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SCHOOL OF MATHEMATICS

HAWKING RADIATION AND BLACK HOLES

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Abstract

In the classical theory of general relativity, black holes can only ever absorb and never emit particles. In this thesis it is shown that a semi-classical treatment of the black hole spacetime formed by a collapsing star leads to black holes emitting particles as if they were black bodies with a temperature of $T_H = \frac{\hbar c^3}{8\pi G k_B M} \approx 6 \times 10^{-8}\,\mathrm{K} \times \frac{M_\odot}{M}$. This is due to the time dependant dynamics of the collapsing star and the fact that the notion of a particle can vary between different observers. Similarities between classical black hole and thermodynamic laws are also discussed, and it is shown that in treating a black hole as a thermodynamic object it will evaporate away in a time proportional to it's mass cubed. This thesis will also demonstrate the Unruh effect and shows that an accelerating observer will see a thermal bath of particles with a Maxwell-Boltzmann distribution. The tools developed to tackle the Unruh effect will prove helpful when studying Hawking radiation.

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1 Introduction

In this thesis, quantum field theory in globally hyperbolic spacetime will be studied. We will consider semi-classical systems by studying quantum effects in a fixed classical background geometry, assuming that quantum fields don't change the geometry of the spacetime. The effects of fixed classical curved space times on quantum fields will be studied, and it will be shown that particles creation comes from time dependant dynamics on the spacetime relative to asymptotically stationary regions of the spacetime as studied in [1, 2, 3, 4, 5]. In particular the time dependant collapse of a star into a black hole will be considered and the effect of this space time on modes coming from past null infinity and going to future null infinity will be studied. This will result in *Hawking radiation* [3]. We will find that the different Killing vectors on stationary space times which are used to define basis modes will lead to these effects. A stationary space time is defined as one which admits an everywhere time like Killing vector field K^a such that $g_{ab}K^aK^b < 0$ as defined in [5]. It will also be shown that as well as non-stationary space times leading to the creation of particles, an accelerating Rindler observer in flat Minkowski space time will observe a thermal bath of particles, and the tools used to show this will help analyse the modes in the space time of a collapsing star. The thermodynamic treatment of black holes will also be considered, and it will be found that by [6], the analogies between thermodynamic laws and classical black hole laws are more than just superficial. An important point to note is that since modes are not normalisable and so not square integrable, we must consider the propagation of wave packets instead, like the Gaussian wave packet as studied here [7]. In considering normalisable wave packets, the integrals we work with will converge and we will get finite numbers of particle emission in the Unruh effect and Hawking radiation. However, when working in a basis of wavepackets the calculations become much more difficult, but since wave packets are simply linear combination of basis modes we can do the calculations in a basis of modes assuming that all integrals converge and then switch to a basis of wave packets before calculating particle number. This is done in the chapter on the Unruh effect and Hawking radiation. In this thesis we will work in units $G = \hbar = c = 1$ unless stated otherwise.

2 | Globally hyperbolic spacetime

Suppose we want to consider quantum field theory on some spacetime (M, g) with M a manifold capturing the geometry of the spacetime and g a metric on M. In this thesis, only globally hyperbolic space times will be considered. Before defining globally hyperbolic spacetime we start with defining some key concepts. [8]

1. The future domain of dependence of a hypersurface Σ is the set of all points for which every causal curve necessarily intersects Σ. This domain is denoted by $D^+(\Sigma)$. Initial data on Σ can be extended used to find data about any point on $D^+(\Sigma)$, in other words the solution to a differential equation on $D^+(\Sigma)$ is determined by initial data on Σ, but the solution outside this domain is not.

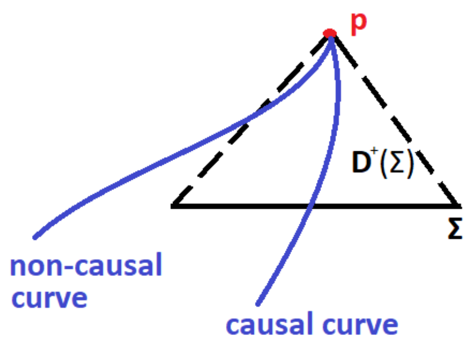

Figure 2.1: The *future domain of dependence* of Σ, with dashed lines being null rays. A point at $p \in D^+(\Sigma)$ is also show, every past causal curve through p must cross Σ.

2. The *past domain of dependence of a hyperspace* Σ is similarly all points p for which any future causal curve passing through p must intersect Σ. This domain is denoted by $D^-(\Sigma)$.

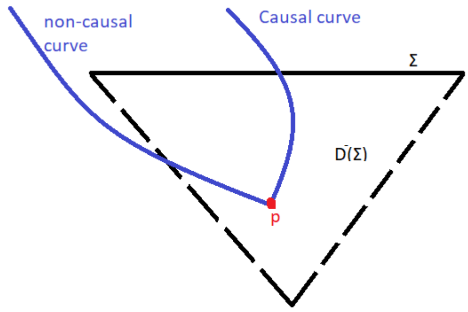

Figure 2.2: The past domain of dependence of Σ, any causal curve which passes through a point p must intersect Σ in the future.

3. Σ is a *Cauchy surface* of (M, g) if $D^+(\Sigma) \bigcup D^-(\Sigma) = M$. (M, g) can have many Cauchy surfaces, for example in $1+1$ Minkowski spacetime every horizontal line $t = C, \forall C \in \mathbb{R}$ is a Cauchy surface ($t = C$ is clearly a Cauchy surface since it is not possible for a causal worldline to "skip" a time t since it must be continuous and it's slope is bounded by the speed of light). In practice this means that if we can have initial data of a partial differential equation on a Cauchy surface we can solve this equation for all points on spacetime. (For a classical particle moving through spacetime if we have it's initial data on a Cauchy surface we can extend it's worldline via the geodesic equation.)

4. A spacetime (M, g) is *globally hyperbolic* if there exists at least one Cauchy surface Σ on it. For example, since we saw how $1+1$ Minkowski space has many Cauchy surfaces, $(\mathbb{R}^{1,1}, \eta)$ is globally hyperbolic. With η given by the $1+1$ Minkowski metric.

3 | Uniformly accelerated observer

This section follows [9] and [10]. In the following arguments, we will consider a $1+1$ Minkowski space time for simplicity. A uniformly accelerated observer will be defined as an observer which is accelerating at a constant rate in their own instantaneous reference frame. According to special relativity [11] such an observer could *not* be uniformly accelerating relative to a fixed inertial reference frame as this would result in the observer eventually surpassing the speed of light relative to that inertial reference frame.

Using the initial definition of a uniformly accelerated observer, we calculate the true world line of such an observer we will call p in our fixed inertial reference frame. We expect p to approach the speed of light as it accelerates but not reach it in our lab frame A with coordinate (t, x). Defining B as the inertial instantaneous rest frame of p with coordinates (t', x') with velocity v relative to the lab frame A (so the velocity dx/dt is not constant, only the velocity v is since it is the relative velocity between two inertial frames). Using Lorentz transformations, finding the velocity of p in A is done by differentiating x' with respect to t' twice to get:

$$\frac{d^2x'}{dt'^2} = \frac{\left(1-v^2\right)^{3/2}}{(1-v(dx/dt))^3}\frac{d^2x}{dt^2} \tag{1}$$

3.1 Rindler Coordinates

Thus we have found the Lorentz transform for the acceleration of p. We define $\frac{d^2x'}{dt'^2} = a = const > 0$ to be a uniformly accelerating observer and $v = \frac{dx}{dt}$ is the instantaneous velocity of the particle with respect to lab frame. This therefore gives us the the following differential equation [10]:

$$\frac{d^2x}{dt^2} = (1-(\frac{dx}{dt})^2)^{3/2}a \tag{2}$$

Solving this equation with $t = 0$ at $v = 0$ simultaneously with the metric $d\tau^2 = dx^2 - dt^2$ gives [9]:

$$t(\tau) = \frac{\sinh(a\tau)}{a} \qquad x(\tau) = \frac{\cosh(a\tau)}{a} \tag{3}$$

We therefore have the worldline of a uniformly accelerating observer p relative to a lab frame in coordinate (t, x). We then perform the transformation to more convenient coordinates:

$$\eta = \frac{\alpha}{a}\tau \qquad \xi = \frac{1}{\alpha}\ln(\frac{\alpha}{a}) \tag{4}$$

We will call these coordinates *Rindler coordinates* and they are related to (t, x) by the following equations:

$$x = \frac{1}{\alpha}e^{\alpha\xi}\cosh(\alpha\eta) \qquad t = \frac{1}{\alpha}e^{\alpha\xi}\sinh(\alpha\eta) \tag{5}$$

The observer p will henceforth be referred to a *Rindler observer*, denoted by R and follows a hyperbolic trajectory and travels along the path $x^\mu x_\mu = \frac{1}{x^2} - \frac{1}{t^2} = \frac{1}{\alpha^2}$. We assumed a is positive so R accelerates in the positive x direction.

These Rindler coordinates, however, only foliate part of space time namely the rightmost quadrant (region 1) of the spacetime diagram, as seen below:

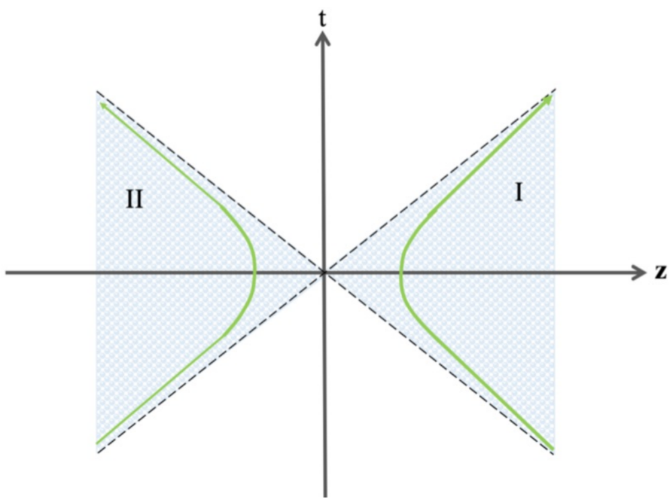

Figure 3.1: A diagram of spacetime showing the right quadrant (region 1) where our Rindler coordinates are defined. Region 2 is covered by Rindler coordinates by taking the negative of the t and x above, however the left and right quadrant cannot be simultaneously be covered by Rindler coordinates since η, ξ have the same range on each quadrant for these coordinates. [12]

We can cover the left wedge of spacetime by taking the negative of these coordinates, where an increase in η now results in the particle moving backwards in time t. These Rindler coordinates, however, cannot be used to cover both wedges simultaneously since both Rindler coordinates have the same range on each. This is an issue since there are clearly no Cauchy surfaces on either wedge (this can be demonstrated by the fact that if you take a line of constant η in the left wedge, this is a perfectly valid causal line but does not enter the right wedge). We can however expand a field ϕ in terms of a basis of solutions on *both* wedges. This will be explored later.

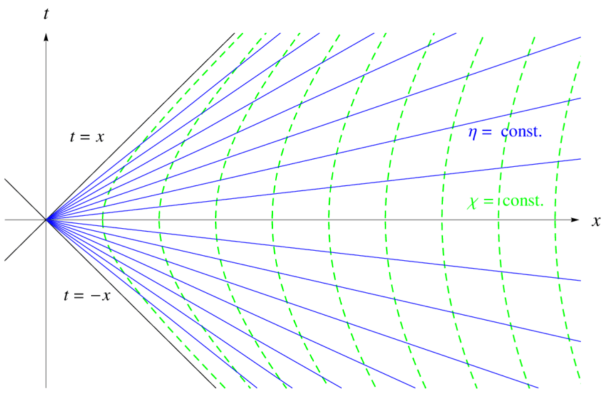

Figure 3.2: The region of spacetime covered by positive Rindler coordinates. Each green dotted line corresponds to a Rindler observer with a uniform acceleration, and lines of constant ξ (in this diagram $\xi = \chi$). Blue lines correspond to lines of constant η. [13]

We can then find the flat metric in terms of these coordinates, substituting into the metric $ds^2 = -dt^2 + dx^2$ gives the metric $ds^2 = e^{2\xi a}(-d\eta^2 + d\xi^2)$.

Since the metric is independent of η then ∂_η is a Killing vector in these coordinates. In (t, x) coordinates this corresponds to the vector:

$$\partial_\eta = \frac{\partial t}{\partial \eta}\partial_t + \frac{\partial x}{\partial \eta}\partial_x = a(x\partial_t + t\partial_x) \tag{6}$$

4 | Quantum field theory globally hyperbolic spacetime

4.1 QFT in globally hyperbolic 3+1 Minkowski spacetime

Taking the metric to have signature $(-,+,+,+)$, consider the massive scalar field Lagrangian density in flat Minkowski spacetime as studied in [14]:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 \tag{1}$$

with ϕ a scalar field. Varying the action and setting it to zero yields the follow massive Klein-Gordon (KG) equation of motion:

$$(\Box - m^2)\phi = 0 \tag{2}$$

with $\Box = \partial_\mu \partial^\mu$. We start by treating this field classically then we will promote it to a quantum field by promoting modes to operators and defining the relevant commutation relations. Start by noting that a basis of modes to the Klein Gordon equation is given by:

$$f = f_0 e^{ik_\mu x^\mu} \tag{3}$$

The arbitrary constant f_0 will be defined later by the commutator relation, and k_μ is a four vector who's negative square must be equal to the m^2 to satisfy the KG equation. $k^\mu k_\mu = -m^2$, $k^\mu = (\omega, k^i)$ and $k_\mu \equiv (-\omega, k_i)$ with $\omega > 0$ the frequency and $k^2 \equiv k^i k_i$ the wave vector yields the equation:

$$m^2 = \omega^2 - k^2 \tag{4}$$

Since the mass is fixed the wavevector determines the frequency up to a sign. We then define positive frequency modes [2] by the action of a time like Killing vector given by:

$$\mathfrak{L}_K g_{\mu v} \equiv K^\sigma \partial_\sigma g_{\mu v} + g_{\sigma v}\partial_\mu K^\sigma + g_{\mu\sigma}\partial_v K^\sigma = 0 \tag{5}$$

$$\mathfrak{L}_K f = -i\omega f \tag{6}$$

where $\mathfrak{L}_K$ is the Lie derivative along the Killing vector field and $g_{\mu\nu}$ is the metric. Any spacetime with a timelike Killing vector is called a *stationary spacetime*. We then define an inner product, requiring that our basis modes are orthogonal:

$$(f,h) = -i\int_\Sigma d^3x\,\left(f\dot{h}^* - \dot{f}h^*\right) \tag{7}$$

On flat Minkowski spacetime, with $\Sigma = \mathbb{R}^3$ a Cauchy surface (in the next section the inner product is show to be independent of the choice of Cauchy surface). For two basis modes with respect to the Killing vector ∂_t of the Minkowski metric $f_k = f_0 e^{-i\omega t + ik_i x^i}, f_{k'} = f_0' e^{-i\omega' t + ik_i' x^i}$ their inner product $(f_k, f_{k'})$ is given by:

$$\begin{aligned}(f_k, f_{k'}) &= -if_0 f_0' \int d^3x \quad i\left(\omega+\omega'\right) e^{-i(\omega-\omega')t} e^{i\left(k_i - k_i'\right)x^i} \\ &= f_0 f_0' \left(\omega+\omega'\right)\delta^3\left(k_i - k_i'\right)(2\pi)^3 e^{-i(\omega-\omega')t}\end{aligned} \tag{8}$$

for $k_i \neq k_i'$ clearly this inner product is just zero, for $k_i = k_i'$ this equation becomes:

$$f_0^2(2\omega)(2\pi)^3\delta^3\left(k_i - k_i'\right) \tag{9}$$

Therefore $f_0 = \frac{1}{(2\pi)^{3/2}\sqrt{2\omega}}$ normalises the inner product for $3+1$ dimensions. In $1+1$ dimensions this normalising factor is found to be $\frac{1}{\sqrt{4\pi\omega}}$. We therefore find the basis modes:

$$f_k = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega}} e^{ik_\mu x^\mu} \tag{10}$$

These bases give us the inner product:

$$(f_k, f_{k'}) = \delta\left(k - k'\right), \quad (f_k, f_{k'}^*) = 0, \quad (f_k^*, f_{k'}^*) = -\delta\left(k - k'\right) \tag{11}$$

Where the equation $(f_k, f_{k'}^*) = 0$ follows obviously from the definition of the inner product. We now have a inner product space of solutions to the Klein-Gordon equation, so we can write the general solution as a linear combination of basis solutions:

$$\phi(x) = \int d^3k(a_k f_k + a_k^* f_k^*) \tag{12}$$

Where a_k, a_k^* are some coefficients. We can canonically quantize this field by promoting these coefficients to operators acting on a Hilbert space and imposing the commutation relations $[a_k, a^\dagger_{k'}] = \delta(k - k')$ yielding:

$$\phi = \int d^3k(a_k f_k + a^\dagger_k f^*_k) \tag{13}$$

We also define a vacuum state $|0\rangle$ by the action of the *annihilation* operator on it $a^\dagger_k |0\rangle = 0, \forall k$, and the expected number of particles of momentum k in a state is given the expectation value of the operator $N_k = a^\dagger_k a_k$.

As derived earlier, the timelike Killing vector for a Rindler observer moving along a line of constant ξ is given by ∂_η, therefore a Rindler observer defines a basis of positive and negative modes $\{h_k, h^*_k\}$ respectively by:

$$\partial_\eta h_k = -i\omega h_k \quad \text{and} \quad \partial_\eta h^*_k = i\omega h^*_k \tag{14}$$

4.2 QFT in curved globally hyperbolic spacetime

For a curved spacetime given by (M, g) that is globally hyperbolic, we can study the scalar field ϕ by considering the action (following [2] and [1]):

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} m^2 \phi^2 \right) \tag{15}$$

Where ∇ is the covariant derivative. In the case of ϕ a scalar field $\nabla_\mu \phi = \partial_\mu \phi$. We analyze quantum fields in a fixed background, in other words quantum fields do not change the background (M, g).

Varying the action and setting it to zero yields the equations of motion:

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu v} \partial_v \right) \phi - m^2 \phi \equiv \Box \phi - m^2 \phi = 0 \tag{16}$$

This is the Klein Gordon equation we are familiar with. For flat spacetime, $\Box \phi = (\partial_\mu \partial^\mu - m^2)\phi$, however for curved spacetime the metric in the above equation will make the equations of motions more complex in general. The inner product of two solutions to in curved spacetime is given by:

$$(\phi_1, \phi_2) = -i \int_\Sigma d^3x \sqrt{\gamma} n^\mu \left(\phi_1 \nabla_\mu \phi^*_2 - \phi^*_2 \nabla_\mu \phi_1 \right) \tag{17}$$

Where γ is the induced metric on the Cauchy surface Σ and n^μ a normal vector to the Cauchy surface. This inner product is also independent under choice of Cauchy surface, so it provides a natural inner product to our space of solutions. This can be seen by the equations:

$$\begin{aligned}
(\phi_1, \phi_2)|_{\Sigma_1} - (\phi_1, \phi_2)|_{\Sigma_2} &= -i \int_{\Omega=\Sigma_1-\Sigma_2} d^3x \sqrt{\gamma} n^\mu \left(\phi_1 \nabla_\mu \phi_2^* - \phi_2^* \nabla_\mu \phi_1\right) \\
&= -i \int_{\partial\Omega} d^4x \sqrt{-g} D^\mu \left(\phi_1 \nabla_\mu \phi_2^* - \phi_2^* \nabla_\mu \phi_1\right) \\
&= -i \int_{\partial\Omega} d^4x \sqrt{-g} \left(\phi_1 m^2 \phi_2^* - \phi_2^* m^2 \phi_1\right) = 0
\end{aligned} \tag{18}$$

Where the second line is obtained using Stokes theorem and the third line simply follows from $(\nabla^\mu \phi_1)(\nabla_\mu \phi_2^*) - (\nabla^\mu \phi_2^*)(\nabla_\mu \phi_1) = 0$ and the equations of motion. The assumption that (M, g) is a stationary spacetime is also made, so there exists some timelike Killing vector field on it, as defined before.

5 | Bogoliubov transformations

In order to relate two different mode bases and annihilation/creation operators, let us first consider a globally hyperbolic stationary $(1+1)$ dimensional spacetime (M, g), and let us again consider the Klein-Gordon field on this spacetime (such that the wavevector k^i is a simply a scalar k because there is only one spacelike coordinate). Consider we had two different Killing vector fields K_f, K_h (for example if we had two different sets of coordinates of a stationary spacetime that permitted different Killing vectors [9], or if we have two stationary spacetimes separated by a non-stationary spacetime as we will later see [5]) then we could define two different bases of positive and negative modes $\{f_k, f_k^*\}$, $\{h_k, h_k^*\}$ such that $\mathfrak{L}_{K_f} f_k = -i\omega f_k$, and $\mathfrak{L}_{K_h} h_k = -i\omega h_k$ with $\omega > 0$. Then each field ϕ can be written in terms of these basis modes [14] as:

$$\phi = \int dk(a_k f_k + a_k^\dagger f_k^*) \tag{1}$$

$$\phi = \int dk(b_k h_k + b_k^\dagger h_k^*) \tag{2}$$

Where the operators satisfy the relations $[a_k, a_k^\dagger] = \delta(k - k')$ and $[b_k, b_k^\dagger] = \delta(k - k')$. The annihilation operators satisfy $a_k |0_a\rangle = 0$ and $b_k |0_b\rangle = 0$ where these two states are the respective vacuum states in each set of coordinates. Since each set of basis coordinates fully spans the the set of solutions of the Klein-Gordon equation, (because they are solutions on a Cauchy surface) we are be able to express each basis element as a linear combination of the other set of bases:

$$h_k = \int dk' A_{kk'} f_{k'} + B_{kk'} f_{k'}^* \tag{3}$$

$$h_k^* = \int dk' A_{kk'}^* f_{k'}^* + B_{kk'}^* f_{k'} \tag{4}$$

For some *Bogoliubov coefficients* $A_{kk'}, B_{kk'}$. Since this is a linear transformation we can write:

$$\begin{pmatrix} h_k \\ h_k^* \end{pmatrix} = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} f_{k'} \\ f_{k'}^* \end{pmatrix} \tag{5}$$

Where repeated indices in matrix multiplication imply integration over indices, for example $Af_{k'} \equiv \int dk' A_{kk'} f_{k'}$ and each vector on the left and right is "infinitely" long since k is a continuous parameter. We also have that by definition of the inner product that $(f_k, f_{k'}) = \delta(k-k')$, $(f_k^*, f_{k'}^*) = -\delta(k-k')$ and $(f_k, f_{k'}^*) = 0$ and $(\alpha f, \beta g) = \alpha\beta^*(f,g)$ therefore [2] we write:

$$\begin{aligned}
\left(h_{k_1}, h_{k_2}\right) &= \delta\left(k_1 - k_2\right) \\
&= \iint dk_3 \left(A_{k_1k_3} f_{k_3} + B_{k_1k_3} f_{k_3}^*\right), \int dk_4 \left(A_{k_2k_4} f_{k_4} + B_{k_2k_4} f_{k_4}^*\right)\Big) \\
= \int k_3 dk_4 \Big(A_{k_1k_3} A_{k_2k_4}^* \left(f_{k_3} f_{k_4}\right) &+ A_{k_1k_3} B_{k_2k_4}^* \left(f_{k_3} f_{k_4}^*\right) + B_{k_1k_3} A_{k_2k_4}^* \left(f_{k_3'}^* f_{k_4}\right) + B_{k_1k_3} B_{k_2k_4}^* \left(f_{k_3}^* f_{k_4}^*\right) \\
&= \iint dk_3 dk_4 A_{k_1k_3} A_{k_2k_4}^* \delta\left(k_3 - k_4\right) - B_{k_1k_3} B_{k_2k_4}^* \delta\left(k_3 - k_4\right) \\
&= \int dk_4 A_{k_1k_4} A_{k_4k_2}^+ - B_{k_1k_4} B_{k_4k_2}^+
\end{aligned} \tag{6}$$

We therefore find in our earlier notation that for normalisable wavepackets: $AA^\dagger - BB^\dagger = \mathbb{I}$.

By the fact that $(h_{k_1}, h_{k_2}^*) = 0$, and using a similar method as before we also find that:

$$\left(h_{k_1}, h_{k_2}^*\right) = 0 \tag{7}$$

$$= \left(\int dk_3 (A_{k_1k_3} f_{k_3} + B_{k_1k_3} f_{k_3}^*), \int dk_4 (A_{k_2k_4}^* f_{k_4}^* + B_{k_2k_4}^* f_{k_4})\right) \tag{8}$$

$$= \int dk_4 A_{k_1k_4} B_{k_4k_2}^t - B_{k_1k_4} A_{k_4k_2}^t \tag{9}$$

And again in our earlier notation for normalisable wavepackets: $AB^t - BA^t = 0$.

These relations allow us to find the inverse of the Bogoliubov transformation matrix since we find that for wavepackets:

$$\begin{pmatrix} AA^\dagger - BB^\dagger & -AB^t + BA^t \\ B^*A^\dagger - A^*B^\dagger & -B^*B^t + A^*A^t \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \tag{10}$$

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} A^\dagger & -B^t \\ -B^\dagger & A^t \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \tag{11}$$

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}^{-1} = \begin{pmatrix} A^\dagger & -B^t \\ -B^\dagger & A^t \end{pmatrix} \tag{12}$$

Since the field ϕ can be expanded in either basis:

$$\phi = \begin{pmatrix} b_k & b_k^\dagger \end{pmatrix} \begin{pmatrix} h_k \\ h_k^* \end{pmatrix} = \begin{pmatrix} a_{k'} & a_{k'}^\dagger \end{pmatrix} \begin{pmatrix} f_{k'} \\ f_{k'}^* \end{pmatrix} \tag{13}$$

$$\phi = \begin{pmatrix} b_k & b_k^\dagger \end{pmatrix} \begin{pmatrix} h_k \\ h_k^* \end{pmatrix} = \begin{pmatrix} a_{k'} & a_{k'}^\dagger \end{pmatrix} \begin{pmatrix} A^\dagger & -B^t \\ -B^\dagger & A^t \end{pmatrix} \begin{pmatrix} h_{k'} \\ h_{k'}^* \end{pmatrix} \tag{14}$$

$$\implies \begin{pmatrix} b_k \\ b_k^\dagger \end{pmatrix} = \begin{pmatrix} A^* & -B^* \\ -B & A \end{pmatrix} \begin{pmatrix} a_{k'} \\ a_{k'}^\dagger \end{pmatrix} \tag{15}$$

finally we can relate the operators:

$$b_k = \int dk' (A_{kk'}^* a_{k'} - B_{kk'} a_{k'}^\dagger) \tag{16}$$

$$b_k^\dagger = \int dk' (-B_{kk'} a_{k'} + A_{kk'} a_{k'}^\dagger) \tag{17}$$

Although we do not yet know the Bogoliubov coefficients we can now calculate the total expected number of particles in a vacuum state associated with a basis of modes $\{f_k, f_k^*\}$, $a_k |0\rangle_a = 0, \forall k$ with respect to another basis of modes $\{h_k, h_k^*\}$ defined by a different Killing vector field $\langle N_k^b \rangle = \langle b_k^\dagger b_k \rangle$ [14] as follows:

$$\langle N_k^b \rangle = \langle 0_a | {N^b}_k | 0_a \rangle = \langle 0_a | b_k^\dagger b_k | 0_a \rangle \tag{18}$$

$$= \int \int dk_1 dk_2 \langle 0_a | \left(-B_{kk_1} a_{k_1} + A_{kk_1} a_{k_1}^\dagger \right) \left(A_{kk_2}^* a_{k_2} - B_{kk_2}^* a_{k_2}^\dagger \right) | 0_a \rangle \tag{19}$$

$$= \int dk_1 \langle 0_a | B_{kk_1} B_{kk_1}^* | 0_a \rangle \tag{20}$$

$$\implies \langle N_k^b \rangle = \int dk' B_{kk'} B_{kk'}^* = \int dk' |B_{kk'}|^2 \tag{21}$$

Since states with an excited kth mode are orthogonal to states with an excited k'th mode for $k \neq k'$. $\langle N_k^b \rangle$ is the expected number of particles with wavevector k in a vacuum with respect to $\{f_k, f_k^*\}$. In our matrix notation, this is $\langle N_k^b \rangle = (BB^\dagger)_{kk}$. These expected particle numbers are nonzero in general! This can be interpreted as one observer's vacuum can be occupied by particles to another observer in general.

6 | The Unruh effect

Now that an equation for the number of particles in a vacuum of one mode basis to another mode basis has been found, we can ask the question: what is the particle number a Rindler observer observes when it observes a Minkowski vacuum? To simplify calculations we consider this question in the context of the massless scalar field in $1+1$ dimensional spacetime. This has action [14]:

$$S = \int d^2x(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi) \tag{1}$$

Varying the action with respect to ϕ and setting this to zero yields the equation $\Box\phi = \partial_\mu\phi\partial^\mu\phi = 0$. The Minkowski basis modes are the same as defined before, and are given by $\{f_k, f_k^*\}$ with vacuum and annihilation operator defined by $a_k\left|0_M\right\rangle = 0\forall k$. In the right Rindler wedge we have that the equation of motion is $\Box\phi = e^{-2\xi}(-\partial_\eta^2 + \partial_\xi^2)\phi = 0$ by substituting the Rindler coordinates into the Klein Gordon equation of motion. This yields the plane wave equation [2]:

$$h_k^R = \frac{1}{\sqrt{4\pi\omega}}e^{ik^\mu x_\mu}, \qquad k^\mu = (\omega, k), \qquad x^\mu = (\eta, \xi) \tag{2}$$

Where the R index indicates these are basis modes on the right wedge only. Positive frequencies defined by the timelike Killing vector we found earlier $\partial_\eta h_k^R = -i\omega h_k^R, \quad \omega > 0$. A field solution on the right wedge can therefore be expanded in this basis of modes as:

$$\phi^R = \int dk(b_k h_k^R + b_k^\dagger h_k^{R*}) \tag{3}$$

Where the R index on ϕ^R denotes that is is only defined on the right wedge of Minkowski space. The vacuum defined by this annihilation operator $b_k\left|0_R\right\rangle = 0, \forall k$ where the subscript R denotes that this is the vacuum defined for a Rindler observer on the right wedge.

Before continuing, one should note that the Rindler modes $\{h_k^R, h_k^{*R}\}$ are not defined on any Cauchy surface of the whole Minkowski space, for reasons explained before. These modes are only defined on a Cauchy surface of the right Rindler wedge and therefore do note form a compete basis of Minkowski spacetime. We can therefore expand any Rindler mode in terms of Minkowski modes (since Minkowski modes are complete over the right Rindler wedge), but the inverse is not true [2]. Rindler modes, in other words, are only complete over a subspace of Minkowski spacetime. However, we can define these Rindler modes over a Cauchy surface of Minkowski space if we define modes by:

$$h_k^R = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu} & \text{in right Rindler wedge} \\ 0 & \text{in left Rindler wedge} \end{cases} \tag{4}$$

then these modes *are* defined on a Cauchy surface of Minkowski spacetime $\eta = t = 0$ but still are not complete on Minkowski spacetime. To make a complete basis we define positive modes on the left wedge with respect to the killing vector on the left wedge $-\partial_\eta$ by:

$$h_k^L = \begin{cases} 0 & \text{in right Rindler wedge} \\ \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu} & \text{in left Rindler wedge} \end{cases} \tag{5}$$

Note that a field expanded in terms of the left and right Rindler basis modes will have vacua given by $|0_L\rangle$ and $|0_R\rangle$ respectively. Expanding a field in Minkowski space in terms of the left *and* right Rindler basis modes will yield the vacuum $|0\rangle_L \otimes |0\rangle_R$. The annihilation operator on the right Rindler vacuum is given by $b_k^L \otimes \mathbb{1}_R$ and the right annihilation operator by $\mathbb{1}_L \otimes b_k^R$. For simplicity the identity in tn the tensor product will be omitted

The left and right Rindler modes form a complete basis of modes of Minkowski space, and the union of these two regions contains a Cauchy surface (namely the horizontal line $t = 0$). [5] We can therefore expand any field in terms of these complete bases:

$$\phi = \int dk (b_k^R h_k^R + b_k^L h_k^L + h.c.) \tag{6}$$

$$\phi = \int dk (a_k f_k + h.c.) \tag{7}$$

Where *h.c.* stands for the Hermitian conjugate of what comes before the letters. Also note that the above modes are not normalizable, so some of the integrals below will not converge. We could switch to a basis of wave packets as outlined in the introduction, however working in a basis of modes makes the calculations easier so we will manipulate the integrals as if they do converge since if we worked in a basis of wavepackets the integral of the linear combinations of these modes would converge [5]. Using our Bogoliubov transformations, we are now ready to ask the question: what does an eternally accelerating Rindler observer in the right wedge see when they observe a Minkowski vacuum? In other words, what is the value of $\langle 0_M | \, b_k^R b_k^{R\dagger} \, |0_M\rangle$? Since we are working with a massless scalar field in flat spacetime, we find that the equation of motion $(-\partial_t^2 + \partial_x^2)\phi = 0$ requires that the wavevector $k_\mu = (-\omega, k)$ must satisfy $\omega = |k| > 0$ and $-\infty < k < \infty$. We can therefore equally integrate over ω when performing Bogoliubov transformations.

Introducing null coordinate $u = t - x$ and $v = t + x$ the Minkowski metric becomes $ds^2 = -dudv$. Wavevectors $k > 0$ correspond to right moving waves:

$$(4\pi\omega)^{-1/2} e^{-i\omega u} \tag{8}$$

whereas wavevectors $k < 0$ correspond to left moving waves:

$$(4\pi\omega)^{-1/2} e^{-i\omega v} \tag{9}$$

On the right wedge with null coordinates $u' = \eta - \xi$ and $v' = \eta + \xi$, the metric becomes $ds^2 = -e^{2a\xi} du' dv'$ and the right and left moving waves are similarly given by $(4\pi\omega)^{-1/2} e^{-i\omega u'}$ and $(4\pi\omega)^{-1/2} e^{-i\omega v'}$ respectively. The null coordinates are related by:

$$\begin{aligned} u &= -a^{-1} e^{-au'} \\ v &= a^{-1} e^{av'} \end{aligned} \tag{10}$$

Since the right Rindler wedge is a subspace of Minkowski spacetime we can expand any Rindler basis mode in terms of Minkowski basis modes (but not visa versa, since some Minkowski modes are defined on the the left Rindler wedge for example).

$$h_\omega^R(u) = \int d\omega' (A_{\omega\omega'} f_{\omega'} + B_{\omega\omega'} f_{\omega'}^*) \tag{11}$$

Using our above definitions of the Minkowski modes in terms of null coordinates we can express h_ω^R as the following integral (limiting our consideration to right moving observers for simplicity):

$$h_\omega^R(u) = \int d\omega' \left(A_{\omega\omega'} \frac{1}{2\pi} \sqrt{\frac{\pi}{\omega'}} e^{-i\omega' u} + B_{\omega\omega'} \frac{1}{2\pi} \sqrt{\frac{\pi}{\omega'}} e^{i\omega u} \right) \tag{12}$$

Note this equation looks a lot like a Fourier transform, in fact if we write the Fourier transform of this mode we get [2]:

$$h^R_\omega(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega' e^{-i\omega' u}\tilde{h}_\omega\left(\omega'\right) \tag{13}$$

$$h^R_\omega(u) = \frac{1}{2\pi}\int_{0}^{+\infty} d\omega' e^{-i\omega' u}\tilde{h}_\omega\left(\omega'\right) + \frac{1}{2\pi}\int_{0}^{+\infty} d\omega' e^{i\omega' u}\tilde{h}_\omega\left(-\omega'\right) \tag{14}$$

Where $\tilde{h}_{\omega'}\left(\omega'\right)$ is just the Fourier transform of $h^R_\omega(u)$, and the second equation is found by splitting the Fourier transform integral into two integrals over $[-\infty, 0]$ and $[0, \infty]$ then taking $\omega' \to -\omega'$ and swapping the integration limits. Now this modified equation *really* looks like our expansion of our right Rindler mode over Minkowski modes for:

$$A_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}}\tilde{h}_\omega\left(\omega'\right), \quad B_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}}\tilde{h}_\omega\left(-\omega'\right). \tag{15}$$

Note that we already have an expression relating $A_{\omega\omega'}$ and $B_{\omega\omega'}$, $AA^\dagger - BB^\dagger = \mathbb{1}$ so if we can relate $\tilde{h}_\omega\left(\omega'\right)$ and $\tilde{h}_\omega\left(-\omega'\right)$ we can determine $B_{\omega\omega'}$ and therefore calculate the expected number of particles a Rindler observers sees in a Minkowksi vacuum, as outlined in the Bogoliubov transformations section.
Finding the desired relation between the above Fourier transforms turns out to be computationally lengthy. We begin with the Fourier transform:

$$\tilde{h}_\omega\left(\omega'\right) = \int_{-\infty}^{+\infty} du e^{i\omega' u} h^R_\omega(u), \tag{16}$$

Using the definition of h^R_ω as defined before on the right and left Rindler wedge, and writing $e^{-i\omega(\eta-\xi)} = e^{-i\omega u'}$ then we find that the condition of "on the right wedge" is equivalent to the condition $u < 0$ and "on the left wedge" is equivalent to $u > 0$. h^R_ω then becomes:

$$h^R_\omega = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{i\frac{\omega}{a}\ln(-au)} & \text{for } u < 0 \\ 0 & \text{for } u > 0 \end{cases} \tag{17}$$

Using that that $u' = -\frac{1}{a}\ln(-au)$, and is only defined in the right Rindler wedge. We therefore find the Fourier transform becomes:

$$\tilde{h}_\omega\left(\omega'\right) = \int_{-\infty}^{0} du e^{i\omega' u}\frac{1}{\sqrt{4\pi\omega}} e^{(i\omega/a)\ln(-au)} \tag{18}$$

Since the function $\ln(-z)$ is multi valued on the complex plane we will give it a branch cut on the positive real axis. $\ln(-z)$ is now holomorphic on the remainder of the complex plane. We can then examine the above integral using the path integral over the path shown below:

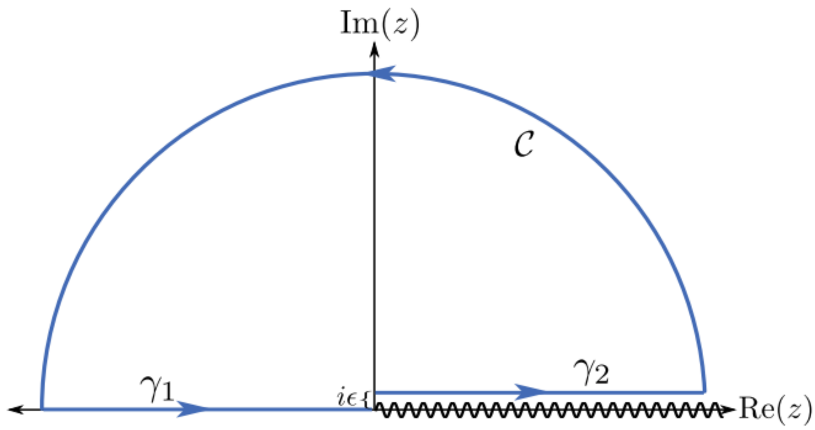

Figure 6.1: The contour over which we will examine $\tilde{h}_\omega(\omega')$, the integral over γ_1 corresponds to $\tilde{h}_\omega(\omega')$. [1]

When taking the contour integral over this closed loop, we omit the integral from 0 to $i\epsilon$ as this will simply go to zero as we take ϵ to zero. The integral over C will also vanish as we take it's radius to infinity by Jordan's lemma [15]. To avoid integrating over the branch cut we integrate over the line which is $i\epsilon$ above the real axis. Since the integrand is holomorphic inside of the contour, there are no poles inside of this contour, and therefore Cauchys formula implies that:

$$0 = \oint due^{i\omega' u} e^{i\frac{\omega}{a}\ln(-au)} = \left\{ \int_{\gamma_1} + \int_{\gamma_2} + \int_C \right\} due^{i\omega' u} e^{i\frac{\omega}{a}\ln(-au)} \tag{19}$$

And therefore the integral over γ_1 (which is clearly just equal to $\tilde{h}_\omega(\omega')$ as defined above) is equal to the negative integral over γ_2. We can then write:

$$\begin{aligned} \tilde{h}_\omega\left(\omega'\right) &= -\frac{1}{\sqrt{4\pi\omega}} \int_{i\epsilon}^{\infty+i\epsilon} due^{i\omega' u} e^{i\frac{\omega}{a}\ln(-au)} \\ &= -\frac{1}{\sqrt{4\pi\omega}} \int_{-\infty-i\epsilon}^{-i\epsilon} due^{-i\omega' u} e^{i\frac{\omega}{a}\ln(au)} \quad (u \leftrightarrow u) \end{aligned} \tag{20}$$

Now taking ϵ to zero and noting that $\ln(-1) = -i\pi$ for this branch cut so $\ln(au) = \ln(-au) - i\pi$ we can write:

$$\begin{aligned} \tilde{h}_\omega\left(\omega'\right) &= -\frac{1}{\sqrt{4\pi\omega}} \int_{-\infty}^{0} due^{-i\omega' u} e^{i\frac{\omega}{a}[\ln(au)-i\pi]} \quad (\epsilon \to 0) \\ &= -e^{\pi\omega/a}\tilde{h}_\omega\left(-\omega'\right) \end{aligned} \tag{21}$$

Since we have related $\tilde{h}_\omega(\omega')$ and $\tilde{h}_\omega(-\omega')$, and looking back to the expressions of the Bogoliubov coefficients in terms of these functions, we can write:

$$A_{\omega\omega'} = -e^{\pi\omega/a} B_{\omega\omega'} \tag{22}$$

And using the equation $AA^\dagger - BB^\dagger = \mathbb{1}$ for normalisable wavepackets we found earlier, and assuming that $B_{\omega\omega'}$ is square integrable by considering the Bogoliubov coefficients for a linear combination of modes that make up a normalisable wave packet, we find that the Rindler observer sees in the Minkowski vacuum:

$$\boxed{\langle N_w \rangle = \int d\omega' |B_{\omega\omega'}|^2 = \frac{1}{e^{2\pi\omega/a} - 1}} \tag{23}$$

Which is non-zero! The notion of a vacuum is therefore different for each observer. Furthermore, it has the same thermal distribution as one would see in a black body radiating object for $T = a/2\pi$. A right Rindler observer will be moving through a thermal bath of particles where a Minkowski observer will observe a vacuum, this effect is called the *Unruh effect* and the tools we used to derive it will come in handy when we examine Hawking radiation.

7 Particle production in non-stationary spacetime

We now know that on a globally hyperbolic spacetime (M, g) when we have two different Killing vectors we can define two sets of positive and negative modes, and in general the positive modes with respect to one of the Killing vectors can be expressed as a linear combination of positive and negative modes with respect to the other Killing vector. When a mode f_k can be expressed as a mixture of positive and negative modes (h_k and h_k^*) this results in a non zero Bogoliubov coefficient $B_{kk'}$ and this results in two observers seeing different numbers of particles in a state. Specifically, we showed that accelerating observers see a thermal bath of bath of particles in the vacuum of a stationary observer. We can then ask the question: can two stationary observers in different regions of a spacetime see different numbers of particles in each others vacua? [16]
In order to define positive and negative modes for the two observers, they must both lie in a stationary spacetime so they can both have a Killing vector with respect to which they can define their modes. As shown in page 3 of [5], for a stationary spacetime it is always possible to choose some coordinates (t, x^i) such that the timelike Killing vector is given by $\partial/\partial t$, and the metric is therefore independent of t. Consider now the metric of a globally stationary spacetime, then the metric is independent of t everywhere and therefore $\partial/\partial t$ is the timelike Killing vector everywhere, all stationary observers on the spacetime define positive modes as a linear combination of the other stationary observers positive modes, and they therefore all see no particles in each others vacua. This is not very interesting.
We can now consider instead the case where we have two regions of stationary spacetime (they must be stationary so we can define positive modes with respect to a Killing vector on them), separated by a non-stationary region of spacetime. We will consider $\mathcal{M}^-$, $\mathcal{M}^0$, and $\mathcal{M}^+$ such that $\mathcal{M}^+ \cup \mathcal{M}^- \cup \mathcal{M}^0 = M$. The spacetime $(\mathcal{M}^+, g)$ defined by $t < t_0$ and $(\mathcal{M}^-, g)$ defined by $t > t_1$ are both stationary, with $\partial/\partial t$ the timelike Killing vector on $(\mathcal{M}^-, g)$. $(\mathcal{M}^0, g)$ is the spacetime defined by $t_0 < t < t_1$ and is not stationary. This spacetime is illustrated below:

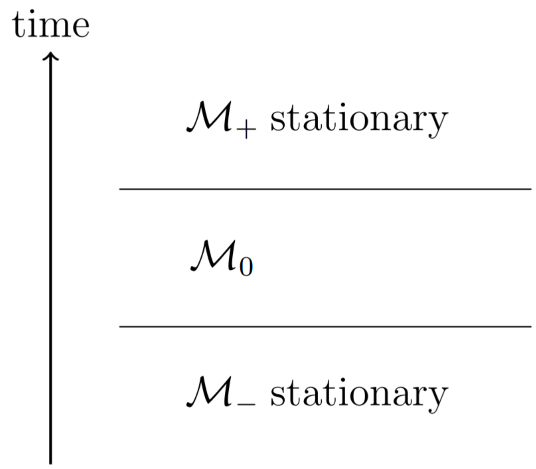

Figure 7.1: The sandwich spacetime (M, g) which is stationary asymptotically in the future and past and non-stationary on some time interval. [5]

The fact that this spacetime is globally hyperbolic implies that any solution to the Klein Gordon equation on a Cauchy surface on any of the regions extends to the entire sandwich spacetime. Now we can consider what an observer on $\mathcal{M}^+$ sees when they look the vacuum of an observer in $\mathcal{M}^-$. Coordinates are chosen such that the metric on $\mathcal{M}^-$ is independent of t, and on $\mathcal{M}^0$ the metric evolves in time such that in general the Killing vector on $\mathcal{M}^+$ will be different.
Now that we have two different regions with different Killing vectors, we can define positive and negative basis modes with respect to them and using the same procedure as in the Bogoliubov transformations section, we find that the expected number of particles seen by an observer on $\mathcal{M}^+$ in the kth mode in a vacuum to an observer on $\mathcal{M}^-$ is $(BB^\dagger)_{kk}$.

8 | Quantum field theory in the spacetime of a collapsing star

In this section we will consider the a massless scalar field in the spacetime of a collapsing star. We will first briefly introduce conformal coordinates for the Schwarzschild spacetime, this section follows [1, 2, 3, 17, 18].

- For the Schwarzschild spacetime with metric $ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$ we first introduce the tortoise coordinate to deal with the divergence at the horizon:

$$r_* = r + 2M \ln\left(\frac{r - 2M}{2M}\right) \tag{1}$$

  This has range $-\infty < r_* < \infty$ over the range $2M < r < \infty$. We then introduce lightcone coordinates:

$$\begin{aligned} u &= t - r_* \\ v &= t + r_*, \\ ds^2 &= -\left(1 - \frac{2M}{r}\right) dudv + r^2 d\Omega_2^2 \end{aligned} \tag{2}$$

  Note that these coordinates are different to the u, v defined in the section on the Unruh effect. Lines of constant v and u correspond to ingoing and outgoing null geodesics respectively. Introducing new coordinates:

$$\begin{gathered} U = -\exp\left(-\frac{u}{4M}\right) \qquad V = \exp\left(\frac{v}{4M}\right) \\ ds^2 = -\frac{32M^3}{r}\exp\left(-\frac{r}{2M}\right) dUdV + r^2 d\Omega_2^2, \end{gathered} \tag{3}$$

  These are called the Kruskal coordinates, and finally introducing conformal coordinates:

$$\begin{aligned} \tilde{U} &= \arctan(U) \\ \tilde{V} &= \arctan(V), \quad -\frac{\pi}{2} < \tilde{U}, \tilde{V} < \frac{\pi}{2} \end{aligned} \tag{4}$$

Null geodesics again correspond to $\tilde{U}$ or $\tilde{V}$ being constant.

Now that we have conformal coordinates to examine the Schwarzchild on a Penrose diagram, we can return to our collapsing star. The Penrose diagram of such a collapsing shell is given by:

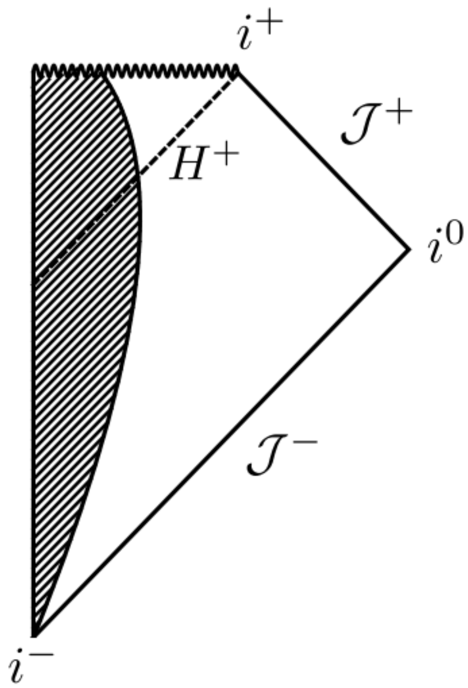

Figure 8.1: The Penrose diagram of the spacetime of a collapsing star. The shaded region corresponds to the inside of the star. It starts at i^- since the star is massive and therefore follows a timelike curve. H^+ is the event horizon of the black hole, i^+ and i^- are future and past timelike infinities respectively, i^0 is the spacelike infinity, and $\mathcal{J}^+$ and $\mathcal{J}^-$ are future and past null infinities respectively. The wavy line is the singularity. [3]

Each point on the above diagram corresponds to a 2-sphere of radius r. This spacetime is also globally hyperbolic since for example $\mathcal{J}^-$ is a Cauchy surface. Birkhoff's theorem states that any spherically symmetric solution of the vacuum field equations have to be stationary and asymptotically flat (Birkhoff's theorem actually states a spherically symmetric spacetime must be static which is a stronger condition than stationary, but we only care about whether or not the spacetime is stationary). The region outside the collapsing star therefore is described by the stationary Schwarzchild metric. The spacetime inside a collapsing star is not stationary since the collapse involves complicated dynamics [1] (the stress-energy tensor inside the star is clearly time dependant and so the metric must be too). This spacetime is, however, stationary at the future and past null infinities $\mathcal{J}^+$ and $\mathcal{J}^-$, so we can define positive and negative modes with respect to a some timelike Killing vectors on these regions. While the basis of modes on $\mathcal{J}^-$ is complete on this spacetime since $\mathcal{J}^-$ is a Cauchy surface, the basis of modes on $\mathcal{J}^+$ is not since causal curves can end on the singularity without passing through $\mathcal{J}^+$. The basis on $H^+ \cup \mathcal{J}^+$ is complete since together these regions form a Cauchy surface. We then define the modes:

- f_ω : positive frequency on $\mathcal{J}^-$
- p_ω : positive frequency on $\mathcal{J}^+$and zero on H^+
- q_ω : positive frequency on $H^+$and zero on $\mathcal{J}^+$.

Since the Killing vector ∂_t is null on H^+, we simply define $\{q_\omega, q^*_\omega\}$ as some arbitrary basis (these basis modes won't matter much anyway). We can then expand any field configuration in either basis $\{f_\omega, f^*_\omega\}$ and $\{p_\omega, q_\omega, p^*_\omega, q^*_\omega\}$.

$$\phi = \int d\omega(a_\omega f_\omega + h.c.) = \int d\omega(b_\omega p_\omega + c_\omega q_\omega + h.c.) \tag{5}$$

We want to then find what an observer at $\mathcal{J}^+$ observes in the vacuum state $|0_{in}\rangle$ with respect to an observer at $\mathcal{J}^-$, where $a_\omega |0_{in}\rangle = 0 \forall \omega$ where the *in* index refers to the fact that we are only interested in incoming modes (towards the origin) coming from $\mathcal{J}^-$. Since $\{f_\omega \, f^*_\omega\}$ is a complete basis, we can expand a the positive mode p_ω as:

$$p_\omega = \int d\omega'(A_{\omega\omega'} f_{\omega'} + B_{\omega\omega'} f^*_{\omega'}) \tag{6}$$

In order to determine $B_{\omega\omega'}$ and find the number of particles observed in $|0_{in}\rangle$ by an observer at future null infinity, we must solve the Klein Gordan equation $\Box\phi = 0$ to find expressions for the positive modes on $\mathcal{J}^-$ and $\mathcal{J}^+$. In the following calculation note that the field configuration will be represented by φ and the polar coordinate by ϕ. Then using the equation:

$$\Box\varphi = \frac{1}{\sqrt{-g}}\partial_\mu \left(g^{\mu v}\sqrt{-g}\partial_v\varphi\right) \tag{7}$$

We find that [2]:

$$\Box\varphi = \partial_t \left(-\left(1 - \frac{2M}{r}\right)^{-1} \partial_t\varphi \right) + \frac{1}{r^2}\partial_r \left(\left(1 - \frac{2M}{r}\right) r^2\partial_r\varphi \right) + \frac{1}{r^2}\Box_{S^2}\varphi, \tag{8}$$

where $\Box_{S^2}\varphi = \left(\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) + \frac{1}{\sin^2\theta}\partial^2_\phi\right)\varphi$ and $\sqrt{-g} = r^2\sin\theta$. Using the ansatz $\varphi = \frac{f(r,t)}{r}Y_l^m(\theta,\phi)$ (note that the $f(r,t)$ in this ansatz and the positive mode f_ω on $\mathcal{J}^-$ are different! When referring to the mode there will always be a lower index ω), and the fact that $\Box_{S^2}Y_l^m = -l(l+1)Y_l^m$, the Klein Gordon equation reduces to:

$$-\left(1-\frac{2M}{r}\right)^{-1}\partial_t^2 f + \frac{2M}{r^2}\left(\partial_r f - \frac{1}{r}f\right) + \left(1-\frac{2M}{r}\right)\partial_r^2 f - \frac{l(l+1)}{r^2}f = 0 \qquad (9)$$

Then using the tortoise coordinate:

$$r^* = r + 2M\ln\left(\frac{r}{2M} - 1\right) \qquad (10)$$

We find that the Klein Gordon equation becomes:

$$\left(-\partial_t^2 + \partial_{r^*}^2\right)f - \left(1-\frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right)f = 0 \qquad (11)$$

Defining the potential $V(r) = \left(1-\frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right)$ we find that:

$$\left(-\partial_t^2 + \partial_{r^*}^2\right)f = V(r)f \qquad (12)$$

Using the ansatz $f(r,t) = e^{-i\omega t}R(r)$ [1], we can again simplify the equation. Note that R, V, f all depend on the value of l of the spherical harmonic, and R also depends on the value of ω. These dependencies are not written explicitly for simplicity. We can then find the equation:

$$\left(\partial_{r_*}^2 + \omega^2\right)R = V(r)R \qquad (13)$$

Note that at $r \to \infty$ and $r \to 2GM$, the potential goes to zero and therefore this potential acts a potential barrier.

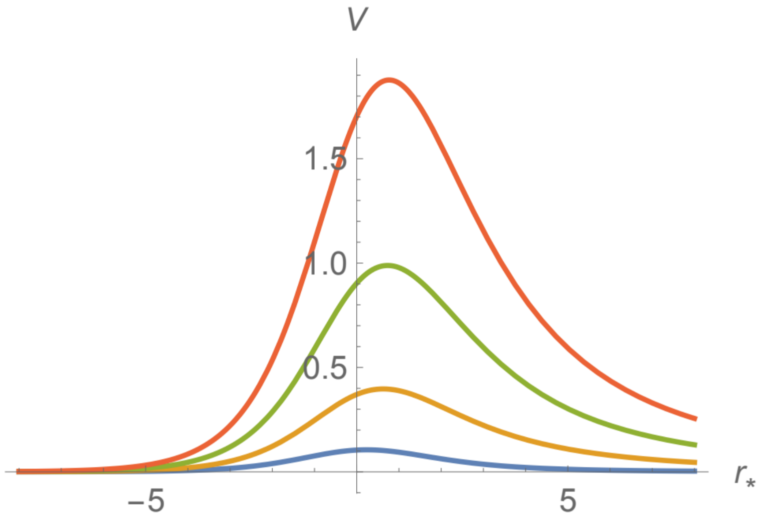

Figure 8.2: The potential barrier for different values of l given by the different colours. Any incoming waves from the left or right will be partially reflected and partially transmitted through the barrier. [4]

The height of this potential barrier is of order l^2 and the region between this barrier and the horizon will be populated by a *thermal atmosphere* of particles mostly with a lower energy than the maximum value of V [4] that are stopped from escaping by the potential barrier. These particles, however are not very relevant to the study of Hawking radiation since they are mostly confined to this region.
Since at infinity, the potential goes to zero, the solutions at $\mathcal{J}^{\pm}$ to this equation $\partial^2_{r_*} R = -\omega^2 R$ are simply plane waves $R = e^{\pm i\omega r_*}$. We can then write the solutions to the Klein Gordon equation on future and past null infinity respectively:

$$
\begin{aligned}
f_{\omega+} &= \frac{1}{\sqrt{2\pi\omega}} e^{-i\omega u} \frac{Y_{lm}}{r} \quad \text{(outgoing)} \\
f_{\omega-} &= \frac{1}{\sqrt{2\pi\omega}} e^{-i\omega v} \frac{Y_{lm}}{r} \quad \text{(ingoing)}
\end{aligned}
$$

$$
\begin{aligned}
p_{\omega+} &= \frac{1}{\sqrt{2\pi\omega}} e^{-i\omega u} \frac{Y_{lm}}{r} \quad \text{(outgoing)} \\
p_{\omega-} &= \frac{1}{\sqrt{2\pi\omega}} e^{-i\omega v} \frac{Y_{lm}}{r} \quad \text{(ingoing)}
\end{aligned}
$$

Where $f_{\omega\pm}$ and $p_{\omega\pm}$ both depend on l, m of the spherical harmonics but since these dependencies won't play a role in the coming calculations we will omit them and only include an ω index. We will only consider the ingoing modes on $\mathcal{J}^-$ since these are the only solutions that will interact with the black hole spacetime, the outgoing modes are not interesting (they simply approach i^0). Since the outgoing modes will no longer be considered we will denote $p_{\omega-}$ simply by p_ω. Similarly we will denote $f_{\omega+}$ by f_ω since the ingoing modes on $\mathcal{J}^+$ are not interesting (ingoing modes on this surface simply approach the singularity without interacting with the collapsing star).

Now that we know what basis modes on $\mathcal{J}^+$ and $\mathcal{J}^-$ look like, we can consider what a mode p_ω is in terms of f_ω and f^*_ω on $\mathcal{J}^-$ by performing a Bogoliubov transformation and therefore determine what $|0_{in}\rangle$ looks like to an observer at future null infinity. First note that p_ω has as support on all of $\mathcal{J}^+$ and it is not possible to normalise so we instead want to consider localised wave packets which we will back propagate to $\mathcal{J}^-$. This can be done in either of two ways. Firstly we can consider the wave packets as a superposition of basis modes p_ω and express each p_ω in terms of the basis modes on $\mathcal{J}^-$, and the wave packet will be the super position of the modes on $\mathcal{J}^-$, and when we refer to modes on $\mathcal{J}^+$ we really mean a linear combination of modes making up a wave packet (in doing this, we assume that all integrals converge nicely and we manipulate them as such). Alternatively, we can choose a new basis such that each basis element is itself a wave packet localised around some retarded time u_0 and peaked around some energy ω_0 on $\mathcal{J}^+$, however this method will be more involved, so we will use the first. These two methods are used to some extent in [18] and [5] respectively.

Let us now consider a mode p_ω on $\mathcal{J}^+$ and back propagate it. As it approaches the potential barrier V, some fraction of it will be reflected and some fraction of it will be transmitted through. The fraction of the mode (which we will call $p^{(1)}_\omega$) that is reflected is not interesting, as it does not enter the non-stationary region of the space time of the collapsing star. $p^{(1)}_\omega$ on $\mathcal{J}^-$ only passed through stationary Schwarzchild spacetime, and so does not contribute to $B_{\omega\omega'}$ by the same argument as was used for the sandwich spacetime. $p^{(1)}_\omega$ therefore does not contribute to Hawking radiation (the frequency of a reflected wave from a stationary source to a stationary potential barrier is the same as the incoming frequency so $\delta(\omega-\omega') = A^{(1)}_{\omega\omega'}$ [1]). The transmitted fraction of p_ω (which we will call $p^{(2)}_\omega$) will continue on into the non-stationary collapsing star region of the spacetime, and expressing this mode on the stationary $\mathcal{J}^-$ will contribute to $B_{\omega\omega'}$. We can therefore write:

$$\begin{aligned} A_{\omega\omega'} &= A^{(2)}_{\omega\omega'} + C\delta(\omega-\omega') \\ B_{\omega\omega'} &= B^{(2)}_{\omega\omega'} \end{aligned} \tag{14}$$

Treating our modes p_ω as normalisable wave packets which we distinguish from the basis modes by switching their lower index to i, since wave packets are square integrable we can define:

$$R_i = \sqrt{\left(p_i^{(1)}, p_i^{(1)}\right)} \quad T_i = \sqrt{\left(p_i^{(2)}, p_i^{(2)}\right)} \tag{15}$$

Where R_i^2 and T_i^2 are the probabilities of the particle corresponding to the wave packet being reflected and transmitted respectively. We therefore have that $R_i^2 + T_i^2 = 1$. We want to determine the properties of the wave packet $p_i^{(2)}$ on $\mathcal{J}^-$ which is made up of a superposition of basis modes. Returning to our consideration of the basis mode $p_\omega^{(2)}$ to simplify calculations, we notice that for large values of u this wave obeys the *Geometric optics* approximation.

8.1 A short note on the Geometric optics approximation

Consider a wave given by the equation $A(x)e^{i\sigma(x)}$, such that A is constant with respect to σ, in other words the phase varies very rapidly with relative to A. Then treating A as a constant and using the wave equation $\Box f = 0$ we find that [19] [2]:

$$\begin{aligned} 0 = \nabla_\mu \nabla^\mu (Ae^{i\sigma}) &= \nabla_\mu (i\nabla^\mu \sigma e^{i\sigma}) \\ i(\nabla_\mu \nabla^\mu \sigma)e^{i\sigma} - (\nabla_\mu \sigma \nabla^\mu \sigma)e^{i\sigma} &= 0 \\ i\sigma\nabla_\mu \nabla^\mu (e^{i\sigma}) - \left(\nabla_\mu \sigma\right)\left(\nabla^\mu \sigma\right)\left(e^{i\sigma}\right) &= 0 \end{aligned} \tag{16}$$

Where the first term on the last line is zero by the wave equation. We therefore find that for A varying slowly with respect to σ:

$$\nabla_\mu \nabla^\mu \sigma = 0 \tag{17}$$

Therefore surfaces of constant phase are null surfaces which can be traced back in time using null geodesics.

Returning to the discussion of back propagating our wave packets to past null infinity, we can then say that wave packets peaked around some frequency corresponding to large u can be traced backwards in time along null geodesics since u increases very quickly here as it diverges as it approaches the event horizon. Notice that as the Kruskal coordinate $U = -e^{-u/4M}$ approaches zero, u increases very rapidly to infinity. Surfaces of constant phase "pile up" near the event horizon.

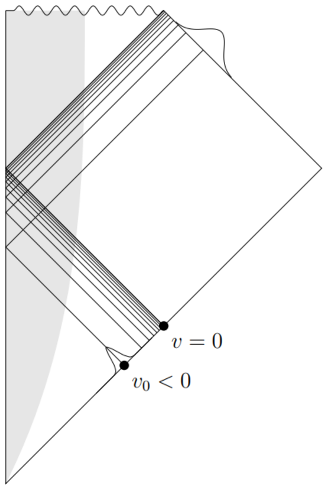

Figure 8.3: A Penrose diagram showing a superposition of modes $p_\omega^{(2)}$ on $\mathscr{J}^+$ forming a wave packet peaked around some $U = -\epsilon$ along with null geodesics corresponding to surfaces of constant phases of the modes. The wave packet back propagated to a point on $\mathscr{J}^-$ is also shown. Our aim is to find v_0 so we can express the modes $p_\omega^{(2)}$ in terms of the basis modes on $\mathscr{J}^-$. [5]

8.2 A short note on deviation vectors

This short section follows closely [5] We define a 1-parameter family of geodesics $\gamma : I \times I' \to M$ where I and I' are both open intervals on $\mathbb{R}$ such that $\forall s\ \gamma(s, \lambda)$ is a geodesic with affine parameter λ, and each value s corresponds to a geodesic on M. Letting U^μ be the tangent vector to the geodesics and S^μ be the tangent vector to curves of constant λ parameterized by s. The vectors S^μ and U^μ are illustrated below. Let us consider the geodesics with coordinate x^μ. Then $U^\mu = \frac{\partial x^\mu}{\partial \lambda}$ and $S^\mu = \frac{\partial x^\mu}{\partial s}$. One could then consider two very nearby geodesics $x^\mu(s + \delta s, \lambda)$ and $x^\mu(s, \lambda)$. Then these two are clearly related by the Taylor expansion $x^\mu(s + \delta s, \lambda) = x^\mu(s, \lambda) + S^\mu \delta s + \mathcal{O}(\delta s^2)$, where S^μ is called a *deviation vector*. Then on the region spanned by our geodesics using a chart (s, λ) we have the vector fields $S = \partial/\partial s$ and $U = \partial/\partial\lambda$. A *geodesic congruence* is defined to be a family of geodesics where on the region $\mathcal{U} \subset \mathcal{M}$ exactly one geodesic passes through each point $p \in \mathcal{U}$ exactly once. Along a geodesic $U \cdot S$ must be constant as proved in [5].

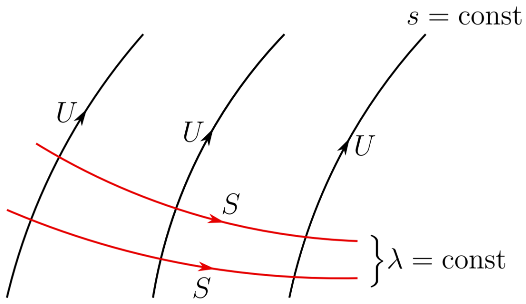

Figure 8.4: The vector fields S^μ and U^μ [5]

8.3 Hawking radiation

Now returning to our collapsing star space time, let us consider a congruence of null geodesics corresponding to each surface of constant phase σ, where the event horizon is a surface of constant phase $\sigma = \infty$. We will call the null geodesic along the event horizon γ. This geodesic also corresponds to the null surface $U = 0$. We can extend γ back in time to some point v on past null infinity $\mathcal{J}^-$. We can also choose this point to be $v = 0$ since the spacetime is invariant under linear transformations $v \to v + c$) [1]. We also consider another null geodesic γ' that ends at $\mathcal{J}^+$ that is very close to γ. Since the null geodesic ends on future null infinity it must correspond to some surface of $U < 0$, as otherwise our geodesic would simply end on the singularity. We likewise extend γ' back to $\mathcal{J}^-$ and we find it lands on some point $v < 0$ since any wave propagating from $v > 0$ simply ends on the singularity. Similar arguments can be made for wave packets, they would be localised about some peak frequency on $\mathcal{J}^+$ and $\mathcal{J}^-$.
Finally, in order to back propagate the mode $p_\omega^{(2)}$, we define the null tangent vector to γ to be T^μ (We replace the tangent U with T to avoid confusion with the Kruskal coordinate) and we also define the null future directed vector S^μ. We then have the following relations hole on the entire null geodesic as proven in [5]:

$$\begin{aligned} T \cdot S &= -1 \\ T \cdot \nabla S^\mu &= 0 \end{aligned} \tag{18}$$

The deviation connecting γ to γ' is then given by ϵS^μ. By spherical symmetry we can simply choose that $S^\phi = S^\theta = 0$. We also can see that the vector field $\partial/\partial V$ is tangent to the outgoing null geodesics since they are defined by constant U. By the equation $T \cdot S = -1$ and the fact that the metric in Kruskal coordinates only has off diagonal terms, (ignoring the coordinates ϕ and θ the metric is 2×2 with zeros on the diagonal and non zero off diagonal terms), S is null and not parallel to T, we have that S must be proportional to $\partial/\partial U$.

$$S = C_1 \frac{\partial}{\partial U} \quad \text{at } \mathcal{J}^+ \tag{19}$$

For some unknown constant C_1. Similarly to section on deviation vectors for $s = U$, with initial $U = 0$ and $\delta U = -\epsilon C_1$, and using the definition of the Kruskal coordinate $U = -e^{-u/4M}$ we find that on γ' the constant phase on this surface of $p_\omega^{(2)}$ is:

$$\sigma = -4M\omega \ln(C_1 \epsilon) \tag{20}$$

We now parallel transport the vectors T and S back along γ to $\mathcal{J}$. Both γ and γ' are ingoing null vectors on $\mathcal{J}^-$, this can be seen in the diagram of the collapsing star space time with surfaces of constant phase. In (u, v) coordinates T now is tangent to the vector field $\partial/\partial u$ since ingoing null geodesics are defined by constant v. Since S is not parallel to T and because the 2×2 metric only has non zero entries on the off diagonal, and the equation $T \cdot S = -1$, we get that:

$$S = C_2 \frac{\partial}{\partial v} \quad \text{at } \mathcal{J}^- \tag{21}$$

For some unknown constant C_2. And so by a similar argument as before, analogously to the section on deviation vectors for $s = v$ and initial $v = 0$, $\delta v = -\epsilon C_2$. Therefore γ' intersects $\mathcal{J}^-$ at $v = -\epsilon C_2$. We can then write the equation for the constant phase of $p_\omega^{(2)}$ on γ' in terms of v by:

$$\sigma = -4M\omega \ln(-\frac{C_1}{C_2} v) \tag{22}$$

Finally we can write the equation for the transmitted fraction of a mode $p_\omega^{(2)}$ in terms of v:

$$p_\omega^{(2)} \approx \begin{cases} 0 & v > 0 \\ \frac{1}{\sqrt{2\pi\omega}} \frac{Y_{lm}}{r} \exp\left(-4\omega M \log(-\frac{C_1}{C_2} v\right) & \text{small } v < 0 \end{cases} \tag{23}$$

We can therefore say that for a mode on $\mathcal{J}^-$ of the form $p_\omega^{(2)} \propto e^{-i\omega u}$ on $\mathcal{J}^+$ will obey:

$$p_\omega^{(2)} \sim \begin{cases} e^{i\omega 4M \ln(-v)} & \text{for} \quad \text{small } v < 0 \\ 0 & \text{for} \quad v > 0 \end{cases} \tag{24}$$

This has the exact same form as the Rindler modes in the coordinates of a non-accelerating observer, so the Bogoliubov analysis is identical except with the acceleration a replaced with $1/4M$. We therefore have that $|A_{\omega\omega'}|^{(2)} = e^{-4M\pi\omega}|B_{\omega\omega'}|$. Analogously to the section on the Unruh effect, a black hole formed from a collapsing star therefore emits radiation with a thermal spectrum with temperature $T = \kappa/2\pi$, where $\kappa = 1/4M$ is the surface gravity of the black hole [20]. Further more, this effect produces a steady flux of particles at late time. Again considering normalisable wave packets, and using the equation found for modes in the Bogoliubov transformation section:

$$(h_{k_1}, h_{k_2}) = \int dk_4 A_{k_1k_4}A^{\dagger}_{k_4k_2} - B_{k_1k_4}B^{\dagger}_{k_4k_2} \tag{25}$$

for some basis modes $h_{k_{1,2}}$. Then using our transmitted normalisable wave packets denoted by $p_i^{(2)}$ (which again are just linear combinations of our basis modes) and the linearity of the inner product we have that:

$$\begin{aligned} T_i^2 = \left(p_i^{(2)}, p_i^{(2)}\right) &= \int dj \left(\left|A_{ij}^{(2)}\right|^2 - \left|B_{ij}\right|^2\right) \\ &= \int dj \left(\left(e^{2\omega_i\pi/\kappa}\right)\left|B_{ij}\right|^2 - \left|B_{ij}\right|^2\right) \\ &= \left(e^{2\omega_i\pi/\kappa} - 1\right)\int dj \left|B_{ij}\right|^2 \\ &= \left(e^{2\omega_i\pi/\kappa} - 1\right)\langle N_i\rangle \end{aligned} \tag{26}$$

$$\implies \langle N_i\rangle = \frac{\Gamma_i}{\left(e^{2\omega_i\pi/\kappa} - 1\right)} \tag{27}$$

Where i denotes the frequency about which our wave packet is peaked, and $\Gamma_i \equiv T_i^2$. Black holes therefore emit radiation as if they were hot black bodies! Recovering constants set to unity earlier, we find that Black holes emit radiation as if they were black bodies with temperature $T = \frac{\hbar c^3}{8\pi G k_B M} \approx 6 \times 10^{-8}\ \mathrm{K} \times \frac{M_\odot}{M}$ which is very cold indeed for a black hole with mass on the order of our sun.

9 | Black hole evaporation

The result that black holes emit Hawking radiation as if they were hot bodies implies that black holes are thermodynamic objects with temperature found earlier. Since they emit particles we expect that they must decrease in mass and and since the radius of a black hole is proportional to the mass, we expect them to shrink. The treatment of black holes as thermodynamic objects implies that the similarities between the first law of black holes and the first law of thermodynamics are more than simply superficial, and Beckenstein [6] suggested that A and κ were not simply analogous to entropy and temperature in these two laws, in fact they were the temperature and entropy of a black hole. The thermodynamic laws and classical black hole laws are nicely outlined in page 46 of [1], and the black hole laws are further studied in [20]. The first law of thermodynamics for a black hole can therefore be written as:

$$dE = T_H dS_{BH} + \Omega_H dJ \tag{1}$$

With the second term related to the angular momentum of the black hole, which disappears for the black holes which we will consider (with charge and angular momentum equal to zero). T_H is the temperature of a black hole found earlier and $S_{BH} = A/4$ is the entropy of a black hole by [6] and $A \propto R^2 \propto M^2$ is the area of a black hole. Hawking radiation implies that this area must get get smaller and so the black hole's entropy must decrease, however the total entropy of the Hawking radiation *and* the black hole $S = S_{radiation} + S_{BH}$ increases, so Hawking radiation obeys the second law of thermodynamics. The classical theory of general relativity suggests that gravitational collapse will result in a black hole which will become stationary and entirely characterized by it's mass, charge and angular momentum, and so the study of the microstates of such black hole would require a new theory of quantum gravity [3].

In our treatment of a black hole as a thermodynamic black body, we can approximate the energy loss of a black hole by considering Stefan's law [5]:

$$\frac{dE}{dt} \approx -CAT^4 \tag{2}$$

For some constant C. Since we found that $T \propto 1/M$, $A \propto R^2 \propto M^2$ for M the mass of the black hole and with A, R the area and radius respectively of a perfectly spherical black hole, and using $E = M$ for a stationary black hole, this gives us:

$$\frac{dM}{dt} \approx C\frac{1}{M^2} \tag{3}$$

Hence the black hole evaporates away completely in time $\tau \propto M^3$, which for a solar mass black hole is a very long time indeed. This approximation is expected to break down as the black hole approaches the Planck mass when quantum mechanical effects are expected to dominate [5].

10 | Conclusions and Future Work

After studying an accelerating observer moving through a Minkowkski vacuum, it was found that such an observer would be surrounded by a thermal bath of particles [9]. This is because of the accelerating observers definition of a positive mode was found to be a mixture of a stationary observer's positive *and* negative modes. A similar thermal production of particles was found in a collapsing star spacetime, where the non-stationary dynamics of the collapsing star were found to be the source of the particle production. In general, it was shown that time dependent space times could lead to particle production as they "mixed" the positive and negative modes of one stationary spacetime as it interacted with the time dependant spacetime before propagating to another stationary spacetime. In the case of the collapsing star spacetime, the modes that passed close to the event horizon of the black hole led to a "mixing" of their positive and negative modes relative to future null infinity [5]. In Hawking's original paper [3], he likened this effect to positive and negative energy particle pair production near the horizon, the anti-particle tunneling across the black hole horizon and due to the Killing vector becoming space like inside of the black hole, this classically forbidden particle can exist as a real particle with a timelike momentum vector. It was also shown that there is evidence in favour of treating black holes as thermodynamic objects, and it was found that in doing so the classical black hole hole laws are violated and the area of a black hole does in fact decrease as it evaporates in a time proportional to it's mass cubed. In this thesis, only the special case of a non-rotating and non-charged black hole was considered so a future project could focus on the particle production of a charged and rotating black hole. A future project could also study how black holes emit particles of higher and higher energy as they evaporate and also different kinds of particles.

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