

# Contents

## Part I Thermodynamics, Statistical Mechanical Models and Phase Transitions

<b>1</b>	<b>Thermodynamics</b>	5
1.1	Formulae and Variables	5
1.2	The Field-Density Representation	8
1.3	The Thermodynamic Limit	9
1.4	Particular Response Functions	10
1.4.1	Magnetic Systems	11
1.4.2	Fluid Systems	11
<b>2</b>	<b>Statistical Mechanics</b>	13
2.1	Distributions	13
2.1.1	Quantum Systems	14
2.1.2	The Connection to Thermodynamics	15
2.2	Variations of the Probability Function	16
2.3	Coupling Representations	17
2.3.1	The Case $n_f = 2$	19
2.4	Lattice Systems	21
2.4.1	Site-Variable Models	22
2.4.2	Edge-Variable Models	22
2.5	Correlation Functions and Symmetry Properties	23
2.5.1	A General Hamiltonian	23
2.5.2	Correlation Functions	24
2.5.3	Symmetry Properties	26
<b>3</b>	<b>A Survey of Models</b>	29
3.1	Upper and Lower Critical Dimensions	29
3.2	The Quantum Heisenberg Model	30
3.2.1	One-Dimensional Chains	31

3.3	Classical Vector Models . . . . .	33
3.4	The Gaussian and Spherical Models . . . . .	34
3.5	Ising Models . . . . .	36
3.5.1	The Spin- $\frac{1}{2}$ Ising Model . . . . .	36
3.5.2	The Spin-1 Ising Model. . . . .	44
3.6	State-Difference Models. . . . .	45
3.6.1	The Classical XY Model . . . . .	45
3.6.2	The Ashkin–Teller Model . . . . .	53
3.6.3	Potts Models . . . . .	54
3.6.4	The Standard Potts Model . . . . .	55
3.7	Chirality . . . . .	57
3.7.1	Chiral Potts Models . . . . .	58
3.7.2	An Extended 3-State Potts Model on the Triangular Lattice . . . . .	59
3.8	Vertex Models . . . . .	60
3.8.1	The Eight-Vertex Model . . . . .	61
3.8.2	The Six-Vertex Model. . . . .	72
3.9	Dimer Models . . . . .	80
3.9.1	The Modified KDP Model Equivalence . . . . .	83
3.9.2	The Ising Model Equivalence. . . . .	85
<b>4</b>	<b>Phase Transitions and Scaling Theory . . . . .</b>	<b>89</b>
4.1	The Geometry of Phase Transitions. . . . .	89
4.1.1	A Two-Dimensional Phase Space . . . . .	90
4.1.2	A Three-Dimensional Phase Space . . . . .	93
4.2	Universality, Fluctuations and Scaling . . . . .	94
4.2.1	Universality . . . . .	95
4.2.2	Scaling for the Ising Model . . . . .	97
4.3	General Scaling Formulation . . . . .	100
4.3.1	The Kadanoff Scaling Hypothesis . . . . .	100
4.3.2	First-Order Transitions. . . . .	104
4.3.3	Effective Exponents . . . . .	105
4.3.4	The Nightingale–’T Hooft Scaling Hypothesis . . . . .	109
4.3.5	Constraints on Scaling. . . . .	110
4.3.6	Scaling Operators and Dimensions . . . . .	115
4.3.7	Correlation Functions . . . . .	118
4.3.8	Variable Scaling Exponents . . . . .	119
4.3.9	Densities and Response Functions. . . . .	121
4.4	Critical Point and Coexistence Curve . . . . .	122
4.4.1	Critical Exponents. . . . .	124
4.4.2	Exponent Inequalities . . . . .	125
4.5	Scaling for a Critical Point. . . . .	125
4.5.1	Scaling Fields for the Critical Point. . . . .	126
4.5.2	Approaches to the Critical Point . . . . .	128

4.5.3	Experimental Variables . . . . .	129
4.5.4	The Density and Response Functions . . . . .	130
4.5.5	Asymptotic Forms. . . . .	131
4.5.6	Critical Exponents and Scaling Laws. . . . .	132
4.5.7	Correlation Scaling at a Critical Point . . . . .	134
4.6	Tricritical Point . . . . .	136
4.7	Scaling for a Tricritical Point . . . . .	141
4.7.1	Scaling Fields for the Tricritical Point . . . . .	141
4.7.2	Tricritical Exponents and Scaling Laws . . . . .	143
4.8	Corrections to Scaling . . . . .	146
4.9	Scaling and Universality . . . . .	148
4.10	Finite-Size Scaling . . . . .	152
4.10.1	The Finite-Size Scaling Field . . . . .	153
4.10.2	The Shift and Rounding Exponents. . . . .	155
4.10.3	Universality and Finite-Size Scaling . . . . .	157
4.11	Conformal Invariance . . . . .	159
4.11.1	From Scaling to the Conformal Group. . . . .	159
4.11.2	Correlation Functions for $d \geq 2$ . . . . .	159
4.11.3	Universal Amplitudes for $d = 2$ . . . . .	161
4.11.4	Schramm–Loewner Evolution. . . . .	163

## Part II Classical Approximation Methods

<b>5</b>	<b>Phenomenological Theory and Landau Expansions . . . . .</b>	<b>169</b>
5.1	Classical Methods. . . . .	169
5.1.1	A First-Order Transition . . . . .	171
5.1.2	Metastability . . . . .	174
5.2	The Van der Waals Equation . . . . .	175
5.3	Landau Expansions with One Order Parameter. . . . .	176
5.3.1	The Spin- $\frac{1}{2}$ Ising Model. . . . .	182
5.4	Landau Expansions with Two Order Parameters . . . . .	183
5.4.1	The Spin-1 Ising Model. . . . .	183
5.4.2	The 3-State Potts Model . . . . .	184
5.5	Landau Theory for a Tricritical Point . . . . .	188
5.5.1	Tricritical Exponents . . . . .	190
5.6	Ginzburg–Landau Theory . . . . .	193
5.6.1	A Critical Point . . . . .	193
5.6.2	The Gaussian Approximation . . . . .	194
5.6.3	Gaussian Critical Exponents. . . . .	197
<b>6</b>	<b>Mean-Field Theory . . . . .</b>	<b>205</b>
6.1	The Ising Ferromagnet. . . . .	205
6.1.1	Mean-Field Fluctuations . . . . .	209

6.2	A Model for Metamagnetism . . . . .	212
6.2.1	The Paramagnetic State . . . . .	215
6.2.2	The Antiferromagnetic State . . . . .	216
6.2.3	A Neighbourhood of the Critical Curve . . . . .	217
6.2.4	The First-Neighbour Antiferromagnet: $\lambda = 0$ . . . . .	221
6.2.5	The First-Order Transition . . . . .	223
6.2.6	A Neighbourhood of the Tricritical Point . . . . .	226
<b>7</b>	<b>Cluster-Variation Methods . . . . .</b>	<b>229</b>
7.1	Improving Mean-Field Theory . . . . .	229
7.2	The KHDeB Hierarchy of Approximations . . . . .	231
7.2.1	Distribution Numbers . . . . .	231
7.2.2	Extensive Quantities . . . . .	233
7.2.3	The Hamiltonian and Free Energy . . . . .	235
7.2.4	The Entropy . . . . .	235
7.2.5	Minimization . . . . .	237
7.2.6	Labelling Configurations . . . . .	238
7.3	The Bethe-Pair Approximation for the Ising Model . . . . .	239
7.4	Reduction to the Mean-Field Approximation . . . . .	241
7.5	3-State Potts Model on a Triangular Lattice . . . . .	243
7.6	A Lattice Model for Fluid Water . . . . .	246

### Part III Exact Results

<b>8</b>	<b>Algebraic Methods . . . . .</b>	<b>259</b>
8.1	The Thermodynamic Limit . . . . .	259
8.2	The Infinite-System Approach . . . . .	261
8.3	Lower Bounds for Phase Transitions: The Peierls Method . . . . .	263
8.3.1	The Simple Lattice Fluid . . . . .	269
8.4	Grand Partition Function Zeros and Phase Transitions . . . . .	271
8.4.1	Ruelle's Theorem . . . . .	273
8.4.2	The Yang-Lee Circle Theorem . . . . .	276
8.4.3	Systems with Pair Interactions . . . . .	279
<b>9</b>	<b>Transformation Methods . . . . .</b>	<b>283</b>
9.1	Related Systems . . . . .	283
9.2	The Wegner Transformation . . . . .	284
9.2.1	Duality for the $\nu$ -State Potts Model . . . . .	286
9.2.2	Duality for the Spin- $\frac{1}{2}$ Ising Model . . . . .	290
9.2.3	The Weak-Graph Transformation . . . . .	291
9.3	The Regular Square-Lattice Eight-Vertex Model . . . . .	293
9.3.1	Symmetry Properties and Transformations . . . . .	294
9.3.2	The Case of Region I . . . . .	297
9.3.3	Regions and Variables . . . . .	301

9.4	The Star-Triangle Transformation . . . . .	304
9.4.1	The $\nu$ -State Potts Model . . . . .	306
9.4.2	The Spin- $\frac{1}{2}$ Ising Model . . . . .	307
<b>10</b>	<b>Edge-Decorated Ising Models . . . . .</b>	<b>311</b>
10.1	Primary and Secondary Sites . . . . .	311
10.2	Super-Exchange or Bond-Dilution. . . . .	312
10.2.1	Critical Properties and Exponent Renormalization. . . . .	314
10.3	A Ferrimagnet . . . . .	319
10.3.1	The Zero-Field Axis . . . . .	320
10.3.2	Non-Zero Field. . . . .	322
10.4	A Competing-Interaction Magnetic Model . . . . .	324
10.5	Decoration with Orientable Molecules . . . . .	326
10.6	A Decorated Lattice Fluid . . . . .	332
10.6.1	Case I: A Single Vapour/Liquid Transition. . . . .	335
10.6.2	Case II: A Water-Like Model. . . . .	337
10.6.3	Case III: Maxithermal, Critical Double and Cuspoidal Points. . . . .	339
<b>11</b>	<b>Transfer Matrices: Incipient Phase Transitions . . . . .</b>	<b>345</b>
11.1	The Transfer Matrix Formulation . . . . .	345
11.1.1	The Eigen Problem . . . . .	346
11.1.2	The Partition Function. . . . .	347
11.1.3	Correlation Functions and Lengths . . . . .	348
11.2	Incipient Phase Transitions. . . . .	353
11.3	Using Symmetry Properties . . . . .	354
11.3.1	Block-Diagonalization of the Transfer Matrix. . . . .	355
11.3.2	Applications. . . . .	359
11.4	Analysis in the Complex Plane: The Wood Method . . . . .	367
11.4.1	Evolution of Partition Function Zeros . . . . .	367
11.4.2	Connection Curves and Cross-Block Curves. . . . .	369
11.4.3	The Spin- $\frac{1}{2}$ Square-Lattice Ising Model . . . . .	372
11.4.4	Critical Points and Exponents. . . . .	376
<b>12</b>	<b>Transfer Matrices: Exactly Solved Models . . . . .</b>	<b>381</b>
12.1	A General Eight-Vertex Model. . . . .	381
12.1.1	A Generalized Star-Triangle Transformation. . . . .	382
12.1.2	The Solution to the GST Transformation and the Elliptic Variable Formulation . . . . .	383
12.1.3	Z-Invariance. . . . .	387
12.1.4	Edge Variables and Matrix Formulation. . . . .	391
12.1.5	Square-Lattice Models. . . . .	393

12.2	Square-Lattice Ising Models . . . . .	394
12.2.1	The Modified Checkerboard Ising Model . . . . .	399
12.2.2	Properties of the Transfer Matrices . . . . .	403
12.2.3	The Reduction to Regular Ising Models . . . . .	407
12.2.4	Transfer Matrix Eigenvectors . . . . .	409
12.2.5	Notational Changes . . . . .	410
12.2.6	Transfer Matrix Eigenvalues . . . . .	412
12.2.7	The Standard Model . . . . .	421
12.3	The Square-Lattice Eight-Vertex Model . . . . .	431
12.3.1	The Low-Temperature Zone $\mathcal{R}_L(\text{I})$ . . . . .	431
12.3.2	The Low-Temperature Zone $\mathcal{R}_L(\text{III})$ . . . . .	433
12.3.3	The Transfer Matrix . . . . .	434
12.3.4	Analysis in Terms of Pauli Matrices . . . . .	437
12.3.5	Analysis of the Transfer Matrix . . . . .	442
12.3.6	The VQ Equation . . . . .	447
12.3.7	The Free Energy and Magnetization . . . . .	475
12.3.8	Critical Behaviour . . . . .	478
12.3.9	The Coupling Form and the Ising Model Limit . . . . .	482
12.3.10	The Six-Vertex Model . . . . .	485
12.3.11	The Eight-Vertex Model and Universality . . . . .	491
<b>13</b>	<b>Dimer Models . . . . .</b>	<b>495</b>
13.1	The Dimer Partition Function . . . . .	495
13.2	Superposition Polynomials and Pfaffians . . . . .	496
13.2.1	The Square-Lattice Case . . . . .	500
13.2.2	The Honeycomb-Lattice Case . . . . .	504
13.3	Vertex and Ising Model Equivalences . . . . .	510
13.3.1	The Five-Vertex Model . . . . .	510
13.3.2	The Honeycomb-Lattice Anisotropic Ising Model . . . . .	512
13.4	K-Type and O-Type Transitions . . . . .	514

## Part IV Series and Renormalization Group Methods

<b>14</b>	<b>Series Expansions . . . . .</b>	<b>521</b>
14.1	The Task and the Methods . . . . .	521
14.2	Moment Expansions . . . . .	524
14.2.1	At Low Temperatures . . . . .	525
14.2.2	At High Temperatures . . . . .	530
14.2.3	Duality for Graphs . . . . .	534
14.3	Cumulant Expansions . . . . .	536
14.3.1	The Low-Temperature Case . . . . .	539
14.3.2	The High-Temperature Case . . . . .	539
14.4	The Finite-Cluster Method . . . . .	540

14.5	The Finite-Lattice Method . . . . .	543
14.5.1	Block-Formation and Accuracy. . . . .	544
14.5.2	Constructing Block Partition Functions . . . . .	548
14.5.3	Calculating the Series . . . . .	551
14.6	The Analysis of Series: Second-Order Transitions. . . . .	555
14.6.1	Late-Term Analysis. . . . .	556
14.6.2	The Ratio Method. . . . .	557
14.6.3	Padé Approximants . . . . .	559
14.6.4	Differential and Algebraic Approximants . . . . .	562
14.7	The Analysis of Series: First-Order Transitions. . . . .	565
<b>15</b>	<b>Real-Space Renormalization Group Theory . . . . .</b>	<b>567</b>
15.1	The Basic Elements of the Renormalization Group . . . . .	567
15.2	RG Transformations and Weight Functions . . . . .	570
15.3	Fixed Points and the Linear Renormalization Group . . . . .	574
15.4	Free Energy and Densities . . . . .	577
15.5	Decimation for the Ising Model . . . . .	579
15.5.1	In One Dimension . . . . .	579
15.5.2	In Two Dimensions. . . . .	585
15.6	The Kosterlitz–Thouless Transition . . . . .	588
15.7	Upper-Bound and Lower-Bound Approximations . . . . .	594
15.7.1	An Upper-Bound Method . . . . .	595
15.7.2	A Lower-Bound Method . . . . .	599
15.8	Finite-Lattice Approximations. . . . .	603
15.9	Variational Approximations . . . . .	607
15.10	Phenomenological Renormalization. . . . .	609
15.10.1	The Square-Lattice Ising Model . . . . .	611
15.10.2	Other Models . . . . .	613
15.10.3	More Than One Coupling . . . . .	614
15.11	Other Renormalization Group Methods . . . . .	615

## Part V Mathematical Appendices

<b>16</b>	<b>Graphs and Lattices . . . . .</b>	<b>619</b>
16.1	Graphs . . . . .	619
16.1.1	Introduction . . . . .	619
16.1.2	The Cyclomatic Number . . . . .	621
16.1.3	Triangulation of Graphs. . . . .	622
16.1.4	Oriented Graphs . . . . .	622
16.1.5	The Dual Graph . . . . .	623
16.2	Lattices . . . . .	624
16.2.1	Types of Regular Lattices . . . . .	624
16.2.2	Lattice Transformations . . . . .	628
16.3	Rapidity Graphs and Lattices . . . . .	633

16.4	Lattice Graphs . . . . .	637
16.4.1	Augmented Graphs and the Whitney Polynomial . . . .	638
16.4.2	Hopping Matrices and the Canonical Flux Distribution . . . . .	638
16.4.3	Embeddings and Topologies. . . . .	639
16.4.4	Lattice Constants . . . . .	640
16.4.5	Partially-Ordered Sequences of Graphs and the T Matrix . . . . .	644
16.4.6	Generating the Partially-Ordered Sequence. . . . .	647
16.4.7	Incorporating Sublattices . . . . .	651
16.4.8	The Guggenheim–McGlashan Approach . . . . .	654
16.4.9	Further Results . . . . .	656
<b>17</b>	<b>Algebra . . . . .</b>	<b>659</b>
17.1	Catastrophe Theory . . . . .	659
17.1.1	Equivalence and Determinancy . . . . .	659
17.1.2	Critical Points, Codimension and Unfoldings . . . . .	662
17.1.3	Symmetry Considerations. . . . .	668
17.2	Matrix Algebra . . . . .	670
17.2.1	Diagonalizability. . . . .	671
17.2.2	Commutativity . . . . .	672
17.2.3	Reducibility . . . . .	673
17.2.4	Theorems of Perron and Frobenius . . . . .	673
17.2.5	Direct Products and Traces. . . . .	675
17.2.6	Defective Matrices . . . . .	675
17.2.7	Groups of Matrices . . . . .	676
17.3	Groups and Representations . . . . .	676
17.3.1	Representations. . . . .	678
17.3.2	Permutation Representations and Equivalence Classes . . . . .	682
17.3.3	Block Diagonalization Within an Equivalence Class. . .	684
17.3.4	Symmetry Groups. . . . .	687
17.4	The Conformal Group . . . . .	691
17.5	Some Transformations in the Complex Plane . . . . .	693
17.6	Algebraic Functions . . . . .	695
17.7	Determinants of Cyclic Matrices. . . . .	700
<b>18</b>	<b>Analysis . . . . .</b>	<b>703</b>
18.1	Fourier Transforms in $d$ Dimensions . . . . .	703
18.1.1	Discrete Finite Lattices . . . . .	703
18.1.2	A Continuous Finite Volume . . . . .	705
18.1.3	A Continuous Infinite Volume . . . . .	707
18.1.4	Integrals Involving Bessel Functions . . . . .	708
18.1.5	Lattice Green's Functions . . . . .	710



18.2	Doubly-Periodic and Quasi-Periodic Functions . . . . .	711
18.3	Elliptic Integrals and Functions. . . . .	714
18.3.1	Elliptic Integrals . . . . .	714
18.3.2	Jacobi Theta Functions . . . . .	717
18.3.3	Jacobi Elliptic Functions . . . . .	720
18.3.4	Transformations in the Elliptic Modulus . . . . .	724
18.3.5	The Modified Amplitude Function . . . . .	726
18.3.6	Nome Series. . . . .	727
18.3.7	Special Results and Functions for Chap. 12 . . . . .	728
18.3.8	Baxter's Modified Theta Functions . . . . .	731
18.4	The Potts Delta Function . . . . .	737
18.4.1	The $\mu = 0$ Case . . . . .	740
18.4.2	The $\mu \neq 0$ Case . . . . .	742
18.5	Padé, Differential and Algebraic Approximants. . . . .	743
18.5.1	Padé Approximants . . . . .	743
18.5.2	Dlog Padé Approximants . . . . .	748
18.5.3	Differential Approximants . . . . .	750
18.5.4	Algebraic Approximants . . . . .	754
	<b>References and Author Index. . . . .</b>	<b>757</b>
	<b>Index . . . . .</b>	<b>783</b>

Equilibrium Statistical Mechanics of Lattice Models

Lavis, D.

2015, XVII, 793 p. 101 illus., Hardcover

ISBN: 978-94-017-9429-9