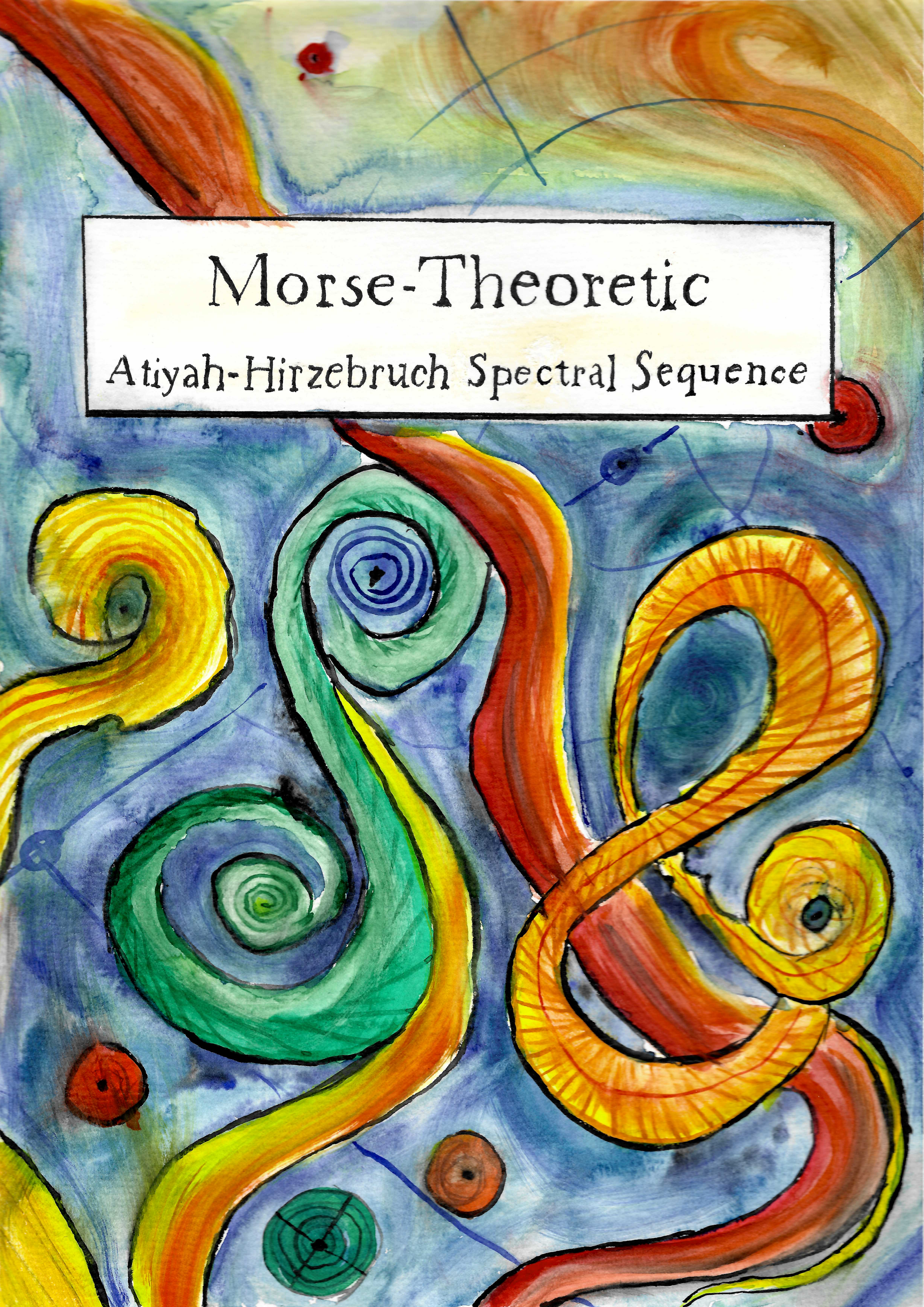
Morse-Theoretic

Atiyah-Hirzebruch Spectral Sequence



Contents

1 Stable and Unstable Manifold theorem

We now want to understand the gradient lines that connect two critical points. Our bigger goal is to count them in a meaningful way such that the boundary operator, defined to be the sum over all counted gradient lines from two critical points with consecutive index, squares up to zero. Therefore, we need the following properties:

1. They need to be finitely many.
2. Some of them need to be counted *negatively*

For the second requirement we will make use of the orientation to give a sign and dimension to give a number. For that we will show that the stable set is the image of a smooth embedding of $\mathbb{R}^{\mathrm{ind}_f(p)}$. For a smooth function $f : M \to \mathbb{R}$ on a finite dimensional compact Riemannian manifold, we have that $-\mathrm{grad}f$ determines a smooth global flow $\varphi_t(x)$. If not said different, $\varphi_t(x)$ will always denote that flow.

Definition 1.1 (stable, unstable manifold)**.** Let $p \in M$ be a critical point of f. Then we define:

1. the **stable set** of p as
$$W(\to p) := \{x \in M | \lim_{t\to\infty} \varphi_t(x) = \text{ß}\}.$$
That is the points that flow towards p.
2. the **unstable manifold** of p as
$$W(p \to) := \{x \in M | \lim_{t\to-\infty} \varphi_t(x) = \text{ß}\}.$$
That is the points that came from p.

Those sets will turn out to be manifolds, which explains the names **(un-)stable manifolds**.

The Main theorem of this chapter will be the:

Theorem 1.2 (Stable and unstable manifold theorem)**.** *Let $f : M \to \mathbb{R}$ be a smooth function on a smooth compact Riemannian manifold (M, g). Let p be a critical point of f. Then the tangent space splits as:*
$$T_pM = T_p^s(M) \bigoplus T_p^u M$$
such that the Hessian is positive definite on T_p^sM and negative definite on T_p^uM. Moreover, the stable and unstable manifolds are submanifolds and images of smooth embeddings:
$$\begin{aligned} E^s : T_p^sM &\to W(\to p) \subset M, \\ E^u : T_p^uM &\to W(p \to) \subset M. \end{aligned}$$

Remark 1.3. The proof is split into several parts. First, we show that p is a hyperbolic fix point of the gradient-induced flow and the Hessian splits into a contracting and expanding part. Then we will make use of analysis to show that there is a neighbourhood of p and a chart, where $W(\to p) \cap B$ is the graph of a Lipschitz function. This will help us define a differentiable structure on the (un-)stable manifold to get the immersions.

Remark 1.4 (Discrete system)**.** We can redefine the (un-)stable manifold for $t > 0$ in a discrete way:

$$W(\to p) = \{x \in M | \lim_{n\to\infty} \varphi_t^n(x) = p\}$$
$$W(p \to) = \{x \in M | \lim_{n\to-\infty} \varphi_t^n(x) = p\}$$

Definition 1.5 (Hyperbolic fixepoints)**.** A fixpoint for a diffeomorphism $\varphi : M \to M$ is called **hyperbolic**, if $\mathrm{d}\varphi|_p : T_pM \to T_pM$ has no complex eigenvalue of length one.

Remark 1.6 (Critical points are hyperbolic fixpoints)**.** If M is a smooth riemannian manifold, $f : M \to \mathbb{R}$ is a Morse function and p is a critical point, then p is a hyperbolic fixpoint of the diffeomorphism φ_t for any $t > 0$.

Proof. We need to differentiate our $-\mathrm{grad}(f)-$induced flow by the starting point. To do this we want to inspect what differential equation $\Phi(t,x) := \frac{\partial}{\partial x}\varphi_t(x)$ solves:

$$\begin{aligned}\frac{\partial}{\partial t}\Phi(t,x)) &= \frac{\partial}{\partial x} - \mathrm{grad}(f)(\varphi_t(x)) \\ &= \Big(- \frac{\partial}{\partial x}\mathrm{grad}(f)\big|_{\varphi_t(x)}\Big)\Phi(t,x))\end{aligned}$$

and

$$\Phi(0,x) = \frac{\partial}{\partial x}\varphi_0(x) = 1 \ .$$

Now since this is a linear differential equation we get a solution by exponentiating:

$$\Phi(t,x) = \mathrm{e}^{-\frac{\partial}{\partial x}\mathrm{grad}(f)\big|_{\varphi_t(x)} t} \ .$$

Now we care for $\Phi(t,p)$ where $t > 0$ and $p \in \mathrm{Crit}(f)$. In this case we have:

$$\frac{\partial}{\partial x}\mathrm{grad}(f)\big|_{\varphi_t(p)} = \frac{\partial}{\partial x}\mathrm{grad}(f)\big|_p = \mathrm{Hess}(f)\big|_p$$

and therefore:

$$\Phi(t,p) = \mathrm{e}^{\mathrm{Hess}(f)\big|_p t}$$

This however does not have an eigenvalue of one, as the Hessian doesn't vanish in non-degenerate critical points. Even more, this is diagonal after a change of coordinates, since the Hessian can be of diagonal form. $\square$

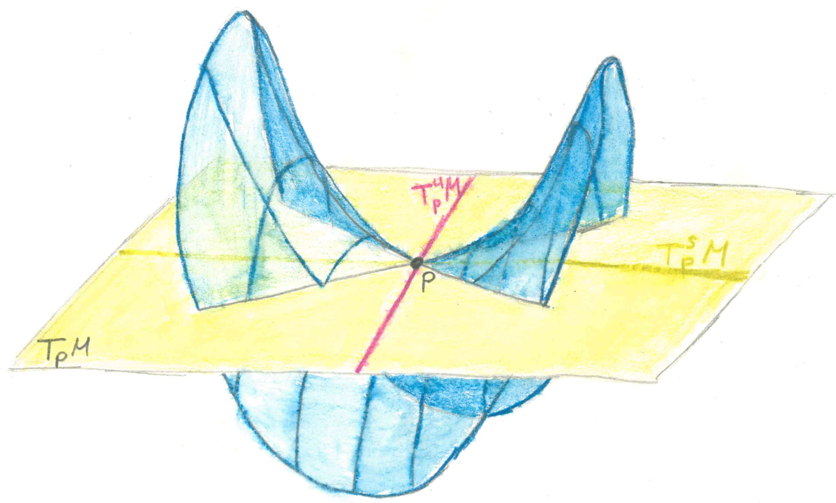


Figure 1: The splitting of the tangent space

Corollary 1.7. By the above, we see that the tangent space T_pM now splits into two parts $T_p^sM \oplus T_p^uM$, such that for $t > 0$:

$$\mathrm{d}\varphi_t\big|_p : T_p^sM \to T_p^sM \text{ is } \textbf{contracting} \text{ with eigenvalue } \lambda < 1$$
$$\mathrm{d}\varphi_t\big|_p : T_p^uM \to T_p^uM \text{ is } \textbf{expanding} \text{ with eigenvalue } \lambda > 1$$

To conclude this, the eigenvalues themselves are not enough, but also the diagonalizability. Or more concrete: If we choose a coordinate chart such that the Hessian is $\mathrm{diag}(-1, ... -1, 1, ...1)$ we know that:

$$\mathrm{d}\varphi_t|_p = \mathrm{diag}(\frac{1}{\mathrm{e}}, ..., \frac{1}{\mathrm{e}}) \oplus \mathrm{diag}(e, ..., e).$$

Then using the Euclidean norm we are a contraction. This situation is depicted in figure 1.

Definition 1.8 ((un-)stable tangend space)**.** For any map $\theta : T_p^sM \oplus T_p^uM \to T_pM$ we define the **stable set** of θ to be:

$$W_r^s(\theta) := \{x \in T_pM | \forall n \geq 0, \theta^n(x) \text{ is defined and } \|\theta^n(x)\| \leq r\}$$

The norm here is arbitrary as we are finite dimensional and therefore all norms are equivalent.

Remark 1.9. In the proof of the (un-)stable manifold theorem, we will map the stable set into such a stable set in the tangent space. So if we could control the structure of such a map, we would win. That is accomplished as the next theorem tells us that this set is the graph of a smooth function, and the projection of the graph onto the domain gives us then a differentiable structure.

Theorem 1.10 (local (un-)stable manifold theorem)**.** *We define:*

$$T := \mathrm{d}\varphi_t\big|_p, \quad T_s := T\big|_{T_p^s M}, \quad T_u := T\big|_{T_p^u M}.$$

Let $\lambda < 1$ such that $\|T_s\| < \lambda$ and $\left\|T_u^{-1}\right\| < \lambda$, where $\|T\| = \sup\{T(x) | \|x\| \le 1\}$ denotes the operator norm. Now for this setting there exists an ε_λ depending only on λ, and there exists a $\delta > 0$ for every $r > 0$ which satisfy: For all $\theta : T_p^s M \times T_p^u M \to T_p M$ that satisfies $\theta - T$ is Lipschitz with Lipschitz constant

$$\mathrm{Lip}(\theta - T) < \varepsilon, \text{ and } \|\theta(0)\| < \delta.$$

In this setting, $W_r^s(\theta)$ is the graph of a Lipschitz function $g : T_p^s M \to T_p^u M$ with $Lip(g) \le 1$ and:

1. *If θ is C^k (k- times smooth differentiable), then so is g.*
2. *If θ is smoothly differentiable ,$\theta(0) = 0$ and $\mathrm{d}\theta\big|_0 = T$, then $g(0) = 0$ and $\mathrm{d}g|_0 = 0$. Hence, $T_p W_r^s(\theta) = T_p^s M$.*

A picture and application of this technical theorem is given in the proof of the global stable and unstable manifold theorem.

Proof. □ fehlt

Proof of theorem 1.2. First we switch to the world of discrete dynamical systems and define $\varphi := \varphi_1 : M \to M$. The idea of the proof is to first find a chart for a small open subset of $W(\to p) = \{x \in M | \lim_{n\to\infty} \varphi^n(x) = p\}$ arround p and then define a global one. This is done by taking a small neighbourhood arround an arbetrary point $x \in W(\to p)$ and moving it via φ to the small neighbourhood of p.

Let U be a small neighborhood of $p \in M$ and $\Psi : U \to T_p M$ be a centered coordinate chart (after identifying $\mathbb{R}^n$and $T_p M$ via $e_i \mapsto \partial q_i$). Now bx corollary 1.7, $T_p M$ splits into $T_p^s M \otimes T_p^u M$ with respect to $\mathrm{d}\varphi\big|_p$. If we inspect the image of $W(\to p) \cap U$ under Φ we realise that it is a small pertubation of $T_p^s M$. This Situation can be seen in figure 2. In fact, it is one that can be controlled using theorem 1.10. For this we define $\theta := \Psi \circ \varphi \circ \Psi^{-1} : T_p M \to T_p M$. Now since θ is smooth we know that it is a lipschitz pertubation of $d\varphi|_p$ and we can apply theorem 1.10 to conclude that there are indices ε and r such that $W_r^s(\theta)$ is the graph of a smooth function. Notice that we might need to

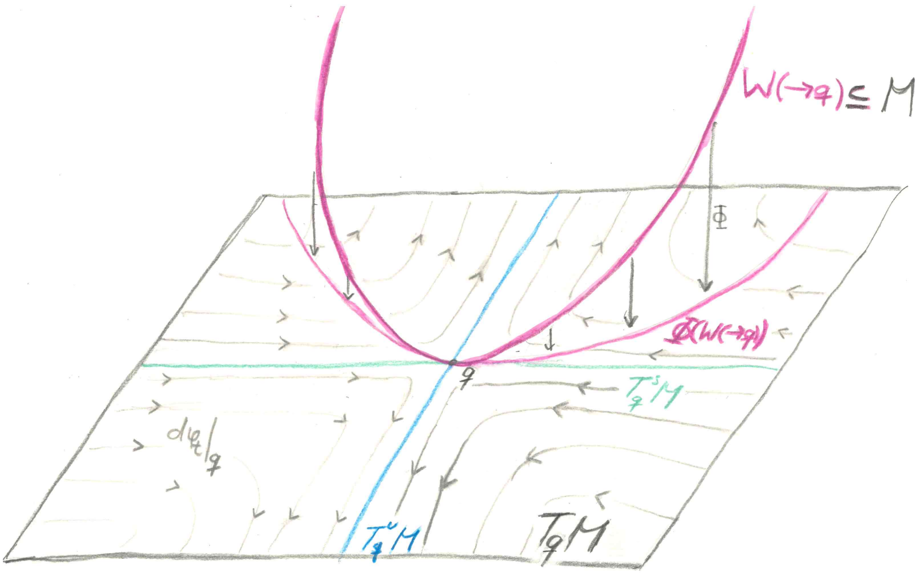


Figure 2: The proof of the (un-)stable manifold theorem
The Manifold is not shown in this image. One should imagen it looking like in figure 1. p can be imagined to be the index sitting at the bottom of the hole of the tilted doughnut.

shrink the domain such that $\mathrm{Lip}(\theta - d\varphi|_p) < \varepsilon$. The next thing we want to do is to see that for a small neighbourhood $U \ni p$

$$W_r^s(\theta) = \Psi(W(\to p) \cap U). \tag{1}$$

This can be done by showing two inclusions:

- ($\subset$): for $x \in W_r^s(\theta)$ we can conclude that $\|\Psi \circ \varphi^n \circ \Psi^{-1}\| < r$ for all $n \in \mathbb{N}$. Now by Bolzano Weierstrass we know that $\lim_{n\to\infty} \Psi \circ \varphi^n \circ \Psi^{-1}$ has at least one accumulation point. And since p is hyperbolic by remark 1.6 we know this point is the origin and is unique. Therefor we conclude $\lim_{n\to\infty} \Psi \circ \varphi^n \circ \Psi^{-1} = 0$ which tells us that $\Psi^{-1}(x) \in W(\to p) \cap U$, where U is a ball inside $\Psi^{-1} B_o(r)$.

- ($\supset$) This direction is almost trivial. Let $y \in W(\to p)$ and $x = \Psi(y)$. Since p is isolated we can choose U small enough such that for all $y \in W(\to p) \cap U$ we have that $\theta^n(\Psi(y)) < r$ for all $n \in \mathbb{N}$. For the last conclusion we might need to shrink U further.

Now we know that $\Psi(W(\to p) \cap U)$ is the graph of a smooth function $g : T_p^s M \to T_p^u M$. Therefor, the projection $\pi(\Psi(W(\to p) \cap U)) \subset T_p^s M$ gives us a chart. To get an Atlas for $W(\to p)$ we can define $U^n := \varphi^{-n}(U)$ for all n to cover $W(\to p)$. This cover comes with charts defined by $(U^n, \pi \circ \Psi \circ \varphi^n)$. For the sake of readability we define the charts $\chi^n := \pi \circ \Psi \circ \varphi^n$. Those charts are compatible since φ is a diffeomorphism. To conclude that we have a submanifold we need to check if the inclusion is an immersion. This however is clear by definition.

Finally, we want to prove that this submanifold is the image of a smooth embedding of $T^s pM$. For this we define $B := W(\to p) \cap U)$ and define $h := \chi \circ \varphi \circ \chi^{-1}$. Since similar Matrices have the same eigenvalues we can conclude that $\mathrm{d}\chi|_0$ and $\mathrm{d}\varphi_p|_{T_p^s M}$ have the same eigenvalues. Therefore, we can introduce an inner product such that the operator norm $\|dh_0\| < 1$. Now let $\alpha \in \mathbb{R}$ and B_0 be a ball centred at 0 such that $\|dh|_x\| \leq \alpha$ for all $x \in B_0$. Now this yields that $h|_{B_0}$ is a contraction, which we can extend to be a contraction $\tilde{h} : T_p^s M \to T_p^s M$. Now again similar as before we define a map $E^s : T_p^s M \to W(\to p)$ by first contracting, then going down and then expanding again. To be more precise, we define $E_n^s : \varphi^{-n} \circ \chi^{-1} \tilde{h}^n$. Finally, we define $E^s(x) = E_n^s(x)$ where $\tilde{h}^n(x) \in B_0$. Obviously, we need to check that this is well-defined: So assume that $n+1$ and n work for x:

$$\begin{aligned}
E_{n+1}^s(x) &= \varphi^{-(n+1)} \circ \chi^{-1} \circ \tilde{h}^{n+1} \\
&= \varphi^{-(n+1)} \circ \chi^{-1} \circ h \circ \tilde{h}^n \\
&= \varphi^{-(n)} \circ \varphi^{-1} \chi^{-1} \circ \underbrace{\chi \circ \varphi \circ \chi^{-1}}_{h} \circ \tilde{h}^n \\
&= \varphi^{-n} \circ \chi^{-1} \circ \tilde{h}^n = E_n^s(x).
\end{aligned}$$

To confirm that E^s gives us an immersion onto $W(\to p)$ we first need to check for smoothness and bijectivity.By definition, we are smooth and by since all functions that compose to E^s_n, the latter is injective for all n. Now if $E^s_n(x) = E^s_m(y)$ with $m > n$, then $\tilde{h}^m(x) \in B_0$, since $\tilde{h}$ is a contraction and by the well definition we have $E^s_m(x) = E^s_m(y)$ which concludes to $x = y$, as the latter are injective for m. The surjectivity is also clear, as for all $x \in W(\to p)$ there is an n such that $\varphi^n(x) \in \chi^{-1}(B_0)$ and with that the point $\tilde{h}^{-n}(\chi(\varphi^n))$ does the job. So the last step is to verify that $\mathrm{d}E^s|_x$ is injective. This follows from $\tilde{h}$ having an injective differential, since it is a contraction. Furthermore, χ^{-1} has an injective differential, as it is the inverse of a chart and φ has an injective differential, since $\mathrm{d}\varphi(x) = e^{-\frac{\partial}{\partial x}|_{\varphi(x)}}$. The latter equality was shown in the proof of remark 1.6. This concludes the proof for the stable set. For the unstable set we replace f by f^-1. □

Corollary 1.11. From the proof of the (un-)stable manifold theorem, we can deduce that $T_pW(\to p) = T^s_pM$.

Proof. For this we just use theorem 1.10 a bit more sophisticated. By the second statement of the theorem, we know that $T_pW^s_r(\theta) = T^s_p$. Notice that we actually talk about an equality here. But since $W^s_r(\theta) = \chi(W(\to p) \cap U)$ we know that $T_pW(\to p) = T^s_p$. □

2 The conley index

We start by defining some needed properties of maps:

Definition 2.1 (Cofibration)**.** Let (X, A) be a topological pair and Y be a topological space. The pair (X, A) satisfies the **homotopy extension property with respect to** Y, if and only if, we can extend homotopies. In other words, for all $f : X \to Y$ and $H : A \times I \to Y$ with $H(x, 0) = f(x)$ there exists a continuous extension $F : X \times I \to Y$ with $F(x, 0) = f(x)$. If a pair (X, A) satisfies the homotopy extension property with respect to any topological space Y, we call (X, A) a **cofibered pair** and the inclusion $i : A \hookrightarrow X$ a **cofibration**.

Definition 2.2. For as topological pair (N, L) define $N/L = N/\sim$ where $x \sim y$ if $x, y \in L$. In other words we contract L to be one point.

Remark 2.3. In the proof of the Morse homology theorem we want to talk about the homology of index pairs as relative homology groups $H^{\text{sing}}_i(N, L)$. However, if the inclusion $L \to N$ is a cofibration we have the natural isomorphism $H^{\text{sing}}_i(N, L) \cong H^{\text{sing}}_i(N/L)$. This is proven in [**LecturesonMorseHomology**] chapter two. To show that something is a cofibration we will use the following fact for metric spaces:

The pair (N, L), where L is closed in N is a cofibration (respectivly the inclusion), if an in N open neighbourhood U of L exists, such that L is a strong deformation retract of U.

This is also presented in chapter two of [**LecturesonMorseHomology**], by showing that such an inclusion admits a Strøm structure. An example of such a cofibration is the pair $(D^n, \partial D^n)$. Finally, a pair that is homeomorphic to a cofibration is again one.

Definition 2.4 (Compact invariant isolated subset)**.** For a flow $\varphi_t : M \to M$ on a locally compact metric space we call a subset $S \subseteq M$ **invariant subspace** if and only if $\varphi_t(S) = S$ for all $t \in \mathbb{R}$. For any subset $N \subseteq M$ we define the **maximal invariant subset**

$$\begin{aligned} I(N) &= \{x \in N | \varphi_t(x) \in N \forall t \in \mathbb{R}\} \\ &= \bigcap_{t \in \mathbb{R}} \varphi_t(N). \end{aligned}$$

A **compact invariant subset** S is called **isolated**, if a compact neighborhood N exists, such that $I(N) = S$.

Definition 2.5 (Index pairs)**.** Let S be an isolated compact invariant subset. A topological pair (N, L) of compact subsets of M where $L \subseteq N$ is called an **index pair of** S, if it satisfies the following:

1. $S = I(\overline{N \setminus L}) \subseteq (N \overset{\circ}{\setminus} L)$.
2. $x \in L$ and $\varphi_{[0,t]}(x) \subseteq N$ implies that $\varphi_{[0,t]}(x) \subseteq L$. We call L **positively invariant in** N. Here $\varphi_{[0,t]}(x) \coloneqq \{\varphi_{\tilde{t}}(x) | \tilde{t} \in [0,t]\}$.
3. For all $x \in N$ such that a t exists, with $\varphi_t(x) \notin N$, there exists a t' with $\varphi_{[0,t']}(x) \subseteq N$ and $\varphi_{t'}(x) \in L$.

This definition captures how the flow lines leave the invariant space S. Due to the third property we call L the **exit set**.

Definition 2.6 (Regular index pairs)**.** We call an index pair (N, L) **regular** if and only if the inclusion $I : L \hookrightarrow N$ is a cofibration.

Remark 2.7. One can show that every isolated compact invariant subset admits an index pair. We won't proof this, as we will always explicitly define such invariant sets and therefore we won't need such a general statement. However, two index pairs of the same invariant set turn out to be homotopy equivalent with a homotopy induced by the flow. The regularities needed for the proof are, that $\varphi_t : M \to M$ is a flow on a locally compact metric space. Those regularities are clearly given in the case of our Riemannian manifolds. We follow a proof from [**LecturesonMorseHomology**] which itself is a reformulation from the proof given by Salomon in [**SalomonConleyIndex**]. In this reformulation the language of the proof is adapted while the main arguments stay the same. This led to some redundant steps and vagueness, that I removed whilst making it more concrete.

Lemma 2.8. *Let N be an isolating neighborhood for the isolated compact invariant set S and let U be a neighbourhood of S. Then there exists a $t > 0$ such that for any $x \in M$ we have:*

$$\varphi_{[-t,t]}(x) \subseteq N \quad \Rightarrow \quad x \in U.$$

Proof. Lets assume this was false. Then for any $t > 0$ there would be a x contradicting the implication. So define $x_n \notin U$ such that $\varphi_{[-n,n]} \subseteq N$. Since $\varphi_0 = \mathrm{id}$ we know that all $x_n \in N$ and therefore by the compactness we would have limit points $x \in \overline{M \setminus U}$, such that $\varphi_{\mathbb{R}}(x) \subseteq N$. Since S was isolated we need to conclude that $x \in S \cap \overline{M \setminus U}$. This is our contradiction as U was a neighbourhood of the closed S and therefore without restriction open making $M \setminus U$ already closed. □

Remark 2.9. Notice that if t satisfies the conditions of lemma 2.8 then every $\tilde{t} \geq t$ also does. This helps us in the next lemma:

Lemma 2.10. *Let (N, L) and $(\tilde{N}, \tilde{L})$ be index pairs for the isolated invariant set S and choose $T \geq 0$ such that the following implications hold for $t \geq T$:*

$$\varphi_{[-t,t]}(x) \subseteq N \setminus L \quad \Rightarrow \quad x \in \tilde{N} \setminus \tilde{L}, \tag{2}$$

$$\varphi_{[-t,t]}(x) \subseteq \tilde{N} \setminus \tilde{L} \quad \Rightarrow \quad x \in N \setminus L. \tag{3}$$

Then the map:

$$h : N/L \times [T, \infty) \to \tilde{N}/\tilde{L}$$

$$([x], t) \mapsto \begin{cases} [\varphi_{3t}(x)] & \text{if } \varphi_{[0,2t]} \subseteq N \setminus L \text{ and } \varphi_{[t,3t]}(x) \subseteq \tilde{N} \setminus \tilde{L} \\ [\tilde{L}] & \text{otherwise,} \end{cases}$$

is continuous.

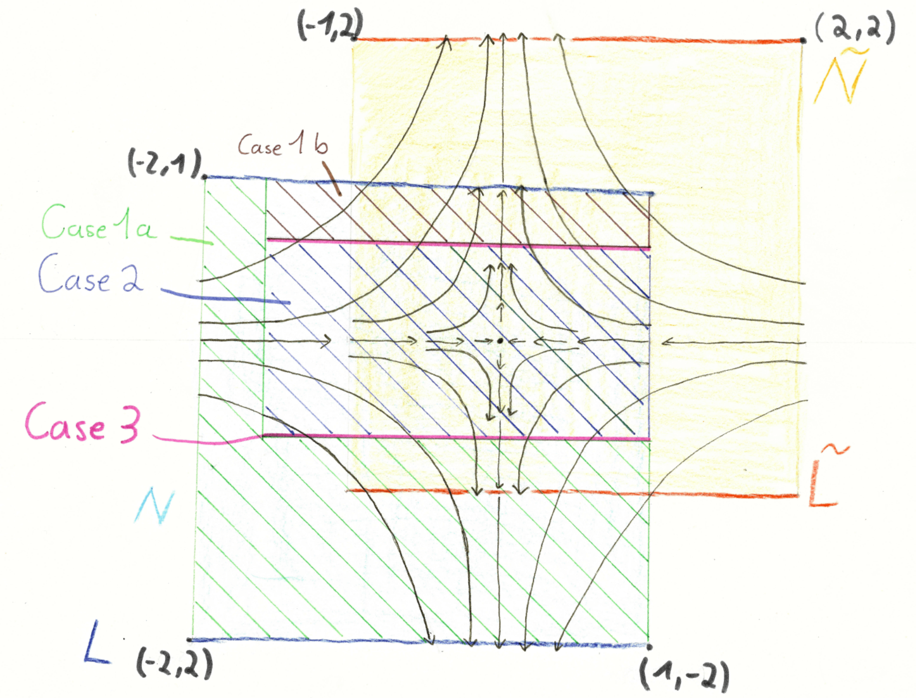


Figure 3: The different cases for the proof of lemma 2.10.
In this picture the flow is induced by the vectorfield $(-x, y)$ and drawn for $t = \frac{1}{3}$. That is due to the ease of calculation and such that the four cases are all visible. In the picture $N = [-2,1]^2$ and $\tilde{N} = [-1,2]^2$. L and $\tilde{L}$ are the horizontal borders.

Proof. We split this proof into three different cases that can be seen in figure 3. We always look at an image point lets say $h([x],t)$, choose an arbitrary neighbourhood U and define a neighbourhood W of $([x],t)$ such that the image of W is contained in U. This clearly gives us locally $\varepsilon-\delta-$continuity, as we are a metric space:
Case one a: $\varphi_{[t,3t]}(x) \nsubseteq \overline{\tilde{N} \smallsetminus \tilde{L}}$. In this case there exists a $t < t^* < 3t$ such that $\varphi_{t^*}(x) \notin \overline{\tilde{N} \smallsetminus \tilde{L}}$. t^* can be taken strictly less then $3t$ since the complement of $\overline{\tilde{N} \smallsetminus \tilde{L}}$ is open. Furthermore, this tells us the existence of a neighbourhood U of $\varphi_{t^*}(x)$ that is disjoint from $\overline{\tilde{N} \smallsetminus \tilde{L}}$. And by the continuity of the flow we have a neighbourhood $W \subseteq M \times [T, \infty]$ such that $(x', t') \in W$ implies that $\varphi_{t^*}(x') \in U$ and $t' < t^* < 3t'$. Thus, $\varphi_{[t',3t']}(x') \nsubseteq \tilde{N} \smallsetminus \tilde{L}$ and therefore $h([x'],t') = [\tilde{L}]$ for all $(x',t') \in W$.

We can argue the same way if $\varphi_{[0,2t]} \nsubseteq (N \smallsetminus L)$ (*Case one b*) and therefore conclude that in this case the map h is continuous. So for the rest of the proof we assume that we are in the first case of the map h or at the boundary:

$$\varphi_{[0,2t]} \subseteq \overline{N \smallsetminus L} \text{ and } \varphi_{[t,3t]}(x) \subseteq \overline{\tilde{N} \smallsetminus \tilde{L}}. \tag{4}$$

Case two: $\varphi_{[t,3t]}(x)$ is disjoint with $\tilde{L}$. Then due to the closure of $\tilde{N}$ and (4) we can conclude that $\tilde{N} \smallsetminus \tilde{L} \supseteq \varphi_{[t,3t]}(x) = \varphi_{[-t,t]}(\varphi_{2t}(x))$ and by the implication (3) we can conclude that $\varphi_{2t}(x) \in N \smallsetminus L$. Since L is the exit set we have that $\varphi_{[0,2t]}(x) \subseteq N \smallsetminus L$. By the above we have that $h([x],t) = [\varphi_{3t}(x)] \in \tilde{N} \smallsetminus \tilde{L}$. As before, due to the continuity of the flow we choose a neighbourhood U of $\varphi_{3t}(x)$ and find a neighbourhood $W \subseteq M \times [T,\infty)$ such that whenever $(x',t') \in W$ we have:

$$\varphi_{[0,2t']} \cap L = \varnothing, \quad \varphi_{[t',3t']}(x') \cap \tilde{L} = \varnothing, \quad \text{and} \quad \varphi_{3t'}(x') \in U.$$

If x' is in N then we have that $\varphi_{[0,2t']}(x')$ is in $N \smallsetminus L$ and similar to before we conclude with (2) that $\varphi_{t'}(x') \in \tilde{N} \smallsetminus \tilde{L}$ and since $\tilde{L}$ is the exit set we have the inclusion $\varphi_{[t',3t']}(x') \subseteq \tilde{N} \smallsetminus \tilde{L}$. Therefore, we have that $h([x'],t') = [\varphi_{3t'}(x')] \in U$ for all $(x',t') \in W$ where $x' \in N$. The continuity of the flow gives us continuity in this area.
Case three: $\varphi_{[t,3t]}(x)$ intersects $\tilde{L}$. Then by (4) and since $\tilde{L}$ is the exit set we have that $\varphi_{3t}(x) \in \tilde{L}$. Now define $[U]$ to be a neighbourhood of $h([x],t) = [\tilde{L}]$ in $\tilde{N}/\tilde{L}$. We want to find a representative of $[U]$, that is an open set of M that reduces to $[U]$ in the quotient space. Let $\pi : \tilde{N} \to \tilde{N}/\tilde{L}$ be the quotient map. A natural choice would be $U := \pi^{-1}([U])$ which is without restriction open in $\tilde{N}$. To make it open in M we can unite it with $M \smallsetminus \tilde{N}$. Now again by the continuity of the flow we have an open neighbourhood $W \subseteq M \times [T,\infty)$ of (x,t) such that whenever $(x',t') \in W$ we have that $\varphi_{3t'}(x') \in U$. But then we have that:

$$h([x'],t') \in \{[\varphi_{3t'}(x')], [\tilde{L}]\} \subseteq [U] \cap [\tilde{L}] = [U].$$

$\square$

Lemma 2.11. *Let $(N,L),(N',L')$ and $(\tilde N,\tilde L)$ be index pairs of S. Choose $T>0$ such that the implications (2) and (3) are satisfied and furthermore choose a $\tilde T$ such that for $t>\tilde T$ we have:*

$$\varphi_{[-t,t]}(x)\subseteq \tilde N\setminus \tilde L \quad\Rightarrow\quad x\in N'\setminus L' \tag{5}$$

$$\varphi_{[-t,t]}(x)\subseteq N'\setminus L' \quad\Rightarrow\quad x\in \tilde N\setminus \tilde L. \tag{6}$$

Now define:

$$h: N/L\times[T,\infty)\to \tilde N/\tilde L$$

$$([x],t)\mapsto\begin{cases}[\varphi_{3t}(x)] & \text{if } \varphi_{[0,2t]}\subseteq N\setminus L \text{ and } \varphi_{[t,3t]}(x)\subseteq \tilde N\setminus\tilde L\\ [\tilde L] & \text{otherwise,}\end{cases}$$

and

$$\tilde h: \tilde N/\tilde L\times[\tilde T,\infty)\to N'/L'$$

$$([x],t)\mapsto\begin{cases}[\varphi_{3t}(x)] & \text{if } \varphi_{[0,2t]}\subseteq \tilde N\setminus \tilde L \text{ and } \varphi_{[t,3t]}(x)\subseteq N'\setminus L'\\ [L'] & \text{otherwise.}\end{cases}$$

Then the following equations hold for $t\geq\max\{T,\tilde T\}$:

$$\tilde h\big(h([x],t),t\big)=\begin{cases}[\varphi_{6t}(x)] & \text{if } \varphi_{[0,4t]}\subseteq N\setminus L \text{ and } \varphi_{[2t,6t]}(x)\subseteq N'\setminus L'\\ [L'] & \text{otherwise.}\end{cases}$$

Proof. The proof is just the equivalence of the two statements:

$$\begin{aligned}&\varphi_{[0,4t]}\subseteq N\setminus L \text{ and } \varphi_{[2t,6t]}(x)\subseteq N'\setminus L'\\ \Leftrightarrow\quad &\varphi_{[0,2t]}(x)\subseteq N\setminus L,\ \varphi_{[t,5t]}(x)\subseteq \tilde N\setminus\tilde L,\ \varphi_{[4t,6t]}(x)\subseteq N'\setminus L'.\end{aligned}$$

„$\Rightarrow$“ Here we need to check three things. The first and the third inclusion are trivially satisfied. For the second we notice that $\varphi_{[0,4t]}\subseteq N\setminus L$ implies that $\varphi_{[t,3t]}(x)\subseteq\tilde N\setminus\tilde L$ by implication (2). Furthermore, $\varphi_{[2t,6t]}(x)\subseteq N'\setminus L'$ implies that $\varphi_{5t}(x)\in\tilde N\setminus\tilde L$ by implication (6). And since $\tilde L$ is the exit set this implies the missing second inclusion.
„$\Leftarrow$“ For the first inclusion notice that $x=\varphi_0(x)\in N$ by definition and $\varphi_{[t,5t]}(x)\subseteq\tilde N\setminus\tilde L$ implies $\varphi_{[2t,4t]}(x)\subseteq N\setminus L$ by implication (3). Using the exit set property we conclude that $\varphi_{[0,4t]}\subseteq N\setminus L$. Finally, $\varphi_{[t,5t]}(x)\subseteq\tilde N\setminus\tilde L$ implies that $\varphi_{[2t,4t]}(x)\subseteq N'\setminus L'$ by (5).

Together with $\varphi_{[4t,6t]}(x)\subseteq N'\setminus L$ this lets us conclude that $\varphi_{[2t,6t]}(x)\subseteq N'\setminus L'$. The additive property of the flow does the rest letting us conclude that we are in the first case of h and h' if and only if the stated property is satisfied. □

Theorem 2.12 (Homotopy equivalence of index pairs)**.** *If S is an isolated compact invariant set and (N,L) and $(\tilde N,\tilde L)$ are two index pairs of S. Then N/L and $\tilde N/\tilde L$ are homotopy equivalent as pointed spaces.*

Proof. Let $h_t : N/L \to \tilde{N}/\tilde{L}$ and $g_t : \tilde{N}/\tilde{L} \to N/L$ be the continuous family of maps from lemma 2.10. By lemma 2.11 we get that

$$g_t \circ h_t([x]) = \begin{cases} [\varphi_{6t}(x)] & \text{if } \varphi_{[0,4t]} \subseteq N \setminus L \text{ and } \varphi_{[2t,6t]}(x) \subseteq N \setminus L \\ [L] & \text{otherwise.} \end{cases}$$

Furthermore, we use lemma 2.10 for $T = 0$. And now we can explicitly write down a homotopy from the identity to $g_t \circ h_t([x])$ as follows:

$$H : N/L \times [0,1] \to N/L$$
$$([x], t') \mapsto \begin{cases} [\varphi_{6t't}(x)] & \text{if } \varphi_{[0,4t't]} \subseteq N \setminus L \text{ and } \varphi_{[2t't,6t't]}(x) \subseteq N \setminus L \\ [L] & \text{otherwise.} \end{cases}$$

This homotopy is continuous by lemma 2.10 and $H(\cdot, 0) = \mathrm{id}$ and $H(\cdot, 1) = g_t \circ h_t$. Similar we show that $h_t \circ g_t$ is homotopic to the identity. With this we have that

$$N/L \simeq \tilde{N}/\tilde{L}.$$

□

Definition 2.13 (The homotopy Conley index)**.** The **Conley index** of an invariant set S is defined as $\pi_1(N, L)$ where (N, L) is a regular index pair. This definition is invariant of the chosen pair and is an invariant of invariant sets.

3 A Morse Theoretic Filtration

In this subchapter we will finally prove the Morse homology theorem. We start by defining some index sets that we will later use in the prove. Again, this prove is due to Solomon and found in [**MorseTheorySalmbon**].

Example 3.1 (Critical points as isolated compact invariant subsets)**.** Let $f : M \to \mathbb{R}$ be a Morse function on a smooth Riemannian manifold (M, g). Let $q \in \mathrm{Crit}_k(f)$. Around q we have coordinate charts $\varphi : U \to T_qM$ (after identifying q_i and ∂q_i) where $\varphi(W(q \to) \cap U) \subseteq T_q^uM$ and $\varphi(W(\to q) \cap U) \subseteq T_q^sM$. With this we can define the balls:

$$\begin{aligned} D_\varepsilon^s &= \{v \in T_q^sM | \, \|v\| \leq \varepsilon\}, \\ D_\varepsilon^u &= \{v \in T_q^uM | \, \|v\| \leq \varepsilon\}. \end{aligned}$$

They give rise to the index pair $N_q := \varphi^{-1}(D^s \times D^u)$ and $L_q = \varphi^{-1}(D^s \times \partial D^u)$. This index pair is in fact regular, since $(N_q, L_q) \cong (D^s \times D^u, D^s \times \partial D^u) \cong (D_\varepsilon^u, \partial D_\varepsilon^u)$. Now we can make the first step towards singular homology, since we know the homology of such a tuple:

$$H_i^{\mathrm{sing}}(N_q, L_q) = H_i^{\mathrm{sing}}(D^k, \partial D^k) = \begin{cases} \mathbb{Z} & \text{if } i = k, \\ 0 & \text{else.} \end{cases} \tag{7}$$

Notice that for $k = 0$ we need to consider $\partial D^k = \emptyset$, to get an index pair. That's why we let ∂ denote the manifold boundary insted of the topological boundary. However, now we can identify for all k:

$$C_k(M, f) = \bigoplus_{\mathrm{Crit}_k(f)} \mathbb{Z} \cong \bigoplus_{q \in \mathrm{Crit}_k(f)} H_k(N_q, L_q; \mathbb{Z}). \tag{8}$$

Furthermore, this isomorphism can be made canonically, by using orientations: Notice for this that (N_q / L_q) is homotopic equivalent to $W(q \to)/(W(q \to) \setminus \{q\})$. Here we get a generator of the $k-$homology group induced by an orientation on $T_p^u M$ that then canonically maps to the $k-$th homology group of (N_q, L_q).

Example 3.2 (A map of homology groups)**.** Let $f : M \to \mathbb{R}$ be a Morse Smale function on a smooth compact Riemannian manifold (M, g). Let $q \in \mathrm{Crit}_k(f)$ and $p \in \mathrm{Crit}_{k-1}(f)$. Assume for a moment we already have an isolated regular compact index pair (N_2, N_0) of $S = W(p \to q) \cup \{p, q\}$. For a $c \in (f(p), f(q))$ we set $N_1 = N_0 \cup (N_2 \cap M^c)$ depicted in figure 4, where M^c is the sublevel set. Clearly now (N_2, N_1) is an index pair for q and (N_1, N_0) is a pair for p. Assuming we have regular index pairs $(N_q.L_q)$ and (N_p, L_p) for p, q we can now define a map:

$$\Delta_k(q \to p) : H_k^{\mathrm{sing}}(N_q, L_q) \to H_{k-1}^{\mathrm{sing}}(N_p, L_p)$$

as the composition:

$$H_k(N_q, L_q; \mathbb{Z}) \xrightarrow{\cong} H_k(N_2, N_1) \xrightarrow{\delta_*} H_{k-1}(N_1, N_0) \xrightarrow{\cong} H_{k-1}(N_p, L_p)$$

where δ_* is the connecting homomorphism, and the other two maps are induced by the homotopic equivalence of two index pairs corresponding to the same isolated compact invariant subset. Putting all together we can define a morphism:

$$\Delta_k : \bigoplus_{q \in \mathrm{Crit}_k(f)} H_k^{\mathrm{sing}}(N_q, L_q) \to \bigoplus_{p \in \mathrm{Crit}_{k-1}(f)} H_{k-1}^{\mathrm{sing}}(N_p, L_p). \tag{9}$$

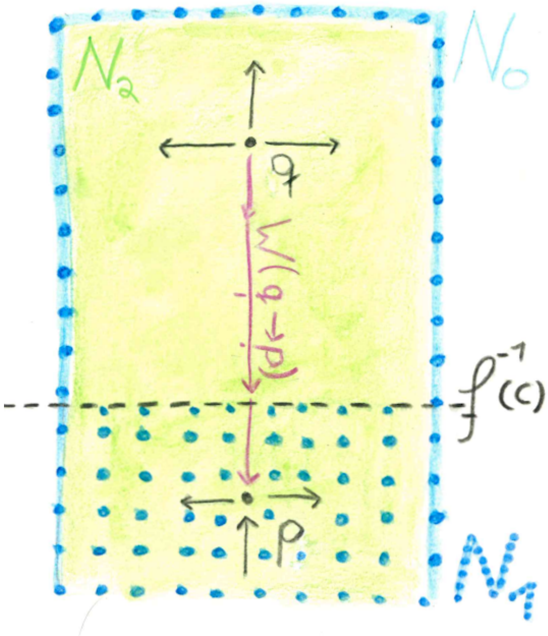


Figure 4: Index pairs for gradient flow lines.

Definition 3.3 (Regular index pairs and a filtration on M)**.** As always let $f : M \to \mathbb{R}$ be a Morse-Smale function on a compact smooth Riemannian manifold (M, g) of dimension m. Let $\varphi_t : M \to M$ be the flow given by $-\mathrm{grad} f$. For $0 \leq j \leq k \leq m$ define:

$$W(k, j) = \bigcup_{j \leq \lambda_p \leq \lambda_q \leq k} W(q \to p).$$

We know that this space is compact. Assume that N is a compact neighbourhood of $W(k, j)$, such that $\mathrm{Crit}(f) \cap N = \mathrm{Crit}(f) \cap W(k, j)$, meaning that N does not contain any new critical points. Then the biggest invariant subspace from definition 2.5 is reference

$$I(N) := \{x \in N | \varphi_t(x) \in N \forall t \in \mathbb{R}\} = W(j, k).$$

This follows from every gradient flow line starting and ending in a critical point. Therefore, $W(k, j)$ is an isolated compact invariant set. By a corollary of the lambda lemma we know that reference

$$W_j^s := \bigcup_{j \leq \lambda_p} W(\to p)$$
$$W_j^u := \bigcup_{\lambda_p \leq j} W(p \to)$$

for all $j = 0, \ldots, m$ are compact as they are finite unions of compact sets. Now we define $N_m := M$ and choose a cofibered compact neighbourhood N_{m-1} of $W_{m-1}^{p\to}$ that is positively invariant and satisfies $N_{m-1} \cap W_m^{\to p} = \varnothing$ One could for example define:

$$N_{m-1} := N_m \setminus \Big(\bigcup_{q \in \mathrm{Crit}_m(f)} \mathring{N}_q \Big),$$

where N_q is taken from example 3.1. Once we check that N_{m-1} is an exit set with respect to N_m we can conclude that (N_m, N_{m-1}) is a regular index pair for $\mathrm{Crit}_m(f)$. The first statement is clear, since every gradient line starting in $N_m \setminus N_{m-1}$ has to pass through N_{m-1}, as they go towards other critical points. This tells us that they have to leave $N_m \setminus N_{m-1}$, which is enclosed by N_{m-1}. The second thing to check can be done by showing that there is a neighbourhood $U \subset N_M$ of N_{m-1} that deforms to N_{m-1}. For this notice that each N_q with $\lambda_q = m$ is homeomorphic to D^m. Now choose a open neighbourhood U_q in the disc of its boundary. By definition this retracts to the boundary with a retraction induced by the flow. And finally define $U := N_{m-1} \cap_{q \in \mathrm{Crit}_m(f)} U_q$. This retracts to N_{m-1} telling us that our index pair is indeed regular.

Now we want to inductively define a filtration: For this we choose a compact cofibered neighborhood N_{m-2} of W_{m-2}^u that is positifely invariant in N_{m-1} and has an empty intersection with $W_{m-1}^{\to p}$. This can be done similar to the construction of N_{m-1} but instead of cutting out neighbourhoods if $q \in \mathrm{Crit}_m(f)$, we can cut out tubular neighborhoods of W_{m-1}^s. This is depicted in figure 5 for the torus. By iterating this process we get a filtration:

$$\varnothing =: N_{-1} \subset N_0 \subset N_1 \subset \cdots \subset N_{m-1} \subset N_m = M, \tag{10}$$

such that (N_k, N_{j-1}) is a regular index pair for $W(k, j)$ for all $0 \leq j \leq k \leq m$.

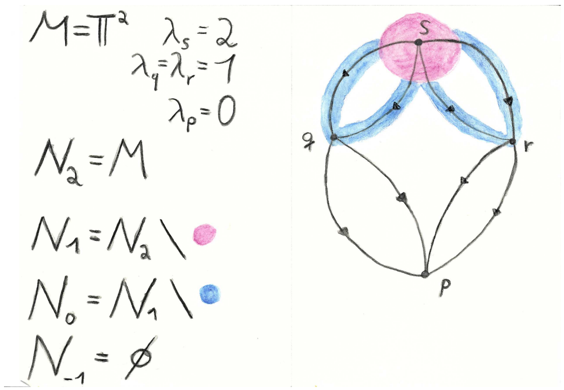


Figure 5: The filtration of the torus.
The picture shows the phase diagram of the torus and schematically depicts the filtration.

4 Orientations with K-theorie

Before we start the hole section, we establish a little reminder on how the K-Groups of the spheres look like: First of all the n-th K-group of a k-Sphere is by definition the reduced K-group of a $n+k$ Sphere. with this we have the general rule:

> If the dimension is **odd**, the K-group matters **not**!

Definition 4.1 (Orientations on Vector Spaces)**.** Let's start with a recall of what an orientation is. For this consideration we start with a vector space $V \cong \mathbb{R}^m$. An orientation of V is the equivalence class of an **ordered basis** $(v_1, \dots v_m)$, where the relation is as follows:
Two basis $v = (v_1, \dots, v_m)$ and $w = (w_1, \dots w_m)$ are equivalent, if the base-change matrix M_w^v has positive determinant. By this relation we get two orientations on $\mathbb{R}^m$. If $m = 0$, we define the orientation to be either 1 or -1. A vector space V together with an orientation $\theta = \overline{v}$ is called an oriented vector space and an ordered basis w of V is called **positive**, if $w \in \theta$, or in other words, if $w \sim v$.

Remark 4.2 (From Real Space to the Sphere)**.** We now want to construct a canonical homeomorphism between S^m and $\overline{\mathbb{R}^m}$. First we use the stereographic projection from the South Pole:

$$\begin{aligned}\pi_S : S^m \setminus \{S\} \subseteq \mathbb{R}^{m+1} &\to \mathbb{R}^m\\ (x_1, \dots, x_m, x_{m+1}) &\mapsto \left(\frac{x_1}{1+x_{m+1}}, \dots, \frac{x_m}{1+x_{m+1}}\right)\end{aligned}$$

with its inverse:

$$\begin{aligned}\pi_S^{-1} : \mathbb{R}^m &\to S^m \setminus \{S\} \subseteq \mathbb{R}^{m+1}\\ (y_1, \dots y_m) &\mapsto \left(\frac{2y_1}{1+\|y\|^2}, \dots, \frac{2y_m}{1+\|y\|^2}, \frac{1-\|y\|}{1+\|y\|}\right)\end{aligned}$$

Those are inverse maps and they give rise to maps between their one-point-compactifications. Hence $\overline{\mathbb{R}^m} \cong S^m$. Lets calculate $\pi_S^{-1} \circ \pi_s$, since we will later need a little bit of insight in those calculations:

$$\begin{aligned}\pi_S^{-1} \circ \pi_s(x_1, \dots, x_{m+1}) &= \pi_S^{-1}\left(\frac{x_1}{1+x_{m+1}}, \dots, \frac{x_m}{1+x_{m+1}}\right)\\ &= \left(\frac{2\frac{x_1}{1+x_{m+1}}}{1+\sum_{i=1}^m(\frac{x_i^2}{(1+x_{m+1})^2})}, \dots, \frac{2\frac{x_m}{1+x_{m+1}}}{1+\sum_{i=1}^m(\frac{x_i^2}{(1+x_{m+1})^2})}, \frac{1-\sum_{i=1}^m(\frac{x_i^2}{(1+x_{m+1})^2})}{1+\sum_{i=1}^m(\frac{x_i^2}{(1+x_{m+1})^2})}\right)\\ &=\end{aligned}$$

Before we keep calculating, we notice the denominator can be rewritten as:

$$1 + \sum_{i=1}^m\left(\frac{x_i^2}{(1+x_{m+1})^2}\right) = \frac{(1+x_{m+1})^2 + \sum_{i=1}^m x_i^2}{(1+x_{m+1})^2}$$

And hence:

$$\frac{2\frac{x_j}{1+x_{m+1}}}{1+\sum_{i=1}^m\left(\frac{x_i^2}{(1+x_{m+1})^2}\right)} = \frac{2\frac{x_j}{1+x_{m+1}}}{\frac{(1+x_{m+1})^2+\sum_{i=1}^m x_i^2}{(1+x_{m+1})^2}} \tag{11}$$

$$= \frac{2x_j(1+x_{m+1})}{(1+x_{m+1})^2+\sum_{i=1}^m x_i^2} \tag{12}$$

$$= \frac{2x_j(1+x_{m+1})}{(1+x_{m+1})^2+\sum_{i=1}^m x_i^2} \tag{13}$$

$$= \frac{2x_j(1+x_{m+1})}{1+2x_{m+1}+\underbrace{\sum_{i=1}^{m+1} x_i^2}_{=1}} \tag{14}$$

$$= x_j \tag{15}$$

And the calculation of the last term is:

$$\begin{aligned}
\frac{1-\sum_{i=1}^m\left(\frac{x_i^2}{(1+x_{m+1})^2}\right)}{1+\sum_{i=1}^m\left(\frac{x_i^2}{(1+x_{m+1})^2}\right)} &= \frac{\frac{(1+x_{m+1})^2-\sum_{i=1}^m x_i^2}{(1+x_{m+1})^2}}{\frac{(1+x_{m+1})^2+\sum_{i=1}^m x_i^2}{(1+x_{m+1})^2}} \\
&= \frac{(1+x_{m+1})^2-\sum_{i=1}^m x_i^2}{(1+x_{m+1})^2+\sum_{i=1}^m x_i^2} \\
&= \frac{1+2x_{m+1}+x_{m+1}^2-\sum_{i=1}^m x_i^2}{1+2x_{m+1}+\underbrace{x_{m+1}^2+\sum_{i=1}^m x_i^2}_{=1}} \\
&= \frac{1+2x_{m+1}+2x_{m+1}^2-\sum_{i=1}^{m+1} x_i^2}{2+2x_{m+1}} \\
&= \frac{x_{m+1}(2+2x_{m+1})}{2+2x_{m+1}} = x_{m+1}
\end{aligned}$$

This concludes the calculation of $\pi_S^{-1} \circ \pi_s = \mathrm{id}$ the other order is similar.

Definition 4.3 (K-theoretic orientations)**.** We define a **K-theoretic orientation** of a vector space V to be a generator of the group $K^m(\overline{V})$. As a sainity check we can confirm: since $\overline{V} \cong S^m$ we are looking at a generator of a group isomorphic to $K(S^{2m}) \cong \mathbb{Z}$, which has two possible generators.

Now V is congruent to $\mathbb{R}^m$ via the coordinate map B_v and that depends on an ordered basis v of v. So the idea is as follows: We have a canonical generator of $K^m(S^m)$ and a map:

$$K^n(S^n) \xrightarrow{(\pi_S^{-1})^*} K^n(\overline{\mathbb{R}^m}) \xrightarrow{B_v^*} K^m(\overline{V})$$

This induces a generator in $K^m(\overline{V})$ that depends on the ordered basis.

Remark 4.4. The K−theoretic orientaion only depends on the orientation.

replace B_v^* mit B_v^-1, extension und so wohl unterscheiden. Vllt den beweis funktoriell aufziehen und faktorisiert über hTop verwenden

Proof. Let v and v' be two equivalent orientations, meanin $\det(M^v_{v'}) > 0$. This means that $M^v_{v'} \simeq \mathrm{id}$, since $\mathrm{GL}(m, \mathbb{R})$ has two path-connected components determined by the determinant. Alternative we can give an homotopy explicit by

$$\begin{aligned} H : \overline{\mathbb{R}^m} \times I &\to \overline{\mathbb{R}^m} \\ (x, t) &\mapsto (1-t)M^v_{v'} + t(\mathrm{id}) \\ (\infty, t) &\mapsto \infty \end{aligned}$$

This is continuos for all t, since it is continuous for all $x \neq \infty$. Furthermore, it is invertible for all t and $x \neq \infty$, because for $x \neq \infty$ we can view H as a continuous map $H(\cdot, t) : I \to \mathrm{GL}(m, \mathbb{R})$ and since I is connected it maps into one connected component and that is $\mathrm{GL}^+(n, \mathbb{R})$. Hence, the composition maps into $\det(H(\cdot, t)) \in \mathbb{R}_{>0}$ for all t. Finally, we have that the restricted map where the points at infinity are excluded is proper and thereby our extended map H is continuos, giving us a continuous homotopy. With this we can conclude, that the diagramm commutes up to homotopy:

$$\begin{tikzcd} \overline{V} \arrow[r, "\mathrm{id}"] \arrow[d, "B_v"'] & \overline{V} \arrow[d, "B_{v'}"] \\ \overline{\mathbb{R}^m} \arrow[r, bend left, "M^v_{v'}"] \arrow[r, bend right, "\mathrm{id}"'] & \overline{\mathbb{R}^m} \end{tikzcd}$$

Hence, the diagram commutes,

$$\begin{tikzcd} K^m(S^m) \arrow[r, "(\pi_S^{-1})^*"] & K^n(\overline{\mathbb{R}^m}) \arrow[r, "B_v^*"] \arrow[d, "\mathrm{id}"] & K^m(\overline{V}) \arrow[d, "\mathrm{id}"] \\ K^m(S^m) \arrow[r, "(\pi_S^{-1})^*"] & K^n(\overline{\mathbb{R}^m}) \arrow[r, "B_{v'}^*"] & K^m(\overline{V}) \end{tikzcd}$$

Thereby the induced generators agree. □

Lemma 4.5 (K-Theoretic Orientations Detect Orientations)**.** *Assume, that v' and v are non-equivalent orientations. Hence,* $\det(M^v_{v'}) < 0$*. Then they induce different generators.*

Proof. With a similar argument to above we can conclude, that $M^v_{v'} \simeq \big((x_1, \dots x_m) \mapsto (x_1, \dots x_{m-1}, -x_m)\big)$ and again there extensions to the one point compactifications are homotypic. Now we define two maps:

$$\begin{aligned} T : S^m &\to S^m \\ (x_1, \dots, x_{m+1}) &\mapsto (-x_1, \dots, x_{m+1}) \end{aligned}$$

and

$$\begin{aligned} S : \overline{R^m} &\to \overline{R^m} \\ (x_1, \dots x_m) &\mapsto (-x_1, \dots x_m) \end{aligned}$$

With those maps we have the commutative diagram:

$$\begin{tikzcd}
S^m & \overline{\mathbb{R}^m} \arrow[l, "\pi_s^{-1}"'] \\
S^m \arrow[u, "T"] & \overline{\mathbb{R}^m} \arrow[l, "\pi_s^{-1}"] \arrow[u, "S"']
\end{tikzcd}$$

And hence since $S \simeq \overline{M_{v'}^{v}}$ we get the diagram, that is commutative up to homotopie:

$$\begin{tikzcd}
S^m & \overline{\mathbb{R}^m} \arrow[l, "\pi_s^{-1}"'] & \overline{V} \arrow[l, "B_{v'}^{-1}"'] \\
S^m \arrow[u, "T"] & \overline{\mathbb{R}^m} \arrow[l, "\pi_s^{-1}"] \arrow[u, "S"] & \overline{V} \arrow[l, "B_v^{-1}"] \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

This induces the commutative diagramm

$$\begin{tikzcd}
K^m(S^m) \arrow[r, "(\pi_S^{-1})^*"] \arrow[d, "T^*"'] & K^m(\overline{\mathbb{R}^m}) \arrow[r, "(B_v^{-1})^*"] \arrow[d, "S^*"] & K^m(\overline{V}) \arrow[d, "\mathrm{id}"] \\
K^m(S^m) \arrow[r, "(\pi_S^{-1})^*"'] & K^m(\overline{\mathbb{R}^m}) \arrow[r, "(B_{v'}^{-1})^*"'] & K^m(\overline{V})
\end{tikzcd} \tag{16}$$

Now we want to see that $T^* = -\mathrm{id}$. For this, we start by looking at $K^m(S^m) = K(S^{2m})$. In this case we define

$$T' : S^m \wedge S^m = S^{2m} \to S^{2m}$$
$$(x_1, \dots x_m, y_1, \dots, y_m) \mapsto (x_1, \dots x_m, -y_1, \dots, y_m)$$

Now by definition $T^* = (T')_*$ Now we have the homotopy between T' and

$$R \wedge \mathrm{id} : S^{2m} \to S^{2m}$$
$$(x_1, \dots x_{2m}) \mapsto (-x_1, \dots x_{2m})$$

because the change of order of bases elements is homotopic to the identity. (This can be seen by reusing the proof of lemma 4.4, since the needed base-change matrix has determinant 1). Now the way we defined $S^{2m} \subseteq \mathbb{R}^{2m+1}$ the map $R \wedge \mathrm{id}$ really is the wedge product of the identity on S^{2m-1} with the map R, that exchanges the South and North Pole. Hence by the lemma we had before , this agrees with the multiplication with -1, and hence the map T^* is the multiplication with -1. Finally, we can compare the induced generators: for this we denote $\beta_v := (B_v^{-1})^* \circ (\pi_S^{-1})^*(\beta)$ to be the generator given by the oriented basis v and $\beta_{v'} := (B_{v'}^{-1})^* \circ (\pi_S^{-1})^*(\beta)$ given by v'. Now by the

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commutativity of the diagram 16 we have that

$$\begin{aligned}
-\beta_{v'} &= -(B_{v'}^{-1})^* \circ (\pi_S^{-1})^*(\beta) \\
&= (B_{v'}^{-1})^* \circ (\pi_S^{-1})^*(-\beta) \\
&= -\beta_{v'} = -(B_{v'}^{-1})^* \circ (\pi_S^{-1})^*(\beta) \\
&= \underbrace{(B_{v'}^{-1})^* \circ (\pi_S^{-1})^* \circ T^*}_{=(B_v^{-1})^* \circ (\pi_S^{-1})^*}(\beta)) \\
&= \beta_v
\end{aligned}$$

□

Definition 4.6 (Oriented Vector Bundles)**.** For a vector bundle $\pi : E \to B$, an orientation is a family $\{\omega_x\}_{x\in B}$ of orientations of the fibers E_x such that there is an atlass Φ with the following propertie:
if $\varphi\Big|_U \to U \times \mathbb{R}^n$ is in Φ, then

$$\varphi_x : (E_x, \omega_x) \to (\mathbb{R}^n, \omega^n)$$

is orientation preserving. ω^n denotes the standars orientation of $\mathbb{R}^n$. If such an orientation exists, we call the bundle **orientable**.

Definition 4.7 (Orientable Manifold)**.** We call a manifold orientable, if its tangent bundle TM is orientable.

Corollary 4.8. A orientable manifold has a atlass, such that for all charts the jacobian of the transition maps is orientation preserving.

Proof. Let $(U_\alpha, \varphi_\alpha)$ be an atlas nof M. Each chart $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$ induces a basis of the Tangend space $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m})$ for all points in U_α. Let Φ be an oriented atlass of the tangend bundle $\pi : TM \to M$. Hence, we get □

5 What are Spheres

In this section we want to inspect representations of homotopical Spheres. To give a little overview we will define the sphere to come with an orientation. To be more specific we have the definition:

Definition 5.1 (The k–-Sphere)**.** Let $\mathbb{R}^k$ be the k-dimensional vector space given as the span of $e_1, \dots e_k$ together with the eucliden metric $\|\cdot\|_2$. Then the set

$$S^k := \{x \in \mathbb{R}^k | \, \|x\|_2 = 1\}$$

is called the k–**sphere**.

This sphere will be our save haven where we will always come back to. In this definition there is no room for "diffrent orientations on a sphere". In our reality we will detect the two "different orientations" of a space A homotopic to a sphere that arise from two constructions as follows: assume that α and β are the maps $A \to S^k$. then there is an endomorphism $f : S^k \to S^k$ such that our diagramm commutes:

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ S^k \arrow[rr, "f"] & & S^k \end{tikzcd}$$

Now $f^* : K^n(S^k) \to K^n(S^k)$ is an isomorphism and hence either the identity or minus the identity.

Now we have a lot of spaces that are homotopic or even homeomorph to the sphere. Just to name a view constructions:

1. The disc in $\mathbb{R}^k$ modulo its boundary $D^k/\partial D^k$.
2. The cone over the boundary of a D^k disc.
3. The resduced and unreduced susopension of a $k-1$ sphere.
4. The one-point compactification of a vector space together with an oriented basis.

We already saw how we can map the one-point compactification of a vector space with an ordered basis to a sphere wia the stereographic projections from the south pole. So lets inspect the other construction

Corollary 5.2. Given the disk $D^k \subseteq \mathbb{R}^k$ we have a homeomorphism

$$\overline{D^k}/\partial D^k \to S^k$$

Proof. We will construct a proper map from the open disk to R^k and by this we get a homeomorphism between their one-point-compaktifications.

$$\begin{aligned} s : D^k &\to \mathbb{R}^k \\ x &\mapsto x \cdot \frac{1}{1 - \|x\|_2} \end{aligned}$$

This is a homeomorphism and proper. Hence, we get a map between the onepoint compactifications

$$\overline{D^k}/\partial D^k \to \mathbb{R}^k \cup \{\infty\}$$

and the latter is homeomorphic to the Sphere. □

Corollary 5.3. Let $\overline{D^k} \cup C\partial D^k$ be the cone over the boundary of the disk. This is homeomorphic to the k–sphere.

Proof. Again we will use the language of onepoint compactifications. First we give a homeomorphism from the "open cone" over the boundary to the open disc. To do this we include our space into R^k as follows:

$$f : \overline{D^k} \cup \partial D^k \times [0,1) \to R^k$$
$$x \mapsto \begin{cases} x & \text{if } x \in \overline{D^k}\,, \\ (1+t)x & \text{if } (x,t) \in \partial D^k \times [0,1)\,. \end{cases}$$

This is a homeomorphism (if we restrict the image to $\{x \in R^k | \, \|x\|_2 < 2\}$). This can be seen as follows: Obviously it is bijective and the contiuouty can be derived from the gluing lemma. To see the contiuoity of the invese we can explicitly describe it:

$$f^{-1} : \{x \in R^k | \, \|x\|_2 < 2\} \to \overline{D^k} \cup \partial D^k \times [0,1)$$
$$x \mapsto \begin{cases} x & \text{if } x \in \overline{D^k}\,, \\ (\frac{x}{\|x\|_2}, (1 - \|x\|_2)) & \text{if } \|x\|_2 \geq 1\,, \end{cases}$$

Again, gluing lemma gives continouity, and a calculating gives us that they are inverse to each other. To see that the map is proper, we check if preimages of compact sets are compact. So let $K \subset \{x \in R^k | \, \|x\|_2 < 2\}$ be compact. Then $K = A \cup B$ where $A = K \cap \overline{D^k}$ and $B = K \cap \{x \in R^k | 1 \leq \|x\|_2 < 2\}$. Now A and B are again compact. (A is the intersection of two compact Hausdorff spaces and B can also be constructed as such a intersection) Since $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ we have to check if f^{-1} is a closed map restricted to $\overline{D^k}$ and $\{x \in R^k | 1 \leq \|x\|_2 < 2\}$. But those restrictions give rise to homeomorphisms and hence they are closed. Now we have a proper homeomorphism which lets us conclude, that the map induced on their onepoint compactifications are also homeomorphisms. After the rescaling:

$$r : \{x \in R^k | \, \|x\|_2 < 2\} \to D^k$$
$$x \mapsto \frac{x}{2}$$

we have a proper homeomorphism $\overline{D^k} \cup \partial D^k \times [0,1) \to D^k$. This lets us deduce the statment, as the onepoint compactification of the disk can be mapped to the sphere as seen in the corollary above. □

We will later use similar maps in our main prove.

Corollary 5.4. We want to understand homöomorphisms between boukettes of spheres.

6 Morsetheoretic Atiyah Hirzebruch Spectralsequeze

Definition 6.1. Let M be a smooth manifold with morse data and

$$\emptyset =: N_{-1} \subset N_0 \subset N_1 \subset \cdots \subset N_{m-1} \subset N_m = M,$$

be a filtrations constructed in 3.3. Then we define the Groups

$$E_r^{p,q}(M) := \frac{\ker\left[K^{p+q}(N_{p+r-1}, N_{p-r}) \to K^{p+q}(N_{p-1}, N_{p-r})\right]}{\ker\left[K^{p+q}(N_{p+r-1}, N_{p-r}) \to K^{p+q}(N_{p}, N_{p-r})\right]}.$$

To define a boundary operator, we proceed as follows: The following diagram commutes (by naturality of the boundary in the triple sequence).

$$\begin{tikzcd}
K^{p+q}(N_{p+r-1}, N_{p-r}) \arrow[r, "\alpha"] \arrow[d, "\delta^*_{triple}"] & K^{p+q}(N_p, N_{p-r}) \arrow[d, "\delta^*_{triple}"] & \\
K^{p+q+1}(N_{p+2r-1}, N_{p+r-1}) \arrow[r, "\beta"] & K^{p+q+1}(N_{p+2r-1}, N_p) \arrow[r] & K^{p+q+1}(N_{p+r-1}, N_p)
\end{tikzcd}$$

Hence, $\beta \circ \delta^*_{triple}$ factors over $K^{p+q}(N_{p+r-1}, N_{p-r})/\ker\alpha$ and because the bottom row is exact (by the triple sequence) we have get a map

$$K^{p+q}(N_{p+r-1}, N_{p-r})/\ker\alpha \to \ker\left[K_{p+q+1}(N_{p+2r-1}, N_p) \to K^{p+q+1}(N_{p+r-1}, N_p)\right]$$

Now since $K^{p+q}(N_{p+r-1}, N_{p-r}) \supseteq \ker\left[K^{p+q}(N_{p+r-1}, N_{p-r}) \to K^{p+q}(N_{p-1}, N_{p-r})\right]$ we can restrict the domain and furthermore compose with the quotient map to get a well-defined map

$$\begin{aligned} d_r^{p,q} :\ E_r^{p,q} &\to E_r^{p+r,q-r+1} \\ [x] &\mapsto [\beta \circ \delta^*_{triple}(x)] \end{aligned}$$

Corollary 6.2 (The first page)**.** We start by analysing the first page

$$\begin{aligned} E_1^{p,q}(M) := \frac{\ker\left[K^{p+q}(N_p, N_{p-1}) \to K^{p+q}(N_{p-1}, N_{p-1})\right]}{\ker\left[\underbrace{K^{p+q}(N_p, N_{p-1}) \to K^{p+q}(N_p, N_{p-1})}_{\mathrm{id}}\right]} &\cong K^{p+q}(N_p, N_{p-1}) \\ &= \tilde{K}^{p+q}(N_p/N_{p-1}) \end{aligned}$$

Now since (N_p, N_{p-1}) is a regular index pair for $\mathrm{Crit}_p(f)$ we have a flow induced homotopiy equivalentce:

$$N_p/N_{p-1} \simeq \Big(\bigcup_{w \in Crit_p(f)} D^u_\varepsilon(w)\Big)/\Big(\bigcup_{w \in Crit_p(f)} \partial D^u_\varepsilon(w)\Big) \cong \bigvee_{w \in \mathrm{Crit}_p(f)} S^p$$

Here, we use the definitions from above

$$D^u_\varepsilon(w) \ := \ \varphi_w^{-1}\big(\{v \in T^u_w M |\, \|v\| \le \varepsilon\}\big),$$

and the last isomorphism is induced by orientations in the unstable manifolds. We can continue our inspection:

$$\begin{aligned} E_1^{p,q} &\cong \tilde{K}^{p+q}(N_p/N_{p-1}) \\ &\cong \bigoplus_{w \in \mathrm{Crit}_p(f)} \tilde{K}^q(pt) = C^k(M, f, g, \mathfrak{o}; \tilde{K}^q(pt)) \end{aligned}$$

In sum we have the chain of horizontal isomorphisms:

$$\begin{tikzcd}
E_1^{p,q} \arrow[r, "\alpha"] \arrow[d, "d_1^{p,q}"] & \tilde{K}^{p+q}(N_p/N_{p-1}) \arrow[r, "\beta"] \arrow[d] & \bigoplus_{w\in \mathrm{Crit}_p(f)} \tilde{K}^q(pt) \arrow[d] & C^p(M,\mathfrak{A},\tilde{K}^q(pt)) \arrow[l] \arrow[d] \\
E_1^{p+1,q} \arrow[r, "\alpha'"] & \tilde{K}^{p+1+q}(N_{p+1}/N_p) \arrow[r, "\beta'"] & \bigoplus_{w\in \mathrm{Crit}_{p+1}(f)} \tilde{K}^q(pt) & C^{p+1}(M,\mathfrak{A},\tilde{K}^q(pt)) \arrow[l]
\end{tikzcd}$$

By naturality of the triple boundary operator, that the second vertical map is the boundary operator. Now on the next rug we can induce two maps. From the left to make the diagram commute and from the right. Do they agree? Or asked differently, is the map induced from the d_1 map the boundary operator of morse homology?

Definition 6.3. Let $q \in \mathrm{Crit}_{k+1}(f)$ and $p \in \mathrm{Crit}_p(f)$. Assume for the moment, that we have a regular index pair (N_2, N_1) for $S := W(q \to p) \cup \{p, q\}$. For $c \in (f(p), f(q))$ we define $N_1 = N_0 \cup (N_2 \cup M^c)$, where M^c denotes the sublevelset. Now (N_2, N_1) is a index pair for q and (N_1, N_0) is one for p. Assume that we are givben any two index pairs (N_q, L_q) and (N_p, L_p) of q and p. Then we define a map

$$\Delta_k(q \to p) : K^k(N_p, L_p) \to K^k(N_q, L_q)$$

as the composition:

$$K^k(N_p, L_p) \xrightarrow{\cong} K^k(N_1, L_0) \xrightarrow{\delta_{triple}} K^k(N_2, L_1) \xrightarrow{\cong} K^k(N_q, L_q)$$

Putting all those together we have the definition of a map:

$$\Delta_k : \bigoplus_{p\in \mathrm{Crit}_k(f)} K^k(N_p, L_p) \to \bigoplus_{q\in \mathrm{Crit}_{k+1}(f)} K^k(N_q, L_q)$$

If we denote the inclusion $i_q : K^k(N_q, L_q) \to \bigoplus_{q\in \mathrm{Crit}_{k+1}(f)} K^k(N_q, L_q)$ we have the above map given by:

$$\Delta_k = \sum_{q\in \mathrm{Crit}_{k+1}(f)} i_q \circ \Big(\bigoplus_{p\in \mathrm{Crit}_f(q)} \Delta_k(q \to p) \Big)$$

all depends on the choice of index pairs?

Lemma 6.4. *Let M be an oriented manifold and $q \in \mathrm{Crit}(f)$. Furthermore, let (N_q, L_q) be a regular index pair. Then an orientation in $T_q^u M$ induces a generator of $K^{\mathrm{ind}(q)}(N_p, L_p)$*

Proof. First, a generator of $T_q^u M$ induces a generator of $K^{\mathrm{ind}(q)}(\overline{T_q^u M})$. Now we need to find a natural map $(N_p/L_p) \to \overline{T_q^u M}$. We do this in steps. First, we define:

$$s : B_q^u(\varepsilon) := \{x \in T_q^u M \mid \|x\| \le \varepsilon\} \to \overline{T_q^u M}$$
$$x \mapsto \begin{cases} \frac{x}{\|x\|-\varepsilon} & \text{if } \|x\| < \varepsilon \\ \infty & \text{else .} \end{cases}$$

This map is can be made into a homomorphism if we make it bijective, meaning we identify all fibers of ∞ giving us the map

$$\begin{aligned} \bar{s} : B_q^u(\varepsilon)/\partial B_q^u(\varepsilon) &\to \overline{T_q^u M} \\ x &\mapsto s(x) \end{aligned}$$

now this map is again a homeomorphisms. Now we need to get a map from $N_p/L_p \to B_q^u(\varepsilon)/\partial B_q^u(\varepsilon)$. We proceed as follows: First we get a diffrent index pair as follows: Let $\varphi : U \to T^u M$ be a morse chart and $D_u^s(\varepsilon)$ be a closed ball of radius ε living in $T_q^s M$. Then $\left(\varphi^{-1}(B_q^u(\varepsilon) \times B_q^s M)\ ,\ \varphi^{-1}(\partial B_q^u(\varepsilon) \times B_q^s M)\right)$ is a regular index pair for q, hence there is a flow induced map

$$\psi :\ (N_p/L_p) \to \left(\varphi^{-1}(B_q^u(\varepsilon) \times B_q^s M) \ /\ \varphi^{-1}(\partial B_q^u(\varepsilon) \times B_q^s M)\right)$$

Furthermore we have the homöomorphism induced from φ:

$$\left(\varphi^{-1}(B_q^u(\varepsilon) \times B_q^s M) \ /\ \varphi^{-1}(\partial B_q^u(\varepsilon) \times B_q^s M)\right) \to \left(B_q^u(\varepsilon) \times B_q^s M \ /\ \partial B_q^u(\varepsilon) \times B_q^s M\right)$$

and the contraction

$$\left(B_q^u(\varepsilon) \times B_q^s M \ /\ \partial B_q^u(\varepsilon) \times B_q^s M\right) \ \to\ B_q^u(\varepsilon)/B_q^s(\varepsilon),$$

which concludes the proof.

The independence of the choice of the chart comes from M being oriented and hence diffrent charts induce the same orientation. Furthermore, in the definition of the map between regular index pairs, we choose a T, but diffrent suitable T give homotopic maps!

show T and T' induced maps between regular index pairs are homotopic

$\square$

Theorem 6.5. *Given the filtration $N_{-1} \subseteq N_0 \subseteq \cdots \subseteq N_{m-1} \subseteq N_m = M$ from 10. The following diagram commutes:*

$$\begin{array}{ccc} C^k(M, \mathfrak{A}, \tilde{K}^l(pt)) & \xrightarrow{\partial^p} & C^{k+1}(M, \mathfrak{A}, \tilde{K}^l(pt)) \\ \downarrow & & \downarrow \\ \bigoplus_{q \in \mathrm{Crit}_k(f)} \tilde{K}^{k+l}(N_q, L_q) & \xrightarrow{\Delta_{k+l}} & \bigoplus_{p \in \mathrm{Crit}_{k+1}(f)} \tilde{K}^{k+l+1}(N_p, L_p) \\ \downarrow & & \downarrow \\ K^{k+l}(N_k, N_{k-1}) & \xrightarrow{\delta_{triple}} & K^{p+l+1}(N_{k+1}, N_k) \end{array}$$

Proof. The lower part should just be homological algebra. The top part is the interesting part! So we assume for the moment, that $q \in \mathrm{Crit}_{p+1}(f)$ and $p \in \mathrm{Crit}_p(f)$ are the only critical points in $f^{-1}([a,b])$, where $a := f^{-1}(p)$ and $b := f^{-1}(q)$. .

this can be archived by alternations of f

Now, we choose the index pairs wisely: First we define the notations:

$$M^t := \{x \in M | f(x) \le t\}\ ,\ M_t := \{x \in M | f(x) \ge t\}$$

and the constants:

$$c \in (a,b)\ ,\ \varepsilon > 0 \text{ small enough } \ ,\ T > 0 \text{ large enough}$$

Now we define the following sets:

$$\begin{aligned} N_q &:= \{x \in M_c | f(\varphi_{-T}(x)) \leq b + \varepsilon\} \\ L_q &:= \{x \in N_q | f(x) = c\} \\ N_p &:= \{x \in M^c | f(\varphi_T(x)) \geq a - \varepsilon\} \\ L_p &:= \{x \in N_p | f(\varphi_T(x)) = a - \varepsilon\} \end{aligned}$$

and with those the sets;

$$\begin{aligned} C &:= N_p \cup N_q \\ B &:= N_p \cup L_q \\ A &:= L_p \cup (L_q \overset{\circ}{-} N_p) \end{aligned}$$

With those we have the following list of facts:

1. (N_q, L_q) is a regular index pair for q.
2. (C, B) is an index pair for q.
3. (N_p, L_p) is a regular index pair for p.
4. (B, A) is an index pair for p

proof the list

We have a contraction

$$c : (N_q, L_q) \to (W(q \to) \cap M_c, W(q \to) \cap f^{-1}(c))$$

Furthermore we show that N_p is a tubular neighbourhood of $W(\to p) \cap M^c$ and hence after analysing L_p we get the isomorphisms to :

Muss noch gezeigt werden, dass das wirrklich deformiert!

$$(N_p, L_p) \cong (\underbrace{D^{k-1}}_{\dim W(p\to)} \times \underbrace{D^{m-k+1}}_{\dim W(\to p)}, \partial D^{k-1} \times D^{m-k+1})$$

Now by compactnes the space $N_p \cap (W(q \to) \cap f^{-1}(c))$ has finitly many connected components $V_1, \dots V_n$, and for each V_j there is a $x_j \in V_j \cap W(q \to p)$. Since N_p is a tubular neighbourhood we get the diffeomorphism

Hier fehlt auch noch ein Beweis

$$\psi_p : N_p \to \underbrace{D^{k-1}}_{\dim W(p\to)} \times \underbrace{D^{m-k+1}}_{\dim W(\to p)}$$

such that

1. $\psi(L_p) = \partial D^{k-1} \times D^{m-k+1}$,

2. $\psi_p(N_p \cap W(\to p) = \{0\} \times D^{m-k+1}$,
3. $\psi(V_j) = D^{k-1} \times \{\theta_j\}$ where $\theta_j \in \partial D^{m-k+1}$.

Using this map we get diffoemorphisms

$$\begin{aligned}\psi_j : V_j &\to D^{k-1}\\ x &\mapsto \pi_1 \circ \psi_p(x) \text{ where } \pi_1 \text{ is the projection onto the first factor.}\end{aligned}$$

This map resticts to a diffeomorphism from $\partial V_j = V_j \cap L_p$ to ∂D^{k-1} Hence, we get the continouos maps. das durchdenken, ob das klar ist

$$(N_p, L_p) \xrightarrow{\pi_1 \circ \psi_p} (D^{k-1}, \partial D^{k-1}) \xleftarrow{\psi_j} (V_j, \partial V_j)$$

and hence

$$K^{k-1}(N_p, L_p) \xleftarrow{(\pi_1 \circ \psi_p)^*} K^{k-1}(D^{k-1}, \partial D^{k-1}) \xrightarrow{(\psi_j)^*} K^{k-1}(V_j, \partial V_j)$$

An orientation of $T_p^u M$ induces a generator in the middle from the left (via $((\pi_1 \circ \psi_p)^*)^{-1}$). This generator can be mapped via $((\psi_j)^*)^{-1}$ to $K^{k-1}(V_j, \partial V_j)$. Another way to get a generator on the right side is from an orientation of $T_q^u M$ as follows: We notice, that

$$T_{x_j} V_j = (-\mathrm{grad}(f)_{x_j})^{\perp} \cap T_{x_j} W(q \to)$$

Hence, a orthogonal oriented basis $(-\mathrm{grad}(f)_{x_j}\ ,\ b_q^u)$ gives a basis (b_q^u) of $T_{x_j} V_j$, and hence a generator of $K^{k-1}()$

different idea

Using parallel transport, we get a basis of $T_{x_j} V_j$ in two ways: First from a basis of $T_p^u M$ via parrallel transport along the flow line of $\varphi_t(x_j)$ and secondly because

$$T_{x_j} V_j = (-\mathrm{grad}(f)_{x_j})^{\perp} \cap T_{x_j} W(q \to) he$$

Hence, a orthogonal oriented basis $(-\mathrm{grad}(f)_{x_j}\ ,\ b_q^u)$ gives a basis (b_q^u) of $T_{x_j} V_j$. We define $n_j = \pm 1$ depending on wheather those agree ore not. (This agrees with the sign in the Morse boundary operator associated to the orbit containing x_j) Now define $S(q \to) := W(q \to) \cap f^{-1}(c) = W^u(q) \cup L_q$. With this we have the commuting diagram

$$\begin{tikzcd}
\bigoplus_j K^k(S(q \to), \overline{S(q \to) \setminus V_j}) \arrow[d] & \\
K^k(S(q \to), \overline{S(q \to) \setminus \bigsqcup_j V_j}) \arrow[r, "\delta_{triple}"] \arrow[d] & K^{k+1}(W(q \to) \cap N_q, S(q \to)) \arrow[d] \\
K^k(B, A) \arrow[r, "\delta_{triple}"] & K^{k+1}(C, B)
\end{tikzcd}$$

Now inspect the map

$$\delta_j : K^k(S(q \to), \overline{S(q \to) \setminus V_j}) \to K^{k+1}(W(q \to) \cap N_q, S(q \to))$$

This map is induced by the follwing maps:

$$\begin{tikzcd}
(W(q \to) \cap N_q) \cup C_1 S(q \to) \arrow[r] \arrow[d, hook] & W(q \to) \cap N_q) \big/ (S(q \arrow[d] \\
\Big((W(q \to) \cap N_q) \cup C_1 S(q \to)\Big) \cup C_2(W(q \to) \cap N_q) \arrow[r] \arrow[d, hook] & S \wedge (S(q \to)) \arrow[d] \\
\Bigg(\Big((W(q \to) \cap N_q) \cup C_1 S(q \to)\Big) \cup C_2(W(q \to) \cap N_q)\Bigg) \cup C_3(C_1 \overline{S(q \to) \setminus V_j}) \arrow[r] & S \wedge (S(q \to)) \big/ (S \wedge \overline{S(q -}
\end{tikzcd}$$

Let β be a generator of the Group

$$K^{k+1}(W(q \to) \cup N_q, W(q \to) \cup L_q)$$

Since The $V_j \subset W(q \to) \cup N_q$ are contractible, without restrictions we have that $\beta\Big|_{V_j} = V_j \times \mathbb{C}^{k+1}$ with the basis (b_j, v_j) such that b_j is an orientation of $T_{x_j} V_j$ and (b_j, v_j) is an oriented basis of $T_{x_j}(W(q \to) \cup N_q)$. Now how does

$$\delta^j_{pair}(\beta) \in K^k$$

Notice how our map is from a $k+1-$ sphere $(W(q \to) \cap N_q) \big/ S(q \to) \cong S^{k+1}$ to a $k+1-$Sphere $S \wedge (V_j \big/ \partial V_j) \cong S^{k+1}$. Hence, we want to figure out, if it is orientation reversing or not. For this we need to really focus on the sphere-isomorphisms. And ask, how the map looks if the diagramm commutes:

$$\begin{tikzcd}
(W(q \to) \cap N_q) \cup C_1 S(q \to) \arrow[r] \arrow[d, hook] & S^{k+1} \arrow[d] \\
\Bigg(\Big((W(q \to) \cap N_q) \cup C_1 S(q \to)\Big) \cup C_2(W(q \to) \cap N_q)\Bigg) \cup C_3(C_1 \overline{S(q \to) \setminus V_j}) \arrow[r] & S^{k+1}
\end{tikzcd}$$

Hence we need to specify the horizontal (and vertical) maps. So we start with the top one:

$$\big(W(q \to) \cap N_q \cup C_1 S(q \to)\big) \to S^{k+1}$$

First we have a map $E^u : T^u_q M \to W(q \to)$ from theorem 1.2 which is a diffeomorphism onto its range. Hence, using its inverse we have a homeopmorphism $W(q \to) \to T^u_q M \cong$

$\mathbb{R}^{k+1}$ where the last congruence is given by an orientation. We can restrict this homeomorphism to $W(q \to) \cap N_q = W(q \to) \cap M_c$ to get a mapping onto a closed subspace of R^{k+1} and after a rescaling this is a closed Disk in R^{k+1}. This follows from $f(E^u(x))$ strictly decreasing if $\|x\|$ decreses, which again follows from the definiton of E^u, which comes from „flowing along the flow downwards "Hence the condition in the image of f being greater or equal to c translates to a condition in the domain of having norm less or equal to someting. w

In fact this is a closed Disc, since the map $E^u(x) = E^u_n(x) = \varphi^n \circ \chi^{-1} \circ \tilde{h}^n(x)$ for a suitible n. (χ is a chart in this context and we can assume that its image is a disc of radius 1) Careful, we swiched back to our discrete setting defining $\varphi := \varphi_1$. Compare the proof of theorem 1.2 to the definitinos. Hence, the inverse is given by $\tilde{h}^{-n}(x) \circ \chi \circ \varphi^{-n}$. Since there is no critical point above c we can assume, that all points in $W(q \to)$ flow below c in finite time. Let $U \subset W(q \to) \cap M_c$ be the domain of χ and ∂U be its boundary. Now for each $x \in D^{k+1}$ we have $\chi^{-1}(\frac{x}{\|x\|}) \in \partial U$ and for each x we define t_x such that $\varphi_{t_x}(\chi^{-1}(\frac{x}{\|x\|})) \in f^{-1}(x)$. With this we define the rescaling

$$\begin{aligned} \chi\big(\varphi^{-n}(W(q \to) \cap M_c)\big) &\to \overline{D^{k+1}} \\ x &\mapsto t_x x \end{aligned}$$

Define t_x such that $f(\varphi_{t_x}(x)) = c$. Now let n be the smallest natural number such $\varphi^{-n}(W(q \to) \cap M_c)$ is contained in the domain of χ. Now $\varphi^{-n}(W(q \to) \cup f^{-1}(c))$

Lets restrict the chart $\chi^0 =: \chi : T^u M \supset U \to W(q \to)$ such that $U = \overline{D^{k+1}}$ is a closed Disc. Now for each

$x \in U \setminus q$ there is a number $t_{0,x} \geq 0$ such that $\varphi_{t_{0,x}}(\chi(x)) \in \chi(\partial U)$. Furthermore for each $x \in U$, there is a number$t_{1,x}$ such that $\varphi_{t_{1,x}}(\chi(x)) \in f^{-1}(c)$ Those numbers all smoothly depend on x.

Now define the map

$$\begin{aligned} \Psi : \overline{D^{k+1}} &\to W(q \to) \cap M_c \\ x &\mapsto \varphi_{(t_{1,x} - t_{0,x})}\chi((x)) \end{aligned}$$

This map is a homeomorphism, since χ is one, and the flow as a map $\varphi : \mathbb{R} \times M \to M$ is continouos. Furthermore there is an inverse given as follows: For each $x \in W(q \to) \cap M_c$ there is a real number $s_{0,x}$ such that $\varphi_{s_{0,x}}(x) \in \chi(\partial U)$, and a real number $s_{1,x}$ such that $\varphi_{s_{1,x}}(x) \in f^{-1}(c)$. Then the map

$$\begin{aligned} \Phi : W(q \to) \cap M_c &\to \overline{D^{k+1}} \\ x &\mapsto \chi^{-1}(\varphi_{s_{0,x} - s_{1,x}})(x) \end{aligned}$$

is the inverse of χ: To see this notice that for $y = \chi(x)$ we have $t_{0,x} = s_{0,y}$, and $t_{1,x} = s_{1,y}$ by definition.

Now call $A := \varphi_{(t_{1,x} - t_{0,x})}\chi(x) = \varphi_{(s_{1,\chi(x)} - s_{0,\chi(y)})}\chi(x)$, then we have:

- $s_{1,A} = s_{0,\chi(x)}$ and,

- $s_{0,A} = -s_{1,\chi(x)} + 2s_{0,\chi(x)}$.

This is because:

$$\varphi_{s_{0,\chi(x)}} \circ \varphi_{(t_{1,x}-t_{0,x})}\chi(x) = \varphi_{s_{0,\chi(x)}} \circ \varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x) = \varphi_{(s_{1,\chi(x)})}\chi((x)) \in f^{-1}(c)\,,$$

and

$$\varphi_{-s_{1,\chi(x)}+2s_{0,\chi(x)}} \circ \varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x) = \varphi_{(s_{0,\chi(x)})}\chi((x)) \in f(\partial U)\,.$$

Hence

$$\begin{aligned}
\Phi \circ \Psi(x) &= \Phi\Big(\varphi_{(t_{1,x}-t_{0,x})}\chi((x))\Big) \\
&= \Phi\Big(\underbrace{\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x)}_{=A}\Big) \\
&= \chi^{-1} \circ \varphi_{s_{0,A}-s_{1,A}}(\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x)) \\
&= \chi^{-1} \circ \underbrace{\varphi_{-s_{1,\chi(x)}+2s_{0,\chi(x)}-s_{0,\chi(x)}}(\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}}_{=\mathrm{id}}\chi(x)) \\
&= x
\end{aligned}$$

And the other way round we have the relations for $B := \chi^{-1}(\varphi_{s_{0,x}-s_{1,x}}(x))$:

$$\begin{aligned}
t_{0,B} &= s_{1,x} \\
t_{1,B} &= -s_{0,x} + 2s_{1,x}\,,
\end{aligned}$$

since we can calculate:

$$\begin{aligned}
\varphi_{s_{1,x}}(\chi(B)) &= \varphi_{s_{1,x}}(\varphi_{s_{0,x}-s_{1,x}}) = \varphi_{s_{0,x}}(x) \in f(\partial U) \\
\varphi_{-s_{0,x}+2s_{1,x}}(\chi(B)) &= \varphi_{-s_{0,x}+2s_{1,x}}(\varphi_{s_{0,x}-s_{1,x}}) = \varphi_{s_{1,x}}(x) \in f^{-1}(c)
\end{aligned}$$

$$\begin{aligned}
\Psi \circ \Phi(x) &= \Psi\big(\chi^{-1}(\varphi_{s_{0,x}-s_{1,x}})(x)\big) \\
&= \varphi_{t_{1,B}-t_{0,B}}\chi \circ \chi^{-1}(\varphi_{s_{0,x}-s_{1,x}})(x) \\
&= \varphi_{-s_{0,x}+2s_{1,x}-s_{1,x}}\varphi_{s_{0,x}-s_{1,x}})(x) \\
&= x
\end{aligned}$$

So now we have a diffeomorpism $\Psi : W(q \rightarrow) \cap N_q \rightarrow \overline{D^{k+1}}$. We extend this to a homeomorphism

$$\begin{aligned}
(W(q \rightarrow) \cap N_q) \cup C_1 S(q \rightarrow) &\rightarrow \overline{D^{k+1}} \cup C(\partial D^{k+1}) \subseteq (\mathbb{R}^{k+1} \times I)\big/\mathbb{R}^{k+1} \times \{1\} \\
(x,t) &\mapsto (\Phi(x), t)
\end{aligned}$$

This map is well defined, since $S(q \to) = W(q \to) \cap f^{-1}(c)$, and if $x \in S(q \to)$, then $s_{1,x} = 0$ and hence:

$$\Phi(x) = \chi^{-1}(\underbrace{\varphi_{s_{0,x}}(x)}_{\in\chi(\partial U)}) \in \partial(U) = \partial(\overline{D^{k+1}})$$

Now we want to construct the map

$$\left(\left((W(q \to) \cap N_q) \cup C_1 S(q \to)\right) \cup C_2(W(q \to) \cap N_q)\right) \cup C_3(C_1\overline{S(q \to) \setminus V_j}) \to S^{k+1} \quad (17)$$

First we contract all unnecesarry parts:

$$\begin{aligned}
&\left(\left((W(q \to) \cap N_q) \cup C_1 S(q \to)\right) \cup C_2(W(q \to) \cap N_q)\right) \cup C_3(C_1\overline{S(q \to) \setminus V_j}) \\
&\to \left(\left((W(q \to) \cap N_q) \cup C_1 S(q \to)\right) \big/ (W(q \to) \cap N_q)\right) \big/ (C_1\overline{S(q \to) \setminus V_j}) \\
&= \left((W(q \to) \cap N_q) \cup C_1 S(q \to)\right) \big/ \left((W(q \to) \cap N_q) \cup (C_1\overline{S(q \to) \setminus V_j})\right) \\
&= \left(C_1 S(q \to)\right) \big/ \left(S(q \to) \cup (C_1\overline{S(q \to) \setminus V_j})\right) \\
&= \left(C_1 V_j\right) \big/ \left(V_j \cap C_1(\partial V_j)\right)
\end{aligned}$$

To summarize, we get a continouos map Θ:

$$\begin{aligned}
&\left(\left((W(q \to) \cap N_q) \cup C_1 S(q \to)\right) \cup C_2(W(q \to) \cap N_q)\right) \cup C_3(C_1\overline{S(q \to) \setminus V_j}) \\
&\to \left(C_1 V_j\right) \big/ \left(V_j \cap C_1(\partial V_j)\right)
\end{aligned}$$

Careful! This map does not neet to be a homotopy equivalence and hence does not give us a continuous inverse up to homotopie. It is a quotiet map induced by collapsing contractible subsets, and hence induces an isomorphism in the K-groups. via ψ we get a map $V_j \to D^k$, which lets us build a map

$$\begin{aligned}
\tilde{\psi} : \left(C_1 V_j\right) \big/ \left(V_j \cap C_1(\partial V_j)\right) &\to D^k \times I \big/ \left(\partial D^k \times I \cup D^k \times \{0,1\}\right) \\
(x,t) &\mapsto \psi(x,t)
\end{aligned}$$

Hence we get the map

Inhalt...

□

Theorem 6.6. *Given the filtration $N_{-1} \subseteq N_0 \subseteq \cdots \subseteq N_{m-1} \subseteq N_m = M$ from 10. The following diagram commutes:*

$$\begin{array}{ccc}
C^k(M,\mathfrak{A},\tilde{K}^l(pt)) & \xrightarrow{\partial^p} & C^{k+1}(M,\mathfrak{A},\tilde{K}^l(pt)) \\
\downarrow & & \downarrow \\
\bigoplus_{p\in\mathrm{Crit}_k(f)} \tilde{K}^{k+l}(N_p,L_p) & \xrightarrow{\Delta_{k+l}} & \bigoplus_{q\in\mathrm{Crit}_{k+1}(f)} \tilde{K}^{k+l+1}(N_q,L_q) \\
\downarrow & & \downarrow \\
K^{k+l}(N_k,N_{k-1}) & \xrightarrow{\delta_{triple}} & K^{p+l+1}(N_{k+1},N_k)
\end{array}$$

Proof. So we assume for the moment, that $q \in \mathrm{Crit}_{k+1}(f)$ and $p \in \mathrm{Crit}_k(f)$ are the only critical points in $f^{-1}([a,b])$, where $a := f^{-1}(p)$ and $b := f^{-1}(q)$. Now, we choose the index pairs wisely: First we define the notations:

$$M^t := \{x \in M | f(x) \le t\}\ ,\ M_t := \{x \in M | f(x) \ge t\}$$

and the constants:

$$c \in (a,b)\ ,\ \varepsilon > 0 \text{ small enough }\ ,\ T > 0 \text{ large enough}$$

Now we define the following sets:

$$\begin{aligned}
N_q &:= \{x \in M_c | f(\varphi_{-T}(x)) \le b + \varepsilon\} \\
L_q &:= \{x \in N_q | f(x) = c\} \\
N_p &:= \{x \in M^c | f(\varphi_T(x)) \ge a - \varepsilon\} \\
L_p &:= \{x \in N_p | f(\varphi_T(x)) = a - \varepsilon\}
\end{aligned}$$

and with those the sets;

$$\begin{aligned}
C &:= N_p \cup N_q \\
B &:= N_p \cup L_q \\
A &:= L_p \cup (L_q \overset{\circ}{-} N_p)
\end{aligned}$$

With those we have the following list of facts:

1. (N_q, L_q) is a regular index pair for q.
2. (C, B) is an index pair for q.
3. (N_p, L_p) is a regular index pair for p.
4. (B, A) is an index pair for p

Since N_p is a tubular neighbourhood of the contractible $W(\to p) \cap M^c$ we get the diffeomorphism:

$$\psi_p : N_p \to \underbrace{\overline{D^{m-k}}}_{\dim W(\to p)} \times \overline{D^k} \subseteq T_p^u M$$

The image here is a subspace of the total space of the trivialized normal bundle. This map satisfies that

1. $\psi(L_p) = \overline{D^{m-k}} \partial D^k$,
2. $\psi_p(N_p \cap W(\to p) = \{0\} \times \overline{D^{m-k}}$,
3. $\psi(V_j) = \{\theta_j\} \times \overline{D^k}$ where $\theta_j \in \partial D^{m-k}$.

Using this map we get diffoemorphisms

$$\begin{aligned} \psi_j : V_j &\to \overline{D^k} \\ x &\mapsto \pi_1 \circ \psi_p(x) \text{ where } \pi_1 \text{ is the projection onto the first factor.} \end{aligned}$$

This map resticts to a diffeomorphism from $\partial V_j = V_j \cap L_p$ to ∂D^{k-1} Hence, an orientation of the unstable tangend space of p induces a map: $V_j \to \overline{D}^k$

das durchdenken, ob das klar ist

Now we want to figure out how the map between the spheres on the right looks if the diagramm commutes:

$$\begin{array}{ccc} (W(q\to) \cap N_q) \cup C_1 S(q\to) & \xrightarrow{\quad\Phi\quad} & S^{k+1} \\ \big\downarrow & & \big\downarrow \\ \Big(\big((W(q\to) \cap N_q) \cup C_1 S(q\to)\big) \cup C_2(W(q\to) \cap N_q)\Big) \cup C_3(C_1 \overline{S(q\to) \setminus V_j}) & \xrightarrow{\Psi} & S^{k+1} \end{array}$$

Hence we need to specify the horizontal (and vertical) maps. So we start with the top one(or rqther its inverse): From the proof of theorem 1.2 we recycle a view maps. Notice how the orientation of $T_q^u M$ gives us a way to identify $T_q^u M$ with $\mathbb{R}^{k+1}$ (the coordinate map) and hence we can restrict the chart $\chi^0 =: \chi : T_q^u M \supset U \to W(q\to)$ such that $U = \overline{D^{k+1}}$ is a closed Disc in $\mathbb{R}^n$.. Now for each $x \in U \setminus q$ there is a number $t_{0,x} \geq 0$ such that $\varphi_{t_{0,x}}(\chi(x)) \in \chi(\partial U)$. Furthermore for each $x \in U$, there is a number$t_{1,x}$ such that $\varphi_{t_{1,x}}(\chi(x)) \in f^{-1}(c)$ Those numbers all smoothly depend on x.

Why is the left an inclusion?

careful, the map χ is just a chart and hence not natural in any sence. But since $W(q\to)$ is orientable, and hence we can make χ and oriented chart.

Now define the map

$$\begin{aligned} \tilde{\Phi} : \overline{D^{k+1}} &\to W(q\to) \cap M_c \\ x &\mapsto \varphi_{(t_{1,x} - t_{0,x})} \chi((x)) . \end{aligned}$$

This map is a homeomorphism, since χ is one, and the flow as a map $\varphi : \mathbb{R} \times M \to M$ is continouos. Furthermore there is an inverse given as follows: For each $x \in W(q\to) \cap M_c$ there is a real number $s_{0,x}$ such that $\varphi_{s_{0,x}}(x) \in \chi(\partial U)$, and a real number $s_{1,x}$ such that $\varphi_{s_{1,x}}(x) \in f^{-1}(c)$. Then the map

$$\begin{aligned} \Phi : W(q\to) \cap M_c &\to \overline{D^{k+1}} \\ x &\mapsto \chi^{-1}(\varphi_{s_{0,x} - s_{1,x}})(x) \end{aligned}$$

is the inverse of $\tilde{\Phi}$: To see this notice that for $y = \chi(x)$ we have $t_{0,x} = s_{0,y}$, and $t_{1,x} = s_{1,y}$ by definition.

Now call $A := \varphi_{(t_{1,x} - t_{0,x})} \chi(x) = \varphi_{(s_{1,\chi(x)} - s_{0,\chi(y)})} \chi(x)$, then we have:

- $s_{1,A} = s_{0,\chi(x)}$ and,
- $s_{0,A} = -s_{1,\chi(x)} + 2s_{0,\chi(x)}$.

This is because:

$$\varphi_{s_{0,\chi(x)}} \circ \varphi_{(t_{1,x}-t_{0,x})}\chi(x) = \varphi_{s_{0,\chi(x)}} \circ \varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x) = \varphi_{(s_{1,\chi(x)})}\chi((x)) \in f^{-1}(c)\,,$$

and

$$\varphi_{-s_{1,\chi(x)}+2s_{0,\chi(x)}} \circ \varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x) = \varphi_{(s_{0,\chi(x)})}\chi((x)) \in f(\partial U)\,.$$

Hence

$$\begin{aligned}
\Phi \circ \tilde{\Phi}(x) &= \Phi\Big(\varphi_{(t_{1,x}-t_{0,x})}\chi((x))\Big) \\
&= \Phi\Big(\underbrace{\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x)}_{=A}\Big) \\
&= \chi^{-1} \circ \varphi_{s_{0,A}-s_{1,A}}(\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x)) \\
&= \chi^{-1} \circ \underbrace{\varphi_{-s_{1,\chi(x)}+2s_{0,\chi(x)}-s_{0,\chi(x)}}(\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}}_{=\mathrm{id}}\chi(x)) \\
&= x
\end{aligned}$$

And the other way round we have the relations for $B := \chi^{-1}(\varphi_{s_{0,x}-s_{1,x}}(x))$:

$$\begin{aligned}
t_{0,B} &= s_{1,x} \\
t_{1,B} &= -s_{0,x} + 2s_{1,x}\,,
\end{aligned}$$

since we can calculate:

$$\begin{aligned}
\varphi_{s_{1,x}}(\chi(B)) &= \varphi_{s_{1,x}}(\varphi_{s_{0,x}-s_{1,x}}) = \varphi_{s_{0,x}}(x) \in f(\partial U) \\
\varphi_{-s_{0,x}+2s_{1,x}}(\chi(B)) &= \varphi_{-s_{0,x}+2s_{1,x}}(\varphi_{s_{0,x}-s_{1,x}}) = \varphi_{s_{1,x}}(x) \in f^{-1}(c)
\end{aligned}$$

$$\begin{aligned}
\tilde{\Phi} \circ \Phi(x) &= \tilde{\Phi}\big(\chi^{-1}(\varphi_{s_{0,x}-s_{1,x}})(x)\big) \\
&= \varphi_{t_{1,B}-t_{0,B}}\chi \circ \chi^{-1}(\varphi_{s_{0,x}-s_{1,x}})(x) \\
&= \varphi_{-s_{0,x}+2s_{1,x}-s_{1,x}}\varphi_{s_{0,x}-s_{1,x}})(x) \\
&= x
\end{aligned}$$

So now we have a diffeomorpism $\Phi : W(q \to) \cap N_q \to \overline{D^{k+1}}$. We extend this to a continuous map

$$(W(q \to) \cap N_q) \cup C_1 S(q \to) \to \overline{D^{k+1}}/\partial\overline{D^{k+1}}$$

by first contracting $C_1S(q\to)$: So now we have a diffeomorpism $\Phi : W(q\to)\cap N_q \to \overline{D^{k+1}}$. We extend this to a continuous map

$$(W(q\to)\cap N_q)\cup C_1S(q\to)\to \overline{D^{k+1}}\cup C_1\partial\overline{D^{k+1}}$$

This is neither a homeopmorphism nor a homotopie equivalence. But it is continouos and since $C_1S(q\to)$ was contractible, it induces an isomorphims in the K-groups. Now since Φ maps $S(q\to)$ to ∂D^{k+1} we can compose the contracting with Φ to get a continouos map that induces an isomorphism in the K-group, and which we will also call Φ:

$$\Phi : (W(q\to)\cap N_q)\cup C_1S(q\to)\to \overline{D^{k+1}}/\partial\overline{D^{k+1}}\,.$$

Now we want to construct the map

$$\Psi : \left(\Big((W(q\to)\cap N_q)\cup C_1S(q\to)\Big)\cup C_2(W(q\to)\cap N_q)\right)\cup C_3(C_1\overline{S(q\to)\setminus V_j})\to S^{k+1} \tag{18}$$

First we contract all unnecesarry parts:

$$\begin{aligned}
&\left(\Big((W(q\to)\cap N_q)\cup C_1S(q\to)\Big)\cup C_2(W(q\to)\cap N_q)\right)\cup C_3(C_1\overline{S(q\to)\setminus V_j})\\
&\to \left(\Big((W(q\to)\cap N_q)\cup C_1S(q\to)\Big)\Big/(W(q\to)\cap N_q)\right)\Big/(C_1\overline{S(q\to)\setminus V_j})\\
&= \Big((W(q\to)\cap N_q)\cup C_1S(q\to)\Big)\Big/\Big((W(q\to)\cap N_q)\cup(C_1\overline{S(q\to)\setminus V_j})\Big)\\
&= \Big(C_1S(q\to)\Big)\Big/\Big(S(q\to)\cup(C_1\overline{S(q\to)\setminus V_j})\Big)\\
&= \big(C_1V_j\big)\big/\big(V_j\cap C_1(\partial V_j)\big)
\end{aligned}$$

To summarize, we get a continouos map Θ:

$$\begin{aligned}
&\left(\Big((W(q\to)\cap N_q)\cup C_1S(q\to)\Big)\cup C_2(W(q\to)\cap N_q)\right)\cup C_3(C_1\overline{S(q\to)\setminus V_j})\\
&\to \big(C_1V_j\big)\big/\big(V_j\cap C_1(\partial V_j)\big)
\end{aligned}$$

Again this is continouos but not even a homotopie equivalence. However, it induces an isomorphism in the K-groups. via ψ we get a map $V_j\to\overline{D^k}$.

$$\begin{aligned}
\tilde{\psi} : \big(C_1V_j\big)\big/\big(V_j\cap C_1(\partial V_j)\big) &\to \overline{D^k}\times I\big/\big(\partial\overline{D^k}\times I\cup\overline{D^k}\times\{0,1\}\big)\\
(x,t)&\mapsto\psi(x,t)
\end{aligned}$$

Now by rescaling the last factor:

$$\tilde{r} : \overline{D^k}\times I\to\overline{D^{k+1}}$$

we get the homomorphism

$$r : \overline{D^k} \times I \big/ \big(\partial \overline{D^k} \times I \cup \overline{D^k} \times \{0,1\}\big) \to \overline{D^{k+1}} \big/ \partial \overline{D^{k+1}}$$
$$(x,t) \mapsto \overline{r(x,t)}\,.$$

To see the well definition, notice how for closed sets we have the equality $\partial(A \times B) = (\partial A \times B) \cup (A \times \partial B)$. In sum we get the map :

$$\Psi : \left(\Big((W(q \to) \cap N_q) \cup C_1 S(q \to) \Big) \cup C_2 (W(q \to) \cap N_q) \right) \cup C_3 (C_1 \overline{S(q \to) \smallsetminus V_j})$$
$$\to \overline{D^{k+1}} \big/ \partial \overline{D^{k+1}}$$

given by $\Psi : r \circ \tilde{\psi} \circ \Theta$ Now we want to ask, how the map

$$\Psi \circ i \circ \Phi^{-1} : \overline{D^{k+1}} \big/ \partial \overline{D^{k+1}} \to \overline{D^{k+1}} \big/ \partial \overline{D^{k+1}}$$

looks like. Our claim is, that this map is homotopic to the identity, if the orientation in $T_{x_j} V_j$ induced from the one in $T_q^u M$ and from $T_p^u M$ agree. To do this we start with the inclusion. (But we need the "Coordinates from Φ maby we can look at $\Phi(x,t)$ and compare it to $\Psi \circ i(x,t)$.

$$i : (W(q \to) \cap N_q) \cup C_1 S(q \to)$$
$$\to \left(\Big((W(q \to) \cap N_q) \cup C_1 S(q \to) \Big) \cup C_2 (W(q \to) \cap N_q) \right) \cup C_3 (C_1 \overline{S(q \to) \smallsetminus V_j})$$

So assume (t,x) lives in the domain. Then $(\Theta \circ i)(t,x) = \overline{(t,x)}$ where the equivalence is given by the collaps of everything but the interior of $C_1 V_j$. Now inspect $\psi(V_j)$. we can identify via a trivialitation of the normal bundle $\psi(V_j)$ with $T_p^u M$ and the orientation of the latter induces a map to D^{k-1}.

□

Corollary 6.7 (The Logic and To-Dos of my Proof)**.** We have the definitions above, of all sets. with those we first want to construct a map of triples

$$t : (A, B, C) \to (N_{p+1}, N_p, N_{p-1})$$

By naturality and definition we then have the commutative diagram:

$$\begin{array}{ccccc}
K^{-p+1}(N_{p+1}, N_p)) & \xrightarrow{t^*} & K^{-p+1}(A,B) & \Longrightarrow & K^{-p+1}(A,B) \\
{\scriptstyle \delta_{triple}} \uparrow & & {\scriptstyle \delta_{triple}} \uparrow & & {\scriptstyle s^*} \uparrow \\
K^{-p}(N_p, N_{p-1}) & \xrightarrow{t^*} & K^{-p}(B,C) & \Longrightarrow & K^{-p+1}(s \vee (B,C))
\end{array}$$

We now have to show that s^* is induced from a continuous map. Then we want to use the maps Φ and Ψ to induce a map $p \simeq \pm\mathrm{id}$ such that the diagram commutes up to homotopie:

$$\begin{tikzcd}
(A,B) \arrow[r, "\Phi"] \arrow[d, "s"] & \bigvee_{j=1}^{l} S^{k+1} \arrow[d, "\oplus_j \delta_j \mathrm{id}"] \\
S \vee (B,C) \arrow[r, "\Psi"] & S^{k+1}
\end{tikzcd}$$

where $\delta_j \in \{-1,1\}$(with +id we denote the identity and with $-id$ we denote a homeomorphism of degree -1, i.e. not homotopic to the identity.) and Φ is a homotopie equivalence. Now since $K^{-p+1}(\bigvee_{j=1}^{l} S^{p+1}) \cong \mathbb{Z}^l$ we can define the following maps. Let β_q be a generator of $K^{-p+1}(A,B)$ and β_p of $K^{-p+1}(S \vee (B,C))$. Then define the koordinate maps

$$\begin{aligned} q_q : K^{-p+1}(A,B) \to \mathbb{Z}; \quad & (\Phi^*)^{-1}(\beta_q) \mapsto 1 \\ q_p : K^{-p+1}(S \vee (B,C)) \to \mathbb{Z}^l; \quad & (\Psi^*)^{-1}(\beta_p) \mapsto \sum_j e_j \,. \end{aligned}$$

With those we get the diagram:

$$\begin{tikzcd}
K^{-p+1}(A,B) & K^{-p+1}(S^{k+1}) \arrow[l, "\Phi^*"'] \arrow[r, "q_q"] & \mathbb{Z} \\
K^{-p+1}(S \vee (B,C)) \arrow[u, "s^*"] & K^{-p+1}(\bigvee_{j=1}^{l} S^{k+1}) \arrow[l, "\Psi^*"] \arrow[u, "g"] \arrow[r, "q_p"] & \mathbb{Z}^l \arrow[u, "h"]
\end{tikzcd}$$

The map h is given by $h : \mathbb{Z}^l \to Z$; $e_i \mapsto \delta_i$ where $\delta_i \in \{-1,1\}$ This all concludes in the final calculation:

$$\begin{aligned} s^*(\beta_p) &= \Phi^* \circ g \circ (\Psi^*)^{-1}(\beta_p) \\ &= \Phi^* \circ q_q^{-1} \circ h \circ q_p \circ (\Psi^*)^{-1}(\beta_p) \\ &= \Phi^* \circ q_q^{-1} \circ h(\sum_j e_j) \\ &= \Phi^* \circ q_q^{-1}(\sum_j \delta_j) \\ &= \sum_j \delta_j \beta_q \end{aligned}$$

Now the hope is that δ_j is the sign that I would get from inspecting the morse boundary operator along a certain flow line. The todo's are:

1. The map t.
2. The map Φ.
3. The map Ψ.
4. Is the map s^* induced?

5. The map h corresponds to a proceedure similar to the Morse boundary.

Once we have done all the above we want to connect the considerations with the boundary operator. To do this we do the proveedure to all pairs $(q, p) \in \mathrm{Crit}(f)_{k+1} \times \mathrm{Crit}(f)_k$. For this we enrich the notaion and call the triple coresponding to such a pair $(A^{q,p}, B^{q,p}, C^{q,p})$. All the maps and spaces are enriched in that way, by adding the pair (q, p) as a superscrit. For T big enough we assume that all $A^{q,p}$ are pairwise disjoint. Then we want a homomorphism (that is a homotopic eqiuvalence)

do wee needc continuity or rather something weaker?

$$\Omega : \bigsqcup_{(q,p)\in\mathrm{Crit}_{k+1}\times\mathrm{Crit}_k} (A^{q,p}, B^{q,p}, C^{q,p}) \to (N_{p+1}, N_p, N_{p-1})$$

$$x \mapsto t^{q,p}(x) \text{ for } x \in A^{q,p}$$

Now, together with the isomorphism

$$\bigoplus_{\mathrm{Crit}_{k+1}} \mathbb{Z} \to K^p(N_{-p+1}, N_p) \ , \ q \mapsto \beta_q \text{ which is a generator of } K^{-p+1}(A^{q,p}, B^{q,p})$$

Fuck das funktioniert alles nicht!