Nonlinear Least Square Estimation

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- Problem description:

Given
$$\begin{cases} y = h(x) + v, \quad v \in N(0, R), \quad y \in \mathbb{R}^{m\times 1} \\ \tilde{x} = 0, \quad x \sim N(\bar{x}_0, P_0) \end{cases}$$

Goal: $\hat{x} = T(y, R, \bar{x}_0, P_0)$

- Recall for linear systems:

$$\begin{cases} y = Hx + v, \quad v \sim N(0,R) \\ \tilde{x} = 0, \quad x \sim N(x_0, P_0) \end{cases} \Rightarrow \underbrace{\hat{x} = \bar{x}_0 + P_0 H^T [H P_0 H^T + R]^{-1} (y - H x_0)}_{\text{Closed Solution. Accurate.}}$$

- What about nonlinear system? The key is linearization.

$$J = (y - h(x))^T R^{-1} (y - h(x)) + (x - \bar{x}_0)^T P^{-1} (x - x_0)$$

$$-\frac{\partial J}{\partial x} = \left(\frac{\partial h(x)}{\partial x}\right)^T R^{-1} (y - h(x)) + P^{-1} (x - x_0) = 0 \qquad (1)$$

$$h(x) = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{bmatrix} \in \mathbb{R}^m, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \quad \underbrace{\frac{\partial h}{\partial x}}_{\text{Jacobian}} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}_{m\times n} \qquad (2)$$

- Define $H = \frac{\partial h}{\partial x}$
- Expand $h(x)$ at $\bar{x}_0$. → Taylor expansion.

$$h(x) = h(x_0) + \frac{\partial h}{\partial x}\Big|_{\bar{x}_0} (x - \bar{x}_0) + \cancel{o(x)} \qquad (3)$$

neglect this higher order term
↳ approximation. → introduce errors.

Substitute (2) and (3) into (1), we obtain

$$\hat{x} = \bar{x}_0 + P_0 H^T [H P H^T + R]^{-1} (y - h(\bar{x}_0)) = \bar{x}_0 + P_0 H_0^T [H P_0 H^T + R^{-1}] (y - h(\bar{x}_0))$$

↳ Approximated, inaccurate.

$$P = E[\tilde{x}\tilde{x}^T] = [H^T R^{-1} H + P_0^{-1}]^{-1}$$

- Solution by iteration: (get closer to the true solution through iteration every time)

$$\begin{cases} \hat{x}_{i+1} = \hat{x}_i + [H_{\hat{x}_i}^T R^{-1} H_{\hat{x}_i} + P_i^{-1}]^{-1} H_{\hat{x}_i}^T R^{-1} (y - h(\hat{x}_i)) \\ P_{i+1} = [H_{\hat{x}_i}^T R^{-1} H_{\hat{x}_i} + P_i^{-1}]^{-1} \end{cases}$$

↳ Approximation, no guarantee on covergence to the solution
A good initial always does the magic.

- Numerical example:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad y = \begin{bmatrix} x_1 x_2 \\ x_1^2 - x_2 \end{bmatrix} + v, \quad v \sim N(0, R), \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x \sim N(\bar{x}_0, P_0), \quad P_0 = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \quad x_0 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad \bar{x}_0 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

- Assume that we take 5 measurements:

$$\tilde{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_5 \end{bmatrix}$$

- Solution:

$$h = \begin{bmatrix} x_1 x_2 \\ x_1^2 - x_2 \end{bmatrix} \quad H = \begin{bmatrix} x_2 & x_1 \\ 2x_1 & -1 \end{bmatrix} \quad \tilde{H} = \begin{bmatrix} H \\ H \\ H \\ H \\ H \end{bmatrix}$$

- Matlab program for this example.