Monte Carlo Applications within the Cox-Ingersoll-Ross Model

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1 Introduction

When analyzing the dynamics of a European call option, many models can be applied to help calculate the option's price; especially, in situations where interest rate volatility is not static. In this analysis, the Cox-Ingersoll-Ross model is used to assess the interest rate of a European call option. Over both a 12-month and 52-week period, this analysis will show how the Cox-Ingersoll-Ross model, as well as the asset pricing model derived from Ito's Lemma, and the pay-off framework for a European option, relate between the time intervals in question. Later, this framework will be compared to a calculation of the option's price using geometric Brownian motion.

The Assumptions of model construction are as follows:

Initial Asset Price: $S(0) = \$50$

Asset Price: $S(t)$

Strike Price: $K = \$50$

Initial interest rate: $r(0) = 7\%$

Interest rate: $r(t)$

Volatility: $\sigma = 13\%$

Maturity time: $T = 1$ year

2 Assessment of 12-Month Option

To begin, an initial sample size of 10,000 was used, as well as the beginning interest rate (r) of 0.07, and the volatility value (σ) of 0.13.

Creating an algorithm of the implicit model, we have:

$$r(t_j) = r(t_{j-1}) + 0.23(0.081 - r(t_{j-1}))\Delta + 0.09\sqrt{r(t_{j-1})\Delta}Z_j$$

Here, interest rate sample paths were created over a 12-month time step basis (see Figure 1).

Using the results of the interest rate sample paths, these indexed results were utilized in the implicit pricing model for the underlying asset of the option (see Figure 2):

$$S(t_j) = S(t_{j-1})e^{\Sigma(r(t_{j-1}) - \frac{\sigma^2}{2})\Delta + \sigma\sqrt{\Delta}X_j}$$

Applying the framework above, the payoff was calculated rendering the value of the option. An additional element of reducing the error tolerance was requested within \$0.05. Utilizing an amplifying constant of 1.1, the required sample size was determined to be 84,304. With the required sample size calculated, the model steps above were repeated, and the value of the option was determined to be \$4.601. In a similar fashion, when asked the value of the option after 1-month, the value was determined to be \$0.899.

3 Assessment of 52-Week Option

Following a near identical cadence to the European call option with a 12-month time step basis, the process repeated using a 52-week time step basis (see Figure 3 and Figure 4). Starting with an initial sample size of 1,000, and reducing the error tolerance within \$0.05, the required sample size was determined to be 26,911. The value of the option was determined to be \$4.582. Lastly, when asked the value of the option after 1-week, the value was determined to be \$0.391.

4 Relationship Between Both Options

When assessing the prices of the options between 12-month and 52-week time steps, the simulation showed that when exercised at maturity, the 12-month option was slightly higher than the 52-week option at \$4.601 compared to \$4.582, respectively. When comparing the value of the options when exercised after their first time step: 1-month for the 12-month option and 1-week for the 52-week option, the option exercised after 1-month was higher than the option exercised after 1-week at \$0.899 and \$0.391, respectively.

Interpreting the comparisons above, it is made clear that options exercised over the same duration vary little in their pricing. While an option may have more time steps, with a duration of equal maturity as we saw with 12-months vs. 52-weeks, the time steps impact the Δ value in both the rate model and the asset model. Since the asset and rate models rely on the previous time step in their own iteration of value, we know Δ decreases when the number of time steps increases.

This in turn scales down the volatility of movement when applied as a product in the rate model:

$$0.23(0.081 - r(t_{j-1}))\Delta$$

And as a product within the summation of the asset model:

$$e^{\Sigma(r(t_{j-1}) - \frac{\sigma^2}{2})\Delta + \sigma\sqrt{\Delta}X_j}$$

Even in the payoff model, Δ will always cancel out the factor of the summation when assessed at maturity, leaving only interest rate:

When $\Delta = \frac{1}{d}$

$$\Delta \sum_1^d r(t_j) = \frac{1}{d}(d)r(t_j) = r(t_j)$$

On the other hand, comparing the situation when the options are exercised after their first time step of 1-month and 1-week, this would change the calculation where the Δ with the smaller time step would take on the larger value of options price.

5 Comparison With Geometric Brownian Motion

When calculating the options using geometric Brownian motion, the value of the 12-month option was \$4.48. Similarly, the value of the 12-month option after 1-month was \$0.894. When assessing the 52-week option, the value was \$4.878. In addition, the value of the 52-week option after 1-week was \$0.395.

Examining the results above, while both the 12-month option after 1-month and 52-week option after 1-week were close to the values in the original model, the 12-month option was lower than its counterpart: \$4.48 vs. \$4.601, while the 52-month option was higher than its counterpart: \$4.582 vs. \$4.878. Mainly due to the original model's error tolerance being within \$0.05 of the true expected value of the option, not having an error tolerance on the GBM model does not produce an optimal sample size to mitigate the error from the true mean. With error minimization not in place, the sample size is lower than in could be. This causes issues since by the law of large numbers, the result makes the variance of the sample mean larger, and the sample mean further from the true mean. Other factors acting on these differences also can be attributed to the fixed interest rate, and the result of only one normal random variable acting on the model.

6 Time Optimization in Code

With sixteen sections of code, as well as an appendix section computing plot outputs, the program runs in about 2.7 seconds. In assessing ways to optimize the code in place, sections computing the algorithms for the Cox-Ross-Ingersoll models, as well as the asset algorithms derived from Ito's Lemma, both use iterative methods. This can cause extra cost on the execution plan of the R output since it uses previously computed values stored in a matrix to compute the current value in question. Instead, this could also be written using an explicit formula to avoid the creation of a loop command to compute values in question.

7 Closing Comments

Overall, going over the examples stated above, it is reasonable to achieve the outputs computed in this exercise. If anything, it may be feasible to apply additional methods using variance reduction techniques to yield a smaller relative error. For example, applying a control variate method by using the same normal random numbers in both the initial pricing model and the GBM model. Here, we could use the initial pricing model as the control variate selected to improve the accuracy of the GBM model in output.

8 Appendix

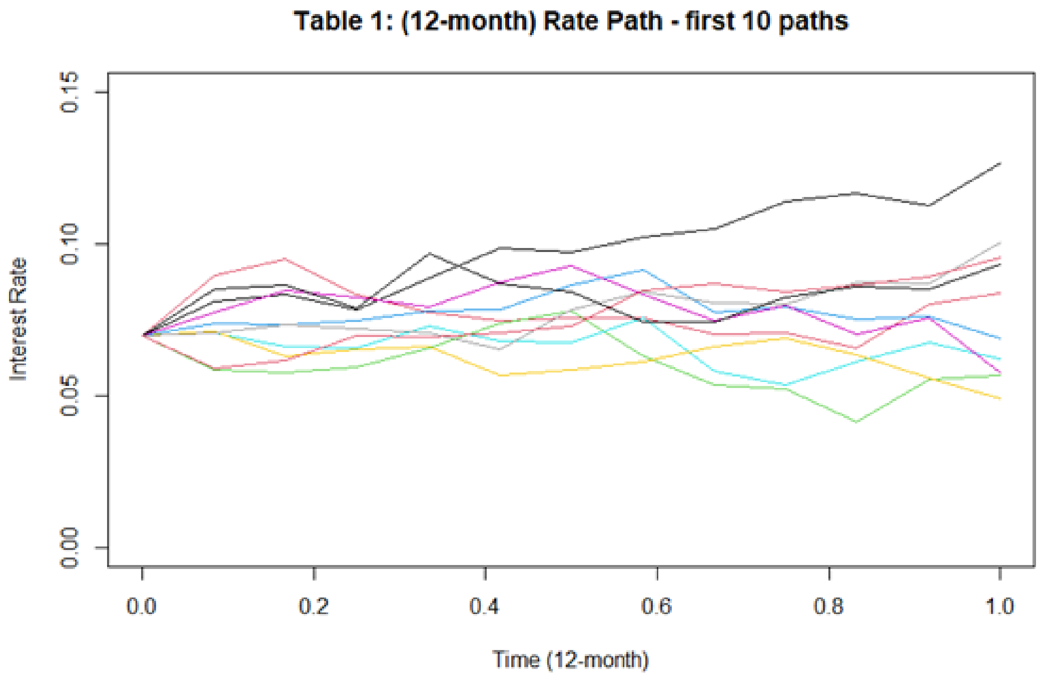

Figure 1: (12-Month) Rate Path - First 10 Paths

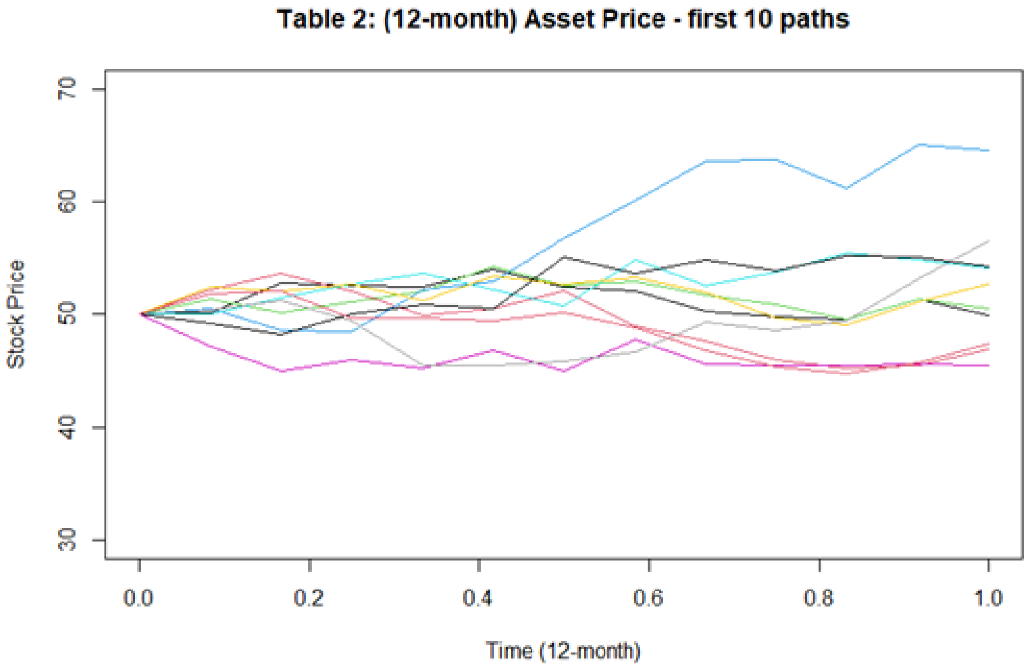

Figure 2: (12-Month) Asset Price - First 10 Paths

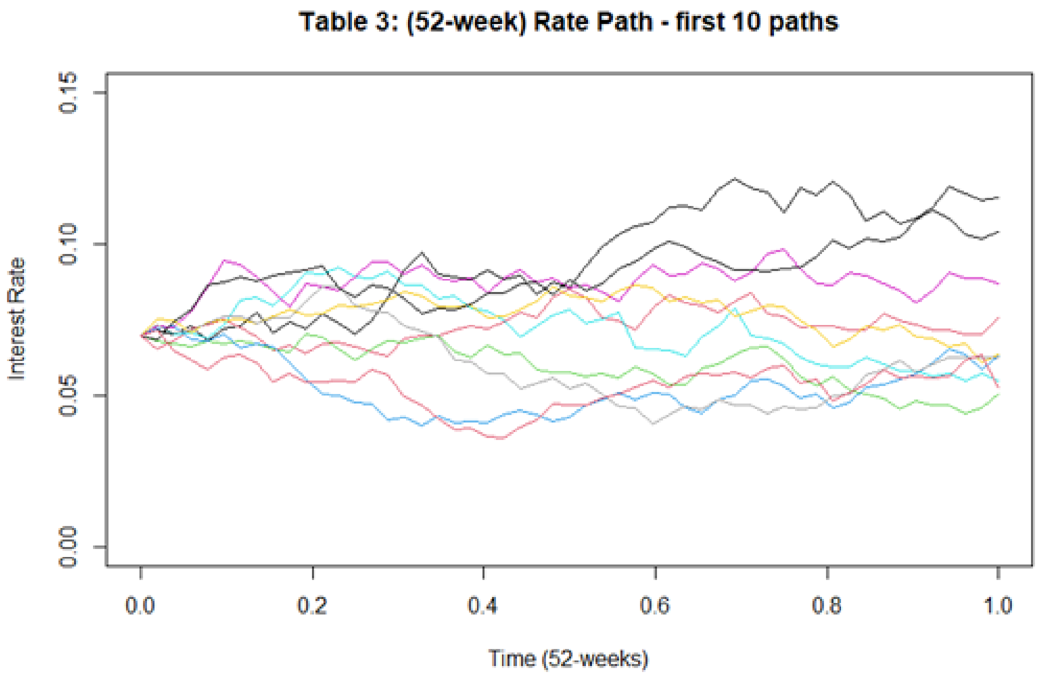

Figure 3: (52-Week) Rate Path - First 10 Paths

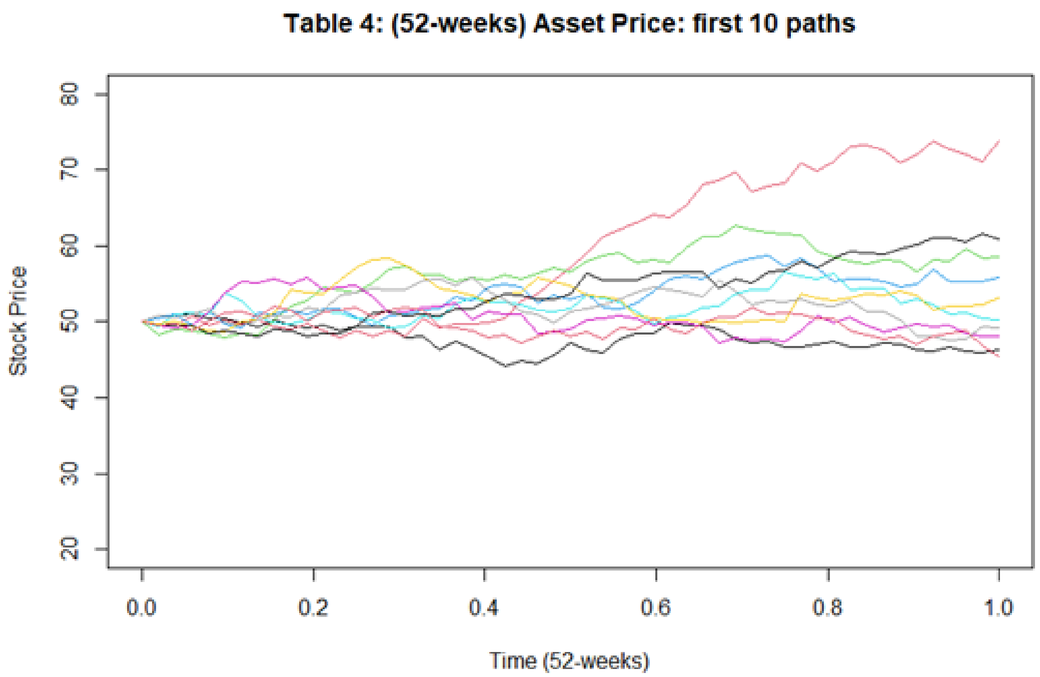

Figure 4: (52-Week) Asset Price - First 10 Paths