Structural Design Optimization

Practice Midterm Solutions

Problem 1

1-B, 2-C, 3-B, 4-E, 5-D, 6-E, 7-C, 8-A, 9-C and 10-A

Problem 2

Solution (Calculus). The bending stress (σ) induced in a rectangular beam at any fiber located a distance y from the neutral axis is obtained by

$$\sigma = \frac{My}{I}$$

where I is the moment of inertia of the cross section about the x- axis.

$$\sigma = \frac{My}{I} = \frac{My}{\frac{1}{12}(2x)(2y)^3} = \frac{3M}{4xy^2}$$

The optimization problem can be stated as following

$$\max_{x,y} \quad f\colon \sigma = \frac{3M}{4xy^2}$$

$$\text{s. t.} \quad \text{g: x}^2 + \text{y}^2 = 4\text{a}^2$$
$$\text{x, y} > 0$$

By plugging $\text{y}^2 = -\text{x}^2 + 4\text{a}^2$ into $= \frac{3M}{4xy^2}$, the objective function f becomes

$$f = \frac{3M}{4x(-x^2 + 4a^2)}$$

and

$$\frac{df}{dx} = \frac{3M((-3x^2 + 4a^2)}{4x^2(-x^2 + 4a^2)^2} = 0$$

$$x^* = \sqrt{\frac{4}{3}}a = \frac{2a}{\sqrt{3}}, \qquad y^* = \frac{2\sqrt{2}a}{\sqrt{3}}$$

Solution (Lagrange multiplier).

The Lagrange function is

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y) = \frac{3M}{4xy^2} + \lambda(x^2 + y^2 - 4a^2)$$

The necessary condition gives

$$\frac{\partial L(x, y, \lambda)}{\partial x} = -\frac{3M}{4x^2y^2} + 2x\lambda = 0$$

$$\frac{\partial L(x, y, \lambda)}{\partial y} = -\frac{3M}{2xy^3} + 2y\lambda = 0$$

$$\frac{\partial L(x, y, \lambda)}{\partial \lambda} = x^2 + y^2 - 4a^2 = 0$$

From the equations above, we can obtain

$$2\lambda = \frac{3M}{2xy^4} = \frac{3M}{4x^3y^2}$$

$$y^2 = 2x^2$$

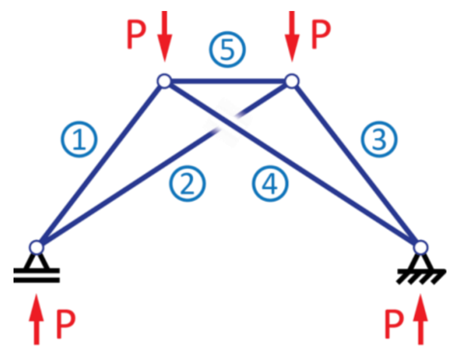
Using $x^2 + y^2 - 4a^2 = 0$ and $y^2 = 2x^2$, we can obtain the optimal dimension of the cross section

$$x^* = \frac{2a}{\sqrt{3}}, \quad y^* = \frac{2\sqrt{2}a}{\sqrt{3}}$$

Problem 3

Solution

The reactions are equal vertical forces P, with no horizontal reactions. The angle relations are:



$$\cos\alpha = \frac{1}{L_1} \qquad \cos\beta = \frac{2}{L_2}$$
$$\sin\alpha = \frac{h}{L_1} \qquad \sin\beta = \frac{h}{L_2}$$

The length of bars $L_1 = L_3$ and $L_2 = L_4$ are function of h:

$$L_1 = \sqrt{1+h^2} \qquad L_2 = \sqrt{4+h^2}$$

Equilibrium at the bottom left (or bottom right) node results in:

$$\begin{aligned} N_1\cos\alpha + N_2\cos\beta &= 0 \\ N_1\sin\alpha + N_2\sin\beta + P &= 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} N_1 &= -P\left(\frac{2L_1}{h}\right) \\ N_2 &= P\left(\frac{L_2}{h}\right) \end{aligned}$$

From symmetry, we know that $N_3 = N_1$ and $N_4 = N_2$. Applying horizontal equilibrium at any top node:

$$-N_1\cos\alpha + N_4\cos\beta + N_5 = 0 \quad \rightarrow \quad N_5 = -\frac{4P}{h}$$

Knowing that $L_3 = L_1$ and $L_4 = L_2$, the volume of the structure (assuming the structure is fully stressed) is:

$$V = \sum_{i=1}^{5} \frac{|N_i|}{\bar{\sigma}} L_i = -\frac{N_1}{\bar{\sigma}}(L_1) + \frac{N_2}{\bar{\sigma}}(L_2) - \frac{N_1}{\bar{\sigma}}(L_1) + \frac{N_2}{\bar{\sigma}}(L_2) - \frac{N_5}{\bar{\sigma}}(L_5)$$
$$V = \frac{2P}{\bar{\sigma}h}(L_1^2) + \frac{P}{\bar{\sigma}h}(L_2^2) + \frac{2P}{\bar{\sigma}h}(L_1^2) + \frac{P}{\bar{\sigma}h}(L_2^2) + \frac{4P}{\bar{\sigma}h}(1)$$
$$V = \frac{P}{\bar{\sigma}h}(4 + 4L_1^2 + 2L_2^2) = \frac{2P}{\bar{\sigma}h}(2 + 2[1+h^2] + [4+h^2])$$
$$V = \frac{2P}{\bar{\sigma}h}(8 + 3h^2)$$

The optimum is found solving for $\frac{dV}{dh} = 0$:

$$\frac{dV}{dh} = \frac{2P}{\bar{\sigma}}\left[\frac{3h^2 - 8}{h^2}\right] = 0$$
$$3 - \frac{8}{h^2} = 0$$
$$h^* = \sqrt{\frac{8}{3}}$$

The second derivative is:

$$\frac{d^2V}{dh^2} = \frac{2P}{\bar{\sigma}}(-2)\left(-\frac{8}{h^3}\right) = \frac{32P}{\bar{\sigma}h^3}$$

The second derivative is positive for $h > 0$, therefore, the point is a minimum as expected.