King's College London

Department of Mathematics

Course CM355Z

NON-LINEAR DYNAMICS

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Contents

Chapter 1

Dynamic Systems

1.1 What is a Dynamic System?

Consider a particle of constant mass m moving on a line so that at time t it is at a point P_t at a distance $x(t)$ from a point O, (Fig. 1.1). Suppose that a force

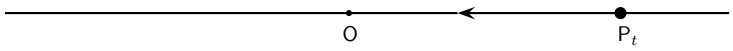


Figure 1.1: A particle moving in simple harmonic motion on a line.

$\boldsymbol{F} = \kappa\overrightarrow{\mathsf{P}_t\mathsf{O}}$ ($\kappa > 0$) is acting on the particle. Then according to *Newton's second law* the equation of motion of the particle is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\kappa x. \tag{1.1}$$

The behaviour of the particle when governed by this equation is called *simple harmonic motion*.

When convenient we shall use the 'dot' notation to signify differentiation with respect to time.[1] Thus

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}(t), \qquad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \ddot{x}(t)$$

and the convenient forms for (1.1) are now the one second-order equation

$$\ddot{x}(t) + \omega^2 x(t) = 0, \tag{1.2}$$

or the pair of coupled first-order equations

$$\dot{x}(t) = v(t), \qquad \dot{v}(t) = -\omega^2 x(t), \tag{1.3}$$

where $\omega^2 = \kappa/m$ and $v(t)$ is the velocity of the particle. This is a simple case

[1] For derivatives of higher than second order this notation becomes cumbersome and will not be used.

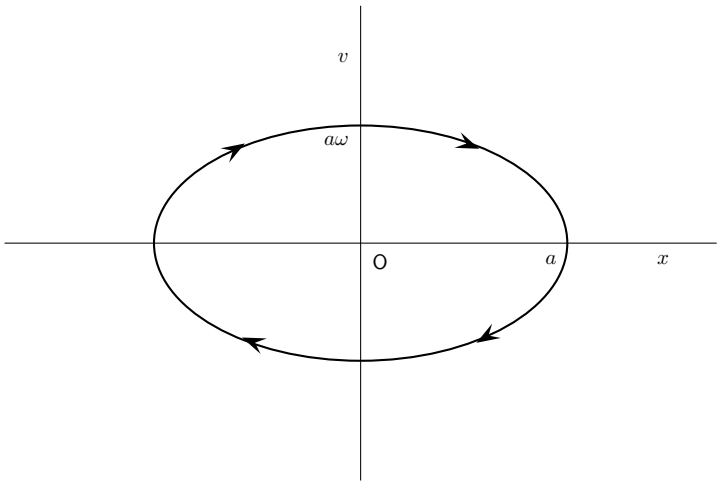


Figure 1.2: The trajectory in phase space for a particle moving with simple harmonic motion.

of a *dynamic system* with two degrees of freedom (x, v). Given that the state of the system at some time $t = 0$ is given by $(x_0, v_0) = (x(0), v(0))$, then the state $(x(t), v(t))$ at time t will be given by solving (1.2) (or equivalently (1.3)). The the set of states for all t will be represented by a path or *trajectory* parameterized by t in the *phase space* Γ_2 of the variables (x, v).

The auxiliary equation for (1.2) is

$$\lambda^2 + \omega^2 = 0, \qquad \text{with roots} \qquad \lambda = \pm \mathrm{i}\omega. \tag{1.4}$$

with solution

$$x(t) = \mathrm{A}\cos(\omega t) + \mathrm{B}\sin(\omega t). \tag{1.5}$$

The motion is *periodic with angular frequency* ω. The period T is the time for it to perform one complete cycle. This is given by $\omega(t + T) = \omega t + 2\pi$. So $T = 2\pi/\omega$.

If the initial conditions are $x(0) = a$, $\dot{x}(0) = 0$, then the solution becomes

$$x(t) = a\cos(\omega t), \quad \text{with the velocity} \quad v(t) = -a\omega\sin(\omega t). \tag{1.6}$$

The particle oscillates about the origin. In the phase space Γ_2 its path is the ellipse

$$\frac{x^2}{a^2} + \frac{v^2}{(a\omega)^2} = 1, \tag{1.7}$$

with motion in the clockwise direction (Fig. 1.2). The time for the phase point (x, v) to pass around the ellipse once is the period T.

We shall now give some more general definitions:

*A **dynamic system** with d degrees of freedom is a set of d variables $x_1, x_2, \ldots, x_d$ (usually, but not always with some particular significance) together with a set of equations which give a deterministic mathematical prescription for the evolution of the variables with time t. The evolution of the **state** $(x_1(t), x_2(t), \ldots, x_d(t))$ of the system with time is given by a **trajectory** in the d-dimensional **phase space** Γ_d of the variables.*

We shall be concerned mainly with systems governed by first-order ordinary differential equations. The standard form for a system with d degrees of freedom is then

$$\begin{aligned} \dot{x}_1(t) &= F_1(x_1, x_2, \ldots, x_d; t), \\ \dot{x}_2(t) &= F_2(x_1, x_2, \ldots, x_d; t), \\ &\vdots \quad \vdots \\ \dot{x}_d(t) &= F_d(x_1, x_2, \ldots, x_d; t). \end{aligned} \tag{1.8}$$

Equations (1.3) are of this type with $d = 2$. It is often convenient to express equations (1.8) in vector form as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{x}; t), \tag{1.9}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)^{\mathrm{T}}$ is an d-dimensional column vector[2] in the phase space Γ_d and $\boldsymbol{F}$ is a family of vector fields on Γ_d parameterized by t.

*A dynamic system of the type (1.8) is **non-linear** if one or more of the functions $F_1, F_2, \ldots, F_d$ is non-linear in one or more of the variables $x_1, \ldots, x_d$.*

1.2 Hamiltonian Systems

*A dynamic system with $2d$ degrees of freedom and variables $x_1, \ldots, x_d, p_1, \ldots, p_d$ is a **Hamiltonian system** if there exists a Hamiltonian function $H(x_1, \ldots, x_d, p_1, \ldots, p_d; t)$ and the evolution is given by the equations*

$$\begin{aligned} \dot{x}_\ell(t) &= \frac{\partial H}{\partial p_\ell}, \\ \dot{p}_\ell(t) &= -\frac{\partial H}{\partial x_\ell}, \end{aligned} \qquad \ell = 1, 2 \ldots, d. \tag{1.10}$$

[2] Our default notation for vectors will be in column form. A superscript 'T' (meaning *transpose*) is used to translate between row and column forms.

From (1.10), the rate of change of H along a trajectory is given by

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \sum_{\ell=1}^{d}\left\{\frac{\partial H}{\partial x_\ell}\dot{x}_\ell(t)+\frac{\partial H}{\partial p_\ell}\dot{p}_\ell(t)\right\}+\frac{\partial H}{\partial t},\\ &= \sum_{\ell=1}^{d}\left\{\frac{\partial H}{\partial x_\ell}\frac{\partial H}{\partial p_\ell}-\frac{\partial H}{\partial p_\ell}\frac{\partial H}{\partial x_\ell}\right\}+\frac{\partial H}{\partial t},\\ &= \frac{\partial H}{\partial t}. \end{aligned} \tag{1.11}$$

If the system is autonomous ($\partial H/\partial t = 0$, see Sect. 1.5) the value of H does not change along a trajectory. It is said to be a *constant of motion*. In the case of many physical systems the value of the Hamiltonian is the total energy the system.

1.3 Conservative Systems

As we have already seen in the case $d = 2$, a system with d variables $x_1, x_2, \ldots, x_d$ determined by second-order differential equations, given in vector form by

$$\ddot{\boldsymbol{x}}(t) = \boldsymbol{G}(\boldsymbol{x}; t), \tag{1.12}$$

where

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_d(t) \end{pmatrix}, \qquad \boldsymbol{G}(\boldsymbol{x}; t) = \begin{pmatrix} G_1(\boldsymbol{x}; t) \\ G_2(\boldsymbol{x}; t) \\ \vdots \\ G_d(\boldsymbol{x}; t) \end{pmatrix}, \tag{1.13}$$

is equivalent to the $2d$-th order dynamical system

$$\dot{\boldsymbol{x}}(t) = \tfrac{1}{m}\boldsymbol{p}(t), \qquad \dot{\boldsymbol{p}}(t) = m\boldsymbol{G}(\boldsymbol{x}; t), \tag{1.14}$$

where

$$\boldsymbol{p}(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_d(t) \end{pmatrix} = m \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_d(t) \end{pmatrix}. \tag{1.15}$$

If there exists a *potential function* $V(\boldsymbol{x}; t)$, such that

$$\boldsymbol{G}(\boldsymbol{x}; t)m = -\boldsymbol{\nabla} V(\boldsymbol{x}; t), \tag{1.16}$$

the system is said to be *conservative*. This is equivalent to the condition that

$$V(\boldsymbol{x}; t) = -\int_{\boldsymbol{x}(0)}^{\boldsymbol{x}(t)} m\boldsymbol{G}(\boldsymbol{x}; t).\mathbf{d}\boldsymbol{r}, \tag{1.17}$$

where the line integral in Γ_d from $\boldsymbol{x}(0)$ to $\boldsymbol{x}(t)$ is independent of the path taken.

By defining

$$H(\boldsymbol{x}, \boldsymbol{p}; t) = \tfrac{1}{2m}\boldsymbol{p}^2 + V(\boldsymbol{x}; t), \tag{1.18}$$

we see that a conservative system is also a Hamiltonian system. In a physical context this system can be taken to represent the motion of a set of $\frac{1}{3}d$ moving in three-dimensional space, with position and momentum coordinates $x_1, x_2, \ldots, x_d$ and $p_1, p_2, \ldots, p_d$ respectively. Then $\frac{1}{2m}\boldsymbol{p}^2$ and $V(\boldsymbol{x}; t)$ are respectively the kinetic and potential energies.

A rather more general case is when, for the system defined by (1.9), there exists a scalar field $U(\boldsymbol{x}; t)$ with

$$\boldsymbol{F}(\boldsymbol{x}; t) = -\boldsymbol{\nabla} U(\boldsymbol{x}; t). \tag{1.19}$$

1.4 Discrete-Time Systems

Although our main interest will be in dynamic systems defined by *differential equations*, it is worth referring to the case where the system is defined by a *difference equation*. This simply corresponds to the situation where 'time' is made discrete and becomes a variable defined on the countable set $n = 0, 1, 2, \ldots$. Then (1.9) is replaced by[3]

$$\mathbf{x}(n+1) = \mathbf{F}[\mathbf{x}(n); n], \qquad n = 0, 1, 2, \ldots. \tag{1.20}$$

In fact, of course, numerical solutions of systems of differential equations are normally calculated by considering the corresponding difference equation. The derivative $\dot{\boldsymbol{x}}(t)$ is replaced by a two (or possibly more) point numerical approximation. Suppose we take $t = n\varepsilon$, with $\varepsilon > 0$ and $\boldsymbol{x}(t) = \boldsymbol{x}(n\varepsilon) = \mathbf{x}(n)$ in (1.9) and use the forward two-point derivative

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \simeq \frac{\boldsymbol{x}(\{n+1\}\varepsilon) - \boldsymbol{x}(n\varepsilon)}{\varepsilon} = \frac{\mathbf{x}(n+1) - \mathbf{x}(n)}{\varepsilon}. \tag{1.21}$$

Then

$$\varepsilon \boldsymbol{F}(\boldsymbol{x}(n\varepsilon); n\varepsilon) = \mathbf{x}(n+1) - \mathbf{x}(n). \tag{1.22}$$

This is a difference equation like (1.20) with

$$\varepsilon \boldsymbol{F}(\boldsymbol{x}(n\varepsilon); n\varepsilon) = \mathbf{F}[\mathbf{x}(n); n] - \mathbf{x}(n), \tag{1.23}$$

and ε as an independent parameter. Different choices of ε may lead to very different behaviours for the equations. Intuitively one may suppose that choosing ε *as small as possible* will lead to behaviour close to that of the underlying differential equation, but there is, of course, a practical limit on accuracy with any

[3] To distinguish between discrete-time and continuous time system we shall use the same letters but a different font.

computing systems and going beyond this will lead to rounding errors. There are also questions of stability. It may be the case that differences in ε, however small they are, lead to large changes in the evolution of (1.23), with none accurately representing the analytic solution of (1.9) which would correspond to the limit $\varepsilon \to 0$.

1.5 Autonomous Systems

A dynamic system of the type (1.8) is **autonomous** *(sometimes called 'stationary') if none of the functions $F_1, F_2, \ldots, F_d$ is an explicit function of t. The time dependence of $\boldsymbol{F}$ in this case enters through the dependence of the variables $x_1(t), \ldots, x_d(t)$ on t.*

It is clear that (1.3) is an autonomous dynamic system. Autonomous systems have the important property that, if the system is at $(x_1^{(0)}, \ldots, x_d^{(0)})$ at time t_0 and $(x_1^{(1)}, \ldots, x_d^{(1)})$ at t_1 then the values $x_1^{(1)}, \ldots, x_d^{(1)}$ are dependent on $x_1^{(0)}, \ldots, x_d^{(0)}$ and $t_1 - t_0$ but not on t_0 and t_1 individually.

In fact being autonomous is not such a severe restraint. A non-autonomous system can be made equivalent to an autonomous system by the following trick. We include the time dimension in the phase space by adding the time line Υ to Γ_d. The path in the $(d+1)$-dimensional space $\Gamma_d \times \Upsilon$ is then given by the dynamical system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{x}, x_t), \qquad \dot{x}_t(t) = 1. \tag{1.24}$$

This is called a *suspended system.*

1.6 Equilibrium Points and Their Stability

In general the determination of the trajectories in phase space, even for autonomous systems, can be a difficult problem. However, we can often obtain a qualitative idea of the phase pattern of trajectories by considering particularly simple trajectories. The most simple of all are the *equilibrium points.*[4] These are trajectories which consist of one single point. If the phase point starts at an equilibrium point it stays there. The condition for $\boldsymbol{x}^*$ to be an equilibrium point of the autonomous system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{x}), \tag{1.25}$$

is

$$\boldsymbol{F}(\boldsymbol{x}^*) = \boldsymbol{0}. \tag{1.26}$$

For the system given by (1.19) it is clear that a equilibrium point is a stationary point of $U(\boldsymbol{x})$ and for the conservative system given by (1.13)–(1.16) equilibrium

[4] Also called, *fixed points*, *critical points* or *nodes.*

points have $\boldsymbol{p} = \mathbf{0}$ and are stationary points of $V(\boldsymbol{x})$. An equilibrium point is useful for obtaining information about phase behaviour only if we can determine the behaviour of trajectories in its neighbourhood. This is a matter of the *stability* of the equilibrium point, which in formal terms can be defined in the following way:

The equilibrium point $\boldsymbol{x}^*$ of (1.25) is said to be ***stable (in the sense of Lyapunov)*** *if there exists, for every $\varepsilon > 0$, a $\delta(\varepsilon) > 0$, such that any solution $\boldsymbol{x}(t)$, for which $\boldsymbol{x}(t_0) = \boldsymbol{x}^{(0)}$ and*

$$|\boldsymbol{x}^* - \boldsymbol{x}^{(0)}| < \delta(\varepsilon), \tag{1.27}$$

satisfies

$$|\boldsymbol{x}^* - \boldsymbol{x}(t)| < \varepsilon, \tag{1.28}$$

for all $t \geq t_0$. If no such $\delta(\varepsilon)$ exists then $\boldsymbol{x}^*$ is said to be ***unstable (in the sense of Lyapunov)****. If $\boldsymbol{x}^*$ is stable and*

$$\lim_{t\to\infty} |\boldsymbol{x}^* - \boldsymbol{x}(t)| = 0. \tag{1.29}$$

it is said to be ***asymptotically stable****. If the equilibrium point is stable and (1.29) holds for every $\boldsymbol{x}^{(0)}$ then it is said to be* ***globally asymptotically stable****. In this case $\boldsymbol{x}^*$ must be the unique equilibrium point.*

There is a warning you should note in relation to these definitions. In some texts the term ***stable*** *is used to mean what we have called 'asymptotically stable' and equilibrium points which are stable (in our sense) but not asymptotically stable are called* ***conditionally*** *or* ***marginally stable****.*

An asymptotically stable equilibrium point is a type of *attractor*. Other types of attractors can exist. For example, a close (periodic) trajectory to which all neighbouring trajectories converge. These more general questions of stability will be discussed in Chap. 3. We now illustrate the ideas described here by returning to the simple harmonic oscillator.

1.7 Damped and Forced Simple Harmonic Oscillators

It is not difficult to see that the simple harmonic system with equations of motion (1.3) is a autonomous Hamiltonian system with momentum $p = mv$ and

$$H(x, p) = \frac{1}{2m}p^2 + \frac{1}{2}\omega^2 x^2. \tag{1.30}$$

It is also a conservative system with $V(x) = \omega^2 x^2/2$.

The point $x = v = 0$ is a stable equilibrium point, but *not* an asymptotically stable equilibrium point. A trajectory which begins close to the equilibrium

point will perform an ellipse about the point without converging to the point or moving away.

1.7.1 A Damped Simple Harmonic Oscillator

Now suppose that motion of the simple harmonic oscillator is slowed down (damped) by a force like viscosity, which is proportional to the velocity. The equation of motion is modified to

$$\begin{aligned} &m\ddot{x}(t) = -\ell v(t) - \kappa x(t), \\ &\text{or equivalently} \\ &\ddot{x}(t) + 2\beta\dot{x}(t) + \omega^2 x(t) = 0, \end{aligned} \tag{1.31}$$

where $\beta = \ell/(2m) > 0$. The auxiliary equation is

$$\lambda^2 + 2\beta\lambda + \omega^2 = 0, \qquad \text{with roots} \qquad \lambda = -\beta \pm \sqrt{\beta^2 - \omega^2}. \tag{1.32}$$

We must consider three cases:

(i) $\underline{\beta > \omega}$. Then both roots of the auxiliary are real and the solution is

$$x(t) = \mathrm{A}\exp(-[\beta+\gamma]t) + \mathrm{B}\exp(-[\beta-\gamma]t) \tag{1.33}$$

where $\gamma = \sqrt{\beta^2 - \omega^2}$. With the initial conditions are $x(0) = a$, $\dot{x}(0) = 0$, the solution becomes

$$\begin{aligned} x(t) &= \frac{a}{2\gamma}\exp(-\beta t)\left\{(\gamma-\beta)\exp(-\gamma t) + (\gamma+\beta)\exp(\gamma t)\right\}, \\ &\quad \text{with the velocity} \\ v(t) &= \frac{a(\beta^2-\gamma^2)}{2\gamma}\exp(-\beta t)\left\{\exp(-\gamma t) - \exp(\gamma t)\right\}, \end{aligned} \tag{1.34}$$

As $t \to \infty$ the solution converges to $x = v = 0$, which is now an asymptotically stable equilibrium point. The path in Γ_2, for $a = 1$, $\omega = 0.6$, $\beta = 0.7$, is shown in Fig. 1.3.

(ii) $\underline{\omega > \beta}$. Then both roots of the auxiliary equation are complex. You can either re-derive the solution from scratch or make the substitution $\gamma = \mathrm{i}\xi$ in (1.34), where $\xi = \sqrt{\omega^2 - \beta^2}$. This gives

$$\begin{aligned} x(t) &= \frac{a}{\xi}\exp(-\beta t)\left\{\xi\cos(\xi t) + \beta\sin(\xi t)\right\}, \\ &\quad \text{with the velocity} \\ v(t) &= \frac{a(\beta^2+\xi^2)}{\xi}\exp(-\beta t)\sin(\xi t). \end{aligned} \tag{1.35}$$

In the limit $\beta \to 0$, $\xi \to \omega$, we recover the undamped solution (1.6). When $\beta > 0$, the solution oscillates with an exponentially decreasing amplitude.

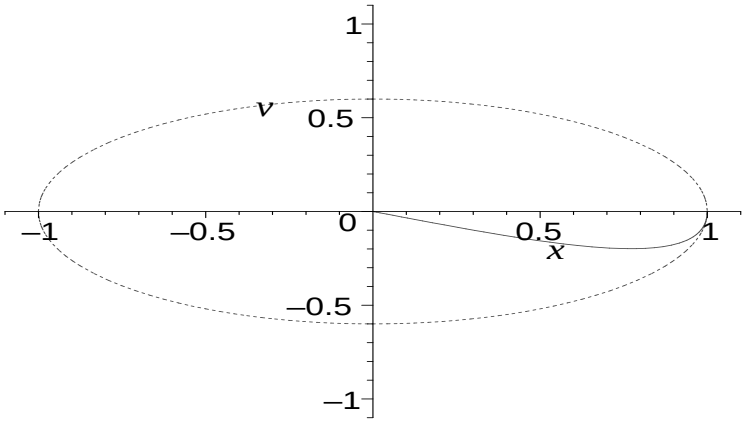


Figure 1.3: The path begins at $(a, 0)$ by following the undamped solution (shown by a broken line) but then converges to the origin.

The path in Γ_2, for $a = 1$, $\omega = 0.6$, $\beta = 0.1$, is shown in Fig. 1.4. Again the origin in Γ_2 is an asymptotically stable equilibrium point but the trajectory approaches it in a spiral.

(iii) $\underline{\omega = \beta}$. Then the roots of the auxiliary are both $\lambda = -\beta$ and the solution is

$$x(t) = [\mathrm{A} + \mathrm{B}t]\exp(-\beta t). \tag{1.36}$$

With the initial conditions are $x(0) = a$, $\dot{x}(0) = 0$, the solution becomes

$$\begin{aligned} x(t) &= a(1 + \beta t)\exp(-\beta t), \\ &\quad \text{with the velocity} \\ v(t) &= -\beta^2 at \exp(-\beta t). \end{aligned} \tag{1.37}$$

The path in Γ_2, for $a = 1$, $\beta = 2$, is shown in Fig. 1.5. Again the origin is a asymptotically stable equilibrium point.

1.7.2 A Forced, Damped Simple Harmonic Oscillator

We now consider a case of the situation where the damped harmonic oscillator is subject to a periodic forcing term. The equation of motion (1.31) becomes

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega^2 x(t) = c\cos(\chi t). \tag{1.38}$$

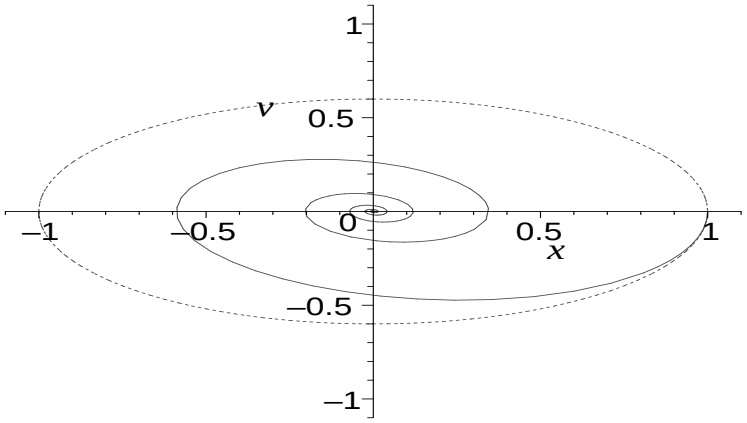


Figure 1.4: The path begins at $(a, 0)$ by following the undamped solution (shown by a broken line) but then spirals into the origin.

The complementary function for this equation is just the same as in the cases we have treated. We shall concentrate on the case $\omega > \beta$ where the complementary function (unforced part of the solution) is periodic

$$x_{\rm c}(t) = \exp(-\beta t)\left\{\mathsf{A}\cos(\xi t) + \mathsf{B}\sin(\xi t)\right\}, \qquad \xi = \sqrt{\omega^2 - \beta^2}. \tag{1.39}$$

The trial function for the particular integral is

$$\mathsf{T}(t) = \mathsf{C}\cos(\chi t) + \mathsf{D}\sin(\chi t), \tag{1.40}$$

and substituting into (1.38) gives the particular solution

$$x_{\rm p}(t) = \frac{c}{\phi}\left\{(\omega^2 - \chi^2)\cos(\chi t) + 2\beta\chi\sin(\chi t)\right\}, \tag{1.41}$$

where

$$\phi = \omega^4 + \chi^4 + 2\chi^2(2\beta^2 - \omega^2). \tag{1.42}$$

Now we apply the initial conditions $x(0) = a$ and $\dot{x}(0) = 0$ to evaluate A and B so that the complementary function becomes

$$x_{\rm c}(t) = \frac{\exp(-\beta t)}{\xi\phi}\left\{\xi[a\phi + c(\chi^2 - \omega^2)]\cos(\xi t) + \beta[a\phi - c(\chi^2 + \omega^2)]\sin(\xi t)\right\} \tag{1.43}$$

When $\beta > 0$ the solution

$$x(t) = x_{\rm c}(t) + x_{\rm p}(t), \qquad \text{with} \qquad v(t) = \dot{x}_{\rm c}(t) + \dot{x}_{\rm p}(t) \tag{1.44}$$

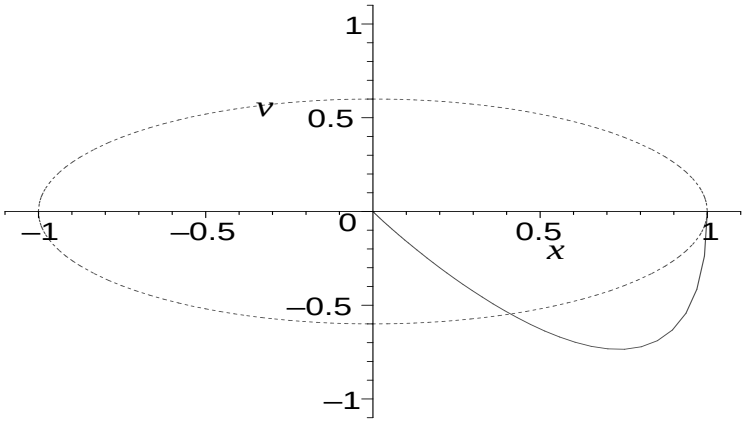


Figure 1.5: The path begins at $(a, 0)$ by following the undamped solution (shown by a broken line) and then converges to the origin as in Fig. 1.3, but in this case the convergence is delayed by the linear t terms.

has a part $x_{\rm c}(t)$ which tends to zero as $t \to \infty$. This is called the *transient contribution* and a part $x_{\rm p}(t)$ which does not attenuate. This is called the *persistent contribution*. In the long-time limit the system tends to an oscillation of the forced frequency. In Γ_2 the solution begins at the point$(a, 0)$ on the ellipse of the 'natural motion' of the oscillator and then converges on the ellipse

$$\frac{x^2}{(\omega^2 - \chi^2)^2} + \frac{v^2}{4\beta^2\chi^2} = \frac{c^2}{\phi^2}. \tag{1.45}$$

Figure 1.6 shows the case where $a = 1$, $c = 2$, $\omega = 0.6$, $\beta = 0.7$ and $chi = 0.5$. The origin in Γ_2 is no longer an equilibrium point but the ellipse (1.45) is an attractor.

1.7.3 A Forced, Undamped Simple Harmonic Oscillator

We now consider the special case $\beta = 0$, when the solution simplifies to

$$x(t) = \left\{a + \frac{c}{(\chi^2 - \omega^2)}\right\}\cos(\omega t) + \frac{c}{(\omega^2 - \chi^2)}\cos(\chi t), \tag{1.46}$$

$$v(t) = -\omega\left\{a + \frac{c}{(\chi^2 - \omega^2)}\right\}\sin(\omega t) - \chi\frac{c}{(\omega^2 - \chi^2)}\sin(\chi t). \tag{1.47}$$

It is clear that the amplitude of the oscillations tends to infinity as χ is 'tuned' to approach ω. This phenomenon is known as *resonance*. For the case $\chi = \omega$ we

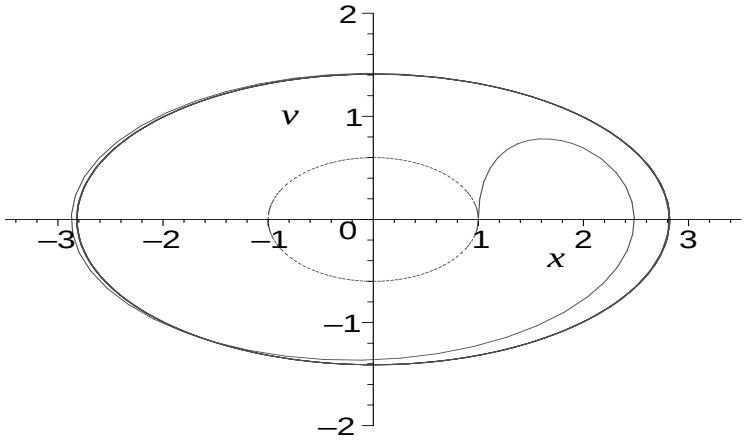


Figure 1.6: A damped force simple harmonic oscillator. The path begins at $(a, 0)$ on the natural ellipse and then converges onto the forced ellipse as $t \to \infty$.

should have taken a different trial function containing a linear term in t. Then we would have obtained the solution

$$x(t) = \frac{1}{2\omega}[2a\omega\cos(\omega t) + ct\sin(\omega t)], \tag{1.48}$$

$$v(t) = \frac{1}{2}[-2a\omega\sin(\omega t) + ct\cos(\omega t)], \tag{1.49}$$

in which the amplitude of the periodic solution increases linearly with t. Away from the resonance case we have, in equations (1.46) and (1.47) a solution which involves contributions with two different angular frequencies ω and χ and periods $T_{\text{N}} = 2\pi/\omega$ and $T_{\text{F}} = 2\pi/\chi$. The possible behaviour divide into two types:

(i) There exist integers n_1 and n_2 such that

$$n_1 T_{\text{N}} = n_2 T_{\text{F}} \qquad \text{or, equivalently} \qquad n_2\omega = n_1\chi. \tag{1.50}$$

Then the period of the solution is $n_1 T_{\text{N}} = n_2 T_{\text{F}}$, where now n_1 and n_2 are the smallest pair of integers which satisfy (1.50). Equation (1.50) can, of course, always we satisfied if ω and χ are rational numbers (fractions or integers) and the case where $a = 1$, $c = 2$, $\omega = \frac{7}{10}$, $\chi = \frac{1}{2}$ is shown in Fig. 1.7.

(ii) There do not exist integers n_1 and n_2 such that (1.50) is satisfied. For this to be the case one or both of ω and χ must be irrational. The curve in

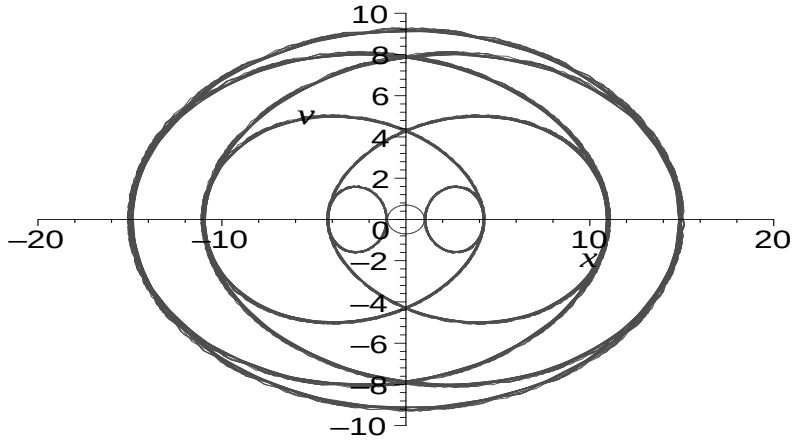


Figure 1.7: An undamped force simple harmonic oscillator where the frequencies are rationally related and the solution is periodic.

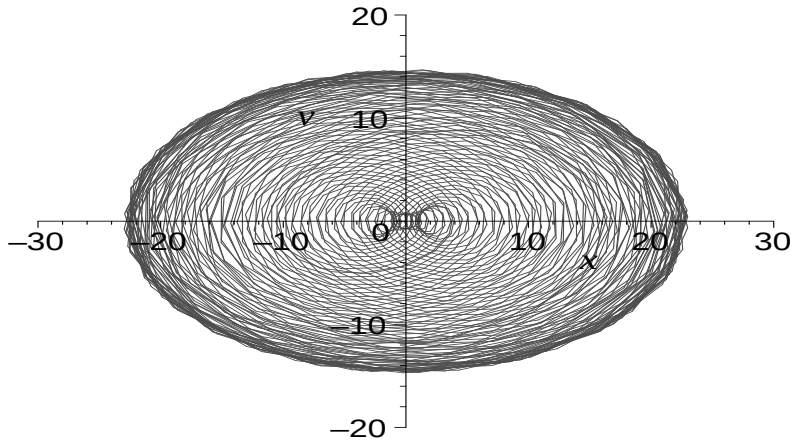


Figure 1.8: An undamped force simple harmonic oscillator where the frequencies are not rationally related and the solution is quasi-periodic.

Γ_2 now never closes. This solution is said to be *quasi-periodic*. The case where $a = 1$, $c = 2$, $\omega = \frac{1}{\sqrt{2}}$, $\chi = \frac{1}{\sqrt{3}}$ is shown in Fig. 1.8.

1.8 One-Variable Autonomous Systems

We first consider a first-order autonomous system. In general a system may contain a number of adjustable parameters $a, b, c, \ldots$ and it is of interest to consider the way in which the equilibrium points and their stability change with changes of these parameters. We consider the equation

$$\dot{x}(t) = F(a, b, c, \ldots, x), \tag{1.51}$$

where $a, b, c, \ldots$ are some (one or more) independent parameters. An equilibrium point $x^*(a, b, c, \ldots)$ is a solution of

$$F(a, b, c, \ldots, x^*) = 0. \tag{1.52}$$

According to the Lyapunov criterion it is *stable* if, when the phase point is perturbed a small amount from x^*, it remains in a neighbourhood of x^*, asymptotically stable if it converges on x^* and *unstable* if it moves away from x^*. We shall, therefore, determine the stability of equilibrium points by linearizing about the point.[5]

Example 1.8.1 Consider the one-variable non-linear system given by

$$\dot{x}(t) = a - x^2. \tag{1.53}$$

The parameter a can vary over all real values and the nature of equilibrium points will vary accordingly.

The equilibrium points are given by $x = x^* = \pm\sqrt{a}$. They exist only when $a \geq 0$ and form the parabolic curve shown in Fig. 1.9. Let $x = x^* + \triangle x$ and substitute into (1.53) neglecting all but the linear terms in $\triangle x$. This gives

$$\frac{\mathrm{d}\triangle x}{\mathrm{d}t} = a - (x^*)^2 - 2x^*\triangle x. \tag{1.54}$$

The right-hand side of (1.54) can be understood either as a Taylor expansion, as far as the linear term, of the right-hand side of (1.53) about $x = x^*$, or as the expansion of the quadratic $(x^* + \triangle x)^2$ with the term $(\triangle x)^2$ neglected.[6] Since $a = (x^*)^2$ this gives

$$\frac{\mathrm{d}\triangle x}{\mathrm{d}t} = -2x^*\triangle x, \tag{1.55}$$

which has the solution

$$\triangle x = \mathrm{C}\exp(-2x^*t). \tag{1.56}$$

Thus the equilibrium point $x^* = \sqrt{a} > 0$ is asymptotically stable (denoted by a continuous line in Fig. 1.9) and the equilibrium point $x^* = -\sqrt{a} < 0$ is unstable

[5] A theorem establishing the formal relationship between this *linear stability* and the Lyapunov criteria will be stated below.

[6] Of course, in cases where the right-hand side of the differential equation is not of some simple polynomial form we shall have to use a Taylor expansion.

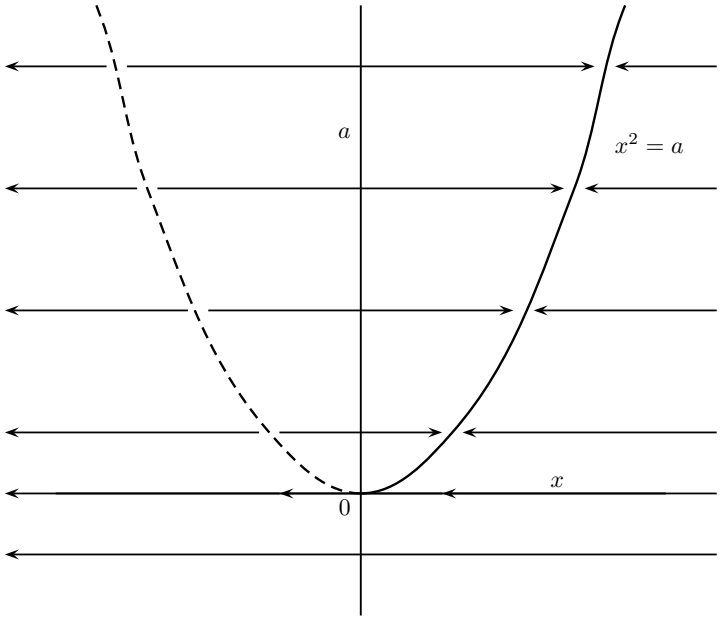


Figure 1.9: The bifurcation diagram for Example 1.8.1. The stable and unstable equilibrium solutions are shown by continuous and broken lines and the direction of the flow is shown by arrows. This is an example of a simple turning point bifurcation.

(denoted by a broken line in Fig. 1.9). When $a \leq 0$ it is clear that $\dot{x}(t) < 0$ so $x(t)$ decreases monotonically from its initial value $x(0)$. In fact for $a = 0$ equation (1.53) is easily solved:

$$\int_{x(0)}^{x} x^{-2} \mathrm{d}x = -\int_{0}^{t} \mathrm{d}t \tag{1.57}$$

gives

$$x(t) = \frac{x(0)}{1 + tx(0)}, \qquad \dot{x}(t) = -\left\{\frac{x(0)}{1 + tx(0)}\right\}^2. \tag{1.58}$$

Then

$$x(t) \to \begin{cases} 0, & \text{as } t \to \infty, \text{ if } x(0) > 0, \\ -\infty, & \text{as } t \to 1/|x(0)|, \text{ if } x(0) < 0. \end{cases} \tag{1.59}$$

In each case $x(t)$ decreases with increasing t. When $x(0) > 0$ it takes 'forever' to reach the origin. For $x(0) < 0$ it attains minus infinity in a finite amount of

time and then 'reappears' at infinity and decreases to the origin as $t \to \infty$. The linear equation (1.55) cannot be applied to determine the stability of $x^* = 0$ as it gives $(\mathrm{d}\triangle x/\mathrm{d}t)^* = 0$. If we retain the quadratic term we have

$$\frac{\mathrm{d}\triangle x}{\mathrm{d}t} = -(\triangle x)^2. \tag{1.60}$$

So including the second degree term we see that $\mathrm{d}\triangle x/\mathrm{d}t < 0$. If $\triangle x > 0$, $x(t)$ moves towards the equilibrium point and, if $\triangle x < 0$, it moves away. In the strict Lyapunov sense the equilibrium point $x^* = 0$ is unstable. But it is 'less unstable' that $x^* = -\sqrt{a}$, for $a > 0$, since there is a path of attraction. It is at the boundary between the region where there are no equilibrium points and the region where there are two equilibrium points. It is said to be *on the margin of stability*. The value $a = 0$ separates the stable range from the unstable range. Such equilibrium points are *bifurcation points*. This particular type of bifurcation is variously called a *simple turning point*, a *fold* or a *saddle-node bifurcation*. Fig.1.9 is the *bifurcation diagram*.

Example 1.8.2 The system with equation

$$\dot{x}(t) = x\{(a + c^2) - (x - c)^2\} \tag{1.61}$$

has two parameters a and c.

The equilibrium points are $x = 0$ and $x = x^* = c \pm \sqrt{a + c^2}$, which exist when $a + c^2 > 0$. Linearizing about $x = 0$ gives

$$x(t) = \mathrm{C}\exp(at) \tag{1.62}$$

The equilibrium point $x = 0$ is asymptotically stable if $a < 0$ and unstable for $a > 0$. Now let $x = x^* + \triangle x$ giving

$$\begin{aligned} \frac{\mathrm{d}\triangle x}{\mathrm{d}t} &= -2\triangle x x^*(x^* - c) \\ &= \mp 2\triangle x\sqrt{a + c^2}\left[c \pm \sqrt{a + c^2}\right]. \end{aligned} \tag{1.63}$$

This has the solution

$$\triangle x = \mathrm{C}\exp\left[\mp 2t\sqrt{a + c^2}\left(c \pm \sqrt{a + c^2}\right)\right]. \tag{1.64}$$

We consider separately the three cases:

$\underline{c = 0.}$
Both equilibrium points $x^* = \pm\sqrt{a}$ are stable. The bifurcation diagram for this case is shown in Fig.1.10. This is an example of a *supercritical pitchfork bifurcation* with one stable equilibrium point becomes unstable and two new stable solutions emerge each side of it. The similar situation with the stability reversed is a *subcritical pitchfork bifurcation*.

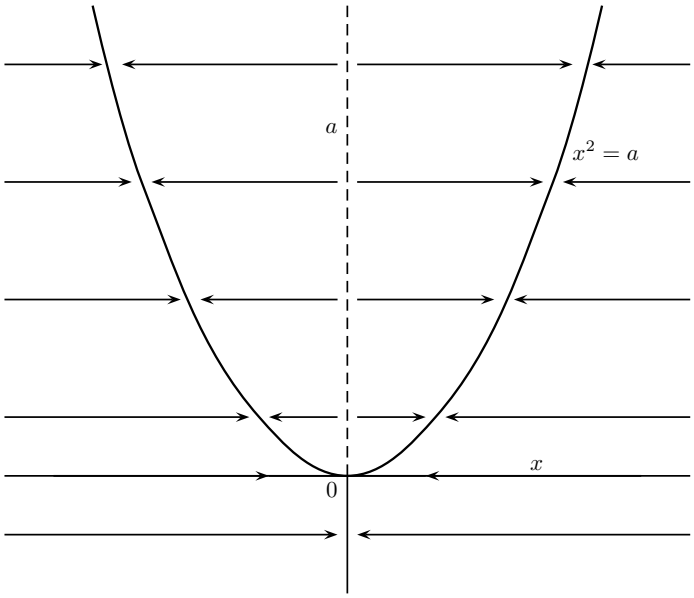


Figure 1.10: The bifurcation diagram for Example 1.8.2, $c = 0$. The stable and unstable equilibrium solutions are shown by continuous and broken lines and the direction of the flow is shown by arrows. This is an example of a supercritical pitchfork bifurcation.

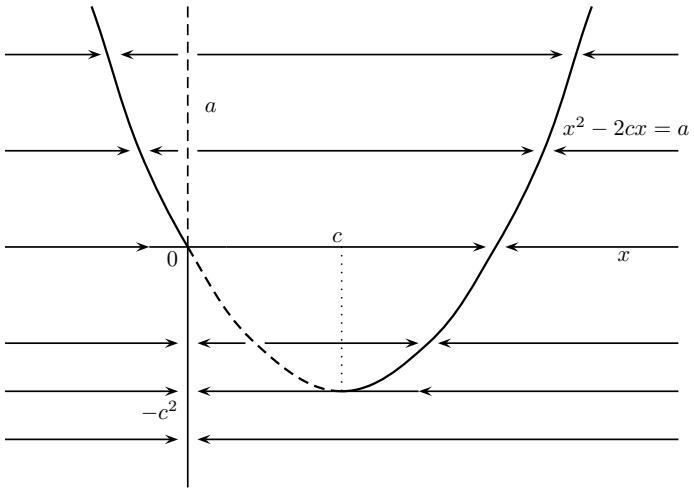


Figure 1.11: The bifurcation diagram for Example 1.8.2, $c > 0$. The stable and unstable equilibrium solutions are shown by continuous and broken lines and the direction of the flow is shown by arrows. This gives examples of both simple turning point and transcritical bifurcations.

$\underline{c > 0.}$
The equilibrium point $x = c + \sqrt{a + c^2}$ is stable. The equilibrium point $x = c - \sqrt{a + c^2}$ is unstable for $a < 0$ and stable for $a > 0$. The point $x = c$, $a = -c^2$ is a simple turning point bifurcation and $x = a = 0$ is a *transcritical bifurcation*. That is the situation when the stability of two crossing lines of equilibrium points interchange. The bifurcation diagram for this example is shown in Fig.1.11.

$\underline{c < 0.}$
This is the mirror image (with respect to the vertical axis) of the case $c > 0$.

Example 1.8.3

$$\dot{x}(t) = cx(b - x). \tag{1.65}$$

This is the *logistic equation*.

The equilibrium points are $x = 0$ and $x = b$. Linearizing about $x = 0$ gives

$$x(t) = \mathrm{C}\exp(cbt) \tag{1.66}$$

The equilibrium point $x = 0$ is stable or unstable according as if $cb <, > 0$. Now let $x = b + \triangle x$ giving

$$\frac{\mathrm{d}\triangle x}{\mathrm{d}t} = -cb\triangle x. \tag{1.67}$$

So the equilibrium point $x = b$ is stable or unstable according as $cb >, < 0$. Now plot the equilibrium points with the flow and stability indicated:

- In the (b, x) plane for fixed $c > 0$ and $c < 0$.
- In the (c, x) plane for fixed $b > 0$, $b = 0$ and $b < 0$.

You will see that in the (b, x) plane the bifurcation is easily identified as transcritical but in the (c, x) plane it looks rather different.

Now consider the difference equation corresponding to (1.65). Writing $\mathsf{x}(n) = x(n\varepsilon)$ and using the two-point forward derivative,

$$\mathsf{x}(n + 1) = \mathsf{x}(n)[(\varepsilon cb + 1) - c\varepsilon\mathsf{x}(n)]. \tag{1.68}$$

Now substituting

$$\mathsf{x} = \frac{(1 - \varepsilon cb)\mathsf{y} + \varepsilon cb}{c\varepsilon} \tag{1.69}$$

into (1.68) gives

$$\mathsf{y}(n + 1) = a\mathsf{y}(n)[1 - \mathsf{y}(n)], \tag{1.70}$$

where

$$a = 1 - \varepsilon cb. \tag{1.71}$$

(1.70) is the usual form of the *logistic difference equation*. The equilibrium points of (1.70), given by setting $\mathsf{y}(n+1) = \mathsf{y}(n) = \mathsf{y}^*$ are

$$\begin{aligned} \mathsf{y}^* = 0 \qquad &\longrightarrow \qquad \mathsf{x}^* = b, \\ \mathsf{y}^* = 1 - 1/a \quad &\longrightarrow \qquad \mathsf{x}^* = 0. \end{aligned} \tag{1.72}$$

Now linearize (1.70) by setting $\mathsf{y}(n) = \triangle\mathsf{y}(n) + \mathsf{y}^*$ to give

$$\triangle\mathsf{y}(n+1) = a(1 - 2\mathsf{y}^*)\triangle\mathsf{y}(n). \tag{1.73}$$

The equilibrium point y^* is stable or unstable according as $|a(1-2\mathsf{y}^*)| <, > 1$. So

- $\mathsf{y}^* = 0$, $(\mathsf{x}^* = b)$ is stable if $-1 < a < 1$, $(0 < \varepsilon cb < 2)$.
- $\mathsf{y}^* = 1 - 1/a$, $(\mathsf{x}^* = 0)$ is stable if $1 < a < 3$, $(-2 < \varepsilon cb < 0)$.

Since the differential equation corresponds to small, positive ε, these stability conditions agree with those derived for the differential equation (1.65). You may know that the whole picture for the behaviour of the difference equation (1.70) involves cycles, period doubling and chaos.[7] Here, however, we are just concerned with the situation for small ε when

$$\mathsf{y} \simeq (c\varepsilon)\mathsf{x}, \qquad\qquad a = 1 - (c\varepsilon)b. \tag{1.74}$$

The whole of the (b, x) plane is mapped into a small rectangle centred around $(1, 0)$ in the (a, y) plane, where a transcritical bifurcation occurs between the equilibrium points $\mathsf{y} = 0$ and $\mathsf{y} = 1 - 1/a$.

1.9 Digression: The Eigen-Problem

Before considering systems of more than variable we need to revise our knowledge of matrix algebra. A $d \times d$ matrix $\boldsymbol{A}$ is said to be *singular* or *non-singular* according as the determinant of $\boldsymbol{A}$, denoted by $\mathrm{Det}\{\boldsymbol{A}\}$, is zero or non-zero. The *rank* of any matrix $\boldsymbol{B}$, denoted by $\mathrm{Rank}\{\boldsymbol{B}\}$, is defined, whether the matrix is square or not, as the dimension of the largest non-singular (square) submatrix of $\boldsymbol{B}$. For the $d \times d$ matrix $\boldsymbol{A}$ the following are equivalent:

(i) The matrix $\boldsymbol{A}$ is non-singular.

(ii) The matrix $\boldsymbol{A}$ has an *inverse* denoted by $\boldsymbol{A}^{-1}$.

(iii) $\mathrm{Rank}\{\boldsymbol{A}\} = d$.

(iv) The set of d linear equations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}, \tag{1.75}$$

[7] Ian Stewart, *Does God Play Dice?*, Chapter 8, Penguin (1990)

where

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}, \qquad \boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{pmatrix}, \tag{1.76}$$

has a unique solution for the variables $x_1, x_2, \ldots, x_d$, for any numbers $c_1, c_2, \ldots, c_d$, given by

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{c}. \tag{1.77}$$

(Of course, when $c_1 = c_2 = \cdots = c_d = 0$ the unique solution is the *trivial solution* $x_1 = x_2 = \cdots = x_d = 0$.)

When $\boldsymbol{A}$ is singular we form the $d \times (d+1)$ *augmented matrix* matrix $\boldsymbol{A}'$ by adding the vector $\boldsymbol{c}$ as a final column. Then the following results can be established:

(a) If

$$\text{Rank}\{\boldsymbol{A}\} = \text{Rank}\{\boldsymbol{A}'\} = m < d \tag{1.78}$$

then (1.75) has an infinite number of solutions corresponding to making an arbitrary choice of $d - m$ of the variables $x_1, x_2, \ldots, x_d$.

(b) If

$$\text{Rank}\{\boldsymbol{A}\} < \text{Rank}\{\boldsymbol{A}'\} \leq d \tag{1.79}$$

then (1.75) has no solution.

Let $\boldsymbol{A}$ be a non-singular matrix. The *eigenvalues* of $\boldsymbol{A}$ are the roots of the d-degree polynomial

$$\text{Det}\{\boldsymbol{A} - \lambda \boldsymbol{I}\} = 0, \tag{1.80}$$

in the variable λ. Suppose that there are d *distinct* roots $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)}$. Then $\text{Rank}\{\boldsymbol{A} - \lambda^{(k)}\boldsymbol{I}\} = d - 1$ for all $k = 1, 2, \ldots, d$. So there is, corresponding to each eigenvalue $\lambda^{(k)}$, a *left eigenvector* $\boldsymbol{v}^{(k)}$ and a *right eigenvector* $\boldsymbol{u}^{(k)}$ which are solutions of the linear equations

$$[\boldsymbol{v}^{(k)}]^{\text{T}}\boldsymbol{A} = \lambda^{(k)}[\boldsymbol{v}^{(k)}]^{\text{T}}, \qquad \boldsymbol{A}\boldsymbol{u}^{(k)} = \boldsymbol{u}^{(k)}\lambda^{(k)}. \tag{1.81}$$

The eigenvectors are unique to within the choice of one arbitrary component. Or equivalently they can be thought of as unique in direction and arbitrary in

length. If $\boldsymbol{A}$ is symmetric it is easy to see that the left and right eigenvectors are the same.[8] Now

$$[\boldsymbol{v}^{(k)}]^{\mathrm{T}}\boldsymbol{A}\boldsymbol{u}^{(j)} = \lambda^{(k)}[\boldsymbol{v}^{(k)}]^{\mathrm{T}}\boldsymbol{u}^{(j)} = [\boldsymbol{v}^{(k)}]^{\mathrm{T}}\boldsymbol{u}^{(j)}\lambda^{(j)} \tag{1.82}$$

and since $\lambda^{(k)} \neq \lambda^{(j)}$ for $k \neq j$ the vectors $\boldsymbol{v}^{(k)}$ and $\boldsymbol{u}^{(j)}$ are orthogonal. In fact since, as we have seen, eigenvectors can always be multiplied by an arbitrary constant we can ensure that the sets $\{\boldsymbol{u}^{(k)}\}$ and $\{\boldsymbol{v}^{(k)}\}$ are *orthonormal* by dividing each for $\boldsymbol{u}^{(k)}$ and $\boldsymbol{v}^{(k)}$ by $\sqrt{\boldsymbol{u}^{(k)} \cdot \boldsymbol{v}^{(k)}}$ for $k = 1, 2, \ldots, d$. Thus

$$\boldsymbol{u}^{(k)} \cdot \boldsymbol{v}^{(j)} = \delta^{\mathrm{Kr}}(k - j), \tag{1.83}$$

where

$$\delta^{\mathrm{Kr}}(k - j) = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \tag{1.84}$$

is called the *Kronecker delta function*. Now form the matrix

$$\boldsymbol{V} = \begin{pmatrix} [\boldsymbol{v}^{(1)}]^{\mathrm{T}} \\ [\boldsymbol{v}^{(2)}]^{\mathrm{T}} \\ \vdots \\ [\boldsymbol{v}^{(d)}]^{\mathrm{T}} \end{pmatrix}, \tag{1.85}$$

which has the left eigenvectors $\boldsymbol{v}^{(k)}$, $k = 1, 2, \ldots, d$ as its rows. In a similar way

$$\boldsymbol{U} = (\boldsymbol{u}^{(1)}\, \boldsymbol{u}^{(2)}\, \cdots\, \boldsymbol{u}^{(d)}) \tag{1.86}$$

has the right eigenvectors as its columns. From the orthonormality condition (1.84)

$$\boldsymbol{V}\boldsymbol{U} = \boldsymbol{I}. \tag{1.87}$$

This means that

$$\boldsymbol{V} = \boldsymbol{U}^{-1}, \qquad \boldsymbol{U} = \boldsymbol{V}^{-1}. \tag{1.88}$$

If $\boldsymbol{A}$ is symmetric $\boldsymbol{U} = \boldsymbol{V}^{\mathrm{T}}$. So the inverse of $\boldsymbol{U}$ (or $\boldsymbol{V}$) is its transpose. A matrix with this property is called *orthogonal*. Now, if we take all the eigenvectors together in (1.81), it can be written

$$\boldsymbol{V}\boldsymbol{A} = \boldsymbol{\Lambda}\boldsymbol{V}, \qquad \boldsymbol{A}\boldsymbol{U} = \boldsymbol{U}\boldsymbol{\Lambda}, \tag{1.89}$$

where $\boldsymbol{\Lambda}$ is the $d \times d$ diagonal matrix with the eigenvalues $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)}$ along the diagonal. From (1.88) and (1.89),

$$\boldsymbol{V}\boldsymbol{A}\boldsymbol{U} = \boldsymbol{U}^{-1}\boldsymbol{A}\boldsymbol{U} = \boldsymbol{\Lambda}. \tag{1.90}$$

The matrix $\boldsymbol{A}$ is diagonalized by the transformation with $\boldsymbol{U}$ (or $\boldsymbol{V}$). When $\boldsymbol{A}$ is symmetric this is an *orthogonal transformation*.

[8] The vectors referred to in many texts simply as 'eigenvectors' are usually the right eigenvectors. But it should be remembered that non-symmetric matrices have two distinct sets of eigenvectors. The left eigenvectors of $\boldsymbol{A}$ are of course the right eigenvectors of $\boldsymbol{A}^{\mathrm{T}}$ and vice versa.

1.10 Linear Autonomous Systems

The autonomous system (1.25) is linear if

$$\boldsymbol{F} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{c}, \tag{1.91}$$

for some $d \times d$ matrix $\boldsymbol{A}$ and a vector $\boldsymbol{c}$ of constants. Thus we have

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{c}, \tag{1.92}$$

An equilibrium point $\boldsymbol{x}^*$, if it exists, is a solution of

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}. \tag{1.93}$$

As we saw in Sect. 1.9 there can be either no solution points, one solution or an infinite number of solutions. We shall concentrate on the case where $\boldsymbol{A}$ is non-singular and there is a unique solution given by

$$\boldsymbol{x}^* = \boldsymbol{A}^{-1}\boldsymbol{c}. \tag{1.94}$$

As in the case of the first-order system we consider a neighbourhood of the equilibrium point by writing

$$\boldsymbol{x} = \boldsymbol{x}^* + \triangle\boldsymbol{x}. \tag{1.95}$$

Substituting into (1.92) and using (1.94) gives

$$\frac{\mathrm{d}\triangle\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}\triangle\boldsymbol{x}. \tag{1.96}$$

Of course, in this case, the 'linearization' used to achieve (1.96) was exact because the original equation (1.92) was itself linear.

As in Sect. 1.9 we assume that all the eigenvectors of $\boldsymbol{A}$ are distinct and adopt all the notation for eigenvalues and eigenvectors defined there. The vector $\triangle\boldsymbol{x}$ can be expanded as the linear combination

$$\triangle\boldsymbol{x}(t) = w_1(t)\boldsymbol{u}^{(1)} + w_2(t)\boldsymbol{u}^{(2)} + \cdots + w_d(t)\boldsymbol{u}^{(d)}, \tag{1.97}$$

of the right eigenvectors of $\boldsymbol{A}$, where, from (1.83),

$$w_k(t) = \boldsymbol{v}^{(k)} \cdot \triangle\boldsymbol{x}(t), \qquad k = 1, 2, \ldots, d. \tag{1.98}$$

Now

$$\begin{aligned} \boldsymbol{A}\triangle\boldsymbol{x}(t) &= w_1(t)\boldsymbol{A}\boldsymbol{u}^{(1)} + w_2(t)\boldsymbol{A}\boldsymbol{u}^{(2)} + \cdots + w_d(t)\boldsymbol{A}\boldsymbol{u}^{(d)} \\ &= \lambda^{(1)}w_1(t)\boldsymbol{u}^{(1)} + \lambda^{(2)}w_2(t)\boldsymbol{u}^{(2)} + \cdots + \lambda^{(d)}w_d(t)\boldsymbol{u}^{(d)} \end{aligned} \tag{1.99}$$

and

$$\frac{\mathrm{d}\triangle\boldsymbol{x}}{\mathrm{d}t} = \dot{w}_1(t)\boldsymbol{u}^{(1)} + \dot{w}_2(t)\boldsymbol{u}^{(2)} + \cdots + \dot{w}_d(t)\boldsymbol{u}^{(d)}. \tag{1.100}$$

Substituting from (1.99) and (1.100) into (1.96) and dotting with $\boldsymbol{v}^{(k)}$ gives

$$\dot{w}_k(t) = \lambda^{(k)}w_k(t), \tag{1.101}$$

with solution

$$w_k(t) = \mathrm{C}\exp\left(\lambda^{(k)}t\right). \tag{1.102}$$

So $\triangle\boldsymbol{x}$ will grow or shrink in the direction of $\boldsymbol{u}^{(k)}$ according as $\Re\left\{\lambda^{(k)}\right\} >, < 0$. The equilibrium point will be unstable if at least one eigenvalue has a positive real part and stable otherwise. It will be asymptotically stable if the real part of every eigenvalue is (strictly) negative. Although these conclusions are based on arguments which use both eigenvalues *and* eigenvectors, it can be seen that knowledge simply of the eigenvalues is sufficient to determine stability. The eigenvectors give the directions of attraction and repulsion.

Example 1.10.1 Analyze the stability of the equilibrium points of the linear system

$$\dot{x}(t) = y(t), \qquad \dot{y}(t) = 4x(t) + 3y(t). \tag{1.103}$$

The matrix is

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix}, \tag{1.104}$$

with $\mathrm{Det}\{\boldsymbol{A}\} = -4$ and the unique equilibrium point is $x = y = 0$. The eigenvalues of $\boldsymbol{A}$ are $\lambda^{(1)} = -1$ and $\lambda^{(2)} = 4$. The equilibrium point is unstable because it is attractive in one direction but repulsive in the other. Such an equilibrium point is called a *saddle-point*.

For a two-variable system the matrix $\boldsymbol{A}$, obtained for a particular equilibrium point, has two eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$. Setting aside special cases of zero or equal eigenvalues there are the following possibilities:

(i) $\lambda^{(1)}$ and $\lambda^{(2)}$ both real and (strictly) positive. $\triangle\boldsymbol{x}$ grows in all directions. This is called an *unstable node*.

(ii) $\lambda^{(1)}$ and $\lambda^{(2)}$ both real with $\lambda^{(1)} > 0$ and $\lambda^{(2)} < 0$. $\triangle\boldsymbol{x}$ grows in all directions, apart from that given by the eigenvector associated with $\lambda^{(2)}$. This, as indicated above, is called a *saddle-point*.

(iii) $\lambda^{(1)}$ and $\lambda^{(2)}$ both real and (strictly) negative. $\triangle\boldsymbol{x}$ shrinks in all directions. This is called a *stable node*.

(iv) $\lambda^{(1)}$ and $\lambda^{(2)}$ conjugate complex with $\Re\{\lambda^{(1)}\} = \Re\{\lambda^{(2)}\} > 0$. $\triangle\boldsymbol{x}$ grows in all directions, but by spiraling outward. This is called an *unstable focus*.

(v) $\lambda^{(1)} = -\lambda^{(2)}$ are purely imaginary. Close to the equilibrium point, the length of $\triangle\boldsymbol{x}$ remains approximately constant with the phase point performing a closed loop around the equilibrium point. This is called an *centre*.

(vi) $\lambda^{(1)}$ and $\lambda^{(2)}$ conjugate complex with $\Re\{\lambda^{(1)}\} = \Re\{\lambda^{(2)}\} < 0$. $\triangle\boldsymbol{x}$ shrinks in all directions, but by spiraling inwards. This is called an *stable focus*.

Example 1.10.2 Analyze the stability of the equilibrium points of the linear system

$$\dot{x}(t) = 2x(t) - 3y(t) + 4, \qquad \dot{y}(t) = -x(t) + 2y(t) - 1. \tag{1.105}$$

This can be written in the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{c}, \tag{1.106}$$

with

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \qquad \boldsymbol{c} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}. \tag{1.107}$$

The matrix is

$$\boldsymbol{A} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \tag{1.108}$$

with $\mathrm{Det}\{\boldsymbol{A}\} = 1$, has inverse

$$\boldsymbol{A}^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}. \tag{1.109}$$

So the unique equilibrium point is

$$\boldsymbol{x}^* = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}. \tag{1.110}$$

Linearizing about $\boldsymbol{x}^*$ gives an equation of the form (1.96). The eigenvalues of $\boldsymbol{A}$ are $2 \pm \sqrt{3}$. Both these numbers are positive so the equilibrium point is an *unstable node*.

1.11 MAPLE for Systems of Differential Equations

In the discussion of systems of differential equations we shall be concerned less with the analytic form of the solutions than with their qualitative structure. As we shall show below, a lot of information can be gained by finding the equilibrium points and determining their stability. It is also useful to be able to plot a trajectory with given initial conditions. MAPLE can be used for this in two (and possibly three) dimensions. Suppose we want to obtain a plot of the solution of

$$\dot{x}(t) = x(t) - y(t), \qquad \dot{y}(t) = x(t), \tag{1.111}$$

over the range $t = 0$ to $t = 10$, with initial conditions $x(0) = 1$, $y(0) = -1$. The MAPLE routine `dsolve` can be used for systems with the equations and the initial conditions enclosed in curly brackets. Unfortunately the solution is returned as a set $\{x(t) = \cdots, y(t) = \cdots\}$, which cannot be fed directly into

the `plot` routine. To get round this difficulty we set the solution to some variable (`Fset` in this case) and extract $x(t)$ and $y(t)$ (renamed as $fx(t)$ and $fy(t)$) by using the MAPLE function `subs`. These functions can now be plotted parametrically. The complete MAPLE code and results are:

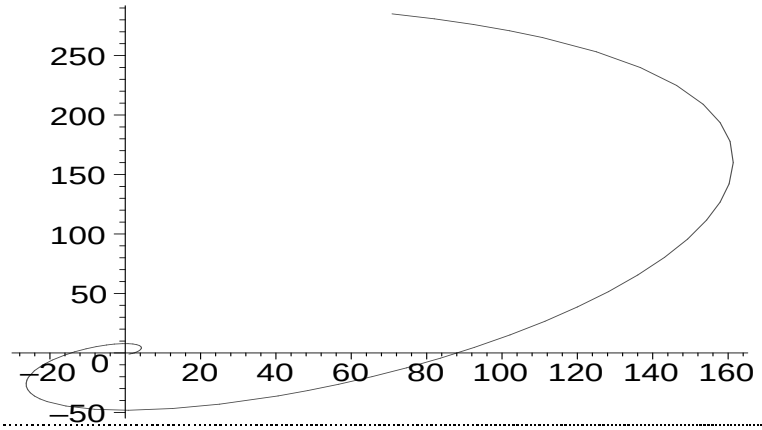
```
> Fset:=dsolve(
> {diff(x(t),t)=x(t)-y(t),diff(y(t),t)=x(t),x(0)=1,y(0)=-1},
> {x(t),y(t)}):

> fx:=t->subs(Fset,x(t)):

> fx(t);
```

$$\frac{1}{3}\,e^{(1/2\,t)}\,(3\cos(\frac{1}{2}\,t\,\sqrt{3})+3\,\sqrt{3}\sin(\frac{1}{2}\,t\,\sqrt{3}))$$

```
> fy:=t->subs(Fset,y(t)):

> fy(t);
```

$$\frac{1}{3}\,e^{(1/2\,t)}\,(3\,\sqrt{3}\sin(\frac{1}{2}\,t\,\sqrt{3})-3\cos(\frac{1}{2}\,t\,\sqrt{3}))$$

```
> plot([fx(t),fy(t),t=0..10]);
```



It is not difficult to see that the eigenvalues of the matrix for the equilibrium point $x = y = 0$ of (1.111) are $\frac{1}{2}(1 \pm \mathrm{i}\sqrt{3})$. The point is an *unstable focus* as shown by the MAPLE plot.

1.12 Linearizing Non-Linear Systems

Consider now the general autonomous system (1.25) and let there by an equilibrium point given by (1.26). To investigate the stability of $\boldsymbol{x}^*$ we again make

the substitution (1.95). Then for a particular member of the set of equations

$$\begin{aligned}\frac{\mathrm{d}\triangle x_\ell}{\mathrm{d}t} &= F_\ell(\boldsymbol{x}^* + \triangle\boldsymbol{x}) \\ &= \sum_{k=1}^{d}\left(\frac{\partial F_\ell}{\partial x_k}\right)^* \triangle x_k + \mathrm{O}(\triangle x_i \triangle x_j),\end{aligned} \tag{1.112}$$

where non-linear contributions in general involve all produces of pairs of the components of $\triangle\boldsymbol{x}$. Neglecting nonlinear contributions and taking all the set of equations gives

$$\frac{\mathrm{d}\triangle\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{J}^*\triangle\boldsymbol{x}, \tag{1.113}$$

where $\boldsymbol{J}^* = \boldsymbol{J}(\boldsymbol{x}^*)$ is the *stability matrix* with

$$\boldsymbol{J}(\boldsymbol{x}) = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} & \cdots & \dfrac{\partial F_1}{\partial x_d} \\ \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial F_2}{\partial x_2} & \cdots & \dfrac{\partial F_2}{\partial x_d} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial F_d}{\partial x_1} & \dfrac{\partial F_d}{\partial x_2} & \cdots & \dfrac{\partial F_d}{\partial x_d} \end{pmatrix}. \tag{1.114}$$

Analysis of the stability of the equilibrium point using the eigenvalues of $\boldsymbol{J}^*$ proceeds in exactly the same way as for the linear case. In fact it can be rigorously justified by the following theorem (also due to Lyapunov):

Theorem 1.12.1 *The equilibrium point $\boldsymbol{x}^*$ is asymptotically stable if the real parts of all the eigenvalues of the stability matrix $\boldsymbol{J}^*$ are (strictly) negative. It is unstable if they are all non-zero and at least one is positive.*

It will be see that the case where one or more eigenvalues are zero or purely imaginary is not covered by this theorem (and by linear analysis). This was the case in Example 1.8.1 at $a = 0$, where we needed the quadratic term to determine the stability.

Example 1.12.1 Investigate the stability of the equilibrium point of

$$\dot{x}(t) = \sin[x(t)] - y(t), \qquad \dot{y}(t) = x(t). \tag{1.115}$$

The equilibrium point is $x^* = y^* = 0$. Using the McLaurin expansion of $\sin(x) = \triangle x + \mathrm{O}(\triangle x^3)$ the equations take the form (1.113), where the stability matrix is

$$\boldsymbol{J}^* = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.116}$$

This is the same stability matrix as for the linear problem (1.111) and the equilibrium point is an *unstable focus*.

Example 1.12.2

$$\dot{x}(t) = -y(t) + x(t)[a - x^2(t) - y^2(t)], \tag{1.117}$$

$$\dot{y}(t) = x(t) + y(t)[a - x^2(t) - y^2(t)]. \tag{1.118}$$

The only equilibrium point for (1.117)–(1.118) is $x = y = 0$. Linearizing about the equilibrium point gives an equation of the form (1.113) with

$$\boldsymbol{J}^* = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}. \tag{1.119}$$

The eigenvalues of $\boldsymbol{J}^*$ are $a \pm \mathrm{i}$. So the equilibrium point is stable or unstable according as $a < 0$ or $a > 0$. When $a = 0$ the eigenvalues are purely imaginary, so the equilibrium point is a *centre*.

We can find two integrals of (1.117)–(1.118). If (1.117) is multiplied by x and (1.118) by y and the pair is added this gives

$$x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t} = (x^2 + y^2)(a - x^2 - y^2). \tag{1.120}$$

With $r^2 = x^2 + y^2$, if the trajectory starts with $r = r_0$ when $t = 0$,

$$2\int_0^t \mathrm{d}t = \begin{cases} \dfrac{1}{a}\displaystyle\int_{r_0}^{r} \left\{\frac{1}{a - r^2} + \frac{1}{r^2}\right\} \mathrm{d}(r^2), & a \neq 0, \\ -\displaystyle\int_{r_0}^{r} \frac{1}{r^4}\mathrm{d}(r^2), & a = 0, \end{cases} \tag{1.121}$$

giving

$$r^2(t) = \begin{cases} \dfrac{ar_0^2}{r_0^2 + \exp(-2at)\{a - r_0^2\}}, & a \neq 0, \\ \dfrac{r_0^2}{1 + 2tr_0^2}, & a = 0. \end{cases} \tag{1.122}$$

This gives

$$r(t) \longrightarrow \begin{cases} 0, & a \leq 0, \\ \sqrt{a}, & a > 0. \end{cases} \tag{1.123}$$

Now let $x = r\cos(\theta)$, $y = r\sin(\theta)$. Substituting into (1.117)–(1.118) and eliminating $\mathrm{d}r/\mathrm{d}t$ gives

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1. \tag{1.124}$$

If θ starts with the value $\theta(0) = \theta_0$ then

$$\theta = t + \theta_0. \tag{1.125}$$

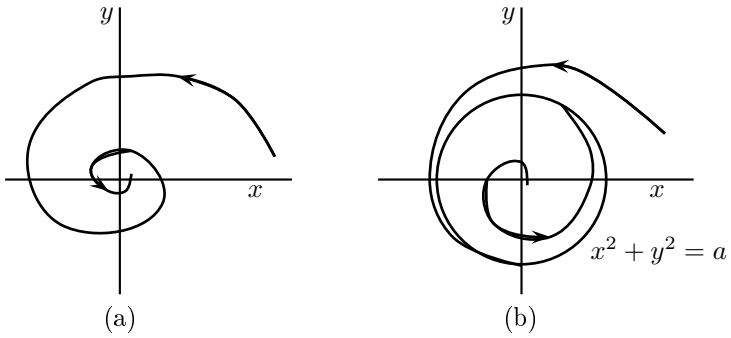


Figure 1.12: A Hopf bifurcation with (a) $a \leq 0$, (b) $a > 0$.

When $a < 0$ trajectories spiral with a constant angular velocity into the origin.

When $a = 0$ linear analysis indicates that the origin is a centre. However, the full solution shows that orbits converge to the origin as $t \to \infty$, with $r(t) \simeq 1/\sqrt{2t}$, which is a slower rate of convergence than any exponential.

When $a > 0$, if $r(0) = r_0 = \sqrt{a}$, $r(t) = \sqrt{a}$. The circle $x^2 + y^2 = a$ is invariant under the evolution of the system. The circle $x^2 + y^2 = a$ is a new type of stable solution called a *limit cycle*. Trajectories spiral, with a constant angular velocity towards the limit cycle circle, either from outside if $r_0 > \sqrt{a}$ or from inside if $r_0 < \sqrt{a}$ see Fig. 1.12. The change over in behaviour at $a = 0$ is an example of the *Hopf bifurcation*. If the behaviour is plotted in the three-dimensional space of $\{a, x, y\}$ then it resembles the supercritical pitchfork bifurcation (Fig. 1.13).

Example 1.12.3

$$\ddot{x}(t) = a - x^2(t), \tag{1.126}$$

which can be written as

$$\dot{x}(t) = y(t), \tag{1.127}$$
$$\dot{y}(t) = a - x^2(t). \tag{1.128}$$

The equilibrium points for (1.127)–(1.128) are given by $x = x^* = \pm\sqrt{a}$, $y = 0$, *when* $a \geq 0$ *and there are no equilibrium points when* $a < 0$.

Before considering the stability of the equilibrium point we obtain an integral of the equations (1.127)–(1.128). Since

$$x^2 \frac{\mathrm{d}x}{\mathrm{d}t} + y \frac{\mathrm{d}y}{\mathrm{d}t} = ay = a \frac{\mathrm{d}x}{\mathrm{d}t}, \tag{1.129}$$

it follows that a trajectory lies on a curve

$$\frac{1}{3}x^3 + \frac{1}{2}y^2 - ax = c. \tag{1.130}$$

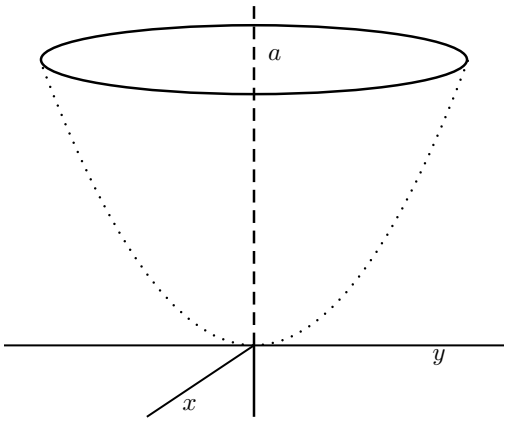
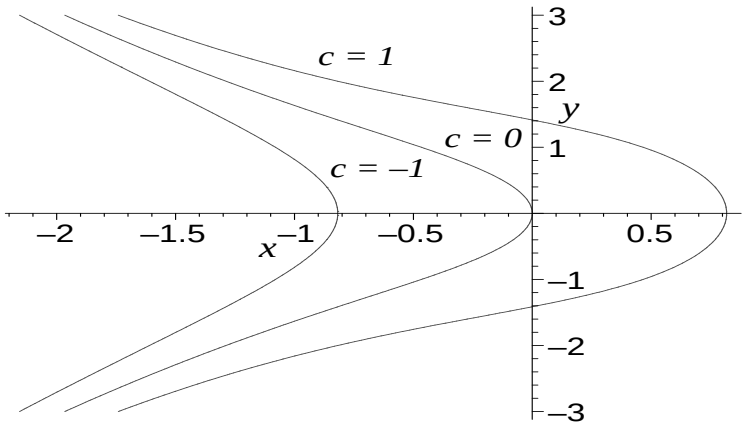


Figure 1.13: A Hopf bifurcation in the space of $\{a, x, y\}$.

for some fixed value of c. The curves are symmetric about the x–axis. Trajectories with $c < 0$ do not cut the y–axis, the trajectory with $c = 0$ passes through the origin and trajectories with $c > 0$ cut the y–axis at $y = \pm\sqrt{2c}$. Curves with $a \neq 0$ cut the x–axis with a vertical tangent. We now consider separately the three ranges of a:

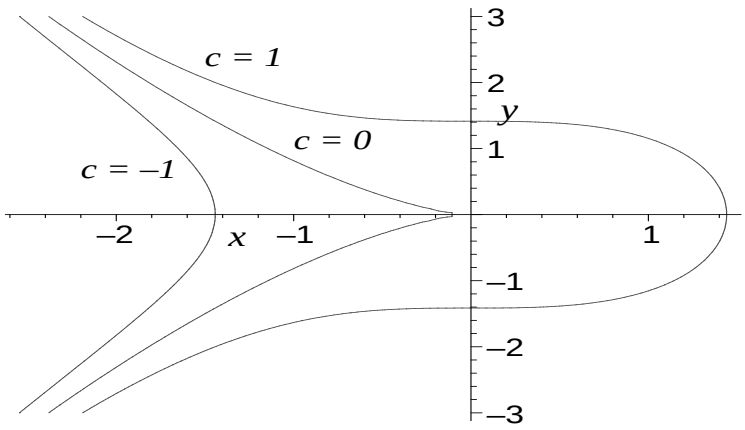
$\underline{a < 0.}$ In this case there are no equilibrium points and $\dot{y}(t) < 0$ for all x and y. A trajectory cuts the x–axis at the roots of $x^3 - 3ax - 3c = 0$. For negative a this cubic has no extrema so a trajectory cut the x–axis only once. Also as $|y|$ increases from zero on a trajectory it follows from (1.130) that x must decrease so the trajectories must be convex to the right. The phase pattern for $a = -1$ can be plotted using MAPLE :

```
> with(plots):

> f:=(x,y,a)->x^3/3+y^2/2-a*x:

> curve:=implicitplot(
> {f(x,y,-1)=-1,f(x,y,-1)=0,f(x,y,-1)=1}, x=-3..1.5,y=-3..3,
> grid=[100,100],labelfont=[TIMES,ITALIC,12]):

> text:=plots[textplot](
> {[-0.8,0.55,`c=-1`], [-0.32,1,`c=0`], [-0.8,2.25,`c=1`]},
> align={ABOVE,RIGHT}, font=[TIMES,ITALIC,14]):

> plots[display]({curve,text});
```



(First run the MAPLE *program without the labelling* $c = -1, 0, 1$ *on the curves. Then add the labels by reading off from the plot the best place for them to be put.)*

$\underline{a = 0.}$ In this case there is one equilibrium point. Curves with $c \neq 0$ are very similar to those for $a < 0$. The curve with $c = 0$ is given by $\frac{1}{3}x^3 + \frac{1}{2}y^2 = 0$ which has a cusp at the origin rather than a vertical tangent. A MAPLE program like the one given above can be used to obtain the plot:



$\underline{a > 0.}$ In this case there are two equilibrium points $x = x^* = \pm\sqrt{a}$, $y = 0$. Linearizing about the equilibrium point $(x^*, 0)$ gives equations of the form (1.113) with

$$\boldsymbol{J}^* = \boldsymbol{J}(x^*, 0) = \begin{pmatrix} 0 & 1 \\ -2x^* & 0 \end{pmatrix}. \tag{1.131}$$

The equilibrium point $(\sqrt{a}, 0)$ has eigenvalues $\lambda^{(1,2)} = \pm \mathrm{i}\tau$, where $\tau = (4a)^{1/4}$, so it is a *centre*. The equilibrium point $(-\sqrt{a}, 0)$ has eigenvalues $\lambda^{(1)} = \tau$, $\lambda^{(2)} = -\tau$, with corresponding right eigenvectors $(1, \tau)^{\mathrm{T}}$ and $(-1, \tau)^{\mathrm{T}}$. So it is a *saddle-point* and the line along which it is attractive is given by $(\triangle x, \triangle y) \sim (-1, \tau)$. Now the trajectory which passes through $(-\sqrt{a}, 0)$ is, from (1.130)

$$\frac{1}{3}x^3 + \frac{1}{2}y^2 - ax = \frac{2}{3}a\sqrt{a}. \tag{1.132}$$

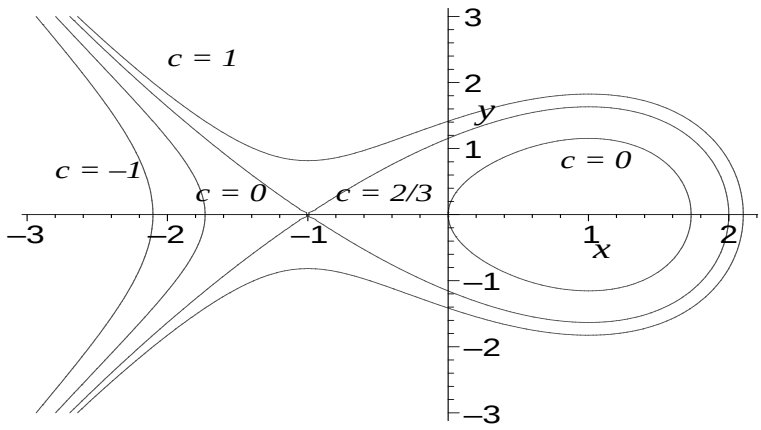
Differentiating

$$\dot{y}(t) = \frac{a - x^2}{y} = \pm\frac{a - x^2}{\sqrt{\frac{4}{3}a\sqrt{a} + 2ax - \frac{2}{3}x^3}}. \tag{1.133}$$

At the equilibrium point $x = -\sqrt{a}$ this expression is undefined and we must substitute $x = \triangle x - \sqrt{a}$. This gives

$$\dot{y}(t) = \frac{a - x^2}{y} = \pm\frac{2\sqrt{a}\triangle x - (\triangle x)^2}{\sqrt{2\sqrt{a}(\triangle x)^2 - \frac{2}{3}(\triangle x)^3}} \simeq \pm\tau\frac{\triangle x}{|\triangle x|}. \tag{1.134}$$

So one of the branches of the curve through the equilibrium point is in the stable direction. The MAPLE plot is:



The closed part of the orbit (1.132), which begins at the saddle-point $(-\sqrt{a}, 0)$ and returns to the same point, has two important properties:

- It separates the closed orbits from the trajectories which go off to infinity and is thus called a *separatrix*.
- It connects the saddle-point to itself. Such a trajectory is called *homoclinic*. (A trajectory connecting different saddle-points together is called *heteroclinic*.)

In general the points where a trajectory cuts the x–axis are given, from (1.130), by

$$f(x) = \tfrac{1}{3}x^3 - ax - c = 0. \tag{1.135}$$

$f(x)$ has a maximum at $x = -\sqrt{a}$ with $f(-\sqrt{a}) = \frac{2}{3}a\sqrt{a} - c$ and a minimum $x = \sqrt{a}$ with $f(\sqrt{a}) = -\frac{2}{3}a\sqrt{a} - c$. So, when $\frac{2}{3}a\sqrt{a} > c > -\frac{2}{3}a\sqrt{a}$, an orbit cuts the x–axis at three points. The upper limit of this range is the separatrix and the case $c = 0$ corresponds to the closed orbit through the origin. As $c \to -\frac{2}{3}a\sqrt{a}$ the closed orbit contracts to a point on the centre $(\sqrt{a}, 0)$. We can calculate the period around a closed orbit from (1.127) and (1.130).

$$\dot{x}(t) = \sqrt{2c + 2ax - \tfrac{2}{3}x^3}. \tag{1.136}$$

If this orbit cuts the x–axis at x_1 and x_2 $(x_1 < x_2)$ then the period T around the orbit is

$$T = 2\int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{2c + 2ax - \frac{2}{3}x^3}}. \tag{1.137}$$

The integrand has a singularity at $x = x_1 = -\sqrt{a}$ on the separatrix; so $T \to \infty$ on this curve, which is the limit of the closed orbits.

Example 1.12.4 *The equation of motion of a simple pendulum of length ℓ swinging under gravity g is*

$$\ddot{\theta}(t) = -a\sin[\theta(t)], \tag{1.138}$$

where θ is the angle the pendulum makes with the downward vertical and $a = g/\ell$. Using the angular velocity ω this equation of motion can be written

$$\dot{\theta}(t) = \omega(t), \tag{1.139}$$
$$\dot{\omega}(t) = -a\sin[\theta(t)]. \tag{1.140}$$

The equilibrium points for (1.139)–(1.140) are $\omega = 0$, $\theta = n\pi$, $n = 0, \pm 1, \pm 2 \ldots$.

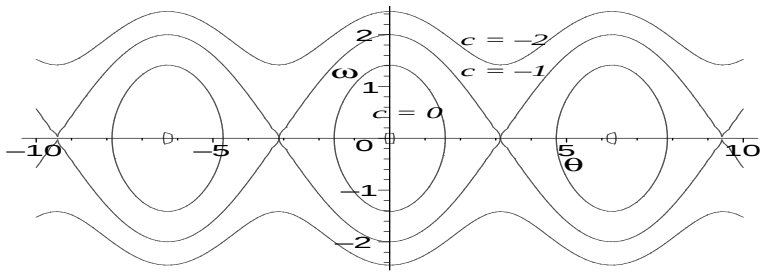
Equations (1.139)–(1.140) can be integrated to give the family of curves

$$\omega^2 = 2a\{\cos(\theta) - c\} \tag{1.141}$$

in the phase space Γ_2 of the variables $\{\theta, \omega\}$ parameterized by c. At the equilibrium point $(\theta^*, 0)$

$$\boldsymbol{J}^* = \begin{pmatrix} 0 & 1 \\ -a\cos(\theta^*) & 0 \end{pmatrix} \tag{1.142}$$

So the eigenvalues are $\lambda^{(1)} = \mathrm{i}\sqrt{a\cos(\theta^*)}$ and $\lambda^{(2)} = -\mathrm{i}\sqrt{a\cos(\theta^*)}$. The equilibrium points $\theta = 2n\pi$, $n = 0, \pm1, \pm2, \ldots$, where the eigenvalues are purely imaginary, are *centres* and the equilibrium points $\theta = (2n+1)\pi$, $n = 0, \pm1, \pm2, \ldots$, where the eigenvalues are real and of opposite sign, are *saddle-points*. A trajectory given by (1.141) cuts the θ–axis at a periodic sequence of points if $|c| \leq 1$ and is the *heteroclinic separatrix* passing through the saddle-points if $c = -1$. If $c = 1$ it collapses into a set of points at the centres. Again using a MAPLE program, like that given above, we obtain curves of (1.141) with $a = 0$:



The period of a closed orbit around the origin which cuts the θ–axis at $\theta = \pm\theta_0 = \pm\arccos(c)$ is given by

$$T = 2\int_{-\theta_0}^{\theta_0} \frac{\mathrm{d}\theta}{\sqrt{2a(\cos(\theta) - c)}}. \tag{1.143}$$

This integral can be expressed in terms of a complete elliptic integral of the first kind (Drazin, p. 28) and the usual formula $T = 2\pi/\sqrt{c} = 2\pi\sqrt{\ell/g}$ for small oscillations can be recovered in the limit $c \to 1$.

1.13 Conservative Systems

For a conservative system with equation

$$\ddot{x}(t) = -V'(x), \tag{1.144}$$

we follow the procedure of Sect. 1.3 and take

$$\dot{x}(t) = y(t), \tag{1.145}$$
$$\dot{y}(t) = -V'(x). \tag{1.146}$$

The equilibrium points are the turning points of $V(a, b, c, \ldots, x)$, appearing in the space Γ_2 of $\{x, y\}$ on the x–axis. Now linearize about the equilibrium point $(x^*, 0)$.

$$\frac{\mathrm{d}\triangle x}{\mathrm{d}t} = \triangle y, \tag{1.147}$$
$$\frac{\mathrm{d}\triangle y}{\mathrm{d}t} = -\triangle x V''(x^*). \tag{1.148}$$

The eigenvalues of the stability matrix are $\pm\mathrm{i}\sqrt{V''(x^*)}$. So x^* is a centre if $V''(x^*) > 0$ and a saddle-point if $V''(x^*) < 0$. These two conditions correspond respectively to the potential function $V(x)$ having a local minimum and maximum respectively at $x = x^*$. From (1.145)–(1.146),

$$y\frac{\mathrm{d}y}{\mathrm{d}t} = -yV'(x) = -\frac{\mathrm{d}x}{\mathrm{d}t}V'(x). \tag{1.149}$$

Integrating this gives

$$\frac{1}{2}y^2 + V(x) = E, \tag{1.150}$$

for constant E. In mechanical terms this is the energy integral for a particle of unit mass, location x and speed y moving under the influence of a potential $V(x)$ with constant energy E. From (1.150)

$$y = \pm\sqrt{2Y(x)}, \tag{1.151}$$

where

$$Y(x) = E - V(x). \tag{1.152}$$

The zeros of $Y(x)$ are the points in Γ_2 where the curve given by (1.150) cuts the x–axis. Let $\tilde{x}$ be such a point about which $Y(x)$ has the Taylor expansion

$$\begin{aligned} Y(x) &= (x-\tilde{x})Y'(\tilde{x}) + \frac{1}{2}(x-\tilde{x})^2 Y''(\tilde{x}) \\ &\qquad + \mathrm{O}((x-\tilde{x})^3), \\ &= -(x-\tilde{x})V'(\tilde{x}) - \frac{1}{2}(x-\tilde{x})^2 V''(\tilde{x}) \\ &\qquad + \mathrm{O}((x-\tilde{x})^3). \end{aligned} \tag{1.153}$$

Then, if $V'(\tilde{x}) \neq 0$,

$$y^2 \simeq -2(x-\tilde{x})V'(\tilde{x}), \tag{1.154}$$

in a neighbourhood of $(\tilde{x}, 0)$. The curve (1.150) is parabolic; convex in the positive x–direction if $V'(\tilde{x}) > 0$ and in the negative x–direction if $V'(\tilde{x}) < 0$.

These cases correspond to extremities of closed orbits of the curve (1.150). Now suppose $V'(\tilde{x}) = 0$ and assume $V''(\tilde{x}) \neq 0$. From (1.151) and (1.145),

$$y \simeq \pm \mathrm{i}(x - \tilde{x})\sqrt{V''(\tilde{x})}, \tag{1.155}$$

$$x(t) \simeq \tilde{x} + c\,\exp\{\pm \mathrm{i}t\sqrt{V''(\tilde{x})}\}. \tag{1.156}$$

where c is a constant. When $V''(\tilde{x}) > 0$ (1.156) again confirms the linear analysis of periodic orbits about a centre. When $V''(\tilde{x}) < 0$ the choice of signs in (1.156) gives the stable and unstable directions from the saddle-point, with the phase point taking an infinite amount of time to reach the saddle-point in the stable direction. This is the same result as the divergence of the integral (1.143) as $\theta \to \arccos(c)$.

Now consider a possible plot (Fig. 1.14) of $Y(x)$ given by (1.152) against x.

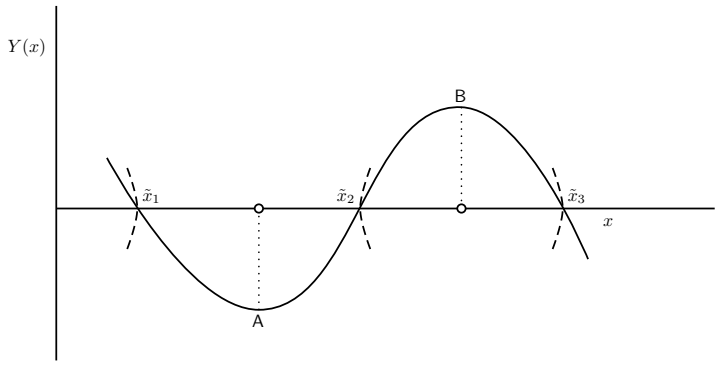


Figure 1.14: A possible plot of $Y(x)$ against x. The shape of parts of trajectories (in the $\{x, y\}$ plane) are shown by broken lines.

The zeros $\tilde{x}_1, \tilde{x}_2, \ldots$ on the x–axis are points like $\tilde{x}$ with $Y'(\tilde{x}) = -V'(\tilde{x}) \neq 0$. The points $\tilde{x}_1$ and $\tilde{x}_3$ are places where $V'(\tilde{x}) = -Y'(\tilde{x}) > 0$, so they correspond to right-hand extremities of closed orbits, while $\tilde{x}_2$ corresponds to a left-hand extremity. The minimum at A is a point where $V'(x) = 0$ and $V''(x) = -Y''(x) < 0$, so it is a saddle-point, whereas the maximum at B is a centre. If A approaches the x–axis $\tilde{x}_1$ and $\tilde{x}_2$ coalesce at a point where $Y'(\tilde{x}) = -V'(\tilde{x}) = 0$, $Y''(\tilde{x}) = -V''(\tilde{x}) > 0$ and the two trajectories merge to produce a crossing point like those shown in the MAPLE plot on page 33, at odd multiples of π. If B approaches the x–axis $\tilde{x}_2$ and $\tilde{x}_3$ coalesce at a point where $Y'(\tilde{x}) = -V'(\tilde{x}) = 0$, $Y''(\tilde{x}) = -V''(\tilde{x}) < 0$ and the orbit between $\tilde{x}_2$ and $\tilde{x}_3$ closes in on the centre like those shown in the MAPLE plot on page 33, at even multiples of π.

Given that there is a closed orbit between $\tilde{x}_2$ and $\tilde{x}_3$, it follows from the symmetry of (1.150) that the time taken between $\tilde{x}_2$ and $\tilde{x}_3$ is half the complete

period. Since $Y(x) > 0$ in the interval $(\tilde{x}_2, \tilde{x}_3)$ the period T for the orbit is given, from (1.151), by

$$T = 2\int_{\tilde{x}_2}^{\tilde{x}_3} \frac{\mathrm{d}x}{\sqrt{2Y(x)}}. \tag{1.157}$$

Problems 1

1) Find out as much as you can about the one-dimensional dynamic systems:

   (i) $\dot{x}(t) = x(t)[a - c - ab\,x(t)]$,

   (ii) $\dot{x}(t) = a\,x(t) - b\,x^2(t) + c\,x^3(t)$,

   You may assume that a and b are non-zero but you can consider the case $c = 0$. You should be able to

   (a) Find the equilibrium points and use linear analysis to determine their stability.

   (b) Draw the bifurcation diagrams in the $\{x, a\}$–plane for the different ranges of b and c.

   (c) Solve the equations explicitly.

2) Verify that the system

   $$\dot{x}(t) = x(t) + \sin[y(t)],$$
   $$\dot{y}(t) = \cos[x(t)] - 2\,y(t) - 1$$

   has an equilibrium point at $x = y = 0$ and determine its type.

3) Find *all* the equilibrium points of

   $$\dot{x}(t) = -x^2(t) + y(t),$$
   $$\dot{y}(t) = 8\,x(t) - y^2(t)$$

   and determine their type.

4) Show that the system given by

   $$\dot{x}(t) = -y + \frac{x(1 - x^2 - y^2)}{\sqrt{x^2 + y^2}}, \qquad \dot{y}(t) = x + \frac{y(1 - x^2 - y^2)}{\sqrt{x^2 + y^2}},$$

   has a stable limit cycle given by $x = \cos(\theta_0 + t)$, $y = \sin(\theta_0 + t)$.

5) A particle moves around a smooth circular wire of radius ℓ which is fixed relative to a vertical plane. Gravity g acts on the particle and the plane rotates with constant angular velocity Ω about a vertical diameter of the circle. The motion of the particle on the circle is given by

$$\ddot{\theta}(t) = \Omega^2 \sin(\theta)\{\cos(\theta) - a\},$$

where θ is the angle the radius to the particle makes with the downward vertical and $a = g/(\Omega^2 \ell) > 0$. Find the equilibrium points in the plane of $\{\theta, \omega\}$ where $\omega(t) = \dot{\theta}(t)$ and give a sketch of the bifurcation diagram in the $\{a, \theta\}$ plane indicating the stability of the equilibrium lines. Find out anything else you can about this problem.

6) Show that the system

$$\dot{x}(t) = -y + x\{f(x,y) - a^2\}^n, \qquad \dot{y}(t) = x + y\{f(x,y) - a^2\}^n,$$

where n is a positive integer and $f(x,y)$ is continuous, can be transformed to

$$\dot{r}(t) = r\{f(r\cos(\theta), r\sin(\theta)) - a^2\}^n, \qquad \dot{\theta}(t) = 1,$$

in terms of polar coordinates given by $x = r\cos(\theta)$, $y = r\sin(\theta)$. Deduce that the equilibrium solution $r = 0$ is stable or unstable according as

$$\{f(0,0) - a^2\}^n <, > 0.$$

With $f(x,y) = x^2 + y^2$ show that the limit cycle $r = a$ is unstable if n is odd and *semistable* if n is even, where 'semistable' means that it is stable from one side and unstable from the other.

7) A system is given by

$$\dot{z}(t) = \mathrm{i}z + zf(|z|),$$

where $z = x + \mathrm{i}y$. Express this formula in polar form. Show that, when

$$f(r) = \begin{cases} \sin\{1/(r^2-1)\} & r \neq 1, \\ 0 & r = 1. \end{cases}$$

the system has limit cycles $r = 1$ and $r = \sqrt{1 + \frac{1}{n\pi}}$ for $n = \pm 1, \pm 2, \ldots$. Determine the stability of the limit cycles and of the equilibrium point $r = 0$.

8) Consider the equation

$$\dot{z}(t) = a + z(b - |z|^2),$$

where z is a complex function of t and a and b are real. Expressing z in the usual polar form $z = r\exp(\mathrm{i}\theta)$ show that

$$\dot{r}(t) = a\cos(\theta) + r(b - r^2), \qquad \dot{\theta}(t) = -\frac{a\sin(\theta)}{r}.$$

Investigate the steady solutions and their stability and sketch their curves in the plane of $\{b, r\}$ for fixed positive, zero and negative a.

Chapter 2

Bifurcations and Catastrophe Theory

2.1 The Classification of Bifurcations

We consider an d-dimensional autonomous system which evolves according to the equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{a}, \boldsymbol{x}), \tag{2.1}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)^{\mathrm{T}}$ and $\boldsymbol{a} = (a_1, a_2, \ldots, a_\eta)^{\mathrm{T}}$ is a vector of independent parameters. The equilibrium points for fixed $\boldsymbol{a}$ are solutions of (2.1) for which $\dot{\boldsymbol{x}}(t) = \boldsymbol{0}$; that is they are the roots of

$$\boldsymbol{F}(\boldsymbol{a}, \boldsymbol{x}) = \boldsymbol{0}. \tag{2.2}$$

The condition (2.2) corresponds to d surfaces in Γ_d which, will in general intersect in one or more points. If the η components of $\boldsymbol{a}$ vary over their allowed ranges then the equilibrium solutions form an η–dimensional *equilibrium surface or surfaces* in the $(d+\eta)$–dimensional space $\Lambda_{d+\eta} = \Pi_\eta \times \Gamma_d$, where Π_η is the space of the parameters $\boldsymbol{a}$.

*A **bifurcation point** or **branch point** is a solution $(\boldsymbol{x}_0, \boldsymbol{a}_0)$ of (2.2) such that the number of solutions $\boldsymbol{x}$ of (2.2) in a small neighbourhood of $\boldsymbol{x}_0$ changes when $\boldsymbol{a}$ varies within a small neighbourhood of $\boldsymbol{a}_0$.*

2.1.1 The One-Dimensional, One-Parameter Case

In this case $d = 1$ with one variable $x(t)$ and $\eta = 1$ with one parameter a. Then (2.1) becomes

$$\dot{x}(t) = F(a, x), \tag{2.3}$$

To discuss the possible bifurcations in this case the following definition and theorem will be needed:

If there exist $P(a',x')$ and $Q(a',x')$, such that

$$\begin{aligned} F(a'+\delta a, x'+\delta x) &= F(a',x') \\ &\quad + P(a',x')\delta a + Q(a',x')\delta x \\ &\quad + \theta(a',x',\delta a,\delta x)\delta a \\ &\quad + \psi(a',x',\delta a,\delta x)\delta x, \end{aligned} \tag{2.4}$$

with

$$\left.\begin{array}{l}\theta(a',x',\delta a,\delta x)\to 0\\ \psi(a',x',\delta a,\delta x)\to 0\end{array}\right\} \quad \text{as } (\delta a,\delta x)\to(0,0) \tag{2.5}$$

then $F(a,x)$ is ***differentiable at*** *(a',x') with*

$$\begin{aligned} P(a',x') &= \left(\frac{\partial F}{\partial a}\right)_{a=a',x=x'} = F_a(a',x'), \\ Q(a',x') &= \left(\frac{\partial F}{\partial x}\right)_{a=a',x=x'} = F_x(a',x'). \end{aligned} \tag{2.6}$$

The Implicit Function Theorem: *If $F(a,x)$ is differentiable and has continuous partial derivatives with $F_x(a,x)\neq 0$ in the closed rectangle $a_1\le a\le a_2$, $x_1\le x\le x_2$ and if $F(a_0,x_0)=0$ at the point (a_0,x_0) in the open rectangle $a_1<a<a_2$, $x_1<x<x_2$, then there exists an interval (a',a'') containing a_0 within which $F(a,x)=0$ defines x as a continuous and differentiable function of a with*

$$\frac{\mathrm{d}x}{\mathrm{d}a} = -\frac{F_a(a,x)}{F_x(a,x)}. \tag{2.7}$$

Suppose that (a_0,x_0) is a point on the curve of equilibrium points given by

$$F(a,x)=0. \tag{2.8}$$

If (a_0,x_0) is a bifurcation point then a small positive or negative change (one but not both) in a will increase the number of solutions in x of (2.8). This means that x_0 must be a multiple root of $F(a_0,x)=0$ and so a necessary, but not sufficient, condition for (a_0,x_0) to be a bifurcation point is that it is a simultaneous solution of (2.8) and

$$F_x(a,x)=0. \tag{2.9}$$

In principle the solutions of the pair of equations (2.8)–(2.9) will give the bifurcation set. Thus for Example 1.8.2 the bifurcation set is given by solving

$$\begin{aligned} &x(x^2 - 2xc - a) = 0 \\ &3x^2 - 4xc - a = 0. \end{aligned} \tag{2.10}$$

These equations give $x = -a/c$ and $a(a + c^2) = 0$, yielding the transcritical bifurcation at $a = x = 0$ and the simple turning point at $a = -c^2$, $x = c$ (see Fig. 1.11).

That these conditions are not *sufficient* to yield a bifurcation is illustrated by the case $F(a, x) = (a - x)^2$. Any values of a and x on the line $x = a$ will satisfy (2.8) and (2.9) but there is no bifurcation. This is the degenerate case of a transcritical bifurcation disappearing when the crossing pair of solutions merge into each other.

We now consider different types of points which can occur on the equilibrium curve. These will include all the simple types of bifurcation. We assume that $F(a, x)$ is infinitely differentiable in both variables. Then the Taylor expansion about (a_0, x_0) in the two variables a and x is

$$\begin{aligned} F(a,x) \quad = \quad & +(a-a_0)F_a(a_0,x_0) + (x-x_0)F_x(a_0,x_0) \\ & + \tfrac{1}{2}(x-x_0)^2 F_{xx}(a_0,x_0) \\ & + (x-x_0)(a-a_0)F_{ax}(a_0,x_0) \\ & + \tfrac{1}{2}(a-a_0)^2 F_{aa}(a_0,x_0) + \cdots. \end{aligned} \tag{2.11}$$

*(a_0, x_0) is **regular point** on the equilibrium curve if $F(a_0, x_0) = 0$ and one or both of $F_a(a_0, x_0)$ and $F_x(a_0, x_0)$ is non-zero. All other points on the equilibrium curve are **singular points**. If (a_0, x_0) is a singular point on the equilibrium curve then it is called a **higher-order singularity** if $F_{xx}(a_0, x_0) = F_{aa}(a_0, x_0) = F_{ax}(a_0, x_0) = 0$.*

Excluding the case where (a_0, x_0) is a higher-order singularity there are the following possibilities:

- (a_0, x_0) is a **regular point** and:

  $F_x(a_0, x_0) \neq 0$. Then, according to the implicit function theorem, (2.8) and (2.11) give the equilibrium curve as an expression for x as a continuous function of a in a neighbourhood around a_0, with $\mathrm{d}x/\mathrm{d}a$ given by (2.7).

  $F_x(a_0, x_0) = 0$. The roles of a and x in the implicit function theorem can now be reversed and the equilibrium curve is given as a expressed as a continuous function of x in a neighbourhood of x_0. From (2.7) $\mathrm{d}a/\mathrm{d}x = 0$ at $x = x_0$, so the equilibrium curve of x as a function of a has a horizontal tangent at (a_0, x_0). The is called a *regular or simple turning point* and it is the only type of bifurcation which is not a singular point. The bifurcation at the origin in Example 1.8.1 is a case of this.

- (a_0, x_0) is a **singular point** with $[F_{ax}(a_0, x_0)]^2 > F_{aa}(a_0, x_0)F_{xx}(a_0, x_0)$. (2.11) as far as quadratic terms has the form

$$F(a, x) = \{\alpha_1(a - a_0) + \beta_1(x - x_0)\}\{\alpha_2(a - a_0) + \beta_2(x - x_0)\}, \tag{2.12}$$

where all the coefficients are real and

$$\begin{aligned} 2\alpha_1\alpha_2 &= F_{aa}(a_0, x_0), \\ 2\beta_1\beta_2 &= F_{xx}(a_0, x_0), \\ \alpha_1\beta_2 + \alpha_2\beta_1 &= F_{ax}(a_0, x_0). \end{aligned} \tag{2.13}$$

The equilibrium curve has two distinct branches through (a_0, x_0) with tangents given by the linear factors in (2.12) and (a_0, x_0) is a *double point.*

Now suppose $\beta_1 \neq 0$ **and** $\beta_2 \neq 0$ and consider the local stability of the first factor at a point

$$x^* = x_0 - \alpha_1(a - a_0)/\beta_1. \tag{2.14}$$

With $x = x^* + \triangle x$, from (2.3) and (2.12),

$$\frac{\mathrm{d}\triangle x}{\mathrm{d}t} = \triangle x(a - a_0)\{\alpha_2\beta_1 - \alpha_1\beta_2\}. \tag{2.15}$$

The curve with tangent $\alpha_1(a - a_0) + \beta_1(x - x_0) = 0$ changes its stability as a increases through a_0 from stable to unstable if $\alpha_2\beta_1 > \alpha_1\beta_2$ and unstable to stable if $\alpha_2\beta_1 < \alpha_1\beta_2$. Since the stability of the second factor is given by reversing the subscripts 1 and 2, it is clear that the stability is also reversed and the bifurcation is a *transcritical point.* This analysis includes the case where either one but not both of α_1 or α_2 is zero. Then one member of the pair of tangents is the vertical line $x = x_0$ as in the case of the transcritical bifurcation at the origin in Example 1.8.2 for $c \neq 0$.

Now suppose $\beta_1 = 0$ **and** $\beta_2 \neq 0$**.** This is the limiting case of the previous situation where the first tangent line is horizontal. The complete curve plotted for a as a function of x has a turning point at (a_0, x_0) and the bifurcation is called a *singular turning point.* The precise form for this bifurcation is revealed by taking higher-order factors. One possibility is the pitchfork bifurcation of Example 1.8.2, with $c = 0$, when the leading terms in $F(a, x)$ about the origin are $x(a - x^2)$. The linear factor is a vertical line and the factor with a horizontal tangent is quadratic.

- (a_0, x_0) is a **singular point** with $[F_{ax}(a_0, x_0)]^2 < F_{aa}(a_0, x_0)F_{xx}(a_0, x_0)$. In this case the set of leading quadratic terms has complex roots and cannot be resolved into real linear factors. There are no points on an equilibrium curve in a neighbourhood of (a_0, x_0) which is an isolated equilibrium point called a *conjugate point*.

- (a_0, x_0) is a **singular point** with $[F_{ax}(a_0, x_0)]^2 = F_{aa}(a_0, x_0)F_{xx}(a_0, x_0)$. In this case the leading term is a product of two identical linear factors. In general

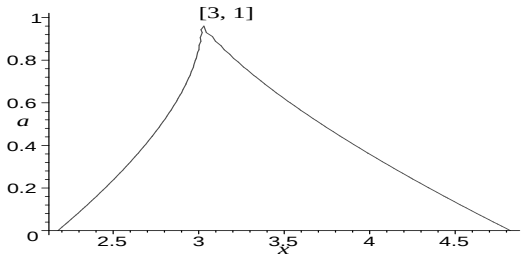
$$F(a, x) = \{\alpha(a - a_0) + \beta(x - x_0)\}^2 + g(a, x), \tag{2.16}$$

where $g(a, x)$ has a zero at (a_0, x_0) and is of at least cubic degree in the variables $(a - a_0)$, $(x - x_0)$. Equation (2.8) has a solution only when $g(a, x) < 0$ and, if $g(a, x)$ changes sign along the line $\alpha(a - a_0) + \beta(x - x_0) = 0$ at (a_0, x_0), the equilibrium curve has a cusp at (a_0, x_0). A simple example is

$$F(a, x) = \{(a - 1) + 2(x - 3)\}^2 + 7(a - 1)^3, \tag{2.17}$$

The curve $F(a, x) = 0$ can be plotted by the following MAPLE code:

```
> with(plots):
> f:=(x,a)->((a-1)+2*(x-3))^2+7*(a-1)^3:
> curve:=implicitplot(f(x,a)=0,x=0..5,
> a=0..4,grid=[100,100], labelfont=[TIMES,ITALIC,12]):
> text:=plots[textplot]
> ([3,1,`(3,1)`],align={ABOVE,RIGHT}, font=[TIMES,ROMAN,12]):
> plots[display]
> ({curve,text});
```



This is a *cusp-point* bifurcation.

2.1.2 The One-Dimensional, Two Parameter Case

In this case (2.3) is replaced by

$$\dot{x}(t) = F(a, b, x), \tag{2.18}$$

and the equilibrium solutions form a surface (or surfaces) in the space Λ_3 of $\{a, b, x\}$. With b set at a fixed value we are taking a slice through the equilibrium surface and on the resulting curve of equilibrium points in the $\{a, x\}$ plane we may see any of the bifurcations described above for a one-dimensional one parameter system. In this situation the parameter b is passive or *irrelevant* to the occurrence of the bifurcation in the sense that it plays no role in the occurrence of the bifurcation. A simple example of this would be the modification

$$\dot{x}(t) = a + b - x^2, \tag{2.19}$$

to Example 1.8.1. There is now a simple turning point bifurcation at $a = -b$, $x = 0$ which gives a picture like Fig. 1.9 in any plane parallel to the x–axis. In a similar way the parameter c is an irrelevant parameter in Examples 1.52 and 1.53.

We now consider an example of a new type of bifurcation which can occur only because of the presence of two parameters.

Example 2.1.1

$$\dot{x}(t) = 4x^3 - 2ax + b. \tag{2.20}$$

The equilibrium points for (2.20) lie on the cubic curve

$$F(a, b, x) = 4x^3 - 2ax + b = 0. \tag{2.21}$$

Taking fixed values of a and b there will in general be three solutions or one solution in x to (2.21) (Fig. 2.1). The boundaries between these regions are given by the curves on the surface where, for fixed a, the tangent is parallel to the x–axis. These curves form lines of simple turning point bifurcations, which from (2.21) are given by

$$\frac{\partial b}{\partial x} = 2a - 12x^2 = 0. \tag{2.22}$$

This gives $x = \pm\sqrt{\frac{a}{6}}$. Now let $x = \pm\sqrt{\frac{a}{6}} + \triangle x$ and, substituting back into (2.20),

$$\frac{\mathrm{d}\triangle x}{\mathrm{d}t} = 4(\triangle x)^3 \pm 12\sqrt{\frac{a}{6}}(\triangle x)^2 + b \mp \frac{4a}{3}\sqrt{\frac{a}{6}}. \tag{2.23}$$

Neglecting the cubic term this becomes a similar situation to that discussed in Example 1.8.1 with a simple turning point bifurcation occurring at

$$b \mp \frac{4a}{3}\sqrt{\frac{a}{6}} = 0, \qquad \text{which is} \qquad 27b^2 = 8a^3. \tag{2.24}$$

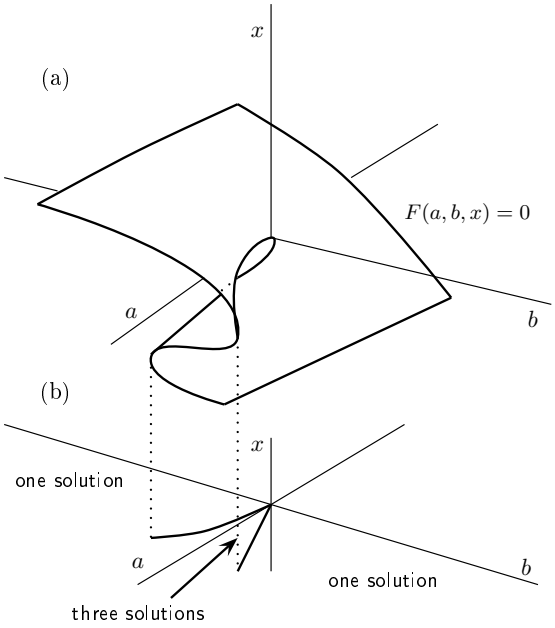


Figure 2.1: (a) The surface $F(a, b, x) = 0$ given by (2.21). (b) The cusp bifurcation set in the plane of $\{a, b\}$.

The curve in the $\{a, b\}$ plane given by (2.24) which is the *bifurcation set* for this example has a cusp at the origin. Both the variables a and b are needed or are *relevant* for the occurrence of this cusp. *It is important to distinguish between this cusp in the bifurcation set in two parameter space and the cusp bifurcation at a single point in the space of the equilibrium curves which is shown in the* MAPLE *figure on page 43.*

2.2 Co-Dimension, Co-Rank and Structural Stability

2.2.1 Co-Dimension

As we have seen the parameters of a system near to a bifurcation can be divided into two sets, those which are relevant to the occurrence of the bifurcation and those which are not. The number of members of the first set is called the *co-dimension* of the bifurcation. The simple turning point of Example 1.8.1 is thus an example of a bifurcation of co-dimension one and the cusp of Example 2.1.1 is a bifurcation of co-dimension two. Another way of understanding this idea is to think of the bifurcation as a geometrical object of dimension $\mathfrak{d}$ in the space Π_η of parameters $\{a, b, c, \ldots\}$. The co-dimension of the bifurcation is then the

number of equations needed to specify the bifurcation. In general this number is $\eta - \mathfrak{d}$. Thus for the simple turning point in Example 1.8.1 $\eta = 1$, $\mathfrak{d} = 0$ giving co-dimension one. In Example 2.1.1 the lines of turning points have $\eta = 2$, $\mathfrak{d} = 1$ so again the co-dimension is one. For the cusp bifurcation, which terminates the lines of turning points in Example 2.1.1, $\eta = 2$ and $\mathfrak{d} = 0$ so the co-dimension for this is two.

2.2.2 Co-Rank

Just as we can divide the *parameters* of a system at a bifurcation into a relevant set and an irrelevant set, we can do the same for the *variables*. The number of relevant variables is called the *co-rank* of the bifurcation. In this case we can be more precise by supposing that $\boldsymbol{x} = \boldsymbol{x}^*(\boldsymbol{a})$ is an equilibrium point for the d–dimensional system given by (2.1). As in Sect. 1.12 we can linearize about $\boldsymbol{x}^*(\boldsymbol{a})$ for a particular value of $\boldsymbol{a}$ to give

$$\frac{\mathrm{d}\triangle\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{J}^*\triangle\boldsymbol{x}, \tag{2.25}$$

where $\boldsymbol{J}^*$ is the stability matrix given by (1.114). With $\boldsymbol{V}$ and $\boldsymbol{U}$ as the $d \times d$ matrices containing the left and right eigenvectors of $\boldsymbol{J}^*$ as rows and columns respectively, as explained in Sect. 1.9, and $\boldsymbol{\Lambda}$ the $d\times d$ diagonal matrix containing the eigenvalues

$$\boldsymbol{J}^* = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}. \tag{2.26}$$

Substituting into (2.25) and operating on the left with $\boldsymbol{V}$ gives

$$\frac{\mathrm{d}\boldsymbol{\Psi}}{\mathrm{d}t} = \boldsymbol{\Lambda}\boldsymbol{\Psi}, \tag{2.27}$$

where

$$\boldsymbol{\Psi} = \boldsymbol{V}\triangle\boldsymbol{x}. \tag{2.28}$$

The $d = 1$ case of this analysis corresponds to the situation where

$$\boldsymbol{J} = \lambda(a, x) = \frac{\partial F}{\partial x}. \tag{2.29}$$

and, as we saw Sect. 2.1.1, the bifurcation set corresponds to the simultaneous solution of $F(a, x) = 0$ and $\lambda(a, x) = 0$. This means that the dimension of the bifurcation set is $\eta + d - 2 = 1 + 1 - 2 = 0$. That is a single point.

For the general case the matrix $\boldsymbol{J}$ has d eigenvalues $\lambda^{(k)}(\boldsymbol{a}, \boldsymbol{x})$, $k = 1, 2, \ldots, d$. At a bifurcation some number ρ ($\leq d$) of these eigenvalues will be zero. The change of variables from $\triangle\boldsymbol{r}$ to $\boldsymbol{\Psi}$ is a linear approximation to a change of variables of which ρ have zero eigenvalues. This means that ρ independent combination of the variables $x_1, x_2, \ldots, x_d$ are relevant to the bifurcation. The *co-rank of the bifurcation* is thus the number ρ of zero eigenvalues at the bifurcation[1].

[1] Since the *rank* of an $d\times d$ matrix is the number of independent rows, which is the number of non-zero eigenvalues, co-rank $= d -$ rank.

2.2.3 Structural Stability

We first consider the one-dimensional case of a function $V(a, b, c, \ldots, x)$ which is a polynomial of degree μ in x with coefficients $a, b, c, \ldots$. By linear and multiplicative scaling of V we can eliminate the constant term and set the coefficient of x^μ to $1/\mu$. For any $V(a, b, c, \ldots, x)$ of this type we now define a set of perturbed polynomials

$$\tilde{V}_p(\varepsilon, a, b, \ldots, x) = \frac{\varepsilon x^p}{p} + V(a, b, c, \ldots, x). \tag{2.30}$$

Then $V(a, b, c, \ldots, x)$ is said to be *structurally stable* if, for all $p > 0$ and for small ε, $\tilde{V}_p(\varepsilon, a, b, \ldots, x)$ has the same x-dependent character (having a non-zero gradient or a maximum or a minimum or a point of inflection) in a neighbourhood of $x = a = b = \cdots = 0$ as $V(a, b, c, \ldots, x)$ does at $x = a = b = c = \cdots = 0$. For each value of μ we begin building a structurally stable polynomial by adding terms to

$$V(x) = \frac{x^\mu}{\mu}. \tag{2.31}$$

For μ even this has a minimum at $x = 0$, for $\mu = 1$ it is a straight line through the origin and for $\mu \geq 3$ and odd there is a point of inflection at the origin. Consider

$$\tilde{V}_p(\varepsilon, x) = \frac{\varepsilon x^p}{p} + \frac{x^\mu}{\mu}, \qquad p \geq \mu. \tag{2.32}$$

This perturbation does not affect the degree of the root at the origin since the first non-zero derivative is still the μ–th with value one. If $p = \mu$ the only effect is a trivial change of coefficient. If $p > \mu$ the large x–value is changed. With p and μ of different parity, or of the same parity with ε negative, this involves new roots far from the origin.

Now consider the possibilities for destabilization with monomial terms with $p < \mu$. (We start with $\mu = 2$ since there is no scope for adding terms for $\mu = 1$, which is structurally stable in a trivial sense.)

- $\underline{\mu = 2,}$ $\tilde{V}_1(\varepsilon, x) = \varepsilon x + \frac{1}{2}x^2$. This simply shifts the minimum to $x = -\varepsilon$ so $x^2/2$ is structurally stable.

- $\underline{\mu = 3,}$ $\tilde{V}_1(\varepsilon, x) = \varepsilon x + \frac{1}{3}x^3$. The point of inflection at $x = 0$ in $V(x)$ has been eliminated leaving no turning points when $\varepsilon > 0$ or split into a maximum and minimum if $\varepsilon < 0$. So $x^3/3$ is structurally unstable. Now consider

  $$V(a, x) = \tfrac{1}{3}x^3 + ax. \tag{2.33}$$

  It is clear that this potential is not destabilized by εx which now just shifts the function a distance ε in the a direction. What about

  $$\tilde{V}_2(\varepsilon, a, x) = \tfrac{1}{2}\varepsilon x^2 + \tfrac{1}{3}x^3 + ax? \tag{2.34}$$

This can be rewritten as

$$\tilde{V}_2(\varepsilon, a, x) = \tfrac{1}{3}\left(x + \tfrac{1}{2}\varepsilon\right)^3 + \left(a - \tfrac{1}{4}\varepsilon^2\right)\left(x + \tfrac{1}{2}\varepsilon\right) + \tfrac{1}{12}\varepsilon(\varepsilon^2 - 6a). \tag{2.35}$$

So a small shift of origin will restore the polynomial to the form (2.33). It follows that (2.33) is structurally stable.

Are there any general conclusions we can draw at this stage? Suppose we want, for some value of μ, to construct the the structurally stable polynomial with minimum co-dimension, that is with the minimum number of parameters as coefficients. It is clear that

$$V(a_1, a_2, \ldots, a_{\mu-1}, x) = \frac{x^\mu}{\mu} + \sum_{k=1}^{\mu-1} a_k \frac{x^k}{k} \tag{2.36}$$

is structurally stable since addition of a perturbation of degree $k \leq \mu$ will just shift the parameter a_k by ε. It is also not difficult to see that the degree $\mu - 1$ monomial can, as in the case $\mu = 3$, be eliminated by a shift in all the remaining parameters and in V and x. It follows that

$$V(a_1, a_2, \ldots, a_{\mu-2}, x) = \frac{x^\mu}{\mu} + \sum_{k=1}^{\mu-2} a_k \frac{x^k}{k} \tag{2.37}$$

is structurally stable and the minimum co-dimension for a μ–degree polynomial is not more that $\mu - 2$. In fact it turns out that (2.37) with degree μ has the maximum degree for a polynomial of co-rank one and co-dimension $\mu - 2$. We shall not prove this general result, but it is worth considering

$$V(a, b, x) = \tfrac{1}{4}x^4 + \tfrac{1}{2}ax^2 + bx. \tag{2.38}$$

We know that this polynomial is unstable if $a = b = 0$. But is it still stable with one but not both of a or b zero? With $V(a, x) = V(a, 0, x)$

$$\tilde{V}_1(\varepsilon, a, x) = \varepsilon x + \tfrac{1}{4}x^4 + \tfrac{1}{2}ax^2. \tag{2.39}$$

The turning points of $\tilde{V}_1(\varepsilon, a, x)$ are given by

$$\frac{\partial \tilde{V}_1}{\partial x} = \varepsilon + x(x^2 + a) = 0. \tag{2.40}$$

For the unperturbed case ($\varepsilon = 0$) and with

$$F(a, x) = -\frac{\partial V}{\partial x}, \tag{2.41}$$

$V(a, x)$ can just be regarded as the *potential* for the pitchfork ($c = 0$) case of Example 1.8.2 and (apart from a trivial reversal of sign for a) the pattern of maxima and minima derived from (2.40) with $\varepsilon = 0$ are just the unstable and stable curves plotted in Fig. 1.11. Now include a small non-zero ε. The picture changes completely and the pitchfork bifurcation structure of potential turning

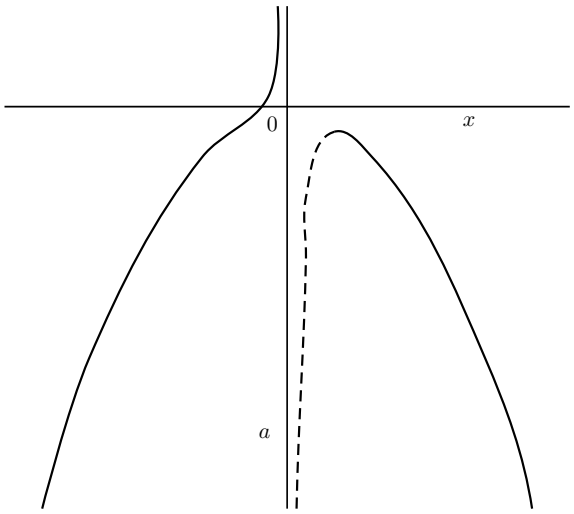


Figure 2.2: The plot of the curves of (2.40) with small positive ε.

points is broken into two disconnected branches (Fig. 2.2). So the function (2.38) is not structurally stable with $b = 0$. With $V(b, x) = V(0, b, x)$

$$\tilde{V}_1(\varepsilon, b, x) = \tfrac{1}{2}\varepsilon x^2 + \tfrac{1}{4}x^4 + bx. \tag{2.42}$$

The turning points of $\tilde{V}_1(\varepsilon, b, x)$ are given by

$$\frac{\partial \tilde{V}_1}{\partial x} = \varepsilon x + (x^3 + b) = 0. \tag{2.43}$$

In this case the potential does not correspond to any kind of bifurcation since for $\varepsilon = 0$ there is only one branch of the curve with a point of inflection at the origin (as a plot of b as a function of x). With non-zero ε the point of inflection is removed to be replaced by a maximum and a minimum. So the function (2.38) is not structurally stable with $a = 0$ and with the two parameters a and b it is the structurally stable quartic one-variable polynomial with smallest co-dimension. In a similar way

$$V(a, b, c, x) = \tfrac{1}{5}x^5 + \tfrac{1}{3}ax^3 + \tfrac{1}{2}bx^2 + cx, \tag{2.44}$$

$$V(a, b, c, d, x) = \tfrac{1}{6}x^6 + \tfrac{1}{4}ax^4 + \tfrac{1}{3}bx^3 + \tfrac{1}{2}cx^2 + dx, \tag{2.45}$$

can be shown to be the lowest degree co-rank one polynomials with co-dimension three and four. If the co-rank is allowed to increase then there are three more structurally stable polynomials with co-dimension not greater than four:

$$V(a, b, c, x, y) = \tfrac{1}{3}x^3 + \tfrac{1}{3}y^3 + cxy - ax - by, \tag{2.46}$$

$$V(a, b, c, x, y) = \tfrac{1}{3}x^3 - xy^2 + c(x^2 + y^2) - ax - by, \tag{2.47}$$

$$V(a, b, c, d, x, y) = x^2 y + \tfrac{1}{4}y^4 + cx^2 + dy^2 - ax - by, \tag{2.48}$$

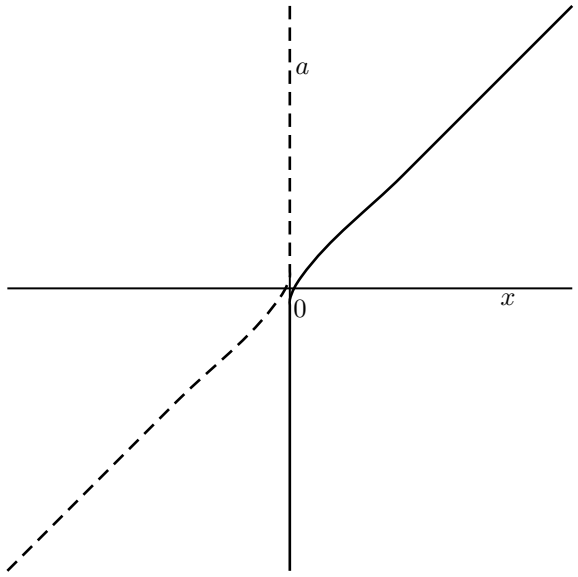


Figure 2.3: The plot of the curves of (2.52) with a small positive ε.

giving in all seven structurally stable polynomials with co-dimension less than or equal to four and degree greater than two. We have seen that the simple turning point bifurcation has the structurally stable potential (2.33) and it is clear that the cusp bifurcation of Example 2.1.1 has the potential

$$V(a,b,x) = -x^4 + ax^2 - bx, \tag{2.49}$$

which with slight changes of parameterization is equivalent (2.38). Thus the cusp bifurcation has a structurally stable potential. We have already seen that the pitchfork bifurcation is not stable and by implication the transcritical bifurcation with

$$F(a,x) = x(a-x), \tag{2.50}$$
$$V(a,x) = \tfrac{1}{3}x^3 - \tfrac{1}{2}ax^2, \tag{2.51}$$

is structurally unstable. This can be seen clearly if we add a term εx to (2.51). Then the equilibrium diagram is given by

$$x(a-x) - \varepsilon = 0. \tag{2.52}$$

With $\varepsilon = 0$ the transcritical bifurcation occurs with the lines $x = 0$ and $x = a$ crossing at the origin and exchanging stability. With $\varepsilon \neq 0$, however small, the bifurcation is removed and the equilibrium points form two non-intersecting branches (Fig. 2.3).

2.3 Bifurcations in More Than One Dimension

In Sect. 2.1 we considered bifurcations with one variable x and up to two parameters. Here we indicate briefly the situation for a system evolving according to (2.1) where $d > 1$. Suppose $(\boldsymbol{a}_0, \boldsymbol{x}_0)$ is an equilibrium point in the $(\eta + d)$–dimensional space $\Lambda_{\eta+d}$ of all the variables and parameters. Then

$$\boldsymbol{F}(\boldsymbol{a}_0, \boldsymbol{x}_0) = 0 \tag{2.53}$$

and the vectorial form of the Taylor expansion (2.11) is

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{a}, \boldsymbol{x}) \quad = \quad & \boldsymbol{J}(\boldsymbol{a}_0, \boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{A}(\boldsymbol{a}_0, \boldsymbol{x}_0)(\boldsymbol{a} - \boldsymbol{a}_0) \\ & + \mathrm{O}(|\boldsymbol{x} - \boldsymbol{x}_0||\boldsymbol{a} - \boldsymbol{a}_0|) \\ & + \mathrm{O}(|\boldsymbol{x} - \boldsymbol{x}_0|^2) + \mathrm{O}(|\boldsymbol{a} - \boldsymbol{a}_0|^2). \end{aligned} \tag{2.54}$$

where

$$\boldsymbol{J}(\boldsymbol{a}, \boldsymbol{x}) = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} & \cdots & \dfrac{\partial F_1}{\partial x_d} \\ \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial F_2}{\partial x_2} & \cdots & \dfrac{\partial F_2}{\partial x_d} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial F_d}{\partial x_1} & \dfrac{\partial F_d}{\partial x_2} & \cdots & \dfrac{\partial F_d}{\partial x_d} \end{pmatrix}. \tag{2.55}$$

is a $d \times d$ matrix and

$$\boldsymbol{A}(\boldsymbol{a}, \boldsymbol{x}) = \begin{pmatrix} \dfrac{\partial F_1}{\partial a_1} & \dfrac{\partial F_1}{\partial a_2} & \cdots & \dfrac{\partial F_1}{\partial a_\eta} \\ \dfrac{\partial F_2}{\partial a_1} & \dfrac{\partial F_2}{\partial a_2} & \cdots & \dfrac{\partial F_2}{\partial a_\eta} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial F_d}{\partial a_1} & \dfrac{\partial F_d}{\partial a_2} & \cdots & \dfrac{\partial F_d}{\partial a_\eta} \end{pmatrix}. \tag{2.56}$$

is a $d \times \eta$ matrix. The differential element $\mathbf{d}\boldsymbol{x}$ of the equilibrium curve at $(\boldsymbol{a}_0, \boldsymbol{x}_0)$ for a differential change $\mathbf{d}\boldsymbol{a}$ in the parameters is given, from (2.54), by taking $\boldsymbol{x} - \boldsymbol{x}_0 \to \mathbf{d}\boldsymbol{x}$, $\boldsymbol{a} - \boldsymbol{a}_0 \to \mathbf{d}\boldsymbol{a}$ and neglecting non-linear terms. This gives

$$\boldsymbol{J}(\boldsymbol{a}_0, \boldsymbol{x}_0)\mathbf{d}\boldsymbol{x} = -\boldsymbol{A}(\boldsymbol{a}_0, \boldsymbol{x}_0)\mathbf{d}\boldsymbol{a}. \tag{2.57}$$

If $\boldsymbol{J}(\boldsymbol{a}_0, \boldsymbol{x}_0)$ is non-singular then $(\boldsymbol{a}_0, \boldsymbol{x}_0)$ is a *regular point* on the equilibrium curve with tangent element

$$\mathbf{d}\boldsymbol{x} = -[\boldsymbol{J}(\boldsymbol{a}_0, \boldsymbol{x}_0)]^{-1}\boldsymbol{A}(\boldsymbol{a}_0, \boldsymbol{x}_0)\mathbf{d}\boldsymbol{a}. \tag{2.58}$$

and, if $\boldsymbol{J}(\boldsymbol{a}_0, \boldsymbol{x}_0)$ is singular, but $\boldsymbol{A}(\boldsymbol{a}_0, \boldsymbol{x}_0)$ has an inverse,

$$\mathbf{d}\boldsymbol{a} = -[\boldsymbol{A}(\boldsymbol{a}_0, \boldsymbol{x}_0)]^{-1}\boldsymbol{J}(\boldsymbol{a}_0, \boldsymbol{x}_0)\mathbf{d}\boldsymbol{x}. \tag{2.59}$$

This is the multi-dimensional version of a *regular turning point*. Otherwise $(\boldsymbol{a}_0, \boldsymbol{x}_0)$ is a *singular point*. As in the case $d = 1$ bifurcations arise both from singular points and regular turning points. They satisfy

$$\mathrm{Det}\{\boldsymbol{J}(\boldsymbol{a}_0, \boldsymbol{x}_0)\} = 0. \tag{2.60}$$

The d equations (2.53) and (2.60) are $(d-1)$-dimensions surfaces in the space $\Lambda_{d+\eta}$. Their intersection is the $(d + \eta - d - 1 = \eta - 1)$–dimensional bifurcation set. Which is simply to say that we can (in principle) eliminate the d variables $x_1, x_2 \ldots, x_d$ between the $d + 1$ equations to give one relationship between the η parameters $a_1, a_2, \ldots, a_\eta$ which is an $(\eta - 1)$–dimensional surface in the η–dimensional space of parameters. An example, for $d = 1$, $\eta = 2$, is the cusp bifurcation set in Fig. 2.1(b).

Example 2.3.1

$$\boldsymbol{F}(a, x, y) = \begin{pmatrix} y - x \\ ax - y - x^2 y \end{pmatrix} \tag{2.61}$$

The equilibrium points for (2.61) lie on the curve in the three-dimensional space which is the intersection of the surfaces

$$y - x = 0, \qquad ax - y - x^2 y = 0 \tag{2.62}$$

and

$$\boldsymbol{J}(a, x, y) = \begin{pmatrix} -1 & 1 \\ a - 2xy & -(1 + x^2) \end{pmatrix} \tag{2.63}$$

The equilibrium curve lies in the plane $x = y$ in the space of the variables $\{a, x, y\}$ and in this plane is given by

$$(a - 1)x - x^3 = 0, \tag{2.64}$$

which is a pitchfork. From (2.60) and (2.63) the bifurcation set is given by solving (2.62) with

$$(1 + x^2) - (a - 2xy) = 0, \tag{2.65}$$

which gives the bifurcation set $a = 1$.

Example 2.3.2 *The system with potential (2.46) has equilibrium set given by*

$$\begin{aligned} -\frac{\partial V}{\partial x} &= -x^2 - cy + a = 0, \\ -\frac{\partial V}{\partial y} &= -y^2 - cx + b = 0, \end{aligned} \tag{2.66}$$

and the bifurcation set is given by eliminating x and y between these equations and

$$\det\{\boldsymbol{J}(a, x, y)\} = \begin{vmatrix} -2x & -c \\ -c & -2y \end{vmatrix} = 4xy - c^2 = 0. \tag{2.67}$$

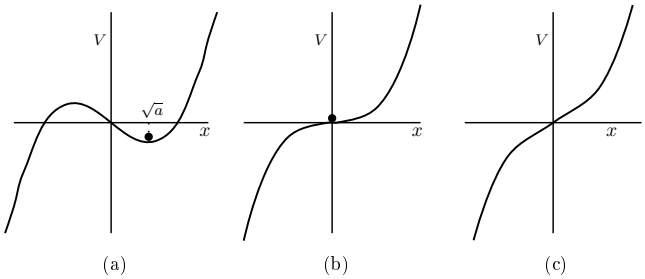


Figure 2.4: The curve $V(a,x) = \frac{1}{3}x^3 - ax$ for (a) $a > 0$, (b) $a = 0$, (c) $a < 0$.

2.4 Catastrophe Theory

This subject, which was initiated by René Thom[2], has been applied to all kinds of situations (conflicts, biological morphogenesis, phase transitions etc.) in which sudden changes occur.

Catastrophe theory is concerned with systems with a set of *state variables* denoted by $x_1, x_2, \ldots, x_d$ and a set of *control variables* denoted by $a_1, a_2, \ldots, a_\eta$. Since time does not enter explicitly into the theory one may suppose that the state variables have reached temporal equilibrium and their values are then smooth functions of the control variables. Changes in the state variables are now caused by changes in the control variables. In general small changes in the control variables lead to small changes in the state variables. However, for some values of the control variables, there is the possibility of a *catastrophe* occurring when a small change in one or more of the control variables leads to a large and discontinuous change in one or more of the state variables. Catastrophe theory is concerned with the classification of the different ways these discontinuous changes can occur. As a simple example of a catastrophe consider a system in which a particle is free to roll on the curve

$$V(a,x) = \tfrac{1}{3}x^3 - ax. \tag{2.68}$$

When $a > 0$ the curve has a local minimum at $x = \sqrt{a}$ and the particle can sit at rest at this minimum (Fig. 2.4(a)). While a remains positive a small change in a will lead to only a small change in the location of the particle. However, at $a = 0$ the maximum and minimum of $V(a,x)$ merge at $x = 0$. The state of the particle becomes precarious (Fig. 2.4(b)) and, when a becomes negative, the catastrophe occurs and the particle is tipped off and falls down to $x = -\infty$ (Fig. 2.4(c)).

Of course we can see that what we are really talking about here is the simple turning point bifurcation, with

$$-\frac{\partial V}{\partial x} = a - x^2. \tag{2.69}$$

[2] *Structural Stability and Morphogenesis*, Benjamin, 1975; for an introduction see P.T. Saunders, *An Introduction to Catastrophe Theory*, Cambridge, 1980.

The maximum and minimum of the potential $V(a,x)$ are the unstable and stable equilibrium states of the particle. The particle moves downwards along the right-hand branch of the parabola in Fig. 1.9 as a is decreased and finally 'drops off' at $a=0$.

This example gives a case where a bifurcation at $x=0$, $a=0$ gives a catastrophe. Now we generalize by considering a smooth potential function $V(\boldsymbol{a},\boldsymbol{x})$, which can be represented approximately in a neighbourhood of the origin by a polynomial and which is linear in the control variables $a_1, a_2, \ldots, a_\eta$. With

$$\boldsymbol{F}(\boldsymbol{a},\boldsymbol{x}) = -\boldsymbol{\nabla} V(\boldsymbol{a},\boldsymbol{x}), \tag{2.70}$$

we have a dynamic system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{a},\boldsymbol{x}). \tag{2.71}$$

When the system has reached equilibrium we can think of its state as a particle lying at a local minimum on the surface of V plotted in the $(d+1)$–dimensional space of the variables $\{V, x_1, x_2, \ldots, x_d\}$. Now the point $\boldsymbol{x}=\boldsymbol{0}$, $\boldsymbol{a}=\boldsymbol{0}$ is a catastrophe if there are paths which can be traced out by varying $\boldsymbol{a}$ near to $\boldsymbol{a}=\boldsymbol{0}$ which lead to discontinuous changes in the equilibrium value of $\boldsymbol{x}$. In the case $d=1$, $\eta=1$ we have already seen that the path through $a=0$ for the potential (2.68) leads to a discontinuous change in equilibrium state. For $d=1$, $\eta=2$ we can think of the path as a small circle around the origin in the plane of the control variables $\{a,b\}$. A discontinuous change in x means that the function $F(a,b,x)=0$ plotted as a surface of x against a and b has a branch-point at the origin, with the equilibrium state changing discontinuously from one branch to another. In Sect. 2.1 we saw that a bifurcation point is just a branch-point. *So catastrophes are bifurcations. But are all bifurcations catastrophes?* The answer is 'no' and we can already produce two examples with co-dimension one, the transcritical bifurcation with $F(a,x)=x(a-x)$ and the pitchfork bifurcation with $F(a,x)=x(a-x^2)$ where passing through $a=0$ does not produce a discontinuous change in x. These are not catastrophes in their own right.[3] On the other hand the cusp bifurcation of Example 2.1.1 does give a discontinuous change in x on a small closed path about the origin in the $\{a,b\}$ plane, either at $b=\sqrt{8a^3/27}$ or $b=-\sqrt{8a^3/27}$ depending on the orientation of the path. So we have two examples of catastrophes:

- The *fold catastrophe* with co-dimension one and co-rank one, which is the simple turning point bifurcation with $F(a,x)=-a-x^2$ (this is just (1.53) with the sign of a reversed) and potential (2.33).

- The *cusp catastrophe* with co-dimension two and co-rank one, which is the cusp bifurcation with $F(a,b,x)=-x^3-ax-b$ (this is just (2.21) with the signs of a, b and x reversed) and potential (2.38).

[3] Although they do make a guest appearance; see Problem Sheet 4.

The distinguishing features of these two cases is that they are structurally stable. In fact it can be shown that all the structural stable polynomial forms give catastrophes. We have already listed these for co-dimension up to four. We can now given them their names as catastrophes.

- (2.44) is the *swallow's tail catastrophe.*
- (2.45) is the *butterfly catastrophe.*
- (2.46) is the *hyperbolic umbilic catastrophe.*
- (2.47) is the *elliptic umbilic catastrophe.*
- (2.48) is the *parabolic umbilic catastrophe.*

The swallow's tail and the butterfly are of co-rank one and co-dimensions three and four respectively. Their names derive from resemblances seen in their bifurcation sets. The umbilics are of co-rank two with the hyperbolic and elliptic being of co-dimension three and the elliptic being of co-dimension four.

2.4.1 Bifurcation Sets Using MAPLE

Given the potential $V(a, b, \ldots, x)$ for a catastrophe of co-rank one, the bifurcation set is given by eliminating x between the two equations

$$\frac{\partial V}{\partial x} = 0, \qquad \frac{\partial^2 V}{\partial x^2} = 0. \tag{2.72}$$

For the swallow's tail $V(a, b, c, x)$ is given by (2.44) and the bifurcation set is obtained by finding the values of a, b and c for which the polynomials

$$\begin{aligned} &x^4 + ax^2 + bx + c = 0, \\ &4x^3 + 2ax + b = 0 \end{aligned} \tag{2.73}$$

have a common solution for x. The simplest way to solve this problem is to construct the *Sylvester determinant*

$$S(a, b, c) = \begin{vmatrix} 1 & 0 & a & b & c & 0 & 0 \\ 0 & 1 & 0 & a & b & c & 0 \\ 0 & 0 & 1 & 0 & a & b & c \\ 4 & 0 & 2\,a & b & 0 & 0 & 0 \\ 0 & 4 & 0 & 2\,a & b & 0 & 0 \\ 0 & 0 & 4 & 0 & 2\,a & b & 0 \\ 0 & 0 & 0 & 4 & 0 & 2\,a & b \end{vmatrix}. \tag{2.74}$$

The bifurcation set is then given by

$$S(a, b, c) = 0. \tag{2.75}$$

Solving this determinant and plotting the results is a fairly complicated task. The easiest way to do it is to use MAPLE. The following is the record of a

MAPLE session which confirms the formula (2.24) for the bifurcation set of the cusp catastrophe and calculates the bifurcation set for the swallow's tail plotting slices through the surface.

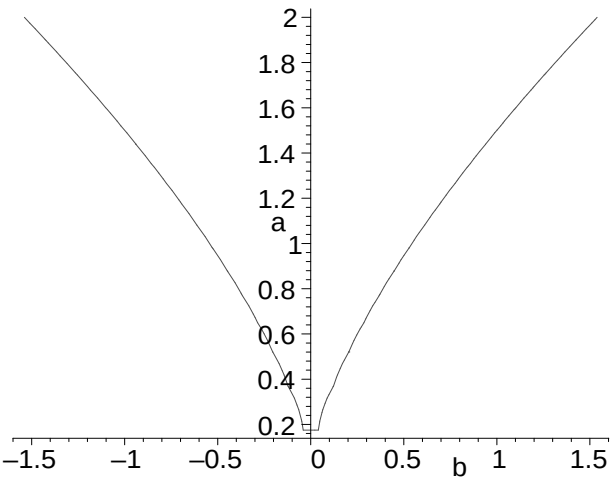
```
> with(linalg,det,matrix):
> with(plots,implicitplot,implicitplot3d):
> # This is the matrix of the Sylvester determinant
> # for the cusp.
> Sc:=(a,b)->matrix([[4,0,-2*a,b,0],[0,4,0,-2*a,b],[12,0,-2*a,0,0],[0,1
> 2,0,-2*a,0],[0,0,12,0,-2*a]]):
> Sc(a,b);
```

$$\begin{bmatrix} 4 & 0 & -2a & b & 0 \\ 0 & 4 & 0 & -2a & b \\ 12 & 0 & -2a & 0 & 0 \\ 0 & 12 & 0 & -2a & 0 \\ 0 & 0 & 12 & 0 & -2a \end{bmatrix}$$

```
> sc:=(a,b)->simplify(det(Sc(a,b))):
> sc(a,b);
```

$$-512\,a^3 + 1728\,b^2$$

```
> #This is the bifurcation set for the cusp.
> # It can be plotted in the {a,b} plane using:
> implicitplot(27*b^2=8*a^3,b=-2..2,a=-0.5..2,grid=[50,50]);
```




```
> # This is the matrix of the Sylvester determinant
> # for the swallow's tail.
```

```
> Sst:=(a,b,c)->matrix([[1,0,a,b,c,0,0],[0,1,0,a,b,c,0]
> ,[4,0,2*a,b,0,0,0],[0,4,0,2*a,b,0,0],[0,0,4,0,2*a,b,0],[0,0,0,4,0,2*a,
> b]]):
```

```
> Sst(a,b,c);
```

$$\begin{bmatrix} 1 & 0 & a & b & c & 0 & 0 \\ 0 & 1 & 0 & a & b & c & 0 \\ 0 & 0 & 1 & 0 & a & b & c \\ 4 & 0 & 2\,a & b & 0 & 0 & 0 \\ 0 & 4 & 0 & 2\,a & b & 0 & 0 \\ 0 & 0 & 4 & 0 & 2\,a & b & 0 \\ 0 & 0 & 0 & 4 & 0 & 2\,a & b \end{bmatrix}$$

```
> sst:=(a,b,c)->simplify(det(Sst(a,b,c))):
```

```
> sst(a,b,c);
```

$$16\,c\,a^4 - 4\,b^2\,a^3 - 128\,c^2\,a^2 + 144\,b^2\,c\,a - 27\,b^4 + 256\,c^3$$

```
> # This is the bifurcation set for the swallow's tail.
```

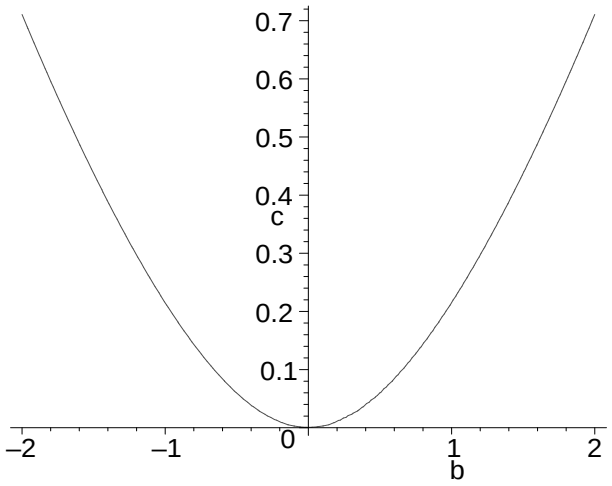
```
> # We can plot various slices through the surface.
```

```
> ssta1:=(b,c)->simplify(sst(1,b,c)):
```

```
> ssta1(b,c);
```

$$16\,c - 128\,c^2 + 144\,b^2\,c + 256\,c^3 - 4\,b^2 - 27\,b^4$$

```
> implicitplot(ssta1(b,c)=0,b=-2..2,c=-0.5..2,grid=[100,100]);
```

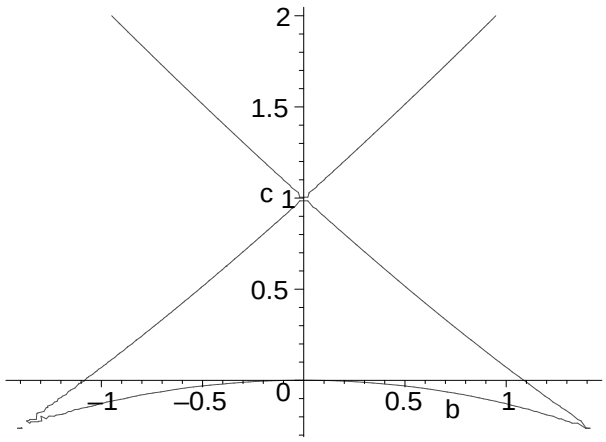



```
> ssta2:=(b,c)->simplify(sst(-2,b,c)):
```

```
> ssta2(b,c);
```

$$256\,c - 512\,c^2 - 288\,b^2\,c + 256\,c^3 + 32\,b^2 - 27\,b^4$$

```
> implicitplot(ssta2(b,c)=0,b=-2..2,c=-2..2,grid=[100,100]);
```



So we see that the bifurcation set for the swallow's tail is given by

$$4ca^4 - b^2a^3 - 32c^2a^2 + 36b^2ca - \frac{27}{4}b^4 + 64c^3 = 0. \tag{2.76}$$

This surface is symmetric under interchange of the sign of b and cuts the $b = 0$ plane in the lines

$$c = 0, \qquad c = \tfrac{1}{4}a^2. \tag{2.77}$$

As we can see from the MAPLE session given above, its intersection with the plane $a = 1$ is given by

$$256c^3 - 128c^2 + 16c + 4(36c - 1)b^2 - 27b^4 = 0. \tag{2.78}$$

This curve passes through $b = c = 0$ and is of a basin shape. Although the point $b = 0$, $c = \frac{1}{4}$, given by (2.77) is a solution of (2.78) it is an isolated point when $a > 0$. Again from the MAPLE session, we see that the intersection of the surface (2.78) with the plane $a = -2$ is given by

$$256c^3 - 512c^2 + 256c + 32(1 - 9c)b^2 - 27b^4 = 0. \tag{2.79}$$

This curve passes through $b = c = 0$, but is now also satisfied by the second solution of (2.77) $b = 0$, $c = 1$, which is a point where the curve intersects itself. The curve has the shape which gives the bifurcation set its name.

Problems 2

1) Consider the roots of $F(\varepsilon, a, x) = 0$, where

$$F(\varepsilon, a, x) = \varepsilon x^2 + x^3 - ax,$$

Show that the pitchfork bifurcation at the origin in the plane of $\{a, x\}$ when $\varepsilon = 0$ becomes a transcritical bifurcation for small $\varepsilon \neq 0$ and that there is a turning point at $a = -\frac{1}{4}\varepsilon^2$, $x = -\frac{1}{2}\varepsilon$. Sketch the equilibrium curves in the $\{a, x\}$ plane for $\varepsilon > 0$.

2) A system is given by

$$\dot{x}(t) = x^3 - 2ax^2 - (b-3)x + c.$$

Find the equation for the bifurcation set, which is the surface in the space of $\{a, b, c\}$ satisfying $F(a, b, c, x) = F_x(a, b, c, x) = 0$. Show that in the plane $a = 1$ the bifurcation set is the curve

$$(27c - 18b + 38)^2 = 4(3b - 5)^3$$

Prove that it has a cusp at $b = \frac{5}{3}$, $c = -\frac{8}{27}$ and sketch the curve. Try sketching curves for other fixed values of a to see how the cusp is affected by variation of a.

3) Show that the cusp bifurcation with

$$V(a, b, x) = \tfrac{1}{4}x^4 + \tfrac{1}{2}ax^2 + bx$$

has pitchfork and transcritical bifurcations in special planes in the $\{a, b, x\}$ space. *(For the second of these you may find it helpful to note that the system considered in Example 1.7.2 has a transcritical bifurcation.)*

4) A two-dimensional system is given by

$$\dot{x}(t) = -x^2 + y^2 - 2cx + a, \qquad \dot{y}(t) = 2xy - 2cy + b.$$

Show that the bifurcation set is given by eliminating x between the polynomials

$$2x^2 + 2xc - a - c^2 = 0,$$

$$4x^4 - 8x^3c + 8c^3x + b^2 - 4c^4 = 0.$$

Either by hard work or by using MAPLE carry out this process and show that the bifurcation set can be expressed in the form

$$27c^8 - 18c^4(a^2 + b^2) + 8c^2a(a^2 - 3b^2) - (a^2 + b^2)^2 = 0.$$

Show that in terms of the polar coordinates $a = r\cos(\theta)$, $b = r\sin(\theta)$ this formula can be expressed in the form

$$(r + c^2)(3c^2 - r)^3 + 8c^2r^3\{\cos(3\theta) - 1\} = 0.$$

Sketch the intersection with a plane of constant c showing that there are cusps at $r = 3c^2$, $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$.

5) Find the equilibrium points of the system

$$\dot{x}(t) = -y - z, \qquad \dot{y}(t) = x + y, \qquad \dot{z}(t) = c + z(x - a).$$

and determine the conditions for their existence. Determine the conditions for the existence of a bifurcation and identify its type.

Chapter 3

Stability

3.1 The Stability of Trajectories

This chapter will be concerned solely with the stability properties of *autonomous* systems. In fact, as we saw in Sect. 1.5, this is not a severe restriction, since a non-autonomous system can be represented as a suspended autonomous system. In this section we consider the general stability properties of a solution $\boldsymbol{x}(t)$ of the dynamical system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{a}, \boldsymbol{x}). \tag{3.1}$$

With $\boldsymbol{x}(t_0) = \boldsymbol{x}^{(0)}$ specifying the solution at time t_0, $\boldsymbol{x}(t)$ defines a *trajectory*[1] in the space Γ_d of the d variables $\{x_1, x_2, \ldots, x_d\}$.

The map ϕ_t: $\Gamma_d \to \Gamma_d$ for all $t \geq 0$ is defined by

$$\phi_t[\boldsymbol{x}(t_0)] = \boldsymbol{x}(t_0 + t) \tag{3.2}$$

and the set of maps $\{\phi_t : t \geq 0\}$ is called a *flow*. Since

$$\phi_{t_1}[\phi_{t_2}[\boldsymbol{x}(t_0)]] = \boldsymbol{x}(t_0 + t_1 + t_2) \qquad t_1, t_2 \geq 0 \tag{3.3}$$

the flow satisfies the conditions

$$\phi_{t_1}\phi_{t_2} = \phi_{t_1+t_2} = \phi_{t_2}\phi_{t_1}. \tag{3.4}$$

It thus has all the properties of an Abelian (commutative) group apart from the possible non-existence of an inverse; it is therefore an *Abelian semigroup*.

An important question concerning a solution $\boldsymbol{x}(t)$ of (3.1) is whether it is *stable*. There are many different definitions of stability in the literature. We shall give two of the most common ones:

[1] Also called the *path* or *orbit*.

The solution $\boldsymbol{x}(t)$ to (3.1), with $\boldsymbol{x}(t_0) = \boldsymbol{x}^{(0)}$, is said to be ***uniformly stable*** *or* ***stable in the sense of Lyapunov*** *if there exists, for every $\varepsilon > 0$, a $\delta(\varepsilon) > 0$, such that any other solution $\tilde{\boldsymbol{x}}(t)$, for which $\tilde{\boldsymbol{x}}(t_0) = \tilde{\boldsymbol{x}}^{(0)}$ and*

$$|\boldsymbol{x}^{(0)} - \tilde{\boldsymbol{x}}^{(0)}| < \delta(\varepsilon), \tag{3.5}$$

satisfies

$$|\boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)| < \varepsilon, \tag{3.6}$$

for all $t \geq t_0$. If no such $\delta(\varepsilon)$ exists then $\boldsymbol{x}(t)$ is said to be ***unstable in the sense of Lyapunov****. If $\boldsymbol{x}(t)$ is uniformly stable and*

$$\lim_{t\to\infty} |\boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)| = 0. \tag{3.7}$$

it is said to be ***asymptotically stable in the sense of Lyapunov****.*

The solution $\boldsymbol{x}(t)$ to (3.1), with $\boldsymbol{x}(t_0) = \boldsymbol{x}^{(0)}$, is said to be ***orbitally stable*** *or* ***stable in the sense of Poincaré*** *if there exists, for every $\varepsilon > 0$, a $\delta(\varepsilon) > 0$, such that, for any other solution $\tilde{\boldsymbol{x}}(t)$, with $\tilde{\boldsymbol{x}}(t_1) = \tilde{\boldsymbol{x}}^{(1)}$ and*

$$|\boldsymbol{x}^{(0)} - \tilde{\boldsymbol{x}}^{(1)}| < \delta(\varepsilon), \tag{3.8}$$

there exists a $t_2(t)$ with

$$|\boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t_2)| < \varepsilon, \tag{3.9}$$

for all $t \geq t_0$. If no such $\delta(\varepsilon)$ exists then $\boldsymbol{x}(t)$ is said to be ***unstable in the sense of Poincaré****. If $\boldsymbol{x}(t)$ is orbitally stable and*

$$\lim_{t\to\infty} |\boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t_2(t))| = 0. \tag{3.10}$$

it is said to be ***asymptotically stable in the sense of Poincaré****.*

It is clear that Lyapunov stability is more restrictive than Poincaré stability, which it implies with $t_1 = t_0$ and $t_2(t) = t$. Lyapunov stability could be characterized by saying that the two solutions are forced to lie in a 'tube' of thickness ε, for $t > t_0$, by the initial condition (3.5) (Fig. 3.1(a)). A cross-section of the tube represents the same time instant on each trajectory. They can be said to have *same histories on the same time scale*. The picture is very similar for Poincaré stability (Fig. 3.1(b)) but in this case the time scales, marked on the trajectories may be different. The two solutions have the *same histories, but not necessarily on the same time scale*. Unless otherwise stated we shall henceforth in the discussion of stability mean stable in the sense of Lyapunov.

For later reference we include at this point the following definitions:

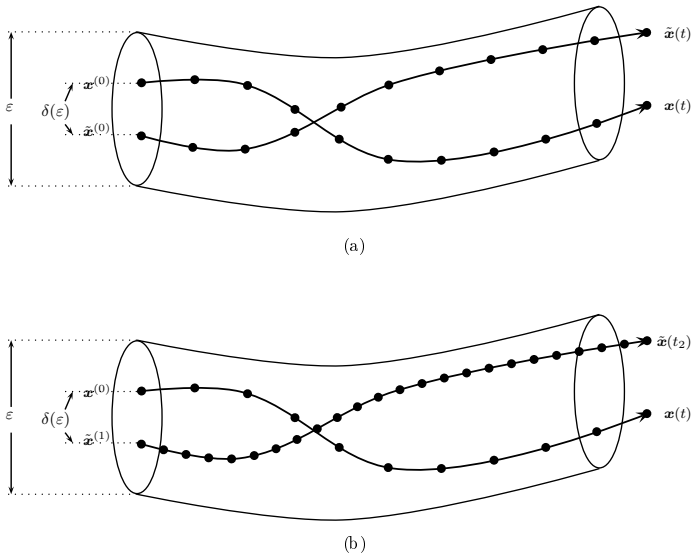


Figure 3.1: Neighbouring trajectories which are stable in (a) the sense of Lyapunov, (b) the sense of Poincaré. Dots on the trajectories indicate equal units of time.

The solution $\boldsymbol{x}(t)$ to (3.1), with $\boldsymbol{x}(t_0) = \boldsymbol{x}^{(0)}$, is a ***periodic solution*** *of period T if, $\boldsymbol{x}(t+T) = \boldsymbol{x}(t)$, for all $t > t_0$, and there does not exist a $T' < T$ with $\boldsymbol{x}(t+T') = \boldsymbol{x}(t)$, for all $t > t_0$.*

A ***cluster (or limit) point*** *$\boldsymbol{x}_\infty$ of the solution $\boldsymbol{x}(t)$ to (3.1), with $\boldsymbol{x}(t_0) = \boldsymbol{x}^{(0)}$, is such that, for all $\tau > 0$ and $\varepsilon > 0$, there exists a $t_1(\varepsilon) > \tau$ with*

$$|\boldsymbol{x}_\infty - \boldsymbol{x}(t_1)| < \varepsilon. \tag{3.11}$$

The set of cluster points is called the ***ω-limit set*** *of the trajectory.*

Given that the solution $\boldsymbol{x}(t)$ to (3.1) is defined for all (positive and negative) t and $\boldsymbol{x}(0) = \boldsymbol{x}^{(0)}$ the ***reverse trajectory*** *$\boldsymbol{x}^R(t)$ is defined by $\boldsymbol{x}^R(t) = \boldsymbol{x}(-t)$. The set of cluster points of the reverse trajectory is called the* ***α-limit set*** *of the trajectory $\boldsymbol{x}(t)$.*

It is clear that the existence of a cluster point $\boldsymbol{x}_\infty$ implies the existence of a sequence $t_1 < t_2 < \cdots < t_n \to \infty$ such that, for the specified trajectory,

$$\boldsymbol{x}(t_n) \to \boldsymbol{x}_\infty, \quad \text{as } n \to \infty. \tag{3.12}$$

Let $\mathfrak{A}$ be the ω-limit set of a particular solution $\boldsymbol{x}(t)$ to (3.1). If there exists a region $\mathcal{D}(\mathfrak{A})$, in Γ_d, which contains $\mathfrak{A}$ and for which the trajectories with $\boldsymbol{x}(0) = \boldsymbol{x}^{(0)}$, for all $\boldsymbol{x}^{(0)}$ in $\mathcal{D}(\mathfrak{A})$, have $\mathfrak{A}$ as their ω-limit set, then $\mathfrak{A}$ is called an ***attractor with basin (or domain)*** *$\mathcal{D}(\mathfrak{A})$. An α-limit with the same property for reverse trajectories is called a* ***repellor****.*

3.2 The Stability of Equilibrium Points

In Sect. 1.6 we defined the stability of an equilibrium point $\boldsymbol{x}^*$. It is now clear that that definition was just for the special case of the stability of a trajectory which consists of the single point $\boldsymbol{x}^*$. An asymptotically stable equilibrium point has a neighbourhood such that every trajectory with $\boldsymbol{x}(0) = \boldsymbol{x}^{(0)}$, and $\boldsymbol{x}^{(0)}$ in the neighbourhood, has $\boldsymbol{x}^*$ as its unique cluster point (and thus the ω-limit set). An asymptotically stable equilibrium point is therefore an attractor with basin consisting of some neighbourhood. Of course, as we shall see, not all attractors are asymptotically stable equilibrium points.

3.2.1 The Lyapunov Direct Method

An interesting method for establishing the stability of an equilibrium point is given by *Lyapunov's first theorem for stability*:

Theorem 3.2.1 *Let $\boldsymbol{x}^*$ be an equilibrium point of (3.1). Suppose that there exists a continuous differentiable function $\mathcal{L}(\boldsymbol{x})$ such that*

$$\mathcal{L}(\boldsymbol{x}^*) = 0 \tag{3.13}$$

and, for some $\mu > 0$,

$$\mathcal{L}(\boldsymbol{x}) > 0, \quad \textit{when } 0 < |\boldsymbol{x}^* - \boldsymbol{x}| < \mu. \tag{3.14}$$

Then $\boldsymbol{x}^*$ is

(i) stable if

$$\boldsymbol{F}(\boldsymbol{a}, \boldsymbol{x}).\boldsymbol{\nabla}\mathcal{L}(\boldsymbol{x}) \leq 0, \quad \textit{when } 0 < |\boldsymbol{x}^* - \boldsymbol{x}| < \mu, \tag{3.15}$$

(ii) asymptotically stable if

$$\boldsymbol{F}(\boldsymbol{a}, \boldsymbol{x}).\boldsymbol{\nabla}\mathcal{L}(\boldsymbol{x}) < 0, \quad \textit{when } 0 < |\boldsymbol{x}^* - \boldsymbol{x}| < \mu, \tag{3.16}$$

(iii) unstable if

$$\boldsymbol{F}(\boldsymbol{a}, \boldsymbol{x}).\boldsymbol{\nabla}\mathcal{L}(\boldsymbol{x}) > 0, \quad \textit{when } 0 < |\boldsymbol{x}^* - \boldsymbol{x}| < \mu. \tag{3.17}$$

Proof: From (3.1) along a trajectory

$$\frac{\mathrm{d}\mathcal{L}(\boldsymbol{x})}{\mathrm{d}t} = \boldsymbol{\nabla}\mathcal{L}(\boldsymbol{x}).\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{a},\boldsymbol{x}).\boldsymbol{\nabla}\mathcal{L}(\boldsymbol{x}). \tag{3.18}$$

From (3.13) and (3.14), $\boldsymbol{x}^*$ is a local minimum of $\mathcal{L}(\boldsymbol{x})$. So we can find an $R > 0$, with $\mu \geq R$, such that, for all $R > |\boldsymbol{x}^* - \boldsymbol{x}_1| > |\boldsymbol{x}^* - \boldsymbol{x}_2| > 0$, $\mathcal{L}(\boldsymbol{x}_1) > \mathcal{L}(\boldsymbol{x}_2)$. Then if (3.15) applies, it follows from (3.18) that a trajectory cannot move further from $\boldsymbol{x}^*$ and, given any $\varepsilon > 0$, (1.28) can be satisfied by choosing $\delta(\varepsilon)$ in (1.27) to be the smaller of ε and R. If the strict inequality (3.16) applies it follows from (3.18) that the trajectory must converge to $\boldsymbol{x}^*$. The condition for $\boldsymbol{x}^*$ to be unstable is established in a similar way.

A function $\mathcal{L}(\boldsymbol{x})$ which satisfies (3.15) is called a *Lyapunov function* and which satisfies (3.16) a *strict Lyapunov function*. The method of establishing stability of an equilibrium point by finding a Lyapunov function is called the *Lyapunov direct method*.

Suppose the dynamical system is given by (2.70)–(2.71) and the function $V(\boldsymbol{a},\boldsymbol{x})$ has a local minimum at $\boldsymbol{x}^*$, for some fixed $\boldsymbol{a} = \boldsymbol{a}^*$. Then the choice

$$\mathcal{L}(\boldsymbol{x}) = V(\boldsymbol{a}^*,\boldsymbol{x}) - V(\boldsymbol{a}^*,\boldsymbol{x}^*), \tag{3.19}$$

satisfies (3.13) and (3.14), with

$$\boldsymbol{F}(\boldsymbol{a}^*,\boldsymbol{x}).\boldsymbol{\nabla}\mathcal{L}(\boldsymbol{x}) = -|\boldsymbol{\nabla}V(\boldsymbol{a}^*,\boldsymbol{x})|^2 < 0. \tag{3.20}$$

So a local minimum of $V(\boldsymbol{a},\boldsymbol{x})$ is, as we might expect, an asymptotically stable equilibrium point. To establish that a local maximum is an unstable equilibrium point simply make the choice

$$\mathcal{L}(\boldsymbol{x}) = V(\boldsymbol{a}^*,\boldsymbol{x}^*) - V(\boldsymbol{a}^*,\boldsymbol{x}). \tag{3.21}$$

Example 3.2.1 *Show that* $(0,0)$ *is a stable equilibrium point of*

$$\dot{x}(t) = -2x - y^2, \qquad \dot{y}(t) = -y - x^2. \tag{3.22}$$

Try

$$\mathcal{L}(x,y) = \alpha x^2 + \beta y^2. \tag{3.23}$$

For α and β positive (3.13) and (3.14) are satisfied and

$$\begin{aligned} \boldsymbol{F}(x,y).\boldsymbol{\nabla}\mathcal{L}(x,y) &= -\{2\alpha x(2x+y^2) + 2\beta y(y+x^2)\} \\ &= -2x^2(2\alpha+\beta y) - 2y^2(\beta + 2\alpha x). \end{aligned} \tag{3.24}$$

So in the neighbourhood $|x| < \beta/(2\alpha)$, $|y| < 2\alpha/\beta$ of the origin (3.15) is satisfied and the equilibrium point is stable.

The problem in this method is to find a suitable Lyapunov function. This in general can be quite difficult. There are, however, two cases where the choice is straightforward:

A conservative system given by ,

$$\ddot{\boldsymbol{x}}(t) = -\boldsymbol{\nabla} V(\boldsymbol{a}, \boldsymbol{x}), \tag{3.25}$$

which in terms of the $2d$ variables $\{x_1, \ldots, x_d, v_1, \ldots, v_d\}$ can be expressed in the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{v}, \qquad \dot{\boldsymbol{v}}(t) = -\boldsymbol{\nabla} V. \tag{3.26}$$

An equilibrium point with $\boldsymbol{a} = \boldsymbol{a}^*$ is given by $\boldsymbol{v} = \boldsymbol{0}$ and a value $\boldsymbol{x} = \boldsymbol{x}^*$ which satisfies $\boldsymbol{\nabla} V = \boldsymbol{0}$. Now try

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{v}) = \tfrac{1}{2}\boldsymbol{v}.\boldsymbol{v} + V(\boldsymbol{a}^*, \boldsymbol{x}) - V(\boldsymbol{a}^*, \boldsymbol{x}^*). \tag{3.27}$$

With

$$\boldsymbol{\nabla}\mathcal{L}(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{\nabla} V \\ \boldsymbol{v} \end{pmatrix} \tag{3.28}$$

$$\boldsymbol{F}(\boldsymbol{a}^*, \boldsymbol{x}).\boldsymbol{\nabla}\mathcal{L}(\boldsymbol{x}) = 0. \tag{3.29}$$

Since, from (3.27), $\mathcal{L}(\boldsymbol{x}^*, 0) = 0$ it follows from (3.29) that the equilibrium point is stable (but not asymptotically stable) if (3.14) holds. From (3.27) this will certainly be the case if $\boldsymbol{x}^*$ is a local minimum of $V(\boldsymbol{a}^*, \boldsymbol{x})$. According to the analysis of Sect. 1.3 such a minimum of the potential is a centre, which is stable in the sense of Lyapunov.

A Hamiltonian system given by (1.10) , in terms of the $2d$ variables $\{x_1, \ldots, x_d, p_1, \ldots, p_d\}$. If the system is autonomous and we have an equilibrium point $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ then, with

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{p}) = H(\boldsymbol{x}, \boldsymbol{p}) - H(\boldsymbol{x}^*, \boldsymbol{p}^*) \tag{3.30}$$

we have, from (1.11)

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \frac{\mathrm{d}H}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{p}).\boldsymbol{\nabla}\mathcal{L}(\boldsymbol{x}, \boldsymbol{p}) = 0. \tag{3.31}$$

The equilibrium point is stable if it is a local minimum of the Hamiltonian. An example where this is true is the equilibrium point at the origin for the simple harmonic oscillator with Hamiltonian (1.30). Even when the equilibrium point is not a local minimum of the Hamiltonian, its form can often be a guide to finding an appropriate Lyapunov function.

Example 3.2.2 *Consider the stability of the equilibrium point at the origin for the system with Hamiltonian*

$$H(a, x_1, x_2, p_1, p_2) = \tfrac{1}{2}\{x_1^2 + x_2^2 + p_1^2 + p_2^2\} + a\{p_1 x_2 - p_2 x_1\}. \tag{3.32}$$

From (1.10) the equations of motion for this system are

$$\begin{aligned} \dot{x}_1(t) &= \frac{\partial H}{\partial p_1} = p_1 + ax_2, & \dot{p}_1(t) &= -\frac{\partial H}{\partial x_1} = -x_1 + ap_2, \\ \dot{x}_2(t) &= \frac{\partial H}{\partial p_2} = p_2 - ax_1, & \dot{p}_2(t) &= -\frac{\partial H}{\partial x_2} = -x_2 - ap_1. \end{aligned} \tag{3.33}$$

The origin is clearly an equilibrium point. However in the plane $x_2 = p_1 = 0$

$$\begin{vmatrix} \dfrac{\partial^2 H}{\partial x_1^2} & \dfrac{\partial^2 H}{\partial x_1 \partial p_2} \\ \dfrac{\partial^2 H}{\partial p_2 \partial x_1} & \dfrac{\partial^2 H}{\partial p_2^2} \end{vmatrix} = 1 - a^2. \tag{3.34}$$

So the origin is a saddle point in this plane when $|a| > 1$. However, the function

$$\mathcal{L}(x_1, x_2, p_1, p_2) = H(0, x_1, x_2, p_1, p_2) \tag{3.35}$$

has a minimum at the origin with

$$\boldsymbol{F}(a, x_1, x_2, p_1, p_2).\boldsymbol{\nabla}\mathcal{L}(x_1, x_2, p_1, p_2) = 0. \tag{3.36}$$

So we have found a Lyapunov function which establishes the stability of the equilibrium point.

3.2.2 Linearization

The Lyapunov criterion for stability of an equilibrium point given by (1.27)–(1.29) is *local* in the sense that a trajectory will wander near to the equilibrium point only in cases where it begins sufficiently close by. In Sects. 1.10 and 1.12 we examined the stability of equilibrium points for systems linearized about an equilibrium point. The criteria for stability that we developed, which are related to the types of eigenvalues of the stability matrix (1.114) at the equilibrium point, apply globally to the linearized equation (1.113) and apply to the full equations (2.1) for infinitesimal disturbances from the equilibrium point. The connection between these conditions for *linear or infinitesimal stability* and the stability conditions given by (1.27)–(1.29) was provided by Thm. 1.12.1. This theorem allows us to use linear analysis to determine the stability (in the Lyapunov and not just the infinitesimal sense) whenever all the eigenvalues have non-zero real parts. Thus it leaves open the question of the stability of a centre. Such a case is the simple harmonic oscillator with equations of motion (1.3). The stability matrix for equilibrium point at the origin has eigenvalues $\pm \mathrm{i}\omega/\sqrt{m}$. We have, however, shown, using the Lyapunov direct method, that this equilibrium point is stable. Another case of interest is Example 1.12.2, where for $a = 0$ the point $x = y = 0$ is a centre. The complete solution shows a slow convergence to the origin. The function

$$\mathcal{L}(x, y) = \tfrac{1}{2}(x^2 + y^2) \tag{3.37}$$

has a minimum at the origin where it is zero and, from (1.117)–(1.118), with $a = 0$,

$$\boldsymbol{F}(x,y).\boldsymbol{\nabla}\mathcal{L}(x,y) = -\{x^2+y^2\}^2. \tag{3.38}$$

So according to Thm. 3.2.1 the origin is asymptotically stable.

We now review and extend our discussion in Sect. 1.12 of two-dimensional autonomous systems given by

$$\dot{x}(t) = F(x,y), \qquad \dot{y}(t) = G(x,y). \tag{3.39}$$

The family of trajectories in the plane Γ_2 of $\{x,y\}$ is given by solving (if it is possible) the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{G(x,y)}{F(x,y)}. \tag{3.40}$$

Now suppose that there is an equilibrium point, which, using if necessary a translation in the variables, can be taken to be at the origin. Linearizing about the equilibrium point

$$F(x,y) = ax + by + \mathrm{O}(x^2+y^2), \quad G(x,y) = cx + dy + \mathrm{O}(x^2+y^2). \tag{3.41}$$

Retaining only linear terms and assuming a *normal mode solution* of the form $x(t) = u_1\exp(\lambda t)$, $y(t) = u_2\exp(\lambda t)$, gives the right eigenproblem

$$\boldsymbol{J}^*\boldsymbol{u} = \boldsymbol{u}\lambda, \tag{3.42}$$

with

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \boldsymbol{J}^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.43}$$

discussed in Sect. 1.9. The general solution to the linearized equations is of the form

$$\begin{aligned} x &= C^{(+)}u_1^{(+)}\exp\{\lambda^{(+)}t\} + C^{(-)}u_1^{(-)}\exp\{\lambda^{(-)}t\}, \\ y &= C^{(+)}u_2^{(+)}\exp\{\lambda^{(+)}t\} + C^{(-)}u_2^{(-)}\exp\{\lambda^{(-)}t\}, \end{aligned} \tag{3.44}$$

where

$$\lambda^{(\pm)} = \tfrac{1}{2}\{p \pm \sqrt{p^2-4q}\}, \tag{3.45}$$

with

$$p = \mathrm{Trace}\{\boldsymbol{J}^*\} = a+d, \qquad q = \mathrm{Det}\{\boldsymbol{J}^*\} = ad-bc, \tag{3.46}$$

are the eigenvalues of $\boldsymbol{J}^*$ with corresponding right eigenvectors $\boldsymbol{u}^{(\pm)} = (u_1^{(\pm)}, u_2^{(\pm)})^{\mathrm{T}}$. It is clear from (3.44) that the topological nature of the trajectories in a neighbourhood of the origin in the $\{x,y\}$ plane is determined by the eigenvalues and right eigenvectors of $\boldsymbol{J}^*$. An equilibrium point for which

each eigenvalue has a non-zero real part is called a *hyperbolic point*; an equilibrium point for which each eigenvalue is purely imaginary is called an *elliptic point*. Since in two dimensions the eigenvalues are either real or a conjugate complex pair, the only alternative to either a hyperbolic or elliptic point is where the eigenvalues are real and one or both are zero. As we have seen above this normally corresponds to a bifurcation.

When $p^2 > 4q$, the eigenvalues are real and unequal. with eigenvectors with real components. There are two directions through the equilibrium point which give straight line trajectories for the linear system. These are given by the two eigenvectors and correspond to taking $C^{(-)} = 0$ and $C^{(+)} = 0$ in (3.44), giving the lines

$$xu_2^{(+)} = yu_1^{(+)}, \tag{3.47}$$

$$xu_2^{(-)} = yu_1^{(-)}. \tag{3.48}$$

We assume, without loss of generality, that $a > d$. Then, if $b = c = 0$, $u_1^{(-)} = u_2^{(+)} = 0$ and the lines (3.47) and (3.48) become respectively the x and y axes. If $c = 0$, but $b \neq 0$, then $u_1^{(-)} = 0$ but $u_2^{(+)} \neq 0$; (3.48) is the y axis but (3.47) is not the x axis. The converse is the case if $b = 0$, $c \neq 0$. Within the class of real unequal eigenvalues there are a number of cases:

(i) $q > 0 > p$, giving $0 > \lambda^{(+)} > \lambda^{(-)}$. From (3.44)

$$\frac{x}{y} = \frac{u_1^{(+)} + (C^{(-)}/C^{(+)}u_1^{(-)} \exp\{(\lambda^{(-)} - \lambda^{(+)})t\}}{u_2^{(+)} + (C^{(-)}/C^{(+)}u_2^{(-)} \exp\{(\lambda^{(-)} - \lambda^{(+)})t\}}$$

$$\to \frac{u_1^{(+)}}{u_2^{(+)}}, \qquad \text{as } t \to \infty. \tag{3.49}$$

So the ultimate approach to the equilibrium point is tangential to the principle direction (3.47), which is called the *strong direction*. This applies to all trajectories in a neighbourhood of the equilibrium point except those lying on the principle direction (3.48) ($C^{(+)} = 0$) (called the *weak direction*). An equilibrium point of this kind is a *stable node*, (Fig. 3.2a).

(ii) $q > 0$, $p > 0$ giving $\lambda^{(+)} > \lambda^{(-)} > 0$. The result (3.49) applies to the linearized equations for all trajectories except for those lying on the weak direction line which tend to infinity on that line. All other trajectories approach the strong line asymptotically at large distances. This prediction applies only to the linearized equations. Non-linear terms in the full equations will probably modify the large distance behaviour. Close to the equilibrium point the trajectories have the same topology as that of the stable mode except that the direction of the flow is reversed. This is an *unstable mode*.

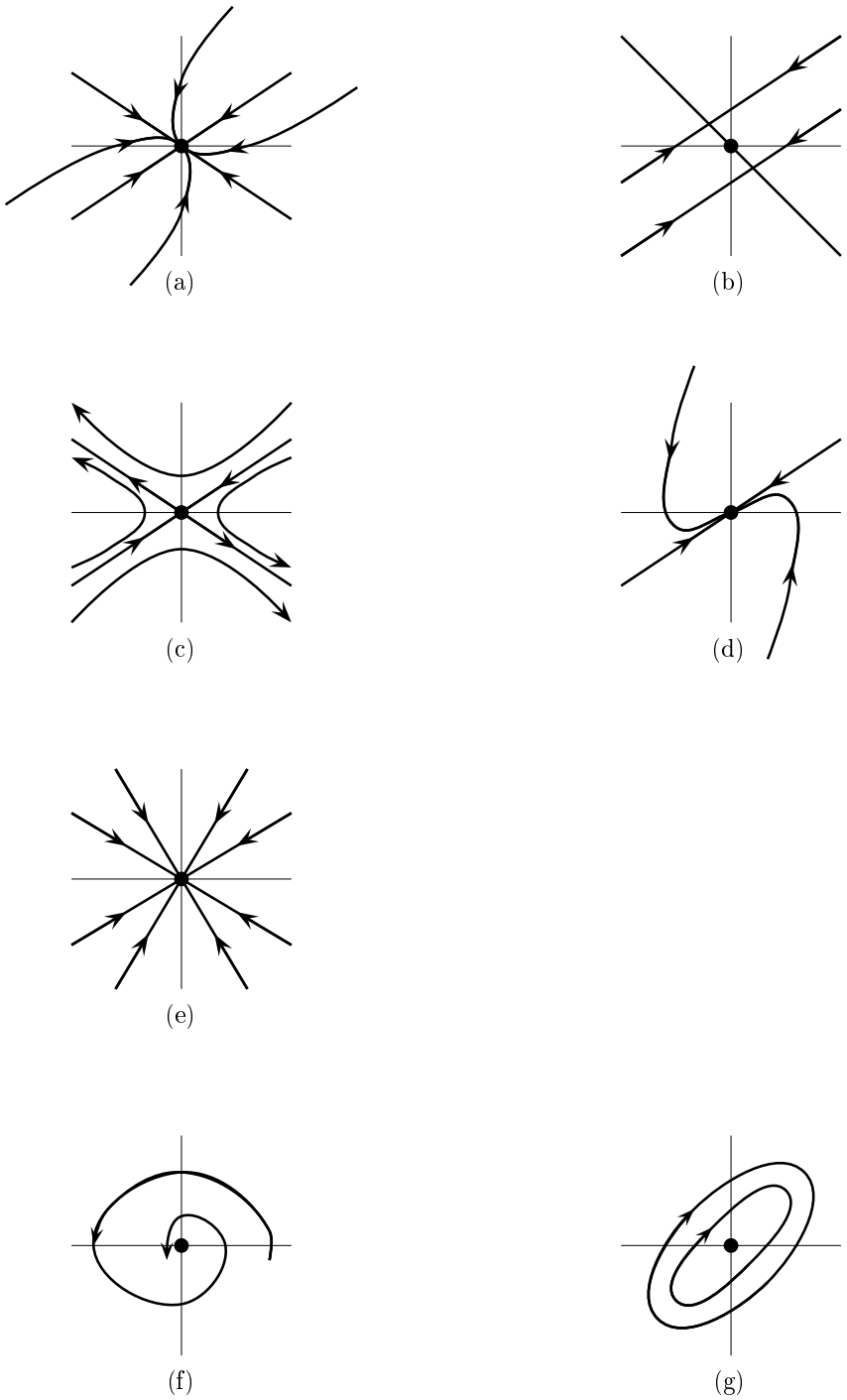


Figure 3.2: Trajectories in a neighbourhood of an equilibrium point: (a) stable node, (b) improper stable node, (c) saddle-point, (d) inflected stable node, (e) perfect stable node, (f) stable focus, (g) centre.

(iii) $q = 0$, $p < 0$ giving $\lambda^{(+)} = 0$, $\lambda^{(-)} = p < 0$. In this case it follows from (3.44) that the trajectories are straight lines which approach the line (3.47) as $t \to \infty$. This is an *improper stable node* (Fig. 3.2b).

(iv) $q = 0$, $p > 0$ giving $\lambda^{(+)} = p > 0$, $\lambda^{(-)} = 0$. In this case it follows from (3.44) that the trajectories are straight lines which retreat from the line (3.48) as t increases. This is an *improper unstable node*.

(v) $q < 0$ giving $\lambda^{(+)} > 0 > \lambda^{(-)}$. This is similar to the case of an unstable mode except that the weak direction is now a direction of approach to the equilibrium point and trajectories near to this line will first be influenced by its attractive power before experiencing the repulsive affect of the strong direction. This is a *saddle-point* (Fig. 3.2c). Again the form of the trajectories may be modified by non-linear terms.

When $p^2 = 4q$, the eigenvalues are real and equal. Within this class there are a number of cases:

(i) $p < 0$, not both $b = 0$ and $c = 0$ giving $\lambda^{(+)} = \lambda^{(-)} < 0$. This case be regarded as the limiting case $q \to p^2/4$ of a stable node. The lines (3.47)–(3.48) degenerate into one linear trajectory of approach. This is called an *inflected stable node* (Fig. 3.2d).

(ii) $p > 0$, not both $b = 0$ and $c = 0$ giving $\lambda^{(+)} = \lambda^{(-)} > 0$. This case be regarded as the limiting case $q \to p^2/4$ of a unstable node. The lines (3.47)–(3.48) degenerate into one linear trajectory of retreat. This is called an *inflected unstable node*.

(iii) $p < 0$, both $b = c = 0$. In this case the equations for x and y are independent and every radial line through the origin is a linear direction of approach. This is called a *sink* or *perfect stable node* (Fig. 3.2e).

(iv) $p > 0$, both $b = c = 0$. Again every radial line is a linear trajectory but now it is a direction of retreat. This is called a *source* or *perfect unstable node*.

When $p^2 < 4q$, the eigenvalues are a conjugate complex pair. $\lambda^{(\pm)} = \frac{1}{2}(p + \mathrm{i}\theta)$, where $\theta = \sqrt{4q - p^2}$. Equations (3.44) still apply but the elements of the eigenvalues are no longer real. However, since x and y are real the solution must be of the form

$$x = C_1 \exp(pt/2) \cos(\gamma_1 + \theta t/2),$$
$$y = C_2 \exp(pt/2) \cos(\gamma_2 + \theta t/2), \tag{3.50}$$

where $C_1, C_2, \gamma_1, \gamma_2$ are constants. There are a number of cases:

(i) $p < 0$. In this case the trajectories spiral into the origin. This is called a *stable focus* (Fig. 3.2f).

(ii) $p > 0$. In this case the trajectories spiral out from the origin. This is called a *unstable focus*.

(iii) $p = 0$. In this case the trajectories form periodic curves around the origin. This is a *centre* (Fig. 3.2g).

A summary of the types of equilibrium points for different regions of the $\{p, q\}$ plane are shown in (Fig. 3.3). The only cases not shown are the sink and source

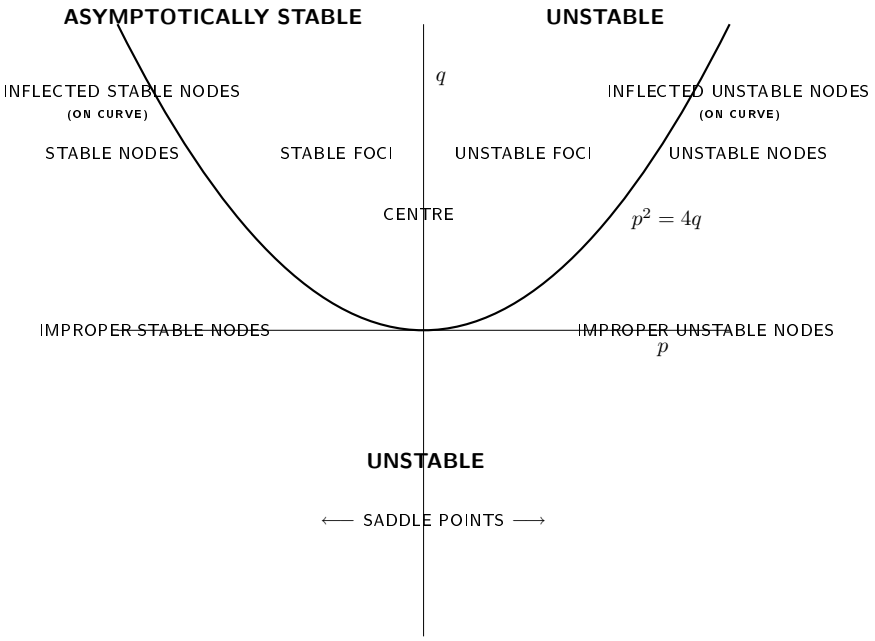


Figure 3.3: Summary in the $\{p, q\}$ plane of the types of equilibrium points.

which also lie on the stable and unstable branches of $p^2 = 4q$.

3.3 Poincaré Maps

For the autonomous system (3.1) a trajectory cannot meet or cross itself in Γ_d unless it is a periodic solution when it forms a simply-connected curve. This is not the case for the non-autonomous, since for a particular $\boldsymbol{a}$ and $\boldsymbol{x}$ it is possible that $\boldsymbol{F}(\boldsymbol{a}, \boldsymbol{x}; t_1) \neq \boldsymbol{F}(\boldsymbol{a}, \boldsymbol{x}; t_2)$ giving $\dot{\boldsymbol{x}}(t_1) \neq \dot{\boldsymbol{x}}(t_2)$ for the same point in space at different times. This situation is simplified by creating the

(autonomous) suspended system, described in Sec. 1.5, so the these two points on the trajectory are at different locations with $x_t = t_1$ and $x_t = t_2$ in the space $\Gamma_d \times \Upsilon$. Henceforth in this section we shall consider only autonomous systems.

This course is mainly concerned with differential equations, although many books on the subject also discuss difference equations (Drazin, Chap. 3). We have already seen that the difference equation

$$\mathbf{x}(n+1) = \mathbf{F}[\boldsymbol{a}, \mathbf{x}(n)], \qquad n = 0, 1, \ldots, \tag{3.51}$$

can be obtained from the differential equation (3.1) by quantizing time. The 'trajectory' in Γ_d will then consist of a sequence of points. An equilibrium point $\mathbf{x}^*$ of (3.51), usually called a *fixed point*, satisfies $\mathbf{x}^* = \mathbf{F}[\boldsymbol{a}, \mathrm{x}^*]$ and there can also be *p-cycles* $\mathbf{x}(1) \to \mathbf{x}(2) \to \cdots \to \mathbf{x}(p) \to \mathbf{x}(1)$.

An alternative method of deriving a discrete time map from a continuous time system is using the *Poincaré map* or *section*. In the space Γ_d take the $(d-1)$–dimensional hypersurface defined by the condition

$$\Pi(\boldsymbol{x}) = 0. \tag{3.52}$$

Now suppose that a particular trajectory cuts the hypersurface (3.52) at times $t_0, t_1, t_2, \ldots$. In cases where an explicit solution can be obtained to the differential system so that we know $\boldsymbol{x}(t_n)$ for all $n = 0, 1, 2, \ldots$, we can define $\mathbf{x}(n) = \boldsymbol{x}(t_n)$, which then gives us a difference map $\mathbf{x}(0) \to \mathbf{x}(1) \to \cdots$. If the succession of points are restricted to those which correspond to passages through the hypersurface in the same sense the construction is called the *Poincaré first-return map*.

Example 3.3.1 *Take the Poincaré section $y = 0$ of the system*

$$\dot{x}(t) = -y + x(a - x^2 - y^2), \qquad \dot{y}(t) = x + y(a - x^2 - y^2). \tag{3.53}$$

This system was investigated in Example 1.12.2. In polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ the solution to this system is given by (1.122) and (1.125).[2] For $a \neq 0$

$$\begin{aligned} r(t) &= \sqrt{\frac{ar^2(0)}{r^2(0) + \exp(-2at)\left\{a - r^2(0)\right\}}}, \\ \theta(t) &= t. \end{aligned} \tag{3.54}$$

The trajectory cuts the plane $y = 0$ at times $t_n = n\pi$, $n = 0, \pm 1, \pm 2 \ldots$. We now define $\theta(n) = \theta(t_n)$, $\mathrm{r}(n) = r(t_n)$ giving $\mathrm{x}(n) = \mathrm{r}(n)\cos[\theta(n)] = \mathrm{r}(n)(-1)^n$. The difference equation relating $\mathrm{x}(n+1)$ and $\mathrm{x}(n)$ can be obtained from (3.54) by replacing $r(0)$ by $\mathrm{r}(n)$ and t by π. So

$$\mathrm{x}(n+1) = -\mathrm{x}(n)\sqrt{\frac{a}{\mathrm{x}^2(n) + \exp(-2a\pi)\left\{a - \mathrm{x}^2(n)\right\}}}. \tag{3.55}$$

[2] Since the system is autonomous we can, without loss of generality, take $t_0 = 0$ and also $\theta(0) = 0$.

The first-return map can be taken to be those points where the trajectory cuts the plane moving in the positive y direction. From the second of equations (3.53) these all occur when $x > 0$ and they could be 'captured' by taking the half-plane $x > 0$, $y = 0$. For $x > 0$, $\theta = \theta(t_n) = 2n\pi$ and (3.55) is modified to

$$\mathrm{x}(n+1) = \mathrm{x}(n)\sqrt{\frac{a}{\mathrm{x}^2(n) + \exp(-4a\pi)\left\{a - \mathrm{x}^2(n)\right\}}}. \tag{3.56}$$

If an equilibrium point of the differential system lies on (3.52) then it will be a fixed point of the discrete map. A periodic trajectory will cut a hypersurface without edges an even number of times and generate a $2p$-cycle in the Poincaré map. In the first-return map it will generate a p-cycle.

Consider now the case of a system where the phase point move on a torus, given in terms the variables $0 \leq \theta < 2\pi$, $0 \leq \phi < 2\pi$ by

$$\begin{aligned} x &= \cos(\theta)\{a + b\cos(\phi)\}, \\ y &= \sin(\theta)\{a + b\cos(\phi)\}, \\ z &= b\sin(\phi), \end{aligned} \tag{3.57}$$

(Fig. 3.4). Suppose now a trajectory is given by $\theta = \alpha t$ and $\phi = \beta t$. This

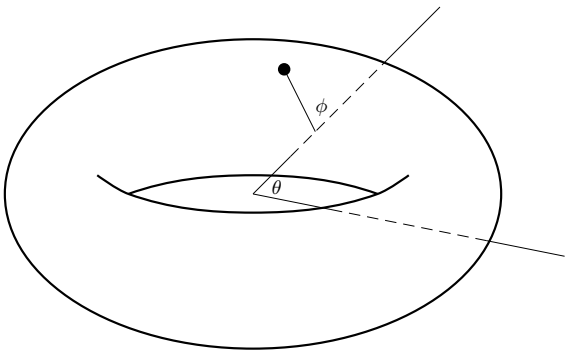


Figure 3.4: A torus in the $\{x, y, z\}$ space.

trajectory winds around the torus. Now consider the first-return map obtain by cutting the torus with the half-plane $y = 0$, $x > 0$. The successive values of θ when the trajectories cut this plane are

$$\theta = 2n\pi, \qquad n = 0, 1, 2, \ldots. \tag{3.58}$$

The corresponding successive values of ϕ are

$$\phi = \left(\frac{\beta}{\alpha}\right)\theta = 2n\pi\left(\frac{\beta}{\alpha}\right). \tag{3.59}$$

The trajectory will be periodic only if, when $\alpha = 2\pi p'$ for some integer p' $\beta = 2\pi q'$ for some integer q'. This is simply the condition

$$\frac{\alpha}{\beta} = \frac{p'}{q'} = \frac{p}{q}, \qquad \text{where } p = p'/s \text{ and } q = q'/s \text{ are coprime integers.} \tag{3.60}$$

Meaning that α/β is a rational number. Such a periodic trajectory cuts the plane $y = 0$ $x > 0$ at the points

$$x = a + b\cos(2n\pi q/p), \qquad z = b\sin(2n\pi q/p). \tag{3.61}$$

It, therefore, generates a p-cycle in the first-return map. When α/β is irrational $\boldsymbol{x}(\theta, \phi)$, with components given by (3.57), is periodic in each of its arguments, but not periodic. The periods are incommensurate and the function is called *quasi-periodic*. It is not difficult to show that the points of the Poincaré map are dense on the circle (3.61)

3.4 The Stability of Periodic Solutions

In Example 1.12.2 we investigated the Hopf bifurcation at which a stable limit cycle emerged from a stable equilibrium point. It is clear that a limit cycle is a type of periodic orbit but we have yet to give a more formal definition. This can be done using the definitions of stability of trajectories given in Sect. 3.1.

The periodic solution $\boldsymbol{x}(t)$ to (3.1) is a ***stable limit cycle*** *if it is asymptotically stable and an* ***unstable limit cycle*** *if it is unstable.*

Just as for trajectories in general the terms stable and unstable can be qualified by the phase 'in the sense of Lyapunov' or 'in the sense of Poincaré' with the former implying the latter. Unless otherwise stated we shall use Lyapunov stability and we shall also concentrate on the autonomous case (3.1). We develop for periodic solutions the analogue of the linearization method of equilibrium points. This is known as *Floquet theory*.

Suppose $\mathring{\boldsymbol{x}}(t)$ is a periodic solution of (3.1) with period T. Thus $\mathring{\boldsymbol{x}}(t) = \mathring{\boldsymbol{x}}(t + T)$. Now consider the trajectory $\boldsymbol{x}(t)$, $0 \leq t \leq T$, where $\boldsymbol{x}(0)$ is near to $\mathring{\boldsymbol{x}}(0)$ and define $\triangle\boldsymbol{x}(t) = \boldsymbol{x}(t) - \mathring{\boldsymbol{x}}(t)$. Then, from (3.1)

$$\frac{\mathrm{d}\triangle\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{a}, \boldsymbol{x}(t)) - \boldsymbol{F}(\boldsymbol{a}, \mathring{\boldsymbol{x}}(t)). \tag{3.62}$$

The Taylor expansion of the right-hand side of (3.62) at fixed t gives

$$\boldsymbol{F}(\boldsymbol{a}, \boldsymbol{x}(t)) - \boldsymbol{F}(\boldsymbol{a}, \mathring{\boldsymbol{x}}(t)) = \boldsymbol{J}(\boldsymbol{a}, \mathring{\boldsymbol{x}}(t))\triangle\boldsymbol{x}(t) + \mathrm{O}(|\triangle\boldsymbol{x}(t)|^2), \tag{3.63}$$

where $\boldsymbol{J}(\boldsymbol{a}, \boldsymbol{x}(t))$ is given by (2.55). Retaining only linear terms,

$$\frac{\mathrm{d}\triangle\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{J}(\boldsymbol{a}, \mathring{\boldsymbol{x}}(t))\triangle\boldsymbol{x}(t). \tag{3.64}$$

Fixing and suppressing reference to $\boldsymbol{a}$, we write

$$\mathring{\boldsymbol{J}}(t) = \boldsymbol{J}(\boldsymbol{a}, \mathring{\boldsymbol{x}}(t)). \tag{3.65}$$

Solving (3.64) is equivalent to looking for a solution $\boldsymbol{w}(t)$ to

$$\dot{\boldsymbol{w}}(t) = \mathring{\boldsymbol{J}}(t)\boldsymbol{w}(t). \tag{3.66}$$

In particular we are interested in the existence of a periodic solution (period T) to (3.66). If such exists then it yields (at least to linear order) a periodic solution $\boldsymbol{x}(t)$ to (3.1) with $\boldsymbol{x}(0)$ close to $\mathring{\boldsymbol{x}}(0)$. To proceed we need a number of results from the theory of differential equations. These will be stated without proofs, which are given in many texts on the theory of ordinary differential equations[3].

(i) The set of solutions $\boldsymbol{w}^{(1)}(t), \boldsymbol{w}^{(2)}(t), \ldots, \boldsymbol{w}^{(r)}(t)$ to (3.66) is *linearly independent* if there exist no constants $c^{(1)}, c^{(2)}, \ldots, c^{(r)}$, which are not all zero and for which $c^{(1)}\boldsymbol{w}^{(1)}(t) + c^{(2)}\boldsymbol{w}^{(2)}(t) + \cdots + c^{(r)}\boldsymbol{w}^{(r)}(t)$ is identically zero for any t.

(ii) If the elements of $\mathring{\boldsymbol{J}}(t)$ are continuous for all t then there exists a set of independent solutions $\boldsymbol{w}^{(1)}(t), \boldsymbol{w}^{(2)}(t), \ldots, \boldsymbol{w}^{(d)}(t)$ to (3.66). This is called a *fundamental set of solutions* and every solution is a linear combination of the members of a fundamental set.

(iii) The set of d-dimensional column vectors $\boldsymbol{w}^{(1)}(0), \boldsymbol{w}^{(2)}(0), \ldots, \boldsymbol{w}^{(d)}(0)$ form an orthogonal set and by choosing suitable linear combinations we can construct a new fundamental set of solutions $\boldsymbol{q}^{(1)}(t), \boldsymbol{q}^{(2)}(t), \ldots, \boldsymbol{q}^{(d)}(t)$, where, for $\ell = 1, 2, \ldots, d$, $\boldsymbol{q}^{(\ell)}(0)$ is the unit vector with zeros everywhere apart from one in the ℓ-th place.

(iv) The $d \times d$ matrix

$$\boldsymbol{Q}(t) = (\boldsymbol{q}^{(1)}(t), \boldsymbol{q}^{(2)}(t), \ldots, \boldsymbol{q}^{(d)}(t)), \tag{3.67}$$

satisfies

$$\dot{\boldsymbol{Q}}(t) = \mathring{\boldsymbol{J}}(t)\boldsymbol{Q}(t), \qquad \boldsymbol{Q}(0) = \boldsymbol{I}, \tag{3.68}$$

and

$$\mathrm{Det}\{\boldsymbol{Q}(t_2)\} \;=\; \mathrm{Det}\{\boldsymbol{Q}(t_1)\} \exp\left\{ \int_{t_1}^{t_2} \mathrm{Trace}\{\mathring{\boldsymbol{J}}(s)\} \mathrm{d}s \right\}, \tag{3.69}$$

which is *Liouville's formula*.

[3] e.g. D. A. Sánchez, *Ordinary Differential Equations and Stability Theory: An Introduction*, W. H. Freeman, 1968.

(v) The solution $\boldsymbol{w}(t)$ to (3.66) which satisfies $\boldsymbol{w}(0) = \boldsymbol{w}_0$, for some $\boldsymbol{w}_0$, can be written

$$\boldsymbol{w}(t) = \boldsymbol{Q}(t)\boldsymbol{w}_0. \tag{3.70}$$

None of these results depends on the periodic property

$$\mathring{\boldsymbol{J}}(t+T) = \mathring{\boldsymbol{J}}(t), \tag{3.71}$$

of $\mathring{\boldsymbol{J}}(t)$, which follows from (3.65) and the fact that $\mathring{\boldsymbol{x}}(t)$ is a periodic solution of period T. Using that property we can now make the following deductions:

- Since the columns of $\boldsymbol{Q}(t)$ form a fundamental set of solutions of (3.66) $\mathrm{Det}\{\boldsymbol{Q}(t)\} \neq 0$ and $\mathrm{Det}\{\boldsymbol{Q}(t+T)\} \neq 0$. Thus the columns of $\boldsymbol{Q}(t+T)$ also form a fundamental set of solutions and, since any solution is a linear combination of a fundamental set,

  $$\boldsymbol{Q}(t+T) = \boldsymbol{Q}(t)\boldsymbol{C}, \tag{3.72}$$

  for some constant $d \times d$ matrix. From (3.72) with $t = 0$ and (3.69) with $t = T$,

  $$\mathrm{Det}\{\boldsymbol{C}\} = \exp\left\{\int_0^T \mathrm{Trace}\{\mathring{\boldsymbol{J}}(s)\}\mathrm{d}s\right\} \neq 0. \tag{3.73}$$

- Suppose that $\lambda^{(k)}$, $k = 1, 2, \ldots, d$ are the eigenvalues of $\boldsymbol{C}$ with right eigenvectors $\boldsymbol{u}^{(k)}$. Thus

  $$\boldsymbol{C}\boldsymbol{u}^{(k)} = \boldsymbol{u}^{(k)}\lambda^{(k)}, \qquad k = 1, 2, \ldots, d. \tag{3.74}$$

  From (3.66), (3.68) and (3.70)

  $$\boldsymbol{w}^{(k)}(t) = \boldsymbol{Q}(t)\boldsymbol{u}^{(k)}, \qquad k = 1, 2, \ldots, d, \tag{3.75}$$

  are solutions of (3.66) with $\boldsymbol{w}^{(k)}(0) = \boldsymbol{u}^{(k)}$.

- From (3.72) and (3.75)

  $$\begin{aligned} \boldsymbol{w}^{(k)}(t+T) &= \boldsymbol{Q}(t+T)\boldsymbol{u}^{(k)} = \boldsymbol{Q}(t)\boldsymbol{C}\boldsymbol{u}^{(k)} = \lambda^{(k)}\boldsymbol{Q}(t)\boldsymbol{u}^{(k)} \\ &= \lambda^{(k)}\boldsymbol{w}^{(k)}(t). \end{aligned} \tag{3.76}$$

  The converse of the development leading to (3.76) is that if, for some solution $\boldsymbol{w}(t)$ of (3.66),

  $$\boldsymbol{w}(t+T) = \lambda\boldsymbol{w}(t), \tag{3.77}$$

  then λ is an eigenvalue of $\boldsymbol{C}$. The proposition that (3.66) with $\mathring{\boldsymbol{J}}(t)$ continuous and satisfying (3.71) has at least one non-trivial solution satisfying (3.77) with $\lambda \neq 0$ is *Floquet's theorem.*

- Although the matrix $\boldsymbol{C}$ was defined, by (3.72), using the fundamental solution matrix $\boldsymbol{Q}(t)$, the eigenvalues are not dependent on this choice. Suppose $\boldsymbol{S}(t)$ is another fundamental solution matrix. There must exist a non-singular matrix $\boldsymbol{Z}$ with $\boldsymbol{S}(t) = \boldsymbol{Q}(t)\boldsymbol{Z}$ and

$$\begin{aligned} \boldsymbol{S}(t+T) &= \boldsymbol{Q}(t+T)\boldsymbol{Z} = \boldsymbol{Q}(t)\boldsymbol{C}\boldsymbol{Z} \\ &= \boldsymbol{S}(t)\boldsymbol{Z}^{-1}\boldsymbol{C}\boldsymbol{Z}. \end{aligned} \tag{3.78}$$

  Comparing (3.78) with (3.72) we see that $\boldsymbol{C}$ has been replaced by $\boldsymbol{Z}^{-1}\boldsymbol{C}\boldsymbol{Z}$, which has the same set of eigenvalues.

- Let

$$\lambda^{(k)} = \exp\left(\sigma^{(k)}T\right). \tag{3.79}$$

  The numbers $\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(d)}$ are called the *characteristic or Floquet exponents* of the linear system (3.66).

- For the solution $\boldsymbol{w}^{(k)}(t)$ to (3.64), defined by (3.75), let

$$\boldsymbol{w}^{(k)}(t) = \boldsymbol{y}^{(k)}(t)\exp\left(\sigma^{(k)}t\right). \tag{3.80}$$

  Then, substituting into (3.76),

$$\boldsymbol{y}^{(k)}(t+T)\exp\left(\sigma^{(k)}\{t+T\}\right) = \lambda^{(k)}\boldsymbol{y}^{(k)}(t)\exp\left(\sigma^{(k)}t\right) \tag{3.81}$$

  and from the definition of $\sigma^{(k)}$

$$\boldsymbol{y}^{(k)}(t+T) = \boldsymbol{y}^{(k)}(t). \tag{3.82}$$

  When $\boldsymbol{w}^{(k)}(t)$ is given the form (3.80), $\boldsymbol{y}^{(k)}(t)$ is periodic, period T.

- Since

$$\mathrm{Det}\{\boldsymbol{C}\} = \prod_{k=1}^{m}\lambda^{(k)}, \tag{3.83}$$

  it follows, from (3.73) and (3.79), that

$$\sum_{k=1}^{m}\sigma^{(k)} \equiv \frac{1}{T}\int_0^T \mathrm{Trace}\{\mathring{\boldsymbol{J}}(s)\}\mathrm{d}s \mod (2\pi\mathrm{i}/T). \tag{3.84}$$

This development now allows us to discuss the stability of the periodic solution $\mathring{\boldsymbol{x}}(t)$ of (3.1). To do so we suppose that the eigenvectors $\boldsymbol{u}^{(k)}$, $k = 1, 2, \ldots, d$ of $\boldsymbol{C}$ form a basis of Γ_d. Then, for any solution $\triangle\boldsymbol{x}(t)$ of (3.64), there exists a set of constants $c^{(k)}$, $k = 1, 2, \ldots, d$ with

$$\triangle\boldsymbol{x}(0) = \sum_{k=1}^{d} c^{(k)}\boldsymbol{u}^{(k)}. \tag{3.85}$$

From (3.75)–(3.76) and (3.79),

$$\triangle\boldsymbol{x}(nT) = \sum_{k=1}^{d} \exp\left(n\sigma^{(k)}T\right) c^{(k)}\boldsymbol{u}^{(k)}, \qquad n = 1, 2, \ldots. \tag{3.86}$$

It follows that:

(i) If $\Re\{\sigma^{(k)}\} < 0$, for $k = 1, 2, \ldots, d$, $\triangle\boldsymbol{x}(nT) \to \triangle\boldsymbol{x}(0)$ as $n \to \infty$, for all choices of $\{c^{(k)}\}$, and $\mathring{\boldsymbol{x}}(t)$ is an asymptotically stable periodic solution, that is a *stable limit cycle*.

(ii) If $\Re\{\sigma^{(k)}\} > 0$, for some k then there exists a choice of $\{c^{(k)}\}$ for which $\triangle\boldsymbol{x}(nT) \to \infty$, as $n \to \infty$. $\mathring{\boldsymbol{x}}(t)$ is an unstable periodic solution, that is a *unstable limit cycle*.

(iii) If for some k', $\sigma^{(k')} = 0$ then the choice of $\triangle\mathring{\boldsymbol{x}}(0)$ with $c^{(k)} = 0$ for $k \neq k'$ gives a periodic orbit close to $\mathring{\boldsymbol{x}}(t)$.

(iv) Purely imaginary Floquet exponents lead to periodic orbits, with periods which are multiples of T, or quasi-periodic orbits rather like those on the torus discussed in Sect. 3.3.

Example 3.4.1 *Suppose that, for $d = 2$, the linearized equations have the form*

$$\frac{\mathrm{d}\triangle x}{\mathrm{d}t} = \triangle y, \qquad \frac{\mathrm{d}\triangle y}{\mathrm{d}t} = -\omega(t)\triangle x, \tag{3.87}$$

where $\omega(t)$ is a real-valued, continuous, periodic function of period T.[4]

Then

$$\mathring{\boldsymbol{J}}(t) = \begin{pmatrix} 0 & 1 \\ -\omega(t) & 0 \end{pmatrix} \tag{3.88}$$

and

$$\mathrm{Trace}\{\mathring{\boldsymbol{J}}(t)\} = 0. \tag{3.89}$$

From (3.84) the Floquet exponents are related by

$$\sigma^{(1)} + \sigma^{(2)} \equiv 0 \mod (2\pi\mathrm{i}/T). \tag{3.90}$$

[4]This system is equivalent to *Hill's equation* $\ddot{z}(t) + \omega(t)z(t) = 0$.

It cannot be the case that both Floquet exponents have negative real part and the periodic solution of a system which leads to the linearized form (3.87) cannot be a stable limit cycle. The alternatives are:

(i) Floquet exponents with real parts of opposite signs which gives an unstable limit cycle.

(ii) Purely imaginary Floquet exponents $\sigma^{(1)} = \sigma^{(2)} = n\pi \mathrm{i}/T$, which gives a periodic solution, period T, if n is even and $2T$, if n is odd.

(iii) Purely imaginary Floquet exponents with other than these special values which give a quasi-period solution.

3.4.1 Periodic Solutions in Two Dimensions

We now consider the case of periodic solutions for two-dimensional autonomous systems given by (3.39), with (3.39) having a unique solution at all points in $\{x, y\}$ which are not equilibrium points ($F(x, y) = G(x, y) = 0$). We state two important results for such systems. The second of these, which is the *Poincaré-Bendixson theorem* will be shown to be a consequence of the first result, which is stated without proof.

Theorem 3.4.1 *If a trajectory of (3.39) has a bounded ω-set, then that set is either an equilibrium point or a periodic trajectory.*

Theorem 3.4.2 *Let $\mathcal{C}$ be a closed, bounded (i.e. compact) subset of the $\{x, y\}$ plane. If there exists a solution $\gamma = \{x(t), y(t)\}$ of (3.39), which is contained in $\mathcal{C}$ for all $t \geq 0$, then it tends either to an equilibrium point or to a periodic solution as $t \to \infty$.*

Proof: Consider the infinite sequence $(x(t_0 + n\varepsilon), y(t_0 + n\varepsilon))$ of points of γ, with $t_0 > 0$, $\varepsilon > 0$, $n = 0, 1, 2, \ldots$. All these points lie in the compact set $\mathcal{C}$ so it follows from the *Bolzano-Weierstrass theorem* that the sequence has at least one limit point. This point must belong to the ω-limit set of γ, which is thus non-empty. From Thm. 3.4.1 this ω-limit set is an equilibrium point or a periodic solution to which γ tends.

It follows from the Poincaré-Bendixson theorem that the existence of a trajectory γ of the type described in the theorem guarantees the existence of either a periodic trajectory or an equilibrium point in $\mathcal{C}$. It is clear that a periodic solution which is the ω-set of γ cannot be an unstable limit cycle, but it also need not be a stable limit cycle.

Example 3.4.2

$$\begin{aligned} \dot{x}(t) &= x(t) - y(t) - x(t)[x^2(t) + 2y^2(t)], \\ \dot{y}(t) &= x(t) + y(t) - y(t)[x^2(t) + y^2(t)]. \end{aligned} \tag{3.91}$$

In polar coordinates (3.91) take the form

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r - r^3\left\{1 + \tfrac{1}{4}\sin^2(2\theta)\right\}, \tag{3.92}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + r^2\sin^2(\theta)\cos(\theta). \tag{3.93}$$

From (3.92)

$$r - \frac{5}{4}r^3 \le \frac{\mathrm{d}r}{\mathrm{d}t} \le r - r^3, \quad \text{for all } \theta, \tag{3.94}$$

and thus

$$\begin{aligned} &\dot{r}(t) < 0, \qquad \text{for all } \theta, \text{ if } r > r_1 = 1, \\ &\dot{r}(t) > 0, \qquad \text{for all } \theta, \text{ if } r < r_2 = 2/\sqrt{5}. \end{aligned} \tag{3.95}$$

So any trajectory with $(x(0), y(0))$ in the annulus

$$\mathcal{C} = \{(x, y) : r_2 \le \sqrt{x^2 + y^2} \le r_1\} \tag{3.96}$$

remains in this region for all $t > 0$. The minimum value of $1 + r^2\sin^2(\theta)\cos(\theta)$ as θ varies at constant r is $1 - 2r^2/(3\sqrt{3})$ and thus

$$\dot{\theta}(t) > 1 - \frac{2r_2^2}{3\sqrt{3}} = 1 - \frac{8}{15\sqrt{3}} \simeq 0.69208. \tag{3.97}$$

So $\dot{\theta}(t)$ is never zero and there are no equilibrium points in $\mathcal{C}$. Thus, from the Poincaré-Bendixson theorem there is at least one periodic orbit.

Problems 3

1) Systems are given by

   (i) $\dot{x}(t) = -x - 2y^2, \qquad \dot{y}(t) = xy - y^3,$

   (ii) $\dot{x}(t) = y - x^3, \qquad \dot{y}(t) = -x^3.$

   Using a trial form of $\mathcal{L}(x, y) = x^n + \alpha y^m$ for the Lyapunov function show (by a judicious choice of n, m and α) that, in each case the equilibrium point $x = y = 0$ is asymptotically stable.

2) A system is given by

   $$\dot{x}(t) = x^2y - xy^2 + x^3, \qquad \dot{y}(t) = y^3 - x^3$$

   Show that $x = y = 0$ is the only equilibrium point and, using a trial form of $\mathcal{L}(x, y) = x^2 + \alpha xy + \beta y^2$ for the Lyapunov function, show that it is unstable.

3) Express

$$\ddot{x}(t) + x(t)\{1 - a|x(t)|\} = 0$$

as a two-dimensional system in the variables x–y and show that

$$\tfrac{1}{2}\{x^2 + y^2\} - \tfrac{1}{3}a|x|^3 = E \tag{3.98}$$

is a constant of motion for any value of the parameter E. Find the equilibrium points and the ranges of a for which they exist. Use linear analysis to determine their types and sketch the bifurcation diagram in the x–a plane. Using (3.98) sketch trajectories in the x–y plane for typical values of a, showing that periodic solutions exist for all a and that the period of the oscillation with amplitude ζ is

$$T = 4\int_0^{\zeta} \frac{\mathrm{d}x}{\sqrt{\zeta^2 - \frac{2}{3}a\zeta^3 - x^2 + \frac{2}{3}ax^3}}.$$

4) Express

$$\ddot{x}(t) + 2a\dot{x}(t) + x(t) + bx^3(t) = 0$$

as a two-dimensional system in the variables x–y and, for $a > 0$, find the equilibrium points for both signs of b. Use linear analysis to determine their types.

For $b > 0$ and $a > 0$, use the Lyapunov function

$$\mathcal{L}(x, \dot{x}) = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 + \tfrac{1}{4}bx^4,$$

to show that $x(t) \to 0$, as $t \to \infty$, for all initial conditions.

5) Show directly from the definitions that the periodic solution $x(t) = a\cos(t)$, $y(t) = -a\sin(t)$ to the system

$$\dot{x}(t) = y(t), \qquad \dot{y}(t) = -x(t)$$

is stable in the Lyapunov sense.

6) Show that the system

$$\ddot{x}(t) + b[\dot{x}^2(t) + x^2(t) - a]\dot{x}(t) + x(t) = 0,$$

can be expressed in the form

$$\dot{r} = b(a - r^2)r\sin^2(\theta), \qquad \dot{\theta} = \tfrac{1}{2}b(a - r^2)\sin(2\theta) - 1,$$

where $x = r\cos(\theta)$, $\dot{x} = r\sin(\theta)$. Deduce that, for $a > 0$, there is a periodic solution $r = \sqrt{a}$, $\theta = t_0 - t$ of period 2π and show that the sum of the Floquet exponents is $-ab$. (This suggests but doesn't prove that the periodic solution is stable if $b > 0$.) Now show that, with $\triangle r = r - \sqrt{a}$,

$$\frac{\mathrm{d}\triangle r}{\mathrm{d}t} = -b\triangle r(\triangle r + \sqrt{a})(\triangle r + 2\sqrt{a})\sin^2(t_0 - t).$$

Hence prove that the periodic solution is stable in the sense of Lyapunov if $b > 0$.

7) Consider the system

$$\dot{x}(t) = F(x, y), \qquad \dot{y}(t) = G(x, y),$$

where F and G are continuous functions of x and y. For the cases

(i) $F(x, y) = x + y - x(x^2 + 2y^2)$, $\quad G(x, y) = -x + y - y(x^2 + 2y^2)$,

(ii) $F(x, y) = -x - y + x(x^2 + 2y^2)$, $\quad G(x, y) = x - y + y(x^2 + 2y^2)$,

show that the origin is the only equilibrium point and determine its type. Express the equations in polar form and show that the system has at least one periodic solution. Determine, using the *Poincaré-Bendixson theorem*, or otherwise, whether it is stable.

Chapter 4

Weakly Nonlinear Systems

4.1 The Lindstedt-Poincaré Method

In Sects. 1.3 and 1.13 we considered the case of conservative systems. Using different variables for the case $d = 2$

$$\dot{\eta}(t) = \xi, \qquad \dot{\xi}(t) = -V'(\eta). \tag{4.1}$$

The equilibrium points are the turning points of $V(\eta)$ appearing in the space of $\{\eta, \xi\}$ on the η–axis. Suppose $\eta = \eta^*$ is such an equilibrium point. Then expanding about the $\eta = \eta^*$

$$V'(\eta) = (\eta - \eta^*)V''(\eta^*) + \psi(\eta - \eta^*), \tag{4.2}$$

where $V''(\eta^*) > 0$ and $\psi(z) = \mathrm{O}(z^2)$. Let $x = \eta - \eta^*$, $y = \xi$ and $\omega_0^2 = V''(\eta^*)$. Then

$$\dot{x}(t) = y, \qquad \dot{y}(t) = -\omega_0^2 x - \psi(x). \tag{4.3}$$

If the non-linear term $\psi(x)$ were neglected then we should have a simple harmonic oscillator with all solutions of period $2\pi/\omega_0$. We now suppose that

$$\psi(x) = \omega_0^2 f(\varepsilon, x), \tag{4.4}$$

where $f(0, x) = 0$. Thus

$$\dot{x}(t) = y, \qquad \dot{y}(t) = -\omega_0^2\{x + f(\varepsilon, x)\}. \tag{4.5}$$

We look for a periodic solution to (4.5) of period $2\pi/\omega(\varepsilon)$. The first step is to replace t by $\tau = \omega(\varepsilon)t$ where $\omega(\varepsilon) = \omega_0 g(\varepsilon)$. This gives

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \tilde{y}, \qquad \{g(\varepsilon)\}^2 \frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tau} = -\{x + f(\varepsilon, x)\}, \tag{4.6}$$

where $\tilde{y} = y/\omega(\epsilon)$. Let

$$\begin{aligned} x(\varepsilon,\tau) &= x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \mathrm{O}(\varepsilon^3), && (4.7)\\ \tilde{y}(\varepsilon,\tau) &= \tilde{y}_0(\tau) + \varepsilon \tilde{y}_1(\tau) + \varepsilon^2 \tilde{y}_2(\tau) + \mathrm{O}(\varepsilon^3), && (4.8)\\ g(\varepsilon) &= 1 + \varepsilon g_1 + \varepsilon^2 g_2 + \mathrm{O}(\varepsilon^3), && (4.9)\\ f(\varepsilon, x(\tau,\varepsilon)) &= \varepsilon f_\varepsilon(\tau) + \varepsilon^2 x_1(\tau) f_{\varepsilon x}(\tau) + \frac{1}{2}\varepsilon^2 f_{\varepsilon\varepsilon}(\tau) + \mathrm{O}(\varepsilon^3). && (4.10) \end{aligned}$$

where

$$\begin{aligned} f_\varepsilon(\tau) &= \frac{\partial f}{\partial \varepsilon}(0, x_0(\tau)), \quad f_{\varepsilon x}(\tau) = \frac{\partial^2 f}{\partial \varepsilon \partial x}(0, x_0(\tau)),\\ f_{\varepsilon\varepsilon}(\tau) &= \frac{\partial^2 f}{\partial \varepsilon^2}(0, x_0(\tau)). && (4.11) \end{aligned}$$

Substituting into (4.6) and equating powers of ε the ε^0 terms give

$$\frac{\mathrm{d}x_0}{\mathrm{d}\tau} = \tilde{y}_0, \qquad \frac{\mathrm{d}\tilde{y}_0}{\mathrm{d}\tau} = -x_0 \tag{4.12}$$

and the ε^1 terms give

$$\frac{\mathrm{d}x_1}{\mathrm{d}\tau} = \tilde{y}_1, \qquad 2g_1\frac{\mathrm{d}\tilde{y}_0}{\mathrm{d}\tau} + \frac{\mathrm{d}\tilde{y}_1}{\mathrm{d}\tau} = -x_1 - f_\varepsilon(\tau). \tag{4.13}$$

The general solution to (4.12) is

$$\begin{aligned} x_0 &= a_0\cos(\tau) + b_0\sin(\tau),\\ \tilde{y}_0 &= b_0\cos(\tau) - a_0\sin(\tau), \end{aligned} \tag{4.14}$$

but for simplicity we shall take $b_0 = 0$. Then substituting into (4.13)

$$\frac{\mathrm{d}x_1}{\mathrm{d}\tau} = \tilde{y}_1, \qquad \frac{\mathrm{d}\tilde{y}_1}{\mathrm{d}\tau} = -x_1 - f_\varepsilon(\tau) + 2g_1a_0\cos(\tau). \tag{4.15}$$

Let us suppose a solution to (4.15) of the form

$$\begin{aligned} x_1 &= a_1\cos(\tau) + b_1\sin(\tau) + X(\tau),\\ \tilde{y}_1 &= b_1\cos(\tau) - a_1\sin(\tau) + X'(\tau). \end{aligned} \tag{4.16}$$

Then the particular integral $X(\tau)$ is a solution of

$$X''(\tau) + X(\tau) = -f_\varepsilon(\tau) + 2g_1a_0\cos(\tau). \tag{4.17}$$

To proceed further we need a particular form for $f_\varepsilon(\tau)$. Suppose, as an example, that $f(\varepsilon, x) = \varepsilon c x^3$. Then (4.17) becomes

$$X''(\tau) + X(\tau) = a_0\left[2g_1 - \tfrac{3}{4}ca_0^2\right]\cos(\tau) - \tfrac{1}{4}ca_0^3\cos(3\tau). \tag{4.18}$$

For which the solution is

$$X(\tau) = a_0\left[g_1 - \tfrac{9}{32}ca_0^2\right]\cos(\tau) + \tfrac{1}{32}ca_0^3\cos(3\tau) + a_0\left[g_1 - \tfrac{3}{8}ca_0^2\right]\tau\sin(\tau). \quad (4.19)$$

Substituting from (4.19) into (4.16) we see that the solution will be periodic only if the final term in (4.19) disappears, for which we need,

$$g_1 = \tfrac{3}{8}ca_0^2. \quad (4.20)$$

We impose the condition that $x(\varepsilon,0) = a_0$, $\tilde{y}(\varepsilon,0) = 0$ and then, from (4.7)–(4.9), (4.14), (4.16), (4.19)–(4.20),

$$x(\varepsilon,t) = a_0\cos(\omega t) + \varepsilon\tfrac{1}{32}ca_0^3\{\cos(3\omega t) + 3\cos(\omega t)\} + \mathrm{O}(\varepsilon^2), \quad (4.21)$$

where

$$\omega(\varepsilon) = \omega_0\left\{1 + \tfrac{3}{8}\varepsilon ca_0^2 + \mathrm{O}(\varepsilon^2)\right\}. \quad (4.22)$$

We have succeeded in obtaining a periodic solution to the equations

$$\dot{x}(t) = y, \qquad \dot{y}(t) = -\omega_0^2 x\{1 + \varepsilon cx^2\}, \quad (4.23)$$

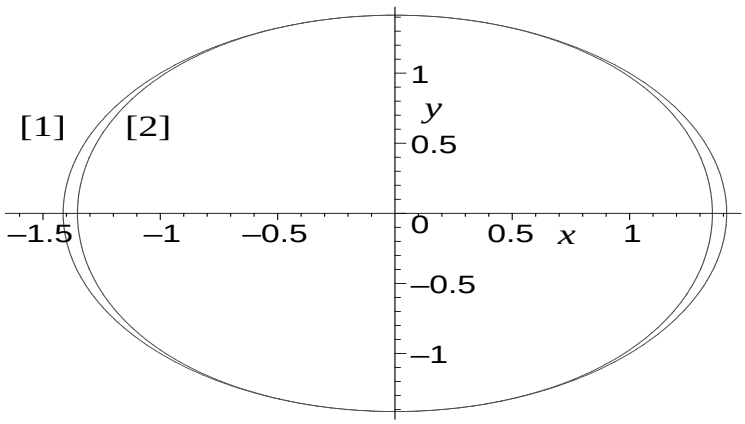
by perturbing the simple harmonic solution. Some insight into this procedure can be gained from the first integral constant of motion. From (4.23)

$$\omega_0^2 x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t} = -\omega_0^2\varepsilon cx^3 y = -\omega_0^2\varepsilon cx^3\frac{\mathrm{d}x}{\mathrm{d}t}, \quad (4.24)$$

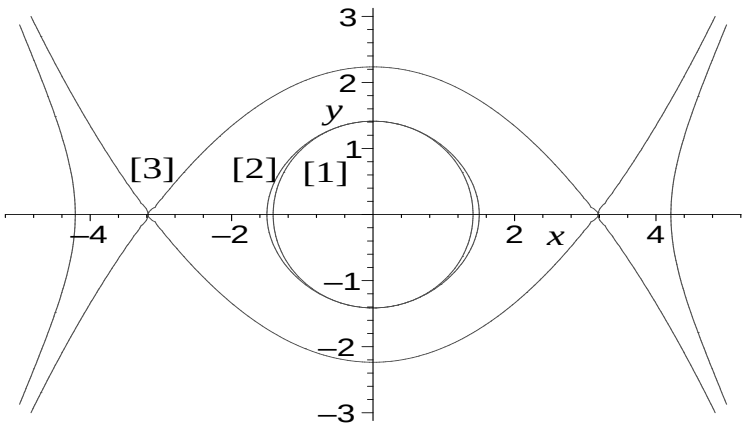
giving

$$\omega_0^2 x^2\left\{\tfrac{1}{2} + \tfrac{1}{4}\varepsilon cx^2\right\} + \tfrac{1}{2}y^2 = E. \quad (4.25)$$

It is convenient to keep $\varepsilon \geq 0$ with $c = \pm 1$. Then with $c = 1$, the only equilibrium point of (4.23) is $x = y = 0$. According to (4.22) the frequency is increased with increasing ε. Curves for (4.25) can be obtained using MAPLE code similar to that given on page 30. Curves of with $c = 1$, $E = \omega_0^2$ are of the form



with (1) $\varepsilon = 0$ and (2) $\varepsilon = 0.1$. The effect of small non-zero ε is to contract the curves in the x–direction. When $c = -1$, (4.22) have saddle-points on the x–axis at $x = 1/\sqrt{-c\varepsilon}$. The set of curves (4.25) has a separatrix through the saddle-points with $E = \omega_0^2/(-4c\varepsilon)$. Curves have the form



where (1) $\varepsilon = 0.0$, $E = \omega_0^2$, (2) $\varepsilon = 0.1$, $E = \omega_0^2$. Now the effect of small ε is to dilate the curve in the x–direction. Curve (3) is the separatrix for $\varepsilon = 0.1$

which has $E = 5\omega_0^2/2$.

This important point about the Lindstedt-Poincaré method is that it allows for perturbations in the angular frequency ω. Without such a perturbation terms of the form $t\sin(\omega t)$ would have been present, preventing the perturbed solution from being periodic. The method can be generalized in various ways. We may for example include a y dependence in the perturbation, so that we have $f(\epsilon, x, y)$ in (4.5).

4.2 The Hopf Bifurcation

As we saw in Example 1.12.2 a Hopf bifurcation occurs when the stability of a focus changes from stable to unstable (supercritical) or unstable to stable (subcritical) with the emergence of a limit cycle, which is stable in the supercritical case and unstable in the subcritical case. We now consider the system given by

$$\dot{x}(t) = -y + ax + xy^2, \qquad \dot{y}(t) = x + ay - x^2. \tag{4.26}$$

The linear terms are the same as those of (1.117)–(1.118) so we might anticipate the occurrence of a Hopf bifurcation, leading to a periodic solution. Since, as in Example 1.12.2, the equilibrium point at $(0, 0)$ is stable or unstable according as $a < 0$ and $a > 0$, the Hopf bifurcation will be supercritical if the periodic orbit occurs for $a > 0$ and subcritical if it occurs for $a < 0$.

We investigate this using a version on the Lindstedt-Poincaré method. In doing so we can, without loss of generality, impose the condition $\dot{x}(0) = 0$. Let $\tau = \omega t$ and $\varepsilon = \sqrt{a}$. Then (4.26) become

$$\omega\frac{\mathrm{d}x}{\mathrm{d}\tau} = -y + \varepsilon^2 x + xy^2, \qquad \omega\frac{\mathrm{d}y}{\mathrm{d}\tau} = x + \varepsilon^2 y - x^2. \tag{4.27}$$

Now substitute the expansions

$$x(\varepsilon, \tau) = \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \varepsilon^3 x_3(\tau) + \mathrm{O}(\varepsilon^4), \tag{4.28}$$

$$y(\varepsilon, \tau) = \varepsilon y_1(\tau) + \varepsilon^2 y_2(\tau) + \varepsilon^3 y_3(\tau) + \mathrm{O}(\varepsilon^4), \tag{4.29}$$

$$\omega(\varepsilon) = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \varepsilon^3\omega_3 + \mathrm{O}(\varepsilon^4), \tag{4.30}$$

into (4.27) and compare coefficients. For ε^1,

$$\frac{\mathrm{d}x_1}{\mathrm{d}\tau} = -y_1, \qquad \frac{\mathrm{d}y_1}{\mathrm{d}\tau} = x_1, \tag{4.31}$$

giving

$$x_1(\tau) = a_1\cos(\tau), \qquad y_1(\tau) = a_1\sin(\tau). \tag{4.32}$$

For ε^2,

$$\omega_1\frac{\mathrm{d}x_1}{\mathrm{d}\tau} + \frac{\mathrm{d}x_2}{\mathrm{d}\tau} = -y_2, \qquad \omega_1\frac{\mathrm{d}y_1}{\mathrm{d}\tau} + \frac{\mathrm{d}y_2}{\mathrm{d}\tau} = x_2 - x_1^2, \tag{4.33}$$

and substituting from (4.32)

$$\frac{\mathrm{d}x_2}{\mathrm{d}\tau} = -y_2 + \omega_1 a_1 \sin(\tau), \tag{4.34}$$

$$\frac{\mathrm{d}y_2}{\mathrm{d}\tau} = x_2 - a_1^2 \cos^2(\tau) - \omega_1 a_1 \cos(\tau). \tag{4.35}$$

Let

$$x_2(\tau) = a_2 \cos(\tau) + X(\tau). \tag{4.36}$$

Then, from (4.34),

$$y_2(\tau) = a_2 \sin(\tau) - X'(\tau) + \omega_1 a_1 \sin(\tau), \tag{4.37}$$

and, substituting into (4.35),

$$X''(\tau) + X(\tau) = \tfrac{1}{2} a_1^2 \{1 + \cos(2\tau)\} + 2\omega_1 a_1 \cos(\tau). \tag{4.38}$$

Thus, from (4.36) and (4.38),

$$x_2(\tau) = a_2 \cos(\tau) + a_1 \omega_1 \{\cos(\tau) + \tau \sin(\tau)\} + \tfrac{1}{6} a_1^2 \{3 - \cos(2\tau)\}. \tag{4.39}$$

This solution will not be periodic unless $\omega_1 = 0$ and applying this condition it follows from (4.39) and (4.37) that

$$\begin{aligned} x_2(\tau) &= a_2 \cos(\tau) + \tfrac{1}{6} a_1^2 \{3 - \cos(2\tau)\} \\ y_2(\tau) &= a_2 \sin(\tau) - \tfrac{1}{3} a_1^2 \sin(2\tau). \end{aligned} \tag{4.40}$$

For ε^3,

$$\frac{\mathrm{d}x_3}{\mathrm{d}\tau} + \omega_2 \frac{\mathrm{d}x_1}{\mathrm{d}\tau} = -y_3 + x_1 + x_1 y_1^2, \tag{4.41}$$

$$\frac{\mathrm{d}y_3}{\mathrm{d}\tau} + \omega_2 \frac{\mathrm{d}y_1}{\mathrm{d}\tau} = x_3 + y_1 - 2x_1 x_2. \tag{4.42}$$

Substituting from (4.32) and (4.40)

$$\frac{\mathrm{d}x_3}{\mathrm{d}\tau} = -y_3 + a_1 \cos(\tau) + a_1 \omega_2 \sin(\tau) + a_1^3 \sin^2(\tau) \cos(\tau), \tag{4.43}$$

$$\begin{aligned} \frac{\mathrm{d}y_3}{\mathrm{d}\tau} &= x_3 + a_1 \sin(\tau) - a_1 \omega_2 \cos(\tau) - 2a_1 a_2 \cos^2(\tau) \\ &\quad - \tfrac{1}{3} a_1^3 \cos(\tau)\{3 - \cos(2\tau)\}. \end{aligned} \tag{4.44}$$

Solving these equations for $x_3(\tau)$ gives

$$\begin{aligned} x_3(\tau) &= a_3 \cos(\tau) + \tfrac{1}{8} a_1 \tau \cos(\tau)\{a_1^2 + 8\} + \tfrac{1}{12} a_1 \tau \sin\tau \{12\omega_2 + 5a_1^2\} \\ &\quad - \tfrac{2}{3} a_1 a_2 \{\cos^2(\tau) - 2\} - \tfrac{1}{48} a_1^3 \{\cos(\tau)\cos(4\tau) \\ &\quad + \sin(\tau)\sin(4\tau) - 12\sin(\tau) + 24\cos(\tau) + 4\sin(\tau)\sin(2\tau) \\ &\quad + 18\sin(\tau)\cos^2(\tau)\}. \end{aligned} \tag{4.45}$$

For the solution to be periodic we must have $a_1 = 2\sqrt{2}\mathrm{i}$, $\omega_2 = 10/3$. The imaginary value of a_1 means that ε is also imaginary and the periodic solution appear for $a < 0$, which means that it is subcritical. To leading order the limit cycle is given by

$$x(a,t) \simeq \sqrt{-8a}\cos(\omega t), \qquad y(a,t) \simeq \sqrt{-8a}\sin(\omega t), \tag{4.46}$$

where

$$\omega \simeq 1 + \frac{10}{3}a. \tag{4.47}$$

4.3 The Krylov, Bogoliubov and Mitropolsky Averaging Method

In Sect. 4.1 we considered the case of a simple harmonic oscillator perturbed by a term which was a function of the spatial variable x. In particular we investigated the case where the perturbation was $\varepsilon c x^3$. In this section we consider a perturbation which is a function of $x(t)$ and $\dot{x}(t)$. That is

$$\ddot{x}(t) + \varepsilon f\left(x(t), \dot{x}(t)\right) + x(t) = 0. \tag{4.48}$$

This includes the case where $f(x,\dot{x}) = \dot{x}$. Then the perturbation is proportional to the speed of the 'particle' and, for $\varepsilon > 0$ it, acts to slow the particle down. This is the way viscosity acts when a particle is moving in a viscous medium, like a simple pendulum swinging in air (or even more so in treacle). This is called *damping*. We could also consider *negative damping*, when $\varepsilon < 0$. With $y(t) = \dot{x}(t)$, (4.48) becomes

$$\dot{x}(t) = y, \qquad \dot{y}(t) = -x - \varepsilon f(x,y). \tag{4.49}$$

In polar coordinates

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\varepsilon\sin(\theta) f(r\cos(\theta), r\sin(\theta)), \tag{4.50}$$

$$\frac{\mathrm{d}(\theta+t)}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t} + 1 = -\frac{\varepsilon\cos(\theta)}{r} f(r\cos(\theta), r\sin(\theta)). \tag{4.51}$$

It can be seen that, when ε is small, $r(t)$ and $\theta(t) + t$ both vary slowly with t. So the motion is close to simple harmonic motion with a circular orbit in the $\{x, y\}$ plane and an angular velocity -1. The *KBM averaging method* consists in going back to (4.48) and supposing that:

(i) $$x(\varepsilon,t) = r\cos(\theta) + \varepsilon u^{(1)}(r,\theta) + \varepsilon^2 u^{(2)}(r,\theta) + \cdots, \tag{4.52}$$
where $u^{(k)}(r, \theta + 2\pi) = u^{(k)}(r,\theta)$ and

$$\int_0^{2\pi} u^{(k)}(r,\theta)\cos(\theta)\mathrm{d}\theta = \int_0^{2\pi} u^{(k)}(r,\theta)\sin(\theta)\mathrm{d}\theta = 0, \qquad k = 1, 2, \ldots. \tag{4.53}$$

(ii)

$$\dot{r}(t) = \varepsilon A^{(1)}(r) + \varepsilon^2 A^{(2)}(r) + \cdots, \tag{4.54}$$

$$\dot{\theta}(t) = -1 + \varepsilon B^{(1)}(r) + \varepsilon^2 B^{(2)}(r) + \cdots. \tag{4.55}$$

The k-th order KBM method consists in retaining terms up to ε^k. We shall now derive the formulae for the first-order method. Substituting into (4.48) and retaining terms up to $\mathrm{O}(\varepsilon)$ gives

$$\frac{\partial^2 u^{(1)}(r,\theta)}{\partial\theta^2} + u^{(1)}(r,\theta) + 2A^{(1)}(r)\sin(\theta) + 2rB^{(1)}(r)\cos(\theta) + f(r\cos(\theta), r\sin(\theta)) = 0. \tag{4.56}$$

Multiplying (4.56) by $\sin(\theta)$ integrating over $[0, 2\pi]$ using (4.53), and then doing the same with $\cos(\theta)$ gives

$$A^{(1)}(r) = -\frac{1}{2\pi}\int_0^{2\pi} \sin(\theta) f(r\cos(\theta), r\sin(\theta))\mathrm{d}\theta, \tag{4.57}$$

$$B^{(1)}(r) = -\frac{1}{2r\pi}\int_0^{2\pi} \cos(\theta) f(r\cos(\theta), r\sin(\theta))\mathrm{d}\theta. \tag{4.58}$$

It will be seen that (4.57)–(4.58) are equivalent to the results obtained by replacing the right-hand sides of (4.50)–(4.51) by their averages over $[0, 2\pi]$. The final task to complete the first-order approximation is to determine a particular integral for (4.56). The complementary function will correspond to substituting the results obtained from integrating (4.54)–(4.55) into the first tem of (4.52).

Example 4.3.1

$$\ddot{x}(t) + 2\varepsilon\dot{x}(t) + x(t) = 0, \tag{4.59}$$

So

$$f(r\cos(\theta), r\sin(\theta)) = 2r\sin(\theta). \tag{4.60}$$

From (4.57)–(4.58),

$$A^{(1)}(r) = -\frac{r}{\pi}\int_0^{2\pi} \sin^2(\theta)\mathrm{d}\theta = -r, \tag{4.61}$$

$$B^{(1)}(r) = -\frac{1}{\pi}\int_0^{2\pi} \sin(\theta)\cos(\theta)\mathrm{d}\theta = 0. \tag{4.62}$$

Substituting results into (4.54)–(4.55) gives, with the initial condition $r(0) = r_0$,

$$r(t) = r_0\exp(-\varepsilon t), \qquad \theta(t) = -t. \tag{4.63}$$

It is now necessary to solve the equation

$$\frac{\partial^2 u^{(1)}(r,\theta)}{\partial\theta^2} + u^{(1)}(r,\theta) = 0 \tag{4.64}$$

which has a solution

$$u^{(1)}(r,\theta) = 0, \tag{4.65}$$

giving

$$x(\varepsilon,t) = r_0 \exp(-\varepsilon t)\cos(t). \tag{4.66}$$

This is an example which can be solved exactly. It just corresponds to the case $a = 0$, $b = 1$, $c = -1$, $d = -2\varepsilon$ of the linear analysis of Sect. 3.2.2. From (3.45) the eigenvalues of the stability matrix are

$$\lambda^{(\pm)} = -\varepsilon \pm \sqrt{\varepsilon^2 - 1}. \tag{4.67}$$

With $-1 < \varepsilon < 1$, this gives, from (3.44),

$$x(\varepsilon,t) = r_0 \exp(-\varepsilon t)\cos\left(t\sqrt{1-\varepsilon^2}\right). \tag{4.68}$$

We see that the first-order KBM method correctly produces the exponential damping and the fact that the linear ε term in θ is zero. An indication of the accuracy of the first-order method is given by the following theorem due to Bogoliubov and Mitropolsky.

Theorem 4.3.1 *If the $R(\varepsilon,t)$ and $\varphi(\varepsilon,t)$ satisfy the equations*

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \varepsilon F(\varepsilon,R,\varphi), \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \Omega(\varepsilon,R) + \varepsilon G(\varepsilon,R,\varphi), \tag{4.69}$$

where $F(\varepsilon,R,\varphi+2\pi) = F(\varepsilon,R,\varphi)$ and $G(\varepsilon,R,\varphi+2\pi) = G(\varepsilon,R,\varphi)$ and $S(\varepsilon,t)$ satisfies

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\varepsilon}{2\pi}\int_0^{2\pi} F(0,R,\varphi)\mathrm{d}\varphi, \qquad S(0) = R(0), \tag{4.70}$$

then there exists a constant C and a sufficiently small value of ε such that

$$|S(t) - R(t)| < C\varepsilon, \qquad \textit{for all } 0 \le t \le 1/\varepsilon. \tag{4.71}$$

4.4 Liénard's Equation

The generic type of the second-order equations considered in this chapter is *Liénard's equation*

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + f(x)\frac{\mathrm{d}x}{\mathrm{d}t} + g(x) = 0, \tag{4.72}$$

where $f(x)$ and $g(x)$ are continuous functions. For this equation we have the theorem:

Theorem 4.4.1 *With*

$$F(x) = \int_0^x f(s)\mathrm{d}s, \tag{4.73}$$

(4.72) has a unique periodic solution, which is asymptotically orbitally stable (asymptotically stable in the sense of Poincaré) if the following conditions are satisfied.

(i) $g(x)$ *is an odd function with* $xg(x) > 0$ *for all* $x \neq 0$.

(ii) $f(x)$ *is an even function.*

(iii) There exists an $a > 0$ *such that:*

(a) $F(x) < 0$ *for* $0 < x < a$.

(b) $F(x) > 0$ *for* $x > a$.

(c) $F(x) = 0$ *only at* $x = 0, \pm a$.

4.5 Duffing's Equation

Duffing's equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \varepsilon\left\{cx^3 + 2\mu\frac{\mathrm{d}x}{\mathrm{d}t}\right\} + x = 0, \tag{4.74}$$

does not satisfy the conditions of Thm. 4.4.1 so we do not anticipate the existence of an asymptotically stable periodic solution. It is, however, a convenient example for the application of the KBM average method. For $\mu = 0$ it gives the case of the non-linear oscillator considered in Sect. 4.1 using the Lindstedt-Poincaré method and for $c = 0$, $\mu = 1$ it gives the case of the damped oscillator of Sect. 4.3. With $y(t) = \dot{x}(t)$, (4.74) gives

$$\dot{x}(t) = y, \qquad \dot{y}(t) = -x - \varepsilon(cx^3 + 2\mu y). \tag{4.75}$$

There is an equilibrium point at $x = y = 0$ with eigenvalues

$$\lambda^{(\pm)} = -\mu\varepsilon \pm \sqrt{\mu^2\varepsilon^2 - 1}, \tag{4.76}$$

for all values of the parameters. For

- $\mu\varepsilon > 1$ it is a stable node,
- $\mu\varepsilon = 1$ it is an inflected stable node,
- $1 > \mu\varepsilon > 0$ it is a stable focus,
- $\mu\varepsilon = 0$ it is a centre,
- $0 > \mu\varepsilon > -1$ it is a unstable focus,

- $\mu\varepsilon = -1$ it is an inflected unstable node,
- $-1 > \mu\varepsilon$ it is an unstable node.

When $\varepsilon c < 0$ there are also equilibrium points at $x = \pm 1/\sqrt{-\varepsilon c}$, $y = 0$ with eigenvalues

$$\lambda^{(\pm)} = -\mu\varepsilon \pm \sqrt{\mu^2\varepsilon^2 + 2}. \tag{4.77}$$

Since the eigenvalues are real and of opposite sign for all values of $\mu\varepsilon$, these equilibrium points are saddle points.

In the notation used in Sect. 4.3

$$f(r\cos(\theta), r\sin(\theta)) = cr^3\cos^3(\theta) + 2\mu r\sin(\theta). \tag{4.78}$$

Substituting into (4.57)–(4.57),

$$A^{(1)}(r) = -\mu r, \qquad B^{(1)}(r) = -\tfrac{3}{8}cr^2. \tag{4.79}$$

Substituting into (4.54)–(4.55) with the initial conditions $r(0) = r_0$, $\theta(0) = 0$ gives

$$\begin{aligned} r(t) &= r_0\exp(-\mu\varepsilon t), \\ \theta(t) &= -t - \tfrac{3}{16}\frac{cr_0^2}{\mu}\{1 - \exp(-2\mu\varepsilon t)\}. \end{aligned} \tag{4.80}$$

Substituting into (4.56)

$$\frac{\partial^2 u^{(1)}(r,\theta)}{\partial\theta^2} + u^{(1)}(r,\theta) = \tfrac{1}{4}cr^3\cos(\theta)\{3 - 4\cos^2(\theta)\}. \tag{4.81}$$

which has the solution

$$u^{(1)}(r,\theta) = \tfrac{1}{8}cr^3\cos^3(\theta). \tag{4.82}$$

Thus, from (4.52),

$$\begin{aligned} x(\varepsilon,t) \quad = \quad & r_0\exp(-\mu\varepsilon t)\cos\left(t + \tfrac{3}{16}\frac{cr_0^2}{\mu}\{1 - \exp(-2\mu\varepsilon t)\}\right) \\ & + \tfrac{1}{8}\varepsilon cr_0^3\exp(-3\mu\varepsilon t)\cos^3\left(t + \tfrac{3}{16}\frac{cr_0^2}{\mu}\{1 - \exp(-2\mu\varepsilon t)\}\right). \end{aligned} \tag{4.83}$$

With $c = 0$, $\mu = 1$ we recover the result (4.66) for the damped oscillator. Expanding the exponentials for small μ and retaining contributions of $\mathrm{O}(\varepsilon)$ gives

$$x(\varepsilon,t) = r_0\cos(\omega t) + \tfrac{1}{32}\varepsilon cr_0^3\{3\cos(\omega t) + \cos(3\omega t)\}. \tag{4.84}$$

where

$$\omega = 1 + \tfrac{3}{8}\varepsilon cr_0^2 \tag{4.85}$$

This agrees (with $\omega_0 = 1$) with the results (4.21)-(4.22) obtained by the Lindstedt-Poincaré method.

4.6 The Van der Pol and Rayleigh Equations

In modelling an electrical circuit with a thermionic valve van der Pol derived an equation of the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \varepsilon(x^2-1)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0 \tag{4.86}$$

and Rayleigh modelled non-linear vibrations with the equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}t^2} + \varepsilon\left\{\frac{1}{3}\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)^3 - \frac{\mathrm{d}w}{\mathrm{d}t}\right\} + w = 0. \tag{4.87}$$

Differentiating this equation with respect to t gives

$$\frac{\mathrm{d}^3w}{\mathrm{d}t^3} + \varepsilon\left\{\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)^2 - 1\right\}\frac{\mathrm{d}^2w}{\mathrm{d}t^2} + \frac{\mathrm{d}w}{\mathrm{d}t} = 0 \tag{4.88}$$

and setting $x(t) = \dot{w}(t)$ recovers (4.86). With $f(x) = \varepsilon(x^2-1)$ and the definition (4.73), $F(x) = \varepsilon x(x^2-3)/3$. So, when $\varepsilon > 0$, van der Pol's equation satisfies the conditions of Thm. 4.4.1 with $a = \sqrt{3}$ and an asymptotically stable periodic solution exists. With $y(t) = \dot{x}(t)$, (4.86) gives

$$\dot{x}(t) = y, \qquad \dot{y}(t) = -x - \varepsilon(x^2-1)y. \tag{4.89}$$

There only equilibrium point is at $x = y = 0$ with eigenvalues

$$\lambda^{(\pm)} = \tfrac{1}{2}\{\varepsilon \pm \sqrt{\varepsilon^2 - 4}\}. \tag{4.90}$$

This is

- a stable node when $\varepsilon < -2$,
- an inflected stable node when $\varepsilon = -2$,
- a stable focus when $-2 < \varepsilon < 0$,
- a centre when $\varepsilon = 0$,
- an unstable focus when $0 < \varepsilon < 2$,
- an inflected unstable node when $\varepsilon = 2$,
- an unstable node for $\varepsilon > 2$.

In a damped system like (4.59) there is a loss of energy due to friction, which causes an exponential approach to the equilibrium point at $x = \dot{x} = 0$. This is the case for van der Pol's equation when $\varepsilon < 0$. However, when $\varepsilon > 0$ the 'friction term' is negative for $|x| < 1$ and the origin is an unstable equilibrium point. When the system is disturbed it *self-excites* and it it only the presence of the x^2 term, leading to positive friction when $|x| > 1$, which prevents it having just an uninteresting exponential growing solution.

We anticipate that the destabilization of the equilibrium point at the origin as ε increases through zero is accompanied by the emergence of a limit cycle. Comparing (4.48) and (4.86) we have

$$f(r\cos(\theta), r\sin(\theta)) = r\sin(\theta)\{r^2\cos^2(\theta) - 1\}. \tag{4.91}$$

Substituting into (4.57)–(4.58) gives

$$A^{(1)}(r) = \tfrac{1}{2}r - \tfrac{1}{8}r^3, \qquad B^{(1)}(r) = 0. \tag{4.92}$$

Substituting into (4.54)–(4.55) with the initial conditions $r(0) = r_0$, $\theta(0) = 0$ gives

$$r(t) = \frac{2r_0\exp(\varepsilon t/2)}{\sqrt{4 + r_0^2\{\exp(\varepsilon t) - 1\}}}, \tag{4.93}$$

$$\theta(t) = -t. \tag{4.94}$$

From (4.56),

$$\frac{\partial^2 u^{(1)}(r,\theta)}{\partial\theta^2} + u^{(1)}(r,\theta) + \tfrac{1}{4}r^3\sin(3\theta) = 0, \tag{4.95}$$

Which has the solution

$$u^{(1)}(r,\theta) = \tfrac{1}{32}r^3\sin(3\theta) \tag{4.96}$$

and

$$x(\varepsilon, t) = r\cos(t) - \tfrac{1}{32}\varepsilon r^3\sin(3t). \tag{4.97}$$

It follows from (4.93) that $r(t) \to 2$ as $t \to \infty$. The stable limit cycle, to $O(\varepsilon)$ is $r = 2$ and the period is 2π. Approach to the limit cycle is from inside if $r_0 < 2$ and from outside if $r_0 > 2$. If the KBM averaging approximation were performed to second-order, the period would acquire an ε dependence and the limit cycle would loose its circularity.

In this model we have an example of a change of stability of an equilibrium point and the emergence of a limit cycle as a parameter passes through a special value. However, this differs from the Hopf bifurcation where the limit cycle grows from nothing. Here the limit cycle springs into existence fully-formed with a radius of the order of two.

4.7 Forced Oscillations

The generic type of equation for a system undergoing *free oscillations* is Liénard's equation (4.72). In this section we consider cases of the non-autonomous modification,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + f(x)\frac{\mathrm{d}x}{\mathrm{d}t} + g(x) = F(t), \tag{4.98}$$

of this equation, where the $F(t) = F(t + 2\pi/\Omega)$ is a periodic *forcing term*. We can think of this as the model for a particle oscillating with possibly damping and non-linear effects, which is subject to an outside periodic force $F(t)$.

The following mathematical results will be useful in our calculations:

- *A particular integral of*

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \alpha^2 x = C\cos(\beta t), \tag{4.99}$$

where $\alpha > 0$ and $\beta > 0$, is

$$x^{(\mathrm{p})}(t) = \frac{C\{(\alpha^2 - \beta^2)\cos(\beta t) + 2\beta\gamma\sin(\beta t)\}}{(\alpha^2 - \beta^2)^2 + 4\beta^2\gamma^2}, \tag{4.100}$$

if $\alpha \neq \beta$ or $\gamma \neq 0$, and

$$x^{(\mathrm{p})}(t) = \frac{Ct\sin(\beta t)}{2\beta}, \tag{4.101}$$

if $\alpha = \beta$ and $\gamma = 0$.

- *For any positive integer n,*

$$\begin{aligned}
\cos^{2n}(\theta) &= \frac{1}{2^{2n}}\left\{\sum_{k=0}^{n-1} 2\begin{pmatrix} 2n \\ k \end{pmatrix}\cos(2[n-k]\theta) + \begin{pmatrix} 2n \\ n \end{pmatrix}\right\}, \\
\cos^{2n-1}(\theta) &= \frac{1}{2^{2n-2}}\left\{\sum_{k=0}^{n-1}\begin{pmatrix} 2n-1 \\ k \end{pmatrix}\cos([2n-2k-1]\theta)\right\},
\end{aligned} \tag{4.102}$$

$$\begin{aligned}
\sin^{2n}(\theta) &= \frac{1}{2^{2n}}\left\{\sum_{k=0}^{n-1} 2(-1)^{n-k}\begin{pmatrix} 2n \\ k \end{pmatrix}\cos(2[n-k]\theta) + \begin{pmatrix} 2n \\ n \end{pmatrix}\right\}, \\
\sin^{2n-1}(\theta) &= \frac{1}{2^{2n-2}}\left\{\sum_{k=0}^{n-1}(-1)^{n+k-1}\begin{pmatrix} 2n-1 \\ k \end{pmatrix}\cos([2n-2k-1]\theta)\right\}.
\end{aligned} \tag{4.103}$$

Example 4.7.1

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega_0^2 x = \Gamma\cos(\Omega t). \tag{4.104}$$

This is just the case of a forced simple harmonic oscillator. Taking, without loss of generality, $\omega_0 > 0$ and $\Omega > 0$, the general solution, if $\Omega \neq \omega_0$, is

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{\Gamma\cos(\Omega t)}{\omega_0^2 - \Omega^2}. \tag{4.105}$$

If Ω is not a rational multiple of ω_0 this solution is quasi-periodic. If $\Omega/p = \omega_0/q$, where p and q are coprime integers, the system is periodic with period

$2\pi p/\Omega = 2\pi q/\omega_0$. The first two terms in (4.105) correspond to the *natural oscillations* of the system and the final term is the *response* of the system to forcing. Suppose we are able to *tune* the forcing term by changing its frequency. Then as $\Omega \to \omega_0$ the amplitude of the response grows without bound. The system approaches *resonance*. Supposing that $A > 0$ then if Ω approaches ω_0 from below the response is in phase with the natural oscillations of the system but if it approaches ω_0 from above the response is out of phase by a phase factor of π. When $\Omega = \omega_0$ (4.105) is replaced by

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{\Gamma t \sin(\omega_0 t)}{2\omega_0}. \tag{4.106}$$

The response is now a secular term which grows without bound as t increase, but which is finite at any given value of t.

Example 4.7.2 *We now modify (4.104) by including the damping term of Example 4.3.1. Thus*

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\varepsilon\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = \Gamma\cos(\Omega t). \tag{4.107}$$

As we saw in Example 4.3.1 the complementary part of the solution of (4.107) is

$$x^{(\mathrm{c})}(t) = \exp(-\varepsilon t)\left\{A\cos\left(t\sqrt{\omega_0^2-\varepsilon^2}\right) + B\cos\left(t\sqrt{\omega_0^2-\varepsilon^2}\right)\right\}. \tag{4.108}$$

and the particular integral is

$$x^{(\mathrm{p})}(\Omega, t) = \frac{\Gamma\{(\omega_0^2-\Omega^2)\cos(\Omega t) + 2\varepsilon\Omega\sin(\Omega t)\}}{(\omega_0^2-\Omega^2)^2 + 4\varepsilon^2\Omega^2}, \tag{4.109}$$

with

$$x(t) = x^{(\mathrm{c})}(t) + x^{(\mathrm{p})}(\Omega, t). \tag{4.110}$$

For $\varepsilon > 0$ the complementary function is called the *transient* part of the solution as it decays with time leaving only the response to forcing given by the particular integral. This term has a *resonance peak*, with amplitude $\Gamma/(2\omega_0\varepsilon)$, when Ω is tuned to the natural frequency[1] of ω_0. We now consider the application of expansion methods, with expansions in terms of a small parameter ε for equations with a forcing term. We distinguish between two cases: *hard forcing* where the forcing term does does not involve ε and *soft or weak forcing* where the forcing term is $\mathrm{O}(\varepsilon)$.

[1] Strictly speaking the parameter for which we usually use the symbol ω or Ω is the *angular frequency* with the actual frequency for an oscillation of period T being $1/T = \omega/2\pi$. We shall, however, when there is no risk of confusion simply use 'frequency' to denote quantities like ω.

4.7.1 The Duffing Equation with a Hard Forcing Term

We use the Lindstedt-Poincaré method to investigate the Duffing equation, with a hard forcing term, no damping and a natural unperturbed frequency ω_0.Thus

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega_0^2\left\{x + \varepsilon c x^3\right\} = \Gamma\cos(\Omega t), \tag{4.111}$$

and with $y(t) = \dot{x}(t)$ this equation becomes

$$\dot{x}(t) = y, \qquad \dot{y}(t) = -\omega_0^2\{x + \varepsilon c x^3\} + \Gamma\cos(\Omega t). \tag{4.112}$$

Apart from the presence of the forcing term these formulae are the special case $f(\varepsilon, x) = \varepsilon c x^3$ of (4.5) and we proceed with the method in the same way. We look for a periodic solution of period $2\pi/\omega(\varepsilon)$. Let

$$\begin{aligned} &\tau = \Omega\omega(\varepsilon)t/\omega_0, \quad \omega(\varepsilon) = \omega_0 g(\varepsilon), \quad \tilde{y} = y\omega_0/\{\Omega\omega(\varepsilon)\}, \\ &\alpha = \omega_0/\Omega, \qquad \tilde{\Gamma} = \Gamma/\Omega^2. \end{aligned} \tag{4.113}$$

Then (4.112) become

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \tilde{y}, \qquad \{g(\varepsilon)\}^2\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tau} = -\alpha^2\{x + \varepsilon c x^3\} + \tilde{\Gamma}\cos(\tau/g(\varepsilon)). \tag{4.114}$$

Let

$$x(\varepsilon,\tau) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \mathrm{O}(\varepsilon^3), \tag{4.115}$$

$$\tilde{y}(\varepsilon,\tau) = \tilde{y}_0(\tau) + \varepsilon\tilde{y}_1(\tau) + \varepsilon^2\tilde{y}_2(\tau) + \mathrm{O}(\varepsilon^3), \tag{4.116}$$

$$g(\varepsilon) = 1 + \varepsilon g_1 + \varepsilon^2 g_2 + \mathrm{O}(\varepsilon^3) \tag{4.117}$$

and substituting into (4.114) the terms of $\mathrm{O}(\varepsilon^0)$ give

$$\frac{\mathrm{d}^2x_0}{\mathrm{d}\tau^2} + \alpha^2 x_0 = \tilde{\Gamma}\cos(\tau). \tag{4.118}$$

As in Sect. 4.1 we impose the condition $\mathrm{d}x/\mathrm{d}t = 0$ at $t = 0$. This condition applies separately to each of the terms in the expansion (4.115) and (4.118) has the solution

$$x_0(\tau) = \begin{cases} a_0\cos(\alpha\tau) + \dfrac{\tilde{\Gamma}\cos(\tau)}{\alpha^2 - 1}, & \alpha \neq 1, \\ a_0\cos(\tau) + \dfrac{\tilde{\Gamma}\tau\sin(\tau)}{2}, & \alpha = 1. \end{cases} \tag{4.119}$$

The terms of $\mathrm{O}(\varepsilon^1)$ give

$$\frac{\mathrm{d}x_1}{\mathrm{d}\tau} = \tilde{y}_1, \qquad \frac{\mathrm{d}\tilde{y}_1}{\mathrm{d}\tau} + 2g_1\frac{\mathrm{d}\tilde{y}_0}{\mathrm{d}\tau} = -\alpha^2(x_1 + c x_0^3) + \tilde{\Gamma}g_1\tau\sin(\tau). \tag{4.120}$$

From (4.118)–(4.120), $x_1(\tau)$ satisfies the equation

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2}+\alpha^2x_1 = g_1\left\{2\alpha^2a_0\cos(\alpha\tau)+\frac{2\tilde{\Gamma}\cos(\tau)}{\alpha^2-1}+\tilde{\Gamma}\sin(\tau)\right\} - \alpha^2c\left\{a_0\cos(\alpha\tau)+\frac{\tilde{\Gamma}\cos(\tau)}{\alpha^2-1}\right\}^3, \qquad \text{if } \alpha\neq 1, \tag{4.121}$$

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2}+x_1 = g_1\{2a_0\cos(\tau)+\tilde{\Gamma}\tau\sin(\tau)\}-c\left\{a_0\cos(\tau)+\frac{\tilde{\Gamma}\tau\sin(\tau)}{2}\right\}^3, \qquad \text{if } \alpha=1. \tag{4.122}$$

We see that at each stage of the expansion process the complementary function obtained at the previous stage will generate new secular terms (of the form $\tau\cos(\alpha\tau)$) unless either the constant (in this case a_0) is set to zero or the coefficients $g_1, g_2, \ldots$ in the expansion of the angular frequency are set to values which eliminate these terms. From (4.102) $\cos^3(\alpha\tau)=\{3\cos(\alpha\tau)+\cos(3\alpha\tau)\}/4$. So the coefficient of $\cos(\alpha\tau)$ on the right-hand side of (4.121) is $2\alpha^2a_0g_1-3\alpha^2ca_0^3/4$. For this to be zero we must have either $a_0=0$ or

$$g_1=\tfrac{3}{8}ca_0^2, \tag{4.123}$$

This is condition (4.20) of Sect. 4.1. In the solution of (4.122) the secular terms generated by factors with $\cos(\tau)$ on the right-hand side are also eliminated by the condition (4.123). Rather than the strategy indicated by (4.123) we shall, for simplicity set the constants $a_0=a_1=\cdots=0$ in the solution. This simply means that the system starts from rest with $x(0)=0$ and is driven by the forcing term from which it acquires the same frequency. This is known as a *synchronous oscillation*. For this situation we do not need perturbations to the angular frequency and $g_1=g_2=\cdots=0$. Then (4.119) becomes

$$x_0(\tau)=\begin{cases}\dfrac{\tilde{\Gamma}\cos(\tau)}{\alpha^2-1}, & \alpha\neq 1,\\[2ex] \dfrac{\tilde{\Gamma}\tau\sin(\tau)}{2}, & \alpha=1,\end{cases} \tag{4.124}$$

and $x_1(t)$ is the solution of

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2}+\alpha^2x_1=\begin{cases}-\dfrac{\alpha^2c\tilde{\Gamma}^3\{3\cos(\tau)+\cos(3\tau)\}}{4(\alpha^2-1)^3}, & \text{if } \alpha\neq 1,\\[2ex] -\frac{1}{32}c\tilde{\Gamma}^3\tau^3\{3\sin(\tau)-\sin(3\tau)\}, & \text{if } \alpha=1.\end{cases} \tag{4.125}$$

From (4.99)–(4.101) we see that in solving (4.125) we must now distinguish two special cases $\alpha=1$, as before, but also $\alpha=3$. Each of these will yield a

resonance contribution in the form of a secular term which becomes large for large τ. In fact if we pursue this method to higher orders in ε a resonance term will arise if $\alpha = 3^n$, $(\omega_0 = 3^n\Omega)$, for some $n = 0, 1, 2, \ldots$. A resonance of the form $\omega_0 = p\Omega$, for $p = 2, 3, \ldots$, is called *ultraharmonic*. That $p = 3^n$ in this case is obviously due to the cubic perturbation. Reverting to the original notation and collecting terms up to $\mathrm{O}(\varepsilon)$, when $\omega_0 \neq 3^n\Omega$,

$$x(\varepsilon, t) = \frac{\Gamma\cos(\Omega t)}{\omega_0^2 - \Omega^2} - \frac{\varepsilon c\Gamma^3\omega_0^2}{4(\omega_0^2 - \Omega^2)^3}\left\{\frac{3\cos(\Omega t)}{\omega_0^2 - \Omega^2} + \frac{\cos(3\Omega t)}{\omega_0^2 - 9\Omega^2}\right\}. \tag{4.126}$$

4.7.2 The Duffing Equation with a Soft Forcing Term

We use the Lindstedt-Poincaré method to investigate the Duffing equation, with a soft forcing term, no damping and a natural unperturbed frequency ω_0. Thus

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega_0^2\left\{x + \varepsilon cx^3\right\} = \varepsilon\Gamma\cos(\Omega t), \tag{4.127}$$

and with $y(t) = \dot{x}(t)$ this equation becomes

$$\dot{x}(t) = y, \qquad \dot{y}(t) = -\omega_0^2\{x + \varepsilon cx^3\} + \varepsilon\Gamma\cos(\Omega t). \tag{4.128}$$

Using the notation defined in (4.113),

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \tilde{y}, \qquad \{g(\varepsilon)\}^2\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tau} = -\alpha^2\{x + \varepsilon cx^3\} + \varepsilon\tilde{\Gamma}\cos(\tau/g(\varepsilon)). \tag{4.129}$$

With the expansions given in (4.115)–(4.117) the terms of $\mathrm{O}(\varepsilon^0)$ in (4.129) give

$$\frac{\mathrm{d}^2x_0}{\mathrm{d}\tau^2} + \alpha^2x_0 = 0. \tag{4.130}$$

Again we impose the condition $\dot{x}(0) = 0$ and (4.130) has the solution

$$x_0(\tau) = a_0\cos(\alpha\tau). \tag{4.131}$$

The terms of $\mathrm{O}(\varepsilon^1)$ give

$$\frac{\mathrm{d}x_1}{\mathrm{d}\tau} = \tilde{y}_1, \qquad \frac{\mathrm{d}\tilde{y}_1}{\mathrm{d}\tau} + 2g_1\frac{\mathrm{d}\tilde{y}_0}{\mathrm{d}\tau} = -\alpha^2(x_1 + cx_0^3) + \tilde{\Gamma}\cos(\tau). \tag{4.132}$$

From (4.130)–(4.132), $x_1(\tau)$ satisfies the equation

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2} + \alpha^2x_1 \quad = \quad \tilde{\Gamma}\cos(\tau) + \alpha^2a_0\cos(\alpha\tau)\left\{2g_1 - \tfrac{3}{4}ca_0^2\right\} - \tfrac{1}{4}\alpha^2ca_0^3\cos(3\alpha\tau). \tag{4.133}$$

The term in $\cos(\alpha\tau)$ on the right-hand side of (4.133) will lead to secular contributions to the solution.

If $\alpha \neq 1$ this term can be eliminated by taking

$$g_1 = \tfrac{3}{8} c a_0^2, \tag{4.134}$$

when (4.133) becomes

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}\tau^2} + \alpha^2 x_1 = \tilde{\Gamma}\cos(\tau) - \tfrac{1}{4}\alpha^2 c a_0^3 \cos(3\alpha\tau), \tag{4.135}$$

with the solution

$$x_1(\tau) = a_1 \cos(\alpha\tau) + \frac{\tilde{\Gamma}\cos(\tau)}{\alpha^2 - 1} + \tfrac{1}{8} c a_0^3 \cos(3\alpha\tau). \tag{4.136}$$

If $\alpha = 1$ then the forcing term has the same frequency as the natural frequency of the system. Terms in $\cos(\tau)$ can be eliminated by taking

$$\Gamma = \tfrac{1}{4}\omega_0^2 a_0 \{3 c a_0^2 - 8 g_1\}, \tag{4.137}$$

giving the solution

$$x_1(\tau) = a_1 \cos(\tau) + \tfrac{1}{32} c a_0^3 \cos(3\tau). \tag{4.138}$$

Then, translating back to the original variables,

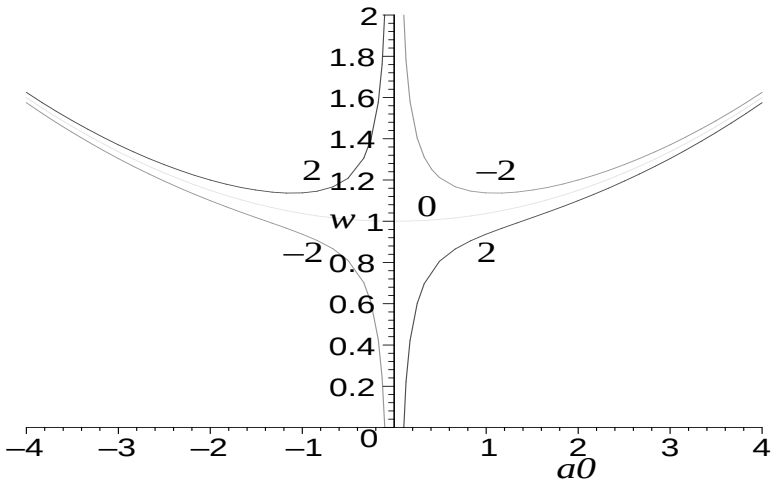
$$x(\varepsilon, t) = (a_0 + \varepsilon a_1)\cos(\omega t) + \tfrac{1}{32}\varepsilon c a_0^3 \cos(3\omega t) + \mathrm{O}(\varepsilon^2), \tag{4.139}$$

where

$$\omega(\varepsilon) = \omega_0 \left\{ 1 + \varepsilon \left(\tfrac{3}{8} c a_0^2 - \frac{\Gamma}{2\omega_0^2 a_0} \right) + \mathrm{O}(\varepsilon^2) \right\}. \tag{4.140}$$

Curves of $\omega(\varepsilon)/\omega_0$ (denoted by w in the plot) against a_0 at fixed Γ, c and ε can be obtained using MAPLE. Here we take $c = 1$ and $\varepsilon = 0.1$ and the curves are labelled with their value of Γ/ω_0^2.

```
> with(plots):
> w:=(epsilon,a0,g,c)->1+epsilon*(3*c*a0^2/8-g/(2*a0)):
> text:=plots[textplot](
> {[-1.0,1.2,‘2‘],
> [0.9,1.2,‘-2‘],[0.9,0.8,‘2‘],[-1.2,0.8,‘-2‘],
> [0.25,1.025,‘0‘]},align={ABOVE,RIGHT},font=[TIMES,ROMAN,12]):
> curve:=plot(
> {w(0.1,a0,2,1),w(0.1,a0,0,1),w(0.1,a0,-2,1)},
> a0=-4..4,w=0..2,labelfont=[TIMES,ITALIC,12]):
> plots[display]({curve,text});
```



For $\Gamma = 0$ the branches $a_0 = 0$ and $a_0 = \sqrt{8(\omega - \omega_0)/(3\varepsilon c\omega_0)}$ form a pitchfork bifurcation. When $\Gamma \neq 0$ the curve breaks into two branches, one giving $a_0 > 0$ and one $a_0 < 0$ (cf. Fig. 2.2).

4.7.3 The Van der Pol Equation with a Weak Forcing Term

We use the KBM averaging method to investigate the van der Pol equation with a weak forcing term and natural frequency ω_0. Thus

$$\ddot{x}(t) + \varepsilon(x^2 - 1)\dot{x}(t) + \omega_0^2 x(t) = \varepsilon\Gamma\cos(\Omega t). \tag{4.141}$$

With $\dot{x}(t) = \omega_0 y(t)$ this becomes

$$\dot{x}(t) = \omega_0 y, \qquad \dot{y}(t) = -\omega_0 x + \varepsilon\left\{(1 - x^2)y + \frac{\Gamma\cos(\Omega t)}{\omega_0}\right\}. \tag{4.142}$$

We now make the same assumptions (4.52)–(4.54) as we did for the autonomous case and replace (4.55) and (4.56) by

$$\dot{\theta}(t) = -\omega_0 + \varepsilon B^{(1)}(r) + \varepsilon^2 B^{(2)}(r) + \cdots, \tag{4.143}$$

$$\omega_0\left\{\frac{\partial^2 u^{(1)}(r,\theta)}{\partial\theta^2} + u^{(1)}(r,\theta)\right\} + 2A^{(1)}(r)\sin(\theta) + 2rB^{(1)}(r)\cos(\theta)$$

$$+ r\sin(\theta)\{r^2\cos^2(\theta) - 1\} = \frac{\Gamma\cos(\Omega t)}{\omega_0}. \tag{4.144}$$

In the autonomous case we obtained $A^{(1)}(r)$ and $B^{(1)}(r)$ by using the orthogonality property (4.53). However, now we have a term which is explicitly dependent on t. A way of solving this in the case where the system is not close to resonance is to write

$$u^{(1)}(r,\theta) = \tilde{u}^{(1)}(r,\theta) + \frac{\Gamma\cos(\Omega t)}{\omega_0^2 - \Omega^2}. \tag{4.145}$$

Inserting this form into (4.144) replaces $u^{(1)}(r,\theta)$ by $\tilde{u}^{(1)}(r,\theta)$ and eliminates the term $\Gamma\cos(\Omega t)/\omega_0$. If we now assume that $\tilde{u}^{(1)}(r,\theta)$ satisfies the orthogonality condition (4.53) the method proceeds as in the autonomous case with the only differences being the extra term in $u^{(1)}(r,\theta)$ and the presence of ω_0. Using (4.97) the solution is now

$$x(\varepsilon,t) = r\cos(\omega_0 t) - \tfrac{1}{32}\varepsilon r^3\sin(3\omega_0 t) + \frac{\varepsilon\Gamma\cos(\Omega t)}{\omega_0^2 - \Omega^2}. \tag{4.146}$$

The more interesting and difficult case is near resonance when $\Omega \simeq \omega_0$. This solution is dominated by the forcing term and the phenomenon is called *entrainment*. To deal with this situation a different approach is needed.

We define $\phi = \Omega t + \theta$, which varies slowly with time near to resonance since, from (4.143),

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \Omega + \frac{\mathrm{d}\theta}{\mathrm{d}t} = \Omega - \omega_0 + \varepsilon B^{(1)}(r) + \mathrm{O}(\varepsilon^2) \simeq \varepsilon B^{(1)}(r). \tag{4.147}$$

Then

$$\cos(\Omega t) = \cos(\phi - \theta) = \cos(\phi)\cos(\theta) + \sin(\phi)\sin(\theta) \tag{4.148}$$

and substituting this into (4.144) and, as for the autonomous case, multiplying successively by $\sin(\theta)$ and $\cos(\theta)$ and integrating over $[0, 2\pi]$ gives

$$\begin{aligned} A^{(1)}(r) &= -\frac{1}{2\pi}\int_0^{2\pi} r\sin^2(\theta)[r^2\cos^2(\theta) - 1]\mathrm{d}\theta + \frac{\Gamma\sin(\phi)}{2\omega_0}, \\ &= \tfrac{1}{8}r(4 - r^2) + \frac{\Gamma\sin(\phi)}{2\omega_0}, \end{aligned} \tag{4.149}$$

$$\begin{aligned} B^{(1)}(r) &= -\frac{1}{2\pi}\int_0^{2\pi} \sin(\theta)\cos(\theta)[r^2\cos^2(\theta) - 1]\mathrm{d}\theta + \frac{\Gamma\cos(\phi)}{2r\omega_0}, \\ &= \frac{\Gamma\cos(\phi)}{2r\omega_0}. \end{aligned} \tag{4.150}$$

From (4.54) and (4.143)

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \tfrac{1}{8}\varepsilon\left\{\frac{4\Gamma\sin(\phi)}{\omega_0} + 4r - r^3\right\}, \tag{4.151}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \Omega + \frac{\mathrm{d}\theta}{\mathrm{d}t} = \Omega - \omega_0 + \frac{\varepsilon\Gamma\cos(\phi)}{2r\omega_0}. \tag{4.152}$$

and from (4.144)

$$\omega_0 \left\{ \frac{\partial^2 u^{(1)}(r,\theta)}{\partial \theta^2} + u^{(1)}(r,\theta) \right\} + \tfrac{1}{4} r^3 \sin(3\theta) = 0, \tag{4.153}$$

which has the solution

$$u^{(1)}(r,\theta) = \frac{r^3 \sin(3\theta)}{32\omega_0}. \tag{4.154}$$

Since $\cos(\theta) = \cos(\phi - \Omega t)$ it follows from (4.52) that

$$x(\varepsilon,t) = \xi(t)\cos(\Omega t) + \zeta(t)\sin(\Omega t) + \frac{\varepsilon r^3 \sin(3\theta)}{32\omega_0} + \mathrm{O}(\varepsilon^2), \tag{4.155}$$

where

$$\xi(t) = r\cos(\phi), \qquad \zeta(t) = r\sin(\phi), \tag{4.156}$$

are called the *van der Pol variables*. From (4.151)–(4.152)

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -\tfrac{1}{2}\varepsilon\sigma\zeta + \tfrac{1}{8}\varepsilon\xi\{4 - \xi^2 - \zeta^2\}, \tag{4.157}$$

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \tfrac{1}{2}\varepsilon\sigma\xi + \tfrac{1}{8}\varepsilon\zeta\{4 - \xi^2 - \zeta^2\} + \varepsilon\gamma. \tag{4.158}$$

where

$$\Omega - \omega_0 = \tfrac{1}{2}\varepsilon\sigma, \qquad \gamma = \frac{\Gamma}{2\omega_0}. \tag{4.159}$$

The equilibrium points in the *van der Pol plane* of the variables $\{\xi, \zeta\}$ are given by

$$\tfrac{1}{2}\zeta\sigma - \tfrac{1}{8}\xi\{4 - \xi^2 - \zeta^2\} = 0, \tag{4.160}$$

$$\tfrac{1}{2}\xi\sigma + \tfrac{1}{8}\zeta\{4 - \xi^2 - \zeta^2\} = -\gamma. \tag{4.161}$$

Squaring and adding these equations gives

$$f(\sigma,\rho) = \sigma^2\rho + \rho(1-\rho)^2 = \gamma^2, \tag{4.162}$$

where

$$\rho = \tfrac{1}{4}\{\xi^2 + \zeta^2\}. \tag{4.163}$$

Periodic trajectories in the van der Pol plane are now given by the positive roots of (4.162). Suppose that $(\mathring{\xi}, \mathring{\zeta})$ is a point on a periodic solution. That is $\mathring{\rho}(\sigma,\gamma) = \{\mathring{\xi}^2 + \mathring{\zeta}^2\}/4$ is a root of (4.162) and $(\mathring{\xi}, \mathring{\zeta})$ satisfy (4.160)–(4.161). Let $\triangle\xi = \xi - \mathring{\xi}$, $\triangle\zeta = \zeta - \mathring{\zeta}$. Substituting into (4.157)–(4.158) and linearizing

$$\frac{\mathrm{d}\triangle\xi}{\mathrm{d}t} = \tfrac{1}{4}\varepsilon\triangle\xi(2 - 2\mathring{\rho} - \mathring{\xi}^2) - \tfrac{1}{4}\varepsilon\triangle\zeta(2\sigma + \mathring{\xi}\mathring{\zeta}), \tag{4.164}$$

$$\frac{\mathrm{d}\triangle\zeta}{\mathrm{d}t} = \tfrac{1}{4}\varepsilon\triangle\xi(2\sigma - \mathring{\xi}\mathring{\zeta}) + \tfrac{1}{4}\varepsilon\triangle\zeta(2 - 2\mathring{\rho} - \mathring{\zeta}^2). \tag{4.165}$$

Then the periodic orbit stability matrix is

$$\mathring{\boldsymbol{J}}(t) = \begin{pmatrix} \frac{1}{4}\varepsilon(2 - 2\mathring{\rho} - \mathring{\xi}^2) & -\frac{1}{4}\varepsilon(2\sigma + \mathring{\xi}\mathring{\zeta}) \\ \frac{1}{4}\varepsilon(2\sigma - \mathring{\xi}\mathring{\zeta}) & \frac{1}{4}\varepsilon(2 - 2\mathring{\rho} - \mathring{\zeta}^2) \end{pmatrix}. \tag{4.166}$$

The eigenvalues of this matrix are

$$\lambda^{(\pm)} = \tfrac{1}{2}\left\{p \pm \sqrt{p^2 - 4q}\right\}, \tag{4.167}$$

where

$$p = \varepsilon(1 - 2\mathring{\rho}), \qquad q = \tfrac{1}{4}\varepsilon^2\left\{\sigma^2 + 1 - 4\mathring{\rho} + 3\mathring{\rho}^2\right\}, \\ p^2 - 4q = \varepsilon^2(\mathring{\rho}^2 - \sigma^2). \tag{4.168}$$

Since these eigenvalues determine the stability of the whole periodic solution, they are, as might be expected dependent only on σ and $\mathring{\rho}$ and not individually on $\mathring{\xi}$ and $\mathring{\zeta}$. Using (4.167)–(4.168) the $\{\sigma, \rho\}$ plane can be divided into regions corresponding to the type of the equilibrium solution Fig. 4.1. When $q < 0$ the equilibrium point is a saddle-point and the curve $q = 0$ separates the region of saddle-points from other types of equilibrium solutions. In the latter region the parts with $p < 0$ and $p > 0$ correspond respectively to stable and unstable solutions and the region is further divided between focii and nodes according as $p^2 < 4q$ and $p^2 > 4q$.

The value of $\mathring{\rho}$, for particular σ and γ is given by a solution of (4.162). The cubic function $f(\sigma, \rho)$, plotted against ρ passes through the origin and tends to infinity for large ρ. It therefore cuts the horizontal line at γ^2 either one or three times for positive ρ. The condition for three positive roots of (4.162) is that the two turning points of $f(\sigma, \rho)$ are at positive values of ρ and lie on opposites sides of the line γ^2. Now

$$\frac{\partial f}{\partial \rho} = \sigma^2 + 1 - 4\rho + 3\rho^2, \tag{4.169}$$

with roots

$$\rho^{(\pm)} = \tfrac{1}{3}\{2 \pm \sqrt{1 - 3\sigma^2}\}, \tag{4.170}$$

where

$$f\left(\sigma, \rho^{(\pm)}\right) = \frac{2}{27}\left\{1 + 9\sigma^2 \pm (3\sigma^2 - 1)\sqrt{1 - 3\sigma^2}\right\}. \tag{4.171}$$

The cubic $f(\sigma, \rho)$ will have real turning points if $3\sigma^2 < 1$ and a point of inflection if $3\sigma^2 = 1$. The former will lead to three positive roots of (4.162) if

$$\frac{2}{27}\left\{1 + 9\sigma^2 - [1 - 3\sigma^2]^{3/2}\right\} < \gamma^2 < \frac{2}{27}\left\{1 + 9\sigma^2 + [1 - 3\sigma^2]^{3/2}\right\}. \tag{4.172}$$

This band of values of γ^2 giving three periodic solutions develops as σ is reduced through $1/\sqrt{3}$ with $\gamma^2 = 8/27$ and there will be three roots on the $\sigma = 0$ axis

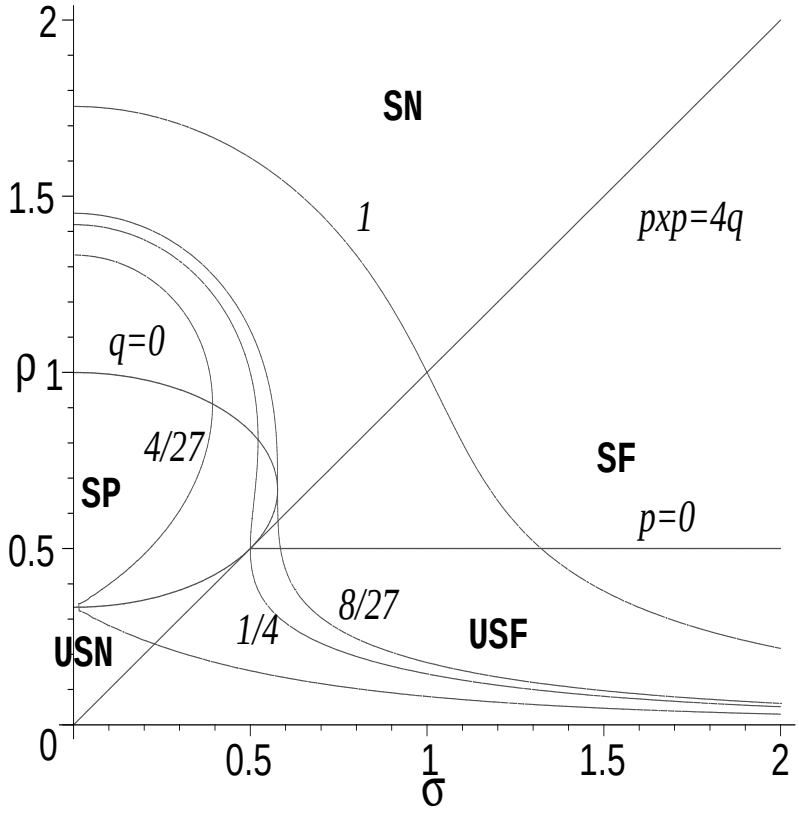


Figure 4.1: The $\{\sigma, \rho\}$ plane divided into regions by the lines $q = 0$, $p = 0$ and $p^2 = 4q$. These regions correspond to the stability types (SN) stable node, (SF) stable focus, (USF) unstable focus, (USN) unstable node and (SP) saddle point, of the periodic solutions in the forced van der Pol equation. Solution curves for (4.162), parameterized and labelled by γ^2, are shown by broken lines.

if $\gamma^2 < \frac{4}{27}$. Solution curves for $\mathring{\rho}$ plotted against σ and parameterized by γ^2 are shown by broken lines in Fig. 4.1. The unforced case is obtained by setting $\sigma = \gamma = 0$ in (4.162). This yields the non-zero solution $\mathring{\rho} = 1$, which gives $r = 2$ agreeing with the result of Sect. 4.6. Since this solution lies on the curve $q = 0$ it is an improper stable node.

4.7.4 Subharmonic Solutions

In Sect. 4.7.1 we consider the Duffing equation for a system with natural frequency ω_0 and a hard forcing term of frequency Ω. We showed that the presence of the cubic term of $\mathrm{O}(\varepsilon)$ led to an expansion in powers of ε which contained *harmonic* terms of wavelength $2\pi/(p\Omega)$ for $p = 3^n$, $n = 1, 2, \ldots$. In this section we again consider the same forms (4.111)–(4.112) for Duffing's equation but we now ask under what conditions on ω_0, c and Γ the solution may contain *subharmonic* terms with wavelengths $2\pi m/\Omega$, for some integer values of m. With $\tau = \Omega t$ and $\tilde{y} = y/\Omega$ (4.112) gives

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \tilde{y}, \qquad \Omega^2 \frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tau} = -\omega_0^2\{x + \varepsilon c x^3\} + \Gamma\cos(\tau). \tag{4.173}$$

Let

$$x(\varepsilon,\tau) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \mathrm{O}(\varepsilon^3), \tag{4.174}$$

$$\tilde{y}(\varepsilon,\tau) = \tilde{y}_0(\tau) + \varepsilon\tilde{y}_1(\tau) + \varepsilon^2\tilde{y}_2(\tau) + \mathrm{O}(\varepsilon^3), \tag{4.175}$$

$$\Omega(\varepsilon) = \Omega_0 + \varepsilon\Omega_1 + \varepsilon^2\Omega_2 + \mathrm{O}(\varepsilon^3) \tag{4.176}$$

and substituting into (4.173) the terms of $\mathrm{O}(\varepsilon^0)$ give

$$\Omega_0^2 \frac{\mathrm{d}^2 x_0}{\mathrm{d}\tau^2} + \omega_0^2 x_0 = \Gamma\cos(\tau). \tag{4.177}$$

which has the solution

$$x_0(\tau) = a_0\cos(\omega_0\tau/\Omega_0) + b_0\sin(\omega_0\tau/\Omega_0) + \frac{\Gamma\cos(\tau)}{\omega_0^2 - \Omega_0^2}. \tag{4.178}$$

In terms of the time variable t this solution will have period $2\pi m/\Omega$, for $m > 1$, if $\Omega_0 = m\omega_0$ giving

$$x_0(\tau) = a_0\cos(\tau/m) + b_0\sin(\tau/m) - \mathcal{G}(m)\cos(\tau). \tag{4.179}$$

where

$$\mathcal{G}(m) = \frac{\Gamma}{\omega_0^2(m^2-1)}. \tag{4.180}$$

The terms of $\mathrm{O}(\varepsilon^1)$ give

$$2m\omega_0\Omega_1 \frac{\mathrm{d}^2 x_0}{\mathrm{d}\tau^2} + m^2\omega_0^2 \frac{\mathrm{d}^2 x_1}{\mathrm{d}\tau^2} = -\omega_0^2\{x_1 + c x_0^3\} \tag{4.181}$$

and substituting from (4.179) gives

$$\begin{aligned}\frac{\mathrm{d}^2 x_1}{\mathrm{d}\tau^2} + \frac{x_1}{m^2} &= \frac{2\Omega_1}{m^3\omega_0}\left\{a_0\cos(\tau/m) + b_0\sin(\tau/m) - m^2\mathcal{G}(m)\cos(\tau)\right\} \\ &\quad - \frac{c}{m^2}\left\{a_0\cos(\tau/m) + b_0\sin(\tau/m) - \mathcal{G}(m)\cos(\tau)\right\}^3.\end{aligned} \tag{4.182}$$

Secular terms will occur in the solution unless the coefficients of $\cos(\tau/m)$ and $\sin(\tau/m)$ on the right of (4.182) are zero. To determine these coefficients we need to expand the final cubic term. In general this is quite complicated because we need not only to reduce all terms to a form with only a single sine or cosine, but we must take into account the fact that, for example $1 - 2/m = 1/m$, when $m = 3$. As an example we consider the particular case $m = 3$. Then the conditions for the coefficients on the right-hand side of (4.182) to be zero are

$$a_0\left\{a_0^2 + b_0^2 + \frac{\Gamma^2}{32\omega_0^4} - \frac{8\omega_0\Omega_1}{9c}\right\} = \frac{\Gamma(a_0^2 - b_0^2)}{8\omega_0^2}, \tag{4.183}$$

$$b_0\left\{a_0^2 + b_0^2 + \frac{\Gamma^2}{32\omega_0^4} - \frac{8\omega_0\Omega_1}{9c}\right\} = -\frac{\Gamma a_0 b_0}{4\omega_0^2}. \tag{4.184}$$

Equation (4.184) has one solution $b_0 = 0$ for which (4.183) gives $a_0 = 0$ or as a root of the quadratic

$$a_0^2 - \frac{\Gamma a_0}{8\omega_0^2} + \frac{\Gamma^2}{32\omega_0^4} - \frac{8\omega_0\Omega_1}{9c} = 0. \tag{4.185}$$

If $b_0 \neq 0$ then by subtracting $a_0\times$ (4.184) from $b_0\times$ (4.183) we have $b_0 = \pm\sqrt{3}a_0$. Then a_0 is a root of the quadratic

$$4a_0^2 + \frac{\Gamma a_0}{4\omega_0^2} + \frac{\Gamma^2}{32\omega_0^4} - \frac{8\omega_0\Omega_1}{9c} = 0, \tag{4.186}$$

which can be expressed in the form

$$(-2a_0)^2 - \frac{\Gamma(-2a_0)}{8\omega_0^2} + \frac{\Gamma^2}{32\omega_0^4} - \frac{8\omega_0\Omega_1}{9c} = 0. \tag{4.187}$$

So if $(\tilde{a}_0^{(\pm)}, 0)$ are the solutions obtained from (4.185) when $b_0 = 0$, the solutions obtained from (4.186) are $(-2\tilde{a}_0^{(\pm)}, \mp 2\sqrt{3}\tilde{a}_0^{(\pm)})$. In each case the nature of the solutions are the same and depend on ω_0, Ω_1, c and Γ.

Problems 4

1) Consider the equation

$$\ddot{x}(t) + x(t)[1 - \varepsilon x(t)] = 0,$$

for an asymmetric spring. Find the equilibrium points and identify their types. Sketch the bifurcation diagram in the $\{\varepsilon, x\}$ plane. Use

(a) the Lindstedt-Poincaré method,

(b) the KBM averaging method,

to find terms up to $\mathrm{O}(\varepsilon)$ in the expansion of the periodic solution $x(\varepsilon, t)$ for which $\dot{x}(\varepsilon, 0) = 0$ and $x(\varepsilon, 0) \simeq a_0 + \varepsilon a_1$.

2) Calculate the synchronous contribution to the solution of

$$\ddot{x}(t) + \omega_0^2\{x(t) - \varepsilon x^4(t)\} = \Gamma\cos(\Omega t),$$

to order $\mathrm{O}(\varepsilon^1)$, when $\omega_0 \neq \Omega, 2\Omega, 4\Omega$, indicating the significance of these special values.
(The method to use is the Lindstedt-Poincaré method, except that, as we saw in Sect. 5.4.1, if only the synchronous part is required, no expansion terms are needed for the frequency.)

3) Use the Lindstedt-Poincaré method to find to $\mathrm{O}(\varepsilon^1)$ the solution of the equation

$$\ddot{x}(t) + \omega_0^2\{x(t) - \varepsilon x^4(t)\} = \varepsilon\Gamma\cos(\Omega t),$$

when $\dot{x}(0) = 0$.

4) Consider the equation

$$\ddot{x}(t) + \omega_0^2\{x(t) + \varepsilon x^2(t)\} = \Gamma\cos(\Omega t).$$

By using the expansion $\Omega = \Omega_0 + \varepsilon\Omega_1 + \ldots$, and looking for subharmonic solutions with $\Omega_0 = 2\omega_0$, find a solution of the form

$$x(t) = A(\varepsilon) + B(\varepsilon)\cos\left(\frac{1}{2}\Omega t\right) + C(\varepsilon)\cos(\Omega t) + +D(\varepsilon)\cos\left(\frac{3}{2}\Omega t\right) + E(\varepsilon)\cos(2\Omega t),$$

evaluating the coefficients to $\mathrm{O}(\varepsilon)$.

5) Describe the assumptions involved in the application of the *Krylov-Bogoliubov-Mitropolsky averaging method* to the equation

$$\ddot{x}(t) + \varepsilon f(x, \dot{x}) + x(t) = \varepsilon\Gamma\cos(\Omega t),$$

where ε is small and positive.

Implement this procedure in the case *van der Pol's equation* where

$$f(x, \dot{x}) = (x^2 - 1)\dot{x}$$

and show that, if $x(0) = r_0 + \mathrm{O}(\varepsilon)$, $\dot{x}(0) = \mathrm{O}(\varepsilon)$, where r_0 is a constant and Ω is not close to unity, the solution to $\mathrm{O}(\varepsilon)$ is

$$x(\varepsilon, t) = r\cos(t) - \tfrac{1}{32}\varepsilon r^3\sin(3t) + \frac{\varepsilon\Gamma\cos(\Omega t)}{1 - \Omega^2},$$

where r is given by

$$\frac{r_0^2(4 - r^2)}{r^2(4 - r_0^2)} = \exp(-\varepsilon t).$$

Chapter 5

Time Series and Chaos

5.1 The Analysis of Time Series

A time series is just sequence of values $x(t_0), x(t_0 + \triangle t), x(t_0 + 2\triangle t), \ldots$, of $x(t)$, for some $\triangle t > 0$. The sequence is often the output of some experiment, or the data collected by some company or survey. As an example, Fig. 5.1 shows the records of a telephone company for the number of newly installed lines, recorded in monthly periods over nine years. As might be expected there is a gradual upward drift of the yearly average and also a roughly periodic behaviour over each yearly period. We should also expect there to be a certain random element (possibly based on global or national economic factors) in the distribution. In fact most work on time series is concerned with systems with a stochastic component. In our discussion we shall, however, be concerned entirely with deterministic systems and those for which the graph of the output data has the overall appearance of some sort of periodicity. This could be something very simple like measuring the displacement of pendulum at regular time intervals $\triangle t$. In this case we know that, if the displacement is fairly small, the data will fit the curve $A\cos\{\omega(t_0 + n\triangle t)\}$ for some A, ω and t_0. We have seen in Sect. 4.7 that if the simple harmonic oscillator has natural frequency ω_0 and is subject to a forcing term of frequency Ω then, if $\omega_0 \neq \Omega$, the solution (4.105) contains terms of frequency ω_0 and Ω. If $p\omega_0 = q\Omega$, where p and q are coprime integers the solution is periodic of period $2\pi p/\Omega = 2\pi q/\omega_0$, but if this is not the case the system will be quasi-periodic. When the system is non-linear and satisfies Duffing's equation we have seen that the response to a forcing term of the form $\Gamma\cos(\Omega t)$, whether it is hard or soft, is to generate terms in the system response which are of frequency $r\Omega$ for positive integers r, which are ultraharmonic terms. We have also seen that subharmonic terms of frequency Ω/r can also be generated by perturbing the forcing.

Suppose now that, instead of trying to find analytic properties of the solution of a non-linear equation, we applied methods of numerical integration to calculate the values of the dependent variable $x(t)$ along a trajectory subject

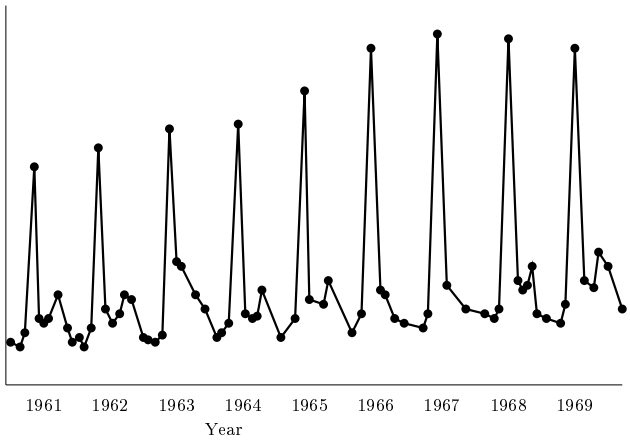


Figure 5.1: The numbers of new lines installed by the Tomasek telephone company in monthly periods from 1961 to 1969.

to certain initial conditions. The result of this process would be a sequence of values $x(t_0), x(t_0 + \triangle t), x(t_0 + 2\triangle t), \ldots$. In other words we will have obtained a time series, no different in kind from that obtained by measuring data from an experiment. The fact that we started with a particular equation would be largely irrelevant. Our task is to analyze the data, based on the general observation that it has an overall periodic-type structure.

A useful approach to analyzing time series is to use Fourier analysis. This use of Fourier methods is a little different from the problem to which such methods are usually applied. In standard applications we are given the analytic form of a function of time $f(t)$, which we know to be of period T. That is $f(t+T) = f(t)$, for all t. We want to resolve $f(t)$ into its harmonic components of periods T/n. That is

$$f(t) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{2\pi nt}{T}\right) + B_n \sin\left(\frac{2\pi nt}{T}\right) \right\}. \tag{5.1}$$

The unknowns in this formula are the coefficients A_0, A_n, B_n, $n = 1, 2, \ldots$. But since

$$\frac{1}{T}\int_0^T \cos\left(\frac{2\pi nt}{T}\right) \mathrm{d}t = \delta^{(\mathrm{Kr})}(n, 0), \tag{5.2}$$

$$\frac{1}{T}\int_0^T \sin\left(\frac{2\pi nt}{T}\right) \mathrm{d}t = 0, \tag{5.3}$$

$$\frac{1}{T}\int_0^T \cos\left(\frac{2\pi nt}{T}\right) \sin\left(\frac{2\pi mt}{T}\right) \mathrm{d}t = 0, \tag{5.4}$$

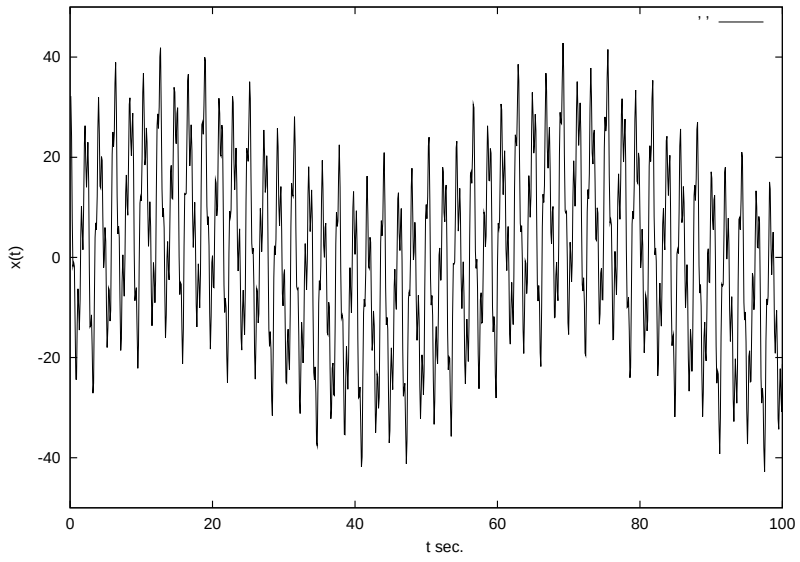


Figure 5.2: A time series plotted over 100 sec.

$$\frac{1}{T}\int_0^T \cos\left(\frac{2\pi nt}{T}\right)\cos\left(\frac{2\pi mt}{T}\right)\mathrm{d}t \;=\; \tfrac{1}{2}\delta^{(\mathrm{Kr})}(n,m), \tag{5.5}$$

$$\frac{1}{T}\int_0^T \sin\left(\frac{2\pi nt}{T}\right)\sin\left(\frac{2\pi mt}{T}\right)\mathrm{d}t \;=\; \tfrac{1}{2}\delta^{(\mathrm{Kr})}(n,m), \tag{5.6}$$

it follows that

$$\frac{1}{T}\int_0^T \cos\left(\frac{2\pi nt}{T}\right) f(t)\mathrm{d}t \;=\; \tfrac{1}{2}A_n, \qquad n=0,1,2,\ldots, \tag{5.7}$$

$$\frac{1}{T}\int_0^T \sin\left(\frac{2\pi nt}{T}\right) f(t)\mathrm{d}t \;=\; \tfrac{1}{2}B_n, \qquad n=1,2,\ldots. \tag{5.8}$$

In the case of time series analysis we have a sequence of data points rather than a functional form and, although we may have indications of periodic behaviour we have no firm knowledge of the period. Indeed the series may be quasi-periodic or chaotic. Consider, as an example, the graph in Fig. 5.2. It has a general periodic structure and seems to have a period of around 55 sec. but this may be deceptive. It may have a much longer period or possibly be quasi-periodic.

In fact, I can reveal that, in this particular case, the graph was plotted[1] from a data file obtained from calculating the values of the function

$$x(t) = 11\sin(t/9) + 20\cos(3t) + 6\cos(5t) + 8\sin(13t), \tag{5.9}$$

[1]The graphs for Figs. 5.2 and 5.3 were obtained using FORTRAN 90 programs.

at intervals of $\triangle t = \frac{1}{10}$ sec. The periods of the successive terms in this expression are $T_0 = 18\pi$, $T_1 = 2\pi/3$, $T_2 = 2\pi/5$ and $T_3 = 2\pi/13$. Since $T_0 = 27T_1 = 45T_2 = 117T_3$, the period of $x(t)$ is $T_0 \simeq 56.549$ sec., quite close to our estimate and (5.9) can be written in the form

$$x(t) = 11\sin\left(\frac{2\pi t}{T_0}\right) + 20\cos\left(\frac{2\pi 27t}{T_0}\right) + 6\cos\left(\frac{2\pi 45t}{T_0}\right) + 8\sin\left(\frac{2\pi 117t}{T_0}\right). \quad (5.10)$$

Thus if we *know* the wavelength of the time series we can use the Fourier method of (5.1), (5.7)–(5.8) to extract the coefficients of the harmonic contributions. In this case the only non-zero coefficients are $B_1 = 11$, $B_{117} = 8$, $A_{27} = 20$ and $A_{45} = 6$. Of course, in practice, we will not have the functional form (otherwise we'd know the answer before we started), but only a data set. The integration will be numerical with a certain amount of error. This question is discussed in more detail below. Of course, we could still attempt to use this approach if we had an approximate estimate of the period. In this case, however, we would find it difficult to detect contributions which were not close to harmonic components of the approximate period.

Instead of attempting to use methods based on an assumed period, we now outline a procedure which relies on data being collected over a long period of time. Consider the transformed function

$$\gamma(\tau;\omega) = \frac{1}{\tau}\int_0^\tau \exp\{\mathrm{i}\,\omega t\}x(t)\mathrm{d}t. \quad (5.11)$$

Now

$$\frac{1}{\tau}\int_0^\tau \cos(\omega_1 t)\sin(\omega_2 t)\mathrm{d}t = \begin{cases} \left[\frac{\cos\{(\omega_1-\omega_2)\tau\}}{2\tau(\omega_1-\omega_2)} - \frac{\omega_2}{\tau(\omega_1^2-\omega_2^2)}\right] & \\ -\frac{\cos\{(\omega_1+\omega_2)\tau\}}{2\tau(\omega_1+\omega_2)}, & \omega_1 \neq \omega_2, \\ \frac{1}{2}\left\{\frac{1}{\tau} - \frac{\cos\{2\omega_1\tau\}}{2\tau\omega_1}\right\}, & \omega_1 = \omega_2, \end{cases} \quad (5.12)$$

$$\frac{1}{\tau}\int_0^\tau \cos(\omega_1 t)\cos(\omega_2 t)\mathrm{d}t = \begin{cases} \frac{\sin\{(\omega_1-\omega_2)\tau\}}{2\tau(\omega_1-\omega_2)} + \frac{\sin\{(\omega_1+\omega_2)\tau\}}{2\tau(\omega_1+\omega_2)}, & \omega_1 \neq \omega_2, \\ \frac{1}{2}\left\{1 + \frac{\sin\{2\omega_1\tau\}}{2\tau\omega_1}\right\}, & \omega_1 = \omega_2, \end{cases} \quad (5.13)$$

$$\frac{1}{\tau}\int_0^\tau \sin(\omega_1 t)\sin(\omega_2 t)\mathrm{d}t = \begin{cases} \frac{\sin\{(\omega_1-\omega_2)\tau\}}{2\tau(\omega_1-\omega_2)} - \frac{\sin\{(\omega_1+\omega_2)\tau\}}{2\tau(\omega_1+\omega_2)}, & \omega_1 \neq \omega_2, \\ \frac{1}{2}\left\{1 - \frac{\sin\{2\omega_1\tau\}}{2\tau\omega_1}\right\}, & \omega_1 = \omega_2 \end{cases} \quad (5.14)$$

and we suppose that τ is large. Then the integral (5.12) is $\mathrm{O}(\tau^{-1})$ even when $\omega_1 = \omega_2$. However, (5.13) and (5.14) both have an $\mathrm{O}(\tau^0)$ term of $\frac{1}{2}$ when $\omega_1 = \omega_2$. Since

$$\frac{\sin\{(\omega_1-\omega_2)\tau\}}{2\tau(\omega_1-\omega_2)} \simeq \frac{1}{2} - \frac{1}{6}(\omega_1-\omega_2)^2\tau^2, \qquad \text{when } \omega_1 \sim \omega_2, \quad (5.15)$$

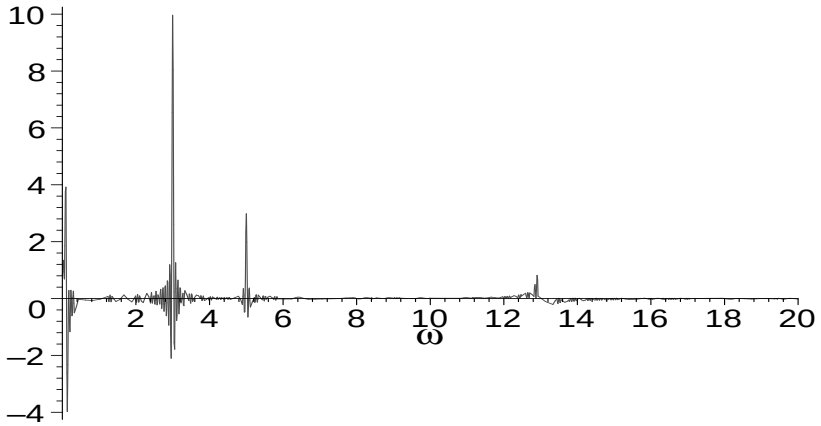
there will be a 'spread', with width $\sim 1/\tau$, around the maximum of $\frac{1}{2}$ at $\omega_1 = \omega_2$.

With this information we can consider the function $\gamma(\tau;\omega)$ computed using $x(t)$ of (5.9). As long as τ is sufficiently large we expect both the real and imaginary parts of $\gamma(\tau;\omega)$ to be almost zero everywhere except near to peaks of height 10 at $\omega = 3$ and 3 at $\omega = 5$ in the real part, and near to peaks of height 5.5 at $\omega = \frac{1}{9}$ and 4 at $\omega = 13$ in the imaginary part. Results computed directly from the functional form with $\tau = 100$ can be obtained using MAPLE. The code for computing real and imaginary parts is:

```
> v1:=t->11*sin(t/9):

> w1:=(tau,omega)->int(v1(t)*cos(omega*t)/tau,t=0..tau):

> u1:=(tau,omega)->int(v1(t)*sin(omega*t)/tau,t=0..tau):

> v2:=t->20*cos(3*t):

> w2:=(tau,omega)->int(v2(t)*cos(omega*t)/tau,t=0..tau):

> u2:=(tau,omega)->int(v2(t)*sin(omega*t)/tau,t=0..tau):

> v3:=t->6*cos(5*t):

> w3:=(tau,omega)->int(v3(t)*cos(omega*t)/tau,t=0..tau):

> u3:=(tau,omega)->int(v3(t)*sin(omega*t)/tau,t=0..tau):

> v4:=t->8*sin(13*t):

> w4:=(tau,omega)->int(v4(t)*cos(omega*t)/tau,t=0..tau):

> u4:=(tau,omega)->int(v4(t)*sin(omega*t)/tau,t=0..tau):

> ww:=(tau,omega)->w1(tau,omega)+w2(tau,omega)+w3(tau,omega)+w4(tau,omega):

> uu:=(tau,omega)->u1(tau,omega)+u2(tau,omega)+u3(tau,omega)+u4(tau,omega):
```

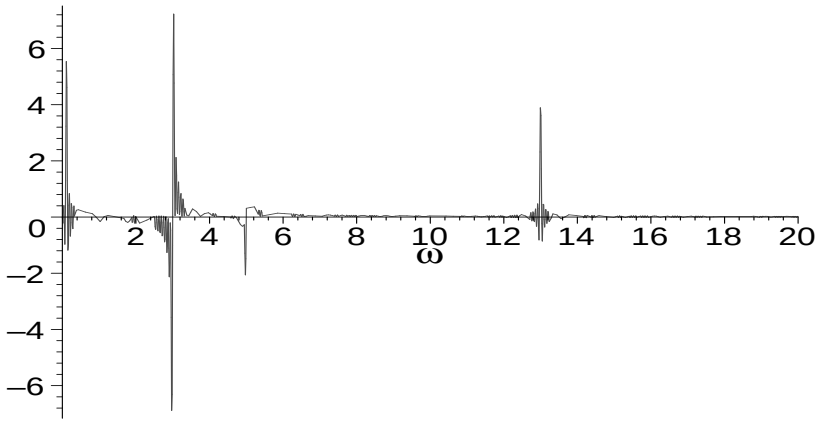
The plot for $\Re\{\gamma(100,\omega)\}$ is then given by:

```
> plot(ww(100,w),w=0..20,labelfont=[SYMBOL,12]);
```



and for $\Im\{\gamma(100,\omega)\}$ by:

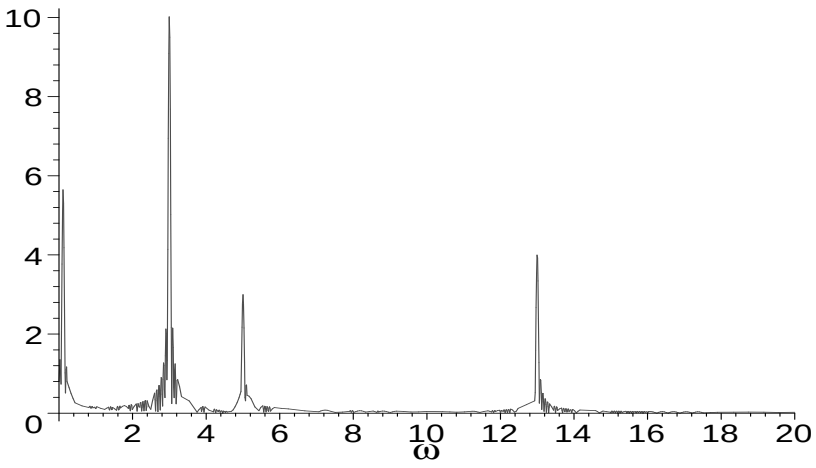
```
> plot(uu(100,w),w=0..20,labelfont=[SYMBOL,12]);
```



It will be seen that the dominant peaks in these graphs are at the points predicted. There are also weaker peaks from the sine contributions in the plot of the $\Re\{\gamma(100,\omega)\}$ and the cosine contributions in the plot of $\Im\{\gamma(100,\omega)\}$.

These arise from the first term in the integral (5.12). Since this term changes sign as ω_2 passes through the value ω_1 we observe that the function has negative and positive values in this region. A more accurate guide to the nature of the function $x(t)$ is the graph of $|\gamma(\tau,\omega)|$. This is called the *spectral function* and its peaks give the *spectrum* of $x(t)$. The spectral function $|\gamma(100,\omega)|$ can be obtained using:

```
> cc:=(tau,omega)->sqrt(ww(tau,omega)*ww(tau,omega)+uu(tau,omega)*uu(tau,omega)):

> plot(cc(100,w),w=0..20,labelfont=[SYMBOL,12]);
```



Of course, the integral form for $\gamma(\tau;\omega)$ given by (5.11) is not appropriate to the analysis of a time series since the only information is a data set[2] $x(0), x(\triangle t)$, $x(2\triangle t),\ldots,x([N-1]\triangle t)$. We need to replace t by $n\triangle t$ and τ by $(N-1)\triangle t$ in (5.11) and approximate the integral by a sum. This gives

$$\gamma(N,\triangle t;\omega)=\frac{1}{N}\sum_{n=0}^{N-1}\exp(\mathrm{i}\,\omega n\triangle t)x(n\triangle t). \tag{5.16}$$

In Fig. 5.3 $|\gamma(1000,1/10;\omega)|$ is plotted from a data file obtained from the function (5.9) rather than by integrating the functional form. Comparison with the MAPLE plot for the spectral function on page 119 and Fig. 5.3 shows that none of the essential properties of the spectrum is lost by using the time series rather than the analytic form. However, use of the formula (5.16) means

[2]Without loss of generality, the starting time t_0 can be set to zero.

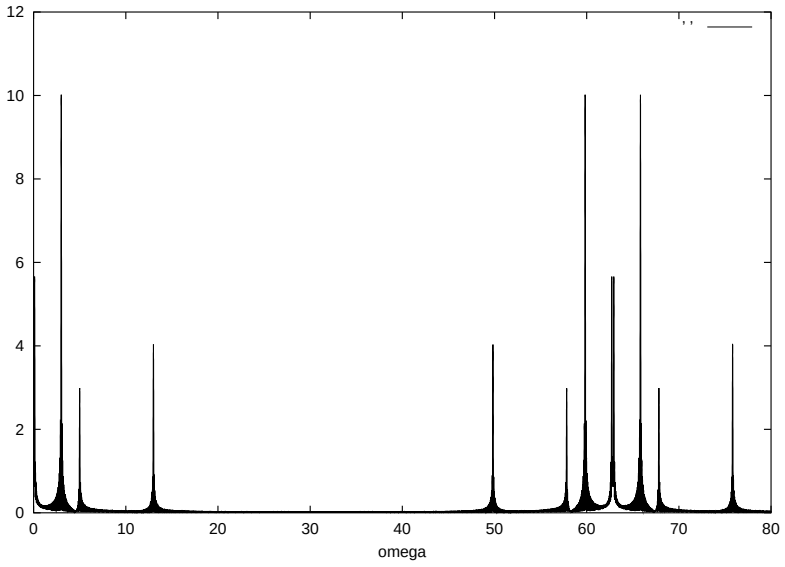


Figure 5.3: The plot of the spectral function $|\gamma(1000, 1/10, \omega)|$, computed from a data file for (5.9).

that $\gamma(N, \triangle t; \omega)$ is periodic in ω with period $2\pi/\triangle t$. The period in this case is $20\pi = 62.83$ as can be clearly seen in Fig. 5.3. If the range of ω were extended in the MAPLE plots derived from the integral formula (5.11) then such periodicity would be seen.

Further consideration of time series will be necessary in relation to the detection of chaotic behaviour in dynamic system.

5.2 Chaos in Dynamic Systems

There are three things to be considered in relation to chaos:

- We need a definition of chaos.
- We need some methods for detecting if a system, either theoretical or experimental, is behaving chaotically.
- We need some idea of what kinds of systems will have the possibility of behaving chaotically.

In fact there are very few attempts in the literature to define chaos in a mathematical sense. The clearest one I know is that given by Devaney[3] for

[3] R. L. Devaney, 1989, *Introduction to Chaotic Dynamic Systems*, Addison Wesley, 2nd Ed. p. 50.

a discrete map $\mathsf{x}(n) \to \mathsf{x}(n+1) = \mathsf{F}[\mathsf{x}(n)]$ on a space $\mathcal{V}$. According to this definition f is chaotic on $\mathcal{V}$ if:

(i) It has sensitive dependence on initial conditions.

(ii) It is topologically transitive.

(iii) Periodic points are dense in $\mathcal{V}$.

Sensitive dependence on initial conditions is just another way of describing unpredictability and this condition is the most important both for discrete and continuous systems. *Topological transitivity* simply means that for any $\mathcal{U}, \mathcal{W} \subset \mathcal{V}$ there will be, under sufficient number of iterations, images of points of $\mathcal{U}$ in $\mathcal{W}$. *Periodic points* occur only for discrete maps (see box below). However, an analogue does exist in the occurrence of subharmonic periodic solutions which give rise to periodic points on a Poincaré section.

At a meeting on Chaos sponsored by the Royal Society in London in 1986, there was[4] a certain unwillingness to come up with a definition of chaos. Eventually the definition proposed was:

Stochastic behaviour occurring in a deterministic system.

In other words the output of the system looks as if it is random in spite of the fact that the system, or equation, generating the output is entirely deterministic.

The best way to detect chaotic output from a system is to observe how the nature of the solution changes when parameters of the system are changed. For these purposes we normally suppose that we have waited a sufficiently long period of time so that transient components of the output have disappeared. This means that the trajectory has reached its attractor. We have already seen that equilibrium points and periodic solutions are attractors and in Example 1.12.2 we saw an example of a Hopf bifurcation between the two. In Example 3.3.1 we considered quasi-periodic motion on a torus and saw that the collection of such trajectories on the torus could be the attractor of a dynamic system. We, therefore, have discovered three types of attractors, equilibrium points, periodic trajectories and quasi-periodic trajectories, none of which is chaotic. What other types of attractors can exist? According to Devaney's definition the chaotic attractor of a difference equation is a region which is topologically transitive and in which periodic points are dense. Below we give a brief discussion of the *logistic map*

$$\mathsf{x}(n+1) = a\mathsf{x}(n)[1 - \mathsf{x}(n)] \tag{5.17}$$

which maps the unit interval into itself when $0 \leq a \leq 4$. As we shall explain, after a sequence of bifurcations, the behaviour becomes chaotic at $a = 3.569946$.

We are, however, in this course concerned with *differential equations* and we speculate about how complicated a differential system needs to be to exhibit

[4] According to Ian Stewart 1989, *Does God Play Dice?* Penguin.

chaotic solutions. In Example 1.8.3 we showed that the differential logistic equation

$$\dot{x}(t) = cx(b - x), \tag{5.18}$$

can be approximated to the logistic map (5.17) with $a = 1 - \varepsilon cb$, where ε is small. This means that a is close to one, and thus outside the chaotic range. This serves to suggest that it may be more difficult, or perhaps impossible, to find chaotic solutions for one-dimensional autonomous systems. That we can restrict our attention to autonomous systems follows from the discussion in Sect. 1.5, where we showed that an d-dimensional non-autonomous system can be made equivalent to a *suspended* $(d+1)$-dimensional autonomous system. A trajectory whose attractor is an equilibrium point, a periodic solution or a quasi-periodic solution is predictable and therefore not chaotic. However, according to the Poincaré-Bendixson theorem (see Sect. 3.4.1) all solutions of a two-dimensional autonomous system which for $t \geq t_0$, for some t_0, are contained in a compact set of the $\{x, y\}$ plane tend to a periodic solution or an equilibrium point. This establishes[5] that chaotic trajectories cannot exist for two-dimensional autonomous systems. This result also holds, of course, for one-dimensional autonomous and non-autonomous systems. We must, therefore, consider, two-dimensional non-autonomous systems or (at least) three-dimensional autonomous systems. The type of attractors of chaotic trajectories are *strange attractors*. Their defining characteristic is that they have a non-integer *fractal dimension*. We shall not have time for a detailed discussion of *fractals*.[6] However, it may be useful to include the definition of fractal dimension (see box).

In fact it is 'almost possible' to define chaos as motion to a strange attractor, except that there is some indication that a strange attractor can sometimes be associated with non-chaotic motion[7] and Hamiltonian systems, although they can be chaotic, do not have attractors.[8]

As we have seen with any time series it is often quite difficult to detect its character just by visual inspection of the graph. We need some other means of 'filtering out' the important qualities associated with different types of behaviour. We have already seen in Sect. 5.1 that a useful tool in this respect is the spectral function. As we shall see it can be used not only to determine the frequencies of periodic components but also indicate the presence of chaos. In addition to this an important test of the presence of chaos is to calculate the *Lyapunov exponents*.

5.2.1 Lyapunov Exponents

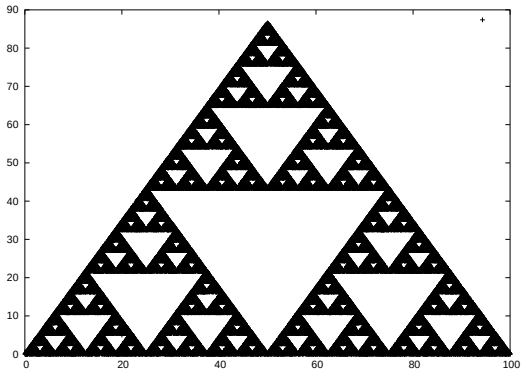
Chaos in a deterministic system implies a sensitive dependence on initial conditions. This means that if two trajectories start close together they will in most

[5] Subject to the restriction of having to consider trajectories contained in a compact set.
[6] A good introduction is that of Hans Lauwerier, 1987, *Fractals*, Penguin.
[7] See F.C. Moon 1992, *Chaotic and Fractal Dynamics*, Wiley, for references.
[8] See E. Ott 1993, *Chaos in Dynamic Systems*, Cambridge, Chapter 7.

Suppose $\mathcal{S}$ is a set of points in d–dimensional space. Let $N(\ell)$ be the minimum number of hypercubes of edge-length ℓ needed to cover $\mathcal{S}$. Then the ***fractal dimension*** *of $\mathcal{S}$ is*

$$\mathcal{D}(\mathcal{S}) = \lim_{\ell \to 0} \frac{\ln\{N(\ell)\}}{\ln\{1/\ell\}}. \tag{5.19}$$

Try this out for a 1×1 square. The number of squares of side $1/n$ needed to cover it is n^2. So $\mathcal{D} = \ln(n^2)/\ln(n) = 2$. In this case you don't even need to take the limit to get the required result. Now consider the case of the ***Sierpinski gasket or sieve****.*



This is constructed by successive removal of the central $\frac{1}{4}$ from an equilateral triangle. In this case if the lengths of the sides of the covering squares goes down by a factor of $\frac{1}{2}$ the number of such squares goes up by a factor of 3. Thus, with $\ell = \left(\frac{1}{2}\right)^n$, $N(\ell) = 3^n$ and the fractal dimension is $\ln(3)/\ln(2) \simeq 1.5849$.
We can define a fractal *as an object with non-integer fractal dimension.*

cases move exponentially away from each other on a small time scale. Thus if $\mathfrak{d}(t_0)$ is a measure of the distance between the phase points on the trajectories at time $t = t_0$ and $\mathfrak{d}(t)$ is the distance at a small, but later, time t

$$\mathfrak{d}(t) = \mathfrak{d}(t_0) \exp[\lambda_{\mathrm{L}}(t - t_0)]. \tag{5.20}$$

If the system is a difference equation then (5.20) is replaced by

$$\mathfrak{d}(n) = \mathfrak{d}(0) \exp[\lambda_{\mathrm{L}} n]. \tag{5.21}$$

The divergence of chaotic orbits must be only a local property because if the system is bounded, as it is in the case of most physical experiments, $\mathfrak{d}(t)$ cannot go to infinity. Thus to define a measure of divergence we must average the exponential growth at a sequence of points along a trajectory. We define the

sequence $t_0, t_1, t_2, \ldots, t_N$, where $t_n = t_0 + n\triangle t$. Then

$$\frac{\mathfrak{d}(t_N)}{\mathfrak{d}(t_0)} = \frac{\mathfrak{d}(t_N)}{\mathfrak{d}(t_{N-1})}\frac{\mathfrak{d}(t_{N-1})}{\mathfrak{d}(t_{N-2})}\cdots\frac{\mathfrak{d}(t_1)}{\mathfrak{d}(t_0)} \tag{5.22}$$

and, from (5.20),

$$\lambda_{\mathrm{L}} = \frac{1}{t_N - t_0}\sum_{n=1}^{N}\ln\left\{\frac{\mathfrak{d}(t_n)}{\mathfrak{d}(t_{n-1})}\right\}. \tag{5.23}$$

Using a similar argument

$$\lambda_{\mathrm{L}} = \frac{1}{N}\sum_{n=1}^{N}\ln\left\{\frac{\mathfrak{d}(n)}{\mathfrak{d}(n-1)}\right\}, \tag{5.24}$$

for a difference equation. With the map of the form $\mathfrak{x}(n) \to \mathfrak{x}(n+1) = \mathsf{F}[\mathfrak{x}(n)]$ this becomes,

$$\lambda_{\mathrm{L}} \simeq \frac{1}{N}\sum_{n=1}^{N}\ln\left|\frac{\mathrm{d}\mathsf{F}[\mathsf{x}]}{\mathrm{d}\mathsf{x}}\right|_{\mathsf{x}=\mathfrak{x}(n)}, \qquad \text{as } N \to \infty. \tag{5.25}$$

Lyapunov exponents give a means of classifying the dilating and contracting characteristics of attractors. For a one-dimensional system the condition for chaos is $\lambda_{\mathrm{L}} > 0$, which, as we have seen, can be the case only for difference equations. In general, in a d–dimensional system, there will be d independent Lyapunov exponents, which measure dilation or contraction in the d independent directions in space and a necessary condition for chaos is that at least one Lyapunov exponent is positive. It must also be the case that at least one Lyapunov is negative, otherwise the set could not be an attractor. For $d = 3$ we have one more exponent which is along the trajectory. It is normally supposed that points *on the same trajectory* do not diverge from each other. This implies a Lyapunov exponent of zero in the direction of the trajectory and we have fixed the parity of all three Lyapunov exponents. Before discussing in detail the calculation of Lyapunov exponents for differential systems, we consider the simple case of the logistic equation.

5.2.2 The Logistic Map

Because this course is intended to be restricted to continuous systems we shall not spend time in a detailed analysis of this system but just summarize the main results. For those of you not familiar with the analysis of discrete systems the main mathematical results are listed here.

For the discrete map $\mathsf{x}(n) \to \mathsf{x}(n+1) = \mathsf{F}[\mathsf{x}(n)]$*:*

(i) A ***fixed point*** x^* *of the mapping is given by* $\mathsf{x}^* = \mathsf{F}[\mathsf{x}^*]$.

(ii) The fixed point x^* *is* ***stable*** *if* $|\mathrm{d}\mathsf{F}/\mathrm{d}\mathsf{x}|^* < 1$, ***unstable*** *if* $|\mathrm{d}\mathsf{F}/\mathrm{d}\mathsf{x}|^* > 1$ *and* ***marginal*** *if* $|\mathrm{d}\mathsf{F}/\mathrm{d}\mathsf{x}|^* = 1$.

(iii) A ***periodic point*** $\mathring{\mathsf{x}}^{(i)}$ *of period* p *is a member of a set* $\mathring{\mathsf{x}}^{(1)} \to \mathring{\mathsf{x}}^{(2)} \to \cdots \to \mathring{\mathsf{x}}^{(p)} \to \mathring{\mathsf{x}}^{(1)}$. *This set of points is called a* ***p-cycle****.* $\mathring{\mathsf{x}}^{(i)}$ *is a fixed point of the iterated mapping*

$$\mathsf{x} = \overbrace{\mathsf{F}\mathsf{F}\cdots\mathsf{F}\mathsf{F}}^{p\,\text{times}}(\mathsf{x}).$$

Using this information it is simple to show that the logistical map has the following properties:

(i) In the range $0 < a < 1$ the map has a single stable fixed point $\mathsf{x} = 0$.

(ii) A transcritical bifurcation occurs at $a = a_0 = 1$ between the fixed points $x = 0$ and

$$\mathsf{x}^* = 1 - \frac{1}{a}. \tag{5.26}$$

(Of course, for $a < 1$, $x^* < 0$.)

(iii) The fixed point x^* is stable for $a_0 < a < a_1 = 3$, when a bifurcation occurs to a two-cycle given by

$$\mathsf{x}^{(\pm)} = \frac{1 + a \pm \sqrt{(a+1)(a-3)}}{2a}. \tag{5.27}$$

(iv) The two-cycle is stable for $a_1 < a < a_2 = 1+\sqrt{6}$, when a bifurcation to a four-cycle occurs.

(v) When $a = 4$ the substitution

$$\mathsf{x} = \sin^2(\pi\theta) \tag{5.28}$$

gives (5.17) in the form

$$\theta(n+1) = \mathcal{NI}(2\theta(n)), \tag{5.29}$$

where $\mathcal{NI}$ denotes 'non-integer part'.

At this point it stops being 'simple to show' and the analysis becomes increasing difficult. However, a mixture of analysis and computing has established the following:

(vi) There is a sequence of *period-doubling* bifurcations at the points $a_3, a_4, a_5, \ldots$, where a_k is the bifurcation from the 2^{k-1}-cycle to the 2^k-cycle.

(vii) $\lim_{k\to\infty} = a_\infty = 3.569946$, and, for $a_\infty < a \leq 4$, the system is chaotic.

(viii) It was shown by Feigenbaum that, with

$$\delta_k = \frac{a_k - a_{k-1}}{a_{k+1} - a_k}, \tag{5.30}$$

$$\lim_{k\to\infty} \delta_k = \delta = 4.6692016. \tag{5.31}$$

The remarkable fact is that the *Feigenbaum number* δ occurs in a wide class of mappings exhibiting period-doubling and not just the logistic map.

(ix) Before a reaches 4, cycles of *all* orders occur. It was shown by Sharkovskii, that if all the positive integers are ordered like

$$\begin{array}{cccccc}
3 \to & 5 \to & 7 \to & 9 \to & 11 \to & \cdots \to \\
6 \to & 10 \to & 14 \to & 18 \to & 22 \to & \cdots \to \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2^n 3 \to & 2^n 5 \to & 2^n 7 \to & 2^n 9 \to & 2^n 11 \to & \cdots \to \\
2^n \to & 2^{n-1} \to & \cdots \to & 4 \to & 2 \to & 1,
\end{array}$$

then the cycles occur in the reverse order. The first odd cycle (of very long period) occurs at $a = 3.6786$ and the three-cycle, which is last, occurs at $a = 3.8284$.

The bifurcation diagram of the logistic equation is shown in Fig. 5.4. The two, four and eight cycles are clearly visible, as is also the 'window' showing the occurrence of the three cycle.

We can determine the onset of chaos by calculating the Lyapunov exponent, which from (5.17) and (5.25) is given, for large N, by

$$\lambda_{\rm L} \simeq \frac{1}{N} \sum_{n=1}^{N} \ln |a[1 - 2\mathrm{x}(n)]|. \tag{5.32}$$

A plot of $\lambda_{\rm L}$ with $N = 1000$ is shown in Fig. 5.5.

Bifurcation points correspond to marginal stability with $\lambda_{\rm L} = 0$ and the first point where the exponent rises to touch the value zero is at the bifurcation point $a = a_2$, when the two-cycle becomes unstable. (More structure with a

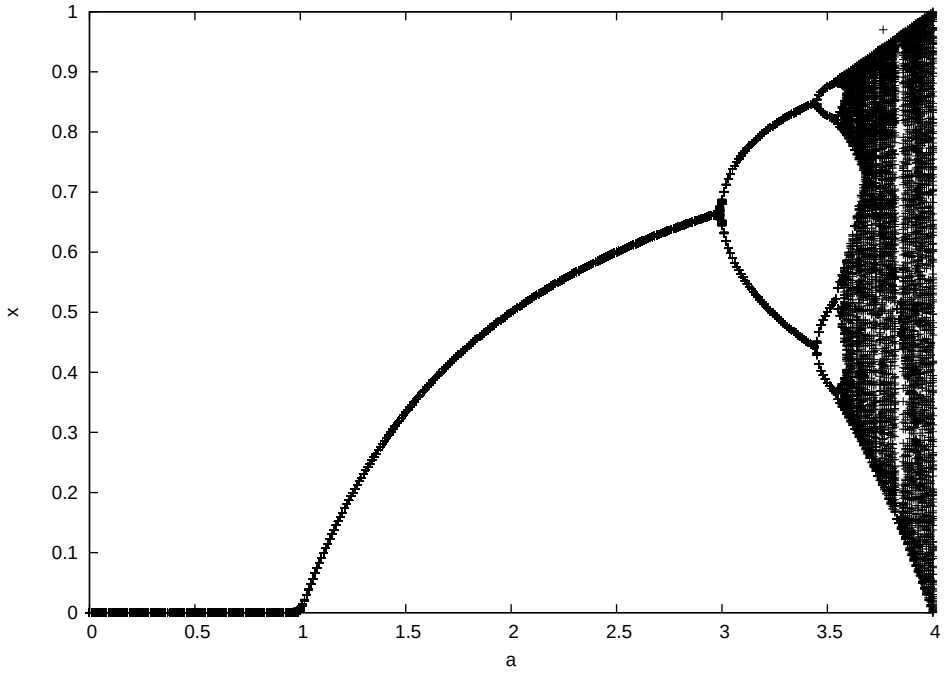


Figure 5.4: The bifurcation diagram for the logistic map (5.17).

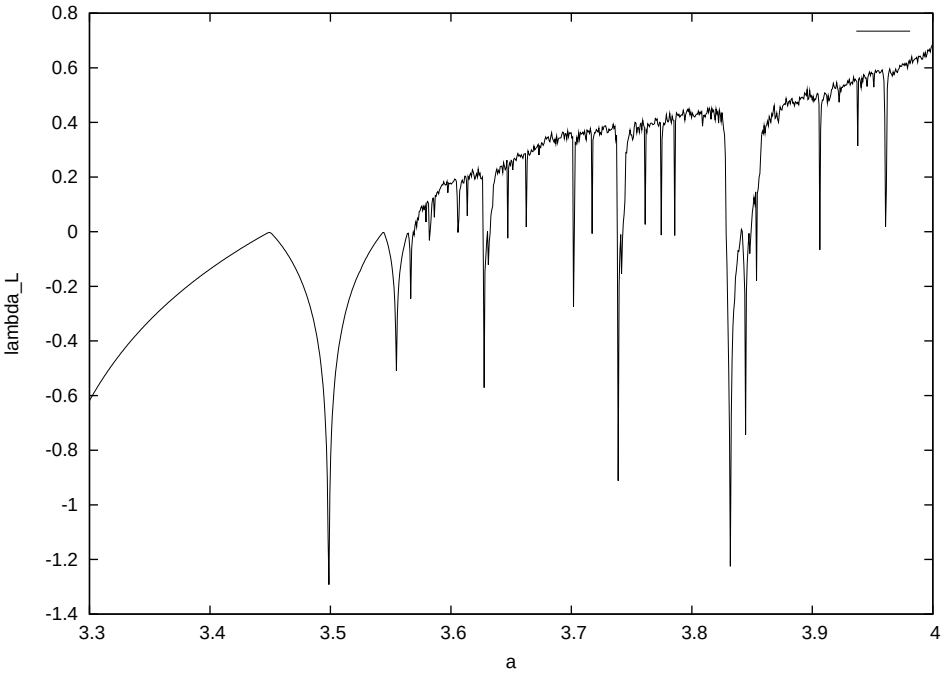


Figure 5.5: The Lyapunov exponent for the logistic equation.

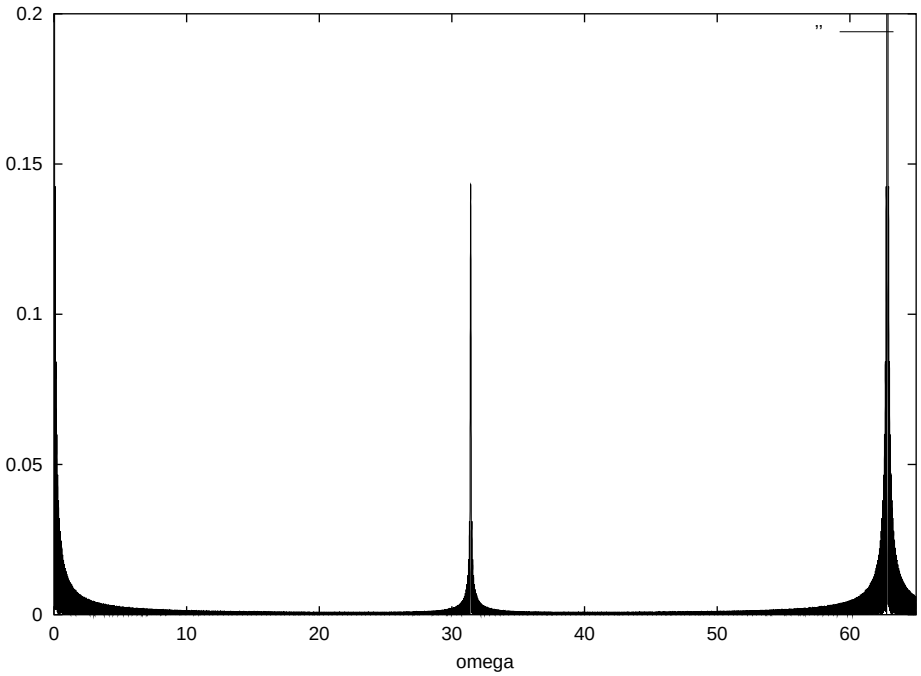


Figure 5.6: The spectral function for the logistic equation with $a = 3.2$.

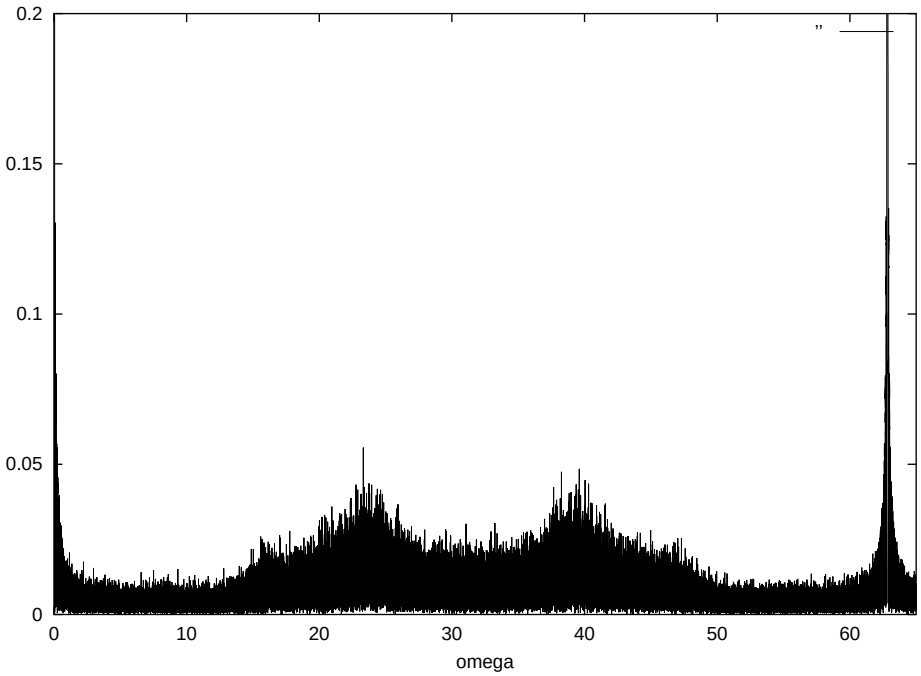


Figure 5.7: The spectral function for the logistic equation $a = 3.9$.

clearer indication of subsequent bifurcation points would have been achieved by using a larger value of N.) The point where the curve first crosses the line $\lambda_{\rm L} = 0$ corresponds to the onset of chaos at $a = a_\infty$. Subsequent dips in value correspond to the occurrence of a new sequence of cycles with the strong dip in the interval $(3.8, 3.9)$ indicating the presence of the three-cycle. An alternative test for the presence of chaos can be made by using the spectral function $|\gamma(N, \triangle t; \omega)|$, which is shown for $a = 3.2$ and $a = 3.9$ in Fig. 5.6 and Fig. 5.7.[9] In each case the sharp maxima correspond to the presence of cycles. The value $a = 3.2$ is in the two-cycle region and the spectral function is close to zero apart from at the cycle frequencies. The value $a = 3.9$ is deep within the chaotic region and the form of the function indicates cycles of all orders.

5.2.3 The Rössler Equations

Consider first the equations

$$\dot{x}(t) = -y - z, \tag{5.33}$$

$$\dot{y}(t) = x + ay. \tag{5.34}$$

For any fixed z, they have the single equilibrium point $x = -az$, $y = -z$ with stability matrix

$$\boldsymbol{J}^* = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}, \tag{5.35}$$

with eigenvalues $\lambda^{(\pm)} = \frac{1}{2}\{a \pm \sqrt{a^2 - 4}\}$. We shall confine out attention to the case $0 < a < 2$, when the equilibrium point is an unstable focus. Now we introduce a third equation

$$\dot{z}(t) = b - zc, \tag{5.36}$$

with $c > b > 0$. In the three-dimensional space of $\{x, y, z\}$ the equilibrium point is now at $x = -ab/c$, $y = -b/c$, $z = b/c$ with stability matrix

$$\boldsymbol{J}^* = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{pmatrix}, \tag{5.37}$$

Two of the eigenvalues are the same as those of the previous case and the third is $\lambda^{(3)} = -c$. So the equilibrium point is attractive in the z–direction. The general solution to (5.36) is

$$z = \mathrm{C}\exp(-ct) + b/c. \tag{5.38}$$

[9] Again we use $\triangle t = 0.1$ giving a period in ω of 20π.

Trajectories converge towards the plane $z = b/c$, while at the same time spiralling outwards in the x and y directions. So this is not a particularly interesting system. Suppose that we now modify (5.36) by adding a non-linear term to give

$$\dot{z}(t) = b + z(x - c). \tag{5.39}$$

Equations (5.33), (5.34) and (5.39) define the *Rössler equations*. This system has two equilibrium points

$$x^{(\pm)} = \tfrac{1}{2}\left\{c \pm \sqrt{c^2 - 4ab}\right\}, \qquad y^{(\pm)} = -x^{(\pm)}/a, \qquad z^{(\pm)} = x^{(\pm)}/a. \tag{5.40}$$

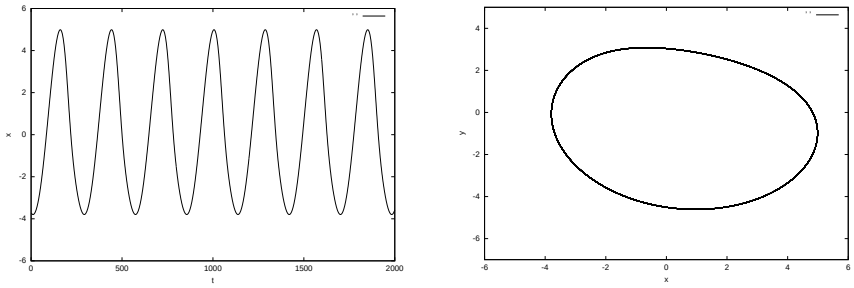
For different values of a, b and c one member of this pair has one real positive eigenvalue and a complex pair with negative real part, and the other has one real negative eigenvalue and a complex pair with positive real part.

Consider (5.39) alone. When the value of x is less than c, z remains stable and this subsystem tends to drive z to a value near to $b/(c-x)$. However, with small b, this quantity is small and (5.33)–(5.34), cause the values of x and y to spiral outwards. The growth in x causes the sign of the $z(x-c)$ term in (5.39) to change. The trajectory leaps upwards. Once z is large the $-z$ term in (5.33) comes into play and forces the value of x downwards again. The whole process then repeats itself. The overall effect of the non-linear term is to confine the attractor to a region around the origin. It is interesting to compute trajectories for this system. To do so it is necessary to use the corresponding difference equations. Take $\mathsf{x}(n) = x(n\triangle t)$, $\mathsf{y}(n) = y(n\triangle t)$ and $\mathsf{z}(n) = z(n\triangle t)$ and replace $\dot{x}(t)$, $\dot{y}(t)$ and $\dot{z}(t)$ by their two-point finite equivalents in (5.33), (5.34) and (5.39). This gives

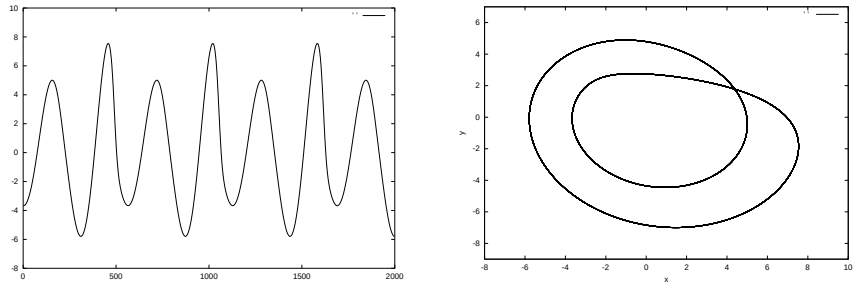
$$\begin{aligned}
\mathsf{x}(n+1) &= \mathsf{x}(n) - \mathsf{y}(n)\triangle t - \mathsf{z}(n)\triangle t,\\
\mathsf{y}(n+1) &= \mathsf{x}(n)\triangle t + \mathsf{y}(n)[1 + a\triangle t],\\
\mathsf{z}(n+1) &= b\triangle t + \mathsf{z}(n)[1 + \{\mathsf{x}(n) - c\}\triangle t].
\end{aligned} \tag{5.41}$$

Using some small (but not too small) value for $\triangle t$, trajectories can now be computed.[10] We consider the case $a = b = 0.2$. Then for values of c less than about 2.83 the projection of the trajectory into the $\{\mathsf{x}, \mathsf{y}\}$ plane is a simple periodic orbit and the output $\mathsf{x}(n)$ is a periodic function, with a single frequency (Fig. 5.8(a): a *simple cycle*). When c is increased through 2.83 the trajectory just fails to close on itself after one circuit and does so after two (Fig. 5.8(b): a *two-cycle*). The period doubles and the frequency halves to a subharmonic. By $c = 4.2$ the process has repeated, leading to an orbit which closes onto itself only after four circuits (Fig. 5.8(c): a *four-cycle*). By $c = 4.35$ (Fig. 5.8(d)) we have an *eight-cycle*. As c is increased period-doubling occurs with increasing frequency until, at a value between 4.35 and 5.0, the system becomes chaotic. The three-dimensional plot of the strange attractor for $c = 5.0$ is shown in Fig. 5.9. You will see that it looks rather like a Möbius strip. On this attractor any

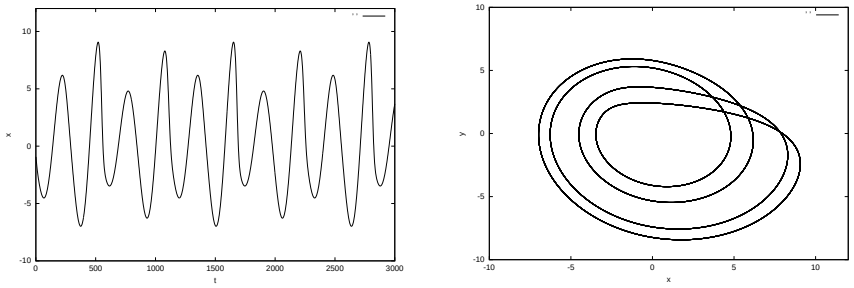
[10] MAPLE is not the most appropriate package for doing this. I used FORTRAN 90 with $\triangle t = 0.02$. It is necessary to run the iteration for a number ($\sim 10^3$, but depending on c) of iterations to eliminate transient behaviour and to ensure that the trajectory has reached the attractor.



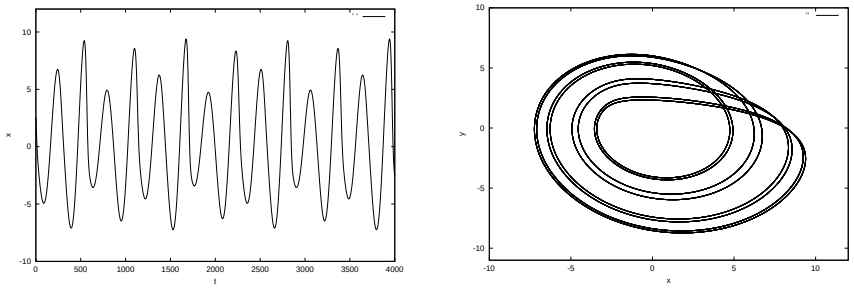
$c = 2.5$; a simple cycle.



$c = 3.5$; a two-cycle.



$c = 4.2$; a four-cycle.



$c = 4.35$; a eight-cycle.

Figure 5.8: A period-doubling sequence for the Rössler equations with $a = b = 0.2$ and increasing values of c.

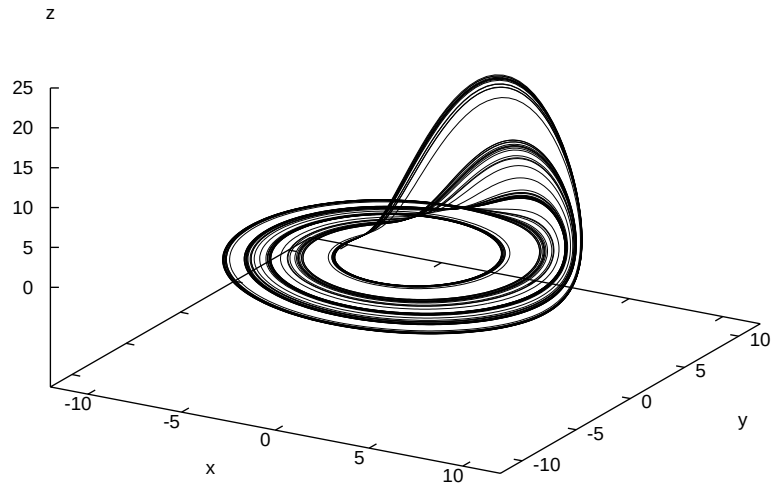


Figure 5.9: The strange attractor for Rössler equations when $a = b = 0.2$, $c = 5.0$.

two trajectories starting at nearby points will diverge exponentially. A brief account of the methods available for calculating Lyapunov exponents for such systems will be given later. For $a = 0.15$, $b = 0.2$ and $c = 10.0$ the three Lyapunov exponents are 0.13, 0.0 and -14.1. The leading exponent of $0.13 > 0$ indicates the system is chaotic. The negative exponent is necessary to hold the attractor together, and the zero exponent is for the direction along a trajectory and indicated that points on the same trajectory maintain their distances apart.

5.2.4 The Lorentz Equations

In this case we have two non-linear terms.

$$\dot{x}(t) = -a(x - y), \tag{5.42}$$

$$\dot{y}(t) = \rho x - y - zx, \tag{5.43}$$

$$\dot{z}(t) = -bz + xy. \tag{5.44}$$

For simplicity we shall take a and b as fixed positive quantities and consider variations in ρ. It will be seen that the transformation $(x, y, z) \to (-x, -y, z)$ leaves the equations unchanged and also there are trajectories which lie on the

z–axis ($x = y = 0$) with $z(t) = z(0)\exp(-bt)$. From (5.42) all equilibrium points must lie on the plane $x = y$ and have either $x = 0$ or $z = \rho - 1$. In the latter case $x^2 = bz$. So the three equilibrium points are

$$x = y = 0, \qquad z = 0, \tag{5.45}$$

$$x = y = \pm\sqrt{b(\rho - 1)}, \quad z = \rho - 1. \tag{5.46}$$

Linearizing about equilibrium point (5.45) gives the stability matrix

$$\boldsymbol{J}^* = \begin{pmatrix} -a & a & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \tag{5.47}$$

with eigenvalues

$$\lambda^{(\pm)} = -\tfrac{1}{2}\left\{1 + a \pm \sqrt{(1+a)^2 + 4a(\rho - 1)}\right\}, \qquad \lambda^{(3)} = -b. \tag{5.48}$$

The first pair of eigenvalues are for eigenvectors lying in the x–y plane and the third is in the z–direction. When $\rho < 1$ the origin is a proper stable node in the x–y plane. It becomes an improper stable node when $\rho = 1$ and a saddle-point when $\rho > 1$. In all cases since we have assumed $b > 0$ it is stable in the z–direction. This linear analysis can be supplemented by using the Lyapunov direct method. Choose the Lyapunov function

$$\mathcal{L}(x, y, z) = \tfrac{1}{2}\{x^2 + ay^2 + bz^2\}. \tag{5.49}$$

This gives

$$\boldsymbol{\nabla}\mathcal{L}.\boldsymbol{F}(x, y, z) = -\tfrac{1}{2}a(1+\rho)(x-y)^2 - \tfrac{1}{2}a(1-\rho)(x^2+y^2) - abz^2. \tag{5.50}$$

which is strictly negative, implying asymptotic stability when $\rho < 1$.

The equilibrium solution (5.46) exists only when $\rho \geq 1$ and the stability matrix is

$$\boldsymbol{J}^* = \begin{pmatrix} a & -a & 0 \\ -1 & 1 & \pm\sqrt{b(\rho-1)} \\ \mp\sqrt{b(\rho-1)} & \mp\sqrt{b(\rho-1)} & b \end{pmatrix} \tag{5.51}$$

and the eigenvalues are solutions of the cubic equation

$$f(\lambda) \equiv \lambda^3 + (a + b + 1)\lambda^2 + b(a + \rho)\lambda + 2ab(\rho - 1) = 0. \tag{5.52}$$

When $\rho > 1$ all the coefficients of this cubic are positive and there are, therefore, no real, positive eigenvalues and there must, of course, be one real negative eigenvalue. The only way for this equilibrium point to be unstable is for there to be a pair of complex roots with positive real part. When $\rho = 1$ (5.52) has roots $\lambda = 0, -b, -(a+1)$. Now suppose that ρ is increased from unity. Since the first eigenvalue is marginal its change, which will be of the order of $\triangle\rho = \rho - 1$,

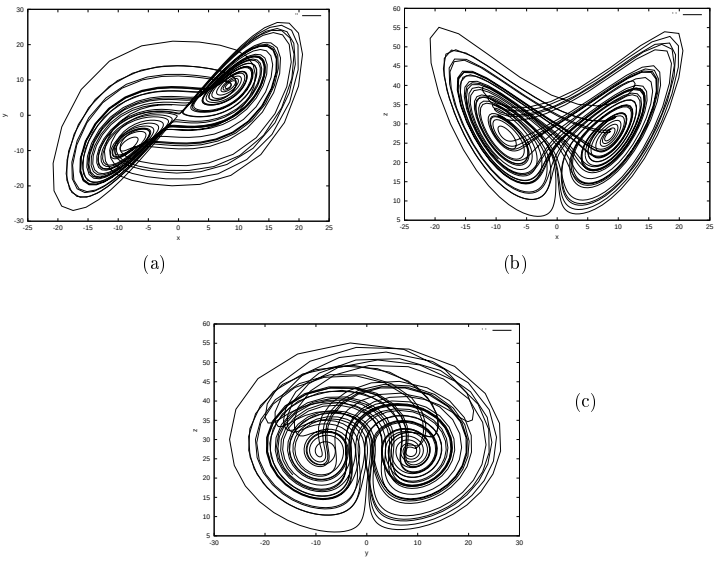


Figure 5.10: The strange attractor for the Lorentz equations shown in projections in (a) the $\{x, y\}$ plane, (b) the $\{x, z\}$ plane, (c) the $\{y, z\}$ plane, with $a = 10$, $b = \frac{8}{3}$, $\rho = 28$.

will determine the stability. Substituting $\lambda = \alpha \triangle \rho$ into (5.52) and solving for the lowest order terms gives $\alpha = -2a/(a+1)$. So the equilibrium points are stable. For them to become unstable, two of the eigenvalues must pass through values where they are purely imaginary. Suppose $\lambda^{(1)} = \mathrm{i}\omega$ and $\lambda^{(2)} = -\mathrm{i}\omega$. Then, since the sum of all three eigenvalues is equal to minus the quadratic coefficient in (5.52), $\lambda^{(3)} = -(a+b+1)$. This must be a root of (5.52) at the value ρ_{c} of ρ where instability sets in. Substituting into (5.52) gives

$$\rho_{\mathrm{c}} = \frac{a(a+b+3)}{a-b-1}. \tag{5.53}$$

Thus instability can occur only if a and b are such that $\rho_{\mathrm{c}} > 1$ and then the equilibrium points will be stable for $1 < \rho < \rho_{\mathrm{c}}$. It is of interest to calculate the eigenvalues of the equilibrium points (5.46) for fixed values of a and b and a range of values of ρ. For $a = 10$ and $b = \frac{8}{3}$, $\rho_{\mathrm{c}} = \frac{470}{19} = 24.737$. Since one eigenvalue is always negative the interest is in the values of the other pair. At $\rho = 1$ one is zero and the other is -11. For ρ near to one all three eigenvalues are real and negative and (with respect to this pair) the equilibrium points are stable nodes. Between $\rho = 1.3$ and 1.4 the pair becomes complex conjugate with negative real parts. The equilibrium points are stable focii. This character persists up to $\rho = \rho_c = 24.737$, when the real parts change sign and we have unstable focii. The passage to chaos in the Lorentz system is very complicated with both period-doubling and period halving. The strange attractor which is

shown in projection for $a = 10$, $b = \frac{8}{3}$ in Fig. 5.10 takes the form of a pair of connected loops around the two equilibrium solutions (5.46). The Lyapunov exponents for $a = 16.0$, $b = 4.0$ and $\rho = 45.92$ are 2.16, 0.0 and -32.4. The particular complexity of this system is evident from the fact that the strange attractor makes its appearance at values of ρ slightly less that ρ_c when the equilibrium points are still stable. It also coexists with limit cycles around the equilibrium points which make their appearance for certain ranges of ρ.

5.3 Lyapunov Exponents and Fractal Dimension

5.3.1 The Transformation of Volumes

Let Γ_d be the phase space of the dynamic system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{x}; t) \tag{5.54}$$

and suppose $\mu(\boldsymbol{x}; t)$ is some *density function* defined on Γ_d. Let $\Upsilon(t) \subset \Gamma_d$ be a volume which moves with the flow of the dynamic system and define the volume integral

$$\mathrm{P}(t) = \int_{\Upsilon(t)} \mu(\boldsymbol{x}; t) \mathrm{d}V, \tag{5.55}$$

A well-known theorem, used in a number of areas including probability theory and fluid dynamics, is that

$$\frac{\mathrm{dP}(t)}{\mathrm{d}t} = \int_{\Upsilon(t)} \left\{ \frac{\partial \mu}{\partial t} + \boldsymbol{\nabla}.[\mu \boldsymbol{F}] \right\} \mathrm{d}V. \tag{5.56}$$

In the special case where $\mu(\boldsymbol{x}; t) = 1$, $\mathrm{P}(t)$ just measures the size of the volume $\Upsilon(t)$. It follows that

$$\boldsymbol{\nabla}.\boldsymbol{F}(\boldsymbol{x}; t) = 0 \tag{5.57}$$

is a necessary and sufficient condition for the flow of the dynamic system to preserve volume. In particular, for the Hamiltonian system defined by (1.10),

$$\boldsymbol{\nabla}.\boldsymbol{F}(\boldsymbol{x}; t) = \sum_{\ell=1}^{d} \left\{ \frac{\partial^2 H}{\partial x_\ell \partial p_\ell} - \frac{\partial^2 H}{\partial p_\ell \partial x_\ell} \right\} = 0. \tag{5.58}$$

So Hamiltonian systems are volume preserving. A system for which $\boldsymbol{\nabla}.\boldsymbol{F}(\boldsymbol{x}; t) < 0$, meaning that volumes shrink with time, is called *dissipative*.

5.3.2 The Lyapunov Spectrum

We now generalize the discussion of Lyapunov exponents given in Sect. 5.2.1 to systems of more than one dimension. Consider first the d–dimensional difference equation

$$\mathbf{x}(n+1) = \mathbf{F}[\mathbf{x}(n)], \tag{5.59}$$

where

$$\begin{aligned}\mathbf{x}(n) &= (\mathsf{x}_1(n), \mathsf{x}_2(n), \ldots, \mathsf{x}_d(n)), \\ \mathbf{F}[\mathbf{x}] &= (\mathsf{F}_1[\mathbf{x}], \mathsf{F}_2[\mathbf{x}], \ldots, \mathsf{F}_d[\mathbf{x}]).\end{aligned} \tag{5.60}$$

Let $\triangle\mathbf{x}(n) = \mathbf{x}(n) - \mathbf{x}(n-1)$. Then

$$\triangle\mathbf{x}(n+1) = \mathbf{J}[\mathbf{x}(n-1)]\triangle\mathbf{x}(n) + \mathrm{O}(|\triangle\mathbf{x}(n)|^2), \tag{5.61}$$

where

$$\mathbf{J}[\mathbf{x}] = \begin{pmatrix} \dfrac{\partial \mathsf{F}_1}{\partial \mathsf{x}_1} & \dfrac{\partial \mathsf{F}_1}{\partial \mathsf{x}_2} & \cdots & \dfrac{\partial \mathsf{F}_1}{\partial \mathsf{x}_d} \\ \dfrac{\partial \mathsf{F}_2}{\partial \mathsf{x}_1} & \dfrac{\partial \mathsf{F}_2}{\partial \mathsf{x}_2} & \cdots & \dfrac{\partial \mathsf{F}_2}{\partial \mathsf{x}_d} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial \mathsf{F}_d}{\partial \mathsf{x}_1} & \dfrac{\partial \mathsf{F}_d}{\partial \mathsf{x}_2} & \cdots & \dfrac{\partial \mathsf{F}_d}{\partial \mathsf{x}_d} \end{pmatrix}. \tag{5.62}$$

Neglecting all but the linear term in (5.61),

$$\triangle\mathbf{x}(n+1) = \mathbf{S}(n)\triangle\mathbf{x}(1), \tag{5.63}$$

where

$$\mathbf{S}(n) = \mathbf{J}[\mathbf{x}(n-1)]\mathbf{J}[\mathbf{x}(n-2)]\cdots\mathbf{J}[\mathbf{x}(0)]. \tag{5.64}$$

Let $\mathbf{\Sigma}(n)$ be the diagonal matrix with the eigenvalues, $\sigma_1(n), \sigma_2(n), \ldots, \sigma_d(n)$, of $\mathbf{S}(n)$ along the diagonal, ordered according to descending magnitude, $\mathbf{V}(n)$ be the matrix with the corresponding left eigenvectors as rows and $\mathbf{U}(n)$ be the matrix with the corresponding right eigenvectors as columns. From Sect. 1.9,

$$\mathbf{V}(n)\triangle\mathbf{x}(n+1) = \mathbf{\Sigma}(n)\mathbf{V}(n)\triangle\mathbf{x}(1), \tag{5.65}$$

The magnitudes $|\,\sigma_1\,(n)|, |\,\sigma_2\,(n)|, \ldots, |\,\sigma_d\,(n)|$ measure the dilations and contractions of the transformation over n steps. As we saw in (5.21) the average of these scale changes are measured by the Lyapunov exponents and for the d–dimensional difference equation system described here we can *define* the d Lyapunov exponents in descending order by

$$\lambda_{\mathrm{L}}^{(\ell)} = \lim_{n\to\infty} \frac{\ln\{\sigma_\ell(n)\}}{n}, \qquad \ell = 1, 2 \ldots, d. \tag{5.66}$$

It will be seen that, in the case $d = 1$, this is equivalent to (5.25).

In the case of the d-dimensional continuous system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{x}), \tag{5.67}$$

we monitor the long-term evolution of an infinitesimal d-dimensional hypersphere of initial conditions. This hypersphere will become a *hyperellipsoid* under

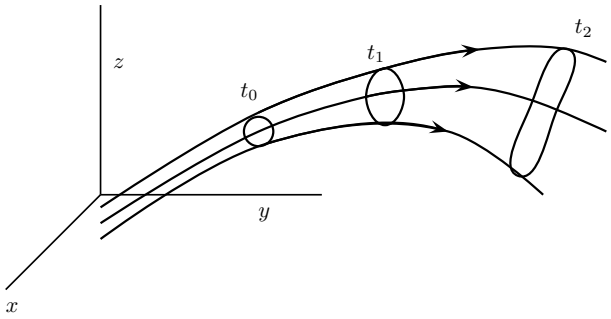


Figure 5.11: The deformation of a sphere of initial conditions under the effect of the flow.

the effect of the deforming nature of the flow. This is shown in Fig. 5.11 for $d = 3$. If $\mathfrak{d}^{(\ell)}(t)$ is the length of the ℓ–th principle axis at time t then a suitable generalization of (5.20) to the case of d Lyapunov exponents is

$$\mathfrak{d}^{(\ell)}(t) = \mathfrak{d}^{(\ell)}(t_0) \exp[\lambda_{\mathrm{L}}^{(\ell)}(t - t_0)], \qquad \ell = 1, 2, \ldots, d. \tag{5.68}$$

The implementation of the procedure implied by (5.68) involves defining the principal axes with an initial hypersphere which is as small as possible and determining their evolution with the non-linear equations. This means determining d neighbouring solutions. As we saw in the one-dimensional case, to obtain the Lyapunov exponents we need to be able to do this over a long period of time, which is not normally practical for a chaotic system.

An alternative approach is to obtain the *fiducial trajectory*, which gives the evolution of the centre of the hypersphere/ellipsoid and then to integrate the linearized equations for d different initial conditions defining an arbitrarily oriented set of d orthonormal vectors. Of course, over a long period of time, even just using the linearized equations, the vectors will diverge in length. They will also reorient themselves towards the direction associated with the largest Lyapunov exponent. The way to deal with this difficulty is by the repeated use of *Gram-Schmidt renormalization* (GSR).

Suppose $\boldsymbol{e}^{(1)}(0), \boldsymbol{e}^{(2)}(0), \ldots, \boldsymbol{e}^{(d)}(0)$ is a set of orthonormal vectors at time $t = 0$ and suppose upon integration over a time period $\triangle t$ they evolve into the set $\tilde{\boldsymbol{e}}^{(1)}(\triangle t), \tilde{\boldsymbol{e}}^{(2)}(\triangle t), \ldots, \tilde{\boldsymbol{e}}^{(d)}(\triangle t)$. These vectors, will in general, no longer be normalized. They will also have all reoriented themselves more towards the direction of the major principal axis, associated with the largest Lyapunov exponent $\lambda_{\mathrm{L}}^{(1)}$. We now apply GSR in such a way as to leave the direction of

$\tilde{\boldsymbol{e}}^{(1)}(\triangle t)$ unaffected. Thus

$$\begin{aligned}
\boldsymbol{e}^{(1)}(\triangle t) &= \frac{\tilde{\boldsymbol{e}}^{(1)}(\triangle t)}{|\tilde{\boldsymbol{e}}^{(1)}(\triangle t)|}, \\
\boldsymbol{e}^{(2)}(\triangle t) &= \frac{\tilde{\boldsymbol{e}}^{(2)}(\triangle t) - \boldsymbol{e}^{(1)}(\triangle t)[\boldsymbol{e}^{(1)}(\triangle t).\tilde{\boldsymbol{e}}^{(2)}(\triangle t)]}{|\tilde{\boldsymbol{e}}^{(2)}(\triangle t) - \boldsymbol{e}^{(1)}(\triangle t)[\boldsymbol{e}^{(1)}(\triangle t).\tilde{\boldsymbol{e}}^{(2)}(\triangle t)]|}, \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\boldsymbol{e}^{(d)}(\triangle t) &= \frac{\tilde{\boldsymbol{e}}^{(d)}(\triangle t) - \sum_{\ell=1}^{d-1} \boldsymbol{e}^{(\ell)}(\triangle t)\left[\boldsymbol{e}^{(\ell)}(\triangle t).\tilde{\boldsymbol{e}}^{(d)}\right]}{\left|\tilde{\boldsymbol{e}}^{(d)}(\triangle t) - \sum_{\ell=1}^{d-1} \boldsymbol{e}^{(\ell)}(\triangle t)\left[\boldsymbol{e}^{(\ell)}(\triangle t).\tilde{\boldsymbol{e}}^{(d)}\right]\right|}
\end{aligned} \tag{5.69}$$

The vector $\boldsymbol{e}^{(1)}(t)$ tends to seek out the direction of most rapid growth and to dilate in proportion to $\exp[\lambda_{\mathrm{L}}^{(1)}t]$. So

$$\lambda_{\mathrm{L}}^{(1)} \simeq \frac{1}{N}\sum_{n=1}^{N} \ln\left|\frac{\tilde{\boldsymbol{e}}^{(1)}(n\triangle t)}{\tilde{\boldsymbol{e}}^{(1)}((n-1)\triangle t)}\right|. \tag{5.70}$$

for large N. The direction of $\boldsymbol{e}^{(2)}(t)$ is orthogonal to $\boldsymbol{e}^{(1)}(t)$, but $\tilde{\boldsymbol{e}}^{(2)}(t)$ is not necessarily in the direction of the second dominant Lyapunov exponent. To obtain $\lambda_{\mathrm{L}}^{(2)}$ we either project $\tilde{\boldsymbol{e}}^{(2)}(t)$ onto the direction of $\boldsymbol{e}^{(2)}(t)$ or observe that the plane of $\tilde{\boldsymbol{e}}^{(1)}(t)$ and $\tilde{\boldsymbol{e}}^{(2)}(t)$ is the same as that of $\boldsymbol{e}^{(1)}(t)$ and $\boldsymbol{e}^{(2)}(t)$. The size $|\tilde{\boldsymbol{e}}^{(1)}(t) \wedge \tilde{\boldsymbol{e}}^{(2)}(t)|$ grows in proportion to $exp[\{\lambda_{\mathrm{L}}^{(1)}t + \lambda_{\mathrm{L}}^{(2)}t\}]$. So

$$\lambda_{\mathrm{L}}^{(1)} + \lambda_{\mathrm{L}}^{(2)} \simeq \frac{1}{N}\sum_{n=1}^{N} \ln\left|\frac{\tilde{\boldsymbol{e}}^{(1)}(n\triangle t) \wedge \tilde{\boldsymbol{e}}^{(2)}(n\triangle t)}{\tilde{\boldsymbol{e}}^{(1)}((n-1)\triangle t) \wedge \tilde{\boldsymbol{e}}^{(2)}((n-1)\triangle t)}\right| \tag{5.71}$$

for large N. In a similar way the first three exponents can be obtained from the growth in size of a volume defined by a triad of vectors.[11] In d–dimensional space the volume of a small hypersphere of radius ε is

$$V(\varepsilon) = \frac{\left\{\Gamma\left(\frac{1}{2}\right)\varepsilon\right\}^d}{\Gamma\left(\frac{1}{2}d+1\right)}, \tag{5.72}$$

where $\Gamma(x)$ is the *gamma function*.[12] On the attractor this volume deforms in a time t into a hyperellipsoid with

$$V(\varepsilon) \to \exp\{(\lambda_{\mathrm{L}}^{(1)} + \lambda_{\mathrm{L}}^{(2)} + \cdots + \lambda_{\mathrm{L}}^{(d)})t\}V(\varepsilon). \tag{5.73}$$

[11] A FORTRAN program for implementing this procedure is given by Wolf, A, Swift J. B., Swinney H. L. and Vastano J. A. (1985) *Physica* D, 285–317.

[12] With the salient properties $x\,\Gamma(x) = \Gamma(x+1)$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

The sum of the Lyapunov exponents will therefore be zero if the system is volume preserving and negative if it is dissipative. Since the basin of attraction of a strange attractor is of the dimension of the space of the system and the attractor itself has a fractal dimension less than d, chaotic systems must be dissipative.

5.3.3 The Dimension of Chaotic Attractors

In Sect. 5.2 we defined fractal dimension and suggested that one (possibly not certain) indication of the presence of chaos was a non-integer fractal dimension of the attractor. The fractal dimension of an attractor $\mathfrak{A}$ is $\mathcal{D}(\mathfrak{A})$ given by (5.19). In principle the fractal dimension of $\mathfrak{A}$ in a space of dimension d could be calculated by covering the space with a hypercubic grid of mesh size ℓ. A trajectory of the system, after transitory factors have disappeared, is then followed and the number $N(\ell)$ of cells of the grid visited by the trajectory over a long period of time is then counted. An approximation to $\mathcal{D}(\mathfrak{A})$ is then given by $-\ln\{N(\ell)\}/\ln\{\ell\}$. Such a procedure is in most cases very difficult to implement. It is also difficult to get an accurate result because of the need to approach in some way the limit of small ℓ.

Although the fractal dimension is related to the number of cells of the grid visited by a trajectory on the attractor, no account is taken of the number of times the trajectory visits a particular cell. A generalization of the fractal dimension $\mathcal{D}(\mathfrak{A})$ of $\mathfrak{A}$ can be made by introducing a probability measure $\mu(\boldsymbol{x})$ over the space of the dynamic system, where $\mu(\boldsymbol{x})\triangle V$ is the probability of finding the phase point of the system in a volume $\triangle V$ around the point $\boldsymbol{x}$. Then label the cells of the grid $s = 1, 2, \ldots$ and define $p_s(\mu, \ell)$ to be the probability of finding the phase point in the s–th cell, obtained by integrating $\mu(\boldsymbol{x})$ over the volume of the cell. The *information entropy* of the probability measure $\mu(\boldsymbol{x})$ is defined by

$$I(\mu, \ell) = -\sum_{s=1}^{N(\ell)} p_s(\mu, \ell) \ln\{p_s(\mu, \ell)\}. \tag{5.74}$$

In information theory this function gives the amount of information necessary to specify the state of the system to within an accuracy of ℓ. The *information dimension* $\mathcal{D}_{\mathrm{I}}(\mu; \mathfrak{A})$ is defined by

$$\mathcal{D}_{\mathrm{I}}(\mu; \mathfrak{A}) = \lim_{\ell \to 0} \frac{I(\mu, \ell)}{\ln\{1/\ell\}}. \tag{5.75}$$

It is clear that when the probability measure is uniform $p_s(\mu, \ell) = 1/N(\ell)$, $I(\mu, \ell) = \ln\{N(\ell)\}$ and the information dimension is equal to the fractal dimension. In general it can be shown that $\mathcal{D}_{\mathrm{I}}(\mu; \mathfrak{A}) \leq \mathcal{D}(\mathfrak{A})$.

Suppose we want to calculate the fractal dimension of the chaotic attractor $\mathfrak{A}$ associated with a difference equation in two dimensions. We cover it in $N(\ell)$ squares of side ℓ. The Lyapunov exponents will satisfy the condition $\lambda_{\mathrm{L}}^{(1)} > 0 > \lambda_{\mathrm{L}}^{(2)}$. Let the map be iterated n times. If we suppose that the

dilation and contraction acts linearly on each square, then each is turned into a parallelogram of average length $\exp\{n\lambda_{\rm L}^{(1)}\}\ell$ and average width $\exp\{n\lambda_{\rm L}^{(2)}\}\ell$. Suppose that we had used a finer grid of squares of side $\exp\{n\lambda_{\rm L}^{(2)}\}\ell$ to cover the attractor. On average we need $\exp\{n[\lambda_{\rm L}^{(1)} - \lambda_{\rm L}^{(2)})]\}$ of the new squares to cover one parallelogram. So we need

$$N(\exp\{n\lambda_{\rm L}^{(2)}\}\ell) = \exp\{n[\lambda_{\rm L}^{(1)} - \lambda_{\rm L}^{(2)})]\}N(\ell), \tag{5.76}$$

squares to cover the attractor. Since, from (5.19),

$$\mathcal{D}(\mathfrak{A}) \simeq -\frac{\ln\{N(\ell)\}}{\ln\{\ell\}} =\simeq -\frac{\ln\{N(\exp\{n\lambda_{\rm L}^{(2)}\}\ell)\}}{\ln\{\exp\{n\lambda_{\rm L}^{(2)}\}\ell\}}, \tag{5.77}$$

it follows, from (5.76) and (5.77), that

$$\mathcal{D}(\mathfrak{A}) = 1 - \frac{\lambda_{\rm L}^{(1)}}{\lambda_{\rm L}^{(2)}}. \tag{5.78}$$

Of course, this analysis cannot be dignified by the title of a proper derivation. Apart from anything else it applies only to difference maps in two dimensions. However, we shall use it as a motivation for defining the *Lyapunov dimension*

$$\mathcal{D}_{\rm L}(\mathfrak{A}) = k + \frac{\lambda_{\rm L}^{(1)} + \lambda_{\rm L}^{(2)} + \cdots + \lambda_{\rm L}^{(k)}}{|\lambda_{\rm L}^{(k+1)}|}, \tag{5.79}$$

where k is the largest value for which $\lambda_{\rm L}^{(1)} + \lambda_{\rm L}^{(2)} + \cdots + \lambda_{\rm L}^{(k)} \geq 0$. In many cases it appears to be true that $\mathcal{D}_{\rm L}(\mathfrak{A}) = \mathcal{D}_{\rm I}(\mathfrak{A})$. From the values given in Sect. 5.2.3 for the Rössler system the dimension of its strange attractor is 2.0092 and for the Lorentz system discussed in Sect. 5.2.4 the dimension is 2.0667.

Problems 5

1) Express the equation

$$\ddot{x}(t) + \mu\dot{x}(t) - x(t) + x^2(t) = 0,$$

as a pair of equations using the second variable $y(t) = \dot{x}(t)$. Find the equilibrium points and determine their linear stability for the different ranges of μ. Show that, when $\mu = 0$,

$$\tfrac{1}{2}\{x^2 - y^2\} = E + \tfrac{1}{3}x^3 \tag{5.80}$$

is an integral of the motion for different values of the parameter E. Either using MAPLE or by hand (and brain) sketch the trajectories given by (5.80). Mark the direction of flow and label the curves with their values of E, identifying the homoclinic trajectory. Using your intuition rather than

undertaking detailed analysis, sketch the form that the corresponding curves take when μ is small and positive or negative. *The breakup of a homoclinic trajectory is often associated with the onset of chaos. This is an example of the breakup of a homoclinic without chaos being involved.*

2) Remember that the equilibrium point $x = y = \sqrt{b(\rho-1)}$, $z = \rho - 1$ of the Lorentz equations becomes unstable as ρ is increased through the value $\rho = \rho_{\rm c}$ where

$$\rho_{\rm c} = \frac{a(a+b+3)}{a-b-1},$$

as long as a and b are such that $\rho_{\rm c} > 1$. With $x_{\rm c} = y_{\rm c} = \sqrt{b(\rho_{\rm c}-1)}$, $z_{\rm c} = \rho_{\rm c} - 1$ define $\triangle x = x - x_{\rm c}$, $\triangle y = y - y_{\rm c}$, $\triangle z = z - z_{\rm c}$ and $\triangle \boldsymbol{r} = (\triangle x, \triangle y, \triangle z)^{\rm T}$. Show that the Lorentz equations can be expressed, without approximation in the form

$$\omega \frac{{\rm d}\triangle \boldsymbol{r}}{{\rm d}\tau} + \boldsymbol{J}^* \triangle \boldsymbol{r} = \boldsymbol{w},$$

where $\tau = \omega t$, for some parameter ω and

$$\boldsymbol{J}^* = \begin{pmatrix} a & -a & 0 \\ -1 & 1 & x_{\rm c} \\ -x_{\rm c} & -x_{\rm c} & b \end{pmatrix}, \qquad \boldsymbol{w} = \begin{pmatrix} 0 \\ (\rho-\rho_{\rm c})(\triangle x + x_{\rm c}) - \triangle x \triangle z \\ \triangle x \triangle y \end{pmatrix}.$$

The matrix $\boldsymbol{J}^*$ is that given by (5.51) in the notes, but evaluated where $\rho = \rho_{\rm c}$. Remember at this point the matrix has two purely imaginary eigenvalues $\pm {\rm i}\,\omega_{\rm c}$ (say) and a third which is equal to $-(a+b+1)$. Show that

$$\omega_{\rm c} = \sqrt{\frac{2ab(a+1)}{a-b-1}}.$$

Let $\boldsymbol{v}$ and $\boldsymbol{u}$ be the left and right eigenvectors for the eigenvalue ${\rm i}\,\omega_{\rm c}$, (with the corresponding eigenvectors for $-{\rm i}\,\omega_{\rm c}$ being their complex conjugates $\overline{\boldsymbol{v}}$ and $\overline{\boldsymbol{u}}$). Assume that:

(i) The eigenvectors satisfy the usual orthnormality condition.

(ii) $|\rho - \rho_{\rm c}| = \varepsilon$, where ε is small.

(iii) $\triangle \boldsymbol{r}$ lies in the plane spanned by $\boldsymbol{u}$ and $\overline{\boldsymbol{u}}$.

(iv) ω and $\triangle \boldsymbol{r}$ have expansions of the form

$$\omega = \omega_{\rm c} + \varepsilon\omega_1 + {\rm O}(\varepsilon^2), \qquad \triangle \boldsymbol{r} = \varepsilon^{\frac{1}{2}} \boldsymbol{p} + \varepsilon \boldsymbol{q} + {\rm O}\left(\varepsilon^{\frac{3}{2}}\right).$$

Show that

$$\boldsymbol{p} = c\boldsymbol{u}\exp(-\mathrm{i}\,\omega t) + \overline{c\boldsymbol{u}}\exp(\mathrm{i}\,\omega t),$$

where c is some complex constant. *Subject to the assumptions made, this establishes the existence of a periodic solution for ρ slightly larger than ρ_{c} and shows that as ρ passes through ρ_{c} there is a Hopf bifurcation.*

3) Show that the Lorentz equations (5.42)–(5.44) can be expressed in the form

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = -a\varepsilon\xi - \eta,$$
$$\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = -\varepsilon\eta - \xi\zeta,$$
$$\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} = -b\varepsilon\zeta + \xi\eta - ab\varepsilon,$$

in terms of the variables $\varepsilon = 1/\sqrt{\rho}$, $\tau = t/\varepsilon$, $\xi = \varepsilon x$, $\eta = \varepsilon^2 a y$, $\zeta = a(\varepsilon^2 z - 1)$. The Lorentz equations in the limit $\rho \to \infty$ are now obtained by setting $\varepsilon = 0$ in these equations. Show that in this limit they have the integrals

$$\tfrac{1}{2}\eta^2 + \tfrac{1}{2}\zeta^2 = \alpha,$$
$$\tfrac{1}{2}\xi^2 - \zeta = \beta.$$

and that

$$\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2 = (2\alpha - \beta^2) - \tfrac{1}{4}\xi^4 + \beta\xi^2.$$

Hence show that, when $\alpha = \frac{9}{8}$, $\beta = \frac{1}{2}$, there is a periodic solution in the $\{\xi, \mathrm{d}\xi/\mathrm{d}\tau\}$ plane with a period (measured in terms of the time parameter τ) of

$$4\int_{-2}^{2} \frac{\mathrm{d}\xi}{\sqrt{(\xi^2+2)(4-\xi^2)}}.$$

4) The *baker's map* is given by

$$\mathsf{x}(n+1) = \begin{cases} \tau_a \mathsf{x}(n), & \text{if } \mathsf{y}(n) < \frac{1}{2}, \\ (1-\tau_b) + \tau_b \mathsf{x}(n), & \text{if } \mathsf{y}(n) > \frac{1}{2}, \end{cases}$$
$$\mathsf{y}(n+1) = \begin{cases} 2\mathsf{y}(n), & \text{if } \mathsf{y}(n) < \frac{1}{2}, \\ 2\mathsf{y}(n) - 1, & \text{if } \mathsf{y}(n) > \frac{1}{2}, \end{cases}$$

where $\tau_a + \tau_b \leq 1$. Given that μ is the probability that the iterated value of $\mathfrak{y}$ is in the range $0 \leq \mathfrak{y} \leq \frac{1}{2}$, determine the Lyapunov exponents, showing that the system is chaotic. Show that the Lyapunov dimension of the attractor is

$$1 - \{\mu \ln(\tau_a) + (1-\mu)\ln(\tau_b)\}^{-1}.$$

Chapter 6

Solutions

6.1 Problems 1

1) (i) The equilibrium points are given by $x = 0$ and $x = x^* = (a - c)/ab$. Linearizing about $x = 0$

$$\frac{\mathrm{d}\triangle x}{\mathrm{d}t} = (a - c)\triangle x,$$

with solution

$$\triangle x = \mathrm{C}\exp[(a - c)t].$$

So this solution is stable if $a < c$ and unstable if $a > c$. Linearize about $x = x^*$

$$\frac{\mathrm{d}\triangle x}{\mathrm{d}t} = (c - a)\triangle x,$$

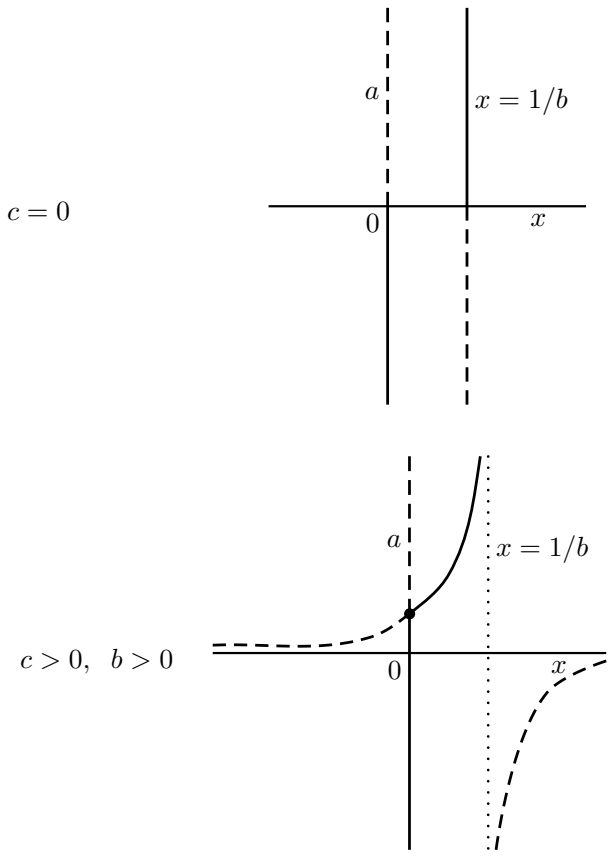
with solution

$$\triangle x = \mathrm{C}\exp[(c - a)t].$$

So this solution is stable if $a > c$ and unstable if $a < c$. There are five different cases:

When $c = 0$, $x^* = 1/b$ and the lines of equilibrium points are parallel to the a-axis. There is no bifurcation but the stability changes at $a = 0$.

When $b > 0$ **and** $c > 0$, there is a *transcritical* bifurcation at $x = 0$, $a = c$ on one branch of $x = x^*(a)$. The second branch is unstable. The case $b < 0$, $c > 0$ is the mirror image of this in the vertical axis.



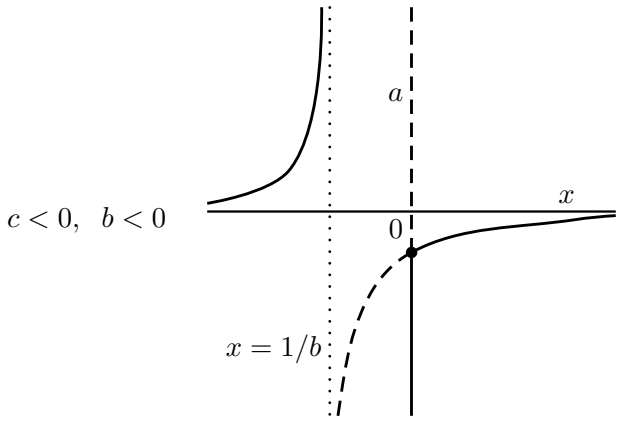
When $b < 0$ and $c < 0$, there is a *transcritical* bifurcation at $x = 0$, $a = c$ on one branch of $x = x^*(a)$. The second branch is stable. The case $b > 0$, $c < 0$ is the mirror image of this in the vertical axis. The equation is separable so

$$\int \frac{\mathrm{d}x}{x(a - c - abx)} = t + \text{constant}.$$

Using partial fractions it is easy to do the integration and the final solution is

$$x(t) = \frac{\mathsf{C}(a - c)\exp[(a - c)t]}{1 + ab\mathsf{C}\exp[(a - c)t]},$$

for some constant C. If $a < c$, $x \to 0$ as $t \to \infty$ and, if $a > c$, $x \to (a - c)/ab$ as $t \to \infty$.



(ii) The equilibrium solutions are $x = 0$ and

$$x = x^* = \begin{cases} a/b, & \text{if } c = 0, \\ \dfrac{b \pm \sqrt{b^2 - 4ac}}{2c}, & \text{if } c \neq 0, \end{cases}$$

Linearizing about $x = 0$

$$\frac{\mathrm{d}\triangle x}{\mathrm{d}t} = a\triangle x.$$

So this equilibrium point is stable if $a < 0$ and unstable if $a > 0$. Linearizing about $x = x^*$

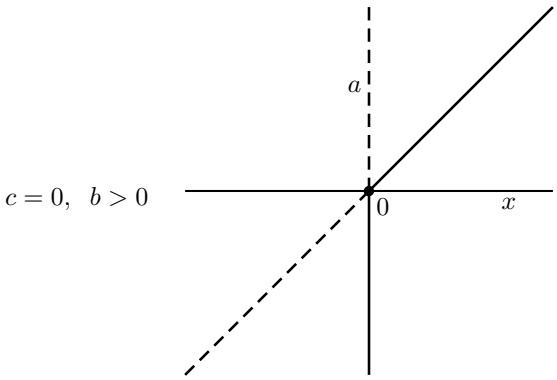
$$\frac{\mathrm{d}\triangle x}{\mathrm{d}t} = x^*(2x^*c - b)\triangle x.$$

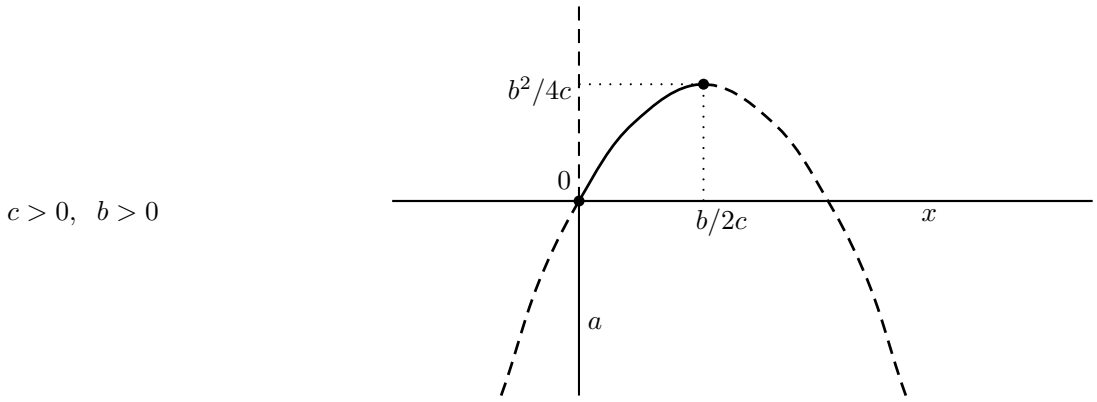
So x^* is stable if $x^*(2x^*c - b) < 0$ and unstable if $x^*(2x^*c - b) > 0$. When $c = 0$ these conditions reduce to $a > 0$ and $a < 0$ respectively.

When $c = 0$ and $b > 0$, there is a transcritical bifurcation at the origin. For $c = 0$ and $b < 0$ the bifurcation diagram is obtained from this by reflection in the vertical axis.

When $c > 0$ and $b > 0$, there is a transcritical bifurcation at the origin and a turning-point bifurcation at $x = b/2c$, $a = b^2/4c$. The case $c > 0$, $b < 0$ is obtained from this by reflection in the vertical axis.

When $c < 0$ and $b < 0$, there are again a transcritical and a turning-point bifurcation at the same locations. The case $c < 0$ and $b > 0$ is obtained from this by reflection in the vertical axis.





Each of these $c \neq 0$ systems of bifurcations goes into a pitchfork bifurcation when $b \to 0$. Denoting the two branches of x^* by $x^{(\pm)}$, the equation can separated into

$$\int \frac{\mathrm{d}x}{x[x - x^{(+)}][x - x^{(-)}]} = \text{constant} + ct$$

Decomposing into partial fractions and integrating gives

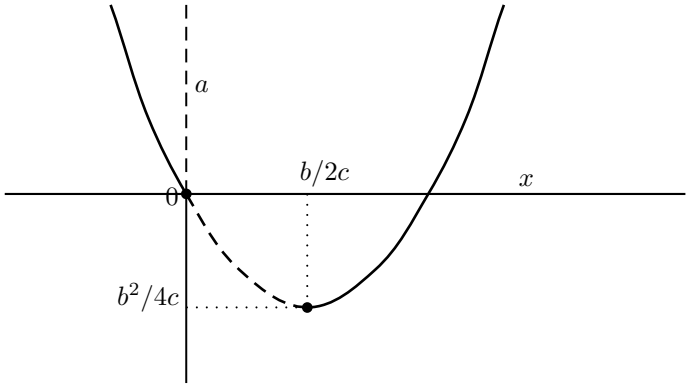
$$\ln \left\{ x^{\alpha} [x - x^{(+)}]^{\gamma^{(+)}} [x - x^{(-)}]^{\gamma^{(-)}} \right\} = \mathsf{C} \exp(ct).$$

where $\alpha = x^{(+)} x^{(-)}$, $\gamma^{(\pm)} = x^{(\pm)}[x^{(\pm)} - x^{(\mp)}]$. The limiting behaviour as $t \to \infty$ can be obtained by considering the various signs of the parameters.

2) The right-hand sides of these two equations are both zero when $x = y = 0$. Now the Taylor expansions of $\sin(x)$ and $\cos(x)$ give

$$\sin(\triangle x) = \triangle x + \mathrm{O}(\triangle x^3), \qquad \cos(\triangle x) = 1 + \mathrm{O}(\triangle x^2).$$

$c < 0, \;\; b < 0$



So when linearized to the same form as ($\star$) we have

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}.$$

This matrix has eigenvalues $\lambda = -2, 1$. The equilibrium point is a *saddle-point.*

3) All the equilibrium points are given by the simultaneous solutions of

$$x^2 = y, \qquad 8x = y^2.$$

This gives $x^4 = 8x$, which has the solutions

$$x = 0, \qquad \text{implying} \qquad y = 0, \tag{6.1}$$

$$x = 2, \qquad \text{implying} \qquad y = 4. \tag{6.2}$$

For (6.1)

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 8 & 0 \end{pmatrix}.$$

This matrix has eigenvalues $\lambda = \pm\sqrt{8}$ giving a *saddle-point.*

For (6.2)

$$\boldsymbol{A} = \begin{pmatrix} -4 & 1 \\ 8 & -8 \end{pmatrix}.$$

This matrix has eigenvalues $\lambda = -6 \pm 2\sqrt{3}$. Both these eigenvalues are negative so the equilibrium point is a *stable node.*

4) $\dfrac{x\mathrm{d}x}{\mathrm{d}t} = -xy + \dfrac{x^2(1-x^2-y^2)}{\sqrt{x^2+y^2}}, \qquad \dfrac{y\mathrm{d}y}{\mathrm{d}t} = xy + \dfrac{y^2(1-x^2-y^2)}{\sqrt{x^2+y^2}}.$

So with $r^2 = x^2 + y^2$

$$\frac{1}{2}\frac{\mathrm{d}r^2}{\mathrm{d}t} = \frac{r^2(1-r^2)}{r},$$

giving

$$\frac{\mathrm{d}r}{\mathrm{d}t} = (1-r^2). \tag{6.3}$$

There is an equilibrium solution with $r = 1$ and with $r = 1 + \triangle r$

$$\frac{\mathrm{d}\triangle r}{\mathrm{d}t} = -2\triangle r.$$

So $r = 1$ is stable. At the point $x = \cos(\theta)$, $y = \sin(\theta)$ on this solution $\dot{\theta}(t) = 1$, so $x = \cos(\theta_0 + t)$ $y = \sin(\theta_0 + t)$ gives the stable limit cycle for any θ_0. Equation (6.3) can be solved to give $r = \tanh(t_0 + t)$.

5) $\qquad \dot{\theta}(t) = \omega, \qquad \dot{\omega}(t) = \Omega^2 \sin(\theta)\{\cos(\theta) - a\}.$

The equilibrium points are given by

(a) $\sin(\theta) = 0$ for which $\theta = 0, \pm\pi, \pm 2\pi, \ldots$.

(b) $\cos(\theta) = a$ which, for $1 \geq a \geq 0$, gives two sets of solutions

$\theta^\star = \pm\theta_0(a) + 2n\pi, \qquad n = 0, \pm 1, \pm 2 \ldots,$

where $\theta_0(a) \to 0$, as $a \to 1$.

First linearize about $n\pi$.

$$\begin{aligned}
\sin(n\pi + \triangle\theta) &= \triangle\theta\,(-1)^n, \\
\cos(n\pi) &= (-1)^n, \\
\frac{\mathrm{d}\triangle\theta}{\mathrm{d}t} &= \triangle\omega, \\
\frac{\mathrm{d}\triangle\omega}{\mathrm{d}t} &= \Omega^2[1 + (-1)^{n+1}a]\triangle\theta.
\end{aligned}$$

So the eigenvalues are $\pm\Omega\sqrt{1 + (-1)^{n+1}a}$. When $a > 1$ *and* n is even the eigenvalues are imaginary and the equilibrium points are *centres*. Otherwise the eigenvalues are real and of different signs so the equilibrium points are *saddle points*.

Linearizing about $\theta^\star = \arccos(a)$ gives

$$\begin{aligned}\frac{\mathrm{d}\triangle\theta}{\mathrm{d}t} &= \triangle\omega,\\ \frac{\mathrm{d}\triangle\omega}{\mathrm{d}t} &= -\Omega^2\sin^2(\theta^\star)\triangle\theta\\ &= -\Omega^2(1-a^2)\triangle\theta.\end{aligned}$$

The eigenvalues are $\pm\mathrm{i}\Omega\sqrt{1-a^2}$. Since these equilibrium points occur only when $a \leq 1$ the eigenvalues are purely imaginary and the equilibrium points are *centres*.

6) $x\dfrac{\mathrm{d}x}{\mathrm{d}t} + y\dfrac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2}\dfrac{\mathrm{d}r^2}{\mathrm{d}t} = r\dfrac{\mathrm{d}r}{\mathrm{d}t}$.
So

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r\{f(r\cos(\theta), r\sin(\theta)) - a^2\}^n,$$

and using

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \cos(\theta)\frac{\mathrm{d}r}{\mathrm{d}t} - r\sin(\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t},$$

gives $\dot\theta(t) = 1$. Linearizing about the origin for r

$$\frac{\mathrm{d}\triangle r}{\mathrm{d}t} = \triangle r\{f(0,0) - a^2\}^n.$$

So the solution is stable or unstable according as $\{f(0,0) - a^2\}^n < 0$ or > 0. Consider now the limit cycle $r = a$.

$$\dot r(t) = r(r^2 - a^2)^n.$$

With $r = a + \triangle r$

$$\frac{\mathrm{d}\triangle r}{\mathrm{d}t} = a(2a\triangle r)^n.$$

If n is odd

$$\begin{aligned}\frac{\mathrm{d}\triangle r}{\mathrm{d}t} &> 0, \qquad \text{when} \qquad \triangle r > 0,\\ \frac{\mathrm{d}\triangle r}{\mathrm{d}t} &< 0, \qquad \text{when} \qquad \triangle r < 0.\end{aligned}$$

So the limit cycle is unstable (in both directions). If n is even $\dfrac{\mathrm{d}\triangle r}{\mathrm{d}t} > 0$ for both signs of $\triangle r$, so the limit cycle is semistable.

7) $z = r\exp(\mathrm{i}\theta)$.

$\dot{z}(t) = \dot{r}(t)\exp(\mathrm{i}\theta) + \mathrm{i}r(t)\dot{\theta}(t)\exp(\mathrm{i}\theta)$.

So

$\dot{r}(t) + \mathrm{i}r(t)\dot{\theta}(t) = \mathrm{i}r(t) + r(t)f(r)$

giving

$\dot{\theta}(t) = 1, \qquad \dot{r}(t) = r(t)f(r).$

Limit cycles are given by

$$\sin\left(\frac{1}{r^2-1}\right) = 0, \quad \text{with solutions } r^\star = \sqrt{1+\frac{1}{n\pi}}, \quad n = \pm 1, \pm 2, \ldots.$$

and $r = 1$. Linearizing about $r^\star$ for the former gives

$$\frac{\mathrm{d}\triangle r}{\mathrm{d}t} = -\frac{2(r^\star)^2\triangle r}{[(r^\star)^2-1]^2}\cos\left(\frac{1}{(r^\star)^2-1}\right) = (-1)^{n+1}\frac{2(r^\star)^2\triangle r}{[(r^\star)^2-1]^2}.$$

So the cycles are stable if n is even and unstable if n is odd. The cycle nearest the origin is $n = -1$ which is unstable. Since $\sin(-1) = -0.84$ the origin is stable.

8) $z = r\exp(\mathrm{i}\theta)$. So

$\dot{z}(t) = \dot{r}(t)\exp(\mathrm{i}\theta) + \mathrm{i}r\dot{\theta}(t)\exp(\mathrm{i}\theta),$

giving

$\dot{r}(t) + \mathrm{i}r\dot{\theta}(t) = a\exp(-\mathrm{i}\theta) + r(b - r^2).$

Taking real and imaginary parts

$$\dot{r}(t) = a\cos(\theta) + r(b-r^2), \qquad \dot{\theta}(t) = -\frac{a\sin(\theta)}{r}.$$

When $a = 0$ This gives

$\dot{r}(t) = r(b-r^2), \qquad \dot{\theta}(t) = 0.$

This is just (with b replacing a) the same as the polar form of Example 2.5.1, yielding the Hopf bifurcation as shown in Fig. 2.5.

When $a \neq 0$ From the second equation $\dot{\theta}(t) = 0$ gives $\theta = 0$ or π. So the equilibrium solutions are

$$r(b - r^2) = -a, \qquad \theta = 0,$$
$$r(b - r^2) = a, \qquad \theta = \pi.$$

Linearizing about the equilibrium solution $(r^\star, \theta^\star)$ gives

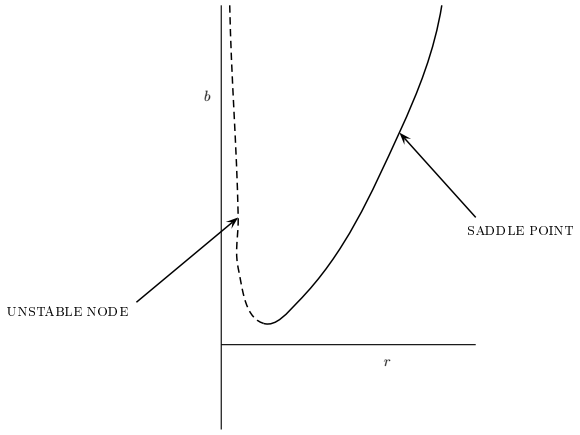
$$\frac{\mathrm{d}\triangle r}{\mathrm{d}t} = \triangle r[b - 3(r^\star)^2], \qquad \frac{\mathrm{d}\triangle\theta}{\mathrm{d}t} = -\frac{\triangle\theta\, a\cos(\theta^\star)}{(r^\star)^2}.$$

The curve in the r–θ plane given by $r(b - r^2) = \pm a$ is

$$b = \pm\frac{a}{r} + r^2.$$

We can now divide the equilibrium solutions into two cases:

- When $a > 0$ and $\theta = \pi$ or $a < 0$ and $\theta = 0$.
  It is clear that this solution is unstable in the θ direction since
  $$-\frac{a\cos(\theta^\star)}{(r^\star)^2} > 0.$$
  The variable $r > 0$ and
  $$b = \frac{|a|}{r} + r^2.$$
  The curve of b as a function of r has a turning point given by
  $$0 = \frac{\mathrm{d}b}{r} = -\frac{|a|}{r^2} + 2r = \frac{r^2 - b}{r + 2r} = \frac{3r^2 - b}{r},$$
  Giving $b = 3r^2$. It follows from the linearized equation for $\triangle r$ that the equilibrium point is stable in the r direction when $r > \sqrt{b/3}$ (giving a *saddle point* when you take into account the instability in the θ direction) and unstable in the r direction when $r < \sqrt{b/3}$ (giving an *unstabler node*. The equilibrium curve is if the form



- When $a < 0$ and $\theta = \pi$ or $a > 0$ and $\theta = 0$.
  It is clear that this solution is stable in the θ direction since

  $$-\frac{a\cos(\theta^\star)}{(r^\star)^2} < 0.$$

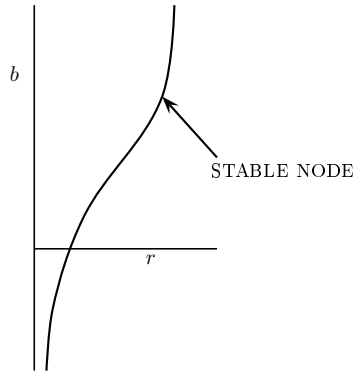
  The variable $r > 0$ and

  $$b = -\frac{|a|}{r} + r^2.$$

  The curve of b as a function of r does not have a turning point and

  $$b - 3(r^\star)^3 = -\left[\frac{|a|}{r} + 2(r^\star)^2\right].$$

  So the equilibrium point is stable in the r direction and is thus a *stable node*. The equilibrium curve is if the form



6.2 Problems 2

1) In this problem we have not been given the equation of motion so we can't deduce the stability. We can, however, if we assume the equation to be of the form[1]

   $$\dot{x}(t) = F(\varepsilon, a, x) = \varepsilon x^2 + x^3 - ax.$$

   When $\varepsilon = 0$ the solutions to $F(0, a, x) = 0$ are $x = 0$ and $a = x^2$. The line of equilibrium points $x = 0$ is stable when $a > 0$ and unstable when $a < 0$. The equilibrium points given by $a = x^2$ are all unstable. So we have a subcritical pitchfork bifurcation. When $\varepsilon \neq 0$ the line of equilibrium

[1] The other possibility is

$$\dot{x}(t) = -F(\varepsilon, a, x) = -\varepsilon x^2 - x^3 + ax,$$

which will simply reverse the stability.

points $x = 0$ remain, with the same stability. The parabola is shifted to $a = \varepsilon x + x^2$, with minimum at $x = -\frac{1}{2}\varepsilon$, $a = -\frac{1}{4}\varepsilon^2$. Now take $x = x^\star + \triangle x$, $a = \varepsilon x^\star + (x^\star)^2$. Then

$$\frac{\mathrm{d}\triangle x}{\mathrm{d}t} = -\triangle x[2x^\star\varepsilon + 3(x^\star)^2 - a] = (2a - \varepsilon x^\star)\triangle x.$$

The line $a = \frac{1}{2}x\varepsilon$ passes through the origin and the minimum of the parabola of equilibrium points. Below the line the equilibrium points on the parabola are stable and above they are unstable. There is a transcritical bifurcation at the origin and a turning point at $x = -\frac{1}{2}\varepsilon$, $a = -\frac{1}{4}\varepsilon^2$. The diagram (with $\varepsilon < 0$) is like Fig. 1.11 with $c = -\frac{1}{2}\varepsilon$ and the stability reversed.

2) Treating the equation as of the form $\dot{x}(t) = F(a, b, c, x)$, the bifurcation set is given by eliminating x between the equations

$$F(a, b, c, x) = x^3 - 2ax^2 - (b - 3)x + c = 0, \qquad (1)$$

$$F_x(a, b, c, x) = 3x^2 - 4ax - (b - 3) = 0. \qquad (2)$$

From (2)

$$x = \tfrac{1}{3}\{2a \pm f(a, b)\}, \quad (3)$$

where

$$f(a, b) = \sqrt{4a^2 + 3(b - 3)}.$$

Subtracting $x\times$ (2) from $3\times$ (1) gives

$$0 = -2ax^2 - 2(b - 3)x + 3c.$$

Eliminating x^2 between this equation and (2) gives

$$x[8a^2 + 6(b - 3)] = 9c - 2a(b - 3).$$

Substituting the values of x given by (3) gives

$$\pm f(a, b)\{8a^2 + 6(b - 3)\} = -16a^3 + 27c - 18a(b - 3).$$

Squaring and substituting for $f(a, b)$ yields

$$\pm\{4a^2 + 3(b - 3)\}\{8a^2 + 6(b - 3)\}^2 = \{16a^3 - 27c + 18a(b - 3)\}^2.$$

When $a = 1$ we have

$$(27c - 18b + 38)^2 = 4(3b - 5)^3.$$

This has a cusp when $3b = 5$, which is at $b = \frac{5}{3}$, $c = -\frac{8}{27}$. Now we can use MAPLE to plot c against b for various values of a.

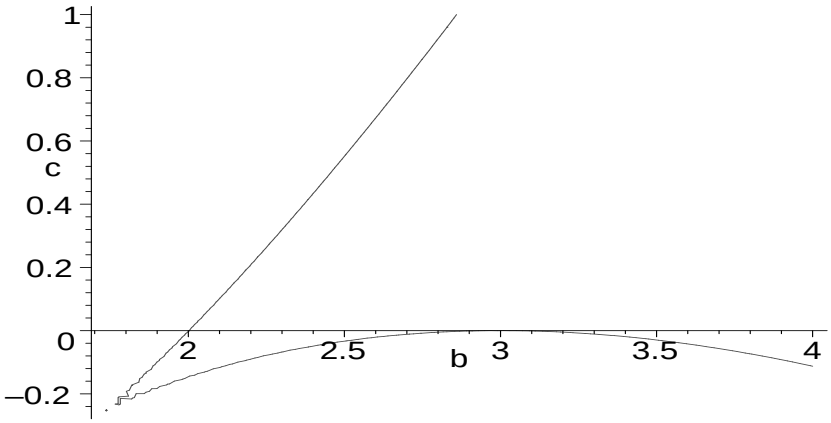
```
> with(plots,implicitplot):
> f:=(a,b)->4*a^2+3*(b-3):
> g:=(a,b)->8*a^2+6*(b-3):
> h:=(a,b,c)->16*a^3-27*c+18*a*(b-3):
> p:=(a,b,c)->f(a,b)*(g(a,b))^2-(h(a,b,c))^2:
> p(a,b,c);
```

$$(4\,a^2+3\,b-9)\,(8\,a^2+6\,b-18)^2-(16\,a^3-27\,c+18\,a\,(b-3))^2$$

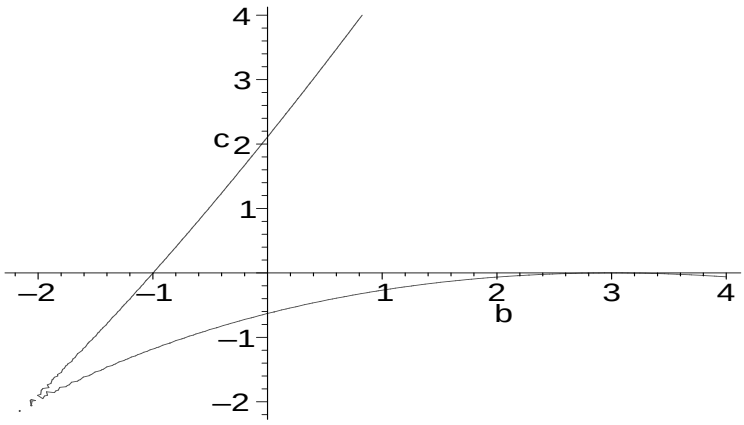
```
> p(1,b,c);
```

$$(-5+3\,b)\,(-10+6\,b)^2-(-38-27\,c+18\,b)^2$$

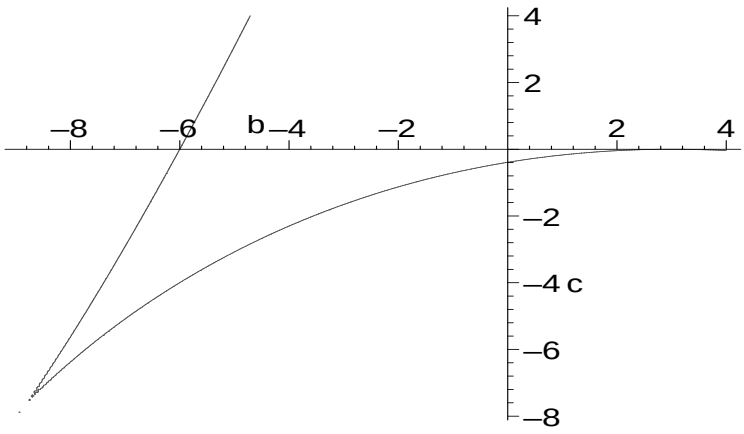
```
> implicitplot(p(1,b,c),b=0..4,c=-1..1,grid=[100,100]);
```



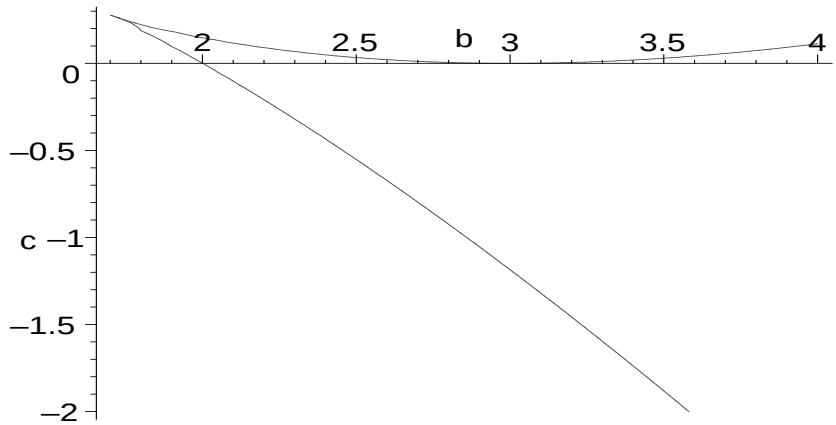

```
> implicitplot(p(2,b,c),b=-6..4,c=-4..4,grid=[100,100]);
```




```
> implicitplot(p(3,b,c),b=-20..4,c=-8..4,grid=[500,100]);
```




```
> implicitplot(p(-1,b,c),b=-1..4,c=-2..4,grid=[100,200]);
```



3) $F(a,b,c,x) = -\dfrac{\partial V}{\partial x} = -x^3 - ax - b \quad (1)$.

The standard form of cubic to produce a pitchfork bifurcation at $x = a = 0$ in the $\{x, a\}$ plane is $x(x^2 + a) = 0$. This can be achieve for (1) in the plane $b = 0$.

The standard form to give transcritical and turning point bifurcations is that given in Example 1.8.2 by the right-hand side of equation (1.61) with $c \neq 0$. It has cubic, quadratic and linear terms but no constant term. We must now transform (1) to this form. Consider

$$\begin{aligned} -(x+\alpha)^3 + 2\beta(x+\alpha)^2 + \gamma(x+\alpha) &= -x^3 + (2\beta - 3\alpha)x^2 \\ &\quad + (4\alpha\beta - 3\alpha^2 + \gamma)x \\ &\quad + (\gamma\alpha + 2\beta\alpha^2 - \alpha^3). \end{aligned}$$

So to eliminate the quadratic term on the right $\beta = 3\alpha/2$ and

$$-x^3 - ax - b = -(x+\alpha)^3 + 3\alpha(x+\alpha)^2 + \gamma(x+\alpha),$$

when α and γ satisfy the relations

$$3\alpha^2 + \gamma = -a$$
$$2\alpha^3 + \gamma\alpha = b.$$

Eliminating γ gives the equation

$$\alpha a - b = -\alpha^3,$$

which, for any number α, is a plane in the $\{x, a, b\}$ space on which transcritical and turning point bifurcations will occur. To locate these bifurcations

$$(x+\alpha)^3 - 3\alpha(x+\alpha)^2 - \gamma(x+\alpha) = (x+\alpha)[x^2 - \alpha x + (\alpha^2 + a)] = 0.$$

The lines of equilibrium points have two branches

$$x = -\alpha$$

and

$$a = -x^2 + \alpha x - \alpha^2.$$

The turning point bifurcation occurs when

$$x = \tfrac{1}{2}\alpha, \qquad a = -\tfrac{3}{4}\alpha^2, \qquad b = \tfrac{1}{4}\alpha^3.$$

The transcritical bifurcation will occur when the two branches cross. That is

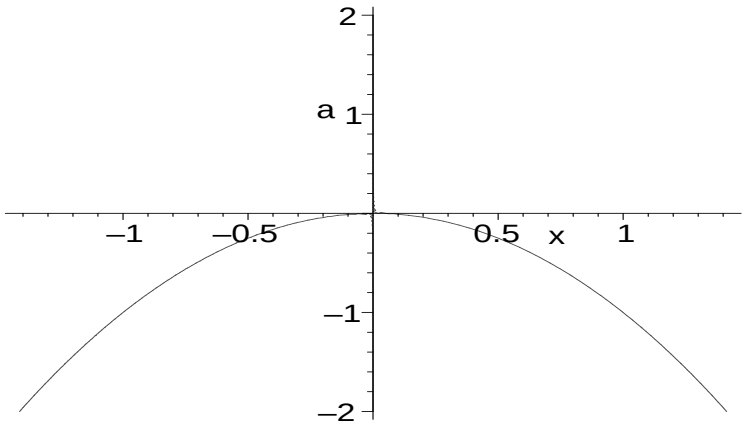
$$x = -\alpha, \qquad a = -3\alpha^2, \qquad b = -2\alpha^3.$$

If for example we choose $\alpha = \frac{2}{3}$, then the plane is

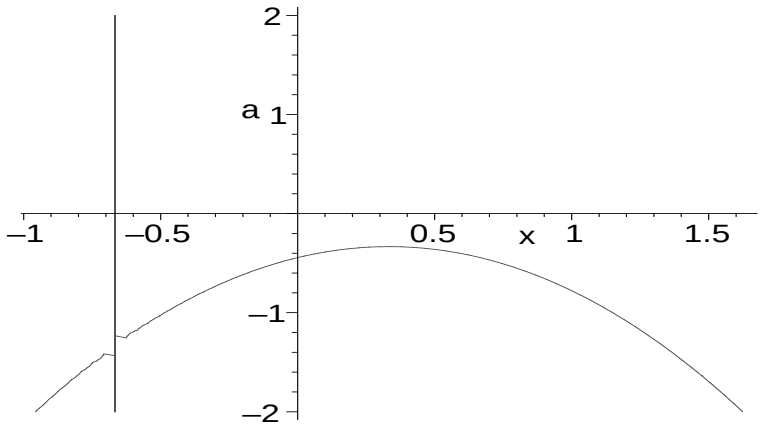
$$18a - 27b + 8 = 0,$$

the turning point occurs at $x = \frac{1}{3}$, $a = -\frac{1}{3}$, $b = \frac{2}{17}$, and the transcritical bifurcation at $x = -\frac{2}{3}$, $a = -\frac{4}{3}$, $b = -\frac{16}{27}$. We can check out results using MAPLE

```
> with(plots,implicitplot):
> F:=(a,b,x)->-x^3-x*a+b:
> implicitplot({F(a,0,x),x
> },x=-2..2,a=-2..2,grid=[100,100]);
```




```
> implicitplot({F(a,-(8+18*a)/27,x),x+2/3},
> x=-2..2,a=-2..2,grid=[100,100]);
```



---

4) For fixed c The equilibrium region is a three-dimensional subspace in the space $\{a, b, c, x, y\}$ which is given by the intersection of

$$0 = -x^2 + y^2 - 2cx + a, \tag{6.4}$$

$$0 \quad = \quad 2xy - 2cy + b. \tag{6.5}$$

The bifurcation set lies in the the equilibrium region and also satisfies the Jacobean condition

$$\begin{vmatrix} -2x - 2c & 2y \\ 2y & 2x - 2c \end{vmatrix} = 0,$$

which is

$$c^2 = x^2 + y^2. \tag{6.6}$$

Now eliminate y between (6.4) and (6.6) to give

$$2x^2 + 2xc - a - c^2 = 0, \tag{6.7}$$

and between (6.5) and (6.6) to give

$$b^2 = 4(x - c)^2(c^2 - x^2).$$

This expands to

$$4x^4 - 8x^3c + 8c^3x + b^2 - 4c^4 = 0. \tag{6.8}$$

Now the hard work starts since, to obtain the bifurcation set x must be eliminated between (6.7) and (6.8). This is most easily done using the Sylvester determinant. The MAPLE program is

```
> with(linalg,det,matrix):
> with(plots,implicitplot):
> S:=(a,b,c)->

> matrix([[2,2*c,-a-c^2,0,0,0],[0,2,2*c,-a-c^2,0,0],[0,0,2,2*c,-a-c^2,0],

> [0,0,0,2,2*c,-a-c^2],[4,-8*c,0,8*c^3,b^2-4*c^4,0],[0,4,-8*c,0,8*c^3,b^2-4*c^4]]):
> S(a,b,c);
```

$$\begin{bmatrix} 2 & 2\,c & -a - c^2 & 0 & 0 & 0 \\ 0 & 2 & 2\,c & -a - c^2 & 0 & 0 \\ 0 & 0 & 2 & 2\,c & -a - c^2 & 0 \\ 0 & 0 & 0 & 2 & 2\,c & -a - c^2 \\ 4 & -8\,c & 0 & 8\,c^3 & b^2 - 4\,c^4 & 0 \\ 0 & 4 & -8\,c & 0 & 8\,c^3 & b^2 - 4\,c^4 \end{bmatrix}$$

```
> s:=(a,b,c)->simplify(det(S(a,b,c))/16):
> s(a,b,c);
```

$$18\,a^2\,c^4 - 8\,c^2\,a^3 - 27\,c^8 + b^4 + 18\,b^2\,c^4 + 2\,b^2\,a^2 + 24\,b^2\,a\,c^2 + a^4$$

```
> # This is different from the answer given
> so
> we check for equivalence.
> g:=(a,b,c)->27*c^8-18*c^4*(a^2+b^2)
> +8*c^2*a*(a^2-3*b^2)-(a^2+b^2)^2;
```

$$g := (a,\,b,\,c) \to 27\,c^8 - 18\,c^4\,(a^2 + b^2) + 8\,c^2\,a\,(a^2 - 3\,b^2) - (a^2 + b^2)^2$$

```
> simplify(s(a,b,c)+g(a,b,c));
```

$$0$$

```
> # So they are the same.
> # Now we translate into polars.
> spolar:=(r,theta,c)->simplify(g(r*cos(theta),r*sin(theta),c)):
> spolar(r,theta,c);
```

$$27\,c^8 - 18\,c^4\,r^2 + 32\,c^2\,r^3\cos(\theta)^3 - 24\,c^2\,r^3\cos(\theta) - r^4$$

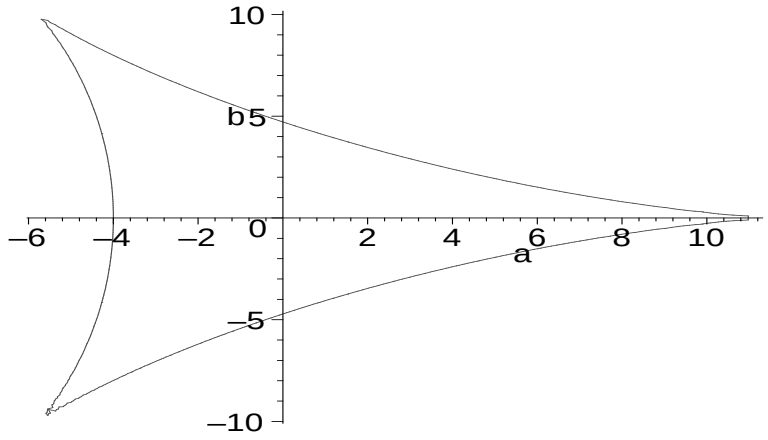
```
> # We again check for equivalence.
> h:=(r,theta,c)->(3*c^2-r)^3*(r+c^2)
> +8*c^2*r^3*(cos(3*theta)-1);
```

$$h := (r,\,\theta,\,c) \to (3\,c^2 - r)^3\,(r + c^2) + 8\,c^2\,r^3\,(\cos(3\,\theta) - 1)$$

```
> expand(spolar(r,theta,c)-h(r,theta,c),trig);
```

$$0$$

```
> implicitplot(s(a,b,2)=0,a=-7..11,b=-10..10,grid=[100,100]);
```



It is clear that

$$(r+c^2)(3c^2-r)^3+8c^2r^3\{\cos(3\theta)-1\}=0 \tag{6.9}$$

has rotational symmetry with period $\theta=\frac{2\pi}{3}$ and that when $\theta=0,\frac{2\pi}{3},\frac{4\pi}{3}$, $r=3c^2$. To see that these point are cusps we take $r=3c^2+\triangle r$ and $\theta=\frac{2n\pi}{3}+\triangle\theta$. Equation (6.9) then gives

$$(\triangle r)^3+(27c^2)[3c^2\triangle\theta]^2=0.$$

This is the standard form for a cusp in the local variables $(\triangle r, 3c^2\triangle\theta)$.

5) The equilibrium points are given by

$$z(x-a)=-c, \qquad y=-x, \qquad z=x. \tag{6.10}$$

So the x coordinates of the equilibrium points are given by real roots of

$$x(x-a)+c=x^2-ax+c=\left(x-\tfrac{1}{2}a\right)^2-\left(\tfrac{1}{4}a^2-c\right)=0,$$

which exist only when $a^2\geq 4c$. The bifurcation set, if it exists, is given by a single equation relating a and c and is obtained by eliminating x, y and z between (6.10) and

$$\begin{vmatrix} 0 & -1 & -1 \\ 1 & 1 & 0 \\ z & 0 & (x-a) \end{vmatrix} = x+z-a=0. \tag{6.11}$$

This gives $x = z = \frac{1}{2}a$, $y = -\frac{1}{2}a$, $a^2 = 4c$. It follows that a bifurcation can occur only when $c \geq 0$. When $c = 0$ there are two lines of equilibrium points in the x–a plane $x = a$, $x = 0$, with a *transcritical bifurcation* at $x = a = 0$; when $c > 0$ the equilibrium curves in the x–a plane are given by

$$a = x + \frac{c}{x}.$$

This is hyperbola with *turning point bifurcations* at $a = \pm 2\sqrt{c}$, $x = \pm\sqrt{c}$.

6.3 Problems 3

1) $\boldsymbol{\nabla}\mathcal{L} = (nx^{n-1}, \alpha m y^{m-1})$. So

   (i) $\boldsymbol{F}.\boldsymbol{\nabla}\mathcal{L} = -nx^n - 2nx^{n-1}y^2 + \alpha m x y^m - \alpha m y^{m+2}$.
   The aim is to make sure that this expression is negative for all signs of x and y. This means eliminating odd degree terms. So we must get rid of the third term and the only way to do it is by arranging that it cancels with the second term. So $n = m = \alpha = 2$ and

   $\boldsymbol{F}.\boldsymbol{\nabla}\mathcal{L} = -2x^2 - 4y^4 < 0$.

   $\mathcal{L}(0,0) = 0$ and $\mathcal{L}(x,y)$ has a minimum at $(0,0)$, so the equilibrium point is *asymptotically stable*.

   (ii) $\boldsymbol{F}.\boldsymbol{\nabla}\mathcal{L} = nx^{n-1}y - nx^{n+2} - \alpha m y^{m-1}x^3$.
   Now we arrange for the first and third terms to cancel by taking $n = 4$, $m = \alpha = 2$. This gives

   $\boldsymbol{F}.\boldsymbol{\nabla}\mathcal{L} = -4x^6 < 0$.

   $\mathcal{L}(0,0) = 0$ and $\mathcal{L}(x,y)$ has a minimum at $(0,0)$, so the equilibrium point is *asymptotically stable*.

2) For the equilibrium points; from the second equation $x^3 = y^3$ giving $x = y$ and then from the first equation $x = y = 0$.

   $\boldsymbol{\nabla}\mathcal{L} = (2x + \alpha y, 2\beta y + \alpha x)$.

   So

   $\boldsymbol{F}.\boldsymbol{\nabla}\mathcal{L} = (2-\alpha)x^4 + 2\beta y^4 + (\alpha - 2)x^2y^2 + (2 + \alpha - 2\beta)x^3 y$.

   For the first three terms to be of only one sign we must take $\alpha = 2$ and we must also eliminate the last term; so $\beta = 2$. Thus

   $\boldsymbol{F}.\boldsymbol{\nabla}\mathcal{L} = 4y^4 > 0$.

   Also $\mathcal{L}(0,0) = 0$ and $\mathcal{L}(x,y) = x^2 + 2xy + 2y^2$ does not change sign in a neighbourhood of $(0,0)$, since $x^2 + 2xy + 2y^2 = 0$ has no real roots. It is therefore always positive and thus $\mathcal{L}(x,y)$ has a minimum at $(0,0)$. So the equilibrium point is *unstable*.

3) $\dot{x}(t) = y(t), \qquad \dot{y}(t) = x(t)\{a|x(t)| - 1\}$

and

$$x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t} = ax|x|\frac{\mathrm{d}x}{\mathrm{d}t}.$$

Integrating

$$\begin{aligned} \tfrac{1}{2}\{x^2 + y^2\} &= E + \begin{cases} \tfrac{1}{3}ax^3, & \text{if } x > 0, \\ -\tfrac{1}{3}ax^3, & \text{if } x < 0, \end{cases} \\ \tfrac{1}{2}\{x^2 + y^2\} - \tfrac{1}{3}a|x|^3 &= E. \end{aligned}$$

The equilibrium points are $x = y = 0$, for all a, $x = \pm 1/a$, $y = 0$, for $a > 0$.

Linearizing about $x = y = 0$,

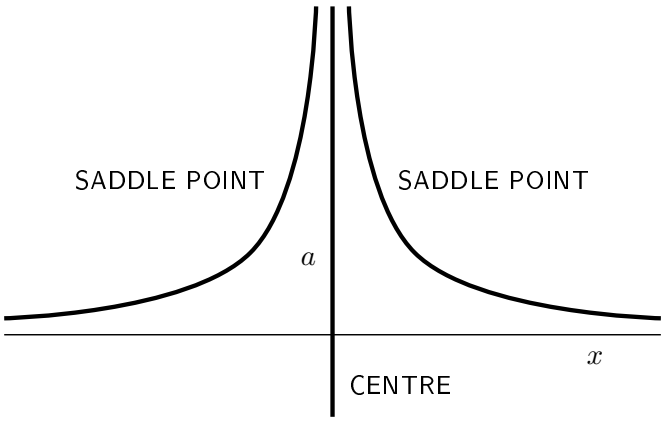
$$\boldsymbol{J}^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which has eigenvalues $\pm$i. So the origin is a *centre*.

Linearizing about $x = \pm 1/a$, $y = 0$, when $a > 0$,

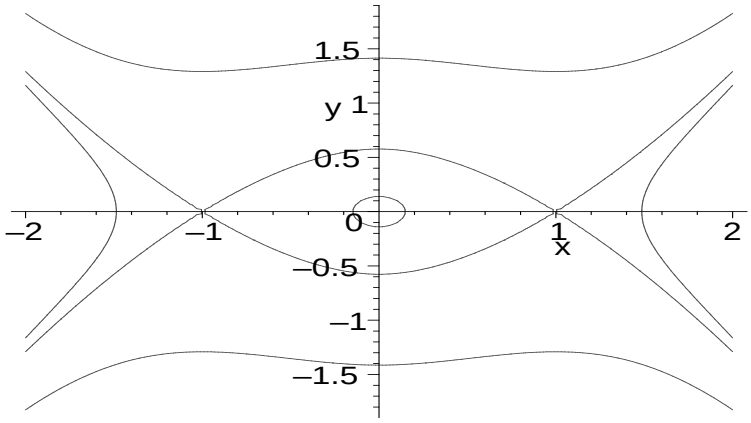
$$\boldsymbol{J}^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

in both cases with eigenvalues ± 1. So the each of these equilibrium points is a *saddle point*.

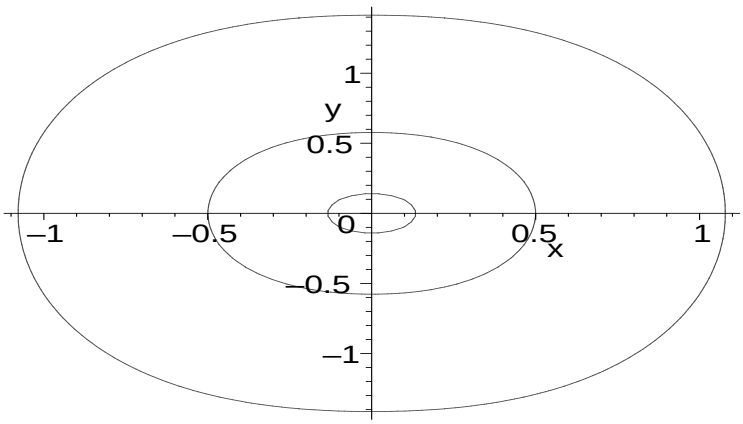


Curves are divided, by the *separatrix*, between closed curves about the centre and open curves with two branches. Since the separatrix passes through both branch-points, its value of E is given by substituting $x = \pm 1/a$, $y = 0$ into $(*)$. This gives $E = 1/(6a^2)$. Curves with $E \leq 1/(6A^2)$ cut the x–axis and for each value of a consist of two open branches and a closed loop. Curves For $E > 1/(6A^2)$ cut the y–axis and for each value of a consist of two open branches.

```
> with(plots):
> f:=(x,y,a,En)->
> x^2/2+y^2/2-a*abs(x^3)/3-En:
> # Try the case a=1, with E=1/100,1/6,1:
> implicitplot(
> {f(x,y,1,1/100),f(x,y,1,1/6),f(x,y,1,1)},
> x=-2..2,y=-2..2,grid=[100,100]);
```




```
> # Try the case a=-1, with E=1/100,1/6,1:
> implicitplot(
> {f(x,y,-1,1/100),f(x,y,-1,1/6),f(x,y,-1,1)},
> x=-2..2,y=-2..2,grid=[100,100]);
```



Periodic solutions (closed curves) exist for all a. Let ζ be the smallest positive root of

$$\tfrac{1}{2}x^2 - \tfrac{1}{3}ax^3 = E.$$

Then

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \sqrt{\tfrac{2}{3}ax^3 + 2E - x^2} \\ &= \sqrt{\zeta^2 - \tfrac{2}{3}a\zeta^3 - x^2 + \tfrac{2}{3}ax^3}.\end{aligned}$$

Integrating over $[-\zeta, \zeta]$ gives $T/2$ and thus the required result.

4) $\dot{x}(t) = y(t), \qquad \dot{y}(t) = -x(t) - bx^3(t) - 2ay(t).$

The equilibrium points are on $y = 0$ and given by

$$x(1 + bx^2) = 0.$$

So they are $x = y = 0$, for all values of the parameters, and $x = \pm 1/\sqrt{-b}$, $y = 0$, when $b < 0$.

Linearizing about $x = y = 0$**,**

$$\boldsymbol{J}^* = \begin{pmatrix} 0 & 1 \\ -1 & -2a \end{pmatrix},$$

which has eigenvalues $-a \pm \sqrt{a^2 - 1}$. So the origin is

- A *stable proper node* if $a > 1$.
- A *stable inflected node* if $a = 1$.
- A *stable focus* if $0 < a < 1$.

Linearizing about $x = \pm 1/\sqrt{-b}$**,** $y = 0$**,**

$$\boldsymbol{J}^* = \begin{pmatrix} 0 & 1 \\ 2 & -2a \end{pmatrix},$$

which has eigenvalues $a \pm \sqrt{a^2 + 2}$. So the equilibrium points are *saddle points.*

Theorem 1.12.1, on page 26, tells us that for $a > 0$ the origin is an asymptotically stable equilibrium point. This can also be established using the Lyapunov direct method. With the given form,

$$\mathcal{L}(x, y) = \tfrac{1}{2}[x^2 + y^2] + \tfrac{1}{4}bx^4$$

for the Lyapunov function,

$$\begin{aligned} \boldsymbol{\nabla}\mathcal{L} &= (x + bx^3, y), \\ \boldsymbol{F}.\boldsymbol{\nabla}\mathcal{L} &= -2ay^2 < 0. \end{aligned}$$

So the origin is an asymptotically stable equilibrium point with $x(t) \to 0$ as $t \to \infty$.

5) The general solution to the equations is

$$x(t) = \mathsf{A}\cos(t) + \mathsf{B}\sin(t), \qquad y(t) = -\mathsf{A}\sin(t) + \mathsf{B}\cos(t).$$

Denote the given periodic solution by

$$\mathring{x}(t) = a\cos(t), \qquad \mathring{y}(t) = -a\sin(t).$$

To show that this is stable we must show that, given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, if

$$\{x(0) - \mathring{x}(0)\}^2 + \{y(0) - \mathring{y}(0)\}^2 = (\mathsf{A} - a)^2 + \mathsf{B}^2 < [\delta(\varepsilon)]^2,$$

then if

$$\begin{aligned}\{x(t)-\mathring{x}(t)\}^2+\{y(t)-\mathring{y}(t)\}^2 &= [(\mathsf{A}-a)\cos(t)+\mathsf{B}\sin(t)]^2 \\ &\quad +[(a-\mathsf{A})\sin(t)+\mathsf{B}\cos(t)]^2<\varepsilon^2.\end{aligned}$$

Expanding this expression gives

$$(\mathsf{A}-a)^2+\mathsf{B}^2<\varepsilon^2.$$

So we just choose $\delta=\varepsilon$.

6) With $y(t)$ defined as $\dot{x}(t)$ we have

$$\dot{x}(t)=y(t), \qquad \dot{y}(t)=by(t)[a-x^2(t)-y^2(t)]-x(t).$$

Then

$$\begin{aligned}r\frac{\mathrm{d}r}{\mathrm{d}t} &= x\frac{\mathrm{d}x}{\mathrm{d}t}+y\frac{\mathrm{d}y}{\mathrm{d}t}=by^2[a-x^2-y^2] \\ &= br^2\sin^2(\theta)[a-r^2].\end{aligned}$$

So

$$\dot{r}=br\sin^2(\theta)(a-r^2).$$

Since $y(t)=\dot{x}(t)$,

$$r\sin(\theta)=\frac{\mathrm{d}[r\cos(\theta)]}{\mathrm{d}t}=\dot{r}\cos(\theta)-\dot{\theta}\,r\sin(\theta).$$

Substituting for $\dot{r}$ gives

$$\dot{\theta}\,r\sin(\theta)=br\sin^2(\theta)\cos(\theta)[a-r^2]-r\sin(\theta).$$

So

$$\dot{\theta}=\tfrac{1}{2}b\sin(2\theta)[a-r^2]-1.$$

Substituting $r=\sqrt{a}$ into the expressions for $\dot{r}$ and $\dot{\theta}$ we have $\dot{r}=0$ and $\dot{\theta}=-1$. So we have a periodic solution

$$\mathring{x}(t)=\sqrt{a}\cos(t_0-t), \qquad \mathring{y}(t)=\sqrt{a}\sin(t_0-t),$$

of period 2π. Now let

$$\triangle x(t)=x(t)-\mathring{x}(t), \qquad \triangle y(t)=y(t)-\mathring{y}(t).$$

Giving

$$\begin{aligned}\frac{\mathrm{d}\triangle x}{\mathrm{d}t} &= \dot{x} - \sqrt{a}\sin(t_0 - t) \\ &= y - \mathring{y} \\ &= \triangle y(t).\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\triangle y}{\mathrm{d}t} &= \dot{y}(t) + \sqrt{a}\cos(t_0 - t) \\ &= by(a - x^2 - y^2) - x - \mathring{x} \\ &= \triangle x + b(\mathring{y} + \triangle y)(a - \mathring{x}^2 - \mathring{y}^2 - 2\mathring{x}\triangle x - 2\mathring{y}\triangle y) \\ &= -\triangle x(1 + 2b\mathring{x}\mathring{y}] - 2b\mathring{y}^2\triangle y \\ &= -\triangle x(t)[1 + ab\sin(2t_0 - 2t)] - 2\triangle y(t)[ab\sin^2(t_0 - t)],\end{aligned}$$

from which

$\mathrm{Trace}\{\mathring{\boldsymbol{J}}(t)\} = -2ab\sin^2(t_0 - t)$.

So the sum of the *Floquet exponents* is

$$\sigma^{(1)} + \sigma^{(2)} = -\frac{1}{2\pi}\int_0^{2\pi} 2ab\sin^2(t_0 - t)\mathrm{d}t = -ab.$$

Substituting $r = \triangle r + \sqrt{a}$ into the differential equation for r, with $\theta = t_0 - t$ we obtain the given equation. For $b > 0$, if $\triangle r > 0$ then

$$\frac{\mathrm{d}\triangle r}{\mathrm{d}t} \leq 0, \qquad \text{over the whole period.}$$

If $\triangle r < 0$ then

$$\frac{\mathrm{d}\triangle r}{\mathrm{d}t} > 0, \qquad \text{if } |\triangle r| < \sqrt{a}.$$

So in either case Lyapunov stability is established for some $\varepsilon > 0$ by choosing δ to be the smaller of ε and $\sqrt{a}$.

7) (i) The equilibrium points are solutions of

$$0 = x + y - x(x^2 + 2y^2), \tag{6.12}$$

$$0 = -x + y - y(x^2 + 2y^2). \tag{6.13}$$

Multiplying (6.12) by y and (6.13) by x and subtracting gives $x^2+y^2 = 0$. So the only equilibrium point is $x = y = 0$. The stability matrix is

$$\boldsymbol{J}^* = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

with eigenvalues $\lambda^{(\pm)} = 1 \pm \mathrm{i}$. So the origin is an *unstable focus*. Now

$$\begin{aligned} r\frac{\mathrm{d}r}{\mathrm{d}t} &= x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t}, \\ &= x^2 + y^2 - (x^2+y^2)(x^2+2y^2). \end{aligned}$$

So

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r - r^3(1+\sin^2(\theta)). \tag{6.14}$$

Also

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \cos(\theta)\frac{\mathrm{d}r}{\mathrm{d}t} - r\sin(\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t} \\ &= r\cos(\theta) + r\sin(\theta) - r\cos(\theta)\{r^2 + r^2\sin^2(\theta)\}, \end{aligned}$$

giving

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -1. \tag{6.15}$$

Equation (6.14) can be expressed in the form

$$\dot{r}(t) = -r(r^2-1) - r^3\sin^2(\theta).$$

So on the circle $r = 1+\delta$ for any $\delta > 0$

$$\dot{r}(t) < 0.$$

Equation (6.14) can also be expressed in the form

$$\dot{r}(t) = 2r\left(\tfrac{1}{2} - r^2\right) + r^3(1-\sin^2(\theta)). \tag{6.16}$$

So on the circle $r = 1/\sqrt{2} - \delta$ for any $1/\sqrt{2} > \delta > 0$

$$\frac{\mathrm{d}r}{\mathrm{d}t} > 0.$$

So the annulus

$$\frac{1}{\sqrt{2}} - \delta \leq r \leq 1 + \delta$$

satisfies the result of the first part of the question and must contain either an equilibrium point or a periodic solution. Since the origin is the only equilibrium point it must contain a periodic solution.

Since, from the Poincaré-Bendixson theorem, the trajectory tends to the periodic solution as $t \to \infty$ it must be stable. Alternatively denote

the periodic solution of (6.15) and (6.16) by $r = \mathring{r}(t)$ and substitute $r = \mathring{r}(t) + \triangle r$ into (6.16) and linearize to give

$$\frac{\mathrm{d}\triangle r}{\mathrm{d}t} = -\{3\mathring{r}^2[1+\sin^2(t)] - 1\}\triangle r.$$

But

$$3\mathring{r}^2[1+\sin^2(t)] - 1 > 3\left(\frac{1}{\sqrt{2}} - \delta\right)^2 [1+\sin^2(t)] - 1 > 0,$$

for sufficiently small δ. So the periodic solution is stable.

(ii) The equilibrium points are solutions of

$$0 = -x - y + x(x^2 + 2y^2), \tag{6.17}$$

$$0 = x - y + y(x^2 + 2y^2). \tag{6.18}$$

Multiplying (6.17) by y and (6.18) by x and subtracting gives $x^2 + y^2 = 0$. So the only equilibrium point is $x = y = 0$. The stability matrix is

$$\boldsymbol{J}^* = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix},$$

with eigenvalues $\lambda^{(\pm)} = -1 \pm \mathrm{i}$. So the origin is a *stable focus*.

AT THIS POINT YOU SHOULD REALIZE THAT THIS IS AN APPLICATION OF THE POINCARÉ-BENDIXSON THEOREM IN THE REVERSE TIME DIRECTION.

$$\begin{aligned} r\frac{\mathrm{d}r}{\mathrm{d}t} &= x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t}, \\ &= -x^2 - y^2 + (x^2+y^2)(x^2+2y^2). \end{aligned}$$

So

$$\dot{r}(t) = -r + r^3(1+\sin^2(\theta)). \tag{6.19}$$

Also

$$\begin{aligned} \cos(\theta)\frac{\mathrm{d}r}{\mathrm{d}t} - r\sin(\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t} &= -r\cos(\theta) - r\sin(\theta) \\ &\quad + r\cos(\theta)\{r^2 + r^2\sin^2(\theta)\}, \end{aligned}$$

giving

$$\dot{\theta}(t) = 1. \tag{6.20}$$

Equation (6.19) can be expressed in the form

$$\dot{r}(t) = r(r^2 - 1) + r^3\sin^2(\theta).$$

So on the circle $r = 1 + \delta$ for any $\delta > 0$

$\dot{r}(t) > 0.$

Equation (6.19) can also be expressed in the form

$$= 2r(r^2 - \frac{1}{2}) + r^3(\sin^2(\theta) - 1). \tag{6.21}$$

So on the circle $r = 1/\sqrt{2} - \delta$ for any $1/\sqrt{2} > \delta > 0$

$\dot{r}(t) < 0.$

So the annulus

$$\frac{1}{\sqrt{2}} - \delta \leq r \leq 1 + \delta$$

satisfies the result of the first part of the question and must contain either an equilibrium point or a periodic solution. Since the origin is the only equilibrium point it must contain a periodic solution.

Since, from the reverse Poincaré-Bendixson theorem, the reverse trajectory tends to the periodic solution as $t \to -\infty$ it must be unstable. Alternatively denote the periodic solution of (6.20) and (6.21) by $r = \mathring{r}(t)$ and substitute $r = \mathring{r}(t) + \triangle r$ into (6.21) and linearize to give

$$\frac{\mathrm{d}\triangle r}{\mathrm{d}t} = \{3\mathring{r}^2[1 + \sin^2(t)] - 1\}\triangle r.$$

But

$$3\mathring{r}^2[1 + \sin^2(t)] - 1 > 3\left(\frac{1}{\sqrt{2}} - \delta\right)^2[1 + \sin^2(t)] - 1 > 0,$$

for sufficiently small δ. So the periodic solution is unstable.

6.4 Problems 4

1) With $y(t)$ denoting $\dot{x}(t)$

$$\dot{x}(t) = y(t), \qquad \dot{y}(t) = -x(t)[1 - \varepsilon x(t)].$$

The equilibrium points are $x = y = 0$ and $x = 1/\varepsilon$, $y = 0$.

Linearizing about $x = y = 0$ the stability matrix is

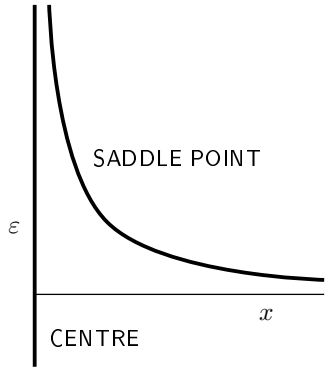
$$\boldsymbol{J}^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with eigenvalues $\pm$i. So the origin is a *centre*.

Linearizing about $x = 1/\varepsilon$, $y = 0$ the stability matrix is

$$\boldsymbol{J}^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with eigenvalues ± 1. So this is a *saddlepoint.*



Although you are not asked to do this it is of interest to find a first integral and plot curves in the x–y plane.

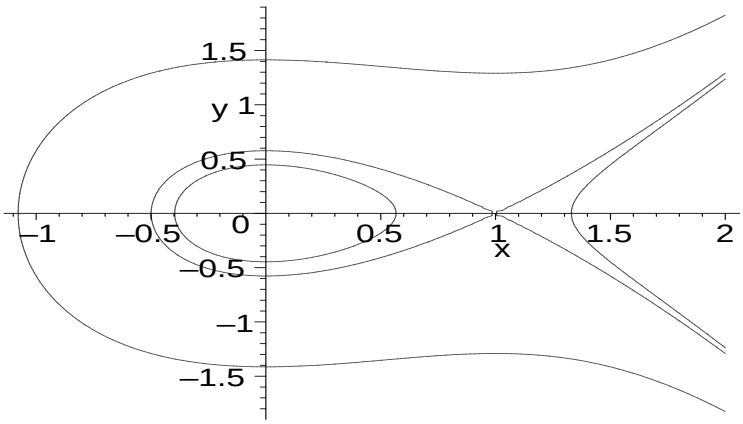
$$x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t} = \varepsilon x^2\frac{\mathrm{d}x}{\mathrm{d}t},$$

with the integral

$$\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 - \tfrac{1}{3}\varepsilon x^3 = E.$$

For a particular ε the curve passes through the saddle point at $x = 1/\varepsilon$, $y = 0$, giving the separatrix with a homoclinic point, when $E = 1/6\varepsilon^2$. We compute the curves for $\varepsilon = 1$ and $E = \frac{1}{10}, \frac{1}{6}, 1$.

```
> with(plots):
> f:=(x,y,epsilon,En)->
> x^2/2+y^2/2-epsilon*x^3/3-En:
> # Try the case epsilon=1, with
> E=1/10,1/6,1:
> implicitplot(
> {f(x,y,1,1/10),f(x,y,1,1/6),f(x,y,1,1)},

> x=-2..2,y=-2..2,grid=[100,100]);
```



(a) Let $\tau = \omega(\varepsilon)t$. Then the equation becomes

$$\omega^2(\varepsilon)\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + x - \varepsilon x^2 = 0.$$

Let

$$x(\varepsilon,\tau) = x_0(\tau) + \varepsilon x_1(\tau) + \mathrm{O}(\varepsilon^2),$$

$$\omega(\varepsilon) = 1 + \omega_1\varepsilon + \mathrm{O}(\varepsilon^2).$$

The ε^0 contribution to the equation is

$$\frac{\mathrm{d}^2x_0}{\mathrm{d}\tau^2} + x_0 = 0,$$

with solution

$$x_0(\tau) = \mathrm{A}_0\cos(\tau) + \mathrm{B}_0\sin(\tau).$$

Since this contribution contains all the $\mathrm{O}(\varepsilon^0)$ part of the solution it follows from the $t = 0$ initial conditions that $\mathrm{B}_0 = 0$ and $\mathrm{A}_0 = a_0$. The ε^1 contribution to the equation is

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2} + 2\omega_1\frac{\mathrm{d}^2x_0}{\mathrm{d}\tau^2} + x_1 - x_0^2 = 0.$$

Substituting for x_0 gives

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2} + x_1 = \tfrac{1}{2}a_0^2[1 + \cos(2\tau)] + 2\omega_1 a_0\cos(\tau).$$

Suppose the solution is of the form

$x_1(\tau) = \mathsf{A}_1 \cos(\tau) + \mathsf{B}_1 \sin(\tau) + X(\tau).$

Then

$X''(\tau) + X(\tau) = \frac{1}{2}a_0^2[1 + \cos(2\tau)] + 2\omega_1 a_0 \cos(\tau).$

A particular solution to this equation is

$X(t) = \frac{1}{2}a_0\omega_1\tau[\cos(\tau) + 2\sin(\tau)] + \frac{1}{2}a_0^2 - \frac{1}{6}a_0^2\cos(2\tau).$

We are interested in finding the periodic contribution. But the first pair of terms involves $\tau\cos(\tau)$ and $\tau\sin(\tau)$, which are not periodic.[2] So to ensure that the solution is periodic we must take $\omega_1 = 0$ so that

$\omega = 1 + \mathrm{O}(\varepsilon^2).$

Also from the initial conditions it follows that $\mathsf{B}_1 = 0$ and

$a_1 = \mathsf{A}_1 + \frac{1}{3}a_0^2.$

Thus we have

$x(\varepsilon, t) = a_0\cos(t) + \varepsilon\left\{\left[a_1 - \tfrac{1}{3}a_0^2\right]\cos(t) + \tfrac{1}{2}a_0^2 - \tfrac{1}{6}a_0^2\cos(2t)\right\}.$

(b) Let

$x(\varepsilon, t) = r\cos(\theta) + \varepsilon u^{(1)}(r, \theta) + \mathrm{O}(\varepsilon^2),$

with

$$\int_0^{2\pi} u^{(1)}(r,\theta)\cos(\theta)\mathrm{d}\theta = \int_0^{2\pi} u^{(1)}(r,\theta)\sin(\theta)\mathrm{d}\theta = 0$$

and

$\dot{r}(t) = \varepsilon A^{(1)}(r) + \mathrm{O}(\varepsilon^2),$

$\dot{\theta}(t) = -1 + \varepsilon B^{(1)}(r) + \mathrm{O}(\varepsilon^2).$

Then

$$\begin{aligned}
\dot{x}(t) &= \dot{r}(t)\cos(\theta) - r\sin(\theta)\dot{\theta}(t) + \varepsilon\left\{\frac{\partial u^{(1)}}{\partial\theta}\dot{\theta}(t) + \frac{\partial u^{(1)}}{\partial r}\dot{r}(t)\right\} \\
\ddot{x}(t) &= \ddot{r}(t)\cos(\theta) - 2\dot{r}(t)\dot{\theta}(t)\sin(\theta) - r\cos(\theta)[\dot{\theta}(t)]^2 - r\sin(\theta)\ddot{\theta}(t) \\
&\quad + \varepsilon\left\{\frac{\partial^2 u^{(1)}}{\partial\theta^2}[\dot{\theta}(t)]^2 + \frac{\partial u^{(1)}}{\partial\theta}\ddot{\theta}(t) + 2\frac{\partial^2 u^{(1)}}{\partial r\partial\theta}\dot{\theta}(t)\dot{r}(t)\right. \\
&\quad \left. + \frac{\partial^2 u^{(1)}}{\partial r^2}[\dot{r}(t)]^2 + \frac{\partial u^{(1)}}{\partial r}\ddot{r}(t).\right\}
\end{aligned}$$

[2] They are called *secular terms*.

The first line of terms contain both $\mathrm{O}(\varepsilon^0)$ and $\mathrm{O}(\varepsilon^1)$ contributions. When we substitute into the equation the $\mathrm{O}(\varepsilon^0)$ contributions cancel and the $\mathrm{O}(\varepsilon^1)$ terms give

$$2\sin(\theta)A^{(1)}(r) + 2r\cos(\theta)B^{(1)}(r) + \frac{\partial^2 u^{(1)}}{\partial\theta^2} + u^{(1)} = r^2\cos^2(\theta).$$

Now

$$A^{(1)}(r) = \frac{1}{2\pi}\int_0^{2\pi} r^2\cos^2(\theta)\sin(\theta)\mathrm{d}\theta = 0,$$

$$B^{(1)}(r) = \frac{1}{2\pi}\int_0^{2\pi} r\cos^3(\theta)\mathrm{d}\theta = 0.$$

So

$$\frac{\partial^2 u^{(1)}}{\partial\theta^2} + u^{(1)} = r^2\cos^2(\theta),$$

which has the solution

$$u^{(1)}(r,\theta) = \mathrm{A}_1\cos(t) + \mathrm{B}_1\sin(t) + \tfrac{1}{6}r^2[3-\cos(2\theta)].$$

Since

$$\dot{r}(t) = \mathrm{O}(\varepsilon^2), \qquad r = \mathrm{A}_0 + \mathrm{O}(\varepsilon^2)$$

$$\dot{\theta}(t) = -1 + \mathrm{O}(\varepsilon^2), \qquad \theta = \mathrm{C}_0 - t + \mathrm{O}(\varepsilon^2)$$

To satisfy the initial conditions $\mathrm{B}_1 - \mathrm{C}_0 = 0$ and

$$x(\varepsilon,t) = \mathrm{A}_0\cos(t) + \varepsilon\left\{\mathrm{A}_1\cos(t) + \tfrac{1}{6}\mathrm{A}_0^2[3-\cos(2t)]\right\}.$$

To satisfy the initial conditions we must now choose $\mathrm{A}_0 = a_0$ and $\mathrm{A}_1 = a_1 - \frac{1}{3}a_0^2$.

2) As was defined in Sect. 4.7.1 the *synchronous* contribution to the solution to an equation with a forcing term is that part with the same frequency as the forcing term. If we are concerned only with this contribution we can neglect perturbations in the frequency. Let

$$x(\varepsilon,t) = x_0(t) + \varepsilon x_1(t).$$

The $\mathrm{O}(\varepsilon^0)$ contribution satisfies

$$\frac{\mathrm{d}^2x_0}{\mathrm{d}t^2} + \omega_0^2 x_0 = \Gamma\cos(\Omega t).$$

The complementary function will not be synchronous since its frequency is ω_0 not Ω. So we just take the particular integral

$$x_0(t) = \begin{cases} \dfrac{\Gamma\cos(\Omega t)}{\omega_0^2 - \Omega^2}, & \Omega \neq \omega_0, \\ \dfrac{\Gamma t\sin(\Omega t)}{2\Omega}, & \Omega = \omega_0. \end{cases}$$

At the excluded value $\Omega = \omega_0$ there is a *resonance* with amplitude growing with t. The $\mathrm{O}(\varepsilon^1)$ contribution satisfies

$$\begin{aligned}\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + \omega_0^2 x_1 &= \omega_0^2 x_0^4 \\ &= \frac{\omega_0^2 \Gamma^4 \cos^4(\Omega t)}{(\omega_0^2 - \Omega^2)^4}.\end{aligned}$$

Since

$$\cos^4(\Omega t) = \tfrac{1}{8}\{\cos(4\Omega t) + 4\cos(2\Omega t) + 3\},$$

$\omega_0 = 2\Omega$ and 4Ω will also give resonances. Excluding these values the particular integral is

$$x_1(t) = \frac{\omega_0^2 \Gamma^4}{8(\omega_0^2 - \Omega^2)^4}\left\{\frac{\cos(4\Omega t)}{\omega_0^2 - 16\Omega^2} + \frac{4\cos(2\Omega t)}{\omega_0^2 - 4\Omega^2} + \frac{3}{\omega_0^2}\right\},$$

and

$$x(t) = \frac{\Gamma\cos(\Omega t)}{\omega_0^2 - \Omega^2} + \frac{\varepsilon\omega_0^2\Gamma^4}{8(\omega_0^2 - \Omega^2)^4}\left\{\frac{\cos(4\Omega t)}{\omega_0^2 - 16\Omega^2} + \frac{4\cos(2\Omega t)}{\omega_0^2 - 4\Omega^2} + \frac{3}{\omega_0^2}\right\}.$$

3) Let

$$\tau = \Omega t\omega(\varepsilon)/\omega_0, \qquad \omega(\varepsilon) = \omega_0 g(\varepsilon),$$
$$\alpha = \omega_0/\Omega, \qquad \tilde{\Gamma} = \Gamma/\Omega^2.$$

Then the equation transforms to

$$g^2(\varepsilon)\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + \alpha^2(x - \varepsilon x^4) = \varepsilon\tilde{\Gamma}\cos[\tau\omega_0/\omega(\varepsilon)].$$

Let

$$x(\varepsilon, \tau) = x_0(\tau) + \varepsilon x_1(\tau) + \mathrm{O}(\varepsilon^2),$$

$$g(\varepsilon) = 1 + \varepsilon g_1 + \mathrm{O}(\varepsilon^2).$$

Then the $\mathrm{O}(\varepsilon^0)$ terms satisfy

$$\frac{\mathrm{d}^2 x_0}{\mathrm{d}\tau^2} + \alpha^2 x_0 = 0.$$

Using the initial condition, the solution is

$$x_0(\tau) = a_0\cos(\alpha\tau).$$

Then the $O(\varepsilon^1)$ terms satisfy

$$\frac{d^2x_1}{d\tau^2} + \alpha^2 x_1 = 2g_1 a_0 \alpha^2 \cos(\alpha\tau) + \alpha^2 a_0^4 \cos^4(\alpha\tau) + \tilde{\Gamma}\cos(\tau).$$

Terms with $\cos(\alpha\tau)$ on the right will give a secular contribution to the solution. Since $\cos^4(\alpha\tau)$ unlike $\cos^3(\alpha\tau)$ does not contain such a term we must take $g_1 = 0$ to eliminate a secular contribution. Then

$$\frac{d^2x_1}{d\tau^2} + \alpha^2 x_1 = \tfrac{1}{8}\alpha^4 a_0^4[\cos(4\alpha\tau) + 4\cos(2\alpha\tau) + 3] + \tilde{\Gamma}\cos(\tau).$$

Using the trial function

$$f(\tau) = A_0 + A_1\cos(2\alpha\tau) + A_3\cos(4\alpha\tau) + A_4\cos(\tau)$$

for the particular solution it follows that

$$x_1(\tau) = a_1\cos(\alpha\tau) + \frac{\tilde{\Gamma}\cos(\tau)}{\alpha^2 - 1} - \tfrac{1}{8}\alpha^2 a_0^4\left[\frac{\cos(4\alpha\tau)}{15\alpha^2} + \frac{4\cos(2\alpha\tau)}{3\alpha^2} - \frac{3}{\alpha^2}\right],$$

giving

$$x(t) = \cos(\omega_0 t)[a_0 + \varepsilon a_1] + \frac{\varepsilon\Gamma\cos(\Omega t)}{\omega_0^2 - \Omega^2} - \tfrac{1}{8}a_0^4\varepsilon\left[\tfrac{1}{15}\cos(4\omega_0 t) + \tfrac{4}{3}\cos(2\omega_0 t) - 3\right].$$

4) Let $\tau = \Omega t$. Then the equation transforms to

$$\Omega^2(\varepsilon)\frac{d^2x}{d\tau^2} + \omega_0^2(x + \varepsilon x^2) = \Gamma\cos(\tau).$$

Let

$$x(\varepsilon,\tau) = x_0(\tau) + \varepsilon x_1(\tau) + O(\varepsilon^2),$$

$$\Omega(\varepsilon) = \Omega_0 + \varepsilon\Omega_1 + O(\varepsilon^2).$$

Then the $O(\varepsilon^0)$ terms satisfy

$$\Omega_0^2\frac{d^2x_0}{d\tau^2} + \omega_0^2 x_0 = \Gamma\cos(\tau),$$

with solution

$$x_0(\tau) = a_0\cos(\omega_0\tau/\Omega_0) + \frac{\Gamma\cos(\tau)}{\omega_0^2 - \Omega_0^2}.$$

If $\Omega_0 = 2\omega_0$ then

$$x_0(\tau) = a_0\cos(\tau/2) - \frac{\Gamma\cos(\tau)}{3\omega_0^2}.$$

Then the $\mathrm{O}(\varepsilon^1)$ terms satisfy

$$2\omega_0\Omega_1\frac{\mathrm{d}^2x_0}{\mathrm{d}\tau^2} + 4\omega_0^2\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2} + \omega_0^2[x_1 + x_0^2] = 0.$$

Substituting for $x_0(\tau)$ gives

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2} + \tfrac{1}{4}x_1 = \frac{\Omega_1}{8\omega_0}\left[a_0\omega_0^2\cos(\tau/2) - \tfrac{4}{3}\Gamma\cos(\tau)\right] - \tfrac{1}{4}\left[a_0\cos(\tau/2) - \frac{\Gamma\cos(\tau)}{3\omega_0^2}\right]^2.$$

Expanding the last term gives

$$\begin{aligned}\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2} + \tfrac{1}{4}x_1 &= \frac{\Omega_1}{8\omega_0}\left[a_0\omega_0^2\cos(\tau/2) - \tfrac{4}{3}\Gamma\cos(\tau)\right] - \tfrac{1}{8}a_0^2[1+\cos(\tau)] \\ &\quad -\frac{\Gamma^2}{72\omega_0^4}[1+\cos(2\tau)] + \frac{\Gamma a_0}{6\omega_0^2}[\cos(3\tau/2)+\cos(\tau/2)].\end{aligned}$$

To eliminate secular terms we must remove the $\cos(\tau/2)$ terms from the non-forcing contribution by setting $\Omega_1 = \frac{4}{3}\Gamma/\omega_0^3$. Then

$$\begin{aligned}\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2} + \tfrac{1}{4}x_1 &= -\tfrac{1}{8}a_0^2[1+\cos(\tau)] - \frac{\Gamma^2}{72\omega_0^4}[1+-16\cos(\tau)+\cos(2\tau)] \\ &\quad + \frac{\Gamma a_0}{6\omega_0^2}\cos(3\tau/2).\end{aligned}$$

This has the solution

$$\begin{aligned}x_1(\tau) &= -a_0^2\left[2 - \tfrac{1}{6}\cos(\tau)\right] - \frac{\Gamma^2}{18\omega_0^4}\left[4 + \tfrac{16}{3}\cos(\tau) - \tfrac{1}{15}\cos(2\tau)\right] \\ &\quad - \frac{\Gamma a_0}{12\omega_0^2}\cos(3\tau/2).\end{aligned}$$

Finally we substitute these results for $x_0(\tau)$ and $x_1(\tau)$ into the expansion for $x(\varepsilon, t)$ with $\tau = \Omega t$ and compare coefficients with those given in the question. We have

$$\begin{aligned}&A(\varepsilon) = -\tfrac{2}{9}\varepsilon\left[9a_0^2 + \frac{\Gamma^2}{\omega_0^4}\right], && B(\varepsilon) = a_0, \\ &C(\varepsilon) = -\frac{\Gamma}{3\omega_0^2} + \tfrac{1}{6}a_0^2\varepsilon - \frac{8\Gamma^2\varepsilon}{27\omega_0^4}, && D(\varepsilon) = -\frac{\varepsilon\Gamma a_0}{12\omega_0^2}, \\ &E(\varepsilon) = \frac{\varepsilon\Gamma^2}{270\omega_0^4}.\end{aligned}$$

5) To apply the *Krylov-Bogoliubov-Mitropolsky averaging method* to the equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \varepsilon f\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) + x = \varepsilon\Gamma\cos(\Omega t),$$

we suppose that:

(i) $x(\varepsilon, t) = r\cos(\theta) + \varepsilon u^{(1)}(r,\theta) + \varepsilon^2 u^{(2)}(r,\theta) + \cdots,$
where $u^{(k)}(r, \theta + 2\pi) = u^{(k)}(r,\theta)$ and

$$\int_0^{2\pi} u^{(k)}(r,\theta)\cos(\theta)\mathrm{d}\theta = \int_0^{2\pi} u^{(k)}(r,\theta)\sin(\theta)\mathrm{d}\theta = 0,$$
$$k = 1, 2, \ldots.$$

(ii)

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \varepsilon A^{(1)}(r) + \varepsilon^2 A^{(2)}(r) + \cdots,$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -1 + \varepsilon B^{(1)}(r) + \varepsilon^2 B^{(2)}(r) + \cdots,$$

The k-th order KBM method consists in retaining terms up to ε^k. We now apply the method to the *Van der Pol equation* with a weak forcing term

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \varepsilon(x^2 - 1)\frac{\mathrm{d}x}{\mathrm{d}t} + x = \varepsilon\Gamma\cos(\Omega t).$$

where Ω is not close to unity. Retaining terms to $\mathrm{O}(\varepsilon)$ it follows from (i) and (ii) that

$$\varepsilon(x^2 - 1) = \varepsilon\{r^2\cos^2(\theta) - 1\},$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = r\sin(\theta) + \varepsilon\left\{A^{(1)}(r)\cos(\theta) - rB^{(1)}(r)\sin(\theta) - \frac{\partial u^{(1)}}{\partial\theta}\right\},$$
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -r\cos(\theta) + \varepsilon\left\{2\sin(\theta)A^{(1)}(r) + 2r\cos(\theta)B^{(1)}(r) + \frac{\partial^2 u^{(1)}}{\partial\theta^2}\right\}.$$

Substituting into Van der Pol equation the terms of $\mathrm{O}(\varepsilon^0)$ cancel and the terms of $\mathrm{O}(\varepsilon^1)$ give

$$\left\{\frac{\partial^2 u^{(1)}(r,\theta)}{\partial\theta^2} + u^{(1)}(r,\theta)\right\} + 2A^{(1)}(r)\sin(\theta) + 2rB^{(1)}(r)\cos(\theta)$$
$$+ r\sin(\theta)\{r^2\cos^2(\theta) - 1\} = \Gamma\cos(\Omega t). \qquad (6.22)$$

We must now eliminate the term with explicit t dependence. We do this by defining

$$u^{(1)}(r,\theta) = \tilde{u}^{(1)}(r,\theta) + \frac{\Gamma\cos(\Omega t)}{1-\Omega^2}$$

and substituting into (6.22) gives

$$\left\{\frac{\partial^2\tilde{u}^{(1)}(r,\theta)}{\partial\theta^2} + \tilde{u}^{(1)}(r,\theta)\right\} + 2A^{(1)}(r)\sin(\theta) + 2rB^{(1)}(r)\cos(\theta) + r\sin(\theta)\{r^2\cos^2(\theta) - 1\} = 0. \tag{6.23}$$

Now multiplying successively by $\sin(\theta)$ and $\cos(\theta)$, integrating over $[0, 2\pi]$ and using the integral results in (i) gives

$$\begin{aligned} A^{(1)}(r) &= -\frac{r}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \sin^2(\theta)\{r^2\cos^2(\theta) - 1\} = \tfrac{1}{8}r(4-r^2), \\ B^{(1)}(r) &= -\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \sin(\theta)\cos(\theta)\{r^2\cos^2(\theta) - 1\} = 0. \end{aligned} \tag{6.24}$$

Since $x(0) = r_0 + \mathrm{O}(\varepsilon^1)$, $\dot{x}(0) = \mathrm{O}(\varepsilon^1)$, $r(0) = r_0$ and $\theta(0) = 0$. From (ii) and (6.24)

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -1, \qquad \text{giving} \qquad \theta = -t,$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \tfrac{1}{8}\varepsilon r(4-r^2),$$

giving

$$\varepsilon t = 8\int_{r_0}^{r} \frac{\mathrm{d}r}{r(4-r^2)} = \ln\left\{\frac{r^2(4-r_0^2)}{r_0^2(4-r^2)}\right\}$$

$$\frac{r_0^2(4-r^2)}{r^2(4-r_0^2)} = \exp(-\varepsilon t).$$

Substituting from (6.24) into (6.23) gives

$$\frac{\partial^2\tilde{u}^{(1)}(r,\theta)}{\partial\theta^2} + \tilde{u}^{(1)}(r,\theta) = -\tfrac{1}{4}r^3\{3\sin(\theta) - 4\sin^3(\theta)\} = -\tfrac{1}{4}r^3\sin(3\theta).$$

This has the solution

$$\tilde{u}^{(1)}(r,\theta) = \tfrac{1}{32}r^3\sin(3\theta).$$

So

$$u^{(1)}(r,\theta) = \tfrac{1}{32} r^3 \sin(3\theta) + \frac{\Gamma\cos(\Omega t)}{1-\Omega^2}$$

and from (i)

$$x(\varepsilon,t) = r\cos(t) - \tfrac{1}{32}\varepsilon r^3 \sin(3t) + \frac{\varepsilon\Gamma\cos(\Omega t)}{1-\Omega^2}.$$

6.5 Problems 5

1) With $\dot{x}(t)$ denoted as $y(t)$

$$\dot{x}(t) = y(t),$$

$$\dot{y}(t) = x(t) - \mu y(t) - x^2(t).$$

The equilibrium points are $(0,0)$ and $(1,0)$.

Linearizing about $x = y = 0$ the stability matrix is

$$\boldsymbol{J}^* = \begin{pmatrix} 0 & 1 \\ 1 & -\mu \end{pmatrix},$$

with eigenvalues $\frac{1}{2}[-\mu \pm \sqrt{\mu^2+4}]$. Since both eigenvalues are real with one positive and one negative, for all μ, this is a *saddle point.*

Linearizing about $x = 1$, $y = 0$ the stability matrix is

$$\boldsymbol{J}^* = \begin{pmatrix} 0 & 1 \\ -1 & -\mu \end{pmatrix},$$

with eigenvalues $\frac{1}{2}[-\mu \pm \sqrt{\mu^2-4}]$. So this equilibrium point is

- A *stable node* if $\mu > 2$.
- A *stable inflected node* if $\mu = 2$.
- A *stable focus* if $0 < \mu < 2$.
- A *centre* if $\mu = 0$.
- A *unstable focus* if $-2 < \mu < 0$.
- A *unstable inflected node* if $\mu = -2$.
- A *unstable node* if $\mu < -2$.

When $\mu = 0$

$$x\frac{\mathrm{d}x}{\mathrm{d}t} = yx, \qquad y\frac{\mathrm{d}y}{\mathrm{d}t} = yx - x^2\frac{\mathrm{d}x}{\mathrm{d}t}.$$

So

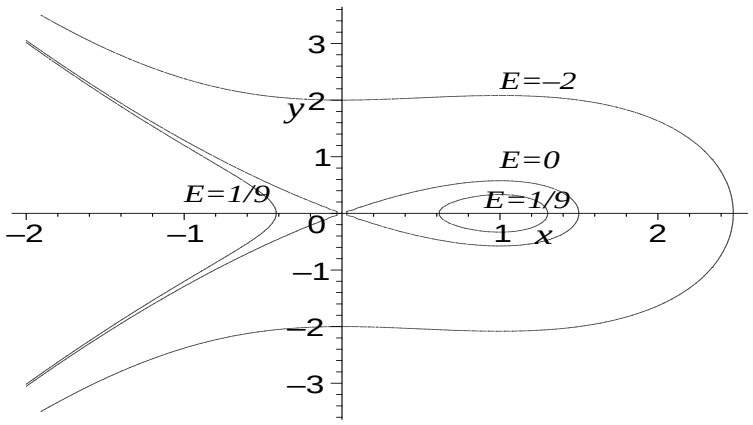
$$x\frac{\mathrm{d}x}{\mathrm{d}t} - y\frac{\mathrm{d}y}{\mathrm{d}t} = x^2\frac{\mathrm{d}x}{\mathrm{d}t},$$

giving

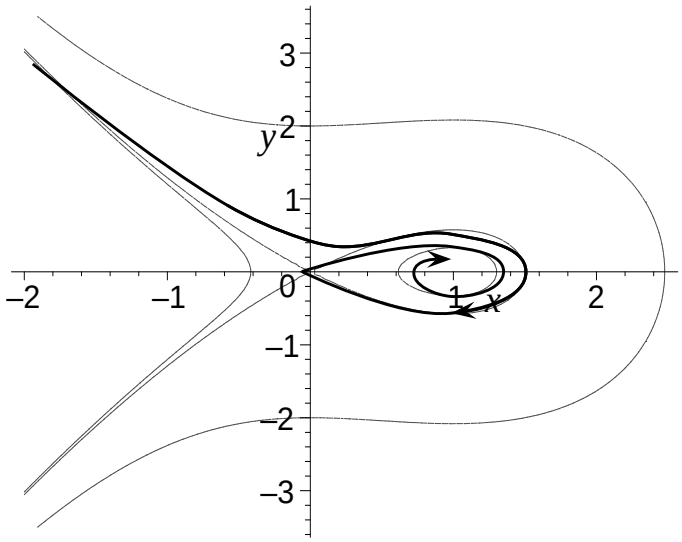
$$\tfrac{1}{2}[x^2 - y^2] = E - \tfrac{1}{3}x^3.$$

The homoclinic trajectory passes through the origin and thus corresponds to $E = 0$. A MAPLE plot for some trajectories is

```
> with(plots):
> f:=(x,y)->(x^2-y^2)/2-x^3/3:
> curve:=
> implicitplot({f(x,y)=-2,f(x,y)=0,f(x,y)=1/9},
> x=-2..3,y=-3.5..3.5,grid=[100,100],
> labelfont=[TIMES,ITALIC,12],linestyle=5,thickness=1):
> text:=
> plots[textplot]({[-1.0,0.2,‘E=1/9‘],
> [0.9,0.1,‘E=1/9‘],[1,0.8,‘E=0‘],[1,2.2,‘E=-2‘]},
> align={ABOVE,RIGHT},font=[TIMES,ITALIC,10]):
> plots[display]({curve,text});
```



The arrows indicating the direction of flow can be added to this diagram using the fact that $\dot{x}(t) > 0$ when $y > 0$. When μ is small and positive the centre at $(1, 0)$ changes to a stable focus. The right-hand part of the homoclinic trajectory breaks into a part spiralling into the focus point and a branch coming from infinity.



When μ is small and negative the figure is obtained by reflecting this diagram in the x–axis and reversing the direction of the arrows.

2) Substituting $x = x_c + \triangle x$, $y = y_c + \triangle y$, $z = z_c + \triangle z$ into the Lorentz equations gives

$$\begin{aligned}
\frac{\mathrm{d}\triangle x}{\mathrm{d}t} &= -a(\triangle x - \triangle y), \\
\frac{\mathrm{d}\triangle y}{\mathrm{d}t} &= \triangle x - \triangle y - x_c\triangle z + (\rho - \rho_c)(x_c + \triangle x) - \triangle x\triangle z, \\
\frac{\mathrm{d}\triangle z}{\mathrm{d}t} &= y_c\triangle x + x_c\triangle y - b\triangle z + \triangle x\triangle y.
\end{aligned}$$

These can be expressed in vector form as

$$\frac{\mathrm{d}\triangle \boldsymbol{r}}{\mathrm{d}t} + \boldsymbol{J}^*\triangle\boldsymbol{r} = \boldsymbol{w},$$

where $\boldsymbol{J}^*$ and $\boldsymbol{w}$ are as given on the question sheet. Now substitute $t = \tau/\omega$ to give the required equation. The eigenvalue equation of $\boldsymbol{J}^*$ is

$$\lambda^3 + (a+b+1)\lambda^2 + b(a+\rho_c)\lambda + 2ab(\rho_c - 1) = 0.$$

Having been given one root the cubic can be factorized and the roots are

$$\lambda = -(a+b+1),$$

$$\lambda = \pm\mathrm{i}\sqrt{b(a+\rho_c)} = \pm\mathrm{i}\sqrt{\frac{2ab(a+1)}{a-b-1}},$$

which identifies ω_c. Let $\boldsymbol{v}$ and $\boldsymbol{u}$ be the left and right eigenvectors of $\boldsymbol{J}^*$ with eigenvalue $\mathrm{i}\omega_c$. Then

$$\boldsymbol{v}^{\mathrm{T}}\boldsymbol{J}^* = \mathrm{i}\omega_c\boldsymbol{v}^{\mathrm{T}}, \qquad \boldsymbol{J}^*\boldsymbol{u} = \boldsymbol{u}\mathrm{i}\omega_c.$$

Let

$$\boldsymbol{p} = \boldsymbol{u}a(\tau) + \bar{\boldsymbol{u}}b(\tau)$$

and substitute into the equation. Since the terms on the right-hand side are of $\mathrm{O}(\varepsilon)$, the terms of $\mathrm{O}(\varepsilon^{1/2})$ give

$$\boldsymbol{u}\frac{\mathrm{d}a}{\mathrm{d}\tau} + \bar{\boldsymbol{u}}\frac{\mathrm{d}b}{\mathrm{d}\tau} + \mathrm{i}[\boldsymbol{u}a - \bar{\boldsymbol{u}}b] = 0.$$

Operating on the left with $\boldsymbol{v}^{\mathrm{T}}$ gives

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} + \mathrm{i}a = 0,$$
$$a(\tau) = c\exp(-\mathrm{i}\tau),$$

and

$$\frac{\mathrm{d}b}{\mathrm{d}\tau} - \mathrm{i}b = 0,$$
$$b(\tau) = c'\exp(\mathrm{i}\tau).$$

Since $\boldsymbol{p}$ is real $c' = \bar{c}$ and we have the required result.

3) Substituting $t = \varepsilon\tau$, $\rho = 1/\varepsilon^2$, $x = \xi/\varepsilon$, $y = \eta/(\varepsilon^2 a)$ and $z = (\zeta + a)/(\varepsilon^2 a)$ into the Lorentz equations to achieve the required forms is straightforward. When $\varepsilon = 0$

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = \eta, \qquad \frac{\mathrm{d}\eta}{\mathrm{d}\tau} = -\xi\zeta, \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}\tau} = \xi\eta$$

and thus

$$\eta\frac{\mathrm{d}\eta}{\mathrm{d}\tau} + \zeta\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} = 0,$$

giving

$$\tfrac{1}{2}\eta^2 + \tfrac{1}{2}\zeta^2 = \alpha,$$

and

$$\xi\frac{\mathrm{d}\xi}{\mathrm{d}\tau} - \frac{\mathrm{d}\zeta}{\mathrm{d}\tau} = 0,$$

giving

$$\tfrac{1}{2}\xi^2 - \zeta = \beta.$$

Also

$$\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2 = 2\alpha^2 - \zeta^2 = 2\alpha - \left(\tfrac{1}{2}\xi^2 - \beta\right)^2 = (2\alpha - \beta^2) - \tfrac{1}{4}\xi^4 + \beta\xi^2.$$

When $\alpha = \frac{9}{8}$, $\beta = \frac{1}{2}$,

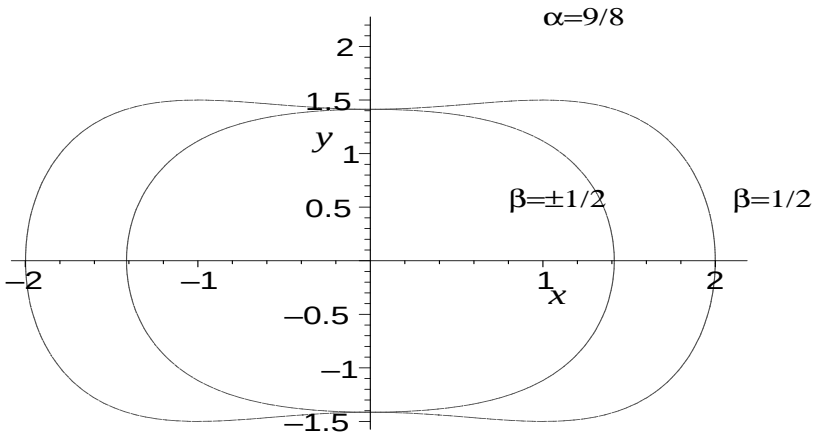
$$\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2 = -\tfrac{1}{4}(\xi^2 + 2)(\xi^2 - 4).$$

The solution is confined to the range $-2 \leq \xi \leq 2$ with $\mathrm{d}xi/\mathrm{d}tau = 0$ at the extremities. The period of the solution will be given by integrating over $[-2, 2]$ and doubling the result. That is

$$4\int_{-2}^{2} \frac{\mathrm{d}\xi}{\sqrt{(\xi^2+2)(4-\xi^2)}}.$$

We can determine the form of the orbits in the ξ–$\mathrm{d}\xi/\mathrm{d}\tau$ plane using MAPLE .

```
> with(plots):
> f:=(x,y,alpha,beta)->

> 2*alpha-beta^2-x^4/4+beta*x^2-y^2:
> curve:=implicitplot({f(x,y,9/8,1/2)=0,f(x,y,9/8,-1/2)=0
> },

> x=-3..3,y=-2.5..2.5,grid=[100,100],labelfont=[TIMES,ITALIC,12],

> linestyle=5,thickness=1):
> text:=plots[textplot](
> {[0.8,0.5,`b=-1/2`],[2.1,0.5,`b=1/2`],[1,2.2,`a=9/8`]},

> align={ABOVE,RIGHT},font=[SYMBOL,10]):
> plots[display]({curve,text});
```



4) For the given transformation the stability matrix is

$$\mathbf{J}[\mathtt{x},\mathtt{y}] = \begin{cases} \begin{pmatrix} \tau_a & 0 \\ 0 & 2 \end{pmatrix}, & \mathtt{y} < \frac{1}{2}, \\ \begin{pmatrix} \tau_b & 0 \\ 0 & 2 \end{pmatrix}, & \mathtt{y} > \frac{1}{2}. \end{cases}$$

Suppose that in n iterations the mapping spends p in the region $\mathtt{y} < \frac{1}{2}$. Then the eigenvalues of $\mathsf{S}(n)$ (defined by (5.64) are 2^n and $\tau_a^p \tau_b^{n-p}$. Then, from (5.66),

$$\lambda_{\mathrm{L}}^{(1)} = \lim_{n\to\infty} \frac{\ln(2^n)}{n} = \ln(2) > 0,$$

$$\lambda_{\mathrm{L}}^{(2)} = \lim_{n\to\infty} \frac{\ln(\tau_a^p \tau_b^{n-p})}{n} = \mu \ln(\tau_a) + (1-\mu)\ln(\tau_b) < 0,$$

where

$$\mu = \lim_{n\to\infty} \left(\frac{p}{n}\right).$$

Since $\lambda_{\mathrm{L}}^{(1)} > 0$ and $\lambda_{\mathrm{L}}^{(2)} < 0$ the system is *chaotic*. In formula (5.79) for the Lyapunov dimension of the attractor we take $k = 1$ to give

$$\mathcal{D}(\mathfrak{A}) = 1 - \{\mu \ln(\tau_a) + (1-\mu)\ln(\tau_b)\}^{-1}.$$