# MATH6004 Numerical Analysis Course Project

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Q. 1 (Numerical Integration) Velocity data for air are collected at different radii from the centerline of a circular 16-cm-diameter pipe as tabulated below: Use numerical integration to determine the mass

| $r$, cm | 0 | 1.00 | 2.00 | 3.00 | 4.00 | 5.00 | 6.00 | 7.00 | 8.00 |
|---|---|---|---|---|---|---|---|---|---|
| $V$, m/s | 10 | 9.80 | 9.60 | 9.30 | 9.06 | 8.68 | 8.18 | 7.41 | 0 |

Table 1: Sampled velocity data.

flow rate, which can be computed as

$$\int_0^R \rho v 2\pi r dr$$

where density $\rho = 1.2kg/m^3$. Express your results in kg/s.

(a) Use Composite Trapezoidal rule;

(b) Use Composite Simpson's rule.

(c) Select a method of your choice that you think is more accurate than the two methods above.

Q. 2 (ODE Initial value problem) As shown in Fig. 1, we have a cubic box containing turbulent flow. The kinetic energy per unit volume is $k$ (dimension: $m^2/s^2$), and the dissipation rate of kinetic energy is $\epsilon$ (dimension $m^2/s^3$). The governing equations are

$$\frac{\partial k}{\partial t} = -\epsilon \tag{1}$$

$$\frac{\partial \epsilon}{\partial t} = -C\frac{\epsilon^2}{k} \tag{2}$$

where $C = 1.83$. At $t_0 = 1\ s$, $k = 1.0\ m^2/s^2$ and $\epsilon = 0.2176\ m^2/s^3$. Use the following numerical method to solve the ODE equation set and predict the kinetic energy $k$ at $t = 5.0\ s$. Comment on the convergence and accuracy of those methods.

(a) Use the Euler method;

(b) Use the modified Euler method;

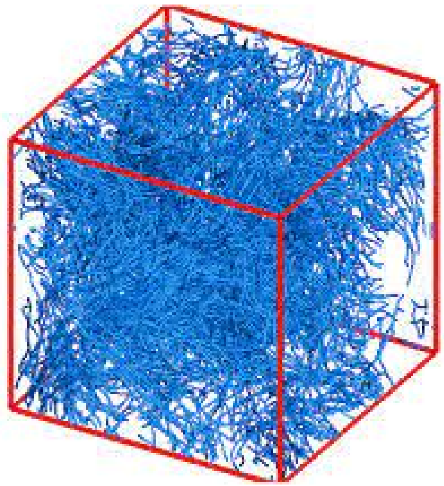

Figure 1: Homogeneous isotropic turbulence in a box

(c) Use the 4-th order Runge-Kutta method.

Q. 3 (Non-linear equations): In a turbulent boundary layer, the time-averaged velocity profile and the wall distance can be written in the following non-dimensional form

$$y_+ = U_+ + e^{-\kappa B}\left[e^{\kappa U_+} - 1 - \kappa U_+ - \frac{1}{2!}(\kappa U_+)^2 - \frac{1}{3!}(\kappa U_+)^3 - \frac{1}{4!}(\kappa U_+)^4\right] \tag{3}$$

where

$$y_+ = \frac{u_\tau y}{\nu}, \qquad U_+ = \frac{U}{u_\tau}, \tag{4}$$
$$\kappa = 0.41, \qquad B = 5.1 \tag{5}$$

In a wind tunnel experiment, we know the air viscosity $\nu = 1.5 \times 10^{-5}$ $m^2/s$ and air density $\rho = 1.25$ $kg/m^3$. At a certain streamwise location, someone measured that at a wall distance of $y = 0.01$ m, the velocity is 21 $m/s$. Given that the relation between $u_\tau$ and the wall shear stress $\tau_{wall}$ can be expressed as

$$u_\tau = \sqrt{\frac{\tau_{wall}}{\rho}} \tag{6}$$

Please try to solve the wall shear stress ($\tau_{wall}$) corresponding to the measurement data within a tolerance of $\epsilon = 10^{-3}$.

(a) Use bisection method;

(b) Use Newton's method;

(c) Try to find a fixed point iteration formula that converges.

(Hint: the first step is to write out a non-linear equation of $f(U_+) = 0$ and solve it using the methods we learned. And then use $U_+$ to compute $u_\tau$ and $\tau_{wall}$.)

Q. 4 (Eigenvalue): For a dynamic system, eigenvalues can be used to determine whether a fixed point (also known as an equilibrium point) is stable or unstable. When all eigenvalues are real, positive, and distinct, the system is unstable; When all eigenvalues are real, negative, and distinct, the system is stable; If the set of eigenvalues for the system has both positive and negative eigenvalues, the fixed point is an unstable saddle point. Here is a matrix that corresponds with a system at a fixed point. Please use QR decomposition to determine how many eigenvalues are positive and how many are negative.

$$A = \begin{pmatrix} 52 & 30 & 49 & 28 \\ 30 & 50 & 8 & 44 \\ 49 & 8 & 46 & 16 \\ 28 & 44 & 16 & 22 \end{pmatrix}$$

Q. 5 (Interpolation and Approximation) For the function of Equation 3 in the previous question, please try the following:

(a) Draw the $y^+$ vs $U^+$ where $y^+$ is the x-axis and $U^+$ is the y-axis, for the range of $[1, 590]$. Plotting the $y^+$ range using the log10 scale.

(b) Choose a set of 5 distinct data points from the function at $y^+ = 1, 25, 100, 300, 590$; construct piecewise linear interpolation, and draw on a figure.

(c) Divide the range $[1, 590]$ into six subintervals and construct a corresponding linear approximation to the function within each interval. Note that each approximation polynomial can be independent of its neighbors. You may adjust the dividing locations to make the approximation more appropriate.

(d) For each subinterval you choose, use the Monomial polynomials up to order 3 to construct least-squares approximation and plot the answers in a figure.

(e) Repeat the previous step by using the Legendre polynomials up to order 3.

Q. 6 (Interpolation and Approximation): We have seen how Runge's phenomenon will affect the interpolation, and now it's time to use what we have learned to fix it (to some extent). Given the Runge function

$$f(x) = \frac{1}{1 + 25x^2}, \quad x \in [-1, 1], \tag{7}$$

(a) Use $N$ equally spaced points in $[-1, 1]$ to interpolate the Runge function, where $N = 15, \ldots, 20$. Take $N = 3$ as an example, we have $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$. Keep your results and we'll ask you to plot them later.

(b) Use roots of $P_{N-1}$, *i.e.*, Legendre polynomial of degree $(N - 1)$, to interpolate the Runge function, where $N = 15, \ldots, 20$. Take $N = 3$ as an example, we have $x_0 = -\sqrt{3/5}$, $x_1 = 0$, and $x_2 = \sqrt{3/5}$.

(c) We'll show that the Runge phenomenon can be reduced by rearranging interpolation points. For each $N$, plot the Runge function, and interpolated polynomials from Q. 6a as well as Q. 6b. As a kind reminder, we recommend you use 201 equally spaced points to plot those functions, no matter how many points you have used to interpolate the function, so that you won't miss something important.

(d) Use least square approximation to approximate the Runge function, with polynomials whose degree is no more than $(N-1)$, where $N = 15, \ldots, 20$. Take $N = 3$ as an example, we want to find $a$, $b$, and $c$ such that

$$\int_{-1}^{1} \left(ax^2 + bx + c - f(x)\right)^2 \mathrm{d}x \tag{8}$$

reaches its minimum. You should calculate the integral analytically.

(e) We'll show that the Runge phenomenon can be reduced by using least square approximations as well. For each $N$, plot the Runge function, the interpolated polynomial from Q. 6a, and the approximated polynomial from Q. 6d.

(f) Now we use a reduced Gauss-Legendre quadrature and do Q. 6d again. For each $N$, do not use the exact integral, but the $N$-point Gauss-Legendre quadrature to approximate the integral. Take $N = 3$ as an example, use

$$\int_{-1}^{1} f x^n \,\mathrm{d}x \approx \frac{5}{9} f\left(-\sqrt{3/5}\right)\left(-\sqrt{\frac{3}{5}}\right)^n + \frac{8}{9} f(0)\cdot 0^n + \frac{5}{9} f\left(\sqrt{3/5}\right)\left(\sqrt{\frac{3}{5}}\right)^n, \quad n = 0, 1, 2 \tag{9}$$

to calculate the integral. For $\int_{-1}^{1} x^{n_1} x^{n_2} \,\mathrm{d}x$, the quadrature is exact so nothing changes.

(g) Finally, we'd like to show you that using Legendre polynomials to interpolate a function is the same as using polynomials and reduced quadrature to approximate a function. For each $N$, plot the Runge function, the interpolated polynomial from Q. 6b, and the approximated polynomial from Q. 6f.

We have plotted them for $N = 20$ for your reference, shown in Figure 2, 3, and 4.

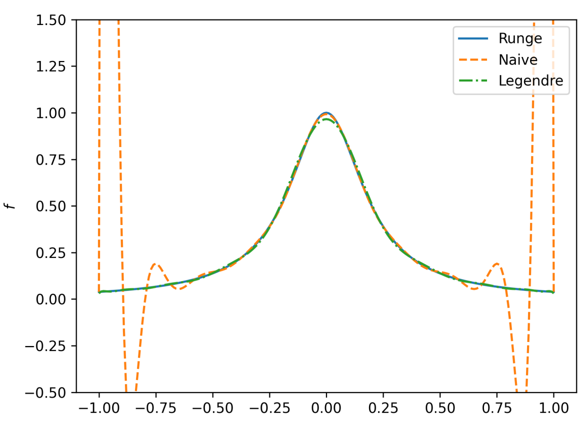


Figure 2: Sample plot for Q. 6c, $N = 20$.

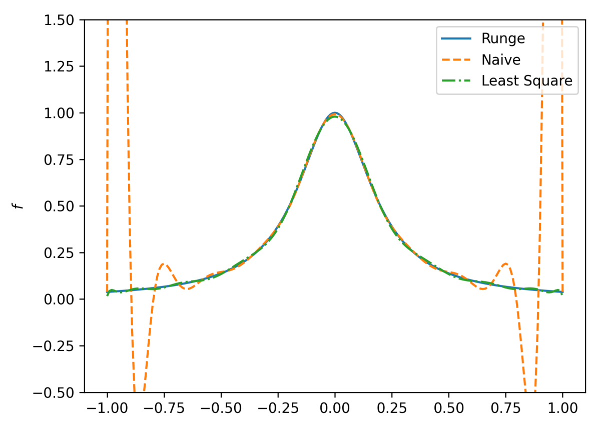


Figure 3: Sample plot for Q. 6e, $N = 20$.

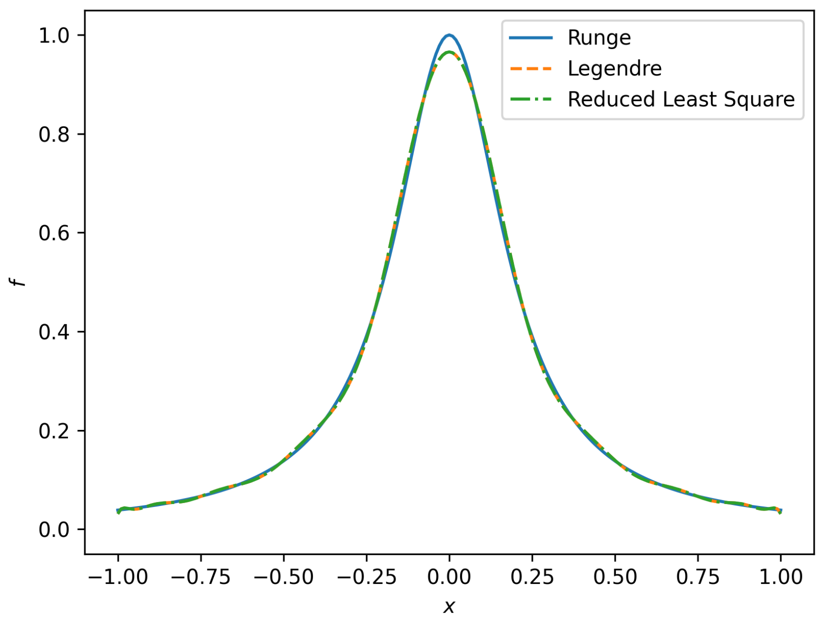


Figure 4: Sample plot for Q. 6g, $N = 20$.