CIVE 546 Structural Design Optimization

(3 units)

Lagrange Multiplier
KKT Conditions
Structural Optimization of Concrete Structures

Instructor: Prof. Yi Shao

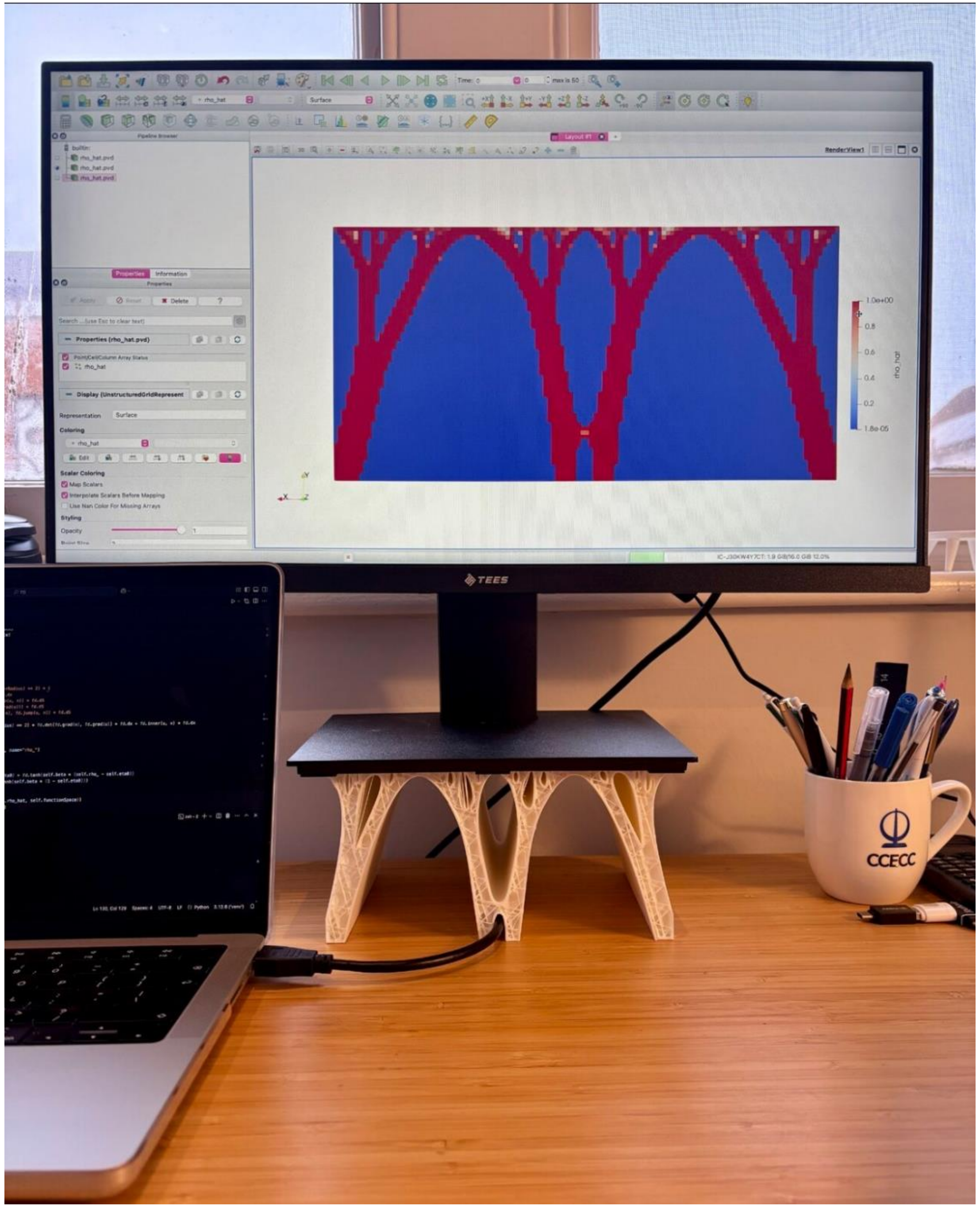
Winter 2025

Administrative announcement

HW2 Assigned

Project 3D printing: another possible on-campus 3D printing service [The factory]

- Free membership
- First 100g of printing free (*each time*)
- Just need to bring your design (e.g., stl file), their expert will help with setting up the printing.
- Will provide drop-in hour during the next few weeks



slido

Please download and install the Slido app on all computers you use

Which of the following conditions are necessary to determine whether a point represents a local minimum of an unconstrained function (select all that applies)

ⓘ Start presenting to display the poll results on this slide.

General form of an optimization problem

Objective function	$\min f(\underline{x})$
	Subject to:
Inequality constraints	$g_j(\underline{x}) \leq 0 \qquad j = 1,,,,p$
Equality constraints	$h_k(\underline{x}) = 0 \qquad k = 1,,,,m$
Box constraints	$x_i^L \leq x_i \leq x_i^U$

Question:

How to determine whether $\underline{x}^*$ represents a local minimum point for $f(\underline{x})$?

Answer:

It depends on Gradient and Hessian

1st order condition: gradient vector is a zero vector

2nd order condition: Hessian matrix is positive semidefinite

Method of Lagrange Multiplier

Let's examine a simple optimization problem with one equality constraint

Lagrangian Multiplier

min

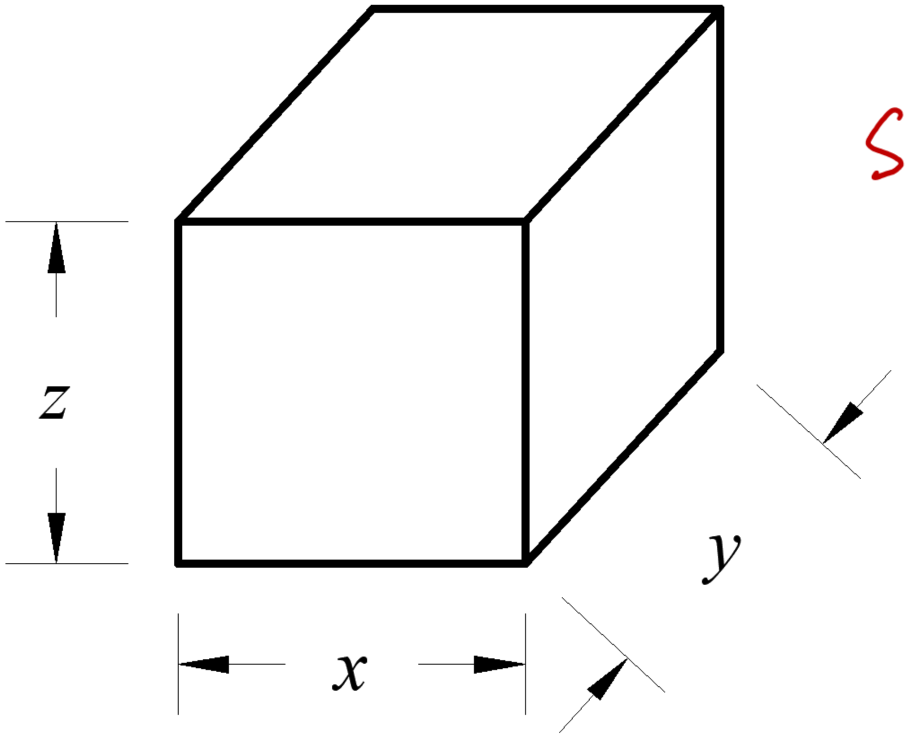
$$\min f(x,y,z) + 0 \Rightarrow \mathcal{L}(x,y,z;\lambda) = f(x,y,z) + \lambda(\underbrace{h(x,y,z)-k}_{=0})$$

Subject to:

$$h(x,y,z) = k \Rightarrow h(x,y,z) - k = 0$$

Example 1

Find the dimension of the box with the largest volume if the total surface area is 64 cm^2



$$\max V(x,y,z) = \underline{xyz}$$

$$\min -V(x,y,z)$$

$$\frac{1}{V(x,y,z)}$$

$$\text{s.t. } 2(xy+yz+xz) = 64$$

$$L(x,y,z;\lambda) = xyz + \lambda(xy+yz+xz-32)$$

$$L_{,x} = yz + \lambda(y+z) = 0$$

$$L_{,y} = xz + \lambda(x+z) = 0$$

$$L_{,z} = xy + \lambda(x+y) = 0$$

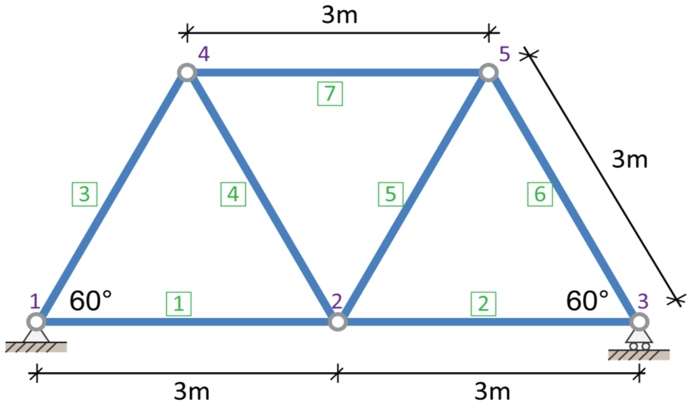
$$L_{,\lambda} = xy+yz+xz-32 = 0$$

$$x = y = z = \sqrt{\frac{32}{3}}$$

$$\lambda = \sqrt{\frac{3}{8}}$$

Example 2

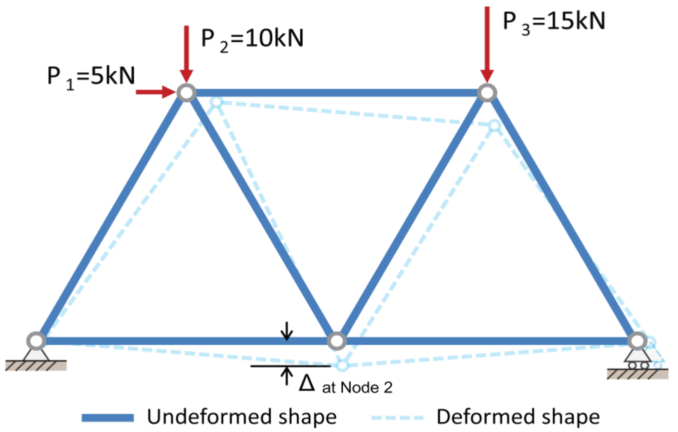
Consider a 7-bar truss structure subjected to concentrated forces acting at the nodes. Size the members in order to minimize the total weight of the truss while ensuring the maximum deflection at node 2 is smaller than 0.1 m. Truss geometry, loading and boundary conditions are shown below.



Material properties

Elastic modulus, $E_i = 20MPa$

Length, L_i = 3m, i =1,…, 7



Review: Method of Virtual Work

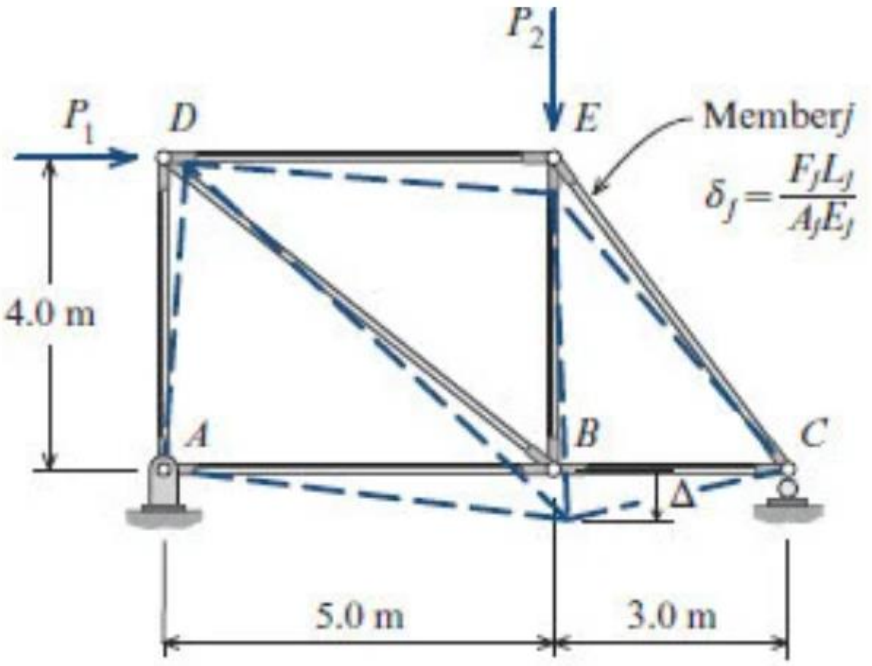
Application to Truss

From CIVE 207 Solid Mechanics

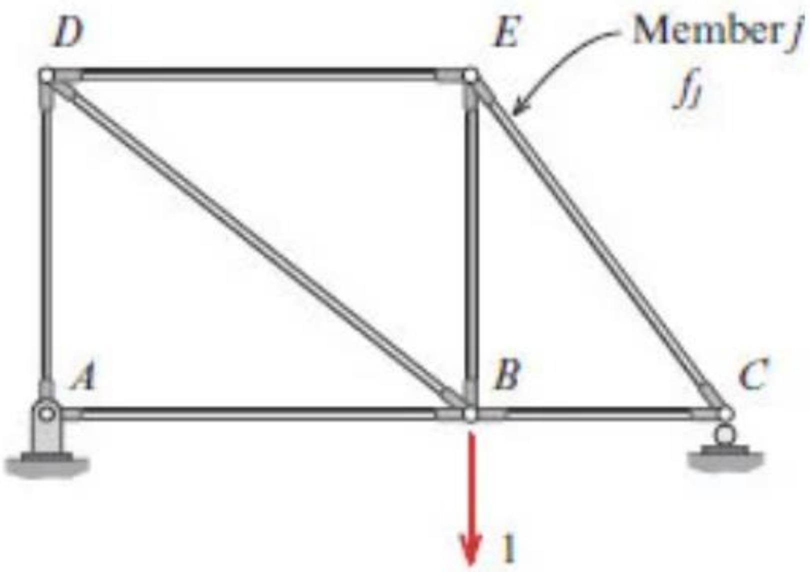
$$W_{ve} = W_{vi}$$

$$W_{ve} = 1 \cdot \Delta \qquad W_{vi} = \sum_j f_j \delta_j = \sum_j f_j \left(\frac{F_j Lj}{A_j E_j} \right)$$

Real system

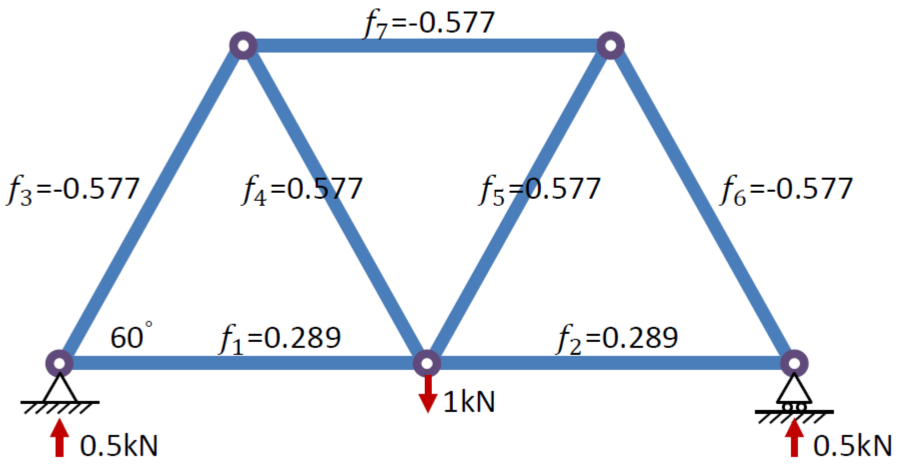


Virtual system

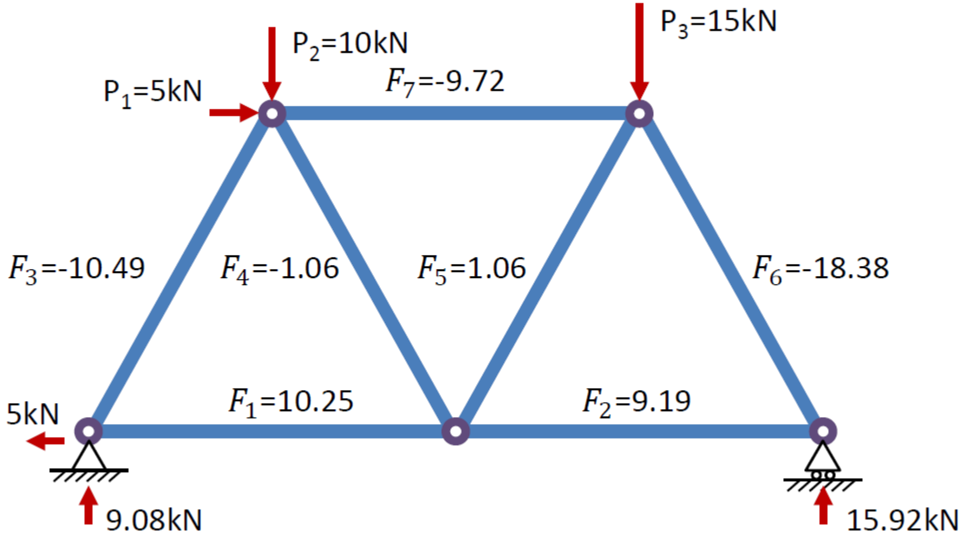


Example 2

Virtual system



Real system



Example 2

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The optimal cross-sectional areas

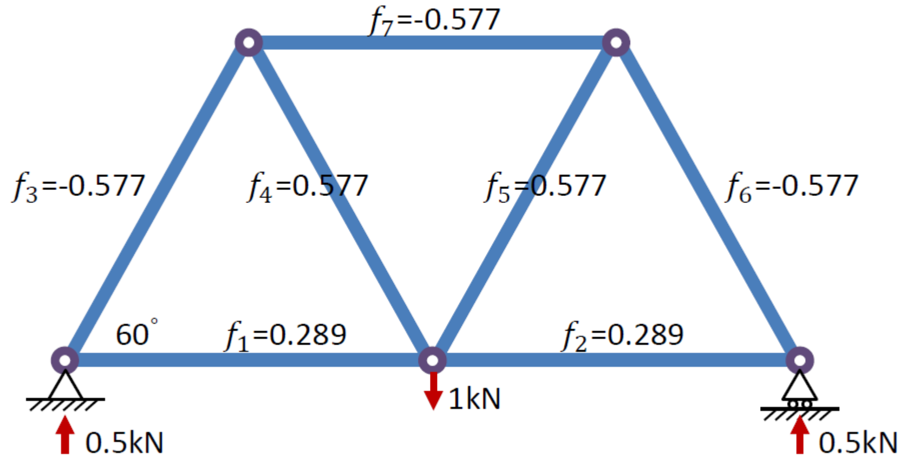
$$A_i^* = \sqrt{\lambda \frac{f_i F_i}{E_i}} = 2.4259 \cdot \sqrt{\frac{f_i F_i}{E_i}}$$

Member	A_i^* (m^2)
1	0.0295
2	0.0280
3	0.0422
4	0.0134
5	0.0134
6	0.0559
7	0.0406

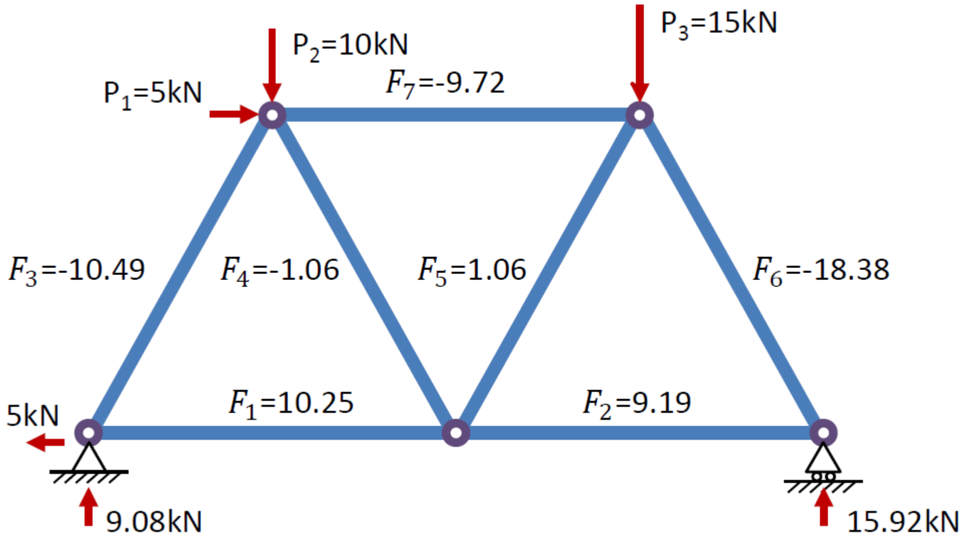
Check vertical displacement

$$1kN \cdot \Delta = \sum_{i=1}^{n} \frac{f_i F_i}{A_i E_i} L_i, \qquad \Delta = \mathbf{0.10}\ m$$

Virtual system



Real system



Method of Lagrange Multiplier

Generalized form for optimization with equality constraints

$\min f(\underline{x})$ $\qquad\qquad$ $\underline{x} = [x_1, , , , , x_n]^T$: n DVs

Subject to:

$h_j(\underline{x}) = 0$ $\qquad\qquad$ $j = 1, , , , , , m$ $\qquad$: m constraints

Lagrangian:

Remark, $m \leq n$

$$L(\underline{x}, \underline{\lambda}) = f(\underline{x}) + \sum_{j=1}^{m} \lambda_j h_j(\underline{x})$$

n+m unknowns → n+m equations

$L_{,x_i} = 0$ $\qquad\qquad$ $i = 1, , , , , , n$ $\qquad$ ***Further inspection***

$L_{,\lambda_j} = 0$ $\qquad\qquad$ $j = 1, , , , , , m$

Method of Lagrange Multiplier

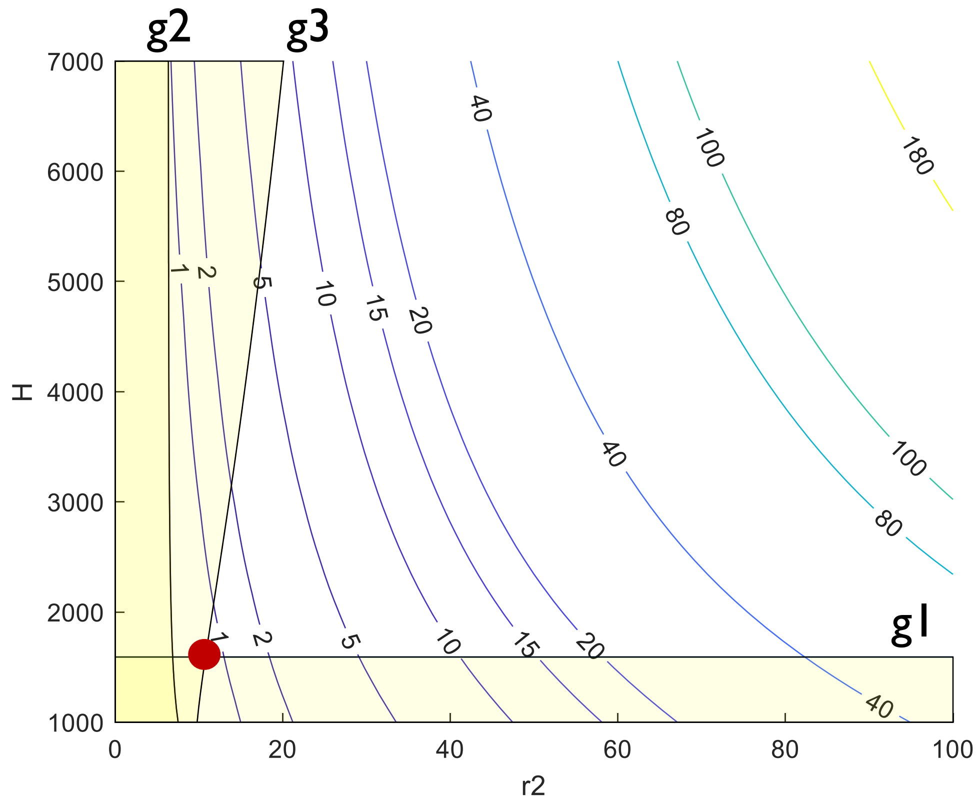
Remarks

Lagrange Multipliers (LM) measures the sensitivity of the optimum design to the changes in each constraint.

When the LM associated with a constraint is ZERO, it means that to a first order approximation, removing this constraint will have NO effect on the optimum value of the objective.

$\lambda_j = 0$ → Inactive constraint

$\lambda_j \neq 0$ → Active constraint



General form of an optimization problem

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	Subject to:
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Box constraints	$x_i^L \leq x_i \leq x_i^U$

How to deal with inequality constraints?

Introducing Slack Variables

Let's examine a simple optimization problem with only inequality constraints

$$L(\underset{\sim}{x}, \underset{\sim}{\lambda}, \underset{\sim}{S}) = f(\underset{\sim}{x}) + \sum_{j=1}^{p} \lambda_j (g_j(\underset{\sim}{x}) + S_j^2)$$

$\min f(\underline{x})$

Subject to:

$g_j(\underline{x}) \leq 0 \qquad j = 1,,,,p$

$+ S_j^2 = 0$

$$\frac{\partial L}{\partial x_i} = 0, \quad i = 1, \cdots, n$$

$$\frac{\partial L}{\partial \lambda_j} = 0, \quad j = 1, \cdots, p$$

Transform the inequality constraints to equality constraints by adding slack variables s_j

$$\frac{\partial L}{\partial S_j} = 2\underbrace{\lambda_j S_j = 0}_{\text{switching}} \quad \begin{cases} \lambda_j = 0 \Rightarrow \text{Inactive} \\ S_j = 0 \Rightarrow \text{Active} \end{cases}$$

Karush–Kuhn–Tucker (KKT) optimality conditions

1:45

Problem: minimize $f(\boldsymbol{x})$, where the design variable vector $\boldsymbol{x} = (x_1, \cdots, x_n)$, subjected to $(s.t.)$ $h_i(\boldsymbol{x}) = 0$, $i = 1 \cdots m$; $g_j(\boldsymbol{x}) \le 0$, $j = 1 \cdots p$.

Let $\boldsymbol{x}^*$ be a regular point of the feasible set that is a local min for $f(\boldsymbol{x})$, subjected to the above constraints. Then there exist LMs $\boldsymbol{\lambda}^*$ ($m + p$ vector) such that the Lagrangian function is stationary wrt x_j, λ_j and s_j at the point $\boldsymbol{x}^*$.

KKT 1) Lagrangian function for the problem written in standard form

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{s}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i h_i(\boldsymbol{x}) + \sum_{j=1}^{p} \lambda_j \left(g_j(\boldsymbol{x}) + s_j^2\right)$$

$$= f(\boldsymbol{x}) + \boldsymbol{\lambda}_E^T h(\boldsymbol{x}) + \boldsymbol{\lambda}_I^T (g(\boldsymbol{x}) + \boldsymbol{s}^2)$$

KKT 2) Gradient conditions

$$\frac{\partial L}{\partial x_k} = \frac{\partial f}{\partial x_k} + \sum_{i=1}^{m} \lambda_i^* \frac{\partial h_i}{\partial x_k} + \sum_{j=1}^{p} \lambda_j^* \frac{\partial g_j}{\partial x_k} = 0, k = 1 \cdots n.$$

$$\frac{\partial L}{\partial \lambda_i} = 0 \ \Rightarrow h_i(\boldsymbol{x}^*) = 0; i = 1 \cdots m.$$

$$\frac{\partial L}{\partial \lambda_j} = 0 \ \Rightarrow \left(g_j(\boldsymbol{x}^*) + s_j^2\right) = 0; j = 1 \cdots p.$$

Karush–Kuhn–Tucker (KKT) optimality conditions

KKT 3) Feasibility check for inequalities

$$s_j^2 \geq 0; \text{ or equivalently } g_j \leq 0; j = 1 \cdots p.$$

KKT 4) Switching conditions

$$\frac{\partial L}{\partial s_j} = 0 \;\Rightarrow \lambda_j^* s_j = 0; j = 1 \cdots p. \quad \left\{ \begin{array}{l} \lambda_j^* = 0 \\ S_j = 0 \end{array} \right.$$

KKT 5) Non-negativity of LMs for inequalities

$$\lambda_j^* \geq 0; \; j = 1 \cdots p.$$

KKT 6) Regularity check

Gradients of active constraints must be linearly independent. In such case, the LMs for the constraints are unique.

Karush–Kuhn–Tucker (KKT) optimality conditions
Example

Use KKT method to find the optimized solution to the following problem

$$\min f(x_1, x_2) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$$

$$s.t.\ g(x) = x_1 + x_2 - 2 \leq 0$$

KKT 1: $L(x_1, x_2, \lambda, S) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2 + \lambda(x_1 + x_2 - 2 + S^2)$ (1)

KKT 2: $L_{,x_1} = 2(x_1 - 1.5) + \lambda = 0$ (2)

$L_{,x_2} = 2(x_2 - 1.5) + \lambda = 0$ (3)

$L_{,\lambda} = x_1 + x_2 - 2 + \underset{\overset{\parallel}{0}}{S^2} = 0$ (4)

Case 1: $S = 0$ ✓

Solve (2)-(4)

$x_1^* = 1,\ x_2^* = 1,\ \underset{\geq 0}{\underline{\lambda^* = 1}},\ S = 0$

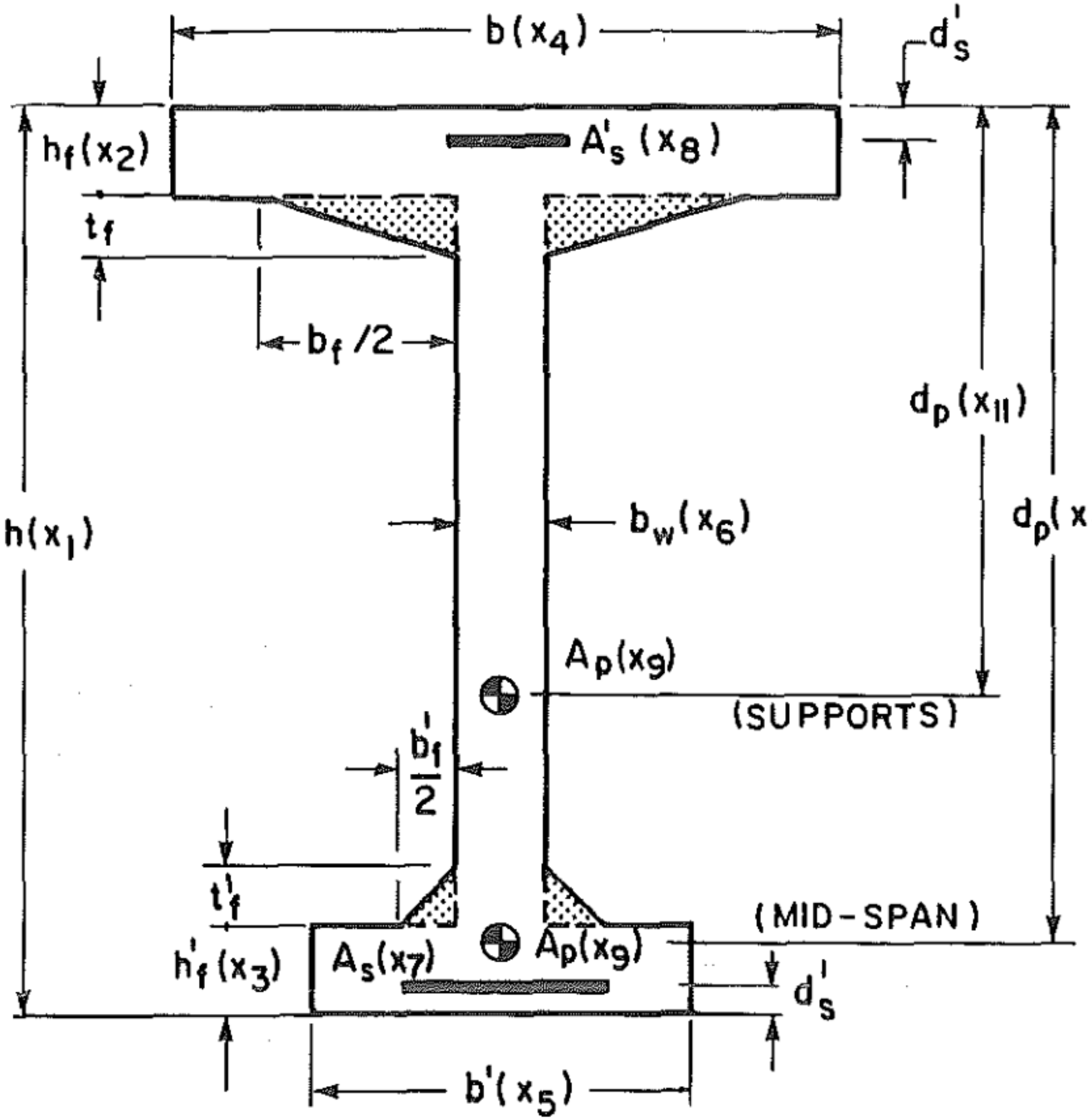
~~Case~~ 2: $\lambda = 0$, Solve (2)-(4)

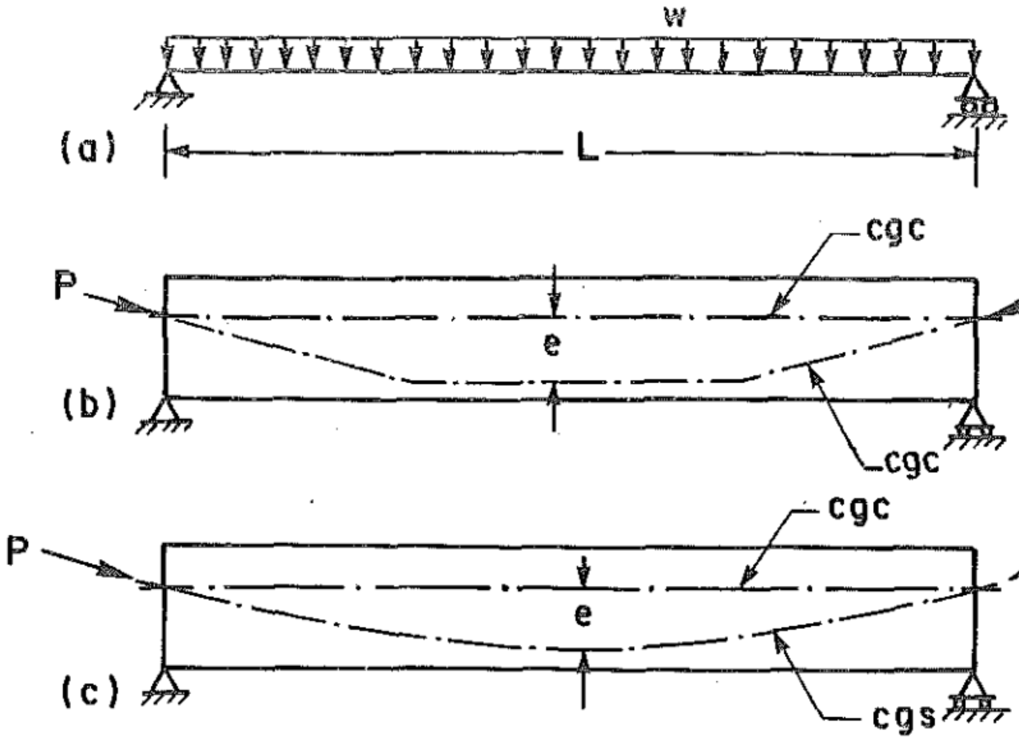
$x_1^* = 1.5,\ x_2^* = 1.5,\ \boxed{S^2 = -1}$ ✗

Structural Design Optimization of Concrete Beams

Problem Setting

Design Variables and Objective Functions

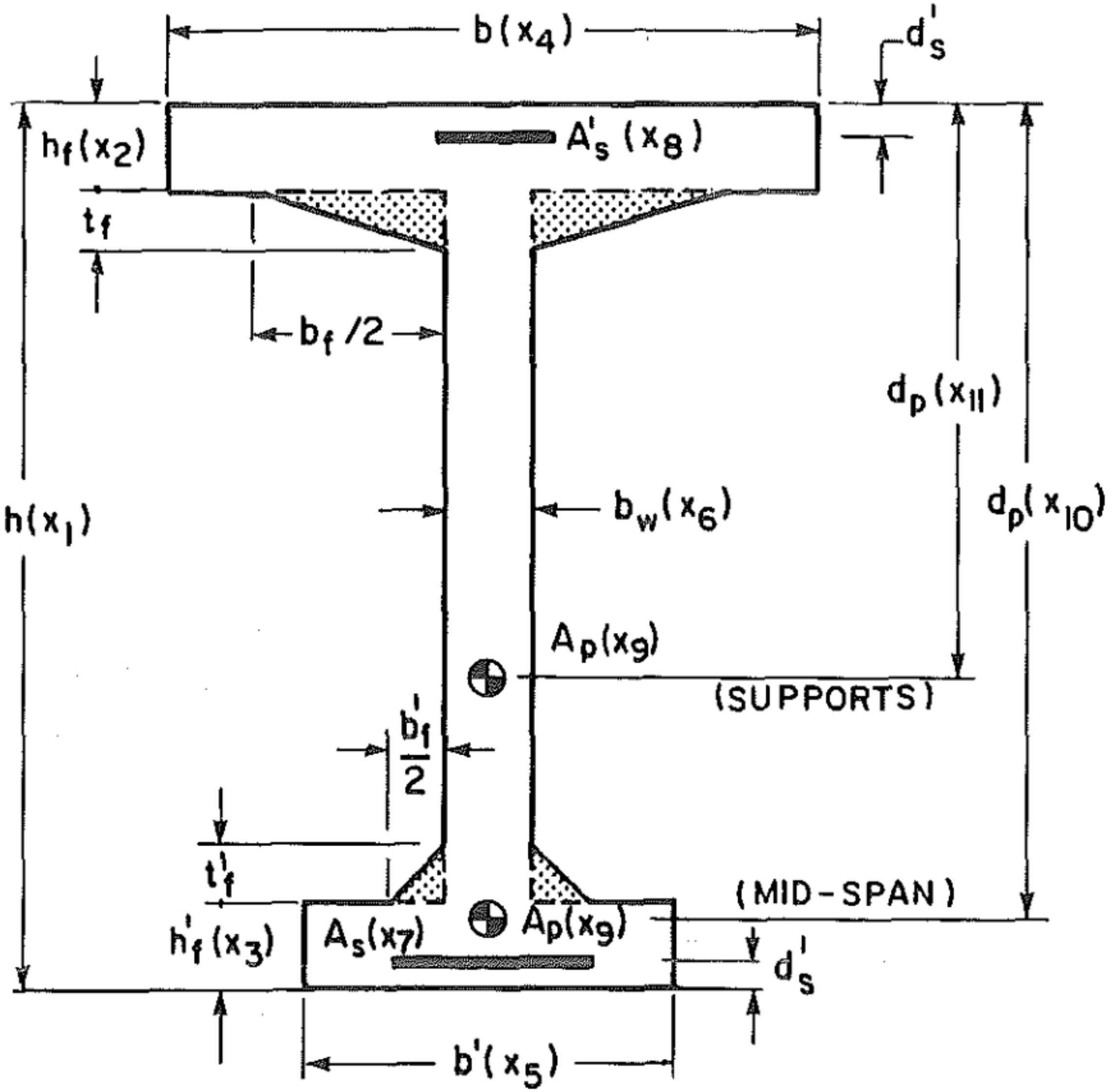




Colin, M. Z., & MacRae, A. J. (1984). Optimization of structural concrete beams. *Journal of structural engineering*, *110*(7), 1573-1588.

Problem Setting

Design Variables and Objective Functions



Design Variables (D.Vs):

- Geometry: x_1-x_6,
- Reinforcement area: x_7-x_9
- Reinforcement locations: x_{10}-x_{11}

Chosen based on your needs!

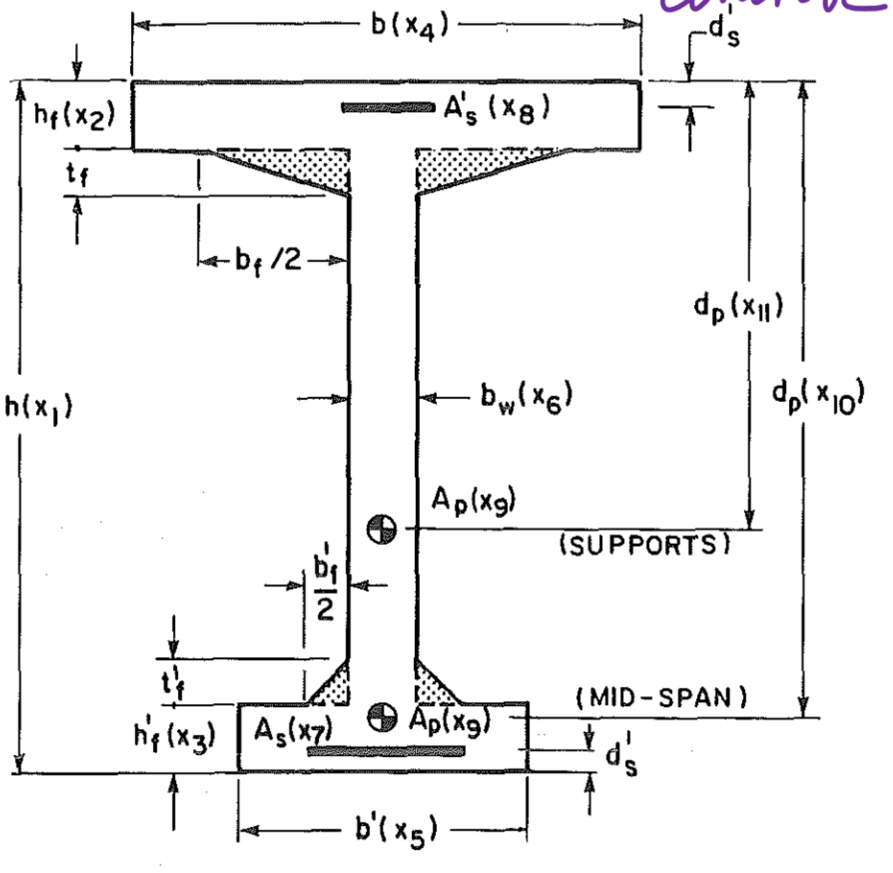
Problem Setting
Design Variables and Objective Functions

Objective Function:

formwork

$$\frac{C}{L} = \underbrace{w_c A_c C_c}_{\text{concrete}} + \underbrace{w_s (A_s + A'_s) C_s}_{\text{Steel}} + \underbrace{w_p A_p C_p}_{\text{prestressing strand.}} + p_f c_f + \frac{C_v}{L}$$

transverse reinforment



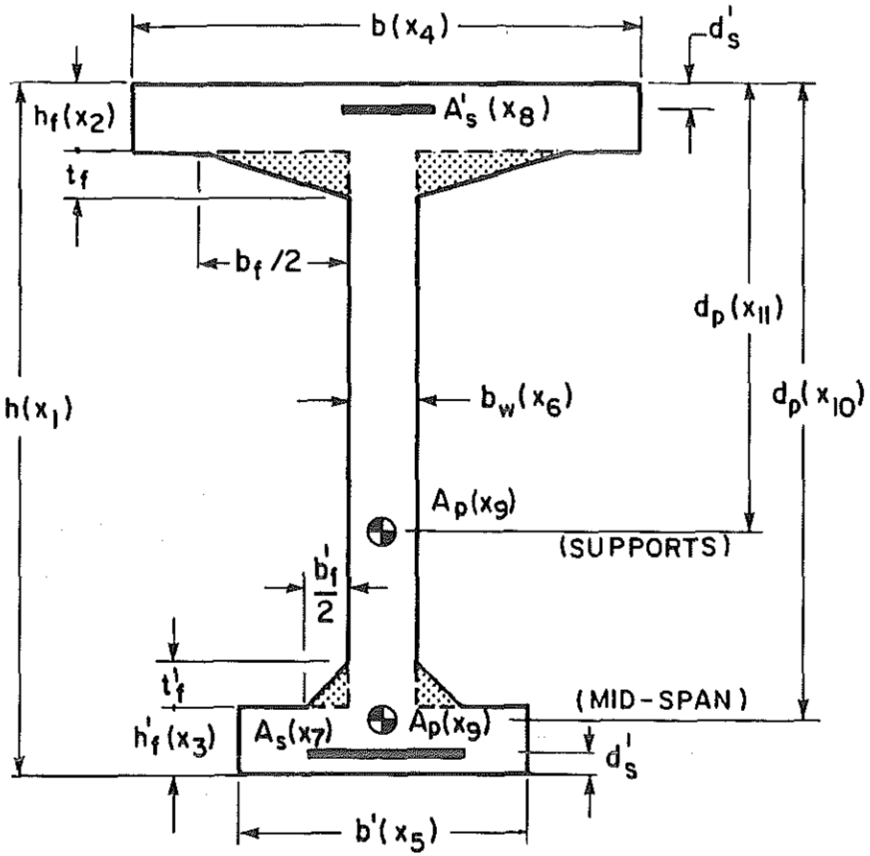
- w, A, C=unit weight, area, unit costs
- subscripts c,s,p,f refer to concrete, steel, prestressing formwork
- p_f=perimeter of the formwork
- C_v=total cost of the transverse reinforcement

Problem Setting

Design Variables and Objective Functions

Objective Function:

$$\frac{C}{L} = w_c A_c C_c + w_s (A_s + A'_s) C_s + w_p A_p C_p + p_f c_f + \frac{C_v}{L}$$



Variations of the objective function

- *Minimum member costs:* C_c, C_s, C_p, $C_f \neq 0$
- *Minimum weight:* $C_c \neq 0$; C_s, C_p, $C_f = 0$
- *Minimum reinforcement:* steel $C_s \neq 0$, $C_c = C_p = C_f = 0$
- *Minimum prestressing:* $\rightarrow C_p \neq 0$, else $= 0$

Problem Setting
Design Constraints

Formulate and simplify according to needs and design codes.

TABLE 1.—Constraint Types for Optimal Design

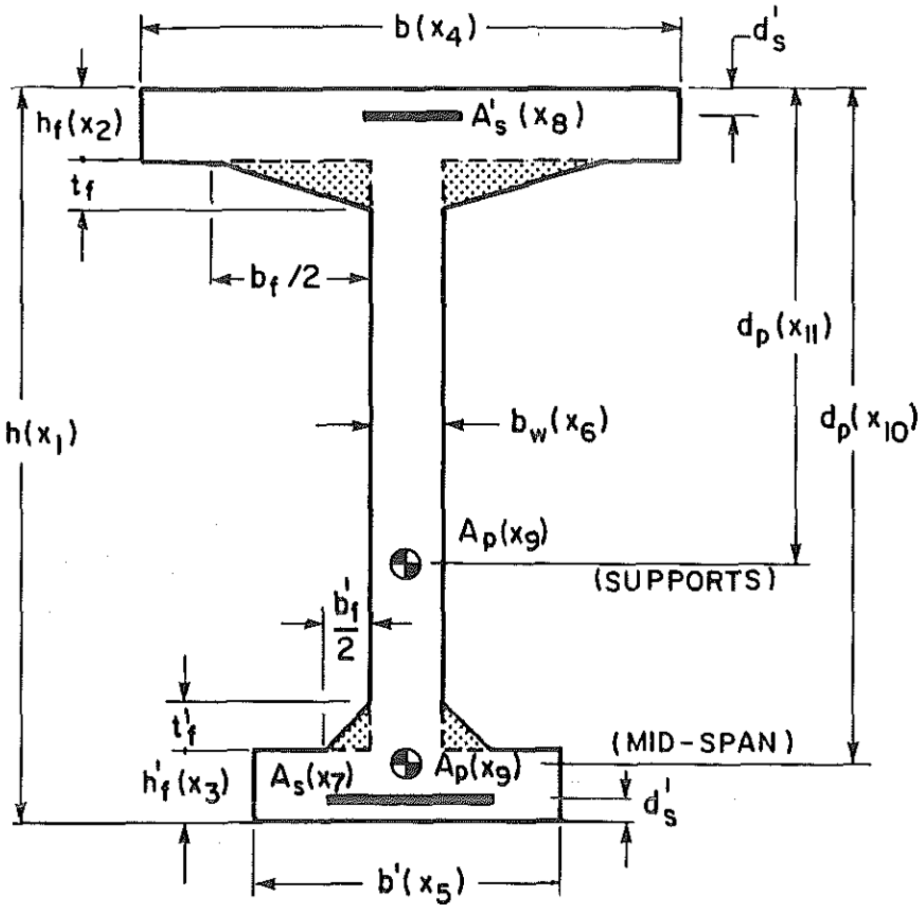
Constraint type (1)	Particulars (2)	Load level (3)	Design Solution RC (4)	PC (5)	PPC (6)
Limit state = SLS (service)					
Effective stresses	Concrete	Transfer		•	•
		Operating			•
		Service		•	•
		Fatigue			•
	Steel A_s	Service			•
		Fatigue			•
	Steel A_p	Transfer		•	•
		Service		•	•
		Fatigue			•
Deflection	Camber	Erection		•	•
	Inst. live load	Service	•	•	•
	Add. long term	Service	•	•	•
Crack width	—	Service			•
Limit state = ULS (ultimate)					
Flexural strength			•	•	•
Ductility factor		Ultimate		•	•
Reinforcement index			•	•	•
Minimum reinforcement	Top, A_s'	Service		•	•
	Bottom, A_s	Transfer		•	•
	Top, A_s'	Construction		•	•
	Bottom, A_s			•	•

Problem Setting
Design Constraints

- Flexural strength

$$\frac{M_u}{\phi M_n} - 1 \leqslant 0$$

  - M_u from factored load combinations: dead loads, live loads, and other effects (creep, shrinkage, etc.)
  - M_n may be simplified and estimated using:



Problem Setting
Design Constraints

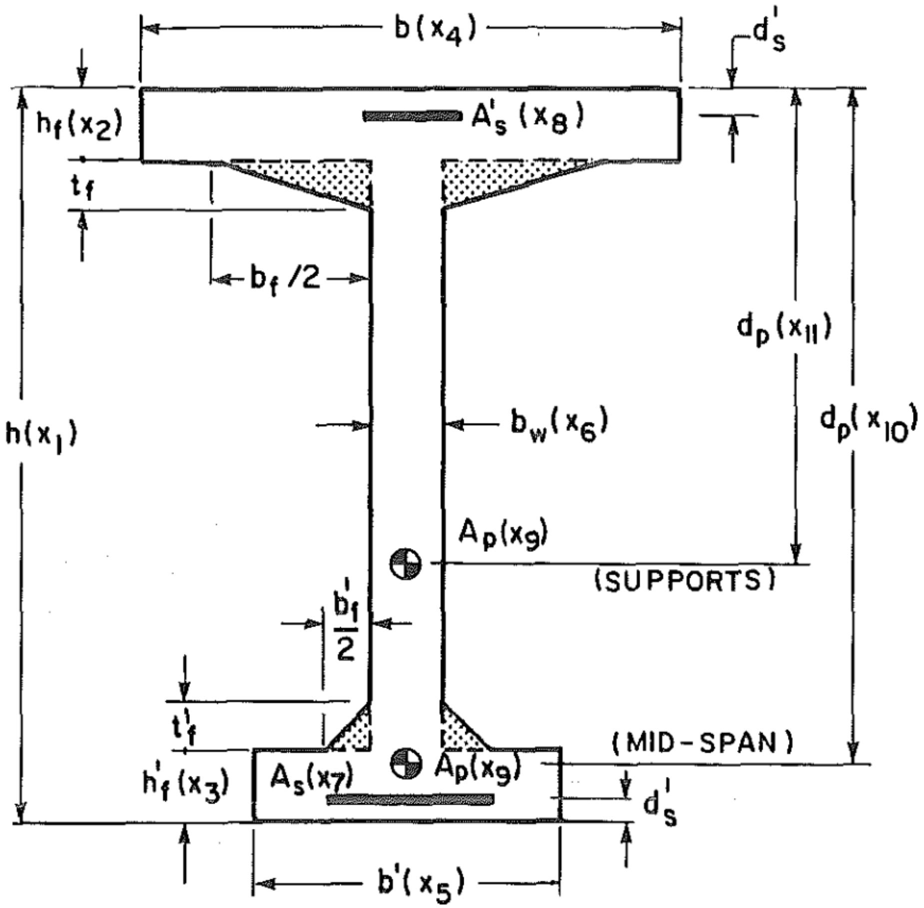
- Shear Strength

$$\frac{V_u}{\phi(V_c + V_s)} - 1 \leqslant 0$$

  - V_u from factored load combinations: dead loads, live loads, and other effects (creep, shrinkage, etc.)

$$V_{ci} = 0.17\lambda\sqrt{f_c'}b_w d$$

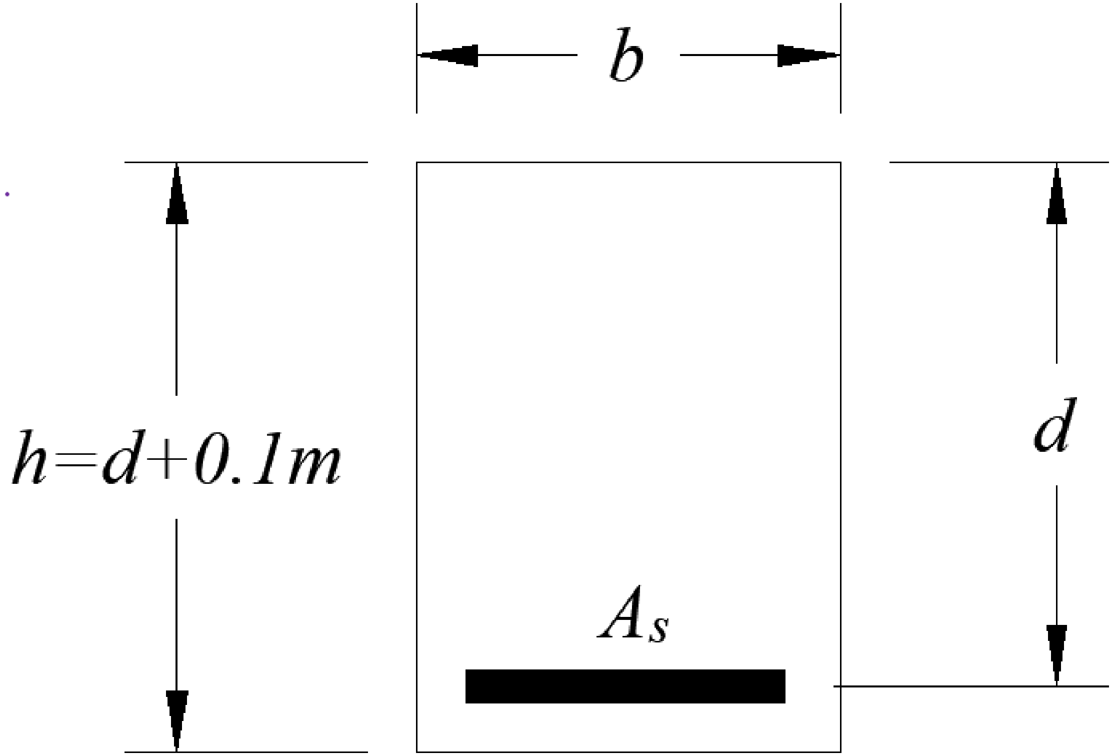
$$V_s = \frac{A_v f_{yt} d}{s}$$



Exercise

Minimize the costs of a singly-reinforced concrete beam with a rectangle cross-section. Consider the costs from material and formwork. The beam needs to resist a bending moment of 500 kN-m.

- Concrete costs: \$200/m^3
- Steel costs: \$7770/m^3
- Formwork: \$14/m^2
  (including all surface area)
- Steel yield strength: 400 MPa
- Concrete compressive strength: 40 MPa



Problem Setting
Formulations

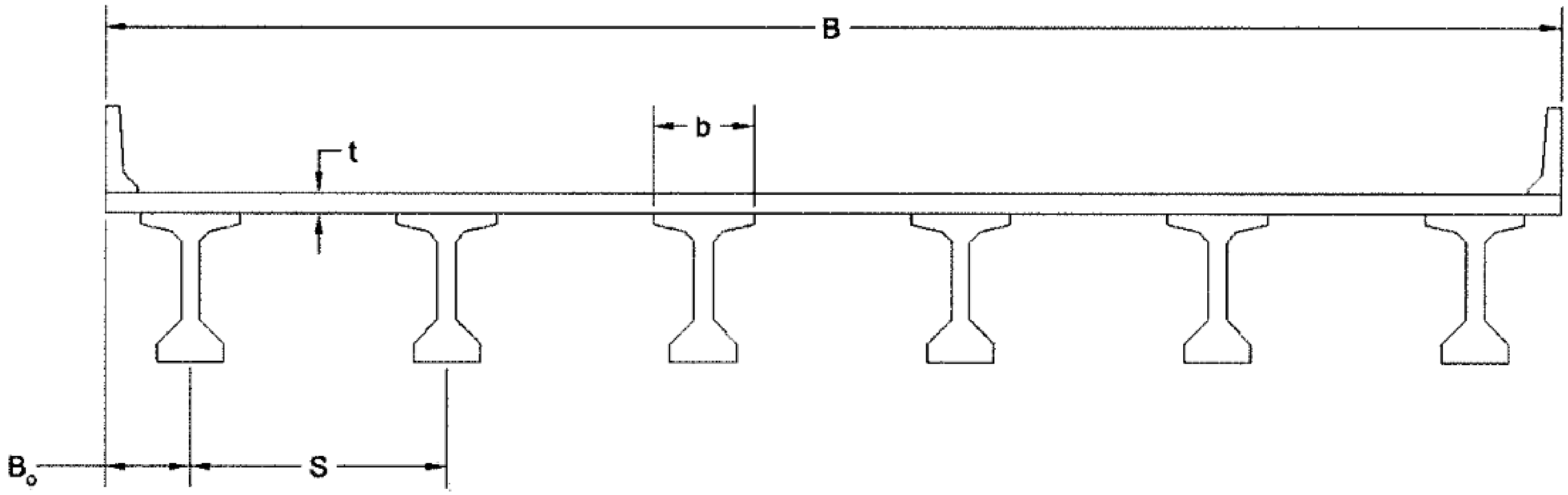
$$\text{minimize} \quad f(\mathbf{x})$$

$$\text{subject to} \quad g_i(\mathbf{x}) \leqslant 0, \quad i = 1 \ldots m$$

$$x_i \in D_i \quad D_i(d_{i1}, d_{i2}, \ldots, d_{in_i}), \quad i = 1 \ldots n_d$$

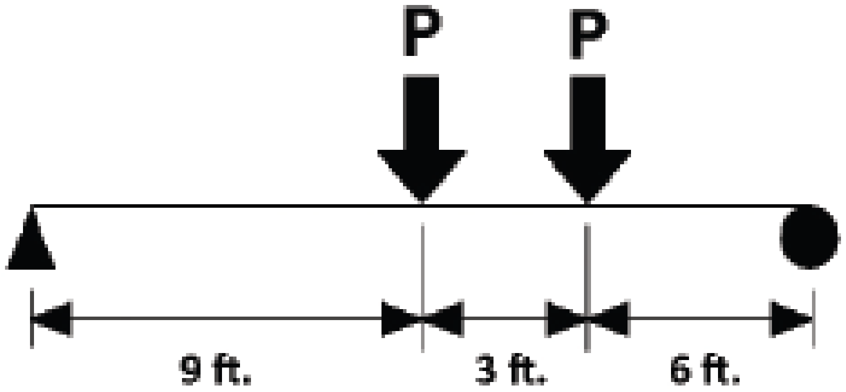
Problem Setting
Various extensions

- One girder to entire bridge
- Consider compressive strength as variable
- Include carbon footprint
- Etc.



One possible project opportunity
PCI Big Beam Competition

1. **Design accuracy**. The beam must carry at least a total factored live load of 32 kips and must not have a total peak applied load of more than 40 kips. The beam shall not crack under the total applied service load of 20 kips. Total applied load is defined as the sum of the two applied point loads. Beams meeting these criteria receive 20 points.
   a. Beams which do NOT hold a total applied load of 32 kip shall be penalized 2 points for each kip, or part of a kip, below 32 kip.
   b. Beams which hold a total applied load of more than 40 kip shall be penalized 1 point for each kip, or part of a kip, above 40 kip.
   c. Beams which crack before a total applied load of 20 kip receive a 5 point penalty.
   d. The load/midspan deflection graph must show a peak load either by post-peak softening or by collapse of the beam. Stopping the test to avoid the overstrength penalty will result in a score of 0 for this category.
2. Lowest cost (calculation must be provided).
3. Lowest weight.



One possible project opportunity
PCI Big Beam Competition

MATERIAL COSTS AND BEAM WEIGHT

The following unit cost shall be used to determine the beam cost. Concrete cost is based on actual strength, not design strength.

Material	Cost	Notes/Instructions
Concrete Cost (yd^3):	\$145/cubic yard (6ksi mix) ≤ \$20 + \$10 (concrete strength ksi) ≤ \$200	Round concrete strength down to nearest ksi
Ultra-High-Performance Concrete	\$400/yd^3	
Prestressing Strand:		Use estimated lengths used in the beam
⅜ in. diameter	\$0.27/ft	
½ in. diameter	\$0.30/ft	
½ in. special	\$0.33/ft	
0.6 in. diameter	\$0.42/ft	
0.7 in. diameter	\$0.55/ft	
Steel:		Use estimated lengths and nominal unit weights in this calculation as provided in the *PCI Design Handbook*
A615/A706	\$0.45/lb	
Welded Wire (deformed or smooth; for shear)	\$0.60/lb	
Epoxy Coated	\$0.50/lb	
A1035	\$0.70/lb	
Plate Steel	\$0.55/lb	
Forming	\$1.25/ft^2 of formwork (include all contact surfaces)	

- There is no need to include cost of steel fabrication, concrete fabrication, curing, inserts, etc. Concrete cost is based on actual strength.
- The beam weight shall be estimated by using the measured unit weight of the concrete or by actually weighing the beam. If the beam weight is estimated, it is estimated based on the gross concrete cross section only, ignoring reinforcement, bearing plates, etc. * Special circumstances or special materials not addressed in these rules must be reviewed by the chair of the committee and/or PCI staff.