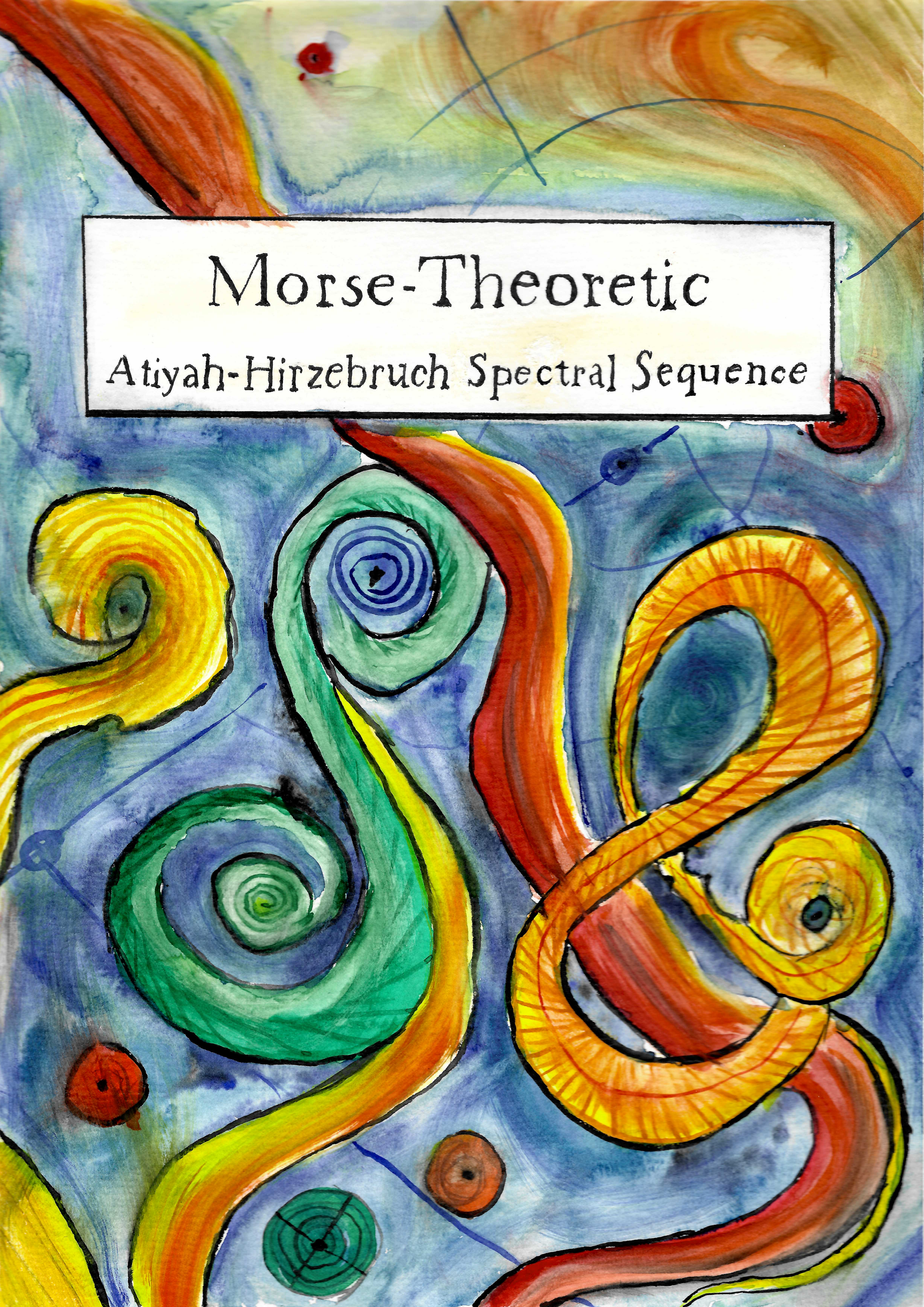
Morse-Theoretic

Atiyah-Hirzebruch Spectral Sequence



Contents

1 Differentiable structures on topological manifolds

Definition 1.1 (Topological Manifold)**.** A second countable Hausdorffspace M is called **topological manifold** of dimension $m \in \mathbb{N}$, if it is locally homeomorphic to $\mathbb{R}^m$. To be precise, if for all $p \in M$ there exists an open neighborhood $U \subseteq M$ of p, an open set $V \subseteq \mathbb{R}^m$ and a map $\varphi : W \to V$ that is a homeomorphism. We call the map $\varphi : U \to V$ a **chart around** p **on** M and φ^{-1} a **local coordinate system around** p **on** M.

Definition 1.2 (Differentiable Manifold)**.** Let M be a topological manifold of dimension M.

1. A **differentiable atlas of class** $r \in \mathbb{N} \cup \{\infty\}$ is a family of charts $\mathfrak{A} = (\varphi_i : U_i \to V_i)_{i \in I}$ such that
   a) $\bigcup_{i \in I} U_i = M$, meaning that (U_i) is an open covering of M.
   b) For every pair $(i,j) \in I^2$ the **transition function**:

   $$\begin{aligned} \varphi_{ij} : \varphi_j(U_i \cap U_j) &\to \varphi_i(U_i \cap U_j) \\ x &\mapsto \left(\varphi_i \circ \varphi_j^{-1}\right)(x) \end{aligned}$$

   is differentiable of class r.

   We call such an atlas a C^r-atlas.

2. Two C^r-atlases $\mathfrak{A}$ and $\mathfrak{B}$ are called **equivalent** if the family $\mathfrak{A} + \mathfrak{B} = (\varphi_i, \varphi_j)_{ij}$ is a C^r-atlas.

A **differentiable structure of class** r on M is an equivalence class c of C^r-atlases. For $r = \infty$ we call the pair (M, c) a smooth manifold.

Corollary 1.3. Every transition functions φ_{ij} $i, j \in I^2$ of a differentiable atlas $\mathfrak{A} = (\varphi_i)_{i \in I}$ is not just a homeomorphism but also a diffeomorphism due to $\varphi_{ji} = \varphi_{ij}^{-1}$

Definition 1.4. Let (M, c) be a differentiable manifold of class r and $U \subseteq M$ open. We call a continuos function

$$f : U \to \mathbb{R}$$

differentiable of class r, if for any one (and hence for all) $(\varphi_i)_{i \in I} = \mathfrak{A} \in c$ the compositions $f \circ \varphi_i^{-1}$ are differentiable of class r. For $r = \infty$ we define:

$$\mathcal{E}(U) = \{f \in U \to \mathbb{R} \text{ continouos} \mid f \text{ is differentiable of class } \infty\}.$$

Corollary 1.5. Let (M, c) be a smooth manifold of dimension m and $U \subseteq M$ be a open subset. With pointwise defined operations, the set $(\mathcal{E}(U), +, \cdot, \circ)$ becomes an $\mathbb{R}$–algebra. Furthermore, $\mathcal{E}$ becomes a sheaf of $\mathbb{R}$-algebras.

Proof. There is not really a need for a proof.However, it might help to work through the definition of a sheaf as a reminder. First, $\mathcal{E}$ is a presheaf, where the restriction in the domain of a function gives the needed restriction homomorphism:

$$\begin{aligned} \mathrm{res}_V^U : \mathcal{E}(U) &\to \mathcal{E}(V) \\ f &\mapsto f\Big|_V . \end{aligned}$$

The required properties of a presheaf are trivial. Furthermore, this gives a sheaf as the requirement of locality is trivial for functions and the property of gluing is also trivial for functions, since differentiability is a local property. □

Definition 1.6. If $p \in M$ is fix, $f \in \mathcal{E}(U)$ and $g \in \mathcal{E}(U')$ such that $p \in U \cap U'$ we say that f and g have the same **germ in** p, if there is another open neighborhood $W \subseteq U \cap U'$ of p such that $f\Big|_w = g\Big|_W$. This defines an equivalence relation $\sim_p$. An equivalence class s of local functions around p is called a **germ in** p. We write $s = f_p$, if $s = [f]$ with $f \in \mathcal{E}(U)$. We write

$$\mathcal{E}_p(M) = \left(\sum_{U \text{open}, p \in U} \mathcal{E}(U) \right) \Big/ \sim_p .$$

For the set of germs and call it the **stalk in** p. Here $\sum$ denotes the co-product (also called sum) in $\top$ and hence the disjoint union.

Corollary 1.7. For a smooth manifold (M, c) the set $\mathcal{E}_p(M)$ inherits an $\mathbb{R}$−algebra structure from the $\mathcal{E}(U)$. Furthermore, it carries a natural (evaluation-)homomorphism:

$$\begin{aligned} \mathrm{eval}_p : \mathcal{E}_p(M) &\to \mathbb{R} \\ f_p &\mapsto f(p) =: f_p(p) \end{aligned}$$

The stalks are also local rings with maximal ideal $\mathfrak{m}_p = \ker(\mathrm{eval}_p)$. Hence, the pair $(M, \mathcal{E})$ gives us a locally ringed space.

Proof. Here, we only need to prove the statement about the locality of the stalks. This follows from $f_p \in \mathcal{E}_p(M)$ being invertible if and only if $f(p) \neq 0$ which is the same as $f_p \notin \ker(\mathrm{eval}_p)$. □

Definition 1.8. Let (M, c) be a smooth manifold of dimension m and $p \in M$. We call an $\mathbb{R}$-linear map $\delta : \mathcal{E}_p(M) \to \mathbb{R}$ a **derivation**, if it satisfies the Leibnitz-rule:

$$\delta(f_p \cdot g_p) = \delta(f_p) g_p(p) + f_p(p) \delta(g_p) \quad \text{for all } f, g \in \mathcal{E}_p(M) .$$

We call $\mathrm{Der}_{\mathbb{R}}(\mathcal{E}_p(M), \mathbb{R})$ the set of derivations and give it a $\mathbb{R}$ vector space structure by pointwise operations. We define the **tangent space of** M **at** p to be the vector space

$$TM_p := \mathrm{Der}_{\mathbb{R}}(\mathcal{E}_p(M), \mathbb{R}) .$$

Corollary 1.9. Let (M, c) be a smooth manifold and $\varphi : U \to V$ be a chart around p with $x_0 = \varphi(p)$ $(\varphi \in \mathfrak{A} \in c)$. Then

$$\xi = \frac{\partial}{\partial x^j}\Big|_p : \mathcal{E}_p(M) \to \mathbb{R}$$
$$f_p \mapsto \xi(f_p) = \frac{\partial}{\partial x^j}\Big|_{x_0}(f \circ \varphi^{-1})$$

is well-defined and a tangent vector. In fact, the family

$$\left(\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^m}\Big|_p\right)$$

defines a basis of TM_p. Hence, the dimension of TM_p is m.

Definition 1.10. For a smooth manifold (M, c) the sum $\sum_p TM_p$ comes with a natural projection

$$\pi : TM \to M$$
$$\xi \mapsto p \text{ where } \xi \in TM_p$$

Furthermore, the **local vector fields** with respect to a chart $\varphi : U \to V$

$$\frac{\partial}{\partial x^j} : U \to \pi^{-1}(U)$$
$$p \mapsto \frac{\partial}{\partial x^j}\Big|_p$$

induce a local trivialization:

$$\pi^{-1}(U) \cong U \times \mathbb{R}^m .$$

We can induce a topology on TM such that all those trivializations are continuos. This then gives an atlas for TM such that we have a $2m$-dimensional manifold. To be precise, the atlas is given by the maps $\pi^{-1}(U_i) \to R^m \times R^m; x \mapsto (\pi(x), q_{\varphi \circ \pi(x)}(x))$ where q_p denotes the coordinate map corresponding to the basis $(\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^m}\big|_p)$ that depends on the chart φ_i. In fact, this yields a smooth manifold and a (smooth) vector bundle of dimension m. We call TM the **tangent bundle**.

Definition 1.11 (The Derivative)**.** Let (M, c) and (M', c') be smooth manifolds (from now on we suppress the differentiable structure in our notation). We call a continuous function $f : M \to M'$ **smooth**, if for all $\varphi \in \mathfrak{A} \in c$ and $\varphi' \in \mathfrak{A}' \in c'$ the maps

$$\varphi' \circ f \circ \varphi^{-1} : V \to V'$$

are smooth. A given smooth function induces a smooth function between the Tangent bundles as follows:

$$Df : TM \to TM' \ , Df_p(\xi)(g_p) = \xi((g \circ f)_p)$$

Here, $\xi \in T_pM$, $g_p \in E_p(M')$.

Corollary 1.12. Let $f : M \to \mathbb{R}$ be smooth and $\varphi : U \to V$ be a chart. Then we can interprete $\mathrm{d}f$ as a one-form. To be prezice asume q to be the coordinate funktion $T\mathbb{R} \to \mathbb{R}$ from the basis induced by the identity as a chart. $v \in \Gamma TM$ we have

$$q \circ \mathrm{d}f(v) = v \cdot f$$

Here on the righd side, f denotes a map $p \mapsto f_p$ such that $v(f)(p) := v_p(f_p)$ is well defined. We keep this notations so vector fields can take funktions as an input.

Proof. We proof this by showing it for the section $\frac{\partial}{\partial x^i}$ and thereby for any, since those sections form a basis of the space of sections as a $C^\infty(M, \mathbb{R})$ vectorspace. Now let $g_p \in \mathcal{E}_p(\mathbb{R})$ and $\varphi(p) = x_0$. Then

$$\mathrm{d}f(\frac{\partial}{\partial x^i}\Big|_p)(g_p) = \frac{\partial}{\partial x^i}\Big|_p(g \circ f)_p = \frac{\partial}{\partial x^i}\Big|_{x_0}(g \circ f \circ \varphi^{-1}) = \frac{\partial}{\partial x}\Big|_p(g_p) \cdot \frac{\partial}{\partial x^i}\Big|_p(f_p)$$

Hence, $\mathrm{d}f(v) = v(f)\frac{\partial}{\partial x}$ letting us conclude the statement. □

Definition 1.13 (A Metric)**.** Let M be a smooth manifold. A section $g \in \Gamma(TM^* \otimes TM^*)$ into the tensor product of the dual Tangend bundle with itself is a **riemannian metric** if the following are satisfied for all $p \in M$:

1. g is **non degenerate**, meaning for a $v_p \in T_pM$ $g_p(v_p, w_p) = 0$ $\forall w_p \in T_pM$ then $v_p = 0$.
2. g is **symmetric**, meaning $g_p(v_p, w_p) = g_p(w_p, v_p)$ $\forall v_p, w_p \in T_pM$.
3. g is **positiv definite**, meaning that $g_p(v_p, v_p) \geq 0 \forall v_p$ and vanishes only for $v_p = 0$.

We cal a touple (M, g) a **riemannian manifold**.

Definition 1.14. Let M, N, L be smooth manifolds and $f : M \times N \to M' \times N'$. Let $X \in \gamma(TM)$ and $Y \in \Gamma(TN)$. We define $\tilde{X} \in \Gamma(T(M \times N))$ as follows. Let $f_p \in \mathcal{E}_p(M, N)$

$$\tilde{X}(f_p) = X((f \circ i_M)_{p_M}) \quad \text{where } p_M = \pi_1(p)\,.$$

Definition 1.15 (The Gradient)**.** Assume that (M, g) is a smooth manifold together with a riemannian metric. Let $f : M \to \mathbb{R}$ be a smooth function. We define its **gradient** to be a vector field $\mathrm{grad}(f) \in \Gamma(TM)$ such that for any vecor field V:

$$q \circ \mathrm{d}f(V) = g(\mathrm{grad}(f), V)\,.$$

Here q denotes the canonical coordinate funtion $TR \to \mathbb{R}$.

Definition 1.16. Let (M, g) be a riemannian manifold and $\varphi : U \to V$ be a chart. We define the smooth functions $g_{ij} : U \to \mathbb{R}$ to be :

$$g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\,.$$

Then if two vectorfields v, w over U are given with $v = \sum v^i \frac{\partial}{\partial x^i}$ and $w = \sum_i w^i \frac{\partial}{\partial x^i}$ we can calculate the metric locally:

$$g(v,w) = \sum_{ij} g_{ij} v^i w^j \; : U \to \mathbb{R}$$

Furthermore, we can invert the matricies $(g_{ij}(p))_{ij}$ for all p (which is a smooth procedure which can be seen by the construction via cramers rule) to get smooth functinos

$$\begin{aligned} g^{ij} : U &\to \mathbb{R} \\ p &\mapsto g^{ij}(p) = \left((g_{kl}(p))_{kl}^{-1}\right)_{ij} \end{aligned}$$

Lemma 1.17. *Let $\varphi : U \to V$ be a chart. Then the gradient has the local form:*

$$\operatorname{grad}(f) = \sum_{i,j} g^{ij} \left(\frac{\partial}{\partial x^i}(f)\right) \frac{\partial}{\partial x^j} =: \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}\,.$$

Here, g^{ij} denotes the smooth functions given by the coordinates of the function $x \mapsto (g_{ij}(x))$ that defines the coordinate representation of the gradient in x.

Proof. Assume that $v, w \in \Gamma(TM)$ such that $v = \sum_i v^i \frac{\partial}{\partial x^i}$ and $w = \sum_i w^i \frac{\partial}{\partial x^i}$. Then $g(v,w) = \sum_{i,j} g_{ij} v^i w^i$. Suppose that $\operatorname{grad}(f) = \sum_j G^j \frac{\partial}{\partial x^j}$ and q is the coordinate map $T\mathbb{R} \to \mathbb{R}$ in each tangend space. Then by definition of the gradient we have:

$$\frac{\partial}{\partial x^j}(f) = q \circ \mathrm{d}f(\frac{\partial}{\partial x^j}) = q \circ \frac{\partial f}{\partial x^j} = g(\operatorname{grad}(f), \frac{\partial}{\partial x^j}) = \sum_{ij} g_{ij} G^i$$

Hence,

$$\begin{aligned} (G^1, \cdots G^m)(g_{ij}) &= \left(\frac{\partial}{\partial x^1}(f), \cdots, \frac{\partial}{\partial x^m}(f)\right) \\ \Leftrightarrow (G^1, \cdots G^m) &= \left(\frac{\partial}{\partial x^1}(f), \cdots, \frac{\partial}{\partial x^m}(f)\right)(g^{ij})\,. \end{aligned}$$

But then we can conclude that $G^j = \sum_i g^{ij} \frac{\partial}{\partial x^i}(f)$.

□

Definition 1.18 (A Dynamical System)**.** Let M be a smooth manifold. A **dynamical system on** M is given by a smooth map $\varphi : \Omega \to M$ such that:

1. $\{0\} \times M \subseteq \Omega \subseteq \mathbb{R} \times M$ open, and for any $p \in M$ the set $I(p) := \pi_1(\Omega \cup \mathbb{R} \times \{p\})$ is connected. (π_1 is the projection onto the first factor)
2. For all $p \in M$ and $t \in I(p)$ we have that $s \in I(\varphi(t,p))$ if and only if $s + t \in I(p)$. Then we have:

$$\varphi(s, (\varphi(t,p))) = \varphi(s+t, p)$$

3. For all p we require $\varphi(0,p) = p$.

We will write $\varphi : t(p) := \varphi(t,p)$ to keep the notation bearable. If for all p $I(p) = \mathbb{R}$ we call the system **global**.

Corollary 1.19. For a global dynamical system φ on a smooth manifold M the map

$$\rho : \mathbb{R} \to \mathrm{Diff},\ t \mapsto \varphi_t$$

is a homomorphism. To see the well definition notice how for all $t \in \mathbb{R}$ φ_{-t} is an inverse of φ_t.

Definition 1.20 (Vector Field of a Dynamical System)**.** Let φ be a dynamical system on a smooth manifold M. Then we call the mapping

$$p \mapsto X(p) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \varphi_t(p) \in T_pM$$

the **associated vecor field**. This is in fact a smooth vector field.

Whilst differentiating leads to a smooth vector field, we can also go the other direktion by locally integrating a vector field to get a flow. To be precise:

Definition 1.21. Inhalt...

Those concepts are inverse to one another or in other words

Theorem 1.22. *Let M be a smooth manifold. Then there is a one to one correspondence between global maximal flows and vector fields.*

Definition 1.23. For Morse theory (or rather Morse homology) the most importend flow is the one associeated to the negative gradient, i.e. for any fixed $p \in M$ we want

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x) = -\mathrm{grad}(f)(\varphi_t(x))$$

2 The conley index

We start by defining some needed properties of maps:

Definition 2.1 (Cofibration)**.** Let (X,A) be a topological pair and Y be a topological space. The pair (X,A) satisfies the **homotopy extension property with respect to** Y, if and only if, we can extend homotopies. In other words, for all $f : X \to Y$ and $H : A \times I \to Y$ with $H(x,0) = f(x)$ there exists a continuous extension $F : X \times I \to Y$ with $F(x,0) = f(x)$. If a pair (X,A) satisfies the homotopy extension property with respect to any topological space Y, we call (X,A) a **cofibered pair** and the inclusion $i : A \hookrightarrow X$ a **cofibration**.

Definition 2.2. For as topological pair (N, L) define $N/L = N/\sim$ where $x \sim y$ if $x, y \in L$. In other words we contract L to be one point.

Remark 2.3. In the proof of the Morse homology theorem we want to talk about the homology of index pairs as relative homology groups $H_i^{\text{sing}}(N, L)$. However, if the inclusion $L \to N$ is a cofibration we have the natural isomorphism $H_i^{\text{sing}}(N, L) \cong H_i^{\text{sing}}(N/L)$. This is proven in [**LecturesonMorseHomology**] chapter two. To show that something is a cofibration we will use the following fact for metric spaces:

The pair (N, L), where L is closed in N is a cofibration (respectivly the inclusion), if an in N open neighbourhood U of L exists, such that L is a strong deformation retract of U.

This is also presented in chapter two of [**LecturesonMorseHomology**], by showing that such an inclusion admits a Strøm structure. An example of such a cofibration is the pair $(D^n, \partial D^n)$. Finally, a pair that is homeomorphic to a cofibration is again one.

Definition 2.4 (Compact invariant isolated subset)**.** For a flow $\varphi_t : M \to M$ on a locally compact metric space we call a subset $S \subseteq M$ **invariant subspace** if and only if $\varphi_t(S) = S$ for all $t \in \mathbb{R}$. For any subset $N \subseteq M$ we define the **maximal invariant subset**

$$\begin{aligned} I(N) &= \{x \in N |\ \varphi_t(x) \in N \forall t \in \mathbb{R}\} \\ &= \bigcap_{t \in \mathbb{R}} \varphi_t(N). \end{aligned}$$

A **compact invariant subset** S is called **isolated**, if a compact neighborhood N exists, such that $I(N) = S$.

Definition 2.5 (Index pairs)**.** Let S be an isolated compact invariant subset. A topological pair (N, L) of compact subsets of M where $L \subseteq N$ is called an **index pair of** S, if it satisfies the following:

1. $S = I(\overline{N \setminus L}) \subseteq (N \overset{\circ}{\setminus} L)$.
2. $x \in L$ and $\varphi_{[0,t]}(x) \subseteq N$ implies that $\varphi_{[0,t]}(x) \subseteq L$. We call L **positively invariant in** N. Here $\varphi_{[0,t]}(x) := \{\varphi_{\tilde{t}}(x) | \tilde{t} \in [0, t]\}$.
3. For all $x \in N$ such that a t exists, with $\varphi_t(x) \notin N$, there exists a t' with $\varphi_{[0,t']}(x) \subseteq N$ and $\varphi_{t'}(x) \in L$.

This definition captures how the flow lines leave the invariant space S. Due to the third property we call L the **exit set**.

Definition 2.6 (Regular index pairs)**.** We call an index pair (N, L) **regular** if and only if the inclusion $I : L \hookrightarrow N$ is a cofibration.

Remark 2.7. One can show that every isolated compact invariant subset admits an index pair. We won't proof this, as we will always explicitly define such invariant sets and therefore we won't need such a general statement. However, two index pairs of the same invariant set turn out to be homotopy equivalent with a homotopy induced by the flow. The regularities needed for the proof are, that $\varphi_t : M \to M$ is a flow on a locally compact metric space. Those regularities are clearly given in the case of our Riemannian manifolds. We follow a proof from [**LecturesonMorseHomology**] which itself is a reformulation from the proof given by Salomon in [**SalomonConleyIndex**]. In this reformulation the language of the proof is adapted while the main arguments stay the same. This led to some redundant steps and vagueness, that I removed whilst making it more concrete.

Lemma 2.8. *Let N be an isolating neighborhood for the isolated compact invariant set S and let U be a neighbourhood of S. Then there exists a $t > 0$ such that for any $x \in M$ we have:*

$$\varphi_{[-t,t]}(x) \subseteq N \quad \Rightarrow \quad x \in U.$$

Proof. Lets assume this was false. Then for any $t > 0$ there would be a x contradicting the implication. So define $x_n \notin U$ such that $\varphi_{[-n,n]} \subseteq N$. Since $\varphi_0 = \mathrm{id}$ we know that all $x_n \in N$ and therefore by the compactness we would have limit points $x \in \overline{M \setminus U}$, such that $\varphi_{\mathbb{R}}(x) \subseteq N$. Since S was isolated we need to conclude that $x \in S \cap \overline{M \setminus U}$. This is our contradiction as U was a neighbourhood of the closed S and therefore without restriction open making $M \setminus U$ already closed. □

Remark 2.9. Notice that if t satisfies the conditions of lemma 2.8 then every $\tilde{t} \geq t$ also does. This helps us in the next lemma:

Lemma 2.10. *Let (N, L) and $(\tilde{N}, \tilde{L})$ be index pairs for the isolated invariant set S and choose $T \geq 0$ such that the following implications hold for $t \geq T$:*

$$\varphi_{[-t,t]}(x) \subseteq N \setminus L \quad \Rightarrow \quad x \in \tilde{N} \setminus \tilde{L}, \tag{1}$$

$$\varphi_{[-t,t]}(x) \subseteq \tilde{N} \setminus \tilde{L} \quad \Rightarrow \quad x \in N \setminus L. \tag{2}$$

Then the map:

$$\begin{aligned} h : N/L \times [T, \infty) &\to \tilde{N}/\tilde{L} \\ ([x], t) &\mapsto \begin{cases} [\varphi_{3t}(x)] & \text{if } \varphi_{[0,2t]} \subseteq N \setminus L \text{ and } \varphi_{[t,3t]}(x) \subseteq \tilde{N} \setminus \tilde{L} \\ [\tilde{L}] & \text{otherwise,} \end{cases} \end{aligned}$$

is continuous.

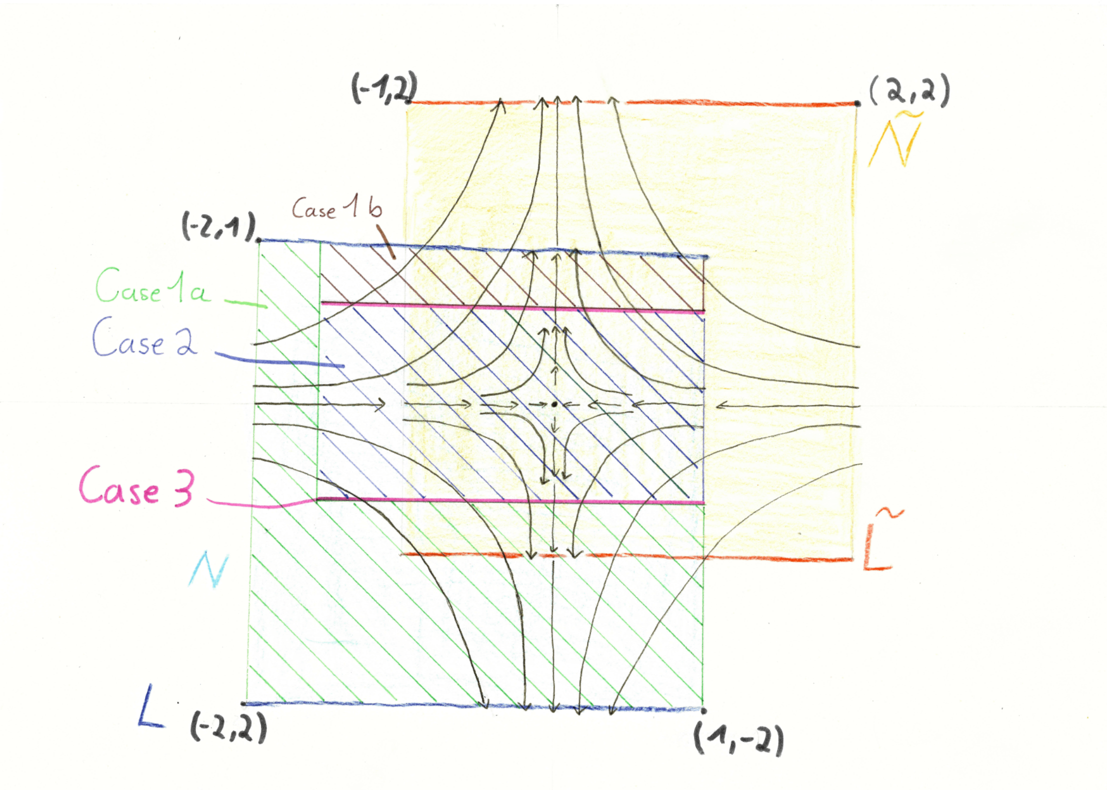


Figure 1: The different cases for the proof of lemma 2.10.
In this picture the flow is induced by the vectorfield $(-x, y)$ and drawn for $t = \frac{1}{3}$. That is due to the ease of calculation and such that the four cases are all visible. In the picture $N = [-2,1]^2$ and $\tilde{N} = [-1,2]^2$. L and $\tilde{L}$ are the horizontal borders.

Proof. We split this proof into three different cases that can be seen in figure 1. We always look at an image point lets say $h([x],t)$, choose an arbitrary neighbourhood U and define a neighbourhood W of $([x],t)$ such that the image of W is contained in U. This clearly gives us locally $\varepsilon - \delta$–continuity, as we are a metric space:

Case one a: $\varphi_{[t,3t]}(x) \notin \overline{\tilde{N} \setminus \tilde{L}}$. In this case there exists a $t < t^* < 3t$ such that $\varphi_{t^*}(x) \notin \overline{\tilde{N} \setminus \tilde{L}}$. t^* can be taken strictly less then $3t$ since the complement of $\overline{\tilde{N} \setminus \tilde{L}}$ is open. Furthermore, this tells us the existence of a neighbourhood U of $\varphi_{t^*}(x)$ that is disjoint from $\overline{\tilde{N} \setminus \tilde{L}}$. And by the continuity of the flow we have a neighbourhood $W \subseteq M \times [T, \infty]$ such that $(x', t') \in W$ implies that $\varphi_{t^*}(x') \in U$ and $t' < t^* < 3t'$. Thus, $\varphi_{[t',3t']}(x') \notin \tilde{N} \setminus \tilde{L}$ and therefore $h([x'], t') = [\tilde{L}]$ for all $(x', t') \in W$.

We can argue the same way if $\varphi_{[0,2t]} \notin (N \setminus L)$ (*Case one b*) and therefore conclude that in this case the map h is continuous. So for the rest of the proof we assume that we are in the first case of the map h or at the boundary:

$$\varphi_{[0,2t]} \subseteq \overline{N \setminus L} \text{ and } \varphi_{[t,3t]}(x) \subseteq \overline{\tilde{N} \setminus \tilde{L}}. \tag{3}$$

Case two: $\varphi_{[t,3t]}(x)$ is disjoint with $\tilde{L}$. Then due to the closure of $\tilde{N}$ and (3) we can conclude that $\tilde{N} \setminus \tilde{L} \supseteq \varphi_{[t,3t]}(x) = \varphi_{[-t,t]}(\varphi_{2t}(x))$ and by the implication (2) we can conclude that $\varphi_{2t}(x) \in N \setminus L$. Since L is the exit set we have that $\varphi_{[0,2t]}(x) \subseteq N \setminus L$. By the above we have that $h([x],t) = [\varphi_{3t}(x)] \in \tilde{N} \setminus \tilde{L}$. As before, due to the continuity of the flow we choose a neighbourhood U of $\varphi_{3t}(x)$ and find a neighbourhood $W \subseteq M \times [T, \infty)$ such that whenever $(x', t') \in W$ we have:

$$\varphi_{[0,2t']} \cap L = \varnothing, \quad \varphi_{[t',3t']}(x') \cap \tilde{L} = \varnothing, \quad \text{and} \quad \varphi_{3t'}(x') \in U.$$

If x' is in N then we have that $\varphi_{[0,2t']}(x')$ is in $N \setminus L$ and similar to before we conclude with (1) that $\varphi_{t'}(x') \in \tilde{N} \setminus \tilde{L}$ and since $\tilde{L}$ is the exit set we have the inclusion $\varphi_{[t',3t']}(x') \subseteq \tilde{N} \setminus \tilde{L}$. Therefore, we have that $h([x'], t') = [\varphi_{3t'}(x')] \in U$ for all $(x', t') \in W$ where $x' \in N$. The continuity of the flow gives us continuity in this area.

Case three: $\varphi_{[t,3t]}(x)$ intersects $\tilde{L}$. Then by (3) and since $\tilde{L}$ is the exit set we have that $\varphi_{3t}(x) \in \tilde{L}$. Now define $[U]$ to be a neighbourhood of $h([x],t) = [\tilde{L}]$ in $\tilde{N}/\tilde{L}$. We want to find a representative of $[U]$, that is an open set of M that reduces to $[U]$ in the quotient space. Let $\pi : \tilde{N} \to \tilde{N}/\tilde{L}$ be the quotient map. A natural choice would be $U := \pi^{-1}([U])$ which is without restriction open in $\tilde{N}$. To make it open in M we can unite it with $M \setminus \tilde{N}$. Now again by the continuity of the flow we have an open neighbourhood $W \subseteq M \times [T, \infty)$ of (x,t) such that whenever $(x', t') \in W$ we have that $\varphi_{3t'}(x') \in U$. But then we have that:

$$h([x'], t') \in \{[\varphi_{3t'}(x')], [\tilde{L}]\} \subseteq [U] \cap [\tilde{L}] = [U].$$

$\square$

Lemma 2.11. *Let $(N, L), (N', L')$ and $(\tilde{N}, \tilde{L})$ be index pairs of S. Choose $T > 0$ such that the implications (1) and (2) are satisfied and furthermore choose a $\tilde{T}$ such that for $t > \tilde{T}$ we have:*

$$\varphi_{[-t,t]}(x) \subseteq \tilde{N} \smallsetminus \tilde{L} \quad \Rightarrow \quad x \in N' \smallsetminus L' \tag{4}$$

$$\varphi_{[-t,t]}(x) \subseteq N' \smallsetminus L' \quad \Rightarrow \quad x \in \tilde{N} \smallsetminus \tilde{L}. \tag{5}$$

Now define:

$$\begin{aligned} h &: N/L \times [T, \infty) \to \tilde{N}/\tilde{L} \\ &([x], t) \mapsto \begin{cases} [\varphi_{3t}(x)] & \text{if } \varphi_{[0,2t]} \subseteq N \smallsetminus L \text{ and } \varphi_{[t,3t]}(x) \subseteq \tilde{N} \smallsetminus \tilde{L} \\ [\tilde{L}] & \text{otherwise,} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \tilde{h} &: \tilde{N}/\tilde{L} \times [\tilde{T}, \infty) \to N'/L' \\ &([x], t) \mapsto \begin{cases} [\varphi_{3t}(x)] & \text{if } \varphi_{[0,2t]} \subseteq \tilde{N} \smallsetminus \tilde{L} \text{ and } \varphi_{[t,3t]}(x) \subseteq N' \smallsetminus L' \\ [L'] & \text{otherwise.} \end{cases} \end{aligned}$$

Then the following equations hold for $t \geq \max\{T, \tilde{T}\}$:

$$\tilde{h}\big(h([x], t), t\big) = \begin{cases} [\varphi_{6t}(x)] & \text{if } \varphi_{[0,4t]} \subseteq N \smallsetminus L \text{ and } \varphi_{[2t,6t]}(x) \subseteq N' \smallsetminus L' \\ [L'] & \text{otherwise.} \end{cases}$$

Proof. The proof is just the equivalence of the two statements:

$$\begin{aligned} &\varphi_{[0,4t]} \subseteq N \smallsetminus L \text{ and } \varphi_{[2t,6t]}(x) \subseteq N' \smallsetminus L' \\ \Leftrightarrow \quad &\varphi_{[0,2t]}(x) \subseteq N \smallsetminus L,\ \varphi_{[t,5t]}(x) \subseteq \tilde{N} \smallsetminus \tilde{L},\ \varphi_{[4t,6t]}(x) \subseteq N' \smallsetminus L'. \end{aligned}$$

„$\Rightarrow$“ Here we need to check three things. The first and the third inclusion are trivially satisfied. For the second we notice that $\varphi_{[0,4t]} \subseteq N \smallsetminus L$ implies that $\varphi_{[t,3t]}(x) \subseteq \tilde{N} \smallsetminus \tilde{L}$ by implication (1). Furthermore, $\varphi_{[2t,6t]}(x) \subseteq N' \smallsetminus L'$ implies that $\varphi_{5t}(x) \in \tilde{N} \smallsetminus \tilde{L}$ by implication (5). And since $\tilde{L}$ is the exit set this implies the missing second inclusion.
„$\Leftarrow$“ For the first inclusion notice that $x = \varphi_0(x) \in N$ by definition and $\varphi_{[t,5t]}(x) \subseteq \tilde{N} \smallsetminus \tilde{L}$ implies $\varphi_{[2t,4t]}(x) \subseteq N \smallsetminus L$ by implication (2). Using the exit set property we conclude that $\varphi_{[0,4t]} \subseteq N \smallsetminus L$. Finally, $\varphi_{[t,5t]}(x) \subseteq \tilde{N} \smallsetminus \tilde{L}$ implies that $\varphi_{[2t,4t]}(x) \subseteq N' \smallsetminus L'$ by (4).

Together with $\varphi_{[4t,6t]}(x) \subseteq N' \smallsetminus L$ this lets us conclude that $\varphi_{[2t,6t]}(x) \subseteq N' \smallsetminus L'$. The additive property of the flow does the rest letting us conclude that we are in the first case of h and h' if and only if the stated property is satisfied. □

Theorem 2.12 (Homotopy equivalence of index pairs)**.** *If S is an isolated compact invariant set and (N, L) and $(\tilde{N}, \tilde{L})$ are two index pairs of S. Then N/L and $\tilde{N}/\tilde{L}$ are homotopy equivalent as pointed spaces.*

Proof. Let $h_t : N/L \to \tilde{N}/\tilde{L}$ and $g_t : \tilde{N}/\tilde{L} \to N/L$ be the continuous family of maps from lemma 2.10. By lemma 2.11 we get that

$$g_t \circ h_t([x]) = \begin{cases} [\varphi_{6t}(x)] & \text{if } \varphi_{[0,4t]} \subseteq N \setminus L \text{ and } \varphi_{[2t,6t]}(x) \subseteq N \setminus L \\ [L] & \text{otherwise.} \end{cases}$$

Furthermore, we use lemma 2.10 for $T = 0$. And now we can explicitly write down a homotopy from the identity to $g_t \circ h_t([x])$ as follows:

$$\begin{aligned} H : N/L \times [0,1] &\to N/L \\ ([x], t') &\mapsto \begin{cases} [\varphi_{6t't}(x)] & \text{if } \varphi_{[0,4t't]} \subseteq N \setminus L \text{ and } \varphi_{[2t't,6t't]}(x) \subseteq N \setminus L \\ [L] & \text{otherwise.} \end{cases} \end{aligned}$$

This homotopy is continuous by lemma 2.10 and $H(\cdot, 0) = \mathrm{id}$ and $H(\cdot, 1) = g_t \circ h_t$. Similar we show that $h_t \circ g_t$ is homotopic to the identity. With this we have that

$$N/L \simeq \tilde{N}/\tilde{L}.$$

□

Definition 2.13 (The homotopy Conley index)**.** The **Conley index** of an invariant set S is defined as $\pi_1(N, L)$ where (N, L) is a regular index pair. This definition is invariant of the chosen pair and is an invariant of invariant sets.

3 A Morse Theoretic Filtration

In this subchapter we will finally prove the Morse homology theorem. We start by defining some index sets that we will later use in the prove. Again, this prove is due to Solomon and found in [**MorseTheorySalmbon**].

Example 3.1 (Critical points as isolated compact invariant subsets)**.** Let $f : M \to \mathbb{R}$ be a Morse function on a smooth Riemannian manifold (M, g). Let $q \in \mathrm{Crit}_k(f)$. Around q we have coordinate charts $\varphi : U \to T_qM$ (after identifying q_i and ∂q_i) where $\varphi(W(q \to) \cap U) \subseteq T_q^uM$ and $\varphi(W(\to q) \cap U) \subseteq T_q^sM$. With this we can define the balls:

$$\begin{aligned} D_\varepsilon^s &= \{v \in T_q^sM | \, \|v\| \le \varepsilon\}, \\ D_\varepsilon^u &= \{v \in T_q^uM | \, \|v\| \le \varepsilon\}. \end{aligned}$$

They give rise to the index pair $N_q := \varphi^{-1}(D^s \times D^u)$ and $L_q = \varphi^{-1}(D^s \times \partial D^u)$. This index pair is in fact regular, since $(N_q, L_q) \cong (D^s \times D^u, D^s \times \partial D^u) \cong (D_\varepsilon^u, \partial D_\varepsilon^u)$. Now we can make the first step towards singular homology, since we know the homology of such a tuple:

$$H_i^{\mathrm{sing}}(N_q, L_q) = H_i^{\mathrm{sing}}(D^k, \partial D^k) = \begin{cases} \mathbb{Z} & \text{if } i = k, \\ 0 & \text{else.} \end{cases} \tag{6}$$

Notice that for $k = 0$ we need to consider $\partial D^k = \varnothing$, to get an index pair. That's why we let ∂ denote the manifold boundary insted of the topological boundary. However, now we can identify for all k:

$$C_k(M, f) = \bigoplus_{\mathrm{Crit}_k(f)} \mathbb{Z} \cong \bigoplus_{q \in \mathrm{Crit}_k(f)} H_k(N_q, L_q; \mathbb{Z}). \tag{7}$$

Furthermore, this isomorphism can be made canonically, by using orientations: Notice for this that (N_q/L_q) is homotopic equivalent to $W(q \to)/(W(q \to) \smallsetminus \{q\})$. Here we get a generator of the $k-$homology group induced by an orientation on $T_p^u M$ that then canonically maps to the $k-$th homology group of (N_q, L_q).

Example 3.2 (A map of homology groups)**.** Let $f : M \to \mathbb{R}$ be a Morse Smale function on a smooth compact Riemannian manifold (M, g). Let $q \in \mathrm{Crit}_k(f)$ and $p \in \mathrm{Crit}_{k-1}(f)$. Assume for a moment we already have an isolated regular compact index pair (N_2, N_0) of $S = W(p \to q) \cup \{p, q\}$. For a $c \in (f(p), f(q))$ we set $N_1 = N_0 \cup (N_2 \cap M^c)$ depicted in figure 2, where M^c is the sublevel set. Clearly now (N_2, N_1) is an index pair for q and (N_1, N_0) is a pair for p. Assuming we have regular index pairs $(N_q.L_q)$ and (N_p, L_p) for p, q we can now define a map:

$$\Delta_k(q \to p) : H_k^{\mathrm{sing}}(N_q, L_q) \to H_{k-1}^{\mathrm{sing}}(N_p, L_p)$$

as the composition:

$$H_k(N_q, L_q; \mathbb{Z}) \xrightarrow{\cong} H_k(N_2, N_1) \xrightarrow{\delta_*} H_{k-1}(N_1, N_0) \xrightarrow{\cong} H_{k-1}(N_p, L_p)$$

where δ_* is the connecting homomorphism, and the other two maps are induced by the homotopic equivalence of two index pairs corresponding to the same isolated compact invariant subset. Putting all together we can define a morphism:

$$\Delta_k : \bigoplus_{q \in \mathrm{Crit}_k(f)} H_k^{\mathrm{sing}}(N_q, L_q) \to \bigoplus_{p \in \mathrm{Crit}_{k-1}(f)} H_{k-1}^{\mathrm{sing}}(N_p, L_p). \tag{8}$$

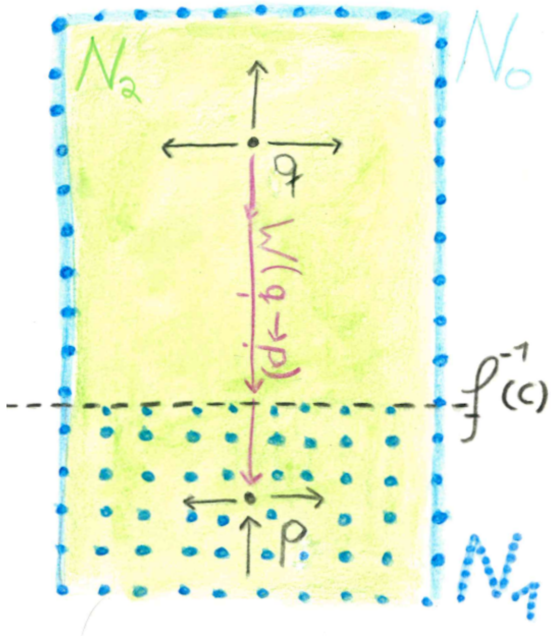


Figure 2: Index pairs for gradient flow lines.

Definition 3.3 (Regular index pairs and a filtration on M)**.** As always let $f : M \to \mathbb{R}$ be a Morse-Smale function on a compact smooth Riemannian manifold (M, g) of dimension m. Let $\varphi_t : M \to M$ be the flow given by $-\mathrm{grad} f$. For $0 \leq j \leq k \leq m$ define:

$$W(k, j) = \bigcup_{j \leq \lambda_p \leq \lambda_q \leq k} W(q \to p).$$

We know that this space is compact. Assume that N is a compact neighbourhood of $W(k, j)$, such that $\mathrm{Crit}(f) \cap N = \mathrm{Crit}(f) \cap W(k, j)$, meaning that N does not contain any new critical points. Then the biggest invariant subspace from definition 2.5 is

reference

$$I(N) := \{x \in N | \varphi_t(x) \in N \forall t \in \mathbb{R}\} = W(j, k).$$

This follows from every gradient flow line starting and ending in a critical point. Therefore, $W(k, j)$ is an isolated compact invariant set. By a corollary of the lambda lemma we know that

reference

$$\begin{aligned} W_j^s &:= \bigcup_{j \leq \lambda_p} W(\to p) \\ W_j^u &:= \bigcup_{\lambda_p \leq j} W(p \to) \end{aligned}$$

for all $j = 0, \ldots, m$ are compact as they are finite unions of compact sets. Now we define $N_m := M$ and choose a cofibered compact neighbourhood N_{m-1} of $W_{m-1}^{p\to}$ that is positively invariant and satisfies $N_{m-1} \cap W_m^{\to p} = \varnothing$ One could for example define:

$$N_{m-1} := N_m \setminus \Big(\bigcup_{q \in \mathrm{Crit}_m(f)} \mathring{N}_q \Big),$$

where N_q is taken from example 3.1. Once we check that N_{m-1} is an exit set with respect to N_m we can conclude that (N_m, N_{m-1}) is a regular index pair for $\mathrm{Crit}_m(f)$. The first statement is clear, since every gradient line starting in $N_m \setminus N_{m-1}$ has to pass through N_{m-1}, as they go towards other critical points. This tells us that they have to leave $N_m \setminus N_{m-1}$, which is enclosed by N_{m-1}. The second thing to check can be done by showing that there is a neighbourhood $U \subset N_M$ of N_{m-1} that deforms to N_{m-1}. For this notice that each N_q with $\lambda_q = m$ is homeomorphic to D^m. Now choose a open neighbourhood U_q in the disc of its boundary. By definition this retracts to the boundary with a retraction induced by the flow. And finally define $U := N_{m-1} \cap_{q \in \mathrm{Crit}_m(f)} U_q$. This retracts to N_{m-1} telling us that our index pair is indeed regular.

Now we want to inductively define a filtration: For this we choose a compact cofibered neighborhood N_{m-2} of W_{m-2}^u that is positifely invariant in N_{m-1} and has an empty intersection with $W_{m-1}^{\to p}$. This can be done similar to the construction of N_{m-1} but instead of cutting out neighbourhoods if $q \in \mathrm{Crit}_m(f)$, we can cut out tubular neighborhoods of W_{m-1}^s. This is depicted in figure 3 for the torus. By iterating this process we get a filtration:

$$\varnothing =: N_{-1} \subset N_0 \subset N_1 \subset \cdots \subset N_{m-1} \subset N_m = M, \tag{9}$$

such that (N_k, N_{j-1}) is a regular index pair for $W(k, j)$ for all $0 \leq j \leq k \leq m$.

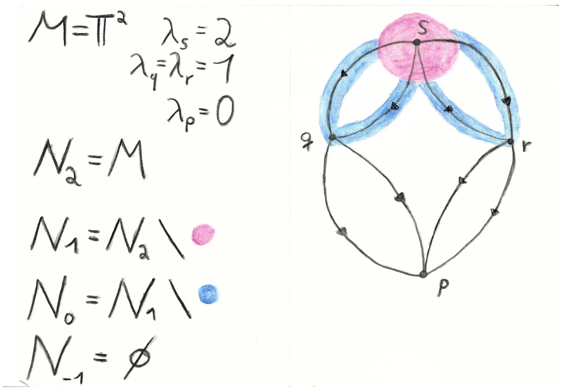


Figure 3: The filtration of the torus.
The picture shows the phase diagram of the torus and schematically depicts the filtration.

Theorem 3.4. *Given the filtration* $N_{-1} \subseteq N_0 \subseteq \cdots \subseteq N_{m-1} \subseteq N_m = M$ *from 9. The following diagram commutes:*

$$\begin{array}{ccc} C^k(M, \mathfrak{A}, \tilde{K}^l(pt)) & \xrightarrow{\partial^p} & C^{k+1}(M, \mathfrak{A}, \tilde{K}^l(pt)) \\ \downarrow & & \downarrow \\ \bigoplus_{p \in \mathrm{Crit}_k(f)} \tilde{K}^{k+l}(N_p, L_p) & \xrightarrow{\Delta_{k+l}} & \bigoplus_{q \in \mathrm{Crit}_{k+1}(f)} \tilde{K}^{k+l+1}(N_q, L_q) \\ \downarrow & & \downarrow \\ K^{k+l}(N_k, N_{k-1}) & \xrightarrow{\delta_{triple}} & K^{p+l+1}(N_{k+1}, N_k) \end{array}$$

Proof. So we assume for the moment, that $q \in \mathrm{Crit}_{k+1}(f)$ and $p \in \mathrm{Crit}_k(f)$ are the only critical points in $f^{-1}([a,b])$, where $a := f^{-1}(p)$ and $b := f^{-1}(q)$. Now, we choose the index pairs wisely: First we define the notations:

$$M^t := \{x \in M | f(x) \leq t\} \ , \ M_t := \{x \in M | f(x) \geq t\}$$

and the constants:

$$c \in (a,b) \ , \ \varepsilon > 0 \text{ small enough } \ , \ T > 0 \text{ large enough}$$

Now we define the following sets:

$$\begin{aligned} N_q &:= \{x \in M_c | f(\varphi_{-T}(x)) \leq b + \varepsilon\} \\ L_q &:= \{x \in N_q | f(x) = c\} \\ N_p &:= \{x \in M^c | f(\varphi_T(x)) \geq a - \varepsilon\} \\ L_p &:= \{x \in N_p | f(\varphi_T(x)) = a - \varepsilon\} \end{aligned}$$

and with those the sets;

$$\begin{aligned} C &:= N_p \cup N_q \\ B &:= N_p \cup L_q \\ A &:= L_p \cup (L_q \overset{\circ}{-} N_p) \end{aligned}$$

With those we have the following list of facts:

1. (N_q, L_q) is a regular index pair for q.
2. (C, B) is an index pair for q.
3. (N_p, L_p) is a regular index pair for p.
4. (B, A) is an index pair for p

Since N_p is a tubular neighbourhood of the contractible $W(\to p) \cap M^c$ we get the diffeomorphism:

$$\psi_p : N_p \to \underbrace{\overline{D^{m-k}}}_{\dim W(\to p)} \times \overline{D^k} \subseteq T_p^u M$$

The image here is a subspace of the total space of the trivialized normal bundle. This map satisfies that

1. $\psi(L_p) = \overline{D^{m-k}} \partial D^k$,
2. $\psi_p(N_p \cap W(\to p) = \{0\} \times \overline{D^{m-k}}$,
3. $\psi(V_j) = \{\theta_j\} \times \overline{D^k}$ where $\theta_j \in \partial D^{m-k}$.

Using this map we get diffoemorphisms

$$\begin{aligned} \psi_j : V_j &\to \overline{D^k} \\ x &\mapsto \pi_1 \circ \psi_p(x) \text{ where } \pi_1 \text{ is the projection onto the first factor.} \end{aligned}$$

This map restics to a diffeomorphism from $\partial V_j = V_j \cap L_p$ to ∂D^{k-1} Hence, an orientation of the unstable tangend space of p induces a map: $V_j \to \overline{D}^k$

das durchdenken, ob das klar ist

Now we want to figure out how the map between the spheres on the right looks if the diagramm commutes:

$$\begin{tikzcd}
(W(q\to) \cap N_q) \cup C_1 S(q\to) \arrow[r, "\Phi"] \arrow[d, hook] & S^{k+1} \arrow[d] \\
\Big(\big((W(q\to) \cap N_q) \cup C_1 S(q\to)\big) \cup C_2(W(q\to) \cap N_q)\Big) \cup C_3(C_1 \overline{S(q\to) \setminus V_j}) \arrow[r, "\Psi"] & S^{k+1}
\end{tikzcd}$$

Hence we need to specify the horizontal (and vertical) maps. So we start with the top one(or rqther its inverse): From the proof of theorem **??** we recycle a view maps. Notice how the orientation of $T_q^u M$ gives us a way to identify $T_q^u M$ with $\mathbb{R}^{k+1}$ (the coordinate map) and hence we can restrict the chart $\chi^0 =: \chi : T_q^u M \supset U \to W(q\to)$ such that $U = \overline{D^{k+1}}$ is a closed Disc in $\mathbb{R}^n$.. Now for each $x \in U \setminus q$ there is a number $t_{0,x} \geq 0$ such that $\varphi_{t_{0,x}}(\chi(x)) \in \chi(\partial U)$. Furthermore for each $x \in U$, there is a number$t_{1,x}$ such that $\varphi_{t_{1,x}}(\chi(x)) \in f^{-1}(c)$ Those numbers all smoothly depend on x.

Why is the left an inclusion?

careful, the map χ is just a chart and hence not natural in any sence. But since $W(q\to)$ is orientable, and hence we can make χ and oriented chart.

Now define the map

$$\begin{aligned} \tilde{\Phi} : \overline{D^{k+1}} &\to W(q\to) \cap M_c \\ x &\mapsto \varphi_{(t_{1,x} - t_{0,x})} \chi((x)) \,. \end{aligned}$$

This map is a homeomorphism, since χ is one, and the flow as a map $\varphi : \mathbb{R} \times M \to M$ is continouos. Furthermore there is an inverse given as follows: For each $x \in W(q\to) \cap M_c$

there is a real number $s_{0,x}$ such that $\varphi_{s_{0,x}}(x) \in \chi(\partial U)$, and a real number $s_{1,x}$ such that $\varphi_{s_{1,x}}(x) \in f^{-1}(c)$. Then the map

$$\begin{aligned} \Phi : W(q \to) \cap M_c &\to \overline{D^{k+1}} \\ x &\mapsto \chi^{-1}(\varphi_{s_{0,x}-s_{1,x}})(x) \end{aligned}$$

is the inverse of $\tilde{\Phi}$: To see this notice that for $y = \chi(x)$ we have $t_{0,x} = s_{0,y}$, and $t_{1,x} = s_{1,y}$ by definition.

Now call $A := \varphi_{(t_{1,x}-t_{0,x})}\chi(x) = \varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x)$, then we have:

- $s_{1,A} = s_{0,\chi(x)}$ and,
- $s_{0,A} = -s_{1,\chi(x)} + 2s_{0,\chi(x)}$.

This is because:

$$\varphi_{s_{0,\chi(x)}} \circ \varphi_{(t_{1,x}-t_{0,x})}\chi(x) = \varphi_{s_{0,\chi(x)}} \circ \varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x) = \varphi_{(s_{1,\chi(x)})}\chi((x)) \in f^{-1}(c)\,,$$

and

$$\varphi_{-s_{1,\chi(x)}+2s_{0,\chi(x)}} \circ \varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x) = \varphi_{(s_{0,\chi(x)})}\chi((x)) \in f(\partial U)\,.$$

Hence

$$\begin{aligned} \Phi \circ \tilde{\Phi}(x) &= \Phi\Big(\varphi_{(t_{1,x}-t_{0,x})}\chi((x))\Big) \\ &= \Phi\Big(\underbrace{\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x)}_{=A}\Big) \\ &= \chi^{-1} \circ \varphi_{s_{0,A}-s_{1,A}}(\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x)) \\ &= \chi^{-1} \circ \underbrace{\varphi_{-s_{1,\chi(x)}+2s_{0,\chi(x)}-s_{0,\chi(x)}}(\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}}_{=\mathrm{id}}\chi(x)) \\ &= x \end{aligned}$$

And the other way round we have the relations for $B := \chi^{-1}(\varphi_{s_{0,x}-s_{1,x}}(x))$:

$$\begin{aligned} t_{0,B} &= s_{1,x} \\ t_{1,B} &= -s_{0,x} + 2s_{1,x}\,, \end{aligned}$$

since we can calculate:

$$\begin{aligned} \varphi_{s_{1,x}}(\chi(B)) &= \varphi_{s_{1,x}}(\varphi_{s_{0,x}-s_{1,x}}) = \varphi_{s_{0,x}}(x) \in f(\partial U) \\ \varphi_{-s_{0,x}+2s_{1,x}}(\chi(B)) &= \varphi_{-s_{0,x}+2s_{1,x}}(\varphi_{s_{0,x}-s_{1,x}}) = \varphi_{s_{1,x}}(x) \in f^{-1}(c) \end{aligned}$$

$$\begin{aligned} \tilde{\Phi} \circ \Phi(x) &= \tilde{\Phi}\big(\chi^{-1}(\varphi_{s_{0,x}-s_{1,x}})(x)\big) \\ &= \varphi_{t_{1,B}-t_{0,B}}\chi \circ \chi^{-1}(\varphi_{s_{0,x}-s_{1,x}})(x) \\ &= \varphi_{-s_{0,x}+2s_{1,x}-s_{1,x}}\varphi_{s_{0,x}-s_{1,x}})(x) \\ &= x \end{aligned}$$

So now we have a diffeomorpism $\Phi : W(q \to) \cap N_q \to \overline{D^{k+1}}$. We extend this to a continuous map

$$(W(q \to) \cap N_q) \cup C_1 S(q \to) \to \overline{D^{k+1}} / \partial \overline{D^{k+1}}$$

by first contracting $C_1 S(q \to)$: So now we have a diffeomorpism $\Phi : W(q \to) \cap N_q \to \overline{D^{k+1}}$. We extend this to a continuous map

$$(W(q \to) \cap N_q) \cup C_1 S(q \to) \to \overline{D^{k+1}} \cup C_1 \partial \overline{D^{k+1}}$$

This is neither a homeopmorphism nor a homotopie equivalence. But it is continouos and since $C_1 S(q \to)$ was contractible, it induces an isomorphims in the K-groups. Now since Φ maps $S(q \to)$ to ∂D^{k+1} we can compose the contracting with Φ to get a continouos map that induces an isomorphism in the K-group, and which we will also call Φ:

$$\Phi : (W(q \to) \cap N_q) \cup C_1 S(q \to) \to \overline{D^{k+1}} / \partial \overline{D^{k+1}} \, .$$

Now we want to construct the map

$$\Psi : \Big(\big((W(q \to) \cap N_q) \cup C_1 S(q \to) \big) \cup C_2 (W(q \to) \cap N_q) \Big) \cup C_3 (C_1 \overline{S(q \to) \setminus V_j}) \to S^{k+1} \quad (10)$$

First we contract all unnecesarry parts:

$$\begin{aligned}
&\Big(\big((W(q \to) \cap N_q) \cup C_1 S(q \to) \big) \cup C_2 (W(q \to) \cap N_q) \Big) \cup C_3 (C_1 \overline{S(q \to) \setminus V_j}) \\
&\to \Big(\big((W(q \to) \cap N_q) \cup C_1 S(q \to) \big) \big/ (W(q \to) \cap N_q) \Big) \big/ (C_1 \overline{S(q \to) \setminus V_j}) \\
&= \big((W(q \to) \cap N_q) \cup C_1 S(q \to) \big) \big/ \big((W(q \to) \cap N_q) \cup (C_1 \overline{S(q \to) \setminus V_j}) \big) \\
&= \big(C_1 S(q \to) \big) \big/ \big(S(q \to) \cup (C_1 \overline{S(q \to) \setminus V_j}) \big) \\
&= \big(C_1 V_j \big) \big/ \big(V_j \cap C_1 (\partial V_j) \big)
\end{aligned}$$

To summarize, we get a continouos map Θ:

$$\begin{aligned}
&\Big(\big((W(q \to) \cap N_q) \cup C_1 S(q \to) \big) \cup C_2 (W(q \to) \cap N_q) \Big) \cup C_3 (C_1 \overline{S(q \to) \setminus V_j}) \\
&\to \big(C_1 V_j \big) \big/ \big(V_j \cap C_1 (\partial V_j) \big)
\end{aligned}$$

Again this is continouos but not even a homotopie equivalence. However, it induces an isomorphism in the K-groups. via ψ we get a map $V_j \to \overline{D^k}$.

$$\begin{aligned}
\tilde{\psi} : \big(C_1 V_j \big) \big/ \big(V_j \cap C_1 (\partial V_j) \big) &\to \overline{D^k} \times I \big/ \big(\partial \overline{D^k} \times I \cup \overline{D^k} \times \{0,1\} \big) \\
(x,t) &\mapsto \psi(x,t)
\end{aligned}$$

Now by rescaling the last factor:

$$\tilde{r} : \overline{D^k} \times I \to \overline{D^{k+1}}$$

we get the homomorphism

$$\begin{aligned} r : \overline{D^k} \times I \big/ \big(\partial \overline{D^k} \times I \cup \overline{D^k} \times \{0,1\}\big) &\to \overline{D^{k+1}} \big/ \partial \overline{D^{k+1}} \\ (x,t) &\mapsto \overline{r(x,t)}\,. \end{aligned}$$

To see the well definition, notice how for closed sets we have the equality $\partial(A \times B) = (\partial A \times B) \cup (A \times \partial B)$. In sum we get the map :

$$\begin{aligned} \Psi : \Bigg(\Big((W(q \to) \cap N_q) \cup C_1 S(q \to)\Big) \cup C_2(W(q \to) \cap N_q)\Bigg) \cup C_3(C_1 \overline{S(q \to) \smallsetminus V_j}) \\ \to \overline{D^{k+1}} \big/ \partial \overline{D^{k+1}} \end{aligned}$$

given by $\Psi : r \circ \tilde{\psi} \circ \Theta$ Now we want to ask, how the map

$$\Psi \circ i \circ \Phi^{-1} : \overline{D^{k+1}} \big/ \partial \overline{D^{k+1}} \to \overline{D^{k+1}} \big/ \partial \overline{D^{k+1}}$$

looks like. Our claim is, that this map is homotopic to the identity, if the orientation in $T_{x_j} V_j$ induced from the one in $T_q^u M$ and from $T_p^u M$ agree. To do this we start with the inclusion. (But we need the "Coordinates from Φ maby we can look at $\Phi(x,t)$ and compare it to $\Psi \circ i(x,t)$.

$$\begin{aligned} i : (W(q \to) \cap N_q) \cup C_1 S(q \to) \\ \to \Bigg(\Big((W(q \to) \cap N_q) \cup C_1 S(q \to)\Big) \cup C_2(W(q \to) \cap N_q)\Bigg) \cup C_3(C_1 \overline{S(q \to) \smallsetminus V_j}) \end{aligned}$$

So assume (t,x) lives in the domain. Then $(\Theta \circ i)(t,x) = \overline{(t,x)}$ where the equivalence is given by the collaps of everything but the interior of $C_1 V_j$. Now inspect $\psi(V_j)$. we can identify via a trivialitation of the normal bundle $\psi(V_j)$ with $T_p^u M$ and the orientation of the latter induces a map to D^{k-1}.

□

Corollary 3.5 (The Logic and To-Dos of my Proof)**.** We have the definitions above, of all sets. with those we first want to construct a map of triples

$$t : (A,B,C) \to (N_{p+1}, N_p, N_{p-1})$$

By naturality and definition we then have the commutative diagram:

$$\begin{tikzcd} K^{-p+1}(N_{p+1}, N_p)) \arrow[r, "t^*"] & K^{-p+1}(A,B) \arrow[r, Rightarrow] & K^{-p+1}(A,B) \\ K^{-p}(N_p, N_{p-1}) \arrow[u, "\delta_{triple}"] \arrow[r, "t^*"] & K^{-p}(B,C) \arrow[u, "\delta_{triple}"] \arrow[r, Rightarrow] & K^{-p+1}(s \vee (B,C)) \arrow[u, "s^*"] \end{tikzcd}$$

We now have to show that s^* is induced from a continuous map. Then we want to use the maps Φ and Ψ to induce a map $p \simeq \pm \mathrm{id}$ such that the diagram commutes up to homotopie:

$$\begin{tikzcd}
(A,B) \arrow[r, "\Phi"] \arrow[d, "s"] & \bigvee_{j=1}^{l} S^{k+1} \arrow[d, "\bigvee_j \delta_j \mathrm{id}"] \\
S \wedge (B,C) \arrow[r, "\Psi"] & S^{k+1}
\end{tikzcd}$$

where $\delta_j \in \{-1,1\}$(with $+\mathrm{id}$ we denote the identity and with $-id$ we denote a homeomorphism of degree -1, i.e. not homotopic to the identity.) and Ψ is a homotopie equivalence. Now since $K^{-p+1}(\bigvee_{j=1}^{l} S^{p+1}) \cong \mathbb{Z}^l$ we can define the following maps. Let β_q be a generator of $K^{-p+1}(A,B)$ and β_p of $K^{-p+1}(S \vee (B,C))$. Then define the koordinate maps

$$\begin{aligned}
q_q : K^{-p+1}(\bigvee_{j=1}^{l} S^{k+1}) \to \mathbb{Z}^l; \quad (\Phi^*)^{-1}(\beta_q) \mapsto \sum_j e_j \\
q_p : K^{-p+1}(S^{k+1}) \to \mathbb{Z}; \quad (\Psi^*)^{-1}(\beta_p) \mapsto 1\,.
\end{aligned}$$

With those we get the diagram:

$$\begin{tikzcd}
K^{-p+1}(A,B) & K^{-p+1}(\bigvee_{j=1}^{l} S_j^{k+1}) \arrow[l, "\Phi^*"'] \arrow[r, "q_q"] & \mathbb{Z}^l \\
K^{-p+1}(s \vee (B,C)) \arrow[u, "s^*"] & K^{-p+1}(S^{k+1}) \arrow[l, "\Psi^*"] \arrow[u, "g"] \arrow[r, "q_p"] & \mathbb{Z} \arrow[u, "h"']
\end{tikzcd}$$

The map h is given by $h : \mathbb{Z} \to Z^l$; $e_i \mapsto \sum_{j=1}^{l} \delta_j$ with the δ_j from above. This all concludes in the final calculation:

$$\begin{aligned}
s^*(\beta_p) &= \Phi^* \circ g \circ (\Psi^*)^{-1}(\beta_p) \\
&= \Phi^* \circ q_q^{-1} \circ h \circ q_p \circ (\Psi^*)^{-1}(\beta_p) \\
&= \Phi^* \circ q_q^{-1} \circ h(\sum_j e_j) \\
&= \Phi^* \circ q_q^{-1}(\sum_j \delta_j) \\
&= \sum_j \delta_j \beta_q
\end{aligned}$$

Now the hope is that δ_j is the sign that I would get from inspecting the morse boundary operator along a certain flow line. The todo's are:

1. The map t.
2. The map Φ.

3. The map Ψ.

4. Is the map s^* induced?

5. The map h corresponds to a proceedure similar to the Morse boundary.

Once we have done all the above we want to connect the considerations with the boundary operator. To do this we do the proveedure to all pairs $(q,p) \in \mathrm{Crit}(f)_{k+1} \times \mathrm{Crit}(f)_k$. For this we enrich the notaion and call the triple coresponding to such a pair $(A^{q,p}, B^{q,p}, C^{q,p})$. All the maps and spaces are enriched in that way, by adding the pair (q,p) as a superscrit. For T big enough we assume that all $A^{q,p}$ are pairwise disjoint. Then we want a homomorphism (that is a homotopic eqiuvalence)

do wee needc continuity or rather something weaker?

$$\Omega : \bigsqcup_{(q,p)\in \mathrm{Crit}_{k+1}\times \mathrm{Crit}_k} (A^{q,p}, B^{q,p}, C^{q,p}) \to (N_{p+1}, N_p, N_{p-1})$$
$$x \mapsto t^{q,p}(x) \text{ for } x \in A^{q,p}$$

Now, together with the isomorphism

$$\bigoplus_{\mathrm{Crit}_{k+1}} \mathbb{Z} \to K^p(N_{-p+1}, N_p) \ , \ q \mapsto \beta_q \text{ which is a generator of } K^{-p+1}(A^{q,p}, B^{q,p})$$

Fuck das funktioniert alles nicht!

Lemma 3.6. *Lets say that $p \in -\mathbb{N}$ is negative for the moment. We have the three spaces*

$$A := N_q \cup N_p \quad , \quad B := L_q \cup N_p \quad , \quad C := L_p \cup (L_q \smallsetminus \mathring{N}_p)\,. \tag{11}$$

By definition we get the triple connecting morphism as the composition of the inclusion with the connecting homomorphism of the pair::

$$\begin{tikzcd}
 & K^p[N_q \cup N_p, L_q \cup N_p] \arrow[r, Rightarrow] & K^p[N_q \cup N_p \cup /(L_q \cup N_p)] \\
 & & K^p[N_q \cup N_p \cup C_1(L_q \cup N_p)] \arrow[u, "(m^*)^{-1}"] \\
K^{p-1}[L_q \cup N_p] \arrow[uur, "\delta_{pair}"] \arrow[r, Rightarrow] & K^p[S_1 \wedge (L_q \cup N_p)] \arrow[r, "\alpha^*"] & K^p[N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)] \arrow[u, "i^*_\alpha"] \\
K^{p-1}[L_q \cup N_p, L_p \cup (L_q \smallsetminus \mathring{N}_p)] \arrow[u, "i^*"] \arrow[r, Rightarrow] & K^p[S_1 \wedge L_q \cup N_p, S_1 \wedge L_p \cup (L_q \smallsetminus \mathring{N}_p)] \arrow[u, "i^*_\beta"] &
\end{tikzcd}$$

The bottom left diagramm commutes by definition. All maps with names $i^*_{something}$ are induced from obvious inclusions. The other maps are defined as follows, where p denotes the choosen point of our pointed spaces:

$$\alpha : N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p) \to S_1 \wedge (L_q \cup N_p)$$
$$x \mapsto \begin{cases} p & \text{if } x \in C_2(N_q \cup N_p) \\ x & \text{else} \end{cases}$$

This map is well defined and continouos, as every point in $S_1 \wedge (L_q \cup N_p)$ is of the form $[(t_1, x)]$ where $x \in L_q \cup N_p$. and those are the points in the domain of α that not get collapsed to p We have the maps:

$$c_\gamma : [N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)] \to \left[\frac{N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \Big/ \frac{C_1(L_p \cup \overline{(L_q \setminus N_p)}) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \right]$$
$$x \mapsto \begin{cases} p & \text{if } x \in C_2(N_q \cup N_p) \quad \cup \quad C_1(L_p \cup \overline{(L_q \setminus N_p)}) \\ x & \text{else} \end{cases}$$

and

$$c_\beta : [S_1 \wedge (L_q \cup N_p)] \to \left[S_1 \wedge L_q \cup N_p \Big/ S_1 \wedge L_p \cup (L_q \setminus \mathring{N}_p) \right]$$
$$x \mapsto \begin{cases} p & \text{if } x \in S_1 \wedge L_p \cup (L_q \setminus \mathring{N}_p) \\ x & \text{else} \end{cases}$$

This map is again well defined and continouos and is also a map of pairs. In fact, both maps α and β can be written as $x \mapsto \overline{x}$ and since the collapsed space is contractible, they induce isomorphism between the K-groups. The map γ is just the identity.

Furthermore, we have the commutative diagramm:

$$\begin{tikzcd} K^p[S_1 \wedge (L_q \cup N_p)] \arrow[r, "\alpha^*"] & K^p[N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)] \\ K^p\left[S_1 \wedge L_q \cup N_p \Big/ S_1 \wedge L_p \cup (L_q \setminus \mathring{N}_p) \right] \arrow[u, "c_\beta^*"] \arrow[r, "\gamma^*"] & K^p\left[\frac{N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \Big/ \frac{C_1(L_p \cup \overline{(L_q \setminus N_p)}) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \right] \arrow[u, "c_\gamma^*"] \end{tikzcd}$$

To see the commutativitie we have to show that

$$c_\beta \circ \alpha = \gamma \circ c_\gamma : N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p) \to \left[S_1 \wedge L_q \cup N_p \Big/ S_1 \wedge L_p \cup (L_q \setminus \mathring{N}_p) \right]$$

since α and β are both of the form $x \mapsto \overline{x} = x$ or p we compare the case distinctions:

- $c_\beta \circ \alpha(x) = p \Leftrightarrow x \in C_1\left(L_p \cup (L_q \setminus \mathring{N}_p)\right) \cup C_2(N_q \cup N_p)$
- $\gamma \circ c_\gamma(x) = p \Leftrightarrow x \in C_2(N_q \cup N_p) \cup C_1(L_p \cup \overline{(L_q \setminus N_p)}) \cup C_2(N_q \cup N_p)$

Which are the same, since $L_q \setminus \mathring{N}_p = \overline{L_q \setminus N_p}$ due to L_q being closed. Now we consider the maps:

$$c : [N_q \cup N_p \cup C_1(L_q \cup N_p)] \to \left[\frac{N_q \cup N_p \cup C_1(L_q \cup N_p)}{\mathrm{cl}\left(N_q \cup N_p \cup C_1(L_q \cup N_p) \setminus \bigcup_{j=1}^{l} C_1 V_j\right)} \right]$$
$$x \mapsto \begin{cases} p & \text{if } x \in \mathrm{cl}(N_q \cup N_p \cup C_1(L_q \cup N_p) \setminus \bigcup_{j=1}^{l} C_1 V_j) \\ x & \text{else} \end{cases}$$

and

$$c_\iota : [N_q \cup N_p \cup C_1(L_q \cup N_p)] \to \left[\frac{N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \Big/ \frac{N_q \cup N_p \cup C_1\left(L_p \cup \operatorname{cl}\left(L_q \setminus \bigcup_{j=1}^{l} V_j\right)\right) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \right]$$

$$x \mapsto \begin{cases} p & \text{if } x \in N_q \cup N_p \cup C_1(L_p \cup \operatorname{cl}\left(L_q \setminus \bigcup_{j=1}^{l} V_j\right)) \cup C_2(N_q \cup N_p) \\ x & \text{else} \end{cases}.$$

We define V_j *to be the the finitly many components* $N_p \cap S(q \to)$ *and hence we can identify the sets* $\overline{L_q \setminus N_p} = \operatorname{cl}(L_q \setminus (L_q \cap N_q)) = \operatorname{cl}(L_q \setminus (\bigcup_{j=1}^{l}))$ *and with this the map* η *in the diagramm below is just the identity:*

$$\begin{tikzcd}
K^p\left[N_q \cup N_p \cup /(L_q \cup N_p)\right] & \\
K^p\left[N_q \cup N_p \cup C_1(L_q \cup N_p)\right] \arrow[u, "(m^*)^{-1}"] & K^p\left[\frac{N_q \cup N_p \cup C_1(L_q \cup N_p)}{\operatorname{cl}\left(N_q \cup N_p \cup C_1(L_q \cup N_p) \setminus \bigcup_{j=1}^{l} C_1 V_j\right)}\right] \arrow[l, "c^*"'] \\
K^p\left[N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)\right] \arrow[u, "i_\alpha^*"] & K^p\left[\frac{C_1(L_q \cup N_p)}{\operatorname{cl}\left(C_1(L_q \cup N_p) \setminus \bigcup_{j=1}^{l} C_1 V_j\right)}\right] \arrow[u, "i_\delta^*"] \\
 & K^p\left[\frac{C_1(L_q)}{\operatorname{cl}\left(C_1(L_q) \setminus \bigcup_{j=1}^{l} C_1 V_j\right)}\right] \arrow[u, "i_\eta^*"] \\
K^p\left[\frac{N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \Big/ \frac{C_1(L_p \cup \overline{(L_q \setminus N_p)}) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)}\right] \arrow[uu, "c_\gamma^*"] & K^p\left[\frac{N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \Big/ \frac{N_q \cup N_p \cup C_1\left(L_p \cup \operatorname{cl}\left(L_q \setminus \bigcup_{j=1}^{l} V_j\right)\right) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)}\right] \arrow[u, "c_\iota^*"] \arrow[l, "\eta^*"']
\end{tikzcd}$$

Now the mamoth task is to show that this beast commutes: For this we preoceed with the same strategy since again all maps are inclusions and collases and hence the maps look like $x \mapsto \overline{x} = x$ *or* p*. For the consideration assume that* $x \neq p$ *in the domain. Then:*

$$\eta \circ c_\gamma \circ i_\alpha(X) = p \Leftrightarrow x \in C_1(L_p \cup \overline{(L_q \setminus N_p)}) \cup C_2(N_q \cup N_p)$$

On the other hand:

$$c_\iota \circ i_\eta \circ i_\delta \circ c(x) = p \Leftrightarrow c(x) = p \text{ or } c_\iota(x) = p$$

$$x \in \mathrm{cl}\left(N_q \cup N_p \cup C_1(L_q \cup N_p) \setminus \bigcup_{j=1}^{l} C_1 V_j\right) \cup N_q \cup N_p \cup C_1(L_p \cup \mathrm{cl}\left(L_q \setminus \bigcup_{j=1}^{l} V_j\right)) \cup C_2(N_q \cup N_p)$$

$$= C_1 \,\mathrm{cl}\left((L_q \cup N_p) \setminus \bigcup_{j=1}^{l} V_j\right) \cup C_1(L_p \cup \mathrm{cl}\left(L_q \setminus \bigcup_{j=1}^{l} V_j\right)) \cup C_2(N_q \cup N_p)$$

$$= C_1 \,\mathrm{cl}\left((L_q) \setminus \bigcup_{j=1}^{l} V_j\right) \cup C_1(L_p \cup \mathrm{cl}\left(L_q \setminus \bigcup_{j=1}^{l} V_j\right)) \cup C_2(N_q \cup N_p)$$

$$= C_1(L_p \cup \mathrm{cl}\left(L_q \setminus \bigcup_{j=1}^{l} V_j\right)) \cup C_2(N_q \cup N_p)$$

And now the conditions agree and hence the big diagramm commutes. Now notice how we have the homeomorphism

$$\Lambda : \left[\frac{C_1(L_q)}{\mathrm{cl}\left(C_1(L_q) \setminus \bigcup_{j=1}^{l} C_1 V_j\right)}\right] \to \bigvee_{j=1}^{l} C_1 V_j \big/ \partial C_1 V_j$$

that is the identity on all $C_1 \mathring{V}_j$. Furthermore we have the compression maps

$$c_j : \bigvee_{j=1}^{l} C_1 V_j \big/ \partial C_1 V_j \to C_1 V_j \big/ \partial C_1 V_j$$

Hence by the additivity of $K-$Theorie, the pair

$$\left(K^p\left[\frac{C_1(L_q)}{\mathrm{cl}\left(C_1(L_q) \setminus \bigcup_{j=1}^{l} C_1 V_j\right)}\right], \left((c_j \circ \Lambda)^*\right)_j\right)$$

satisfies the coproduct universal property.

So consider the maps

$$c^* \circ i_\delta^* \circ i_\eta^* \circ (c_j \circ \Lambda)^* : C_1 V_j \big/ \partial C_1 V_j \to K^p \left[N_q \cup N_p \cup C_1(L_q \cup N_p)\right]$$

This map is induced from

$$N_q \cup N_p \cup C_1(L_q \cup N_p) \to C_1 V_j \big/ \partial C_1 V_j$$
$$x \mapsto \begin{cases} x & \text{if } x \in C_1(\mathring{V}_j), \\ p & \text{else }. \end{cases}$$

For all $C_1 V_j$ we need an homeomorphism to S^{k+1}:

$$r : C_1 V_j \big/ \partial C_1 V_j \to I \times D^k \big/ \partial (I \times D^k)$$
$$(t, x) \mapsto (t, \psi(x))$$

For this we use the following lemma:

Lemma 3.7. *ALternatifly, we dont need a homeomorphism $V_j \to T_p^u M$, rater from a subset of V_j (maby a ball contained in V_j). then anything outside of V_j can be disregarded. or rather define V_j for a T big enough to be a "ball arround x_j that lifes in our currently defined V_j.*

Lemma 3.8 (The map $V_j \to D^{k-1}$)**.** *We construct such a map by constructing a map $\psi : V_j \to T_p^u M$ with certain properties . We first notice, that the normal bundle of $W(\to p)$ is trivializable. Hence there is a family of sections $\left(s^1, \dots, s^{\mathrm{ind} p}\right)$ that are pointwise linearly independent (even orthonormal).We can now contract $W(\to p)$ to p via c_{x_j} since the stable sets are open discs: To be spesific we have the map*

die eigenschaften hier nennen

$$c_{x_j} : W(\to p) \times I \to W(\to p)$$

that satisfies $c_{x_j}(\cdot, 0) = id$ and $c_{x_j}(\cdot, 1) = \mathbf{const_{x_j}}$. Using this we collaps the normal bundle by definig the map fibrewise on the basis:

$$C_{x_j} : NW(\to p) \times I \to NW(\to p)$$
$$(s_l(p), p) \mapsto (s_l(c_{x_j}(p)), c_{x_j}(p))$$

Now we make use of the Tubular neighbourhood theorem to get a immersion

$$\Theta : NW(\to p) \to M\,,$$

and define $\mathrm{Tub}_p^s := \mathrm{im}\Theta$. Assume T to be big enough such that $V_j \subseteq \mathrm{Tub}_p^s$. With this we transprot the contraction to x_j:

$$\Theta \circ C_{x_j} \circ \Theta^{-1} : \mathrm{Tub}_p^s \times I \to \mathrm{Tub}_p^s.$$

Since transversal intersection (or in a chart non vanisching partial derivative) is a property that is locally stable we can assume, that the map $\Theta \circ C_{x_j} \circ \Theta^{-1}\big|_{V_j}$ is injective by using the implicit function theorem in a chart that contains V_j. With this we get a map from V_j to a open subspace of the normal space in x_j. Now we need to show that we can define a homeomorphism from V_j to the normal space. This can be shown by showing that V_j is open and starshaped. Assum that for all T the V_j is not star shaped. Hence, for fixed ε there will always be a T and a $x \in V_j$ such that there exists a $0 < \lambda < 1$ with $\varphi_T(\lambda x) < a - \varepsilon$. From this we will construct a contradiction to the compactnes. First we take a secuence of T_i that approaches infinity. Since for the limit of T_i the set N_p approaches $W(\to p) \cap M^c$ we can assume that there is a series δ_i approaching zero, such *that* $N_p \subseteq W(\to p) \times B_{\delta_i}$*. The last set denotes the immersed normal bundle restricted to the discs of radius* δ_i *in each fibre. Now since the contraction was defined via a orthonormal frame, the set* V_j *gets mapped to the normal space in* x_j *and is contanied in a ball of radius* δ_i*. By assumption for each* T_i *there is a* $\tilde{x}_i \in V_j$ *and a* λ_i *such that* $\lambda_i \tilde{x}_i \notin V_j$*. Define* $x_i := \lambda_i \tilde{x}_i$*. Then if* i *approaches infinity,* δ_i *approaches* 0 *and hence* x_i *approaches* x_j*, whilst never reaching it. Said differently: there are infinitly many different* x_i*. All those come with different flow lines* γ_i *between different critical points.*