1.

In class, we assumed that

$$\frac{d}{dx}f(x) \approx \frac{\left[f(x) + f'(x)\,dx + \frac{1}{2}f''(x)\,dx^2\right](1+g_+\varepsilon) - f(x)\left[1+g_0\varepsilon\right]}{dx}$$

where we introduced g_+, g_0, ε to account for numerical precision and taking into consideration the first and second Taylor terms.

When we have two points:

$$\bar{f}(x\pm\delta) = \left[f(x) \pm f'(x)\,\delta + \cancel{\frac{1}{2}f''(x)\,\delta^2} \pm \frac{1}{6}f'''(x)\,\delta^3 + \cancel{\frac{1}{24}f''''(x)\,\delta^4}\right](1+g_\pm\varepsilon) + O(\delta^5)$$

$$\bar{f}(x\pm2\delta) = \left[f(x) \pm f'(x)\,2\delta + \cancel{\frac{1}{2}f''(x)\,4\delta^2} \pm \frac{1}{6}f'''(x)\,8\delta^3 + \cancel{\frac{1}{24}f''''(x)\,16\delta^4}\right](1+g_{\pm\pm}\varepsilon) + O(\delta^5)$$

from $\pm\delta$:

Divide by 2δ.

$$f'\cancel{2\delta} + \frac{1}{3}f'''\,\delta^{\cancel{3}2} + \underbrace{\cancel{f'(2\delta)g\varepsilon}}_{\text{Assumed to be }\sim10^{-16}\text{ in class}} + \underbrace{\frac{1}{3}f'''\,\delta^{3}g_+\varepsilon}_{\text{the new term}} + f\,g_+\varepsilon/2\delta + \frac{1}{5!}f^{(5)}\,\delta^5$$

from $\pm2\delta$:

Divide by 4δ

$$f'\cancel{4\delta} + \frac{8}{3}f'''\,\delta^{2} + \cancel{f'(4\delta)g\varepsilon} + \underbrace{\frac{8}{3}f'''\,\delta^{3}g_{++}\varepsilon}_{\text{new term}} + f\,g_{++}\varepsilon/4\delta + \frac{1}{5!}\cdot\cancel{64}\,16\,f^{(5)}\,\delta^{\cancel{5}4}.$$

We want to cancel the next $\sim O(\delta^2)$ term (not coming from machine precision). We can weight the $f'|_{\pm\delta}$ by 4 and $f'|_{\pm 2\delta}$ by 1 and take the difference:

$$f' = \left[4 f'|_{\pm\delta} - f'|_{\pm 2\delta}\right] \cdot \frac{1}{3}$$

which is the same as the central difference:

$$f' \simeq \frac{f_{-2\delta} + 8 f_{+\delta} - 8 f_{-\delta} - f_{+2\delta}}{12\delta} .$$

b) The leading term in error results in:

$$4 f g \varepsilon_1 / 2\delta - f g \varepsilon_2 / 4\delta + \frac{4}{5!} f^{(5)} \delta^5 / 2\delta - \frac{16}{5!} f^{(5)} \delta^5 / 4\delta]$$

$$\sim \frac{f g \varepsilon}{\delta} + \frac{1}{100} f^{(5)} \delta^4 \quad \text{(error term)}.$$

$$\frac{d\varepsilon_f}{d\delta} \sim \frac{f g \varepsilon}{\delta^2} - f^{(5)} \delta^3 \equiv 0$$

$$\Longrightarrow \delta \sim \left(\frac{f \varepsilon}{f^{(5)}}\right)^{1/5}$$