Photometric Stereo

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1 Definitions and Notations

Suppose we have M images that are indexed by i, and each image has N pixels that are indexed by j. The intensity of j-th pixel in i-th image is therefore denoted as $I_{i,j}$.

At j-th pixel, denote $\mathbf{n}_j$ the surface normal, ρ_j the albedo, and $\mathbf{b}_j = \rho_j \mathbf{n}_j$ the scaled normal. Note that these quantities are the same across all images and therefore independent of i.

Each image comes from a different lighting environment, which has a different rendering function $\mathcal{R}_i(\cdot)$, written as

$$I_{i,j} = \rho_j \mathcal{R}_i(\mathbf{n}_j), \tag{1}$$

and we consider following rendering models in this note.

1. Directional lighting (distant point light source)

$$\mathcal{R}_i(\mathbf{n}) = \max\{\boldsymbol{\ell}_i^\top \mathbf{n},\ 0\}. \tag{2}$$

2. Directional lighting plus an ambient component (first order spherical harmonics[1])

$$\mathcal{R}_i(\mathbf{n}) = \max\{\boldsymbol{\ell}_i^\top \mathbf{n} + \alpha_i,\ 0\}. \tag{3}$$

We also define

$$\lambda_i = \|\boldsymbol{\ell}_i\|, \quad \widehat{\boldsymbol{\ell}}_i = \frac{\boldsymbol{\ell}_i}{\lambda_i}, \tag{4}$$

where λ_i is the strength of i-th lighing, and $\widehat{\boldsymbol{\ell}}_i$ is a unit vector for direction of i-th lighting.

The goal of photometric stereo is to recover the scene property $\{\rho_j, \mathbf{n}_j\}$ from the intensity measurements $I_{i,j}$, given (fully or partially) calibrated lighting environment $\mathcal{R}_i(\cdot)$.

2 Fully Calibrated Lighting

2.1 Directional lighting

For notational simplicity, we first only consider pixels that are not in shadow, and drop the $\max\{\cdot,\ 0\}$ part of the rendering function

$$I_{i,j} = \rho_j \boldsymbol{\ell}_i^\top \mathbf{n}_j = \boldsymbol{\ell}_i^\top \mathbf{b}_j. \tag{5}$$

[1] The first order spherical harmonics model sometimes does not have the $\max\{\cdot,\ 0\}$ part.

Define $\mathbf{I}_j \in \mathbb{R}^{M\times 1}$ the (vertical) concatenation of all $I_{i,j}$ for $1 \le i \le M$, and $\boldsymbol{L} \in \mathbb{R}^{3\times M}$ the (horizontal) concatenation of all $\boldsymbol{\ell}_i$. Then we have

$$\mathbf{I}_j = \boldsymbol{L}^\top \mathbf{b}_j, \tag{6}$$

where $\mathbf{I}_j$ are measured intensities, $\boldsymbol{L}^\top$ are calibrated lighting parameters, and $\mathbf{b}_j$ is the unknown scene property, which can be easily solved in the least squares manner

$$\mathbf{b}_j = (\boldsymbol{L}^\top)^\dagger\, \mathbf{I}_j = (\boldsymbol{L}\boldsymbol{L}^\top)^{-1}\, \boldsymbol{L}\, \mathbf{I}_j. \tag{7}$$

And finally, we can recover ρ_j and $\mathbf{n}_j$ as

$$\rho_j = \|\mathbf{b}_j\|, \quad \mathbf{n}_j = \frac{\mathbf{b}_j}{\rho_j}. \tag{8}$$

2.2 Directional lighting with an ambient component

The rendering equation in this model can be written as

$$I_{i,j} = \rho_j \left(\boldsymbol{\ell}_i^\top \mathbf{n}_j + \alpha_i\right) = \boldsymbol{\ell}_i^\top \mathbf{b}_j + \alpha_i \|\mathbf{b}_j\|. \tag{9}$$

Then we can formulate estimation of scene property as the following optimization problem

$$\min_{\mathbf{b}_j} \quad \sum_{i=1}^{M} \left(I_{i,j} - \boldsymbol{\ell}_i^\top \mathbf{b}_j - \alpha_i \|\mathbf{b}_j\|\right)^2. \tag{10}$$

Unfortunately, we do not have a fast (non-iterative) algorithm for solving this problem yet.

3 Partially Calibrated Lighting

In reality, we can usually calibrate part of the lighting property accurately. For example, using a chrome sphere, we are able to calibrate the lighting directions $\widehat{\boldsymbol{\ell}}_i$, but not their strengths λ_i. Therefore, we also need to estimate the unknown lighting parameters λ_i from the measurements $I_{i,j}$.

Since the number of images M is much smaller than the number of pixels N, the number of unknown lighting parameters is also much smaller than number of scene property parameters. We define a cost function on lighting parameters

$$\mathrm{cost}(\{\lambda_i\}) = \min_{\{\rho_j, \mathbf{n}_j\}} \sum_{i=1}^{M} \sum_{j=1}^{N} \left(I_{i,j} - \rho_j \mathcal{R}_i\left(\mathbf{n}_j\right)\right)^2, \tag{11}$$

where the minimization can be solved as a fully-calibrated problem described in previous section. We can then perform a optimization on this cost to find the optimal lighting parameters $\{\lambda_i\}$.

3.1 Directional lighting

We give the specific form of lighting strength estimation under the directional lighting model. The cost function to be minimized is

$$\mathrm{cost}\left(\{\lambda_i\}\right) = \min_{\{\rho_j, \mathbf{n}_j\}} \sum_{i=1}^{M} \sum_{j=1}^{N} \left(I - \lambda_i \rho_j \widehat{\boldsymbol{\ell}}_i^\top \mathbf{n}_j\right)^2 = \sum_{j=1}^{M} \min_{\mathbf{b}_j} \|\mathbf{I}_j - \boldsymbol{L}\left(\boldsymbol{\lambda}\right)^\top \mathbf{b}_j\|^2, \tag{12}$$

where

$$\boldsymbol{L}\left(\boldsymbol{\lambda}\right) = \left[\begin{array}{ccc} \lambda_1 \widehat{\boldsymbol{\ell}}_1 & \cdots & \lambda_M \widehat{\boldsymbol{\ell}}_M \end{array}\right]. \tag{13}$$

The minimization over $\mathbf{b}_j$ inside the sum can be analytically solved by a linear least squares, resulting in a cost function

$$\text{cost}\left(\{\lambda_i\}\right) = \sum_{j=1}^{M} \|\mathbf{I}_j - \boldsymbol{L}\left(\boldsymbol{\lambda}\right)^\top \left(\boldsymbol{L}(\boldsymbol{\lambda})\boldsymbol{L}(\boldsymbol{\lambda})^\top\right)^{-1} \boldsymbol{L}(\boldsymbol{\lambda})\, \mathbf{I}_j\|^2. \tag{14}$$

This is a non-linear least squares problem that can be solved with a Levenberg-Marquardt algorithm. For completeness, we provide the gradient (Jacobian) of the cost function.

First, the partial derivative of matrix $\boldsymbol{L}\left(\boldsymbol{\lambda}\right)$ over λ_i is simply

$$\frac{\partial \boldsymbol{L}}{\partial \lambda_i} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \widehat{\boldsymbol{\ell}}_i & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}. \tag{15}$$

Now we define

$$\mathbf{f}_j = \boldsymbol{L}^\top \left(\boldsymbol{L}\boldsymbol{L}^\top\right)^{-1} \boldsymbol{L}\, \mathbf{I}_j - \mathbf{I}_j, \tag{16}$$

and take partial derivative of $\mathbf{f}_j$ with respect to λ_i

$$\begin{aligned}
\frac{\partial \mathbf{f}_j}{\partial \lambda_i} =& \frac{\partial}{\partial \lambda_i}\left(\boldsymbol{L}^\top \left(\boldsymbol{L}\boldsymbol{L}^\top\right)^{-1} \boldsymbol{L}\right)\mathbf{I}_j && (17)\\
=& \left(\frac{\partial \boldsymbol{L}^\top}{\partial \lambda_i}\left(\boldsymbol{L}\boldsymbol{L}^\top\right)^{-1}\boldsymbol{L} + \boldsymbol{L}^\top \frac{\partial \left(\boldsymbol{L}\boldsymbol{L}^\top\right)^{-1}}{\partial \lambda_i}\boldsymbol{L} + \boldsymbol{L}^\top\left(\boldsymbol{L}\boldsymbol{L}^\top\right)^{-1}\frac{\partial \boldsymbol{L}}{\partial \lambda_i}\right)\mathbf{I}_j && (18)\\
=& \left(\frac{\partial \boldsymbol{L}^\top}{\partial \lambda_i}\left(\boldsymbol{L}\boldsymbol{L}^\top\right)^{-1}\boldsymbol{L} + \boldsymbol{L}^\top \frac{\partial \left(\boldsymbol{L}\boldsymbol{L}^\top\right)^{-1}}{\partial \lambda_i}\boldsymbol{L} + \boldsymbol{L}^\top\left(\boldsymbol{L}\boldsymbol{L}^\top\right)^{-1}\frac{\partial \boldsymbol{L}}{\partial \lambda_i}\right)\mathbf{I}_j && (19)\\
=& \left(\frac{\partial \boldsymbol{L}^\top}{\partial \lambda_i}\left(\boldsymbol{L}\boldsymbol{L}^\top\right)^{-1}\boldsymbol{L} - \boldsymbol{L}^\top\left(\boldsymbol{L}\boldsymbol{L}^\top\right)^{-1} \frac{\partial \left(\boldsymbol{L}\boldsymbol{L}^\top\right)}{\partial \lambda_i}\left(\boldsymbol{L}\boldsymbol{L}^\top\right)^{-1}\boldsymbol{L} + \boldsymbol{L}^\top\left(\boldsymbol{L}\boldsymbol{L}^\top\right)^{-1}\frac{\partial \boldsymbol{L}}{\partial \lambda_i}\right)\mathbf{I}_j. && (20)
\end{aligned}$$

The last step is based on the fact that for any invertible matrix $\boldsymbol{A}$ that depends on a parameter t, we have [1]

$$\frac{d\boldsymbol{A}^{-1}}{dt} = -\boldsymbol{A}^{-1}\frac{d\boldsymbol{A}}{dt}\boldsymbol{A}^{-1}. \tag{21}$$

In fact, this calculation can be used to estimate or improve any/all component(s) of the lighting matrix $\boldsymbol{L}$, not restricting to its strength $\boldsymbol{\lambda}$.

References

[1] Wikipedia, "Plagiarism — Wikipedia, the free encyclopedia," 2014, [Online; accessed 08-February-2014]. [Online]. Available: http://en.wikipedia.org/wiki/Invertible_matrix