CIVE 546 Structural Design Optimization

(3 units)

A Truss Optimization Example and Optimality Condition for Unconstrained Nonlinear Optimization

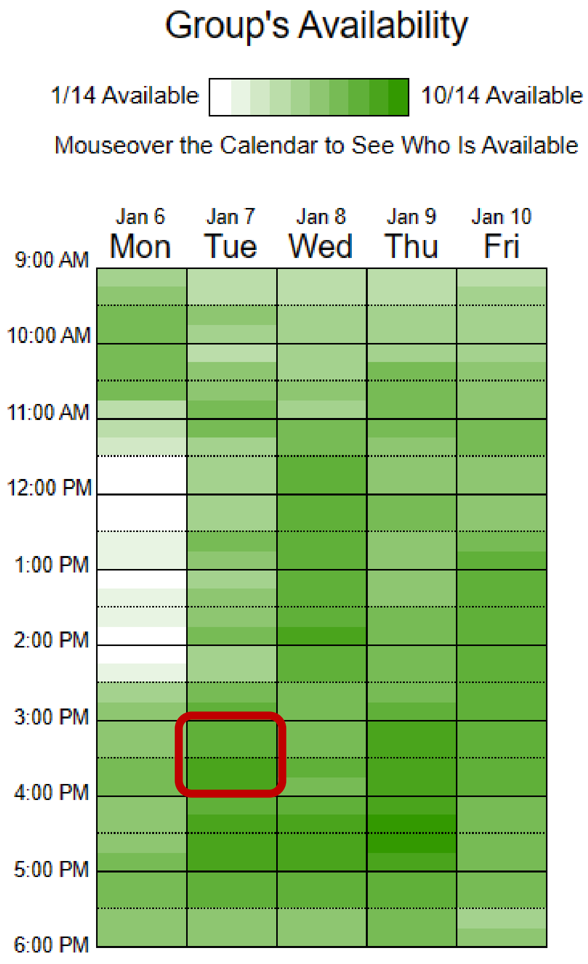
Instructor: Prof. Yi Shao

Class always starts at 11:40am

Winter 2025

Administrative announcements

- Tuesdays 3-4pm or by appointment
  MD480
- HW 1 assigned, recommended completion date: Jan 27
- Project 3D printing
  - Another possible on-campus 3D printing service [The factory]
  - The Schulich library 3D printing service cannot guarantee time-sensitive deliveries
- Exercises are in MATLAB. For project, you may try to find and use other TopOpt program in python.

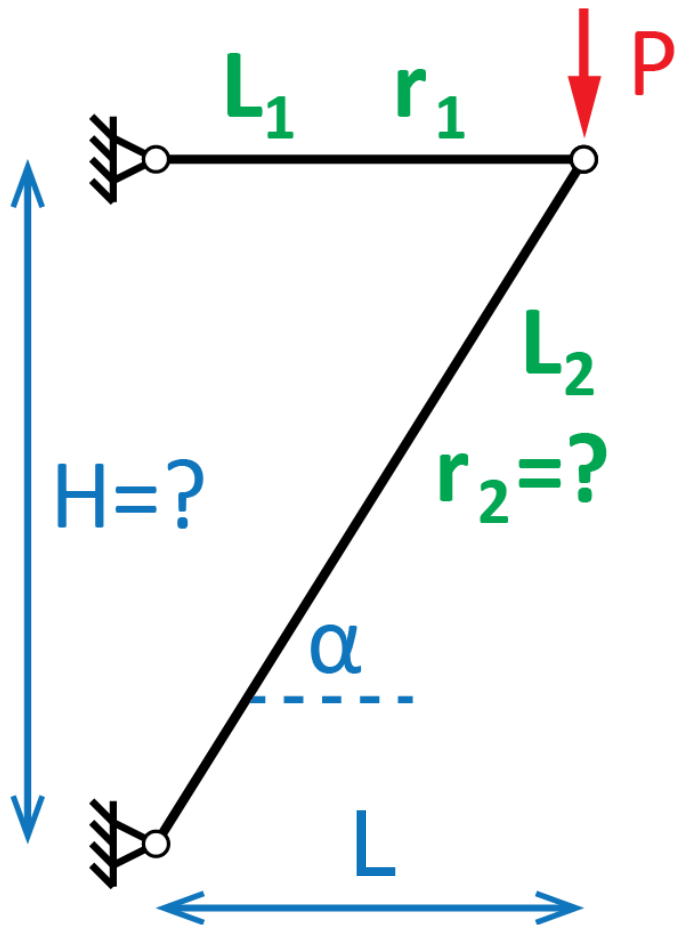


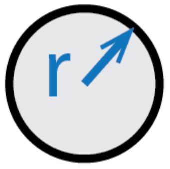
General form of an optimization problem

Objective function	$\min f(\underline{x})$
	Subject to:
Inequality constraints	$g_j(\underline{x}) \leq 0 \qquad j = 1,,,,p$
Equality constraints	$h_k(\underline{x}) = 0 \qquad k = 1,,,,m$
Box constraints	$x_i^L \leq x_i \leq x_i^U$

A Simple yet Powerful Example

Two-bar truss optimization





Cross-section

Design the 2-bar Truss of r_2 (radius of cross-section of bar 2) and H to minimize volume of material such that the stress in the bars are below the yield and buckling stress

$$0 \leq r_2 \leq 100\, mm$$
$$1000 \leq H \leq 7000\, mm$$

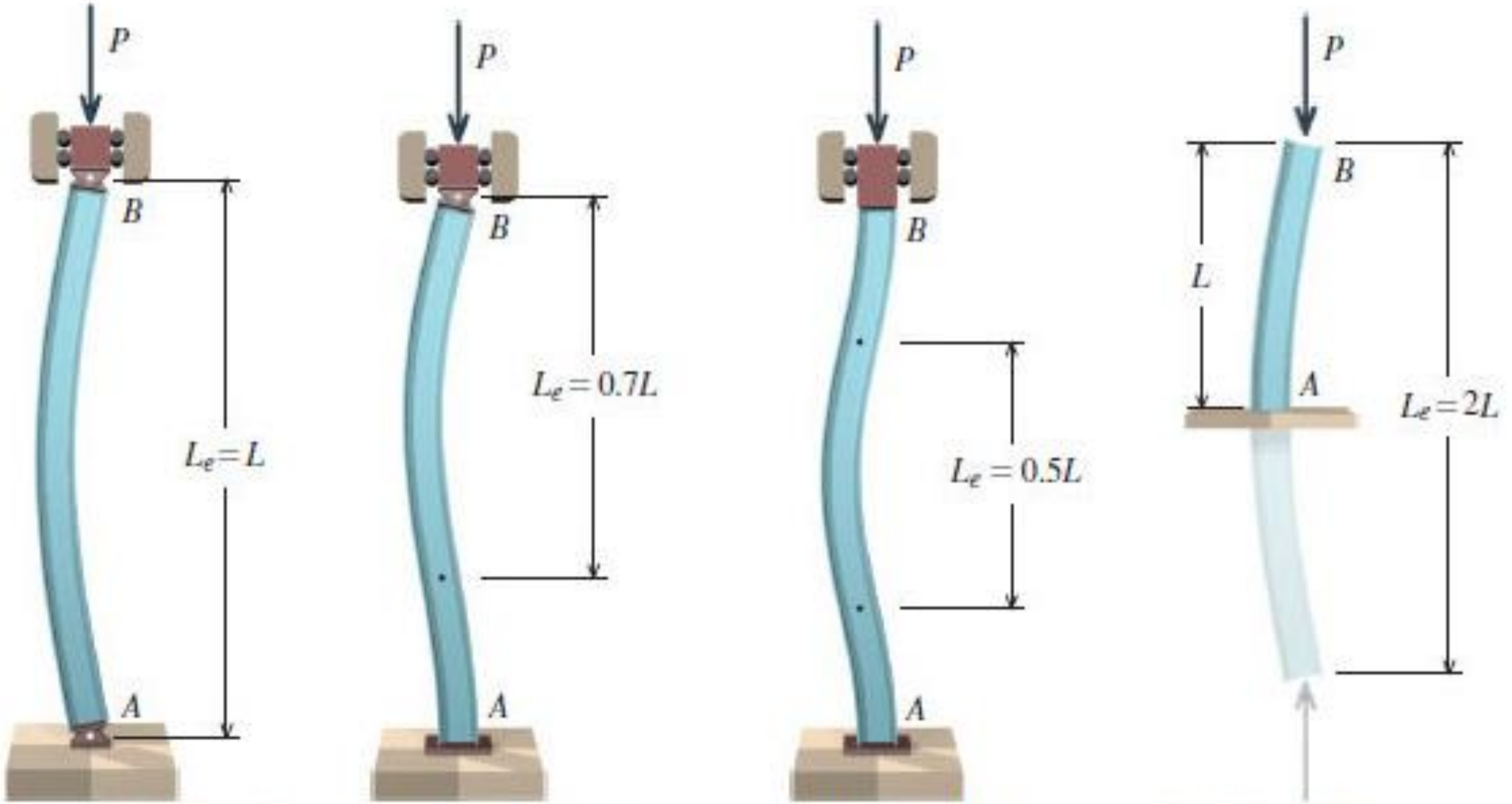
$r_1 = 5\ mm$
$L = 1000\ mm$
$P = 50{,}000\ N$
$E = 200{,}000\ N/mm^2$
$\sigma_y = 400\ N/mm^2$

From CIVE 207 Solid Mechanics

Columns and Buckling with Different End Conditions

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2}$$

$$L_e = KL$$



(*a*) Pinned–pinned column: $K = 1$

(*b*) Fixed–pinned column: $K = 0.7$

(*c*) Fixed–fixed column: $K = 0.5$

(*d*) Fixed–free column: $K = 2$

MATLAB Code

- We will use the optimizer in MATLAB called **fmincon**
  - Function minimizer with constraints
  - See also **fminunc** (name should make it obvious)
- Usage

```
[X,FVAL,EXITFLAG]=fmincon(FUN,X0,A,B,Aeq,Beq,LB,UB,NONLCON,OPTIONS)
```

- FUN: Objective function $f(x)$
- X0: Starting point x_0
- Linear inequalities are expressed $Ax \leq B$
- Linear equalities are expressed $A_{eq}x = B_{eq}$
- LB: Lower bound for variable x
- UB: Upper bound for variable x
- NONLCON: Nonlinear constraints function $C(x) \leq 0$ and $C_{eq}(x) = 0$
  - This function should return both, C and Ceq

- Note that x can be a vector (not a single variable)

MATLAB Code

```
clear all;clc;close all;
%x(1):r2
%x(2):H
r1=5;
L=1000;
P=50000;
sigma_y=400;
E=200000;

GetVolume = @(x)pi*x(1)^2*sqrt(x(2)^2+L^2);

LB = [0,1000];
UB = [100,7000];

function [c,ceq] = GetConstraints(x,r1,L,P,sigma_y,E)
c = [P*sqrt(x(2)^2+L^2)/(x(2)*pi*x(1)^2*sigma_y)-1
4*P*(L^2+x(2)^2)^(3/2)/(E*x(2)*pi^3*x(1)^4)-1];
ceq = [];
End
```

Clear variables, screen and close all windows

⇒ Given values

⇒ obj

Box constraints

Non

Given

≤0 → g_2

≤0 → g_3

MATLAB Code (Continued)

```
A = [0 -pi*r1^2*sigma_y/(P*L)];
b = [-1];
```

$\Rightarrow g_1$

```
Aeq = [];
beq = [];

x0 = [100,7000];

[x, fval] = fmincon(GetVolume,x0,A,b,Aeq,beq,LB,UB,@(x)GetConstraints(x,r1,L,P,sigma_y,E))
```

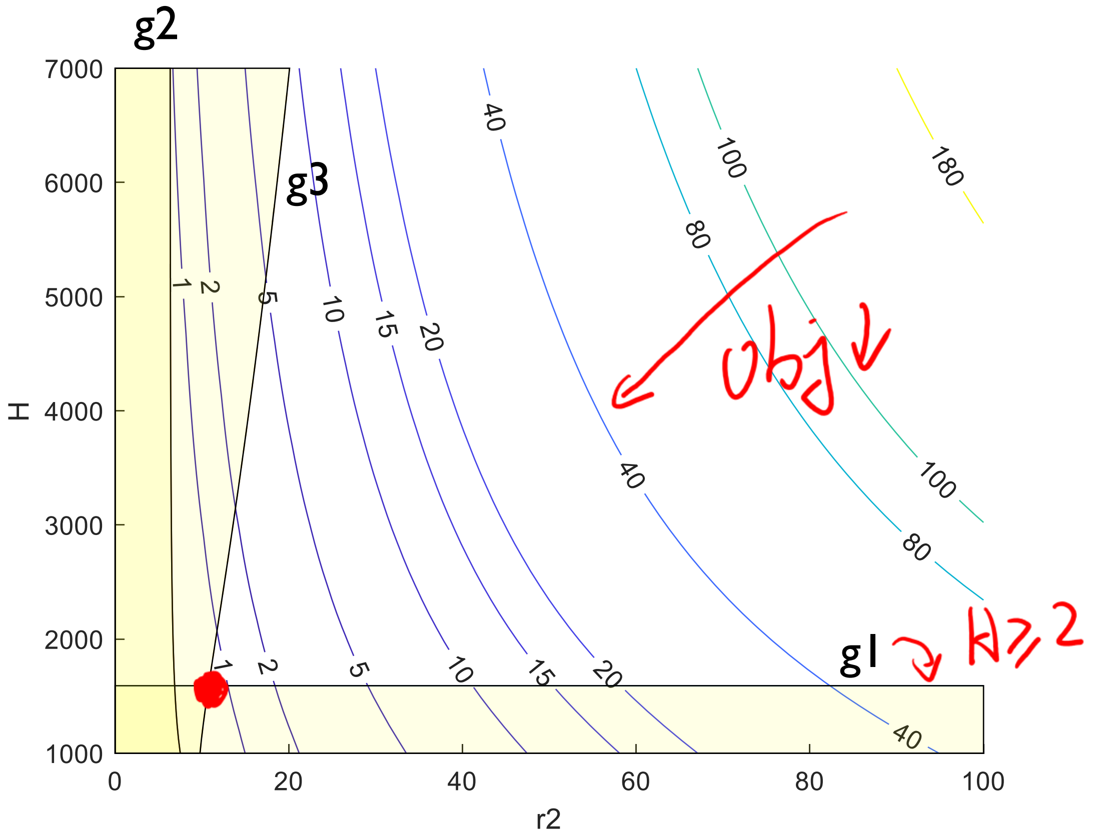
Remark:

- Use help function to understand the usage of different functions
- Use plot for line figures
- Use contour for contour figures

Results

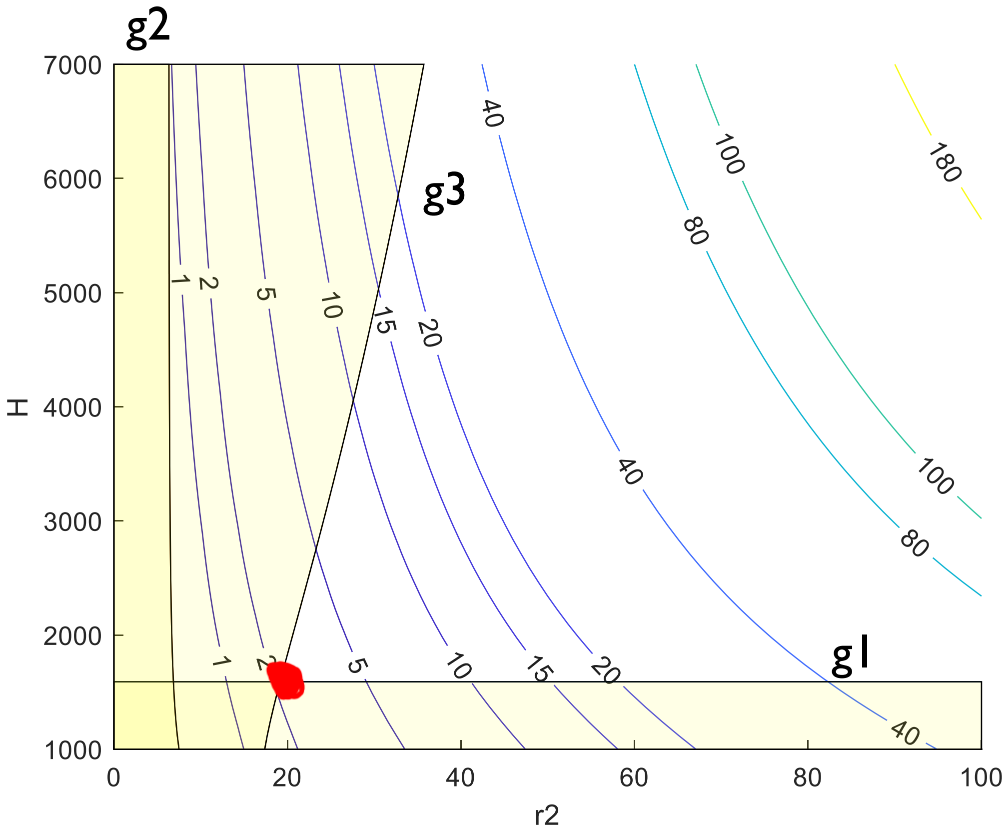
Effects of changing E

σ_y=400 N/mm^2;
E=200,000 N/mm^2



Optimal point: r2=10.8 mm, H=1591.5mm

σ_y=400 N/mm^2;
E=20,000 N/mm^2



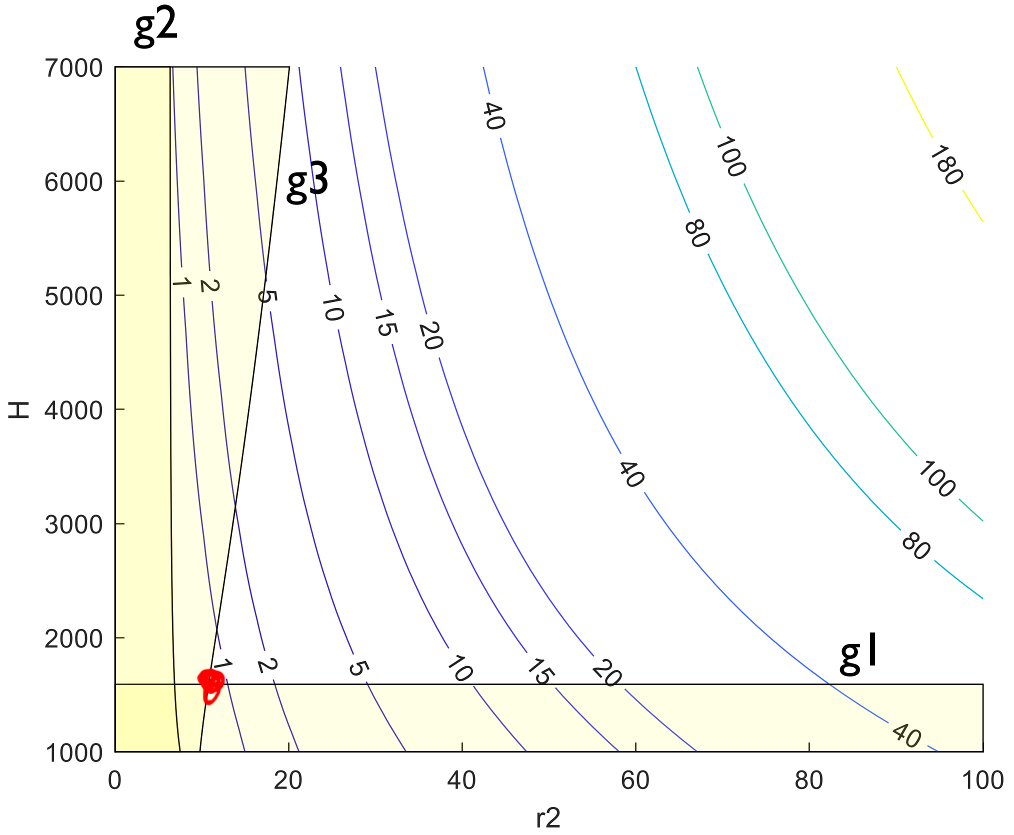
Optimal point: r2=19.2 mm, H=1591.5mm

Results

Effects of changing E

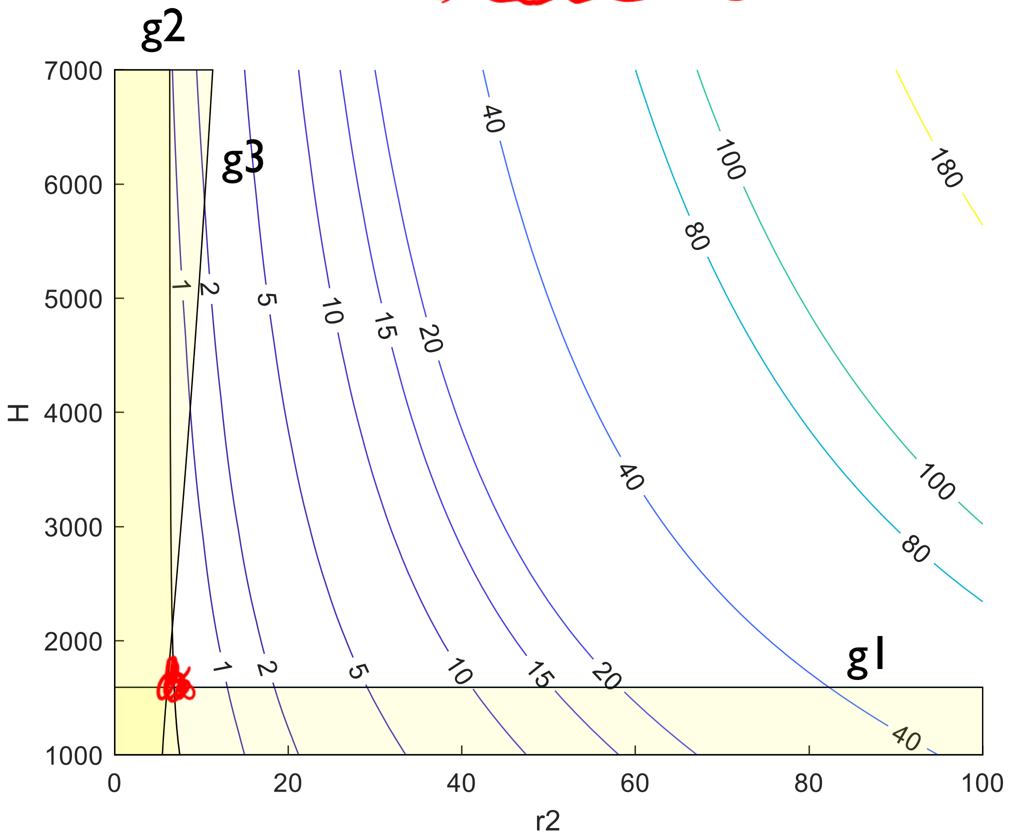
12:45

$\sigma_y = 400\ N/mm^2;$
$E = 200{,}000\ N/mm^2$



Optimal point: r2=10.8 mm, H=1591.5mm

$\sigma_y = 400\ N/mm^2;$
$E = 2000{,}000\ N/mm^2$

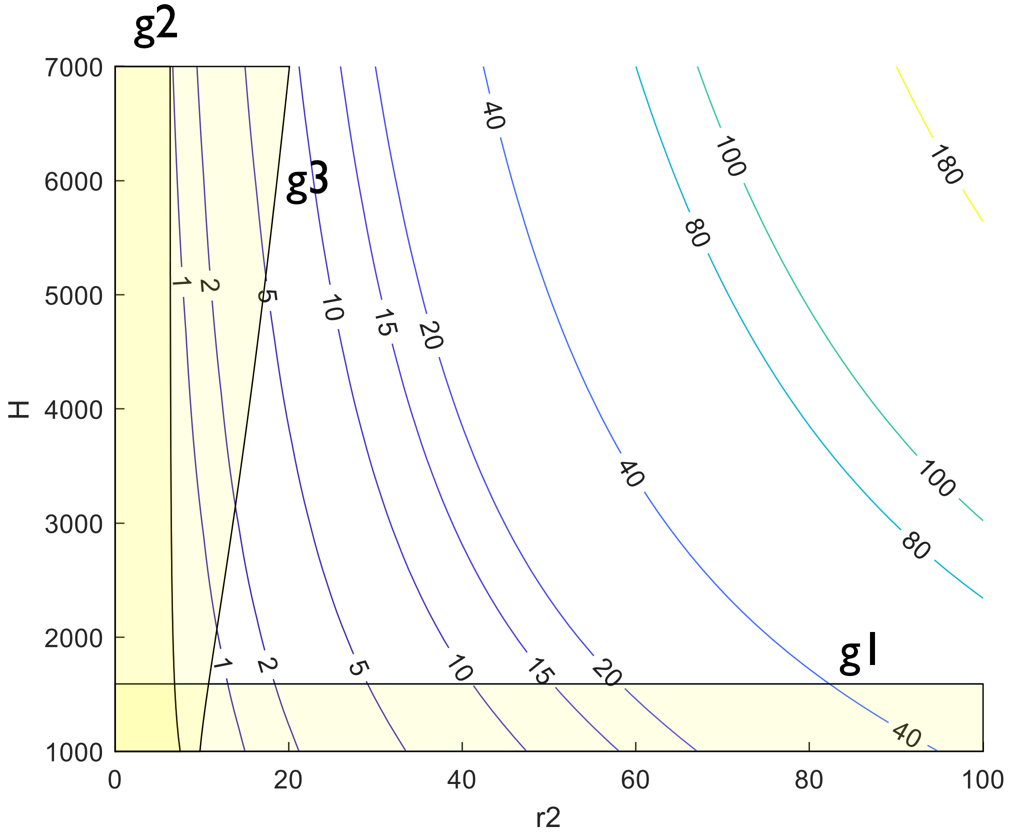


Optimal point: r2=6.9 mm, H=1591.5mm

Results

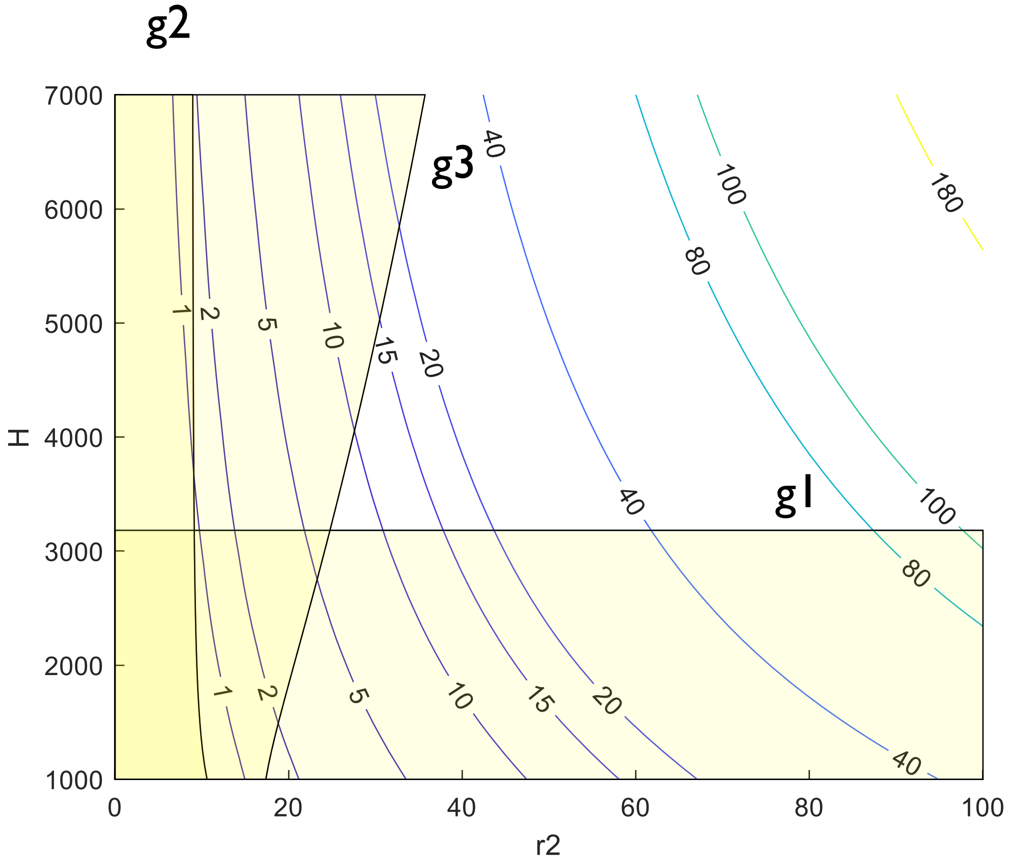
Effects of changing σ_y

$\sigma_y = 400\ N/mm^2;$
$E = 200{,}000\ N/mm^2$



Optimal point: r2=10.8 mm, H=1591.5mm

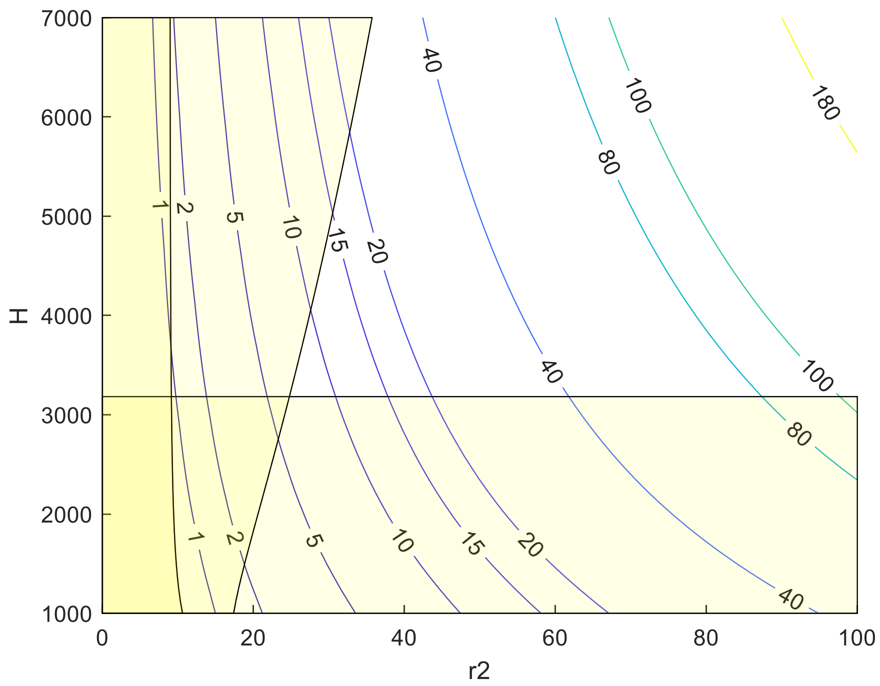
$\sigma_y = 200\ N/mm^2;$
$E = 20{,}000\ N/mm^2$



Optimal point: r2=24.8 mm, H=3183.1mm

How can we recognize/certify if a point is a (local) minimum point in an optimization problem?

Optimality condition!



General form of an optimization problem

Objective function	$\min f(\underline{x})$
	Subject to:
Inequality constraints	$g_j(\underline{x}) \leq 0 \qquad j = 1,,,,p$
Equality constraints	$h_k(\underline{x}) = 0 \qquad k = 1,,,,m$
Box constraints	$x_i^L \leq x_i \leq x_i^U$

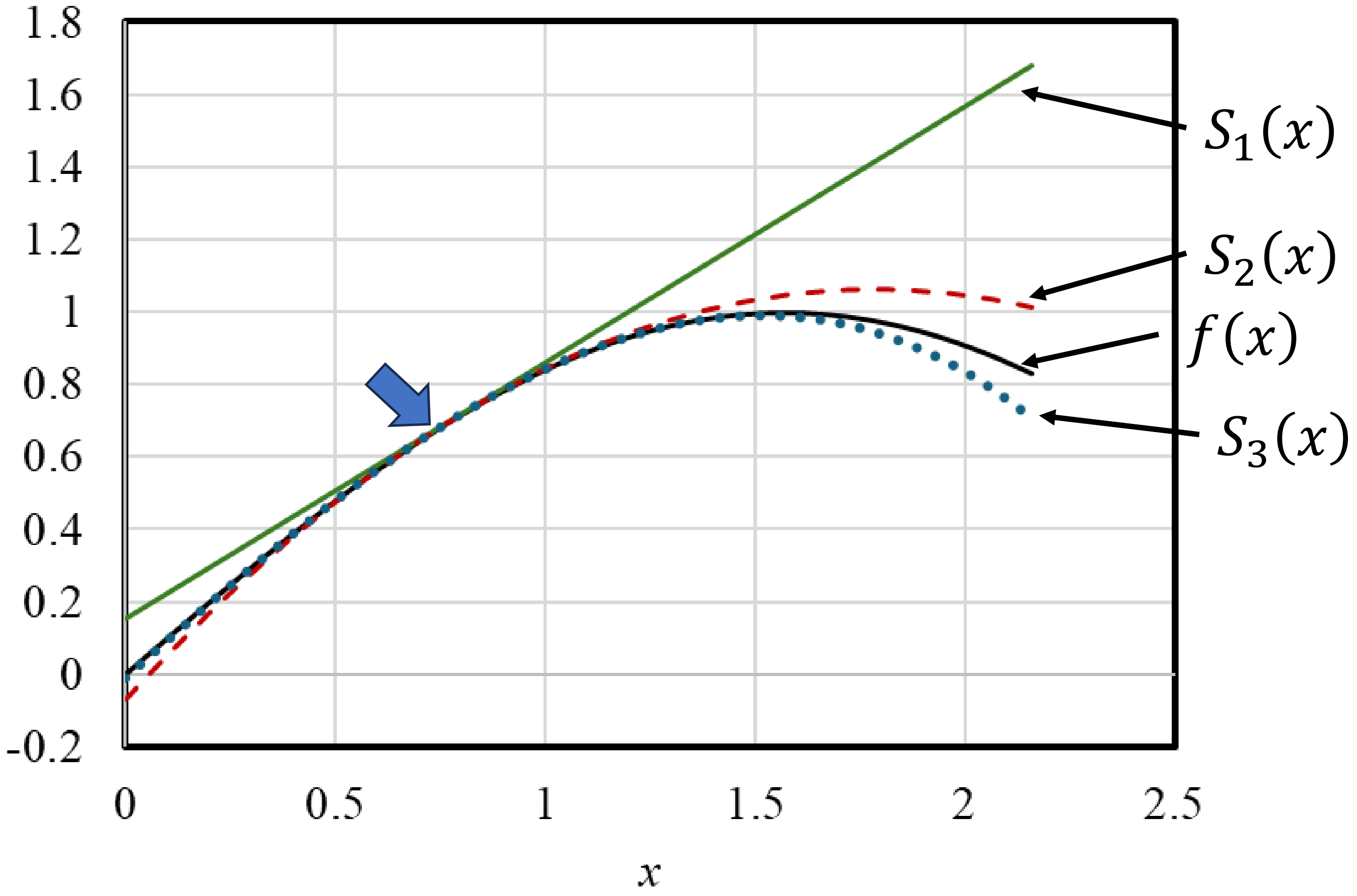
Temporarily ignore

To understand the optimality condition, let's first ignore constraints and look at Unconstrained Optimization using Calculus

Example: Taylor series for a function of 1 variable
Approximate $f(x) = \sin x$ around $x_0 = \pi/4$ to the first 3 orders

Example: Taylor series for a function of 1 variable

Approximate $f(x) = \sin x$ around $x_0 = \pi/4$ to the first 3 orders



Generalization for a function of n variables
Approximate $f(\underline{x})$ around $\underline{x_0}$

Example: Taylor series for a function of 2 variable

Approximate $f(\underline{x}) = \sin x_1 \cdot \sin x_2$ around $\underline{x_0} = (\pi/4, \pi/4)$ to the first 2 orders

Example: Taylor series for a function of 2 variable

Approximate $f(\underline{x}) = \sin x_1 \cdot \sin x_2$ around $\underline{x_0} = (\pi/4, \pi/4)$ to the first 2 orders

Remark:

Let's check $f(\underline{x})$, and $S(\underline{x})$ at $(\pi/5,\ 3\pi/10)$, around 0.22 Euclidean distance from $\underline{x_0}$

Exact: $f(\pi/5,\ 3\pi/10)$=0.47553

Approx: $S(\pi/5,\ 3\pi/10)$=0.47533

Only 0.05% error

1:40

Question:

How to determine whether $\underline{x}^*$ represents a local minimum point for $f(\underline{x})$?

Answer:

It depends on Gradient and Hessian

1st order condition: gradient vector is a zero vector

2nd order condition: Hessian matrix is positive semidefinite

Mathematical definition:

If for any non-zero real $\underline{x}$

$\underline{x}^T \underline{H}\ \underline{x} > 0, \underline{H}$ is positive definite

$\underline{x}^T \underline{H}\ \underline{x} \geq 0, \underline{H}$ is positive semidefinite

Method for assessing:

$\underline{H}$ is positive definite if and only if all its eigenvalues are positive

$\underline{H}$ is positive semidefinite if and only if all its eigenvalues are nonnegative

Eigenvalues and Eigen-vectors

Let $\underline{H}$ be a n x n matrix, if there exists a real number λ and n x1 vector $\underline{v}$, so

$$\underline{H}\ \underline{v} = \lambda\ \underline{v}$$

λ: eigenvalue

$\underline{v}$: eigenvectors

Example:

Calculate the gradient and Hessian of function:

$$f(\underline{x}) = 100 \cdot \left(x_2 - x_1^2\right)^2 + (1 - x_1)^2$$

Show that $\underline{x^*} = (1,1)$ is a minimum point