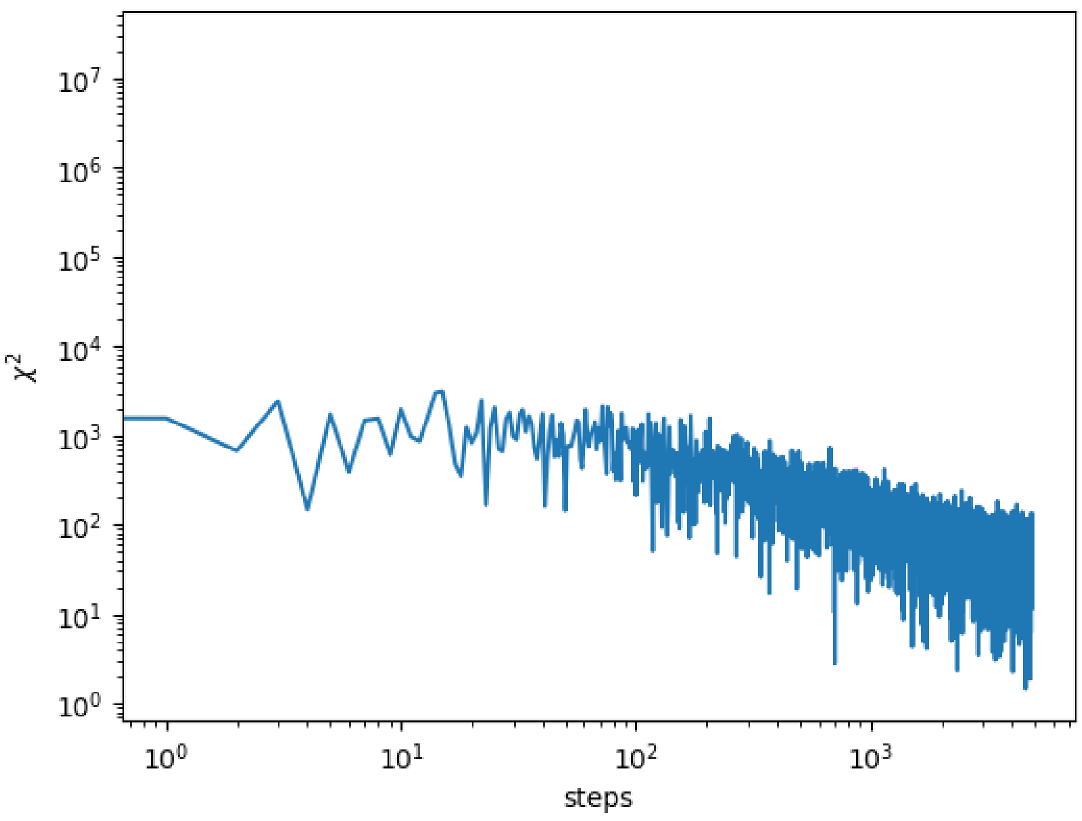
[260]:
```
chain = np.load('obj/tau_chain.npy')
chiChain = np.load('obj/tau_chiChain.npy')
```

9 Comparison of Results

[262]:
```
plt.plot(chiChain[250:])
plt.ylabel("$\chi^2$")
plt.xlabel("steps")
plt.show()
```



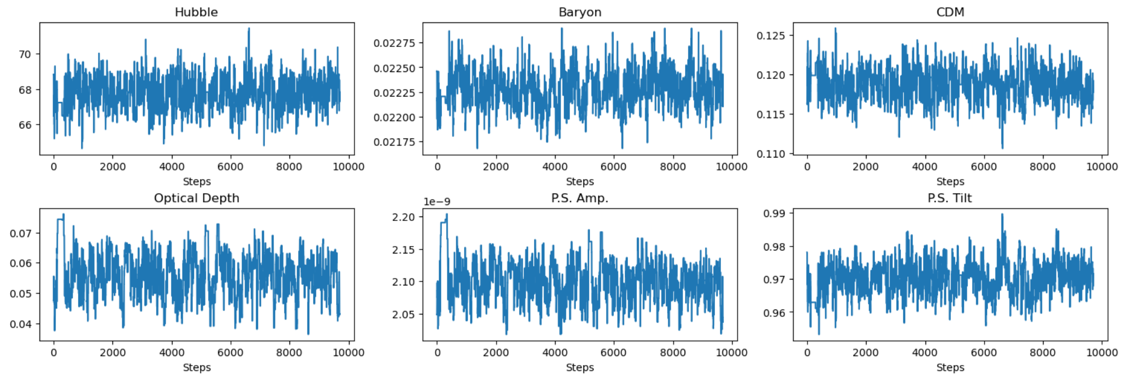
[267]:
```
plt.loglog(np.abs(np.fft.rfft(chiChain[300:])))
plt.ylabel("$\chi^2$")
plt.xlabel("steps")
plt.show()
```



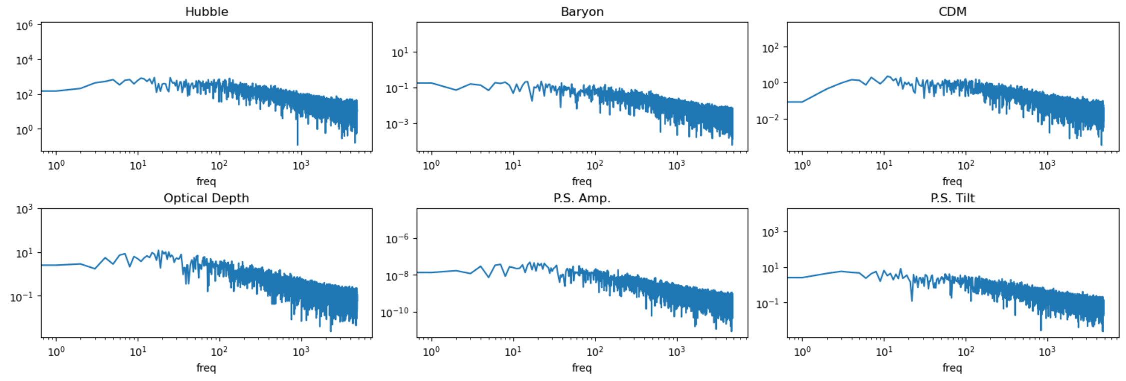
10 Chain Convergence

Looking at FFT of the variables, we see that the spectrum is independent of frequency up to around $\frac{10000}{100} = 100$ steps, below which we see local correlated noise.
The fourth and fifth variables are correlated, and their convergence is weaker than the others (to be dealt with in Question 4)

[273]:
```
plt.ion()
fig = plt.figure(figsize = (15,5),constrained_layout = True)
axes = fig.subplots(2,3).flatten()

for i in np.arange(6):
    axes[i].plot(chain[300:, i])
    axes[i].set_xlabel('Steps')
    axes[i].set_title(f"{p_names[i]}")
```



[274]:
```
fig = plt.figure(figsize = (15,5),constrained_layout = True)
axes = fig.subplots(2,3).flatten()

for i in np.arange(6):
    axes[i].loglog(np.abs(np.fft.rfft(chain[300:, i])))
    axes[i].set_xlabel('freq')
    axes[i].set_title(f"{p_names[i]}")
```



10.1 Reasonable rejection rate

Between $20\% - 25\%$

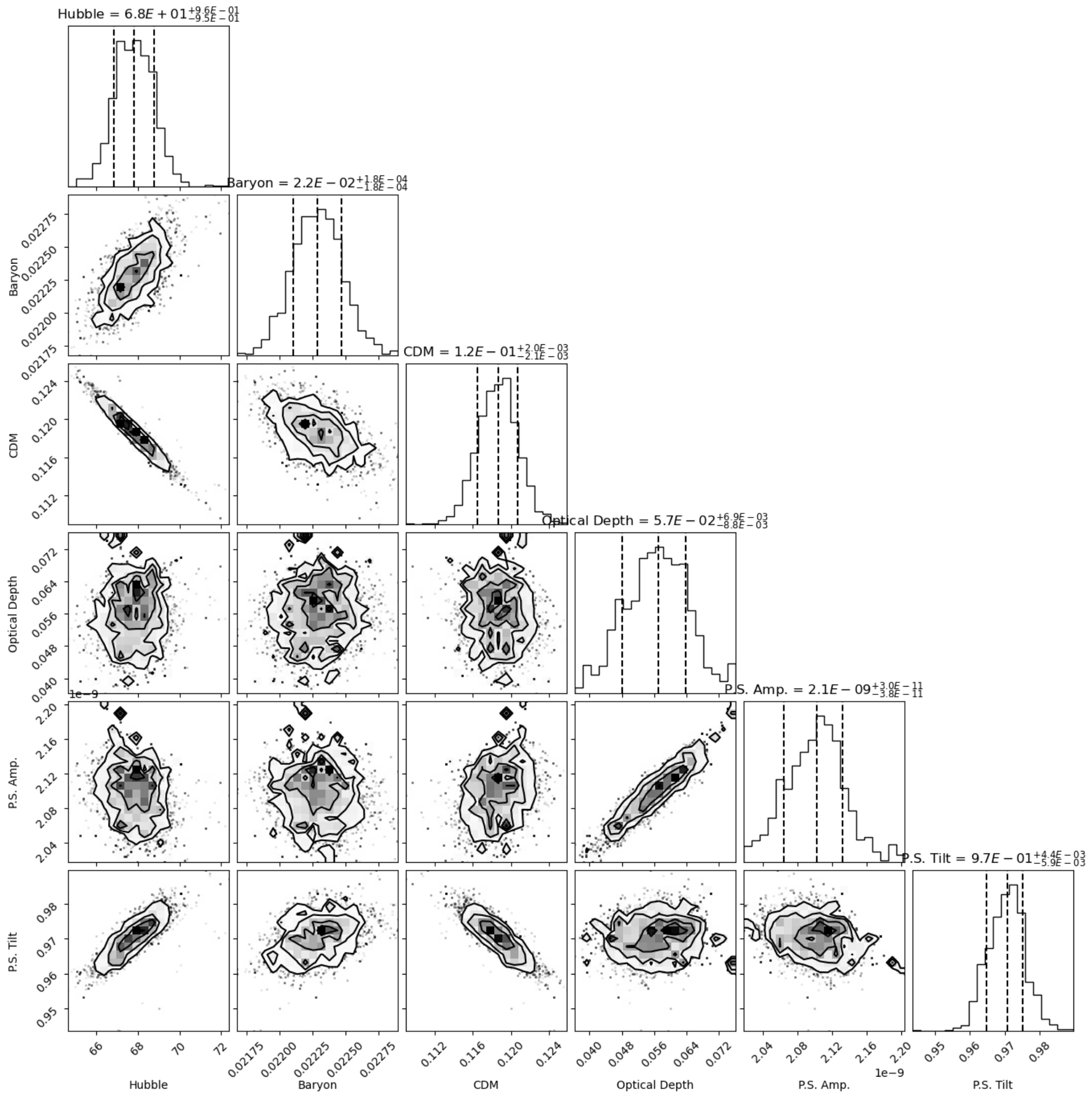
[275]:
```
# rejected steps
acc_rate = (len(chain) - np.sum(np.logical_and(np.diff(chain[:,0])==0,np.
 ↪diff(chain[:,1])==0)))/len(chain)

print('Acceptance rate is :', acc_rate)
```

```
Acceptance rate is : 0.2003
```

11 Corner Plots

[276]:
```
plt.ioff()
corner.corner(chain,
    labels=p_names, quantiles=[0.16, 0.5, 0.84],
    show_titles=True, title_fmt = '.1E')
```

[276]:



[277]:
```
taup = np.mean(chain, axis = 0)
taup_err = np.std(chain,axis = 0)
```

```
[278]: p_names = ['Hubble', "Baryon", "CDM", "Optical Depth", "P.S. Amp.", "P.S. Tilt"]
print("IMPORTANCE SAMPLING:")
for i, n in enumerate(p_names):
    print(f"\t{n} = ".ljust(20), f"{ip[i]:.2E}", "+/-", f"{ip_err[i]:.2E}")

p_names = ['Hubble', "Baryon", "CDM", "Optical Depth", "P.S. Amp.", "P.S. Tilt"]

print("\n\nNew MCMC:")
for i, n in enumerate(p_names):
    print(f"\t{n} = ".ljust(20), f"{taup[i]:.2E}", "+/-", f"{taup_err[i]:.2E}")
```

```
IMPORTANCE SAMPLING:
        Hubble =            6.88E+01 +/- 1.23E+00
        Baryon =            2.25E-02 +/- 2.26E-04
        CDM =               1.17E-01 +/- 2.79E-03
        Optical Depth =     5.49E-02 +/- 6.24E-03
        P.S. Amp. =         2.08E-09 +/- 2.99E-11
        P.S. Tilt =         9.75E-01 +/- 7.06E-03



New MCMC:
        Hubble =            6.78E+01 +/- 9.47E-01
        Baryon =            2.23E-02 +/- 1.89E-04
        CDM =               1.19E-01 +/- 2.13E-03
        Optical Depth =     5.65E-02 +/- 7.72E-03
        P.S. Amp. =         2.10E-09 +/- 3.41E-11
        P.S. Tilt =         9.70E-01 +/- 5.26E-03
```

12 Comparison

The two results agree with each other within their 3σ intervals. The errorbars are roughly the same.

```
[288]: np.vstack([np.load('obj/tau_chiChain.npy'),np.load('obj/tau_chain.npy').T]).T
```

```
[288]: (10000, 7)
```

```
[341]: np.savetxt('planck_chain_tauprior.txt',
        np.vstack([np.load('obj/tau_chiChain.npy'),np.load('obj/tau_chain.npy').
 ↪T]).T)

np.savetxt('planck_chain.txt',
        np.vstack([np.load('obj/chiChain.npy'),np.load('obj/chain.npy').T]).T)
```

```
[344]: np.loadtxt('planck_chain_tauprior.txt')
```

```
[344]: array([[3.27220367e+03, 6.90000000e+01, 2.20000000e-02, …,
                6.00000000e-02, 2.10000000e-09, 9.50000000e-01],
               [3.27220367e+03, 6.90000000e+01, 2.20000000e-02, …,
                6.00000000e-02, 2.10000000e-09, 9.50000000e-01],
               [3.27220367e+03, 6.90000000e+01, 2.20000000e-02, …,
                6.00000000e-02, 2.10000000e-09, 9.50000000e-01],
               …,
               [2.58095968e+03, 6.78226018e+01, 2.21042111e-02, …,
                4.30829877e-02, 2.03693630e-09, 9.71568803e-01],
               [2.58095968e+03, 6.78226018e+01, 2.21042111e-02, …,
                4.30829877e-02, 2.03693630e-09, 9.71568803e-01],
               [2.58095968e+03, 6.78226018e+01, 2.21042111e-02, …,
                4.30829877e-02, 2.03693630e-09, 9.71568803e-01]])
```

```
[ ]:
```