4.

a) This is just geometric series:

$$\sum_{n=0}^{r-1} x^n = \frac{1-x^r}{1-r} \Longrightarrow \sum_{x=0}^{N-1}\left(e^{-i\frac{2\pi k}{N}}\right)^x = \frac{1-\exp(-2\pi i k \cancel{\tfrac{N}{N}})}{1-\exp(-2\pi i k/N)}.$$

b)

$$\lim_{k\to 0} \frac{1-\exp(-2\pi i k)}{1-\exp(-2\pi i k/N)} \overset{\text{L'Hopital}}{=} \lim_{k\to 0}\frac{\cancel{2\pi i}\exp(-2\pi i k)}{\frac{1}{N}\cancel{2\pi i}\exp(-2\pi i k/N)} \to N.$$

If k is an integer $\exp(-2\pi i k) = 1$. But not an integer multiple of N. Hence, the numerator $\to 0$ but denominator doesn't.

$$\frac{\cancelto{0}{1-\exp(-2\pi i k)}}{1-\exp(-2\pi i k/N)} = 0.$$

c) Let $f(x) = \sin\left(\frac{2\pi k' x}{N}\right) = \frac{e^{i\frac{2\pi k' x}{N}} - e^{-i\frac{2\pi k' x}{N}}}{2i}.$

$$\mathcal{F}[f(x)][k] = \sum_{x=0}^{N-1} \exp(-2\pi i k x/N)\, f(x)$$

$$= \frac{1}{2i}\left(\sum_{x=0}^{N-1}\exp\left(-\frac{2\pi i(k-k')}{N}x\right) - \sum_{x=0}^{N-1}\exp\left(-\frac{2\pi i(k+k')}{N}x\right)\right)$$

$$= \frac{N}{2i}\left[\delta(k-k') - \delta(k+k')\right]$$