Benchmarking mixed-integer and set-based formulations for a state task network (STN) scheduling problem

1. Background

The State Task Network (STN) representation is a widely used modeling approach for optimal multipurpose batch production scheduling. Depending on the problem size and modeling aspects included, this problem may yield a challenging Mixed-Integer Programming (MIP) problem. Optimal STN scheduling relies on three key components that are used to define the scheduling problem: tasks (e.g., manufacture product A), the units available to perform these tasks, and states that define the feasible interconnections between tasks, i.e., states act as initial, final and intermediate storage between operations.

2. Problem statement

The decisions that need to be optimized in a STN scheduling problem can be summarized as:

1. *Timing* (discrete): start execution time of each task in each unit
2. *Amounts* (continuous): amount of material processed in each task

While the above decisions are common in scheduling, STN problems differentiate from others due to the consideration of states, representing storage facilities in which incoming materials from previous task executions can be stored until they are released for subsequent task executions. This means that there is a mass balance inherently embedded within the scheduling problem. These mass balances are needed to formulate constraints that ensure that, at a given time point, enough materials are available for the execution of subsequent tasks and storage facilities do not overflow. Mathematically, storage modeling requires the following intermediate expressions:

1. *Storage state:* at every point in time and every storage facility, the current storage state is equal to the state at a previous time point plus incoming materials from finished tasks minus released materials to starting tasks.

The constraints that are generally present in the problem are:

1. *Non-overlapping*: *timing* variables are constrained such that a given unit can only process one task at a time.
2. *Task capacity*: the *amounts* of material processed in tasks must be within lower and upper task capacity limits.
3. *State tracking*: at every point in time, *storage states* must be within lower and upper storage capacity limits.

General challenges

The key characteristics that make STN scheduling problems challenging are:

1. *Optional tasks:* in its more general form, a STN problem has optional tasks, i.e., the number of times a task is executed in each unit is initially unknown.
2. *Ordering of tasks:* the order of execution of tasks is also unknown and difficult to infer from a general STN.
3. *User-defined network:* freedom in the user-defined network structure, potentially resulting in the following situations that make *state tracking* non-trivial:
   - *Material mixing or splitting:* storage facilities are allowed to accept incoming/releasing materials from/to multiple tasks simultaneously.
   - *Recycles:* tasks are not interconnected by states one after the other. Instead, a task is allowed to accept incoming/releasing of material from/to any different tasks in the network, potentially leading to materials that are recirculated in the operation.

3.1. General STN problem: formulations

While many formulations are available in the literature, the MIP shown in section S3 of the supplementary material is the most widely used model for this problem. Nonetheless, set-based representations (i.e., Hexaly) of this problem have not been studied. Therefore, we propose the following set-based formulations that are equivalent to the original MIP formulation (see sections S4-S7 in the supplementary material for further details):

1. MInP(1): a Mixed-Interval Program (MInP) that uses Intervals instead of binaries to represent *timing* decisions with the following characteristics:
   - Opposed to MIP: *Amount* variables are defined for each interval. *Storage state* expressions are expressed accordingly.
   - Opposed to MIP: disjunctions (i.e., "or" constraints) are used to represent *optional tasks.*
   - Opposed to MIP: *Non-overlapping* constraints are expressed directly in terms of interval variables, without introducing list variables.

2. MInLiP(1): a Mixed Interval-List Program (MInLiP) with the same variables in MInP(1) plus auxiliary list variables with the following characteristics:
   - As in MInP(1): *Amount* variables are defined for each interval. *Storage state* expressions are expressed accordingly.
   - Opposed to MInP(1): *list* variables are included in constraints to represent *optional tasks.*
   - Opposed to MInP(1): *Non-overlapping* constraints are expressed in terms of interval and list variables.

3. MInP(2): a Mixed-Interval Program (MInP) that uses Intervals instead of binaries to represent *timing* decisions with the following characteristics:
   - As in MIP: *Amount* variables are defined for each point in time. *Storage state* expressions also remain unchanged.
   - Opposed to MIP: disjunctions (i.e., "or" constraints) are used to represent *optional tasks.*

- Opposed to MIP: *Non-overlapping* constraints are expressed directly in terms of interval variables, without introducing list variables.
- Opposed to MIP: disjunctions are included in *task capacity* constraints to connect time-indexed *amount* variables to intervals.

4. MInLiP(2): a Mixed Interval-List Program (MInLiP) with the same variables in MInP(2) plus auxiliary list variables with the following characteristics:
   - As in MInP(2): *Amount* variables are defined for each point in time. *Storage state* expressions also remain unchanged.
   - As in MInP(2): disjunctions are included in *task capacity* constraints to connect time-indexed *amount* variables to intervals.
   - Opposite to MInP(2): *list* variables are included in constraints to represent *optional tasks.*
   - Opposite to MInP(2): *Non-overlapping* constraints are expressed in terms of interval and list variables.

3.2. General STN problem: test problem

The objective function considered is the minimization of the negative of the profit (i.e., profit maximization). More details about the problem are provided below:

1. *Sets:* the STN consists of *tasks* (e.g., reaction and separation) denoted by index $i \in I$, *units* (e.g., batch reactor and distillation column) denoted by index $j \in J$, and *states* denoted by index $k \in K$ (e.g., intermediate storage and final products). The production environment is defined in terms of sets IK^-/IK^+ which contain the available task-state pairs (i,k) when task i consumes/produces material from/to state k. Set $IJ^\times$ is defined to represent the feasible task to units mapping through tuples (i,j). A discrete time grid is defined through set $T = \{0,1,2,\dots,n_T\}$ with index t.

2. *Parameters:* the parameters of the problem include the revenue from selling the material in state k at the end of the scheduling horizon (ω_k); the cost of running task i ($\alpha_{i,j}$); the processing time of task i in unit j ($\tau_{i,j}^p$); the maximum (β_j^{max}) and minimum (β_j^{min}) capacity of unit j; the maximum storage capacity of state k (γ_k^{max}), the initial storage in state k (γ_k^{init}), and the fraction of material in state k consumed ($\rho_{i,k}^-$) or produced ($\rho_{i,k}^+$) by task i.

3. *Case study:* This case study was adapted from the work by Velez and Maravelias [1]. It considers the STN and the parameters shown in Figure 1, where blocks denote tasks and circles denote states. It consists of five tasks $I = \{T1, T2, \dots, T5\}$, nine states $K = \{S1, S2, \dots, S9\}$, and four units $J = \{U1, U2, U3, U4\}$. Figure 1 also presents the task to units mapping ($IJ^\times$), the maximum capacity of states (γ_k^{max}), the initial condition of states (γ_k^{init}), and the maximum and minimum unit capacities ($\beta_j^{max}/\beta_j^{min}$). This illustration also presents the values of $\rho_{i,k}^+$ and $\rho_{i,k}^-$ as percentages. Note that the products of this process are $S8$ and $S9$, while the inputs are $S1, S2$ and $S3$.

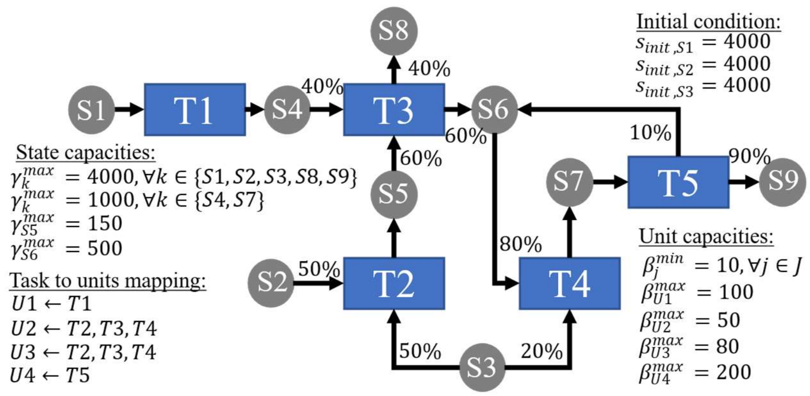

Figure 1. STN diagram

For simplicity, the units of the problem are omitted. Table 1 shows the values of the fixed processing times ($\tau_{i,j}^p$) and cost coefficients ($\alpha_{i,j}$) for each task to unit pair in $IJ^\times$. In addition, the revenue from selling materials in state k at the end of the scheduling horizon is assumed to be $\omega_{S8} = 3$, $\omega_{S9} = 4$, and $\omega_k = 0, \forall k \in K \setminus \{S8, S9\}$.

Table 1. Processing times and cost coefficients.

$IJ^\times$	$\tau_{i,j}^p$	$\alpha_{i,j}$
$(T1, U1)$	1	10
$(T2, U2)$	1	15
$(T2, U3)$	2	30
$(T3, U2)$	1	5
$(T3, U3)$	3	25
$(T4, U2)$	1	5
$(T4, U3)$	5	20
$(T5, U4)$	2	20

3.3. General STN problem: results

The different formulations discussed above were tested using a single test problem with 20 accuracy levels in the time discretization, i.e., with a scheduling horizon of η_f =120 [units of time] and a desired discretization of $\delta_f = 1$ [units of time], each accuracy level corresponds to a different number of discrete time points. This leads to 20 “virtual” test problems with different complexities in the number of scheduling variables. All problems were modeled and solved in Hexaly 14 with a time limit of 300 seconds and a random seed of 1.

Performance profiles using different dimensionless qualities (DQ) including time, gap, objective value and lower bound are shown in Figures 2-5. These plots can be analyzed as follows:

1) If there is a formulation (*) whose performance line is always above others, then formulation (*) is the best formulation with respect to that DQ.
2) Otherwise, there is no clear winer, and the plots should be analyzed further.

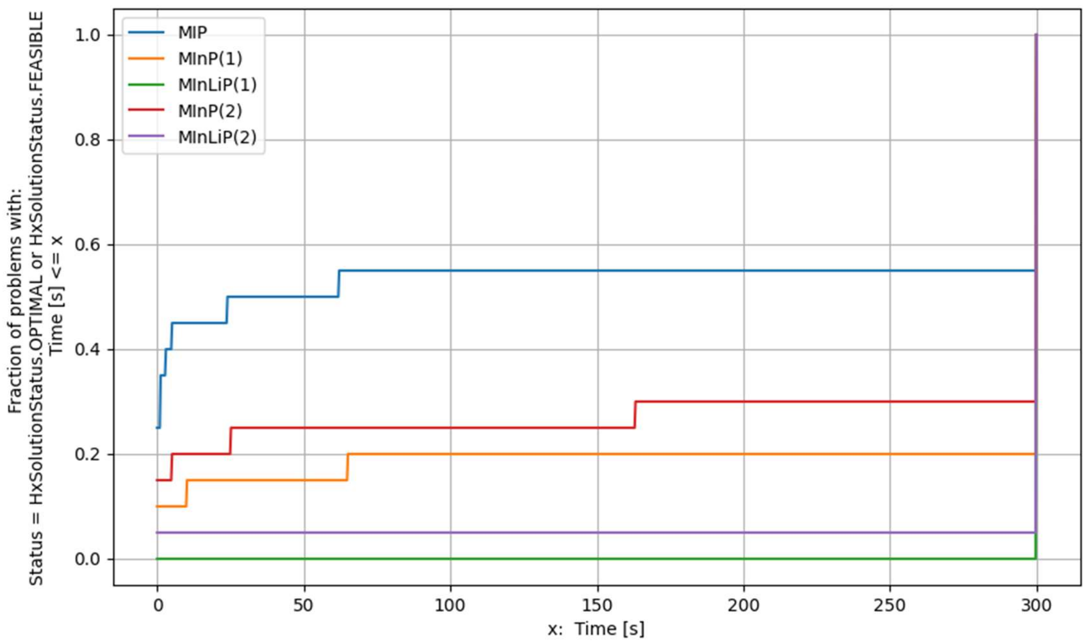

Figure 2. performance profile for the solution time.

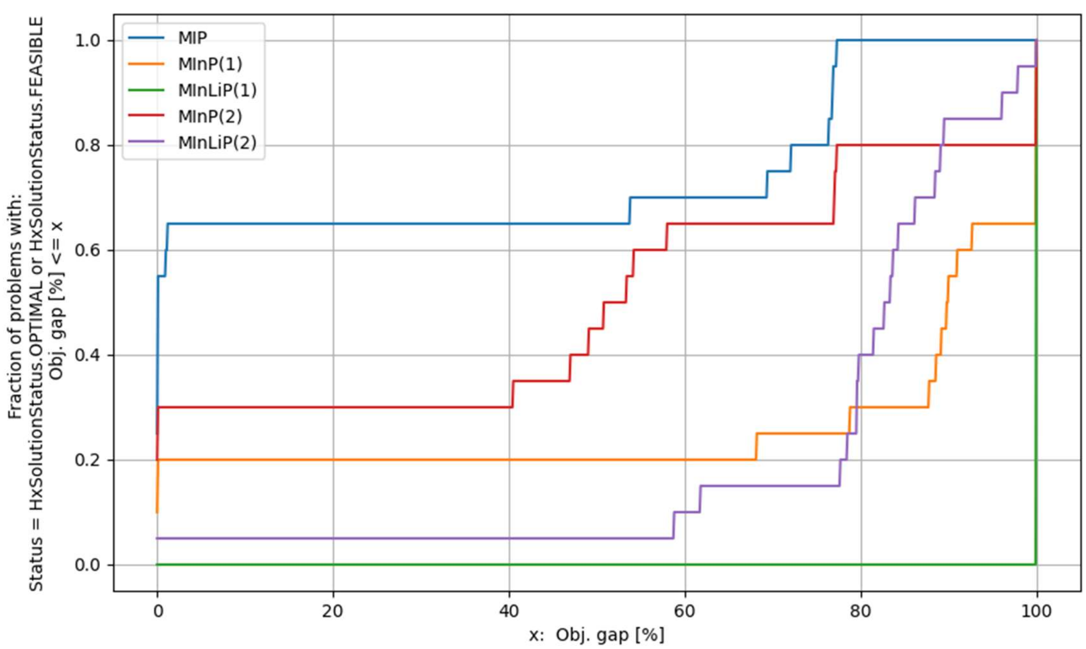

Figure 3. performance profile for the optimality gap.

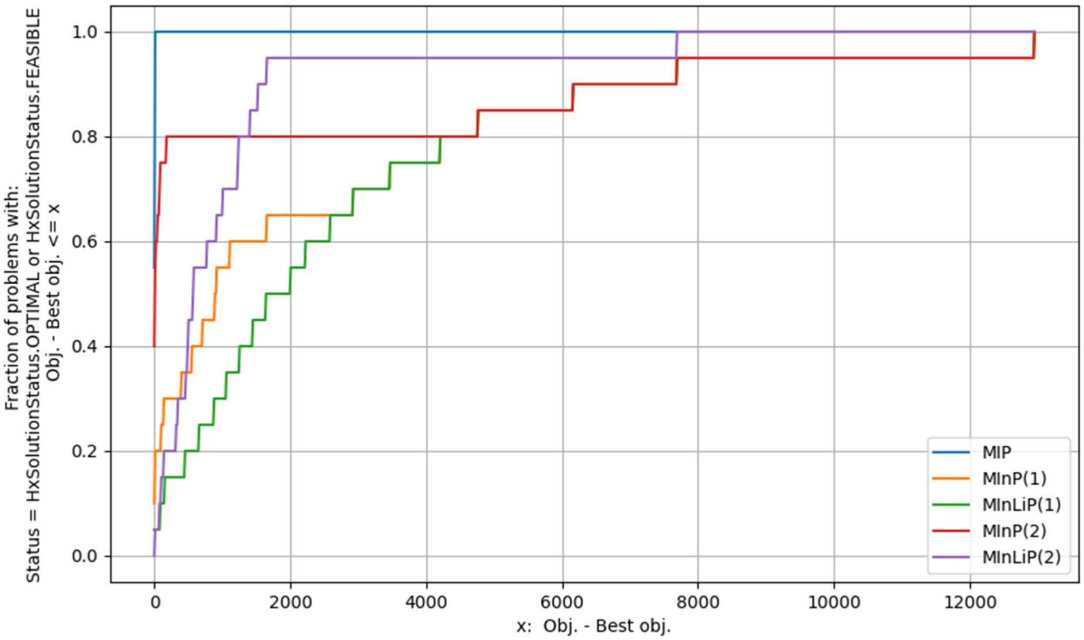

Figure 4. performance profile for the objective value.

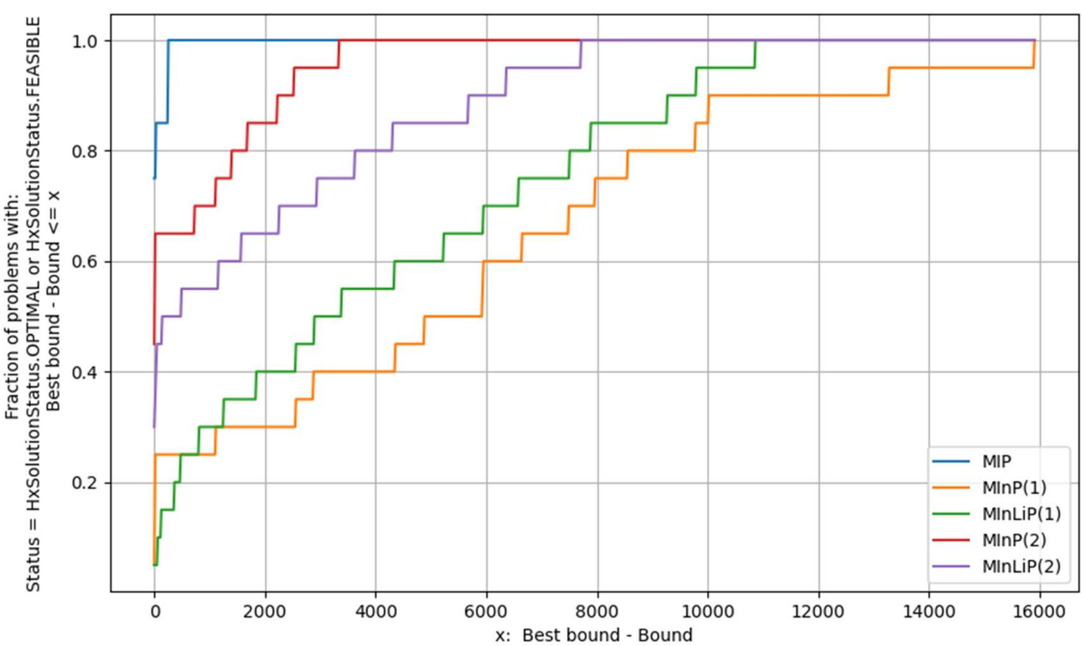

Figure 5. performance profile for the lower bound.

Figures 2-5 show that, for the test problems considered, the MIP formulation beats the set-based formulations with respect to every DQ. In this regard, the main avenue of future work is (if possible) to model optional tasks in such a way (or the best way possible) that Hexaly can exploit problem structures more efficiently. Thus, I am also wondering if, with the modeling tools currently available, there is a better formulation than the ones I have proposed.

4.1. STN problem without optional tasks: formulations

Aiming to avoid modeling difficulties with optional tasks, the proposed set-based models have been modified to consider a simpler case without optional tasks, i.e., the number of times each task is executed in each unit is known. These formulations are shown in sections S8-S12 of the supplementary material.

4.2. STN problem without optional tasks: test problem

Simplified test problems in which the optimal number of times each task is executed in each unit were fixed to the optimal MIP solutions are considered.

4.3. STN problem without optional tasks: results

The same computational experiments proposed above were considered, resulting in Figures 6-9.

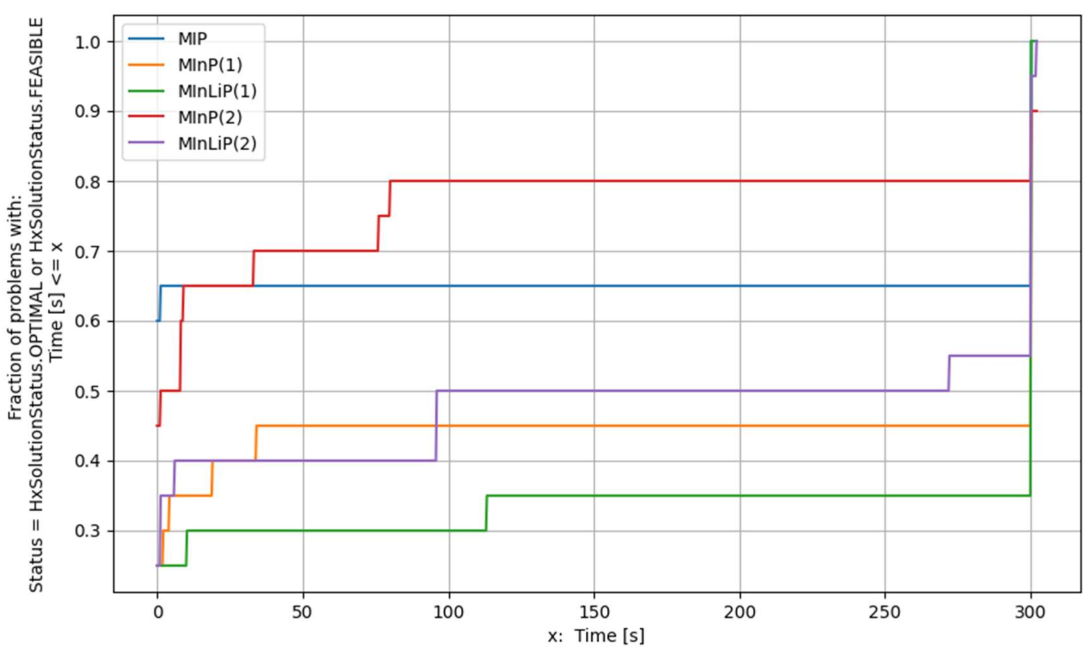

Figure 6. Performance profile for the solution time.

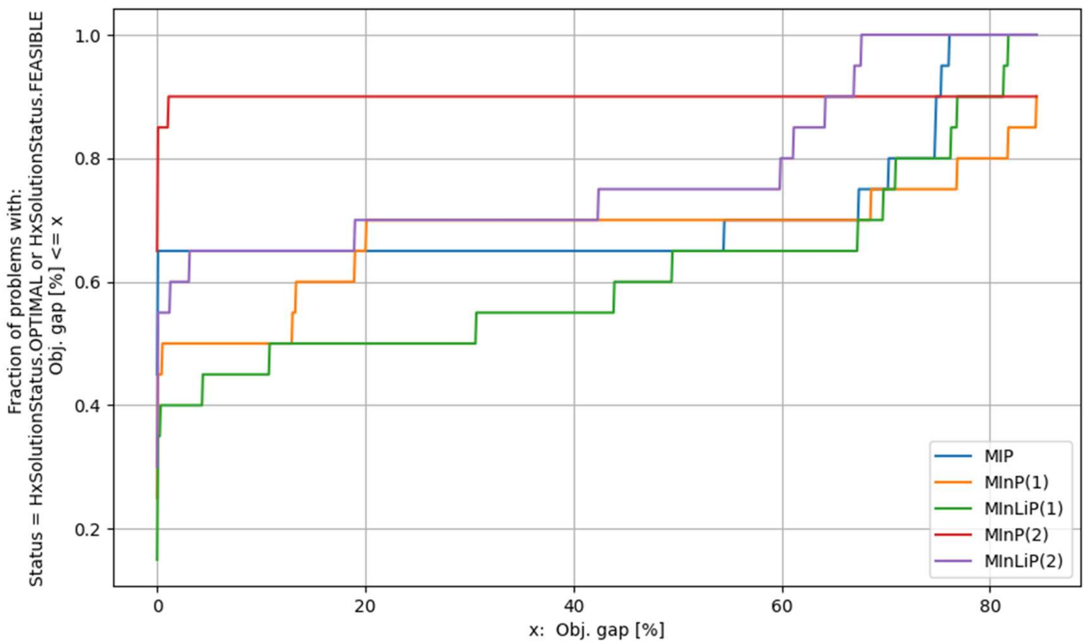

Figure 7. performance profile for the optimality gap.

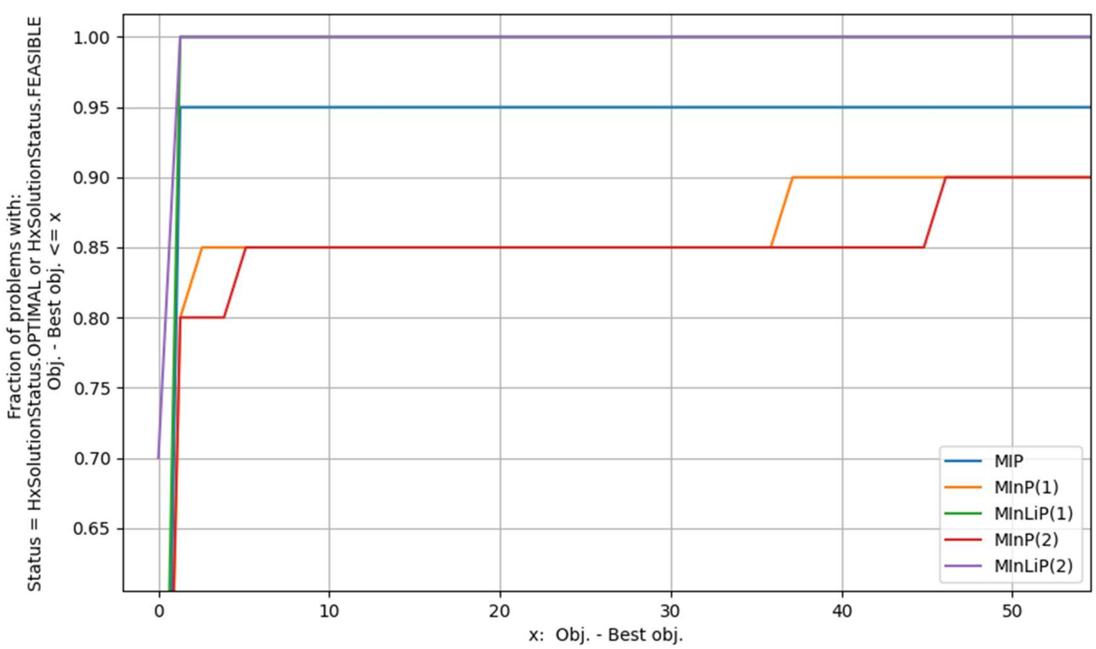

Figure 8. performance profile for the objective value.

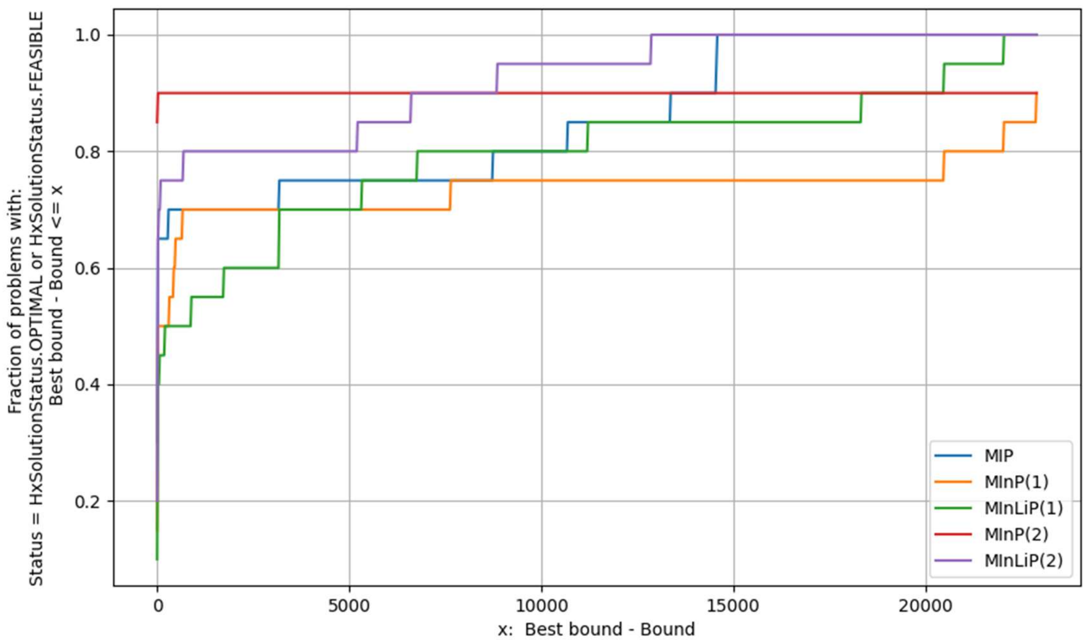

Figure 9. performance profile for the lower bound.

In terms of computational time (Figure 6), the MIP and MInP(2) are the best, with MIP being the first at solving a large fraction of problems fast, and MInP(2) being the one that solves the largest fraction of problems within the established time limit. In terms of closing the optimality gap (Figure 7), assuming that optimality gaps below 10%, 20% or even 40% are acceptable, then MInP(2) is again the best formulation. In terms of the objective function quality, the MinLiP(1) and MinLiP(2) formulation are the first at ensuring that 100% of the test problems solved have an objective close to the best-known objective. In terms of finding a good lower bound, MInP(2) is better than any other formulation at finding tight lower bounds, i.e., lower bounds whose absolute difference with the best known lower bound is less than 5000.

References

[1] S. Velez and C. T. Maravelias, “Reformulations and Branching Methods for Mixed-Integer Programming Chemical Production Scheduling Models,” *Ind. Eng. Chem. Res.*, vol. 52, no. 10, pp. 3832–3841, Mar. 2013, doi: 10.1021/ie303421h.

Supplementary material

S1. Notation

A : set

$\mathbb{P}(A)$: Power set of A, i.e., set of subsets of A.

$X \in \mathbb{P}(A)$: set variable

$\mathbb{S}_{|A|}$: symmetric group of degree $|A|$

$x \in \mathbb{S}_{|A|}$: list variable

$[\![n_1..n_2]\!]$: set of consecutive integer n_1 to n_2

$[m^{start}..m^{end}) \in [\![n_1..n_2]\!]$: interval variable

$\lfloor\cdot\rfloor$: Floor operator

$\sum_{\substack{i\in I \\ i\in I'}} x_i$: Conditional summation over $i \in I$ if condition $i \in I'$ is satisfied.

$$\mathbf{if}(cond) = \begin{cases} 1, cond = True \\ 0, cond = False \end{cases}$$

S2. Problem data

<u>Sets</u>

$i \in I$ *(tasks)*

$j \in J$ *(units)*

$k \in K$ *(states, i.e., storage)*

$T = \{0,1,2, \dots, n_T\}$, *time grid*

$IJ^{\times} = \{(i,j) | task\ i \in I\ can\ be\ performed\ in\ j \in J\}$

$IK^{+} = \{(i,k)\ |\ task\ i \in I\ produces\ material\ to\ state\ k \in K\}$

$IK^{-} = \{(i,k)\ |\ task\ i \in I\ consumes\ material\ from\ state\ k \in K\}$

$q \in Q_{i,j} = \{1,2, \dots, \lfloor n_T/\tau_{i,j} \rfloor\}$ *(Set of executions of task i in unit j)*

$I_j = \left\{0,1,\ldots,\left(\sum_{\substack{i\in I\\(i,j)\in IJ^{\times}}} |Q_{i,j}|\right)-1\right\}$ *(Set of identifiers for all tasks that can be performed in unit j)*

<u>Sequences</u>

$o_j(p): I_j \longmapsto \{i \in I \mid (i,j) \in IJ^{\times}\} \times Q_{i,j}$
(Fixed sequence $o_j = \langle o_j(p) \rangle_{p\in I_j} = \langle o_j(p) \mid p \in I_j\rangle$ that maps from I_j to $\{i \in I \mid (i,j) \in IJ^{\times}\} \times Q_{i,j}$)

<u>For fixed batching-assignment variables</u>

$\psi_{i,j}$ (Number of times task i is performed in unit j)

$q \in Q_{i,j} = \{1,2,\ldots,\psi_{i,j}\}$ *(Set of executions of task i in unit j)*

$N_j = \left\{0,1,\ldots,\left(\sum_{\substack{i\in I\\(i,j)\in IJ^{\times}}} \psi_{i,j}\right)-1\right\}$ *(Set of identifiers for all tasks that can be performed in unit j)*

$o_j(p): N_j \longmapsto \{i \in I \mid (i,j) \in IJ^{\times}\} \times Q_{i,j}$
(Fixed sequence $o_j = \langle o_j(p) \rangle_{p\in N_j} = \langle o_j(p) \mid p \in N_j\rangle$ that maps from N_j to $\{i \in I \mid (i,j) \in IJ^{\times}\} \times Q_{i,j}$)

Set data is updated to consider only those tasks that need to be scheduled in the different sets of the formulation.

S3. MIP (general)

Variables

$x_{i,j,t} \in \{0,1\},\ \forall (i,j) \in IJ^{\times}, \forall t \in T$

$b_{i,j,t} \in \left[0, \beta_j^{max}\right] \subset \mathbb{R},\ \forall (i,j) \in IJ^{\times}, \forall t \in T$

Expressions

$$s_{k,0} = \gamma_k^{init} - \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{-}}} \rho_{i,k}^{-} b_{i,j,0}, \forall k \in K$$

$$s_{k,t} = s_{k,t-1} + \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{+},\ t-\tau_{i,j}\geq 0}} \rho_{i,k}^{+} b_{i,j,t-\tau_{i,j}} - \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{-}}} \rho_{i,k}^{-} b_{i,j,t}, \forall k \in K, \forall t \in T \backslash \{0\}$$

Constraints

$$\sum_{\substack{i\in I \\ (i,j)\in IJ^{\times}}} \sum_{\substack{t'\in T \\ t-\tau_{i,j}+1\leq t'\leq t}} x_{i,j,t'} \leq 1,\ \forall j \in J, \forall t \in T$$

$$\beta_j^{min} x_{i,j,t} \leq b_{i,j,t} \leq \beta_j^{max} x_{i,j,t},\ \forall (i,j) \in IJ^{\times}, \forall t \in T$$

$$\gamma_k^{min} \leq s_{k,t} \leq \gamma_k^{max}, \forall k \in K, \forall t \in T$$

Objective

$$\min - \sum_{k\in K} \omega_k s_{k,n_T} + \sum_{(i,j)\in IJ^{\times}} \sum_{t\in T} \alpha_{i,j} x_{i,j,t}$$

S4. MInP(1) (general)

Variables

$[\mathfrak{x}_{j,i,q}^{start}..\mathfrak{x}_{j,i,q}^{end}) \in [\![0..n_T]\!],\ \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

$\mathfrak{b}_{j,i,q} \in \left[0, \beta_j^{max}\right] \subset \mathbb{R},\ \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

Expressions

$$s_{k,0} = \gamma_k^{init} - \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{-}}} \sum_{q\in Q_{i,j}} \mathbf{if}\left(\mathfrak{x}_{j,i,q}^{start} = 0\right)\rho_{i,k}^{-}\mathfrak{b}_{j,i,q}, \forall k \in K$$

$$s_{k,t} = s_{k,t-1} + \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{+},\ t-\tau_{i,j}\geq 0}} \sum_{q\in Q_{i,j}} \mathbf{if}\left(\mathfrak{x}_{j,i,q}^{end} = t\right)\rho_{i,k}^{+}\mathfrak{b}_{j,i,q} - \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{-}}} \sum_{q\in Q_{i,j}} \mathbf{if}\left(\mathfrak{x}_{j,i,q}^{start} = t\right)\rho_{i,k}^{-}\mathfrak{b}_{j,i,q},$$

$\forall k \in K, \forall t \in T\setminus\{0\}$

Constraints

$$\begin{pmatrix} \mathfrak{x}_{j,i,q}^{end} - \mathfrak{x}_{j,i,q}^{start} = \tau_{i,j} \\ \beta_j^{min} \leq \mathfrak{b}_{j,i,q} \leq \beta_j^{max} \end{pmatrix} \vee \begin{pmatrix} \mathfrak{x}_{j,i,q}^{end} - \mathfrak{x}_{j,i,q}^{start} = 0 \\ \mathfrak{b}_{j,i,q} = 0 \end{pmatrix}, \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$$

$$\sum_{\substack{i\in I \\ (i,j)\in IJ^{\times}}} \sum_{q\in Q_{i,j}} \mathbf{if}(t \geq \mathfrak{x}_{j,i,q}^{start}) - \mathbf{if}(t \geq \mathfrak{x}_{j,i,q}^{end}) \leq 1,\ \forall j \in J, \forall t \in T$$

$$\gamma_k^{min} \leq s_{k,t} \leq \gamma_k^{max}, \forall k \in K, \forall t \in T$$

Objective

$$\min - \sum_{k\in K} \omega_k s_{k,n_T} + \sum_{(i,j)\in IJ^{\times}} \sum_{q\in Q_{i,j}} \mathbf{if}(\mathfrak{x}_{j,i,q}^{end} - \mathfrak{x}_{j,i,q}^{start} > 0)\alpha_{i,j}$$

S5. MInLiP(1) (general)

Variables

$\left[\mathfrak{x}_{j,i,q}^{start}..\mathfrak{x}_{j,i,q}^{end}\right) \in [\![0..n_T]\!],\ \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

$\mathfrak{o}_j = \langle \mathfrak{o}_j(0), \mathfrak{o}_j(1), \ldots, \mathfrak{o}_j(|\tilde{I}_j| - 1)\rangle \in \mathbb{S}_{|\tilde{I}_j|}, \tilde{I}_j \in \mathbb{P}(I_j), \forall j \in J$

$\mathfrak{b}_{j,i,q} \in [0, \beta_j^{max}] \subset \mathbb{R},\ \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

Expressions

$$s_{k,0} = \gamma_k^{init} - \sum_{\substack{(i,j)\in IJ^{\times}\\(i,k)\in IK^{-}}} \sum_{q\in Q_{i,j}} \mathbf{if}\left(\mathfrak{x}_{j,i,q}^{start} = 0\right)\rho_{i,k}^{-}\mathfrak{b}_{j,i,q}, \forall k \in K$$

$$s_{k,t} = s_{k,t-1} + \sum_{\substack{(i,j)\in IJ^{\times}\\(i,k)\in IK^{+},\ t-\tau_{i,j}\geq 0}} \sum_{q\in Q_{i,j}} \mathbf{if}\left(\mathfrak{x}_{j,i,q}^{end} = t\right)\rho_{i,k}^{+}\mathfrak{b}_{j,i,q} - \sum_{\substack{(i,j)\in IJ^{\times}\\(i,k)\in IK^{-}}} \sum_{q\in Q_{i,j}} \mathbf{if}\left(\mathfrak{x}_{j,i,q}^{start} = t\right)\rho_{i,k}^{-}\mathfrak{b}_{j,i,q},$$
$$\forall k \in K, \forall t \in T\backslash\{0\}$$

Constraints

$$\left[\mathfrak{x}_{j,o_j(\mathfrak{o}_j(p-1))}^{start}..\mathfrak{x}_{j,o_j(\mathfrak{o}_j(p-1))}^{end}\right) < \left[\mathfrak{x}_{j,o_j(\mathfrak{o}_j(p))}^{start}..\mathfrak{x}_{j,o_j(\mathfrak{o}_j(p))}^{end}\right), \forall j \in J, \forall p \in \{1, \ldots, |\mathfrak{o}_j| - 1\}$$

$$\mathfrak{x}_{j,o_j(p)}^{end} - \mathfrak{x}_{j,o_j(p)}^{start} = \mathbf{if}\left(p \in \mathfrak{o}_j\right)\tau_{j,o_j(p)}, \forall p \in I_j$$

$$\mathbf{if}(p \in \mathfrak{o}_j)\beta_j^{min} \leq \mathfrak{b}_{j,o_j(p)} \leq \mathbf{if}\left(p \in \mathfrak{o}_j\right)\beta_j^{max}, \forall p \in I_j$$

$$\gamma_k^{min} \leq s_{k,t} \leq \gamma_k^{max}, \forall k \in K, \forall t \in T$$

Objective

$$\min - \sum_{k\in K} \omega_k s_{k,n_T} + \sum_{(i,j)\in IJ^{\times}} \sum_{q\in Q_{i,j}} \mathbf{if}(\mathfrak{x}_{j,i,q}^{end} - \mathfrak{x}_{j,i,q}^{start} > 0)\alpha_{i,j}$$

S6. MInP(2) (general)

Variables

$[\mathfrak{x}^{start}_{j,i,q}..\mathfrak{x}^{end}_{j,i,q}) \in [\![0..n_T]\!],\ \forall (i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

$b_{i,j,t} \in \left[0, \beta_j^{max}\right] \subset \mathbb{R},\ \forall (i,j) \in IJ^{\times}, \forall t \in T$

Expressions

$s_{k,0} = \gamma_k^{init} - \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{-}}} \rho^{-}_{i,k} b_{i,j,0}\,, \forall k \in K$

$s_{k,t} = s_{k,t-1} + \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{+},\ t-\tau_{i,j}\geq 0}} \rho^{+}_{i,k} b_{i,j,t-\tau_{i,j}} - \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{-}}} \rho^{-}_{i,k} b_{i,j,t}\,, \forall k \in K, \forall t \in T\backslash\{0\}$

Constraints

$\left(\mathfrak{x}^{end}_{j,i,q} - \mathfrak{x}^{start}_{j,i,q} = \tau_{i,j}\right) \vee \left(\mathfrak{x}^{end}_{j,i,q} - \mathfrak{x}^{start}_{j,i,q} = 0\right), \forall (i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

$\sum_{\substack{i\in I \\ (i,j)\in IJ^{\times}}} \sum_{q\in Q_{i,j}} \mathbf{if}(t \geq \mathfrak{x}^{start}_{j,i,q}) - \mathbf{if}(t \geq \mathfrak{x}^{end}_{j,i,q}) \leq 1,\ \forall j \in J, \forall t \in T$

$\mathbf{if}\left(\bigvee_{q\in Q_{i,j}} \begin{pmatrix} \mathfrak{x}^{start}_{j,i,q} = t \\ \mathfrak{x}^{end}_{j,i,q} - \mathfrak{x}^{start}_{j,i,q} \neq 0 \end{pmatrix}\right) \beta_j^{min} \leq b_{i,j,t} \leq \mathbf{if}\left(\bigvee_{q\in Q_{i,j}} \begin{pmatrix} \mathfrak{x}^{start}_{j,i,q} = t \\ \mathfrak{x}^{end}_{j,i,q} - \mathfrak{x}^{start}_{j,i,q} \neq 0 \end{pmatrix}\right) \beta_j^{max},\ \forall (i,j) \in IJ^{\times}, \forall t \in T$

$\gamma_k^{min} \leq s_{k,t} \leq \gamma_k^{max}, \forall k \in K, \forall t \in T$

Objective

$\min - \sum_{k\in K} \omega_k s_{k,n_T} + \sum_{(i,j)\in IJ^{\times}} \sum_{q\in Q_{i,j}} \mathbf{if}(\mathfrak{x}^{end}_{j,i,q} - \mathfrak{x}^{start}_{j,i,q} > 0)\alpha_{i,j}$

S7. MInLiP(2) (general)

Variables

$[\mathfrak{x}_{j,i,q}^{start} .. \mathfrak{x}_{j,i,q}^{end}) \in [\![0..n_T]\!],\ \forall (i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

$\mathfrak{o}_j = \langle \mathfrak{o}_j(0), \mathfrak{o}_j(1), \dots, \mathfrak{o}_j(|\tilde{I}_j| - 1) \rangle \in \mathbb{S}_{|\tilde{I}_j|}, \tilde{I}_j \in \mathbb{P}(I_j), \forall j \in J$

$b_{i,j,t} \in [0, \beta_j^{max}] \subset \mathbb{R},\ \forall (i,j) \in IJ^{\times}, \forall t \in T$

Expressions

$$s_{k,0} = \gamma_k^{init} - \sum_{\substack{(i,j) \in IJ^{\times} \\ (i,k) \in IK^{-}}} \rho_{i,k}^{-} b_{i,j,0}, \forall k \in K$$

$$s_{k,t} = s_{k,t-1} + \sum_{\substack{(i,j) \in IJ^{\times} \\ (i,k) \in IK^{+},\ t-\tau_{i,j} \geq 0}} \rho_{i,k}^{+} b_{i,j,t-\tau_{i,j}} - \sum_{\substack{(i,j) \in IJ^{\times} \\ (i,k) \in IK^{-}}} \rho_{i,k}^{-} b_{i,j,t}, \forall k \in K, \forall t \in T \backslash \{0\}$$

Constraints

$$\left[\mathfrak{x}_{j,o_j(\mathfrak{o}_j(p-1))}^{start} .. \mathfrak{x}_{j,o_j(\mathfrak{o}_j(p-1))}^{end}\right) < \left[\mathfrak{x}_{j,o_j(\mathfrak{o}_j(p))}^{start} .. \mathfrak{x}_{j,o_j(\mathfrak{o}_j(p))}^{end}\right), \forall j \in J, \forall p \in \{1, \dots, |\mathfrak{o}_j| - 1\}$$

$$\mathfrak{x}_{j,o_j(p)}^{end} - \mathfrak{x}_{j,o_j(p)}^{start} = \mathbf{if}(p \in \mathfrak{o}_j) \tau_{j,o_j(p)}, \forall p \in I_j$$

$$\mathbf{if}\left(\bigvee_{q \in Q_{i,j}} \begin{pmatrix} \mathfrak{x}_{j,i,q}^{start} = t \\ \mathfrak{x}_{j,i,q}^{end} - \mathfrak{x}_{j,i,q}^{start} \neq 0 \end{pmatrix}\right) \beta_j^{min} \leq b_{i,j,t} \leq \mathbf{if}\left(\bigvee_{q \in Q_{i,j}} \begin{pmatrix} \mathfrak{x}_{j,i,q}^{start} = t \\ \mathfrak{x}_{j,i,q}^{end} - \mathfrak{x}_{j,i,q}^{start} \neq 0 \end{pmatrix}\right) \beta_j^{max},\ \forall (i,j) \in IJ^{\times}, \forall t \in T$$

$$\gamma_k^{min} \leq s_{k,t} \leq \gamma_k^{max}, \forall k \in K, \forall t \in T$$

Objective

$$\min - \sum_{k \in K} \omega_k s_{k,n_T} + \sum_{(i,j) \in IJ^{\times}} \sum_{q \in Q_{i,j}} \mathbf{if}(\mathfrak{x}_{j,i,q}^{end} - \mathfrak{x}_{j,i,q}^{start} > 0) \alpha_{i,j}$$

S8. MIP (simplified)

Variables

$x_{i,j,t} \in \{0,1\},\ \forall (i,j) \in IJ^{\times}, \forall t \in T$

$b_{i,j,t} \in \left[0, \beta_j^{max}\right] \subset \mathbb{R},\ \forall (i,j) \in IJ^{\times}, \forall t \in T$

Expressions

$$s_{k,0} = \gamma_k^{init} - \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{-}}} \rho_{i,k}^{-} b_{i,j,0}\,, \forall k \in K$$

$$s_{k,t} = s_{k,t-1} + \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{+},\ t-\tau_{i,j}\geq 0}} \rho_{i,k}^{+} b_{i,j,t-\tau_{i,j}} - \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{-}}} \rho_{i,k}^{-} b_{i,j,t}\,, \forall k \in K, \forall t \in T \backslash \{0\}$$

Constraints

$$\sum_{\substack{i\in I \\ (i,j)\in IJ^{\times}}} \sum_{\substack{t'\in T \\ t-\tau_{i,j}+1\leq t'\leq t}} x_{i,j,t'} \leq 1,\ \forall j \in J, \forall t \in T$$

$$\beta_j^{min} x_{i,j,t} \leq b_{i,j,t} \leq \beta_j^{max} x_{i,j,t},\ \forall (i,j) \in IJ^{\times}, \forall t \in T$$

$$\gamma_k^{min} \leq s_{k,t} \leq \gamma_k^{max}, \forall k \in K, \forall t \in T$$

$$\sum_{t\in T} x_{i,j,t} = \psi_{i,j},\ \forall (i,j) \in IJ^{\times}$$

Objective

$$\min -\sum_{k\in K} \omega_k s_{k,n_T} + \sum_{(i,j)\in IJ^{\times}} \alpha_{i,j} \psi_{i,j}$$

S9. MInP(1) (simplified)

Variables

$[\mathfrak{x}_{j,i,q}^{start}..\mathfrak{x}_{j,i,q}^{end}) \in [\![0..n_T]\!],\ \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

$\mathfrak{b}_{j,i,q} \in \left[\beta_j^{min}, \beta_j^{max}\right] \subset \mathbb{R},\ \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

Expressions

$$s_{k,0} = \gamma_k^{init} - \sum_{\substack{(i,j)\in IJ^{\times}\\(i,k)\in IK^{-}}} \sum_{q\in Q_{i,j}} \mathbf{if}\left(\mathfrak{x}_{j,i,q}^{start} = 0\right)\rho_{i,k}^{-}\mathfrak{b}_{j,i,q}, \forall k \in K$$

$$s_{k,t} = s_{k,t-1} + \sum_{\substack{(i,j)\in IJ^{\times}\\(i,k)\in IK^{+},\ t-\tau_{i,j}\geq 0}} \sum_{q\in Q_{i,j}} \mathbf{if}\left(\mathfrak{x}_{j,i,q}^{end} = t\right)\rho_{i,k}^{+}\mathfrak{b}_{j,i,q} - \sum_{\substack{(i,j)\in IJ^{\times}\\(i,k)\in IK^{-}}} \sum_{q\in Q_{i,j}} \mathbf{if}\left(\mathfrak{x}_{j,i,q}^{start} = t\right)\rho_{i,k}^{-}\mathfrak{b}_{j,i,q},$$
$$\forall k \in K, \forall t \in T\backslash\{0\}$$

Constraints

$$\mathfrak{x}_{j,i,q}^{end} - \mathfrak{x}_{j,i,q}^{start} = \tau_{i,j}, \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$$

$$\sum_{\substack{i\in I\\(i,j)\in IJ^{\times}}} \sum_{q\in Q_{i,j}} \mathbf{if}(t \geq \mathfrak{x}_{j,i,q}^{start}) - \mathbf{if}(t \geq \mathfrak{x}_{j,i,q}^{end}) \leq 1,\ \forall j \in J, \forall t \in T$$

$$\gamma_k^{min} \leq s_{k,t} \leq \gamma_k^{max}, \forall k \in K, \forall t \in T$$

Objective

$$\min - \sum_{k\in K} \omega_k s_{k,n_T} + \sum_{(i,j)\in IJ^{\times}} \sum_{q\in Q_{i,j}} \alpha_{i,j}$$

S10. MInLiP(1) (simplified)

Variables

$\left[\mathfrak{x}_{j,i,q}^{start}..\mathfrak{x}_{j,i,q}^{end}\right) \in [\![0..n_T]\!],\ \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

$\mathfrak{o}_j = \langle \mathfrak{o}_j(0), \mathfrak{o}_j(1), \dots, \mathfrak{o}_j(|\widetilde{N}_j| - 1)\rangle \in \mathbb{S}_{|\widetilde{N}_j|}, \widetilde{N}_j \in \mathbb{P}(N_j), \forall j \in J$

$\mathfrak{b}_{j,i,q} \in [\beta_j^{min}, \beta_j^{max}] \subset \mathbb{R},\ \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

Expressions

$s_{k,0} = \gamma_k^{init} - \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{-}}} \sum_{q\in Q_{i,j}} \mathbf{if}\left(\mathfrak{x}_{j,i,q}^{start} = 0\right)\rho_{i,k}^{-}\mathfrak{b}_{j,i,q}, \forall k \in K$

$s_{k,t} = s_{k,t-1} + \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{+},\ t-\tau_{i,j}\geq 0}} \sum_{q\in Q_{i,j}} \mathbf{if}\left(\mathfrak{x}_{j,i,q}^{end} = t\right)\rho_{i,k}^{+}\mathfrak{b}_{j,i,q} - \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{-}}} \sum_{q\in Q_{i,j}} \mathbf{if}\left(\mathfrak{x}_{j,i,q}^{start} = t\right)\rho_{i,k}^{-}\mathfrak{b}_{j,i,q},$
$\forall k \in K, \forall t \in T\backslash\{0\}$

Constraints

$\mathfrak{x}_{j,i,q}^{end} - \mathfrak{x}_{j,i,q}^{start} = \tau_{i,j}, \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

$|\mathfrak{o}_j| = \sum_{\substack{i\in I \\ (i,j)\in IJ^{\times}}} \psi_{i,j}, \forall j \in J$

$\left[\mathfrak{x}_{j,o_j(\mathfrak{o}_j(p-1))}^{start}..\mathfrak{x}_{j,o_j(\mathfrak{o}_j(p-1))}^{end}\right) < \left[\mathfrak{x}_{j,o_j(\mathfrak{o}_j(p))}^{start}..\mathfrak{x}_{j,o_j(\mathfrak{o}_j(p))}^{end}\right), \forall j \in J, \forall p \in \{1, \dots, |\mathfrak{o}_j| - 1\}$

$\gamma_k^{min} \leq s_{k,t} \leq \gamma_k^{max}, \forall k \in K, \forall t \in T$

Objective

$\min - \sum_{k\in K} \omega_k s_{k,n_T} + \sum_{(i,j)\in IJ^{\times}} \sum_{q\in Q_{i,j}} \alpha_{i,j}$

S11. MInP(2) (simplified)

Variables

$[\mathfrak{x}_{j,i,q}^{start}..\mathfrak{x}_{j,i,q}^{end}) \in [\![0..n_T]\!],\ \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

$b_{i,j,t} \in \left[0, \beta_j^{max}\right] \subset \mathbb{R},\ \forall(i,j) \in IJ^{\times}, \forall t \in T$

Expressions

$s_{k,0} = \gamma_k^{init} - \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{-}}} \rho_{i,k}^{-} b_{i,j,0}\,, \forall k \in K$

$s_{k,t} = s_{k,t-1} + \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{+},\ t-\tau_{i,j}\geq 0}} \rho_{i,k}^{+} b_{i,j,t-\tau_{i,j}} - \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{-}}} \rho_{i,k}^{-} b_{i,j,t}\,, \forall k \in K, \forall t \in T\backslash\{0\}$

Constraints

$\mathfrak{x}_{j,i,q}^{end} - \mathfrak{x}_{j,i,q}^{start} = \tau_{i,j}, \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

$\sum_{\substack{i\in I \\ (i,j)\in IJ^{\times}}} \sum_{q\in Q_{i,j}} \mathbf{if}(t \geq \mathfrak{x}_{j,i,q}^{start}) - \mathbf{if}(t \geq \mathfrak{x}_{j,i,q}^{end}) \leq 1,\ \forall j \in J, \forall t \in T$

$\mathbf{if}\left(\bigvee_{q\in Q_{i,j}} \left(\mathfrak{x}_{j,i,q}^{start} = t\right)\right) \beta_j^{min} \leq b_{i,j,t} \leq \mathbf{if}\left(\bigvee_{q\in Q_{i,j}} \left(\mathfrak{x}_{j,i,q}^{start} = t\right)\right) \beta_j^{max},\ \forall(i,j) \in IJ^{\times}, \forall t \in T$

$\gamma_k^{min} \leq s_{k,t} \leq \gamma_k^{max}, \forall k \in K, \forall t \in T$

Objective

$\min - \sum_{k\in K} \omega_k s_{k,n_T} + \sum_{(i,j)\in IJ^{\times}} \sum_{q\in Q_{i,j}} \alpha_{i,j}$

S12. MInLiP(2) (simplified)

Variables

$[\mathfrak{x}_{j,i,q}^{start}..\mathfrak{x}_{j,i,q}^{end}) \in [\![0..n_T]\!],\ \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$

$\mathfrak{o}_j = \langle \mathfrak{o}_j(0), \mathfrak{o}_j(1), \dots, \mathfrak{o}_j(|\widetilde{N}_j| - 1)\rangle \in \mathbb{S}_{|\widetilde{N}_j|}, \widetilde{N}_j \in \mathbb{P}(N_j), \forall j \in J$

$b_{i,j,t} \in [0, \beta_j^{max}] \subset \mathbb{R},\ \forall(i,j) \in IJ^{\times}, \forall t \in T$

Expressions

$$s_{k,0} = \gamma_k^{init} - \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{-}}} \rho_{i,k}^{-} b_{i,j,0}, \forall k \in K$$

$$s_{k,t} = s_{k,t-1} + \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{+},\ t-\tau_{i,j}\geq 0}} \rho_{i,k}^{+} b_{i,j,t-\tau_{i,j}} - \sum_{\substack{(i,j)\in IJ^{\times} \\ (i,k)\in IK^{-}}} \rho_{i,k}^{-} b_{i,j,t}, \forall k \in K, \forall t \in T\setminus\{0\}$$

Constraints

$$\mathfrak{x}_{j,i,q}^{end} - \mathfrak{x}_{j,i,q}^{start} = \tau_{i,j}, \forall(i,j) \in IJ^{\times}, \forall q \in Q_{i,j}$$

$$|\mathfrak{o}_j| = \sum_{\substack{i\in I \\ (i,j)\in IJ^{\times}}} \psi_{i,j}, \forall j \in J$$

$$\mathbf{if}\left(\bigvee_{q\in Q_{i,j}} \left(\mathfrak{x}_{j,i,q}^{start} = t\right)\right)\beta_j^{min} \leq b_{i,j,t} \leq \mathbf{if}\left(\bigvee_{q\in Q_{i,j}} \left(\mathfrak{x}_{j,i,q}^{start} = t\right)\right)\beta_j^{max},\ \forall(i,j) \in IJ^{\times}, \forall t \in T$$

$$\left[\mathfrak{x}_{j,o_j(\mathfrak{o}_j(p-1))}^{start}..\mathfrak{x}_{j,o_j(\mathfrak{o}_j(p-1))}^{end}\right) < \left[\mathfrak{x}_{j,o_j(\mathfrak{o}_j(p))}^{start}..\mathfrak{x}_{j,o_j(\mathfrak{o}_j(p))}^{end}\right), \forall j \in J, \forall p \in \{1, \dots, |\mathfrak{o}_j| - 1\}$$

$$\gamma_k^{min} \leq s_{k,t} \leq \gamma_k^{max}, \forall k \in K, \forall t \in T$$

Objective

$$\min - \sum_{k\in K} \omega_k s_{k,n_T} + \sum_{(i,j)\in IJ^{\times}} \sum_{q\in Q_{i,j}} \alpha_{i,j}$$