CIVE 546 Structural Design Optimization

(3 units)

Final Review

Instructor: Prof. Yi Shao

Class always starts at 11:40am

Winter 2025

General form of an optimization problem

Objective function	$\min f(\underline{x})$
	Subject to:
Inequality constraints	$g_j(\underline{x}) \leq 0 \qquad j = 1, , , , p$
Equality constraints	$h_k(\underline{x}) = 0 \qquad k = 1, , , , m$
Box constraints	$x_i^L \leq x_i \leq x_i^U$

Question:

How to determine whether $\underline{x}^*$ represents a local minimum point for $f(\underline{x})$?

Answer:

It depends on Gradient and Hessian

1st order condition: gradient vector is a zero vector

2nd order condition: Hessian matrix is positive semidefinite

Karush–Kuhn–Tucker (KKT) optimality conditions

Problem: minimize $f(\boldsymbol{x})$, where the design variable vector $\boldsymbol{x} = (x_1, \cdots, x_n)$, subjected to $(s.t.)$ $h_i(\boldsymbol{x}) = 0$, $i = 1 \cdots m$; $g_j(\boldsymbol{x}) \le 0$, $j = 1 \cdots p$.

Let $\boldsymbol{x}^*$ be a regular point of the feasible set that is a local min for $f(\boldsymbol{x})$, subjected to the above constraints. Then there exist LMs $\boldsymbol{\lambda}^*$ ($m + p$ vector) such that the Lagrangian function is stationary wrt x_j, λ_j and s_j at the point $\boldsymbol{x}^*$.

KKT 1) Lagrangian function for the problem written in standard form

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{s}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i h_i(\boldsymbol{x}) + \sum_{j=1}^{p} \lambda_j \left(g_j(\boldsymbol{x}) + s_j^2\right)$$
$$= f(\boldsymbol{x}) + \boldsymbol{\lambda}_E^T h(\boldsymbol{x}) + \boldsymbol{\lambda}_I^T \left(g(\boldsymbol{x}) + \boldsymbol{s}^2\right)$$

KKT 2) Gradient conditions

$$\frac{\partial L}{\partial x_k} = \frac{\partial f}{\partial x_k} + \sum_{i=1}^{m} \lambda_i^* \frac{\partial h_i}{\partial x_k} + \sum_{j=1}^{p} \lambda_j^* \frac{\partial g_j}{\partial x_k} = 0, k = 1 \cdots n.$$

$$\frac{\partial L}{\partial \lambda_i} = 0 \;\Rightarrow h_i(\boldsymbol{x}^*) = 0; i = 1 \cdots m.$$

$$\frac{\partial L}{\partial \lambda_j} = 0 \;\Rightarrow \left(g_j(\boldsymbol{x}^*) + s_j^2\right) = 0; j = 1 \cdots p.$$

Karush–Kuhn–Tucker (KKT) optimality conditions

KKT 3) Feasibility check for inequalities

$$s_j^2 \geq 0; \text{ or equivalently } g_j \leq 0; j = 1 \cdots p.$$

KKT 4) Switching conditions

$$\frac{\partial L}{\partial s_j} = 0 \ \Rightarrow \lambda_j^* s_j = 0; j = 1 \cdots p.$$

KKT 5) Non-negativity of LMs for inequalities

$$\lambda_j^* \geq 0; \ j = 1 \cdots p.$$

KKT 6) Regularity check

Gradients of active constraints must be linearly independent. In such case, the LMs for the constraints are unique.

Karush–Kuhn–Tucker (KKT) optimality conditions w/o slack

Problem: minimize $f(\boldsymbol{x})$, where the design variable vector $\boldsymbol{x} = (x_1, \cdots, x_n)$, subjected to $(s.t.)$ $h_i(\boldsymbol{x}) = 0,\ i = 1 \cdots m;\ g_j(\boldsymbol{x}) \le 0,\ j = 1 \cdots p.$

KKT 1) Lagrangian function definition

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i h_i(\boldsymbol{x}) + \sum_{j=1}^{p} \lambda_j g_j(\boldsymbol{x})$$

KKT 2) Gradient conditions

$$\frac{\partial L}{\partial x_k} = 0;\ \frac{\partial f}{\partial x_k} + \sum_{i=1}^{m} \lambda_i^* \frac{\partial h_i}{\partial x_k} + \sum_{j=1}^{p} \lambda_j^* \frac{\partial g_j}{\partial x_k} = 0, k = 1 \cdots n.$$

Karush–Kuhn–Tucker (KKT) optimality conditions w/o slack

KKT 3) Feasibility check

$$h_i(\boldsymbol{x}^*) = 0; i = 1 \cdots m; \; g_j(\boldsymbol{x}^*) \leq 0, j = 1 \cdots p.$$

KKT 4) Switching conditions

$$\lambda_j^* g_j(\boldsymbol{x}^*) = 0, j = 1 \cdots p.$$

KKT 5) Non-negativity of LMs for inequalities

$$\lambda_j^* \geq 0; \; j = 1 \cdots p.$$

KKT 6) Regularity check

Gradients of active constraints must be linearly independent. In such case, the LMs for the constraints are unique.

Convexity

Convex function: Theorem

A function of $f(\underline{x})$ defined on a convex set S is convex if
For any two points $\underline{x_1}, \underline{x_2} \in S$

$$f\left(\alpha \underline{x_1} + \beta \underline{x_2}\right) \leq \alpha f\left(\underline{x_1}\right) + \beta f\left(\underline{x_2}\right)$$

$$\text{for } \alpha + \beta = 1, \alpha \geq 0, \beta \geq 0$$

A function of $f(\underline{x})$ defined on a convex set S is convex iff (if an only if) its hessian matrix $\underline{H}$ is positive semi-definite or positive definite at all points in the set S

Convexity

Convex set: Convex feasible domain

Let the feasible domain S be defined by the constraints of the general optimization problem defined in the standard form

$$S = \{\underline{x} | h_i(\underline{x}) = 0, i = 1,,,m; g_j(\underline{x}) \leq 0, j = 1,,,p\}$$

S is a convex set if $g_j(\underline{x})$ are convex AND $h_i(\underline{x})$ are linear

Remark:

- Feasible domain defined by ANY nonlinear equality constraints is always non-convex
- Feasible domain defined by linear equality or inequality constrains is always convex

Unconstrained Minimization

General Descent Method

$$\underline{x}^{k+1} = \underline{x}^{k} + \alpha^{k}\underline{\Delta x}^{k}$$

So that $\quad f(\underline{x}^{k+1}) < f(\underline{x}^{k})$

Where, k is iteration number

$\underline{\Delta x}^{k}$ is the step or search direction

α^{k} is the step size or step length

Procedure:

given a starting point $\underline{x}^{0} \in$ **dom** f.

Repeat

1. Determine a descent direction $\underline{\Delta x}^{k}$.
2. Line search. Choose a step size α^{k}.
3. Update. $\underline{x}^{k+1} = \underline{x}^{k} + \alpha^{k}\underline{\Delta x}^{k}$.

Until stopping criterion is satisfied.

Procedure:

given a starting point $\underline{x}^{0} \in$ **dom**

Repeat

1. Determine a descent direction

2. Line search. Choose a step size

3. Update. $\underline{x}^{k+1} = \underline{x}^{k} + \alpha^{k}\underline{\Delta x}^{k}$.

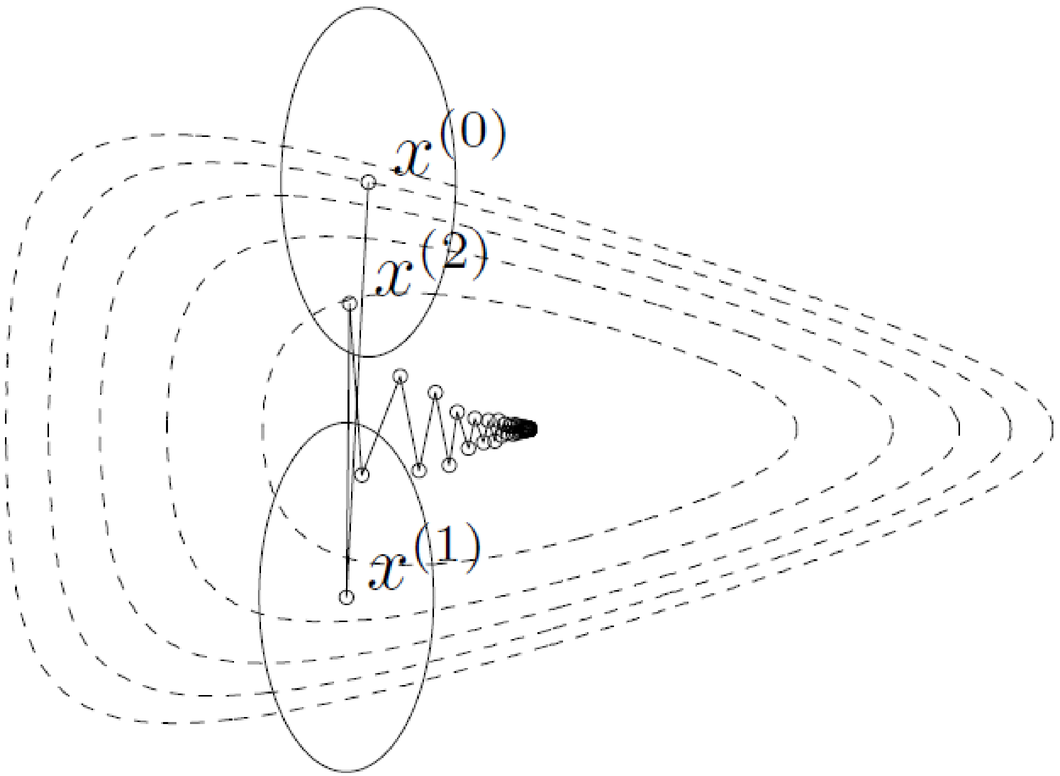
Until stopping criterion is satisfi

Unconstrained Minimization

Steepest Descent Method (First-order method)

$$\underline{x}^{k+1} = \underline{x}^{k} + \alpha^{k}\underline{\Delta x}^{k}$$

Where $\underline{\Delta x}^{k} = -\frac{\nabla f(\underline{x}^{k})}{|\nabla f(\underline{x}^{k})|}$: steepest descent direction



Challenging if the condition number of the hessian matrix is large, which indicates an elongated design space.
Condition number is the ratio of the largest eigenvalue to the smallest eigenvalue

Unconstrained Minimization

Newton's Method (Second-order method)

$$\underline{x^{k+1}} = \underline{x^k} + \alpha^k \underline{\Delta x^k}$$

Where $\underline{\Delta x^k} = -\nabla^2 f\left(\underline{x^k}\right)^{-1} \nabla f\left(\underline{x^k}\right)$