ASSIGNMENT 2

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*Reflection questions encourage you to think about how mathematics is done. This is an important ingredient of success. Reflection questions contribute to your **engagement grade**.*

1. Consider the differences between the types of questions on the previous written assignment, Assignment 1, and the types of questions on your high school math assignments. In one or two paragraphs, describe one major difference, and explain how you might approach assignments in this course differently. Be specific and provide details.

   Answers
   High school mathematics often focuses on individual rules and strategies for solving homogeneous sets of equations. Within this framework of memorization and application of set algorithms, math can have no concept of interdisciplinary application and the value of reasoning beyond what is explicitly taught is similarly undermined. Therefore, math assignments in high school often involve repeating the exact process taught in class to solve a set of problems that are identical in most respects. On the contrary, the previous written assignment was focused on our application of calculus concepts to physics. Specifically, rather than simply getting the correct answer to an equation, we were asked to focus on understanding the implications of mathematical concepts on real-life physical phenomena.

A *capacitor* is a basic circuit element that can store charge.

A simple form of capacitor is a pair of conducting plates separated by a dielectric (non-conducting) material. There are other geometries possible such are rolling thin conducting plates and a malleable separating dielectric material into a cylindrical shape, all the while preserving the separation of the two plates.

By attaching a battery, say, to the capacitor, we can store charge from the battery in the capacitor for use at a later time. The capacitor can then become a source of current in the circuit, at least until the capacitor has discharged.

2. (★★★☆) Suppose we have a capacitor of capacitance C in a circuit with both a resistor of resistance R and a battery with emf $\mathcal{E}$. All three of these values are constant. This circuit is shown in Figure 1. Note there is a current I, which is equal to the rate at which the charge Q changes in time t; that is, $I = dQ/dt$. We wish to study $Q(t)$, the charge on the capacitor as a function of time.

   We can apply Ohm's Law to this circuit to find that

   $$\mathcal{E} - IR - \frac{Q}{C} = \mathcal{E} - \frac{dQ}{dt}R - \frac{Q}{C} = 0.$$

   Note that this is a differential equation for the function $Q(t)$, which we rearrange as

   $$\frac{dQ}{dt} = \frac{\mathcal{E}}{R} - \frac{Q}{RC}. \tag{1}$$

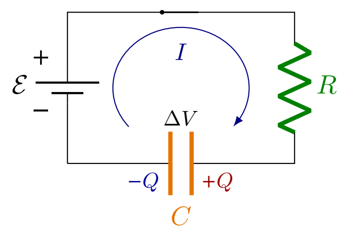


Figure 1: An RC-circuit with an EMF (battery)

(a) Let us assume the capacitor has no charge on it at time $t = 0$. That is, $Q(0) = 0$. Confirm that

$$Q(t) = C\mathcal{E}\left(1 - e^{-t/RC}\right)$$

is a solution of the differential equation (3) with the initial condition $Q(0) = 0$.

(b) Sketch a graph of $Q(t)$.

(c) What is $\lim_{t\to\infty} Q(t)$? What can you conclude about the charge on the capacitor from this result?

(d) Find an expression for the current $I(t)$.

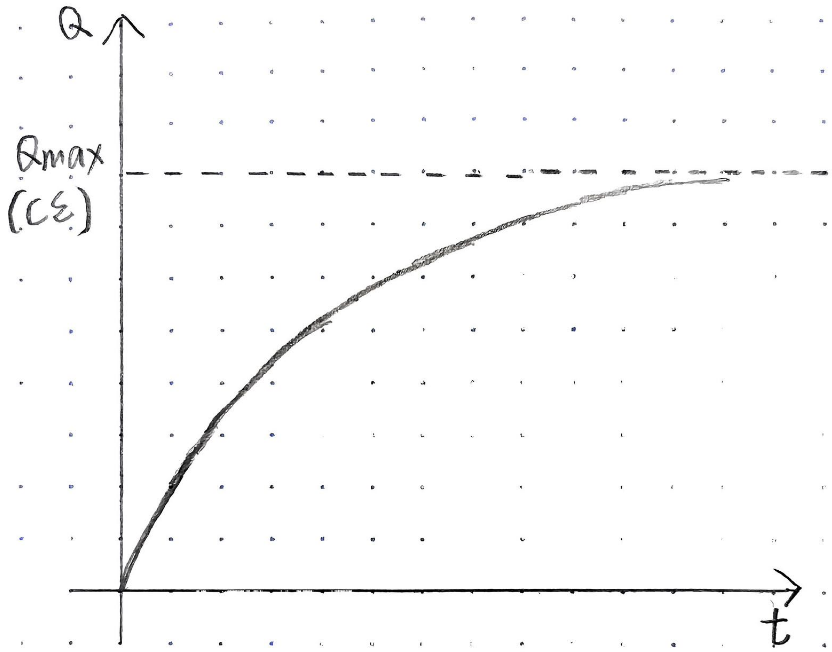
(e) What can you conclude about the behaviour of this circuit after a long time passes?

Answers

(a) If we plug in the equation $Q(t) = C\mathcal{E}\left(1 - e^{-t/RC}\right)$ into the equation $\frac{dQ}{dt} = \frac{\mathcal{E}}{R} - \frac{Q}{RC}$ and the equivalence remains true, then we will have confirmed the solution for the differential equation.

$$\begin{aligned}
\frac{dQ}{dt} &= \frac{\mathcal{E}}{R} - \frac{Q}{RC} \\
\frac{d}{dt}[C\mathcal{E}(1 - e^{-t/RC})] &= \frac{\mathcal{E}}{R} - \frac{C\mathcal{E}(1 - e^{-t/RC})}{RC} \\
\frac{\mathcal{E}e^{-t/RC}}{R} &= \frac{\mathcal{E}}{R} - \frac{\mathcal{E} - \mathcal{E}e^{-t/RC}}{R} \\
\frac{\mathcal{E}e^{-t/RC}}{R} &= \frac{\mathcal{E}}{R} - \frac{\mathcal{E}}{R} + \frac{\mathcal{E}e^{-t/RC}}{R} \\
\frac{\mathcal{E}e^{-t/RC}}{R} &= \frac{\mathcal{E}e^{-t/RC}}{R} \\
L.S. &= R.S.
\end{aligned}$$

(b) The graph of Q as a function of t is as follows:



(c) When we take $\lim_{t\to\infty} C\mathcal{E}\left(1-e^{-t/RC}\right)$, we can immediately substitute t for ∞ which gives us $C\mathcal{E}\left(1-e^{-\infty/RC}\right)$. Since the constants R and C are finite divisors to ∞, they can be omitted as they do not affect the calculation. After removing RC from the term $-\infty/RC$, we are left with $C\mathcal{E}\left(1-e^{-\infty}\right)$. Now, because $e^{-\infty}=0$, we are left with $C\mathcal{E}$.In short, $\lim_{t\to\infty} Q(t) = C\mathcal{E}$.

As time t passes, we know that $Q(t)$ will increase because $Q(t)$ is the constant $C\mathcal{E}$ multiplied by the coefficient $1-e^{-t/RC}$. Since $e^{x}\geq 0$ for all real values of x, the highest that this coefficient can be is where $e^{-t/RC}=0$ and the coefficient is therefore 1. This is what the limit of $Q(t)$ as $t\to-\infty$ represents and is also the highest possible value of $Q(t)$ for any value of t. Furthermore, since our limit is given in terms of $\mathcal{E}$ and C wherein $\mathcal{E}$ represents the voltage and C represents the capacitance, we can conclude that their product $\lim_{t\to\infty} Q(t) = C\mathcal{E}$ represents the most charge that the capacitor in question can possibly hold in this circuit.

(d) Since we know that $I=\frac{dQ}{dt}$, current I as a function of time t is simply the derivative of $Q(t)$ with respect to time. In fact, we have already inadvertently gotten $I(t)$ in our equivalence in 2.a.

$$\begin{aligned} I &= \frac{dQ}{dt} \\ I(t) &= \frac{d}{dt}Q(t) \\ &= \frac{d}{dt}[C\mathcal{E}(1-e^{-t/RC})] \\ &= \frac{\mathcal{E}e^{-t/RC}}{R} \end{aligned}$$

(e) As more and more time passes, the circuit will continue to build charge in the capacitor at a logarithmic rate that gradually approaches $C\mathcal{E}$. In short, the charge Q of the battery cell is being transferred to the capacitor at an ever-slowing rate over time.

3. (★★★☆) Suppose now we remove the battery from the circuit, as shown in Figure 2.

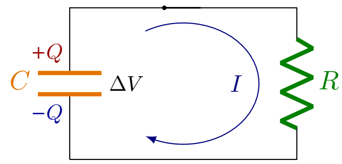


Figure 2: An RC circuit with no EMF (battery)

(a) Write down the differential equation that comes form Ohm's Law for the circuit shown in Figure 2.

(b) Suppose that the capacitor is the value of Q you found in Question 1(c); call this value $Q_{\max}$. Write down the solution $Q(t)$ to the differential equation you found in part (a) satisfying the initial condition $Q(0) = Q_{\max}$.

(c) Sketch a graph of $Q(t)$.

(d) Find an expression for the current $I(t)$.

(e) What can you conclude about the behaviour of this circuit after a long time passes?

Answers

(a) Since the battery has been removed from the circuit, we must remove $\mathcal{E}$ which represents the electromotive force provided by the battery to the circuit from the equation for Ohm's law which we were provided in question 2. This leaves us with the equation $\frac{dQ}{dt} = -\frac{Q}{RC}$.

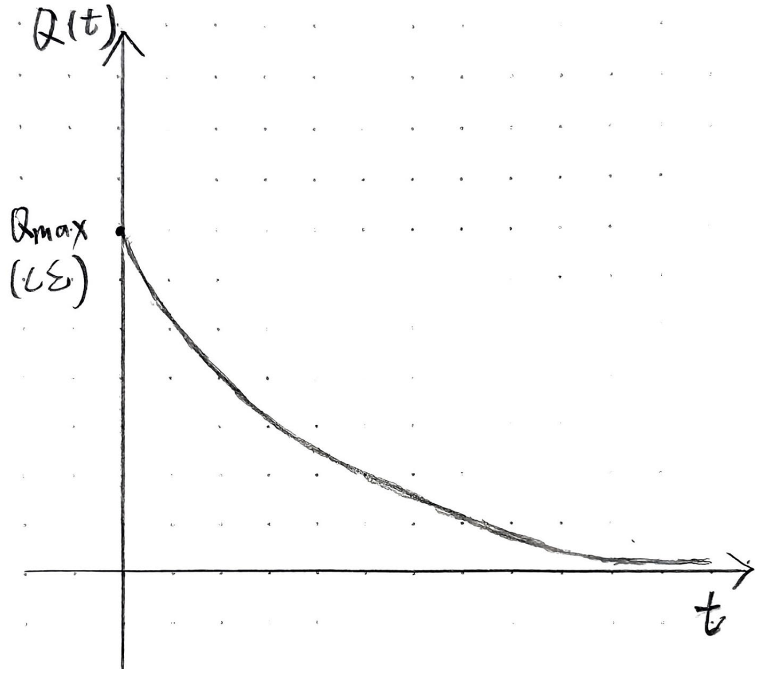
(b) The rate of discharge of a capacitor is the inverse of its rate of charge in the sense that, supposing functions $Q_{charging}(t)$ and $Q_{discharging}(t)$ representing the charge and discharge over time of a capacitor, their relationship is that $Q_{discharging}(t) = Q_{max} - Q_{charging}(t)$ where Q_{int} represents the initial charge of the capacitor. Since the capacitor the figure 2 circuit is discharging, the equation should then be:

$$\begin{aligned} Q(t) &= C\mathcal{E} - C\mathcal{E}(1 - e^{-t/RC}) \\ &= C\mathcal{E} - C\mathcal{E} + C\mathcal{E}e^{-t/RC} \\ &= C\mathcal{E}e^{-t/RC} \end{aligned}$$

To confirm that this satisfies the initial condition $Q(0) = Q_{max}$:

$$\begin{aligned} Q(0) &= C\mathcal{E}e^{-t/RC} \\ &= C\mathcal{E}e^{0} \\ &= C\mathcal{E} \\ &= Q_{max} \end{aligned}$$

(c) The graph of Q as a function of t for the circuit in figure 2 is as follows:



(d) Student answer goes here.

(e) Student answer goes here.

4. (★★★☆) The quantity $\tau = RC$ that appears in the solutions for $Q(t)$ in these RC-circuits is a time constant characteristic of these circuits; it has dimensions (units) of time and is a useful quantity to consider in practical applications.

   (a) Calculate $Q(\tau)$ for the RC circuit without the battery assuming the initial charge on the capacitor is $Q_0 \neq 0$. Now, calculate $Q(\tau)$ for the circuit with the battery assuming the initial charge on the capacitor is 0. What does τ measure in each of these cases?

   (b) Engineers generally consider a charging capacitor starting with zero charge to be fully charged after a time period of 5τ. They also consider a fully charged but discharging capacitor to be fully discharged after a time period of 5τ. Use calculations to help you explain why this practice can be considered reasonable?

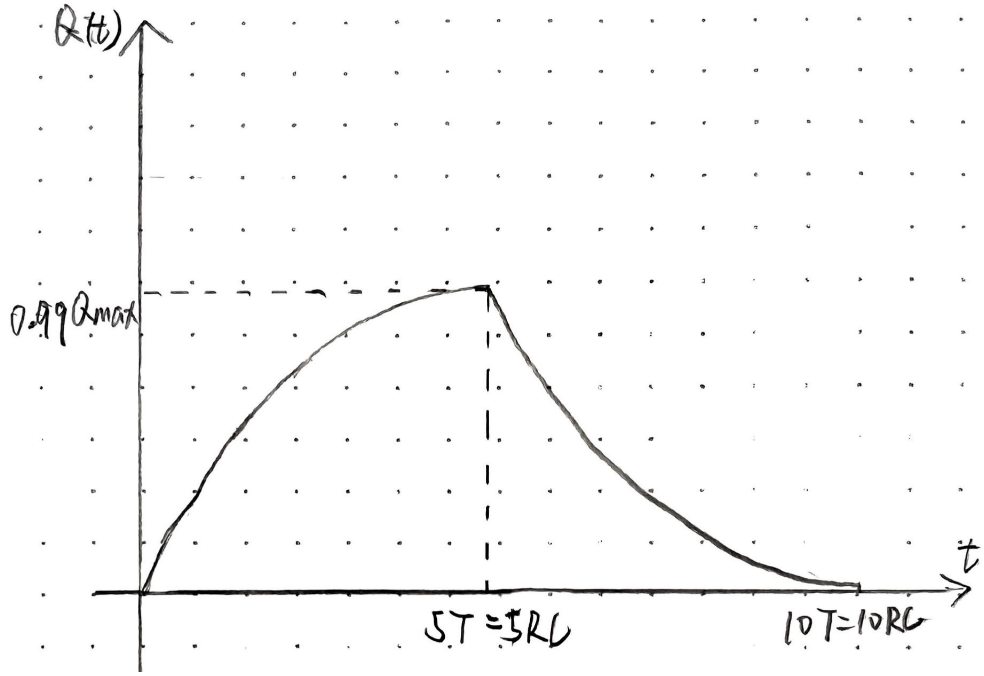
   (c) Assume the initial charge on a capacitor is zero and it is put in series with a resistor and an emf. Sketch the graph of the piecewise function giving the charge $Q(t)$ on this capacitor that results if you first charge the capacitor for a time period of 5τ in this circuit, and then remove the emf from the circuit, leaving only the resistor and capacitor in series, and measure the charge on the capacitor for a period 5τ afterwards. Assume you can remove the emf instantaneously from the circuit.

Answers

(a) Student answer goes here.

(b) Student answer goes here.

(c) The graph of piecewise function $Q(t)$ representing the continuous alternation between charge and discharge of the capacitor in intervals of $\tau = 5RC$ is as follows:



Q(t)
0.99Qmax
t
5T =5RC
10T=10RC