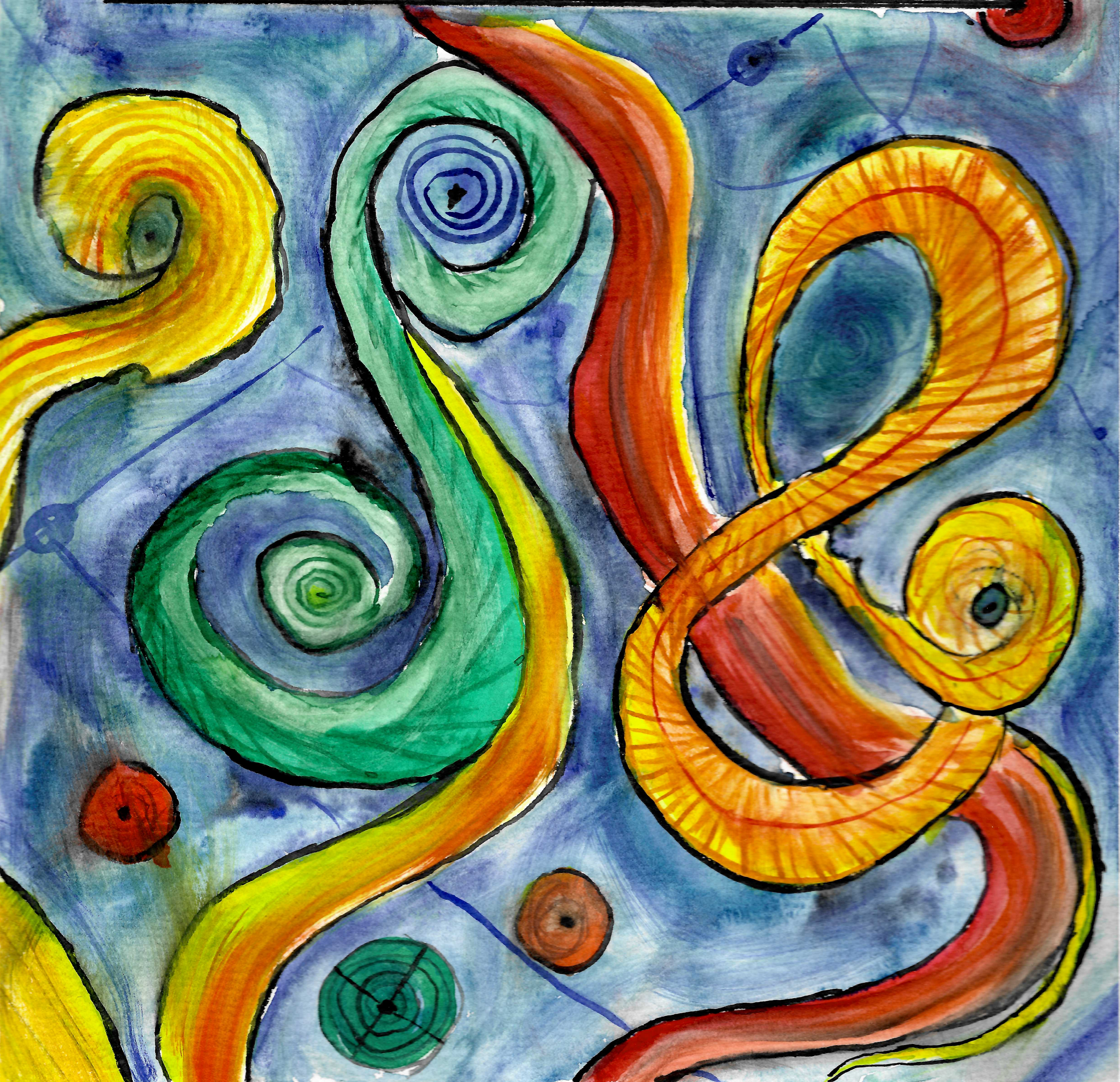
Morse-Theoretic

Atiyah-Hirzebruch Spectral Sequence



Contents

1 Vector Bundles

We follow the book of Atiyah [**atiyah1989k**].

1.1 Basic Definitions

Definition 1.1 (Family of vector spaces)**.** Let X be a topological Space. A **family of vector spaces over** X is a topological space E, equipped with :

(i) a continuous function $\pi_E \to X$,

(ii) a finite dimensional vector space structure on each fiber $E_x := \pi^{-1}(x)$ that is compatible with the topological structure on $E_x \subseteq E$.

We call the map π the projection , E the total space and X the base space.

Definition 1.2 (Section)**.** A **section** of a family of vector spaces $\pi : E \to X$ is a continuous map $s : X \to E$ such that $\pi s = \mathrm{id}$.

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ X \arrow[ur, "s"] \arrow[r, "\mathrm{id}"'] & X \end{tikzcd}$$

Definition 1.3 (Homomorphism of families of vector spaces)**.** A **homomorphism** form one family $\pi : E \to X$ to another family $\tilde{\pi} : F \to X$ with the same base space is a continuous map $\varphi : E \to F$ such that:

(i) $\tilde{\pi}\varphi = \pi$, i.e.
$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\pi"] & F \arrow[dl, "\tilde{\pi}"] \\ x & \end{tikzcd}$$

(ii) for each $x \in X$ $\varphi_x : E_x \to F_x$ is linear w.r.t. the vector space structure.

In the case that φ is bijective (and thereby a point-wise vector space isomorphism) and has an continuous inverse we call it an **isomorphism** and call E and F isomorphic.

Definition 1.4 (Pullback of a family of vector spaces)**.** If $\pi : E \to Y$ is a family of vector spaces, X a topological space and $f : X \to Y$ a continuous function. Then we define the induced the pullback of the family $f^*\pi : f^*(E) \to X$ as follows: $f^*(E)$ is the subspace of $(x, e) \in X \times E$ such that $f(x) = \pi(e)$ or in other words: e needs to be in the fiber above $f(x)$ with the obvious projection.

Corollary 1.5. If $g : Z \to Y$ is continuous, there is a natural isomorphism

$$g^* f^*(E) \cong (g \circ f)^*(E).$$

Definition 1.6 (Trivial bundle)**.** If a family $\pi : E \to X$ is isomorphic to a product family $\tilde{\pi} : X \times V \to X$ we call it a **trivial family**.

Definition 1.7 (Vector bundle)**.** If for every $x \in X$ there is a neighborhood U containing x such that $E\Big|_Y := i^*(E)$, where $i : U \hookrightarrow X$ is the inclusion,is trivial we call the family a **vector bundle**. Notice that if $f : Y \to X$ we get the induced vector bundle $f^*(E)$.

Corollary 1.8. Notice how the function $x \mapsto \dim(E_x)$ is locally constant.

Corollary 1.9. Since the topology of the bundle needs to be compatible with the vector space we can conclude that the space of global sections denoted by $\Gamma(X)$ is itself a vector space.

Definition 1.10 (Compact-open topology)**.** Let X, Y be topological spaces. Then the set of continuous maps $C(X,Y)$ can be equipped with the topology that has

$$\{V(K,U) | K \subseteq X \text{ compact }, U \subseteq Y \text{ open}\} \text{ where } V(K,U) := \{f \in C(X,Y) | \ f(K) \subseteq U\}$$

as a subbasis. This coincides with the Topology of $\hom(V,W)$ as a subset of $\mathbb{C}^n$ after identification with the corresponding Matrix.

Definition 1.11 (Homomorphism Bundle)**.** Let W, V be vector spaces and $E = V \times X$,$F = W \times X$ the product bundles. Now any homomorphism $\varphi : E \to F$ determines a map $\Phi : X \to Hom(V \times W)$ by sending $x \mapsto (\varphi_x)$. With respect to the compact open topology, this Φ is continuous. Furthermore, a homomorphism continuous map $\Phi : X \to \hom(V,W)$ determines a homomorphism $\varphi : V \to W$:

Proof. Let $\{e_i\}$ and $\{f_i\}$ be basis of V and W. Then each $\Phi(x)$ is a matrix $\Phi(x)_{i,j}$, such that:

$$\Phi(x)e_i = \sum_j \Phi(x)_{i,j} f_j \, .$$

Now Φ is continuous if and only if $\Phi_{i,j}$ is continuous. Since $\varphi(x,v) = (x, \Phi(x)v)$ this is equivalent to φ being continuous. □

Theorem 1.12 (Isomorphic vector bundles)**.** *Let E and F be vector bundles and $\varphi : E \to F$ be a homomorphism. Then φ is an Isomorphism if and only if all φ_x are isomorphisms.*

Proof. To show that it is an Isomorphism we need an continuous inverse, respectifly we need to show that f is a homeomorphism. First of all we can deduce, that φ is bijective: Assume it wasn't injective, than there would be a fibre in which φ_x is not injective. Assume that it wasn't surjective. Then there would be a $w \in F$ that has no preimage which tells us that in the fiber containing w there is no isomorphism. Hence there is a unique inverse and we need to check if it is continuous, which is a local condition. Therefore we can assume that E and F are product bundles. Then we have a continuous map $\Phi : X \to \hom(V,W)$. Furthermore, since φ_x are Isomorphisms we can restrict the image to $\mathrm{iso}(V,W)$ which is an open subset of $\hom(V,W)$. Now we can define the map $x \mapsto \Phi(x)^{-1}$ which is continuous since taking inverses is continuous. Thereby, we get by the corollary above a continuous map $\psi : F \to W$ by sending $(x,w) \mapsto (x, \Phi(x)^{-1}(W))$, which obviously is the inverse. □

1.2 Operations on vector bundles

Definition 1.13 (Continuous functor)**.** Let T be an covariant endofunctor in the category of finite dimensional vector spaces. We call T a **continuous functor**, if for all V, W the map $T : \hom(V, W) \to \hom(T(V), T(W))$ is continuous. If E is a vector bundle we define the set:

$$T(E) := \bigcup_{x \in X} T(E_x)\,.$$

Now we want to equip this with a topology this such that $T(E)$ is a bundle over the same base space as E.

Definition 1.14 (Constructing new bundles)**.** We start by assuming that $E = X \times V$ in that case we define the topology on $T(E)$ as the product topology $X \times T(V)$. Now suppose that $F = X \times W$ is another product bundle and $f : E \to F$ is a homomorphism. Let $\Phi : X \to \hom(V, W)$ be the corresponding map. Now $T\Phi : X \to \hom(T(V), T(W))$ is continuous by assumption. Thereby, $T(\varphi) : X \times T(V) \to X \times T(W)$ is continuous. If φ is an isomoprhism, then $T(\varphi)$ is also an isomorphism. That is because it is continuous and an isomorphism fiber-wise.

Now assume that E is trivial. Then we can choose an isomophism $E \to X \times V$ and induce a topology on $T(E)$ by pulling it back via $Z(\alpha) : T(E) \to X \times T(V)$. This topology does not depend on α by the above. For general Vector bundles we do this construction locally.

If $f : X \to Y$ is continuous, there is a natural isomorphism

$$f^*T(E) \cong Tf^*(E)$$

Corollary 1.15. We can identify $\Gamma \hom(E, F)$ with $\hom(E, F)$ since for any section $s : X \to \hom(E, F)$ we can define the homomophism $\tilde{s} : E \to F$ by sending $v \mapsto s(v)$. This is a homomorphism since $\tilde{\pi} s(v) = \pi(v)$.

1.3 Sub-bundles and quotient bundles

Definition 1.16 (Sub-bundles)**.** Let $\pi : E \to X$ be a bundle. A **sub-bundle** is a subset of E which is a bundle in the induced topology and vector-structure.

Definition 1.17 (Mono- and epimorphism)**.** A homomorphism $\varphi : E \to F$ is a **monomorphism** is all φ_x are monomorphisms (i.e. injective). Respectively it is called an **epimorphism** if all φ_x are epimorphisms(i.e. surjectiv).

Corollary 1.18. If F is a sub-bundle of E, then the inclusion $\varphi : F \to E$ is a monomorphism.

Lemma 1.19. *If $\varphi : E \to F$ is a monomorphims, then $\varphi(E)$ is a sub-bundle and $\varphi : E \to \varphi(E)$ is an isomorphism.*

Proof. Since $\varphi : E \to \varphi(E)$ is bijective, it is an isomorphism if $\varphi(E)$ is a sub-bundle. Since then, we have isomorphisms fibrewise. (compare 1.12). So we need to check that $\varphi(E)$ is a sub-bundle which is a local property. Thereby we can assume that E, F are product bundles. Define $F = X \times V$ and let $x \in X$, let W_x be a subspace complementary to $\varphi_x(V)$. Then $G = X \times W_x \subseteq F$. Now define

$$\begin{aligned} \theta : E \oplus G &\to F \\ (a, b) &\mapsto \varphi(E) + i(b) \quad \text{where } i\text{- denotes the inclusion } G \hookrightarrow F. \end{aligned}$$

This is an isomorphism and thereby, there exists and open neighbourhood U of x such that $\theta\big|_U$ is an isomorphism. That is because $\mathrm{iso}(V, W)$ is open in $\mathrm{hom}(V, W)$. Since E is a sub-bundle of $E \otimes G$ we have that $\theta(E) = \varphi(E)$ is a sub-bundle of F. □

Corollary 1.20. In the proof, we have shown more: locally a sub-bundle is a direct summand. Furthermore, we have shown, that the set of points x where φ_x is a monomorphims, is open.

Definition 1.21 (Quotient bundle)**.** If F is a sub-bundle of E we define the **quotient bundle** E/F as the union of all spaces E_x/F_x induced with the quotient topology.

Corollary 1.22. The quotient bundle is a bundle because locally F is a direct summand and thereby E/F is locally trivial.

Definition 1.23 (Strict homomorphisms)**.** A homomorphism $\varphi : E \to F$ is a **strict homomorphism**, if the function $x \mapsto \dim(\ker(\varphi_x))$ is locally constant.

Proposition 1.24. If $\varphi : E \to F$ is a strict homomorphism, then

(i) $\ker(\varphi) = \bigcup_x \ker(\varphi_x)$ is a sub-bundle of E,

(ii) $\mathrm{im}\varphi = \bigcup_x \mathrm{im}(\varphi_x)$ is a sub-bundle of F,

(iii) $\mathrm{coker}(\varphi) = \bigcup_x \mathrm{coker}(\varphi_x)$ is a bundle in the quotient structure.

Proof. Obviously, (ii) implies (iii), since the cokernel is defined to be the quotient.

$$F/\mathrm{im}(\varphi)$$

We start by proving (ii): Since the problem is local, we can assume $E = X \times V$ for some V. We then choose W_x to be complementary to $\ker(\varphi_x)$ in V and define $G = X \times W_x$. By φ we get an homomorphism

$$\psi : G \to E$$

such that ψ_x is a monomorphism. Thus, ψ is a monomorphism in some neighbourhood U of x and thereby, we have that $G(G)\Big|_U$ is a sub-bundle of $E\Big|_U$. However, $\psi(G) \subseteq \varphi(E)$ and since $\dim(\varphi(E_y))$ is constant for all y we have:

$$\begin{aligned}\dim(\psi(G_y)) &= \dim(\psi(G_x))\\ &= \dim(\varphi(F_x))\\ &= \dim(\varphi(F_y))\end{aligned}$$

for all $y \in U$, we have the equality: $\psi(G)\Big|_U = \varphi(E)\Big|_U$. Thus, $\varphi(E)$ is a sub-bundle of F.

To show (i), we proceed as follows: since φ is strict, $\varphi^* : F^* \to E^*$ is also. This follows from the connection of the dimensions $\dim(\ker(\varphi_x^*)) = \dim(W) - \dim(V) + \dim(\ker(\varphi_x))$, where W denotes the fibres in the product bundle F. But now

$$E^* \to \operatorname{coker}(\varphi^*)$$

is en epimorphism and thereby

$$\operatorname{coker}(\varphi^*)^* \to E^{**}$$

is a monomorphism. Furthermore, for all x we have the natural commutative diagram:

$$\begin{tikzcd}
\ker(\varphi_x) \arrow[r] \arrow[d, "\cong"] & E_x \arrow[d, "\cong"] \\
\operatorname{coker}(\varphi_x^*)^* \arrow[r, hook] & E_x^{**}
\end{tikzcd}$$

where the verticals are Isomorphisms. But then by lemma 1.19 $\ker(\varphi)$ is a sub-bundle of E^{**} and hence of E. □

Corollary 1.25. Notice that the argument above shows, that around x the rank can only increase, because ψ is a monomorphism. Hence, $\operatorname{rank}(\varphi_x)$ is **upper semi-continuous in** x (even if φ is not strict).

Definition 1.26 (Projection operator)**.** A **projection operator** $P : E \to E$ is a homomorphism such that $P^2 = P$.

Lemma 1.27. *Any projection operator $P : E \to E$ determines a direct sum decomposition $E = (PE) \oplus (1 - P)E$*

Proof. Notice how $1-P$ maps each x to the kernel of P. Furthermore $\operatorname{im}(P)+\operatorname{im}(1-P) = E$. That is because $v = P(v) + (v - P(v))$. Hence, $\operatorname{rank}(P_x) + \operatorname{rank}(1 - P) = \dim(E_x)$. Furthermore, $\operatorname{rank}(P_x)$ and $\operatorname{rank}(1 - P)$ are upper semi-continuous functions in x and thereby locally constant. Finally the above shows that $\ker(P) = (1-P)(E)$ we have the decomposition. □

Definition 1.28 (Metrics on Bundles)**.** A **metric** on a bundle E is a section $h : X \to \text{herm}(E)$ such that $h(x)$ is positive definite for all x. herm(E) denotes the vector space of of hermitian forms (symmetric sesquilinear). A bundle together with a metric is a **Hermitian bundle**.

Lemma 1.29 (Metric induced complement)**.** *Suppose that F is a sub-bundle of E. Then a metric provides a definite complementary sub-bundle.*

Proof. For each x we have the orthonormal projection $P_x : E_x \to F_x$ inducing a map $P : E \to F$. This is continuous: Assume, that F is trivial with the sections $f_1, \cdots, f_n$ such that (f_i) give fiber-wise a orthonormal basis. Then, for $v \in F_x$ we have

$$P_x(v) = \sum_i h_x(v, f_i(x)) f_i(x) .$$

now h is continuous and thereby we have a projection operator. □

1.4 Vector bundles on compact spaces

From now an all our base spaces are **compact Hausdorff**.

Definition 1.30 (Support of a funktion)**.** If $f : X \to V$ is continuous and vector valued we call $\overline{f^{-1}(V \setminus \{0\})}$ the support of f.

We need the following fact:

Fact 1.31 (Tietze Extension Theorem)**.** Let X be a normal space (that is T4, i.e. two disjoint close sets have disjoint open neighborhoods), and $Y \subset X$ be a closed subspace, V a real vector space and $f : Y \to V$ a continuous map. Then there exists a continuous map $g : X \to V$ such that $g\big|_Y = f$.

Fact 1.32 (Existence of Partitions of Unity)**.** Let X be a compact Hausdorff space, $\{U_i\}$ a finite open covering. Then there exist continuous maps $f_i : X \to \mathbb{R}$ such that:

(i) $f_i(x) \geq 0$ for all $x \in X$

(ii) $\text{supp}(f_i) \subseteq U_i$

(iii) $\sum_i f_i(X) = 1$ for all $x \in X$.

Lemma 1.33. *Let X be compact Hausdorff, $Y \subseteq X$ closed subspace, and E a bundle over X. Then any section $Y \to E\big|_Y$ can be extended to X.*

Proof. let $s \in \Gamma(E\big|_Y)$. By the Tietze extension theorem we can deduce, that for each $x \in X$ there exists an open set U containing X and a $t \in \Gamma(E\big|_U)$ such that $t\big|_{U \cap Y} = s\big|_{U \cap Y}$. Now by the compactness we can restrict to a finite sub-covering $\{U_\alpha\}$ of those open sets with the associated sections t_α. Let $\{p_\alpha\}$ be a corresponding partition of unity. Then we define $s_\alpha \in \Gamma(E)$ as :

$$S_\alpha(X) = \begin{cases} p_\alpha(x)t_\alpha(x) & \text{if } x \in U_\alpha \\ 0 & \text{else} \end{cases}$$

Finally the sum $\sum_\alpha S_\alpha$ is a section that restricts to s. □

Lemma 1.34 (Extension of Isomorphisms of Homomorphisms)**.** *Let Y be a closed subspace of a compact Hausdorff space X, and let E, F be two vector bundles over X. If $f : E\big|_Y \to F\big|_Y$ is an isomorphism, then there exists an open set U containing Y and an extension $\tilde{f} : E\big|_U \to F\big|_U$ which is an isomorphism.*

Proof. By the definition 1.11 we can identify f with a local section into $\hom(E, F)$ and by lemma 1.33 extend it to global section. Now let U be the open set of those points for which the map reaches isomorphisms. Then U is open and contains Y. □

Lemma 1.35 (Induced Bundles are Homotopy Invariant)**.** *Let Y be a compact Hausdorff space, $f_t : Y \to X$ $(t \in I := [0,1])$ a homotopy and E a vector bundle over X. Then*

$$f_0^* E \cong f_1^* E\,.$$

Proof. Let $f : Y \times I \to X$ be the homotopy and $\pi : Y \times I \to Y$ be the projection. Now apply 1.34 to the two bundles f^*E and $\pi^* f_t^* E$ for the subspace $Y \times \{t\}$. Here, there is an obvious isomorphism, which thereby is extendable to an open neighborhood $U \supset Y \times \{t\}$. Hence, every point (x, t) has an open neighborhood contained in U and by the product topology this contains again an open neighborhood of (x, t) of the form $V_x \times (t - \delta_x, t + \delta_x)$. Now by the compactness we can restrict to finitely many and thereby the minimum of those finite δ_x is obtained letting us deduce the existence of a δ, such that $U \supseteq U \times (t - \delta, t + \delta)$. Now we compare two induced bundles: $f_t^*(E)$ and $f_{t+\varepsilon}^*(E)$, where $\varepsilon < \delta$. Denote the two inclusions

$$\begin{aligned} i_t : Y &\hookrightarrow Y \times (t - \delta, t + \delta) \subseteq Y \times I \\ y &\mapsto (y, t) \end{aligned}$$

$$\begin{aligned} i_{t+\varepsilon} : Y &\hookrightarrow Y \times (t - \delta, t + \delta) \subseteq Y \times I && (1) \\ y &\mapsto (y, t + \varepsilon) && (2) \end{aligned}$$

With those we have that

$$\begin{aligned} f_{t+\varepsilon}^*(E) &\cong i_{t+\varepsilon}^* f^*(E) \\ &\cong i_{t+\varepsilon}^*(\pi^* f_t^*(E)) \\ &\cong (\pi \circ i_{t+\varepsilon})^* f_t^*(E) \\ &= (\pi \circ i_t)^* f_t^*(E) \\ &= i_t^*(\pi^* f_t^*(E)) \\ &= i_t^*(f^*(E)) \cong f_t^*(E) \end{aligned}$$

Hence, the function $f_t^* E \mapsto [f_t^* E]$ that maps each bundle to its isomorphism class is locally constant and thereby it is constant on the connected I. Hence:

$$f_0^* E = f_1^* E \,.$$

□

Definition 1.36 (The Set of Isomorphism Classes)**.** We define $\mathrm{Vect}(X)$ to be the set of isomorphism classes of vector bundles on X and $\mathrm{Vect}_n(X)$ the set of isomorphism classes of rank n. $\mathrm{Vect}(X)$ is an abelian semi-group under the operation $\oplus$ and in Vect_nthere is a naturally distinguished element: the class of the trivial bundle.

Lemma 1.37. 1. *If $f : X \to Y$ is a homotopy equivalence, $f^* : \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ is bijective.*

2. *If X is contractible, every bundle over X is trivial and* $\mathrm{Vect}(X) \cong \mathbb{N}_0$

Proof. This is an immediate consequence of lemma 1.35. □

Lemma 1.38. *If E is a bundle over $X \times I$, and $\pi : X \times I \to X \times \{0\}$ is the projection, E is isomorphic to $\pi^*(E\big|_{X\times\{0\}})$*

Proof. This follows form $X \times I \simeq X$ □

Definition 1.39 (Trivialisation of a Vector Bundle)**.** Ix Y is a closed subspace of X, E is a vector bundle over X and $\alpha : E\big|_Y \to Y \times V$ is an isomorphism. Then α is a **trivialisation of E over Y**. Furthermore, let $\pi : Y \times V \to V$ denote the projection and define an equivalence relation on $E\big|_Y$ by

$$e \sim e' \;\Leftrightarrow \pi\alpha(e) = \pi\alpha(e') \,.$$

We extend this relation by the identity on $E\big|_{X\setminus Y}$ and we define E/α denote the quotient space of E defined by that relation. This then is a bundle. For the local triviality we need to check only at the base point Y/Y of X/Y. But this is clear since we can extend α to an isomorphism $\tilde{\alpha} : E\big|_U \to U \times V$ via lemma 1.34. But then $\tilde{\alpha}$ induces an isomorphism

$$(E\big|_U)/\alpha \cong (U/Y) \times V$$

proofing the local triviality.

Lemma 1.40. *A trivialization of α of a bundle E over $Y \subseteq X$ defines a bundle E/α over X/Y. The isomorphism class of E/α depends only on the homotopy class of α.*

Proof. The first part is shown in the definition 1.39. Now suppose that α_0 and α_1 are homotopic trivializations of E over Y. This means that we have a trivialization β of $E \times I$ over $Y \times I \subset X \times V$ that induces the α_i at the end points. Let $f : (X/Y) \times I \to (X \times I)/(X \times I)$ be the natural map. Then $f^*((E \times I)/\beta)$ is a bundle on $(X/Y) \times I$ who's restriction to $(X/Y) \times \{i\}$ is E/α_i $(i = 0, 1)$. Hence by lemma 1.35 we have

$$E/\alpha_1 \cong E/\alpha_2$$

□

Lemma 1.41. *Let $Y \subseteq X$ be a closed contractible subspace. Then $f : X \to X/Y$ induces a bijection*

$$f^* : \mathrm{Vect}(X/Y) \to \mathrm{Vect}(X) .$$

Proof. We will construct an inverse of f^* as follows. Let E be a bundle on X. By lemma 1.37 we have that $E\Big|_Y$ is trivial and thereby a trivialization $\alpha : E\Big|_Y \to Y \times V$ exists and two such trivializations differ by an automorphism of $Y \times V$ that can be identified with a map $Y \to \mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$. But since V and $\mathrm{GL}(V)$ are contractible, α is unique up to homotopy and thereby the isomorphism class of $E\Big|_Y$ is determined by E. Thereby, the map

$$\begin{aligned} \mathrm{Vect}(X) &\to \mathrm{Vect}(X/Y) \\ E &\mapsto E\Big|_\alpha \end{aligned}$$

is well defined. Furthermore it is a two-sided inverse and thereby f^* is a bijection. □

Definition 1.42 (Glueing and Clutching of Vector Bundles)**.** Let

$$X = X_1 \cup X_2 , \quad A = X_1 \cap X_1 ,$$

and all spaces be compact. Assume that E_i is a vector bundle over X_i and $\varphi : E_1\Big|_A \to E_2\Big|_A$ is an isomorphism. Then we define an vector bundle $E_1 \cap_\varphi E_2$ on X as follows: We topologize the space $E_1 \cap_\varphi E_2$ as the quotient of $E_1 + E_2/\sim$, where $\sim$ means the identification of points with its image under φ. Identifying X with $X_1 + X_2$ in the quotient space we get the natural projection $p : E_1 \cap_\varphi E_2 \to X$ and the fibers have a natural vector space structure. It remains to show that this is locally trivial. This is obvious outside of A so let $a \in A$ and v_1 be a closed neighbourhood of a in X_1 such that $E_1\Big|_{V_1}$ is trivial giving us the isomorphisms

$$\begin{aligned} \theta_1 : E_1\Big|_{V_1} &\to V_1 \times \mathbb{C}^n \\ \theta_1^A : E_1\Big|_{V_1 \cap A} &\to (V_1 \cap A) \times \mathbb{C}^n \end{aligned}$$

And finally define

$$\theta_2^A : E_2\Big|_{V_1\cap A} \to (V_1 \cap A) \times \mathbb{C}^n$$

by composition with φ.Let V_2 be a neighborhood of a in X_2 such that

$$\theta_2 : E_2\Big|_{V_2} \to V_2 \times \mathbb{C}^n$$

is an extension of θ_2^A. Finally the pair (θ_1, θ_2) defines a well defines isomorphism

$$\theta_1 \cap_\varphi \theta_2 : E_1 \cap_\varphi E_2 \to (V_1 \cap V_2) \times \mathbb{C}^n\,.$$

This proves the local triviality.

Corollary 1.43 (Elementary Properties of the Gluing and Clutching Construction)**.** Elementary properties of the gluing and clutching construction are:

1. If E is a bundle over X, then the identity defines ans isomorphism and

$$E_1 \cup_{\mathrm{id}_A} E_2 \cong E.$$

2. If $\beta_i : E_i \to E_i'$ are isomorphisms on X_i and $\varphi'\beta_1 = \beta_2\varphi$, then

$$E_1 \cup_\varphi E_2 \cong E_1' \cup_{\varphi'} E_2'\,.$$

3. If (E_i, φ) and (E_i', φ') are two "clutching data" on X_i, then

$$(E_1 \cup_\varphi E_2) \oplus (E_1' \cup_{\varphi'} E_2') \cong E_1 \oplus E_1' \bigcup_{\varphi\oplus\varphi'} E_2 \oplus E_2'\,,$$
$$(E_1 \cup_\varphi E_2) \otimes (E_1' \cup_{\varphi'} E_2') \cong E_1 \otimes E_1' \bigcup_{\varphi\otimes\varphi'} E_2 \otimes E_2'\,,$$
$$(E_1 \cup_\varphi E_2)^* \cong E_1^* \bigcup_{(\varphi^*)^{-1}} E_2^*\,.$$

$(-)^*$ denotes the dual here.

Lemma 1.44. *The isomorphism class of $E_1 \cup_\varphi E_2$ only depends on the homotopy class of the isomorphism $\varphi : E_1\big|_A \to E_2\big|_A$.*

Proof. A homotopy of isomorphisms $E_1\big|_A \to E_2\big|_A$ is an isomorphism

$$\Phi : \pi^* E_1\Big|_{A\times I} \to \pi^* E_2\Big|_{A\times I}$$

where $\pi : X \times I \to X$ is the projection. This is just a reformulation of the definition of a homotopy in the category of vector bundles. Denote

$$\begin{aligned} f_t : X &\to X \times I \\ x &\mapsto (x, t)\,, \end{aligned}$$

and

$$\varphi_t : E_1\Big|_A \to E_2\Big|_A$$

be the isomorphism induced from Φ by f_t. Then

$$E_1 \cup_{\varphi_t} E_2 \cong f_t^*(\pi^* E_1 \cup_\Phi \pi^* E_2)\,.$$

Finally f_0 and f_1 are homotopic and therefore by lemma 1.35:

$$E_1 \cup_{\varphi_0} E_2 \cong E_1 \cup_{\varphi_1} E_2$$

□

Definition 1.45 (Homotopy Classes of Maps)**.** We denote by $[X, Y]$ the homotopy class of maps $X \to Y$.

Definition 1.46 (Suspension of Topological Spaces)**.** For a Topological space X we denote the **suspension** of X by

$$S(X) := (X \times [0,1])/(X \times \{0\})\big/(X \times \{1\})$$

Lemma 1.47. *For any X, there is a canonical bijection*

$$\mathrm{Vect}_n(S(X)) \cong [X, \mathrm{GL}(n, \mathbb{C})]$$

Proof. First we write

$$S(X) = C^+(X) \cap C^-(X)\,,$$

where

$$C^+(X) = ([0, \frac{1}{2}] \times X)/(\{0\} \times X)\,,$$
$$C^-(X) = ([\frac{1}{2}, 1] \times X)/(\{1\} \times X)\,.$$

Then $C^+(X) \cap C^-(X) = X$. If E is any n–dimensional bundle over $S(X)$ then $E\Big|_{C^+(X)}$ and $E\Big|_{C^-(X)}$ are trivial (by the contractability of the base space). Denote with

$$\alpha^\pm : E\Big|_{C^\pm(X)} \to C^\pm(X) \times V$$

such isomorphisms. Then $\alpha^+\Big|_X \circ (\alpha^-\Big|_X)^{-1}$ is a bundle map from the product bundle $X \times V$ to itself and can be identified with a map $\alpha : X \to \mathrm{iso}(V) = \mathrm{GL}(n, \mathbb{C})$. This construction is independent from the choice of $\alpha^{\pm}$ (after mapping it to its homotopy class), because the homotopie classes of $\alpha^{\pm}$ are unique. Hence we have a natural map

$$\theta : \mathrm{Vect}_n(S(X)) \to [X, \mathrm{GL}(n, \mathbb{C})]$$

The trivial bundle gets mapped to the class of the trivial map. The clutching construction from definition 1.42 gives an inverse of this by identifying an $\alpha : X \to \mathrm{GL}(n, \mathbb{C})$ with an homeomorphism $X \times V \to X \times V$. Then we can glue the two sides of the suspension together. Clearly those maps are inverse to each other. □

Lemma 1.48. *Let E be any bundle over X. Then there exists a (Hermitian) metric on E*

Proof. A Metric on a Vector space V defines a metric on the product bundle $X \times V$ by the constant section. Now let $\{U_\alpha\}$ be a finite open cover of X such that $E\Big|_{U_\alpha}$ is trivial, and let h_α be a metric for $E\Big|_{U_\alpha}$. Let $\{p_\alpha\}$ be a partition of unity with $\mathrm{supp}(p_\alpha) \subset U_\alpha$. Define

$$k_\alpha(x) = \begin{cases} p_\alpha(X) h_\alpha(x) & \text{if } x \in U_\alpha\,, \\ 0 & \text{else }\,. \end{cases}$$

Finally, define $k = \sum_\alpha p_\alpha$. Since k_α is positive semi-definite and by the properties of a partition of unity, k is positive definite. □

Definition 1.49. A sequence of vector bundle homomorphisms

$$\longrightarrow E \longrightarrow F \longrightarrow$$

is called exact, if it is fiber-wise exact.

Corollary 1.50. If $0 \longrightarrow E' \xrightarrow{\varphi'} F \xrightarrow{\varphi''} E'' \longrightarrow 0$ is exact, then there is an isomorphism

$$E \cong E' \oplus E''\,.$$

Proof. If we give E a metric we have an isomorphism $E \cong E' \oplus (E')^\perp$. However, $(E')^\perp \cong E''$. □

Definition 1.51 (Ample Subspace)**.** A subspace $V \subseteq \Gamma(E)$ is said to be **ample** if

$$\begin{aligned} \varphi : X \times V &\to E \\ (x, s) &\mapsto s(x) \end{aligned}$$

is a surjection.

Lemma 1.52. *If E is any bundle over a compact Hausdorff space X, then $\Gamma(E)$ contains a finite dimensional ample subspace.*

Proof. let $\{U_\alpha$ be a finite open covering of X such that $E\big|_{U_\alpha}$ is trivial for which α and let $\{p_\alpha\}$ be a partition of unity with $\operatorname{supp} p_\alpha \subseteq U_\alpha$. Since $E\big|_{U_\alpha}$ is trivial we can find a finite-dimensional ample subspace $V_\alpha \subseteq \Gamma(E\big|_{U_\alpha})$. This could for example be the subspace generated by the $e_i : X \to V$ that send x to (x, e_i). Now define

$$\begin{aligned} \theta_\alpha : V_\alpha &\to \Gamma(E) \\ v_\alpha &\mapsto \theta_\alpha(v_\alpha) \end{aligned}$$

such that

$$\theta_\alpha(v_\alpha)(x) = \begin{cases} p_\alpha(x)v_\alpha(x) & \text{if } x \in U_\alpha, \\ 0 & \text{else.} \end{cases}$$

Finally the maps θ_α define a homomorphism

$$\theta : \prod_\alpha V_\alpha \to \Gamma(E)$$

and the image of θ is a finite dimension al subspace of $\Gamma(E)$. Furthermore, for each $x \in X$ there is an α such that $p_\alpha(X) > 0$. Hence, the map:

$$\theta_\alpha(V_\alpha) \to E_x$$

is surjective. □

Corollary 1.53. If E is any bundle, there exists an epimorphism $\varphi : X \times \mathbb{C}^m \to E$ for some $m \in \mathbb{N}$.

Proof. This is just lemma 1.52. □

Corollary 1.54. If E is any bundle, there exists a bundle F such that $E \oplus F$ is trivial.

Proof. Consider the exact sequence $0 \longrightarrow \ker(\varphi) \xrightarrow{i} X \times \mathbb{C}^m \xrightarrow{\varphi} E \longrightarrow 0$ By corollary 1.50 the statement is proven. □

Definition 1.55 (Grassmann Manifold)**.** If V is any vector space and n any integer, the set $\mathrm{G}_n(V)$ is the set of all subspaces of V of **codimension** n. If V is given a Hermitian metric, each element of $\mathrm{G}_n(V)$ defines a projection operator. Hence, we have a map $\mathrm{G}_n(V) \to \operatorname{End}(V)$, where the latter denotes the set of endomorphisms on V. This gives $G_n(V)$ its topology.

Suppose that E is a bundle over a space X, V is a vector space, and $\varphi : X \times V \to E$ an epimorphism. The map

$$\begin{aligned} \Phi : X &\to \mathrm{G}_n(V) \\ x &\mapsto \ker(\varphi_x) \end{aligned}$$

is called the induced map by φ, where n is the dimension of E. This map is continuous for any metric on V.

Definition 1.56 (Classifying Bundle over $\mathrm{G}_n(V)$)**.** Let V be a vector space and let $F \subseteq \mathrm{G}_n(V) \times V$ be the sub-bundle consisting of all points (g, v) such that $v \in g$. We call this the **tautological bundle**. Now define

$$E := (\mathrm{G}_n(V) \times V)/F\,.$$

Then E is called the **classifying Bundle over** $G_n(V)$.

Corollary 1.57. If E' is a bundle over X and $\varphi : X \times V \to E'$ an epimorphism, then if $\Phi : X \to \mathrm{G}_n(V)$ is the map induced by φ, we have

$$E' \cong \Phi^*(E)$$

where E is the classifying bundle.

Proof. To see this we show that we have an isomorphism, i.e. a homeomorphism that is fiber-wise an isomorphism. So take $x \in X$. Then:

$$\Phi^*(E)_x = E_{\Phi(x)} = E_{\ker(\varphi_x)} = V/\ker(\varphi_x) \cong \mathrm{im}(\varphi_x) = E'_x$$

Hence fiber-wise this is an isomorphism, that smoothly depends on x. To see that this is a homeomorphism we just need to write it down. Dentote with g_x the isomorphism in the fiber over x. Then the map is:

$$\begin{aligned} E' &\to \Phi^*(E) \\ e &\mapsto (\pi(e), g_{\pi(e)}(e)) \end{aligned}$$

□

Lemma 1.58. *The topology on* $\mathrm{G}_n(V)$ *does not depend on the metric on* V

Proof. Suppose that h and h' are two metrics on V. Let $\mathrm{G}_n(V_h)$ be the set $\mathrm{G}_n(V)$ with the topology induced by h. We then have the epimorphism $\mathrm{G}_n(V_h) \times V \to E$, where E is the classifying bundle. This induces the identity map $\mathrm{G}_n(V_h) \to \mathrm{G}_n(V_{h'})$, which is continuous and thereby the topology does not depend on the metric. □

Definition 1.59. We start with the natural projections onto the first $m-1$ factors

$$\begin{aligned} \pi_m : \mathbb{C}^m &\to \mathbb{C}^{m-1} \\ (z_1, \ldots, z_m) &\mapsto (z_1, \ldots, z_{m-1})\,. \end{aligned}$$

These induce continuous maps

$$\begin{aligned} i_{m-1} : \mathrm{G}_n(\mathbb{C}^{m-1}) &\to \mathrm{G}_n(\mathbb{C}^m) \\ w &\mapsto \pi_m(w)^{-1} \cong w \oplus \underbrace{\langle z_m \rangle}_{:=\mathbb{C}_m}\,. \end{aligned}$$

Define E_m to be the classifying bundle over $\mathrm{G}_n(\mathbb{C}^m)$, then

$$i^*_{m-1}(E_m) \cong E_{m-1}$$

This can be easyly seen, as fiberwise:

$$i^*_{m-1}(E_m)_x = (E_m)_{i_{m-1}(x)} = \left(\mathbb{C}^{m-1} \oplus \mathbb{C}_m\right)/\left(x \oplus \mathbb{C}_m\right) \cong \left(C^{m-1}\right)/x = (E_{m-1})_x$$

Theorem 1.60. *The map*

$$\varinjlim_{m} [X, G_n(\mathbb{C}^m)] \to \mathrm{Vect}_n(X)$$

induced by $f \mapsto f^*(E_m)$ is an isomorphism for all compact Hausdorff spaces X.

Proof. We prove this statement by constructing an inverse map. So we start with a vector bundle E over X. Then there exist an m such that $X \times \mathbb{C}^m \to E$ is an epimorphism. Let $\Phi : X \to E_m$ be the induced map by φ. Now we need to show, that there is a m large enough, such that the homotopy class of Φ does not depend on the choice of φ.

Suppose that $\varphi_i : X \times \mathbb{C}^{m_i} \to E$ are two epimorphisms for $i = 0, 1$. Let $\Phi_i : X \to \mathrm{G}_n(\mathbb{C}^{m_i})$ be the induced maps. Now define

$$\begin{aligned} \psi_t : X \times \mathbb{C}^{m_0} \times \mathbb{C}^{m_1} &\to E \\ (x, v_0, v_1) &\mapsto (1-t)\varphi_0(x, v_0) + t\varphi_1(x, v_1)\,. \end{aligned}$$

This is again an epimorphism for all t. After identify $\mathbb{C}^{m_0} \oplus \mathbb{C}^{m_1} = \mathbb{C}^{m_0+m_1}$ via $v_0 + v_1 \mapsto (v_0, v_1)$ then

$$f_0 = j_0\Phi_0 \quad , \quad f_1 = Tj_1\Phi_1\,,$$

where

$$j_i : \mathrm{G}_n(\mathbb{C}^{m_i}) \to \mathrm{G}_n(\mathbb{C}^{m_0+m_1})$$

is the inclusion and

$$T : \mathrm{G}_n(\mathbb{C}^{m_0+m_1}) \to \mathrm{G}_n(\mathbb{C}^{m_0+m_1})$$

is the map permuting the coordinates and thereby homotopic to the identity. Hence,

$$j_1\Phi_1 \simeq f_1 \simeq f_0 = j_0\Phi_0.$$

By definition of the direct limit this concludes the independence of the choice. □

Lemma 1.61 (Tensor Produkt)**.** *Let V denote a finite dimensinoal $\mathbb{C}$−vector space and X a topological space. With V^X we denote the vector space of smooth functions from X to V and with $C(X)$ the space of complex valued smooth functions on X. for this we have a natural isomorphism*

$$V^X \to C(X) \otimes V$$

Proof. We will make use of the universal property of tensor products as follows: First define

$$\begin{aligned} \varphi : C(X) \times V &\to V^X \\ (f, v) &\mapsto (x \mapsto f(x) \cdot v) \end{aligned}$$

By the universal property we get a commutative diagram:

$$\begin{tikzcd} C(X)\times V \arrow[r] \arrow[dr, "\varphi"'] & C(X)\otimes V \arrow[d, "\tilde{\varphi}"] \\ & V^X \end{tikzcd}$$

By the uniqueness we have:

$$\begin{aligned} \tilde{\varphi} : C(X)\otimes V &\to V^X \\ \sum_i (f_i \otimes v_i) &\mapsto \left(x \mapsto \sum_i f_i(x)\cdot v_i\right) \end{aligned}$$

Now this is an isomorphism since it is bijective.

□

Remark 1.62. Let $C(X)$ denote the ring of complex-valued functions on X (with point-wise addition and multiplication). If E is a vector bundle, then $\Gamma(E)$ is a $C(X)$–module under point-wise multiplication. Moreover, a homomorphism $\varphi : E \to F$ determines a $C(X)$–module homomorphism

$$\begin{aligned} \Gamma(\varphi) : \Gamma(E) &\to \Gamma(F) \\ s &\mapsto \varphi \circ s \end{aligned}$$

Hence, Γ denotes a Functor from the category of vector bundles over X denoted with $\mathbf{Vect}_X$ and the category of $C(X)$–modules denoted with $\mathbf{Mod}_{C(X)}$ If E is trivial of dimension n, then $\Gamma(E)$ is free of rank n. If F is also trivial, then

$$\Gamma\, : \mathrm{hom}(E,F) \to \mathrm{hom}_{C(X)}(\Gamma(E),\Gamma(F))$$

is a bijection. After choosing two isomorphisms

$$E \cong X\times Y \quad , \quad F \cong X \times W$$

we have:

$$\begin{aligned} \mathrm{hom}(E,F) \cong \mathrm{hom}_{\mathbb{C}}(V,W)^X &\cong C(X)\otimes \mathrm{hom}_{\mathbb{C}}(V,W) \\ &\cong \mathrm{hom}_{C(X)}(\Gamma(V),\Gamma(W))\,. \end{aligned}$$

where $\mathrm{hom}_{\mathbb{C}}(V,W)^X$. Thus we have an fully faithful functor from the category of free vector bundles to the category of finitely generated free $C(X)$–modules. Now we need to check for essentially surjectivity, i.e. that every free $C(X)$–module is isomorphic to $\Gamma(E)$ for some vector bundle E. Such a module is always isomorphic to $C(X)^n$ and this is (by the product topology of $\mathbb{C}^n$ isomorphic to the set $C(x,\mathbb{C}^n)$ as a $\mathbb{C}$ module. Finally, the latter is $\Gamma(X\times\mathbb{C}^m)$. Now we want to get this equivalence down to get:

Theorem 1.63. *There is an equivalence between the category of finitely generated projective modules over $C(X)$ and the category of vector bundles over X induced by Γ.*

$$\mathbf{Pmod}(C(X)) \sim \mathbf{Vect}(X)$$

Proof. □ fehlt noch

1.5 Additional Strucures

Definition 1.64 (Bilinear Form)**.** Let V be a vector bundle. An element $T \in \hom(V \otimes V, 1)$ is a **billinear form**. If T induces a non-degenerate $T_x \in \hom(V_x \otimes V_x, \mathbb{C})$ for all X we call T **non-degenerate bilinear form on** X. Alternatively we can think of T as an element of $\mathrm{iso}(V, V^*)$. A pair (V, T) will be called a self-dual bundle.

Definition 1.65. If $T \in \hom(V \otimes V, 1)$ is symmetric, i.e. T_x is symmetric we call (V, T) **orthogonal**. If T_x is skew symmetric we call it a symplectic bundle.

Definition 1.66. If $T \in \mathrm{iso}(V, \overline{V})$, where $\overline{V}$ denotes the complex conjugate bundle. We call such a bundle a **self conjugate bundle**. We can think of T as anti-linear. If $T^2 = \mathrm{id}$ we call (V, T) a **real** bundle. The subbundle $\mathrm{aut}(T) \subset V$ has the structure of a real vector bundle and We can identify

$$\mathrm{aut}(T) \otimes_{\mathbb{R}} \mathbb{C} \cong V\,.$$

If $T^2 = -\mathrm{id}$ we call (V, T) a **quaternion** bundle. We can define a quaternion vector space structure on each V_x by putting $j(v) = Tv$.

Lemma 1.67. *If V has a hermitian metric h we get an isomorphism $\overline{V} \to V^*$ and hence we can turn self-conjugate bundles into self dual bundles. The same works for*

1. *orthogonal and real*
2. *symplectic and quaternion.*

Definition 1.68. Let F, G be two continuous functors on the category of vector spaces. Then by an $F \to G$ bundle we mean a pair (V, T) where V is a vector bundle and $T \in \mathrm{iso}(F(V), G(V))$. If

$$F(V) = G(V) = V \otimes V \otimes \cdots \otimes V \quad m \text{ times}$$

We call $m \to m$ an $m-$bundle and $t \in \mathrm{fix}(mV)$. But be careful! The lemma 1.35 does not hold for general $F \to G$ bundles. Hence, we need to redefine what "isomorphic" means for $F \to G$ bundles.

Remark 1.69. A $m-$bundle should be thought of as a **mod m vector bundle** over $S(X)$.

ich hab keinen plan warum...

1.6 G-bundles over G-spaces

Definition 1.70 (G-space)**.** Let G be a topological group. By $G-$**space** we mean a topological space X together with a continuous action of G on X.

A $G-$**map** between two $G-$spaces is a map commuting with the action of G. Finally, a $G-$space E is a $G-$**vector bundle** over the $G-$space X if:

(i) E is a vector bundle over X,

(ii) the projection $E \to X$ is a G–map

(iii) for each $g \in G$ the map $E_x \to E_{g(x)}$ is a vector space homomorphism.

Example 1.71. If G is the trivial group, then every vector bundle is a G–vector bundle.

If X is one point, then X is always a G–space and a G–vector bundle is a finite dimensional representation of G, since there is a map $g \mapsto \hom(V, V)$. Thereby, if we view G–modules as Modules over the group G with the Abelien group V carrying a vector space structure we gained a generalization of G–modules and vector bundles. Due to technical simplification, we restrict to G being a **finite group**.

Definition 1.72 (Extreme Kindes of G-spaces)**.** There are two special or extreme kinds of G–spaces:

(i) X is a **free** G–space if: $g \neq 1 \Rightarrow g(X) \neq x$

(ii) X is a **trivial** G–space if $g(X) = x$ for all $x \in X$ and $g \in G$

Corollary 1.73. Suppose that E is a G–vector bundle over X. If X is a free G–space and define X/G to be its orbit space. Then $\pi : X \to X/G$ is a finite covering map.

Furthermore, E is necessarily a free G–space because if $g(e) = e$ for some $e \in E$, then $\pi(g(e)) = \pi(e)$ which is equivalent to $g(\pi(e)) = \pi(e)$ and the freeness of X concludes that $g = 1$. The orbit space G/E has a natural vector bundle structure over X/G. Which is well,defined by the second condition of a G–vector bundle and the locally freenes comes from the fact, that $E/G \to X/G$ is locally isomorphic to $E \to X$. Conversely, if V is a vector bundle over X, then $\pi * (E)$, where $\pi : E \to E/G$, is a vector bundle over X. Hence:

Proposition 1.74. If X is a G–free, vector bundles over X correspond bijectively to vector bundles over X/G by

$$E \mapsto E/G$$

Fact 1.75. Any finite representation is the direct sum of irreducible representations. Hence is V is a representation of G, we have a unique (up to order) decomposition:

$$V \cong \sum_{i=1}^{k} n_i V_i \,.$$

Corollary 1.76. Now for any two G−modules (i.e. representation spaces of G) V, W we have the vector space $\hom_G(V, W)$ of G−homomorphisms. We then have:

$$\hom_G(V_i, V_j) = \begin{cases} 0 & i \neq 0, \\ \mathbb{C} & i = j \,. \end{cases}$$

Hence, for any V we have that the natural map

$$\sum V_i \otimes \hom_G(V_i, V) \to V$$

is a G−isomorphism. Now assume that E is any G−bundle over the trivial G−space X. Then we can define the homomorphism $\mathrm{Av} \in \mathrm{End}(E)$ by

$$\mathrm{Av}(e) := \frac{1}{|G|} \sum_{g \in G} g(e) \,.$$

This is a Projection operator, as:

$$\mathrm{Av}\mathrm{Av}(e) = \frac{1}{|G|^2} \sum_{g_1 \in G, g_2 \in G} g_1(g_2(e))$$

But since for a fixed g_1 the map $g \mapsto g_1 g$ is a bijection in G we have:

$$\frac{1}{|G|^2} \sum_{g_1 \in G, g_2 \in G} g_1(g_2(e)) = \frac{1}{|G|^2} |G| \sum_{g \in G} g(e) = \mathrm{Av}(e) \,.$$

Hence the image of Av is a sub-bundle denoted by E^G.

2 K-Theory

Definition 2.1 (Universal property of the Grothendieck Group)**.** If we have any Abelian semigroup $(H, +)$ with 0 we define the Grothendieck Group $\mathcal{G}(H)$ together with a semigroup homomorphism $\alpha : H \to \mathcal{G}(H)$ to be the Abelian Group satisfying the following universal condition: If G is any Abelian group with a semigroup homomorphism $\beta : H \to G$ there is a unique homomorphism $\kappa : \mathcal{G}(H) \to G$ such that $\beta = \kappa\alpha$.

$$\begin{tikzcd} H \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathcal{G}(H) \arrow[d, "\kappa"] \\ & G \end{tikzcd}$$

Remark 2.2. All works also, if H does not contain a 0 but we don't need that case here.

Remark 2.3. The Grothendiek Group exists and is unique up to isomorphism.

Proof. The uniqueness is clear by yoga. The existence can be proven by the following two ways:

1. We take the quotient $\mathrm{F}(H)/\mathrm{E}(H)$, where F denotes the free Abelian group with $+$ and $\mathrm{E}(H)$ is the group generated by elements of the form $a + a' - (a \oplus a')$ where $\oplus$ is the addition in H.
2. Denote $\Delta : H \to H \times H$ be the diagonal homomorphism of semigroups, and let $\mathrm{K}(H)$ be the cosets of $\Delta(H)$ in $H \times H$. Then $\mathrm{K}(A) = H \times H/\Delta(H)$ is a semigroup. But interchanging the factors induces an inverse, and hence it is a group.

Whilst both constructions work, we continue our inspection of the second. Denote $\alpha_H : H \to \mathrm{K}(H)$ to be the composition

$$a \mapsto (a, 0) \mapsto (a, 0) + \Delta(H)\,.$$

Now assume $\beta : H \to H'$ is a semigroup homomorphism. We then have the commutative diagram

$$\begin{tikzcd}
H \arrow[r, "\mathrm{id}\times 0"] \arrow[d, "\beta"] & H \times H \arrow[r, "\pi"] \arrow[d, "\beta\times\beta"'] \arrow[dr, "\pi'\circ(\beta\times\beta)"] & \mathrm{K}(H) \arrow[d, "\mathrm{K}(\beta)"] \\
H' \arrow[r, "\mathrm{id}\times 0"] & H' \times H' \arrow[r, "\pi'"] & \mathrm{K}(H')
\end{tikzcd}$$

where $\mathrm{K}(\beta)$ is the map induced by $\pi' \circ (\beta \times \beta)$ via the universal property of quotients. Hence, K is a functor and if H' is an Abelian group, we get the universal property, because $\alpha_{H'}$ is the identity. ☐

Remark 2.4. Notice if H is a semi-ring (a semigroup with an associative and distributive multiplication) the result gives a ring.

Definition 2.5 (K-Group)**.** For any topological space X we have the semigroup $(\mathrm{Vect}(X), \oplus)$ with the trivial bundle being the neutral element. We define the K−Group of X to be $\mathrm{K}(\mathrm{Vect}(X))$ and write just $\mathrm{K}(X)$. Every element of $\mathrm{K}(X)$ is of the form

$$\overline{(E,F)} = \overline{(E,0)} + \overline{(0,F)} = \overline{E} - \overline{F}$$

Theorem 2.6 (Additivity)**.** *Let $X \sqcup Y$ be a disjoint sum. Then we have a natural isomorphism*

$$K(X) \oplus K(Y) \to K(X \sqcup Y)$$

Proof. Let $\overline{(E_X, F_X)} \in K(X)$. We identify the bundle E_X over X with the bundle E over $X \sqcup Y$ that satisfies $E\big|_X = E_X$ and $E\big|_Y = 0$. Similar, we let $\overline{(E_Y, F_Y)} \in K(Y)$. With the identification above we construct the map

$$\Phi : K(X) \oplus K(Y) \to K(X \sqcup Y)$$
$$\left(\overline{(E_X, F_X)}, \overline{(E_Y, F_Y)}\right) \mapsto \overline{\left(E_X \oplus E_Y, F_X \oplus F_Y\right)}$$

Now subjectivity and infectivity are just an easy calculation. ☐

Remark 2.7. Let E, F and G be vector bundles such that $F \oplus G$ is trivial. We write $\underline{n}$ for the trivial $n-$bundle. Then

$$\overline{E} - \overline{F} = \overline{E} + \overline{G} - (\overline{F} + \overline{G}) = \overline{E \oplus F} - \overline{\underline{n}}$$

Hence every element if $\mathrm{K}(X)$ is of the form $\overline{F} - \overline{\underline{n}}$

Remark 2.8. Suppose that $\overline{E} = \overline{F}$ for two bundles E, F. Then there is a bundle G such that $E \oplus G \cong F \oplus G$ because it follows immediately, that there exists a G such that:

$$(E, 0) + (G, G) = (F, 0)$$

But since $(G, G) = (G, 0) - (G, 0)$ we can conclude the statement. Now define G' such that $G \oplus G' = \underline{n}$. Then we can deduce that $E \oplus \underline{n} = F \oplus \underline{n}$. This motivates the definition:

Definition 2.9 (Stably Equivalent Bundles)**.** We call two bundles E, F **stably equivalent** if there exists a trivial bundle, such that the addition with it makes them equivalent. It follows that $\overline{E} = \overline{F}$ if and only if E and F are stably equivalent.

Definition 2.10. We get a directed set by the map $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X)$ by adding a trivial line bundle. And with this we can define the direct limit

$$\varinjlim_n \mathrm{Vect}_n(X)\,.$$

Lemma 2.11. *For a compact connected space X:*

$$\mathrm{K}(X) \cong \mathbb{Z} \times \varinjlim_n \mathrm{Vect}_n(X)$$
$$\overline{F} - \overline{\underline{n}} \mapsto (n, F)$$

Proof. First we check for well definition. If $\overline{F} - \overline{\underline{n}} = \overline{E} - \overline{\underline{m}}$ and $n \geq m$ we have $\overline{F} = \overline{E} + \overline{\underline{n-m}}$ and thereby E and F are stably equivalent. Now the semigroup structure is trivial, just a matter of commutativity. The map is bijective by definition 2.9. ☐

Now we want to give a homotopy-theoretic interpretation of $\mathrm{K}(X)$. Suppose $f : X \to Y$ is continuous, then the map $f^* : \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ induce a homomorphism $\mathrm{K}(Y) \to \mathrm{K}(X)$ that only depends on the homotopy class of f.

3 Cohomology theory properties of K

First we start by purely formal statements for which we need a few definitions:

Definition 3.1. We need the following categories:

- **TopC** is the category of compact spaces.

- $\mathbf{TopC^0}$ is the category of pointed compact spaces.
- $\mathbf{TopC^2}$ is the category of pairs of compact spaces.

Here we have the natural functors:

- $\mathbf{TopC^2} \to \mathbf{TopC^0}$ by sending a pair $(X,Y) \mapsto (X/Y,[Y])$ with obvious morphisms. If $Y = \varnothing$ this is the disjoint union of X with a point.
- $\mathbf{TopC} \to \mathbf{TopC^2}$ by sending $X \mapsto (X,\varnothing)$ again with obvious morphisms.
- $\mathbf{TopC} \to \mathbf{TopC^0}$ as the composition.

If X is in $\mathbf{TopC^0}$ with base-point x_0 we define $\tilde{K}(X)$ as the kernel of the map $i^* : K(X) \to K(x_0)$, where i is the inclusion of the base-point.

Corollary 3.2. If $c : X \to x_0$ is the collapsing map sending each point to the base-point, then we get a splitting

$$K(X) \cong \tilde{K}(X) \oplus K(x_0)\,.$$

Proof. We have the exact sequence of Rings:

$$0 \longrightarrow \tilde{K}(X) \xrightarrow{inclusion} K(X) \xrightarrow{i^*} K(x_0) \longrightarrow 0$$

where $c^* : K(x_0) \to K(X)$ is a right inverse of i^*, and hence the sequence splits and $K(X) \cong \tilde{K}(X) \oplus K(x_0)$. The splitting is natural with respect to maps in $\mathbf{TopC^0}$, because for a map of pointed spaces $f : X \to Y$ we get commutativity:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{K}(X) \arrow[r, hook] & K(X) \arrow[r, "i^*"'] & K(x_0) \arrow[r] \arrow[l, bend right, "c_X^*"'] & 0 \\
0 \arrow[r] & \tilde{K}(Y) \arrow[r, hook] & K(Y) \arrow[r] \arrow[u, "f^*"] & K(y_0) \arrow[r] \arrow[u, "(f|_{x_0})^*"'] \arrow[l, bend left, "c_Y^*"] & 0
\end{tikzcd}$$

So let $k \in K(y_0)$. Then since K is a contravariant functor we know that

$$c_X^*(f\Big|_{x_0})^*(k) = (f\Big|_{x_0} \circ c_x)^*(k)\,.$$

But since pointed maps send basepoints we have that $f\Big|_{x_0} \circ c_x = C_y \circ f\Big|_{x_0}$ and hence we can conclude the commutativity. □

Corollary 3.3. By the above we can conclude that $\tilde{K}$ is a functor from $\mathbf{TopC^0}$ to $\mathbf{Ring}$.

Proof. The proof is somewhat fundamental.First we notice that we have a funcor from $\mathbf{TopC^0}$ to the category of short exact sequences of rings. Since the right inverse respectifly the splitting is natural by the above, we have a functor to the short exact sequences together with right inverses. Here we have an endofuncor that sends each such sequence to a splitted one:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "id"] & B \arrow[r, "g"] \arrow[d, "\Phi"] & C \arrow[r] \arrow[d, "id"] \arrow[l, bend right, "r"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "\mathrm{id}\oplus 0"] & A \oplus C \arrow[r, "\pi_2"] & C \arrow[r] \arrow[l, bend left, "0\oplus id"] & 0 \end{tikzcd}$$

Where Φ is constructed as follows: first we get a left inverse of f given by $l(b) = f^{-1}(b - r \circ g(b))$ and with this we define $\Phi(b) = l(b) \oplus g(b)$. Finally, we have a functor that sends such a sequence to its middle therm. □

Corollary 3.4. If $X \in \mathbf{TopC}$, we define $\tilde{K}(X)$ to be the composition of $\tilde{K}$ with the natural map from $\mathbf{TopC} \to \mathbf{TopC^0}$. Then, since $K(\varnothing) = 0$ we have $K(X) \cong \tilde{K}(X^0)$ where X^0 denotes the pointed X.

Definition 3.5. For (X, Y) in $\mathbf{TopC^2}$ we define $K(X, Y) := \tilde{K}(X/Y)$. This gives a contravariant functor.

Corollary 3.6. The pointed sum $\bigvee$ denotes the coproduct in the categorie of pointed topological spaces.

Proof. Let (X, x_0) and (Y, y_0) be pointed topological spaces. Then $(X \vee Y)$ together with the inclusions

$$i_X : X \to (X \vee Y)$$
$$i_Y : X \to (X \vee Y)$$

satisfies that for any other (Z, z_0) with two cpointed continouos maps

$$f_X : X \to Z$$
$$f_Y : Y \to Z$$

there exists exactly one pointet continouos map

$$f_X \vee f_Y : X \vee Y \to Z$$

such that

$$f_X \vee f_Y \circ i_X = f_X \quad \text{and} \quad f_X \vee f_Y \circ i_Y = f_Y$$

. The existence of this function is clear and the uniquness also. □

Definition 3.7 (Smash Product)**.** First we define the wedge product in $\mathbf{TopC^0}$ as follows: For X, Y we define $X \vee Y := X \cup Y/\sim$ where $\sim$ identifies the base points. With this we define the smash product as

$$X \wedge Y := (X \times Y)/(X \vee Y)$$

Proposition 3.8. For pointed spaces X, Y, Z we have natural isomorphisms:

$$X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$$

From now on, we can view the spheres as pointed spaces by taking the base point to be the one.

Corollary 3.9. Obviously, $S^n \cong S^1 \wedge \cdots \wedge S^1$ n−times. To see this, just realize that $S^n = I^n/\partial$.

Definition 3.10. For a pointed space X, we define $S^1 \vee X$ to be the reduced suspension and the n-th iterated suspension is naturally isomorphic to $S^n \vee X$ and we write $S^n X$. Furthermore, by mapping f to $\overline{\mathrm{id} \times f}$ we get an endofuncor in $\mathbf{TopC^0}$.

Definition 3.11 (Lower K-groups)**.** For $n \geq 0$ we define:

$$\begin{array}{lcll} \tilde{K}^{-n}(X,p) & = & \tilde{K}(S^n X) & \text{für} \quad (X,p) \in Ob(\mathbf{TopC^0}), \\ K^{-n}(X,Y) & = & \tilde{K}^{-n}(X/Y,[Y]) & \text{für} \quad (X,Y) \in Ob(\mathbf{TopC^2}) \\ K^{-n}(X) & = & K^{-n}(X,\varnothing) & \text{für} \quad X \in Ob(\mathbf{TopC})\ . \end{array}$$

All those definitions give contravariant functors into the category of Rings that factor over the category **hTopC**.

Definition 3.12. For $X \in Ob(\mathbf{TopC})$ we define

$$CX := (I \times X)\big/(\{0\} \times X)$$

and for $f \in \hom(X,Y)$ we define

$$\begin{aligned} Cf : CX &\to CY \\ \overline{(t,x)} &\mapsto \overline{(t,f(x))} \end{aligned}$$

This gives a functor from **TopC** to $\mathbf{TopC^0}$. We define the **unreduced suspension of X** to be $CX/X := CX/\{1\} \times X$.

Corollary 3.13. Notice, that we can include $I \hookrightarrow XC/X$ and by this identification if we collapse I to a point we get $(CX/X)\big/I = SX$. But then since $SX \simeq CX/X$ we have an isomorphism between $\tilde{K}(SX) \cong K(CX,X)$. This is because the latter is by definition $\tilde{K}(CX/X)$ and since $\tilde{K}$ factors over **hTopC** we are done. hence, the use of SX for the reduced and unreduced suspension leads to no problem.

Corollary 3.14. If $(X,Y) \in ob(\mathbf{TopC^2})$ we define $X \cup CY$ to be

$$X \sqcup CY\big/(Y \sim \{1\} \times Y) \in Ob(\mathbf{TopC^0})\,.$$

Notice that there is a natural (in the sence that this is a natural transformation between the two functors) isomorphism:

$$(X \cup CY)\big/X \cong CY/Y\,. \tag{3}$$

Hence we have for $Y \in ob(\mathbf{TopC^0})$:

$$\begin{aligned} K(X \cup CY, X) &\cong K(CY, Y) \\ &\cong \tilde{K}(SY) \\ &\cong \tilde{K}^{-1}(Y),. \end{aligned}$$

Lemma 3.15. *Let $(X, Y) \in ob(\mathbf{TopC^2})$ and define $i : (Y, \varnothing) \to (X, \varnothing)$ and $j : (Y, \varnothing) \to (X, Y)$ to be the inclusions. Then we have an exact sequence:*

$$K(X, Y) \xrightarrow{j^*} K(X) \xrightarrow{i^*} K(Y)$$

Here we identify $K(X) = K(X, \varnothing)$ and the same with $K(Y)$ such that we have well defined induced functions from the same functor.

Proof. First, we show that $\mathrm{im}(i^*) \subseteq \ker(j^*)$. To see this, notice that $i^* \circ j^* = (j \circ i)^*$, and $j \circ i$ factors over (Y, Y):

$$\begin{tikzcd} (Y, \varnothing) \arrow[r, "i"] \arrow[d] & (X, \varnothing) \arrow[d, "j"] \\ (Y, Y) \arrow[r] & (X, Y) \end{tikzcd}$$

Hence, $j^* \circ i^*$ factors over the trivial Ring and thereby is trivial.

Now suppose $\xi \in \ker(i^*)$. The intuition now is that if we have a vector bundle that gets mapped to 0 this means that it is trivial over Y which lets us define a vector bundle over the quotient X/Y by a trivialization:

We write $\xi = [E] - [n]$ for E being a vector bundle over X. (Compare remark 2.8). Since $i^*\xi = 0$ it follows that $i^*([E] - [n]) = [EvY] - [n\big|_Y] = 0$ and hence E and n are stably equivalent, meaning there exists n $m \in \mathbb{N}$ such that

$$E\Big|_Y \oplus \underline{m} = E \oplus \underline{m}\Big|_Y = n \oplus m\,.$$

In other words: There exists a trivialization α of $E \oplus \underline{m}\Big|_Y$. This trivialization defines a bundle $(E \oplus m)/\alpha$ over X/Y and hence an element:

$$\eta = [(E \oplus m)/\alpha] - [n \oplus m] \in \tilde{K}(X/Y) = K(X, Y)$$

With this we can calculate

$$\begin{aligned} j^*(\eta) &= [E \oplus m] - [n \oplus m] \\ &= [E] - [n] = \xi\,. \end{aligned}$$

$\square$

Corollary 3.16. Let $X = A \vee B$. Then the sequence:

$$0 \longrightarrow \tilde{K}(B) \xrightarrow{j^*} \tilde{K}(A \vee B) \xrightarrow{i^*} \tilde{K}(A) \longrightarrow 0$$

where $i : A \to A \vee B$ is the inclusion and $j : A \vee B \to B$ is the collapsing of A, is exact.

Proof. Let $X = A \vee B$ and $Y = A$. Then $X/Y \cong B$ and under this homeomorphism j from lemma 3.15 becomes the map j from this corollary. All that is left is to show that i^* is surjective, and j^* is injective. To see that i^* is surjective, consider the map $q' : A \vee B \to A$ by collapsing B. Then $i^* \circ q'^* = (q' \circ i)^* = (\mathrm{id})^*$. Hence, i^* has a canonical right inverse and is thereby injective. If we define $i_B : B \to A \vee B$ to be the inclusion, we hence that: $i_B^* \circ j^* = (j \circ i_B)^* = \mathrm{id}$ and hence j^* is injective, leaving us with the statement. □

Theorem 3.17 (Additivity)**.** *The proof of the corollary above gives us not just a sort exact sequence, but a canonically splitting short exact sequence. Hence we get a canonical isomorphism:*

$$\tilde{K}(X \vee Y) \cong \tilde{K}(X) \oplus \tilde{K}(Y).$$

The isomorphism in fact is of the form: $i_X^* \oplus i_Y^*$, where i_X and i_Y are the inclusions of X, Y into the sum $X \vee Y$. Furthermore, an inverse is given by $\psi : \tilde{K}(X) \oplus \tilde{K}(Y) \to \tilde{K}(X \vee Y)$ by $\psi(a_x, a_y) = c_X^*(a_x) + c_Y^*(a_Y)$.

Proof. The isomorphism in general from a splitting short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[l, bend left, "r"] \arrow[r] & 0 \end{tikzcd}$$

is constructed by using a left inverse $l : B \to A$ defined as $l(b) := f^{-1}(b - r \circ g(b))$. The final isomorphism $\Phi : B \to A \oplus B$ is then given by $l \oplus g$. In our case l has a special form, since we can include X via i_X and get a left inverse of j^*: $i_X^* \circ j^* = (j \circ i_X)^* = (\mathrm{id}_X)^*$. Now an inverse of $\Phi := i_X^* \oplus i_Y^*$ is given by

$$\begin{aligned} \Psi : \tilde{K}(X) \oplus \tilde{K}(Y) &\to \tilde{K}(X \vee Y) \\ (a_X, a_Y) &\mapsto c_X^*(a_X) + y_Y^*(a_Y) \end{aligned}$$

To see this we calculate the compositions:

$$\begin{aligned} \Phi \circ \Psi(a_X, a_Y) &= \big(i_X^*(c_X^*(a_X) + c_Y^*(a_X)), i_Y^*(c_X^*(a_X) + c_Y^*(a_X))\big) \\ &= \Big(\underbrace{(c_X \circ i_X)}_{=\mathrm{id}}{}^*(a_X) + \underbrace{(c_Y \circ i_X)}_{=0}{}^*(a_X) \quad , \quad \underbrace{(c_X \circ i_Y)}_{=0}{}^*(a_X) + \underbrace{(c_Y \circ i_Y)}_{=\mathrm{id}}{}^*(a_X)\Big) \\ &= (a_X, a_Y) \end{aligned}$$

Hence this is a right inverse of Φ and thereby the inverse. (To see this notice how :

$$\begin{aligned} \Phi \circ \Psi = \mathrm{id} = \Phi \circ \Phi^{-1} &\qquad \Phi \text{ from the right} \\ \Phi \circ \Psi \circ \Phi = \Phi &\qquad \Psi^{-1} \text{ from the left} \\ \Psi \circ \Phi = \mathrm{id} & \end{aligned}$$

, and is thereby the left inverse) □

Corollary 3.18. With ne notation from above, the pair $(\tilde{K}(X \vee Y), (c_X^*, c_Y^*))$ satisfies the universal property of the coproduct.

Proof. let G be a group and

$$f_X : \tilde{K}(X) \to G$$
$$f_Y : \tilde{K}(Y) \to G$$

be two homomorphisms. Then the map $h := f_X \circ i_X^* + f_X \circ i_X^*$ satisfies

$$h \circ c_X^* = f_X \quad \text{and} \quad h \circ c_Y^* = f_Y$$

The uniqueness is an easy diagramm yoga. □

Corollary 3.19. If $(X, Y) \in ob(\mathbf{TopC^2})$ and $Y \in ob(\mathbf{TopC^0})$ such that X is pointed by using the base point y_0 from Y. Then the sequence

$$K(X, Y) \xrightarrow{j^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(Y)$$

is exact.

Proof. We have the commutative diagram with exact top row:

$$\begin{tikzcd}
K(X,Y) \arrow[r] \arrow[ddr] & K(X) \arrow[r] \arrow[d, "\cong"] & K(Y) \arrow[d, "\cong"] \\
 & \tilde{K}(X) \oplus K(y_0) \arrow[r] \arrow[d, two heads] & \tilde{K}(Y) \oplus K(y_0) \arrow[d] \\
 & \tilde{K}(X) \arrow[r] & \tilde{K}(Y)
\end{tikzcd}$$

. The middle horizontal map is a map that respects the grading, since we have the same base point. By this we can do a bit of diagram yoga concluding the statement. □

Theorem 3.20. *For $(X, Y) \in ob(\mathbf{TopC^2})$ we have a natural exact sequence that is infinite to the left:*

$$\cdots \longrightarrow K^{-2}(Y) \xrightarrow{\delta} K^{-1}(X, Y) \xrightarrow{j^*} K^{-1}(X) \xrightarrow{i^*} K^{-1}(Y)$$

$$\xrightarrow{\delta} K^0(X, Y) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(Y)$$

Proof. It is sufficient to show that if $Y \in ob(\mathbf{TopC^0})$, we have an exact sequence:

$$\tilde{K}^{-1}(X) \xrightarrow{i^*} \tilde{K}^{-1}(Y) \xrightarrow{\delta} K^0(X, Y) \xrightarrow{j^*} \tilde{K}^0(X) \xrightarrow{i^*} \tilde{K}^0(Y) \tag{4}$$

Because, if (4) holds, we can replace (X, Y) by $(S^n X, S^n Y)$ to get the sequence that is infinite to the left. Furthermore, if we replace the pair of unpointed spaces (X, Y) by (X^0, Y^0), where X^0 denotes the pointed space with base point $\varnothing$ (from corollary 3.4) we get the sequence from the theorem. By corollary 3.19 we get the exactness at the last to groups. To get the exactnes at the other spots we apply the same corollary to the pairs:

- $(X \cup CY, X)$
- $((X \cup CY) \cup CX, X \cup CY)$

The first gives us an exact sequence $K(X \cup CY, X) \xrightarrow{m^*} \tilde{K}(X \cup CY) \xrightarrow{k^*} \tilde{K}(X)$ Since CY is contractible we have an isomorphism induced by the collapsing map $X \cup CY \simeq X/Y$:

$$\pi^* : \tilde{K}(X/Y) \to \tilde{K}(X \cup CY).$$

Furthermore, the composition $k^* \circ \pi^* = (\pi \circ k)^* = j^*$ Now define

$$\theta : K(X \cup CY, X) \to \tilde{K}^{-1}(Y)$$

be the isomorphism defined in corollary 3.14. With this we define:

$$\delta := (\pi^*)^{-1} \circ m^* \circ \theta^{-1} : K^{-1}(Y) \to K(X, Y)$$

This gives us the commutative diagram, where the vertical maps are all isomorphisms and the top row is exact.

$$\begin{tikzcd}
K(X \cup CY, X) \arrow[r, "m^*"] \arrow[d, "\theta"] & \tilde{K}(X \cup CY) \arrow[r, "k^*"] & \tilde{K}(X) \arrow[d, "\mathrm{id}"] \\
K^{-1}(Y) \arrow[r, "\delta"] & K(X, Y) \arrow[r, "j^*"] \arrow[u, "\pi^*"] & \tilde{K}(X)
\end{tikzcd}$$

Hence, the sequence 4 is exact in the middle. Finally, we look at the pair $((X \cup C_1 Y) \cup C_2 X, X \cup C_1 Y)$. The C_i is just for us to distinguish the cones. First we get again an exact sequence:

$$K\big((X \cup C_1 Y) \cup C_2 X, X \cup C_1 Y\big) \xrightarrow{l^*} \tilde{K}\big((X \cup C_1 Y) \cup C_2 X\big) \xrightarrow{s^*} \tilde{K}(X \cup C_1 Y)$$

Our goals is now to show that we commute with the start of the exact sequence (4), hence to finde vertical maps between the top rows such that the diagram commutes:

$$\begin{tikzcd}
K\big((X \cup C_1 Y) \cup C_2 X, X \cup C_1 Y\big) \arrow[r, "l^*"] \arrow[d, "?"] & \tilde{K}\big((X \cup C_1 Y) \cup C_2 X\big) \arrow[r, "s^*"] \arrow[d, "?"] & \tilde{K}(X \cup C_1 Y) \arrow[d, "?"] \\
\tilde{K}^{-1}(X) \arrow[r, "i^*"] & \tilde{K}^{-1}(Y) \arrow[r, "\delta"] & K(X, Y) \\
K(C_2 X, X) \arrow[r, "i^*"] \arrow[u] & K(C_2 Y, Y) \arrow[u] &
\end{tikzcd}$$

the bottom row is added for clearification on what inclusion induces i^*. We start with the right square and notice that by definition we get a commutative diagramm:

$$\begin{tikzcd}
K(X \cup C_1Y \cup C_2X) \arrow[r, "s^*"] \arrow[d, "(\tilde{\pi}^*)^{-1}"] & \tilde{K}(X \cup C_1Y) \arrow[d, "\mathrm{id}"] \\
K(X \cup C_1Y, X) \arrow[r, "m^*"] \arrow[d, "\theta"] & \tilde{K}(X \cup C_1Y) \arrow[d, "(\pi^*)^{-1}"] \\
K^{-1}(Y) \arrow[r, "\delta"] & K(X, Y)
\end{tikzcd}$$

The bottom square commutes by definition. In the top square, the map $\tilde{\pi}$ denotes the collapsing map $X \cup C_1Y \cup C_2X \to (X \cup C_1Y)\big/X$. To see the commutativity in the top square notice how $\tilde{\pi}^* \circ s^* = (s \circ \tilde{\pi})^*$ and the maps m and $s \circ \tilde{\pi}$ coincide and hence the top square commutes. For the left square we run into a Problem, since the map l^* induces a map λ by the "wrong" inclusion (compared to i^*):

$$\begin{tikzcd}
K\big((X \cup C_1Y) \cup C_2X, X \cup C_1Y\big) \arrow[r, "l^*"] \arrow[d] & \tilde{K}\big((X \cup C_1Y) \cup C_2X\big) \arrow[d] \\
\tilde{K}(C_2X\big/X) \cong \tilde{K}^{-1}(X) \arrow[r, "\lambda"] & \tilde{K}(C_1Y\big/Y) \cong \tilde{K}^{-1}(Y)
\end{tikzcd}$$

To resolve that problem we introduce the double cone on Y $C_1Y \cup C_2Y$, that fits into the commutative diagram:

$$\begin{tikzcd}
X \cup C_1Y \cup C_2X \arrow[rr, Rightarrow] \arrow[dd] & & C_1Y/Y \arrow[r, Rightarrow] & SY \\
& C_1Y \cup C_2Y \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] \arrow[dl] & & \\
C_2X/X & & C_2Y/Y \arrow[ll] \arrow[r, Rightarrow] & SY
\end{tikzcd}$$

The double arrows always induce isomorphisms in the K–Rings. This setup induces a diagramm that is **not commutative**:

$$\begin{tikzcd}
& K(C_1Y/Y) \arrow[dl] & \tilde{K}(SY) \arrow[l] \arrow[dd, Rightarrow] \\
K(C_1Y \cup C_2Y) & & \\
& K(C_2Y/Y) \arrow[ul] & \tilde{K}(SY) \arrow[l]
\end{tikzcd}$$

However, we want to show that it is commutative up to sign. For this we need the lemma:

Lemma 3.21. *Let Y be a pointed space. Define*

$$\begin{aligned} T : S^1 &\to S^1 \\ t &\mapsto 1 - t \end{aligned}$$

and let $T \wedge 1 : SY \to SY$ be the map induced by T on S^1 and the identity on Y. Then:

$$\begin{aligned} (T \wedge 1)^* : \tilde{K}(SY) &\to \tilde{K}(SY) \\ y &\mapsto -y \end{aligned}$$

To prove the lemma, we do the following construction: for each $f : Y \to \mathrm{GL}(n, \mathbb{C})$ we define E_f to be the corresponding vector bundle over SY from lemma 1.47. Then the map $f \mapsto [E_f] - [n]$ induces a Group homomorphism

$$\varinjlim_n [Y, \mathrm{GL}(n, \mathbb{C})] \to \tilde{K}(SY)$$

where the group structure comes from $\mathrm{GL}(n, \mathbb{C})$. This map is a bijection by lemma 1.47 together with lemma 2.11. Now we want to check if this is a group homomorphism. For this, let

$$f, g : Y \to \mathrm{GL}(n, \mathbb{C})$$

be given. Then $f \cdot g(y) = f(y) \cdot g(y)$ Now the latter map can be incuded into $[Y \to \mathrm{GL}(2n, \mathbb{C})]$ where it is homotopic to $f(y) \oplus g(y)$. The $\oplus$ denotes the block matrix $\begin{pmatrix} f(y) & 0 \\ 0 & g(y) \end{pmatrix}$. This homotopy $\begin{pmatrix} f(y) & 0 \\ 0 & g(y) \end{pmatrix} \simeq \begin{pmatrix} f(y)g(y) & 0 \\ 0 & 1 \end{pmatrix}$ is given by:

$$\rho_t(y) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} cos(t) & \sin(t) \\ sin(t) & \cos(t) \end{pmatrix}, \qquad t \in (0, \frac{\pi}{2})$$

Hence: the induced elements in $\tilde{K}(SY)$ from $f \cdot g$ and $f \oplus g$ agree, where the latter gives the sum. With this we have the following commutative diagramm:

$$\begin{array}{ccc} \tilde{K}(SX) & \longleftarrow & \varinjlim_n [Y, \mathrm{GL}(n, \mathbb{C})] \\ {\scriptstyle (T\wedge 1)^*}\downarrow & & \downarrow {\scriptstyle f \mapsto f^{-1} (pointwise)} \\ \tilde{K}(SX) & \longleftarrow & \varinjlim_n [Y, \mathrm{GL}(n, \mathbb{C})] \end{array}$$

To see why this commutes, notice how we defined the map by gluing via f, where $T \wedge 1$ changes the place of the upper and lower vector bundle. Hence the gluing function needed is now the inverse.

With this we have that the map from lemma 3.21 corresponds to inverting in the homotopy group and hence it itself is the inversion. This construction however gives us a now commuting diagram from above:

$$\begin{tikzcd}
& K(C_1Y/Y) \arrow[dl] & \tilde{K}(SY) \arrow[l] \arrow[dd, Rightarrow, "x\mapsto -x"] \\
K(C_1Y \cup C_2Y) & & \\
& K(C_2Y/Y) \arrow[ul] & \tilde{K}(SY) \arrow[l]
\end{tikzcd}$$

Hence we have a commutativ diagramm up tp the sign:

$$\begin{tikzcd}
K\big((X\cup C_1Y)\cup C_2X, X\cup C_1Y\big) \arrow[r, "l^*"] \arrow[d] & \tilde{K}\big((X\cup C_1Y)\cup C_2X\big) \arrow[r, "s^*"] \arrow[d] & \tilde{K}(X\cup C_1Y) \arrow[d] \\
\tilde{K}^{-1}(X) \arrow[r, "-i^*"] & \tilde{K}^{-1}(Y) \arrow[r, "\delta"] & K(X,Y)
\end{tikzcd}$$

Then, the lower horizontal is exact and hence the lower horizontal with i^* replacing $-i^*$ is exact. But this concludes the proof. □

Theorem 3.22 (Excision)**.** *For any pair (X,A) and $U \subseteq A$ such that $\bar{U} \subseteq \mathring{U}$ the inclusion*

$$i : (X \setminus U, A \setminus U) \to (X,A)$$

induces an isomorphism

$$i^* : K^{-n}(X,A) \to K^{-n}(X \setminus U, A \setminus U)$$

Proof. Define $X_1 := X \setminus U$ and $X_2 = A$. Then $X = X_1 \cup X_2 = \mathring{X}_1 \cup \mathring{X}_2$. With that we have that $X_1/(/X_1 \cap X_2) \cong X/X_2$. Thus:

$$K^{-n}(X_1, X_1 \cap X_2) = \tilde{K}^{-n}((X_1/X_1 \cap X_2) = \tilde{K}^{-n}((X/X_2) = K^{-n}(X, X_2).$$

□

Proposition 3.23. The map $\delta : K^{-n}(Y) \to K^{-n+1}(X,Y)$ is a natural transformation between the functors

- K^{-n+1} from $\mathbf{TopC^2}$ to $\mathbf{Ab}$
- $K^{-n} \circ R$ from $\mathbf{TopC^2}$ to $\mathbf{Ab}$, where R is an endofunctor in $\mathbf{TopC^2}$ sending $(X,A) \mapsto (A, \varnothing)$

Proof. Let $f : (X,A) \to (Y,B)$ be a map of pairs. then we need to check if the diagramm

$$\begin{tikzcd}
(X,A) \arrow[d, "f"] & K^{-n}(X,A) \arrow[r, "\delta"] & K^{-n+1}(A,\varnothing)) \\
(Y,B) & K^{-n}(Y,B) \arrow[r, "\delta"] \arrow[u, "f^*"] & K^{-n+1}(B,\varnothing)) \arrow[u, "(f|_A)^*"']
\end{tikzcd}$$

commutes. This however is the case since δ is defined via the pullback of the inclusion (m^*) together with a the natural isomorphisms θ and π^* ☐

Corollary 3.24. If $Y \subseteq X$ is a retract. Then for all $n \geq 0$, the sequence

$$K^{-n}(X,Y) \xrightarrow{j^*} K^{-n}(X) \xrightarrow{i^*} K^{-n}(Y)$$

is a splitting short exact sequence. This splitting is naturally dependent on the retraction map. The kernel of the map i^* is $K^{-n}(X,Y)$.

Theorem 3.25 (Additivity). *If X and Y are pointed topological spaces and $X \vee Y$ denotes the topological sum, we have: $\tilde{K}^{-n}(X \vee Y) \cong \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(Y)$*

Proof. for $n = 0$ we have: $\tilde{K}(X \vee Y, Y) = \tilde{K}(X \vee Y/Y) = \tilde{K}(X)$, and we can applie the preceeding corollary.

Since for the reduced suspension, we have that $S(X \vee Y) \cong SY \vee SY$ and hence we can use the argument again if we replace X with SX and Y with SY. ☐

Proof. Notice, that if Y is a retraction of X, then SY is a retraction of SX. This is true since S is an endofunctor and retraction means that the inclusion has a left inverse. This property, however, gets translated by the functor. Now since K is functorial this induces a right inverse of i^* which again lets us deduce that i^* is surjective and with that we have that δ is zero. ☐

Corollary 3.26. Let X, Y be objects in $\mathbf{TopC^0}$. Then we have the projection maps

$$\pi_X : X \times Y \to Y\,, \quad \pi_Y : X \times Y \to Y\,.$$

Those maps induce isomorphisms for all $n \geq 0$:

$$\tilde{K}^{-n}(X \times Y) \cong \tilde{K}^{-n} \oplus \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(Y)$$

Proof. X is a retract of $X \times Y$ and Y is a retract of $(X \times Y)/X$ (for the quotient to be natural, we include X at the base point of Y). With this we apply corollary 3.24 twice:

$$\begin{aligned} \tilde{K}^{-n}(X \times Y) \;&\cong \tilde{K}^{-n}(X \times Y/X) && \oplus \tilde{K}^{-n}(X) \\ &\cong \tilde{K}^{-n}(Y) \oplus \underbrace{\tilde{K}^{-n}((X \times Y/X)/Y)}_{\tilde{K}^{-n}(X \wedge Y)} && \oplus \tilde{K}^{-n}(X)s \end{aligned}$$

☐

Corollary 3.27. Let's inspect the isomorphism induced by the splitting for $n = 0$:

$$\begin{tikzcd} \tilde{K}(X \times Y/X) \arrow[r, "j^*"] & \tilde{K}(X \times Y) \arrow[r, "(\tilde{i}_Y)^*"] \arrow[l, bend left, "l"] & \tilde{K}(Y) \end{tikzcd}$$

Here we have the following maps:

- $i_y : Y \to X \times Y$
- $\tilde{i}_Y : Y \to (X \times Y/X)$
- $j : (X \times Y) \to (X \times Y/X)$
- $l : \tilde{K}(X \times Y) \to \tilde{K}(X \times Y/X)$ is a left inverse of j^*

Now we have that $\tilde{i}_Y = j \circ i_Y$ and with this we can deduce:

$$\begin{aligned} \tilde{i_Y}^* \circ l &= (j \circ i_Y)^* \circ l \\ &= i_Y{}^* \circ j^* \circ l \\ &= i_Y^* \end{aligned}$$

Hence, in the proof above the isomorphism has the form:

$$\begin{aligned} \Phi : \tilde{K}(X \times X) \to \tilde{K}(X \wedge Y) \oplus \tilde{K}(Y) \oplus \tilde{K}(X) \\ \Phi = g \oplus i_Y^* \oplus i_X^* \end{aligned}$$

which lets us induce the diagram

$$\begin{tikzcd} & \tilde{K}(X \times Y) \arrow[d, "l \oplus i_X{}^*"] \arrow[dr, "i_Y{}^* \oplus i_X{}^*"] & \\ \tilde{K}(X \wedge Y) \arrow[r] & \tilde{K}(X, Y/X) \oplus \tilde{K}(X) \arrow[r] & \tilde{K}(Y) \oplus \tilde{K}(X) \end{tikzcd}$$

where the bottom is exact.If we define $\tilde{j} : (X \times Y)/X \to X \wedge Y$, the kernel of $i_Y{}^* \oplus i_X{}^*$ is isomorphic to $\tilde{K}(X \wedge Y)$ via the mapping $j^* \circ \tilde{j}^* : \tilde{K}(X \wedge X) \to \tilde{K}(X \times Y)$. Which is induced by the natural quotient. Hence we gain a short exact sequence that splits:

$$0 \longrightarrow \tilde{K}(X \wedge Y) \xrightarrow{j^*} \tilde{K}(X \times Y) \xrightarrow{(i_X)^* \oplus (i_Y)^*} \tilde{K}(X) \oplus \tilde{K}(Y) \longrightarrow 0$$

Definition 3.28 (Exterior Product)**.** The **exterior product** $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ is defined via the two projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ by:

$$\begin{aligned} \mu : K(X) \otimes K(Y) \to K(X \times Y) \\ (a \otimes b) \mapsto (\pi_X)^*(a) \cdot (\pi_Y)^*(b), \end{aligned}$$

where $\cdot$ denotes the multiplication in the K-group, which was induced by the tensor Product. This μ is a ring homomorphism, since:

$$\begin{aligned} \mu((a \otimes b)(c \otimes d)) &= \mu(ac \otimes bd) \\ &= \pi_X^*(ac)\pi_Y^*(bd) \\ &= \pi_X^*(a)\pi_X^*(c)\pi_Y^*(b)\pi_Y^*(d) \\ &= \pi_X^*(a)\pi_Y^*(b)\pi_X^*(c)\pi_Y^*(d) \\ &= \mu(a \otimes b)\mu(c \otimes d) \end{aligned}$$

Definition 3.29. If we consider $\tilde{K}(X) = \ker(K(X) \to K(x_0))$ we have a mapping

$$\begin{aligned} \tilde{K}(X) \otimes \tilde{K}(Y) &\to K(X \times Y) \\ a \otimes b &\mapsto \mu(a \otimes b) \end{aligned}$$

Now for such a $a \otimes b$ we have:

$$\begin{aligned} (i_X)^* \mu(a \otimes b) &= (i_X)^* \Big((\pi_X)^*(a) \cdot (\pi_Y)^*(b) \Big) \\ &= (i_X)^*(\pi_X)^*(a) \cdot (i_X)^*(\pi_Y)^*(b) \\ &= (i_X)^*(\pi_X)^*(a) \cdot \underbrace{(\underbrace{\pi_Y \circ i_X}_{X \to \{y_0\}})^*(b)}_{=0} \qquad = 0\,, \end{aligned}$$

where the zero follows from $\pi_Y \circ i_X$ factoring over $\{y_0\} \to Y$ and b is in the kernel of that induced map (since it is in $\tilde{K}(Y)$. Similarly, $(i_Y)^* \mu(a \otimes b) = 0$ and by that, we get a pairing:

$$\tilde{K}(X) \otimes \tilde{K}(Y) \to \ker((i_X)^* \oplus (i_Y)^*) \cong \tilde{K}(X \wedge Y)$$

The last isomorphism is natural in the sense that it is induced by the pullback along the quotient. Since $S^n \wedge X \wedge S^m \wedge Y = S^{n+m} \wedge X \wedge Y$ we get the paring for all negative reduced K−groubs and if we are not pointed we can replace X by $(X/\varnothing)$ to get a unreduced version:

$$K^{-n}(X) \otimes K^{-m}(Y) \to K^{-n-m}(X \times Y)$$

This fits into the diagram:

Corollary 3.30.

4 Differentiable structures on topological manifolds

Definition 4.1 (Topological Manifold)**.** A second countable Hausdorffspace M is called **topological manifold** of dimension $m \in \mathbb{N}$, if it is locally homeomorphic to $\mathbb{R}^m$. To be precise, if for all $p \in M$ there exists an open neighborhood $U \subseteq M$ of p, an open set $V \subseteq \mathbb{R}^m$ and a map $\varphi : W \to V$ that is a homeomorphism. We call the map $\varphi : U \to V$ a **chart around p on M** and φ^{-1} a **local coordinate system around p on M.**

Definition 4.2 (Differentiable Manifold)**.** Let M be a topological manifold of dimension M.

1. A **differentiable atlas of class** $r \in \mathbb{N} \cup \{\infty\}$ is a family of charts $\mathfrak{A} = (\varphi_i : U_i \to V_i)_{i \in I}$ such that
   a) $\bigcup_{i \in I} U_i = M$, meaning that (U_i) is an open covering of M.

b) For every pair $(i,j) \in I^2$ the **transition function**:

$$\begin{aligned}\varphi_{ij} : \varphi_j(U_i \cap U_j) &\to \varphi_i(U_i \cap U_j)\\ x &\mapsto \left(\varphi_i \circ \varphi_j^{-1}\right)(x)\end{aligned}$$

is differentiable of class r.

We call such an atlas a C^r-atlas.

2. Two C^r-atlases $\mathfrak{A}$ and $\mathfrak{B}$ are called **equivalent** if the family $\mathfrak{A} + \mathfrak{B} = (\varphi_i, \varphi_j)_{ij}$ is a C^r-atlas.

A **differentiable structure of class** r on M is an equivalence class c of C^r-atlases. For $r = \infty$ we call the pair (M, c) a smooth manifold.

Corollary 4.3. Every transition functions φ_{ij} $i,j \in I^2$ of a differentiable atlas $\mathfrak{A} = (\varphi_i)_{i \in I}$ is not just a homeomorphism but also a diffeomorphism due to $\varphi_{ji} = \varphi_{ij}^{-1}$

Definition 4.4. Let (M, c) be a differentiable manifold of class r and $U \subseteq M$ open. We call a continuos function

$$f : U \to \mathbb{R}$$

differentiable of class r, if for any one (and hence for all) $(\varphi_i)_{i \in I} = \mathfrak{A} \in c$ the compositions $f \circ \varphi_i^{-1}$ are differentiable of class r. For $r = \infty$ we define:

$$\mathcal{E}(U) = \{f \in U \to \mathbb{R} \text{ continouos} \mid f \text{ is differentiable of class } \infty\}.$$

Corollary 4.5. Let (M, c) be a smooth manifold of dimension m and $U \subseteq M$ be a open subset. With pointwise defined operations, the set $(\mathcal{E}(U), +, \cdot, \circ)$ becomes an $\mathbb{R}$–algebra. Furthermore, $\mathcal{E}$ becomes a sheaf of $\mathbb{R}$-algebras.

Proof. There is not really a need for a proof.However, it might help to work through the definition of a sheaf as a reminder. First, $\mathcal{E}$ is a presheaf, where the restriction in the domain of a function gives the needed restriction homomorphism:

$$\begin{aligned}\mathrm{res}_V^U : \mathcal{E}(U) &\to \mathcal{E}(V)\\ f &\mapsto f\big|_V.\end{aligned}$$

The required properties of a presheaf are trivial. Furthermore, this gives a sheaf as the requirement of locality is trivial for functions and the property of gluing is also trivial for functions, since differentiability is a local property. □

Definition 4.6. If $p \in M$ is fix, $f \in \mathcal{E}(U)$ and $g \in \mathcal{E}(U')$ such that $p \in U \cap U'$ we say that f and g have the same **germ in** p, if there is another open neighborhood $W \subseteq U \cap U'$ of p such that $f\big|_w = g\big|_W$. This defines an equivalence relation $\sim_p$. An equivalence class

s of local functions around p is called a **germ in** p. We write $s = f_p$, if $s = [f]$ with $f \in \mathcal{E}(U)$. We write

$$\mathcal{E}_p(M) = \left(\sum_{U \mathrm{open}, p \in U} \mathcal{E}(U) \right) \Big/ \sim_p \; .$$

For the set of germs and call it the **stalk in** p. Here $\sum$ denotes the co-product (also called sum) in $\top$ and hence the disjoint union.

Corollary 4.7. For a smooth manifold (M, c) the set $\mathcal{E}_p(M)$ inherits an $\mathbb{R}$–algebra structure from the $\mathcal{E}(U)$. Furthermore, it carries a natural (evaluation-)homomorphism:

$$\begin{aligned} \mathrm{eval}_p : \mathcal{E}_p(M) &\to \mathbb{R} \\ f_p &\mapsto f(p) =: f_p(p) \end{aligned}$$

The stalks are also local rings with maximal ideal $\mathfrak{m}_p = \ker(\mathrm{eval}_p)$. Hence, the pair $(M, \mathcal{E})$ gives us a locally ringed space.

Proof. Here, we only need to prove the statement about the locality of the stalks. This follows from $f_p \in \mathcal{E}_p(M)$ being invertible if and only if $f(p) \neq 0$ which is the same as $f_p \notin \ker(\mathrm{eval}_p)$. □

Definition 4.8. Let (M, c) be a smooth manifold of dimension m and $p \in M$. We call an $\mathbb{R}$-linear map $\delta : \mathcal{E}_p(M) \to \mathbb{R}$ a **derivation**, if it satisfies the Leibnitz-rule:

$$\delta(f_p \cdot g_p) = \delta(f_p) g_p(p) + f_p(p) \delta(g_p) \quad \text{for all } f, g \in \mathcal{E}_p(M) \, .$$

We call $\mathrm{Der}_{\mathbb{R}}(\mathcal{E}_p(M), \mathbb{R})$ the set of derivations and give it a $\mathbb{R}$ vector space structure by pointwise operations. We define the **tangent space of** M **at** p to be the vector space

$$TM_p := \mathrm{Der}_{\mathbb{R}}(\mathcal{E}_p(M), \mathbb{R}) \, .$$

Corollary 4.9. Let (M, c) be a smooth manifold and $\varphi : U \to V$ be a chart around p with $x_0 = \varphi(p)$ $(\varphi \in \mathfrak{A} \in c)$. Then

$$\begin{aligned} \xi = \frac{\partial}{\partial x^j}\Big|_p : \mathcal{E}_p(M) &\to \mathbb{R} \\ f_p &\mapsto \xi(f_p) = \frac{\partial}{\partial x^j}\Big|_{x_0} (f \circ \varphi^{-1}) \end{aligned}$$

is well-defined and a tangent vector. In fact, the family

$$\left(\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^m}\Big|_p \right)$$

defines a basis of TM_p. Hence, the dimension of TM_p is m.

Definition 4.10. For a smooth manifold (M,c) the sum $\sum_p TM_p$ comes with a natural projection

$$\begin{aligned}\pi : TM &\to M\\ \xi &\mapsto p \text{ where } \xi \in TM_p\end{aligned}$$

Furthermore, the **local vector fields** with respect to a chart $\varphi : U \to V$

$$\begin{aligned}\frac{\partial}{\partial x^j} : U &\to \pi^{-1}(U)\\ p &\mapsto \frac{\partial}{\partial x^j}\Big|_p\end{aligned}$$

induce a local trivialization:

$$\pi^{-1}(U) \cong U \times \mathbb{R}^m .$$

We can induce a topology on TM such that all those trivializations are continuos. This then gives an atlas for TM such that we have a $2m$-dimensional manifold. To be precise, the atlas is given by the maps $\pi^{-1}(U_i) \to R^m \times R^m; x \mapsto (\pi(x), q_{\varphi \circ \pi(x)}(x))$ where q_p denotes the coordinate map corresponding to the basis $(\frac{\partial}{\partial x^1}\big|_p, ..., \frac{\partial}{\partial x^m}\big|_p)$ that depends on the chart φ_i. In fact, this yields a smooth manifold and a (smooth) vector bundle of dimension m. We call TM the **tangent bundle**.

Definition 4.11 (The Derivative)**.** Let (M,c) and (M',c') be smooth manifolds (from now on we suppress the differentiable structure in our notation). We call a continuous function $f : M \to M'$ **smooth**, if for all $\varphi \in \mathfrak{A} \in c$ and $\varphi' \in \mathfrak{A}' \in c'$ the maps

$$\varphi' \circ f \circ \varphi^{-1} : V \to V'$$

are smooth. A given smooth function induces a smooth function between the Tangent bundles as follows:

$$Df : TM \to TM' \ , Df_p(\xi)(g_p) = \xi((g \circ f)_p)$$

Here, $\xi \in T_pM$, $g_p \in E_p(M')$.

Corollary 4.12. Let $f : M \to \mathbb{R}$ be smooth and $\varphi : U \to V$ be a chart. Then we can interprete $\mathrm{d}f$ as a one-form. To be prezice asume q to be the coordinate funktion $T\mathbb{R} \to \mathbb{R}$ from the basis induced by the identity as a chart. $v \in \Gamma TM$ we have

$$q \circ \mathrm{d}f(v) = v \cdot f$$

Here on the righd side, f denotes a map $p \mapsto f_p$ such that $v(f)(p) := v_p(f_p)$ is well defined. We keep this notations so vector fields can take funktions as an input.

Proof. We proof this by showing it for the section $\frac{\partial}{\partial x^i}$ and thereby for any, since those sections form a basis of the space of sections as a $C^\infty(M,\mathbb{R})$ vectorspace. Now let $g_p \in \mathcal{E}_p(\mathbb{R})$ and $\varphi(p) = x_0$. Then

$$\mathrm{d}f(\frac{\partial}{\partial x^i}\Big|_p)(g_p) = \frac{\partial}{\partial x^i}\Big|_p(g\circ f)_p = \frac{\partial}{\partial x^i}\Big|_{x_0}(g\circ f\circ\varphi^{-1}) = \frac{\partial}{\partial x}\Big|_p(g_p)\cdot\frac{\partial}{\partial x^i}\Big|_p(f_p)$$

Hence, $\mathrm{d}f(v) = v(f)\frac{\partial}{\partial x}$ letting us conclude the statement. □

Definition 4.13 (A Metric)**.** Let M be a smooth manifold. A section $g \in \Gamma(TM^*\otimes TM^*)$ into the tensor product of the dual Tangend bundle with itself is a **riemannian metric** if the following are satisfied for all $p \in M$:

1. g is **non degenerate**, meaning for a $v_p \in T_pM$ $g_p(v_p,w_p) = 0$ $\forall w_p \in T_pM$ then $v_p = 0$.
2. g is **symmetric**, meaning $g_p(v_p,w_p) = g_p(w_p,v_p)$ $\forall v_p,w_p \in T_pM$.
3. g is **positiv definite**, meaning that $g_p(v_p,v_p) \geq 0 \forall v_p$ and vanishes only for $v_p = 0$.

We cal a touple (M,g) a **riemannian manifold**.

Definition 4.14 (The Gradient)**.** Assume that (M,g) is a smooth manifold together with a riemannian metric. Let $f : M \to \mathbb{R}$ be a smooth function. We define its **gradient** to be a vector field $\mathrm{grad}(f) \in \Gamma(TM)$ such that for any vecor field V:

$$q\circ \mathrm{d}f(V) = g(\mathrm{grad}(f),V)\,.$$

Here q denotes the canonical coordinate funtion $TR \to \mathbb{R}$.

Definition 4.15. Let (M,g) be a riemannian manifold and $\varphi : U \to V$ be a chart. We define the smooth functions $g_{ij} : U \to \mathbb{R}$ to be :

$$g_{ij} = g(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})\,.$$

Then if two vectorfields v,w over U are given with $v = \sum v^i\frac{\partial}{\partial x^i}$ and $w = \sum_i w^i\frac{\partial}{\partial x^i}$ we can calculate the metric locally:

$$g(v,w) = \sum_{ij} g_{ij}v^iw^j \;: U \to \mathbb{R}$$

Furthermore, we can invert the matricies $(g_{ij}(p))_{ij}$ for all p (which is a smooth procedure which can be seen by the construction via cramers rule) to get smooth functinos

$$\begin{aligned} g^{ij} &: U \to \mathbb{R} \\ &p \mapsto g^{ij}(p) = \big((g_{kl}(p))^{-1}_{kl}\big)_{ij} \end{aligned}$$

Lemma 4.16. *Let $\varphi : U \to V$ be a chart. Then the gradient has the local form:*

$$\operatorname{grad}(f) = \sum_{i,j} g^{ij}\left(\frac{\partial}{\partial x^i}(f)\right)\frac{\partial}{\partial x^j} =: \sum_{i,j} g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}\,.$$

Here, g^{ij} denotes the smooth functions given by the coordinates of the function $x \mapsto (g_{ij}(x))$ that defines the coordinate representation of the gradient in x.

Proof. Assume that $v, w \in \Gamma(TM)$ such that $v = \sum_i v^i \frac{\partial}{\partial x^i}$ and $w = \sum_i w^i \frac{\partial}{\partial x^i}$. Then $g(v,w) = \sum_{i,j} g_{ij} v^i w^i$. Suppose that $\operatorname{grad}(f) = \sum_j G^j \frac{\partial}{\partial x^j}$ and q is the coordinate map $T\mathbb{R} \to \mathbb{R}$ in each tangend space. Then by definition of the gradient we have:

$$\frac{\partial}{\partial x^j}(f) = q \circ \mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = q \circ \frac{\partial f}{\partial x^j} = g\left(\operatorname{grad}(f), \frac{\partial}{\partial x^j}\right) = \sum_{ij} g_{ij} G^i$$

Hence,

$$\begin{aligned}(G^1, \cdots G^m)(g_{ij}) &= \left(\frac{\partial}{\partial x^1}(f), \cdots, \frac{\partial}{\partial x^m}(f)\right)\\ \Leftrightarrow (G^1, \cdots G^m) &= \left(\frac{\partial}{\partial x^1}(f), \cdots, \frac{\partial}{\partial x^m}(f)\right)(g^{ij})\,.\end{aligned}$$

But then we can conclude that $G^j = \sum_i g^{ij} \frac{\partial}{\partial x^i}(f)$.

□

5 Stable and Unstable Manifold theorem

We now want to understand the gradient lines that connect two critical points. Our bigger goal is to count them in a meaningful way such that the boundary operator, defined to be the sum over all counted gradient lines from two critical points with consecutive index, squares up to zero. Therefore, we need the following properties:

1. They need to be finitely many.
2. Some of them need to be counted *negatively*

For the second requirement we will make use of the orientation to give a sign and dimension to give a number. For that we will show that the stable set is the image of a smooth embedding of $\mathbb{R}^{\operatorname{ind}_f(p)}$. For a smooth function $f : M \to \mathbb{R}$ on a finite dimensional compact Riemannian manifold, we have that $-\operatorname{grad} f$ determines a smooth global flow $\varphi_t(x)$. If not said different, $\varphi_t(x)$ will always denote that flow.

Definition 5.1 (stable, unstable manifold)**.** Let $p \in M$ be a critical point of f. Then we define:

1. the **stable set** of p as
$$W(\to p) := \{x \in M \mid \lim_{t\to\infty} \varphi_t(x) = ß\}.$$
That is the points that flow towards p.

2. the **unstable manifold** of p as

$$W(p \to) := \{x \in M | \lim_{t \to -\infty} \varphi_t(x) = ß\}.$$

That is the points that came from p.

Those sets will turn out to be manifolds, which explains the names **(un-)stable manifolds**.

The Main theorem of this chapter will be the:

Theorem 5.2 (Stable and unstable manifold theorem)**.** *Let $f : M \to \mathbb{R}$ be a smooth function on a smooth compact Riemannian manifold (M, g). Let p be a critical point of f. Then the tangent space splits as:*

$$T_pM = T_p^s(M) \bigoplus T_p^u M$$

such that the Hessian is positive definite on $T_p^s M$ and negative definite on $T_p^u M$. Moreover, the stable and unstable manifolds are submanifolds and images of smooth embeddings:

$$\begin{aligned} E^s : T_p^s M &\to W(\to p) \subset M, \\ E^u : T_p^u M &\to W(p \to) \subset M. \end{aligned}$$

Remark 5.3. The proof is split into several parts. First, we show that p is a hyperbolic fix point of the gradient-induced flow and the Hessian splits into a contracting and expanding part. Then we will make use of analysis to show that there is a neighbourhood of p and a chart, where $W(\to p) \cap B$ is the graph of a Lipschitz function. This will help us define a differentiable structure on the (un-)stable manifold to get the immersions.

Remark 5.4 (Discrete system)**.** We can redefine the (un-)stable manifold for $t > 0$ in a discrete way:

$$\begin{aligned} W(\to p) &= \{x \in M | \lim_{n \to \infty} \varphi_t^n(x) = p\} \\ W(p \to) &= \{x \in M | \lim_{n \to -\infty} \varphi_t^n(x) = p\} \end{aligned}$$

Definition 5.5 (Hyperbolic fixepoints)**.** A fixpoint for a diffeomorphism $\varphi : M \to M$ is called **hyperbolic**, if $d\varphi|_p : T_pM \to T_pM$ has no complex eigenvalue of length one.

Remark 5.6 (Critical points are hyperbolic fixpoints)**.** If M is a smooth riemannian manifold, $f : M \to \mathbb{R}$ is a Morse function and p is a critical point, then p is a hyperbolic fixpoint of the diffeomorphism φ_t for any $t > 0$.

Proof. We need to differentiate our $-\mathrm{grad}(f)$–induced flow by the starting point. To do this we want to inspect what differential equation $\Phi(t,x) := \frac{\partial}{\partial x}\varphi_t(x)$ solves:

$$\begin{aligned}\frac{\partial}{\partial t}\Phi(t,x)) &= \frac{\partial}{\partial x} - \mathrm{grad}(f)(\varphi_t(x)) \\ &= \Big(- \frac{\partial}{\partial x}\mathrm{grad}(f)\big|_{\varphi_t(x)}\Big)\Phi(t,x))\end{aligned}$$

and

$$\Phi(0,x) = \frac{\partial}{\partial x}\varphi_0(x) = 1 \ .$$

Now since this is a linear differential equation we get a solution by exponentiating:

$$\Phi(t,x) = \mathrm{e}^{-\frac{\partial}{\partial x}\mathrm{grad}(f)\big|_{\varphi_t(x)} t} \ .$$

Now we care for $\Phi(t,p)$ where $t > 0$ and $p \in \mathrm{Crit}(f)$. In this case we have:

$$\frac{\partial}{\partial x}\mathrm{grad}(f)\big|_{\varphi_t(p)} = \frac{\partial}{\partial x}\mathrm{grad}(f)\big|_p = \mathrm{Hess}(f)\big|_p$$

and therefore:

$$\Phi(t,p) = \mathrm{e}^{\mathrm{Hess}(f)\big|_p t}$$

This however does not have an eigenvalue of one, as the Hessian doesn't vanish in non-degenerate critical points. Even more, this is diagonal after a change of coordinates, since the Hessian can be of diagonal form. □

Corollary 5.7. By the above, we see that the tangent space T_pM now splits into two parts $T_p^sM \oplus T_p^uM$, such that for $t > 0$:

$$\mathrm{d}\varphi_t\big|_p : T_p^sM \to T_p^sM \text{ is } \textbf{contracting} \text{ with eigenvalue } \lambda < 1$$
$$\mathrm{d}\varphi_t\big|_p : T_p^uM \to T_p^uM \text{ is } \textbf{expanding} \text{ with eigenvalue } \lambda > 1$$

To conclude this, the eigenvalues themselves are not enough, but also the diagonalizability. Or more concrete: If we choose a coordinate chart such that the Hessian is $\mathrm{diag}(-1, ... - 1, 1, ...1)$ we know that:

$$\mathrm{d}\varphi_t|_p = \mathrm{diag}(\frac{1}{\mathrm{e}}, ..., \frac{1}{\mathrm{e}}) \oplus \mathrm{diag}(e, ..., e).$$

Then using the Euclidean norm we are a contraction. This situation is depicted in figure 1.

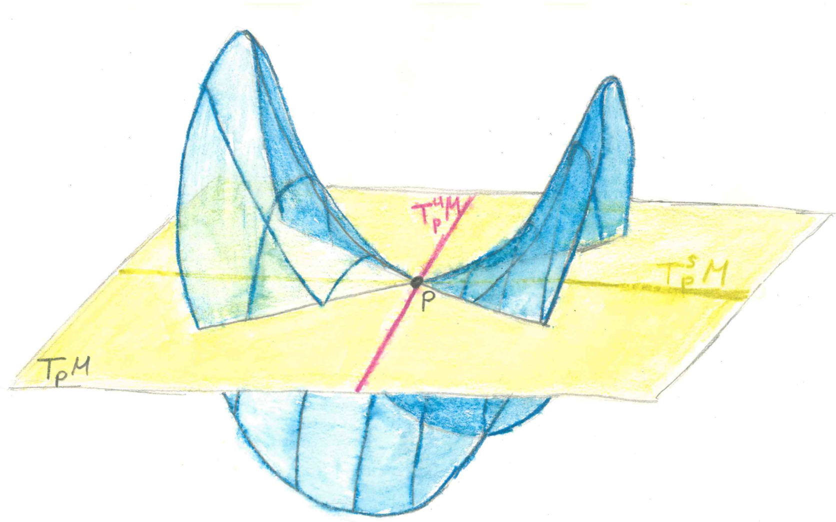


Figure 1: The splitting of the tangent space

Definition 5.8 ((un-)stable tangend space)**.** For any map $\theta : T_p^sM \bigoplus T_p^uM \to T_pM$ we define the **stable set** of θ to be:

$$W_r^s(\theta) := \{x \in T_pM | \forall n \geq 0, \theta^n(x) \text{ is defined and } \|\theta^n(x)\| \leq r\}$$

The norm here is arbitrary as we are finite dimensional and therefore all norms are equivalent.

Remark 5.9. In the proof of the (un-)stable manifold theorem, we will map the stable set into such a stable set in the tangent space. So if we could control the structure of such a map, we would win. That is accomplished as the next theorem tells us that this set is the graph of a smooth function, and the projection of the graph onto the domain gives us then a differentiable structure.

Theorem 5.10 (local (un-)stable manifold theorem)**.** *We define:*

$$T := \mathrm{d}\varphi_t\big|_p, \quad T_s := T\big|_{T_p^sM}, \quad T_u := T\big|_{T_p^uM}.$$

Let $\lambda < 1$ such that $\|T_s\| < \lambda$ and $\left\|T_u^{-1}\right\| < \lambda$, where $\|T\| = \sup\{T(x) | \|x\| \leq 1\}$ denotes the operator norm. Now for this setting there exists an ε_λ depending only on λ, and there exists a $\delta > 0$ for every $r > 0$ which satisfy: For all $\theta : T_p^sM \times T_p^uM \to T_pM$ that satisfies $\theta - T$ is Lipschitz with Lipschitz constant

$$\mathrm{Lip}(\theta - T) < \varepsilon, \text{ and } \|\theta(0)\| < \delta.$$

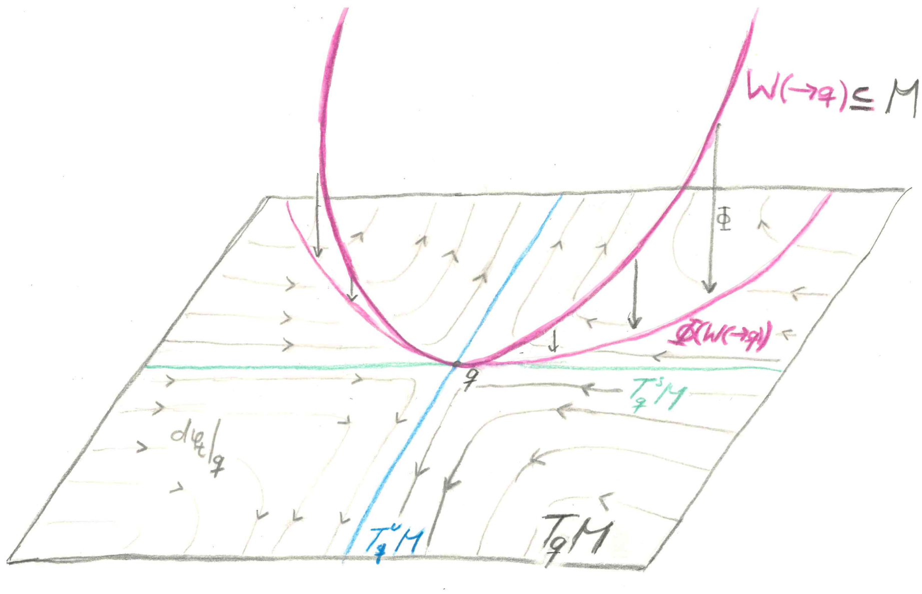


Figure 2: The proof of the (un-)stable manifold theorem
The Manifold is not shown in this image. One should imagen it looking like in figure 1. p can be imagined to be the index sitting at the bottom of the hole of the tilted doughnut.

In this setting, $W_r^s(\theta)$ is the graph of a Lipschitz function $g : T_p^s M \to T_p^u M$ with $Lip(g) \leq 1$ and:

1. *If θ is C^k (k- times smooth differentiable), then so is g.*
2. *If θ is smoothly differentiable ,$\theta(0) = 0$ and $\mathrm{d}\theta\big|_0 = T$, then $g(0) = 0$ and $\mathrm{d}g|_0 = 0$. Hence, $T_p W_r^s(\theta) = T_p^s M$.*

A picture and application of this technical theorem is given in the proof of the global stable and unstable manifold theorem.

Proof. □ fehlt

Proof of theorem 5.2. First we switch to the world of discrete dynamical systems and define $\varphi := \varphi_1 : M \to M$. The idea of the proof is to first find a chart for a small open subset of $W(\to p) = \{x \in M | \lim_{n\to\infty} \varphi^n(x) = p\}$ arround p and then define a global one. This is done by taking a small neighbourhood arround an arbetrary point $x \in W(\to p)$ and moving it via φ to the small neighbourhood of p.

Let U be a small neighborhood of $p \in M$ and $\Psi : U \to T_pM$ be a centered coordinate chart (after identifying $\mathbb{R}^n$and T_pM via $e_i \mapsto \partial q_i$). Now bx corollary 5.7, T_pM splits into $T_p^sM \otimes T_p^uM$ with respect to $\mathrm{d}\varphi\big|_p$. If we inspect the image of $W(\to p) \cap U$ under Φ we realise that it is a small pertubation of T_p^sM. This Situation can be seen in figure 2. In fact, it is one that can be controlled using theorem 5.10. For this we define $\theta := \Psi \circ \varphi \circ \Psi^{-1} : T_pM \to T_pM$. Now since θ is smooth we know that it is a lipschitz pertubation of $d\varphi|_p$ and we can apply theorem 5.10 to conclude that there are indices ε and r such that $W_r^s(\theta)$ is the graph of a smooth function. Notice that we might need to shrink the domain such that $\mathrm{Lip}(\theta - d\varphi|_p) < \varepsilon$. The next thing we want to do is to see that for a small neighbourhood $U \ni p$

$$W_r^s(\theta) = \Psi(W(\to p) \cap U). \tag{5}$$

This can be done by showing two inclusions:

- ($\subset$): for $x \in W_r^s(\theta)$ we can conclude that $\left\|\Psi \circ \varphi^n \circ \Psi^{-1}\right\| < r$ for all $n \in \mathbb{N}$. Now by Bolzano Weierstrass we know that $\lim_{n\to\infty} \Psi \circ \varphi^n \circ \Psi^{-1}$ has at least one accumulation point. And since p is hyperbolic by remark 5.6 we know this point is the origin and is unique. Therefor we conclude $\lim_{n\to\infty} \Psi \circ \varphi^n \circ \Psi^{-1} = 0$ which tells us that $\Psi^{-1}(x) \in W(\to p) \cap U$, where U is a ball inside $\Psi^{-1}B_o(r)$.

- ($\supset$) This direction is almost trivial. Let $y \in W(\to p)$ and $x = \Psi(y)$. Since p is isolated we can choose U small enough such that for all $y \in W(\to p) \cap U$ we have that $\theta^n(\Psi(y)) < r$ for all $n \in \mathbb{N}$. For the last conclusion we might need to shrink U further.

Now we know that $\Psi(W(\to p) \cap U)$ is the graph of a smooth function $g : T_p^sM \to T_p^uM$. Therefor, the projection $\pi(\Psi(W(\to p) \cap U)) \subset T_p^sM$ gives us a chart. To get an Atlas for $W(\to p)$ we can define $U^n := \varphi^{-n}(U)$ for all n to cover $W(\to p)$. This cover comes with charts defined by $(U^n, \pi \circ \Psi \circ \varphi^n)$. For the sake of readability we define the charts $\chi^n := \pi \circ \Psi \circ \varphi^n$. Those charts are compatible since φ is a diffeomorphism. To conclude that we have a submanifold we need to check if the inclusion is an immersion. This however is clear by definition.

Finally, we want to prove that this submanifold is the image of a smooth embedding of T^spM. For this we define $B := W(\to p) \cap U)$ and define $h := \chi \circ \varphi \circ \chi^{-1}$. Since similar Matrices have the same eigenvalues we can conclude that $\mathrm{d}\chi|_0$ and $\mathrm{d}\varphi_p|_{T_p^sM}$ have the same eigenvalues. Therefore, we can introduce an inner product such that the operator norm $\|dh_0\| < 1$. Now let $\alpha \in \mathbb{R}$ and B_0 be a ball centred at 0 such that $\|dh|_x\| \leq \alpha$ for all $x \in B_0$. Now this yields that $h|_{B_0}$ is a contraction, which we can extend to be a contraction $\tilde{h} : T_p^sM \to T_p^sM$. Now again similar as before we define a map $E^s : T_p^sM \to W(\to p)$ by first contracting, then going down and then expanding again. To be more precise, we define $E_n^s : \varphi^{-n} \circ \chi^{-1}\tilde{h}^n$. Finally, we define $E^s(x) = E_n^s(x)$ where $\tilde{h}^n(x) \in B_0$. Obviously, we need to check that this is well-defined: So assume that $n + 1$

and n work for x:

$$\begin{aligned}
E^s_{n+1}(x) &= \varphi^{-(n+1)} \circ \chi^{-1} \circ \tilde{h}^{n+1} \\
&= \varphi^{-(n+1)} \circ \chi^{-1} \circ h \circ \tilde{h}^{n} \\
&= \varphi^{-(n)} \circ \varphi^{-1}\chi^{-1} \circ \underbrace{\chi \circ \varphi \circ \chi^{-1}}_{h} \circ \tilde{h}^{n} \\
&= \varphi^{-n} \circ \chi^{-1} \circ \tilde{h}^{n} = E^s_n(x).
\end{aligned}$$

To confirm that E^s gives us an immersion onto $W(\to p)$ we first need to check for smoothness and bijectivity.By definition, we are smooth and by since all functions that compose to E^s_n, the latter is injective for all n. Now if $E^s_n(x) = E^s_m(y)$ with $m > n$, then $\tilde{h}^m(x) \in B_0$, since $\tilde{h}$ is a contraction and by the well definition we have $E^s_m(x) = E^s_m(y)$ which concludes to $x = y$, as the latter are injective for m. The surjectivity is also clear, as for all $x \in W(\to p)$ there is an n such that $\varphi^n(x) \in \chi^{-1}(B_0)$ and with that the point $\tilde{h}^{-n}(\chi(\varphi^n))$ does the job. So the last step is to verify that $\mathrm{d}E^s|_x$ is injective. This follows from $\tilde{h}$ having an injective differential, since it is a contraction. Furthermore, χ^{-1} has an injective differential, as it is the inverse of a chart and φ has an injective differential, since $\mathrm{d}\varphi(x) = e^{-\frac{\partial}{\partial x}|_{\varphi(x)}}$. The latter equality was shown in the proof of remark 5.6. This concludes the proof for the stable set. For the unstable set we replace f by f^-1. □

Corollary 5.11. From the proof of the (un-)stable manifold theorem, we can deduce that $T_pW(\to p) = T^s_pM$.

Proof. For this we just use theorem 5.10 a bit more sophisticated. By the second statement of the theorem, we know that $T_pW^s_r(\theta) = T^s_p$. Notice that we actually talk about an equality here. But since $W^s_r(\theta) = \chi(W(\to p) \cap U)$ we know that $T_pW(\to p) = T^s_p$. □

6 The conley index

We start by defining some needed properties of maps:

Definition 6.1 (Cofibration)**.** Let (X, A) be a topological pair and Y be a topological space. The pair (X, A) satisfies the **homotopy extension property with respect to** Y, if and only if, we can extend homotopies. In other words, for all $f : X \to Y$ and $H : A \times I \to Y$ with $H(x, 0) = f(x)$ there exists a continuous extension $F : X \times I \to Y$ with $F(x, 0) = f(x)$. If a pair (X, A) satisfies the homotopy extension property with respect to any topological space Y, we call (X, A) a **cofibered pair** and the inclusion $i : A \hookrightarrow X$ a **cofibration**.

Definition 6.2. For as topological pair (N, L) define $N/L = N/\sim$ where $x \sim y$ if $x, y \in L$. In other words we contract L to be one point.

Remark 6.3. In the proof of the Morse homology theorem we want to talk about the homology of index pairs as relative homology groups $H_i^{\text{sing}}(N,L)$. However, if the inclusion $L \to N$ is a cofibration we have the natural isomorphism $H_i^{\text{sing}}(N,L) \cong H_i^{\text{sing}}(N/L)$. This is proven in [**LecturesonMorseHomology**] chapter two. To show that something is a cofibration we will use the following fact for metric spaces:

The pair (N,L), where L is closed in N is a cofibration (respectivly the inclusion), if an in N open neighbourhood U of L exists, such that L is a strong deformation retract of U.

This is also presented in chapter two of [**LecturesonMorseHomology**], by showing that such an inclusion admits a Strøm structure. An example of such a cofibration is the pair $(D^n, \partial D^n)$. Finally, a pair that is homeomorphic to a cofibration is again one.

Definition 6.4 (Compact invariant isolated subset)**.** For a flow $\varphi_t : M \to M$ on a locally compact metric space we call a subset $S \subseteq M$ **invariant subspace** if and only if $\varphi_t(S) = S$ for all $t \in \mathbb{R}$. For any subset $N \subseteq M$ we define the **maximal invariant subset**

$$\begin{aligned} I(N) &= \{x \in N | \ \varphi_t(x) \in N \forall t \in \mathbb{R}\} \\ &= \bigcap_{t \in \mathbb{R}} \varphi_t(N). \end{aligned}$$

A **compact invariant subset** S is called **isolated**, if a compact neighborhood N exists, such that $I(N) = S$.

Definition 6.5 (Index pairs)**.** Let S be an isolated compact invariant subset. A topological pair (N,L) of compact subsets of M where $L \subseteq N$ is called an **index pair of** S, if it satisfies the following:

1. $S = I(\overline{N \setminus L}) \subseteq (N \overset{\circ}{\setminus} L)$.
2. $x \in L$ and $\varphi_{[0,t]}(x) \subseteq N$ implies that $\varphi_{[0,t]}(x) \subseteq L$. We call L **positively invariant in** N. Here $\varphi_{[0,t]}(x) \coloneqq \{\varphi_{\tilde{t}}(x) | \tilde{t} \in [0,t]\}$.
3. For all $x \in N$ such that a t exists, with $\varphi_t(x) \notin N$, there exists a t' with $\varphi_{[0,t']}(x) \subseteq N$ and $\varphi_{t'}(x) \in L$.

This definition captures how the flow lines leave the invariant space S. Due to the third property we call L the **exit set**.

Definition 6.6 (Regular index pairs)**.** We call an index pair (N,L) **regular** if and only if the inclusion $I : L \hookrightarrow N$ is a cofibration.

Remark 6.7. One can show that every isolated compact invariant subset admits an index pair. We won't proof this, as we will always explicitly define such invariant sets and therefore we won't need such a general statement. However, two index pairs of the same invariant set turn out to be homotopy equivalent with a homotopy induced by the flow. The regularities needed for the proof are, that $\varphi_t : M \to M$ is a flow on a locally compact metric space. Those regularities are clearly given in the case of our Riemannian manifolds. We follow a proof from [**LecturesonMorseHomology**] which itself is a reformulation from the proof given by Salomon in [**SalomonConleyIndex**]. In this reformulation the language of the proof is adapted while the main arguments stay the same. This led to some redundant steps and vagueness, that I removed whilst making it more concrete.

Lemma 6.8. *Let N be an isolating neighborhood for the isolated compact invariant set S and let U be a neighbourhood of S. Then there exists a $t > 0$ such that for any $x \in M$ we have:*

$$\varphi_{[-t,t]}(x) \subseteq N \quad \Rightarrow \quad x \in U.$$

Proof. Lets assume this was false. Then for any $t > 0$ there would be a x contradicting the implication. So define $x_n \notin U$ such that $\varphi_{[-n,n]} \subseteq N$. Since $\varphi_0 = \mathrm{id}$ we know that all $x_n \in N$ and therefore by the compactness we would have limit points $x \in \overline{M \setminus U}$, such that $\varphi_{\mathbb{R}}(x) \subseteq N$. Since S was isolated we need to conclude that $x \in S \cap \overline{M \setminus U}$. This is our contradiction as U was a neighbourhood of the closed S and therefore without restriction open making $M \setminus U$ already closed. □

Remark 6.9. Notice that if t satisfies the conditions of lemma 6.8 then every $\tilde{t} \geq t$ also does. This helps us in the next lemma:

Lemma 6.10. *Let (N, L) and $(\tilde{N}, \tilde{L})$ be index pairs for the isolated invariant set S and choose $T \geq 0$ such that the following implications hold for $t \geq T$:*

$$\varphi_{[-t,t]}(x) \subseteq N \setminus L \quad \Rightarrow \quad x \in \tilde{N} \setminus \tilde{L}, \tag{6}$$

$$\varphi_{[-t,t]}(x) \subseteq \tilde{N} \setminus \tilde{L} \quad \Rightarrow \quad x \in N \setminus L. \tag{7}$$

Then the map:

$$\begin{aligned} h : N/L \times [T, \infty) &\to \tilde{N}/\tilde{L} \\ ([x], t) &\mapsto \begin{cases} [\varphi_{3t}(x)] & \text{if } \varphi_{[0,2t]} \subseteq N \setminus L \text{ and } \varphi_{[t,3t]}(x) \subseteq \tilde{N} \setminus \tilde{L} \\ [\tilde{L}] & \text{otherwise,} \end{cases} \end{aligned}$$

is continuous.

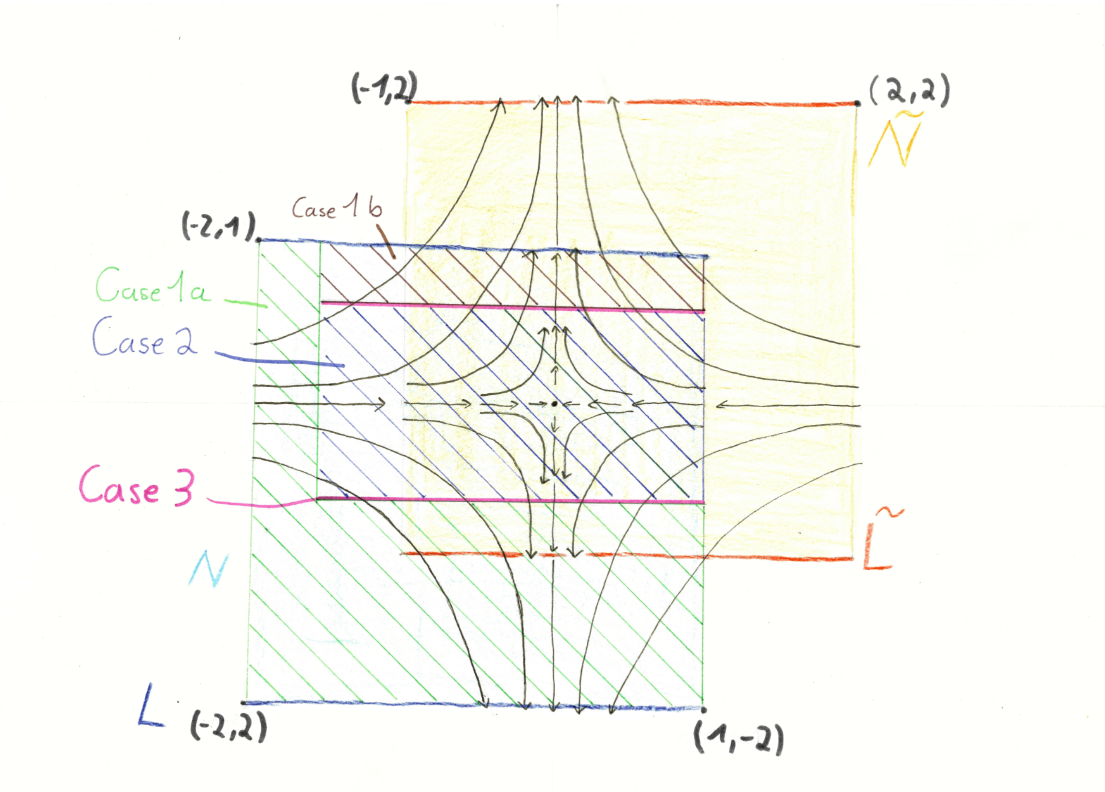


Figure 3: The different cases for the proof of lemma 6.10.
In this picture the flow is induced by the vectorfield $(-x, y)$ and drawn for $t = \frac{1}{3}$. That is due to the ease of calculation and such that the four cases are all visible. In the picture $N = [-2, 1]^2$ and $\tilde{N} = [-1, 2]^2$. L and $\tilde{L}$ are the horizontal borders.

Proof. We split this proof into three different cases that can be seen in figure 3. We always look at an image point lets say $h([x],t)$, choose an arbitrary neighbourhood U and define a neighbourhood W of $([x],t)$ such that the image of W is contained in U. This clearly gives us locally $\varepsilon - \delta$–continuity, as we are a metric space:
Case one a: $\varphi_{[t,3t]}(x) \nsubseteq \overline{\tilde{N} \setminus \tilde{L}}$. In this case there exists a $t < t^* < 3t$ such that $\varphi_{t^*}(x) \notin \overline{\tilde{N} \setminus \tilde{L}}$. t^* can be taken strictly less then $3t$ since the complement of $\overline{\tilde{N} \setminus \tilde{L}}$ is open. Furthermore, this tells us the existence of a neighbourhood U of $\varphi_{t^*}(x)$ that is disjoint from $\overline{\tilde{N} \setminus \tilde{L}}$. And by the continuity of the flow we have a neighbourhood $W \subseteq M \times [T, \infty]$ such that $(x', t') \in W$ implies that $\varphi_{t^*}(x') \in U$ and $t' < t^* < 3t'$. Thus, $\varphi_{[t',3t']}(x') \nsubseteq \tilde{N} \setminus \tilde{L}$ and therefore $h([x'],t') = [\tilde{L}]$ for all $(x',t') \in W$.

We can argue the same way if $\varphi_{[0,2t]} \nsubseteq (N \setminus L)$ (*Case one b*) and therefore conclude that in this case the map h is continuous. So for the rest of the proof we assume that we are in the first case of the map h or at the boundary:

$$\varphi_{[0,2t]} \subseteq \overline{N \setminus L} \text{ and } \varphi_{[t,3t]}(x) \subseteq \overline{\tilde{N} \setminus \tilde{L}}. \tag{8}$$

Case two: $\varphi_{[t,3t]}(x)$ is disjoint with $\tilde{L}$. Then due to the closure of $\tilde{N}$ and (8) we can conclude that $\tilde{N} \setminus \tilde{L} \supseteq \varphi_{[t,3t]}(x) = \varphi_{[-t,t]}(\varphi_{2t}(x))$ and by the implication (7) we can conclude that $\varphi_{2t}(x) \in N \setminus L$. Since L is the exit set we have that $\varphi_{[0,2t]}(x) \subseteq N \setminus L$. By the above we have that $h([x],t) = [\varphi_{3t}(x)] \in \tilde{N} \setminus \tilde{L}$. As before, due to the continuity of the flow we choose a neighbourhood U of $\varphi_{3t}(x)$ and find a neighbourhood $W \subseteq M \times [T, \infty)$ such that whenever $(x',t') \in W$ we have:

$$\varphi_{[0,2t']} \cap L = \varnothing, \quad \varphi_{[t',3t']}(x') \cap \tilde{L} = \varnothing, \quad \text{and} \quad \varphi_{3t'}(x') \in U.$$

If x' is in N then we have that $\varphi_{[0,2t']}(x')$ is in $N \setminus L$ and similar to before we conclude with (6) that $\varphi_{t'}(x') \in \tilde{N} \setminus \tilde{L}$ and since $\tilde{L}$ is the exit set we have the inclusion $\varphi_{[t',3t']}(x') \subseteq \tilde{N} \setminus \tilde{L}$. Therefore, we have that $h([x'],t') = [\varphi_{3t'}(x')] \in U$ for all $(x',t') \in W$ where $x' \in N$. The continuity of the flow gives us continuity in this area.
Case three: $\varphi_{[t,3t]}(x)$ intersects $\tilde{L}$. Then by (8) and since $\tilde{L}$ is the exit set we have that $\varphi_{3t}(x) \in \tilde{L}$. Now define $[U]$ to be a neighbourhood of $h([x],t) = [\tilde{L}]$ in $\tilde{N}/\tilde{L}$. We want to find a representative of $[U]$, that is an open set of M that reduces to $[U]$ in the quotient space. Let $\pi : \tilde{N} \to \tilde{N}/\tilde{L}$ be the quotient map. A natural choice would be $U := \pi^{-1}([U])$ which is without restriction open in $\tilde{N}$. To make it open in M we can unite it with $M \setminus \tilde{N}$. Now again by the continuity of the flow we have an open neighbourhood $W \subseteq M \times [T, \infty)$ of (x,t) such that whenever $(x',t') \in W$ we have that $\varphi_{3t'}(x') \in U$. But then we have that:

$$h([x'],t') \in \{[\varphi_{3t'}(x')], [\tilde{L}]\} \subseteq [U] \cap [\tilde{L}] = [U].$$

□

Lemma 6.11. *Let* $(N, L), (N', L')$ *and* $(\tilde{N}, \tilde{L})$ *be index pairs of* S. *Choose* $T > 0$ *such that the implications (6) and (7) are satisfied and furthermore choose a* $\tilde{T}$ *such that for* $t > \tilde{T}$ *we have:*

$$\varphi_{[-t,t]}(x) \subseteq \tilde{N} \setminus \tilde{L} \quad \Rightarrow \quad x \in N' \setminus L' \tag{9}$$

$$\varphi_{[-t,t]}(x) \subseteq N' \setminus L' \quad \Rightarrow \quad x \in \tilde{N} \setminus \tilde{L}. \tag{10}$$

Now define:

$$h : N/L \times [T, \infty) \to \tilde{N}/\tilde{L}$$
$$([x], t) \mapsto \begin{cases} [\varphi_{3t}(x)] & \textit{if } \varphi_{[0,2t]} \subseteq N \setminus L \textit{ and } \varphi_{[t,3t]}(x) \subseteq \tilde{N} \setminus \tilde{L} \\ [\tilde{L}] & \textit{otherwise,} \end{cases}$$

and

$$\tilde{h} : \tilde{N}/\tilde{L} \times [\tilde{T}, \infty) \to N'/L'$$
$$([x], t) \mapsto \begin{cases} [\varphi_{3t}(x)] & \textit{if } \varphi_{[0,2t]} \subseteq \tilde{N} \setminus \tilde{L} \textit{ and } \varphi_{[t,3t]}(x) \subseteq N' \setminus L' \\ [L'] & \textit{otherwise.} \end{cases}$$

Then the following equations hold for $t \geq \max\{T, \tilde{T}\}$*:*

$$\tilde{h}\big(h([x], t), t\big) = \begin{cases} [\varphi_{6t}(x)] & \textit{if } \varphi_{[0,4t]} \subseteq N \setminus L \textit{ and } \varphi_{[2t,6t]}(x) \subseteq N' \setminus L' \\ [L'] & \textit{otherwise.} \end{cases}$$

Proof. The proof is just the equivalence of the two statements:

$$\begin{aligned} & \varphi_{[0,4t]} \subseteq N \setminus L \text{ and } \varphi_{[2t,6t]}(x) \subseteq N' \setminus L' \\ \Leftrightarrow \quad & \varphi_{[0,2t]}(x) \subseteq N \setminus L, \ \varphi_{[t,5t]}(x) \subseteq \tilde{N} \setminus \tilde{L}, \ \varphi_{[4t,6t]}(x) \subseteq N' \setminus L'. \end{aligned}$$

„$\Rightarrow$" Here we need to check three things. The first and the third inclusion are trivially satisfied. For the second we notice that $\varphi_{[0,4t]} \subseteq N \setminus L$ implies that $\varphi_{[t,3t]}(x) \subseteq \tilde{N} \setminus \tilde{L}$ by implication (6). Furthermore, $\varphi_{[2t,6t]}(x) \subseteq N' \setminus L'$ implies that $\varphi_{5t}(x) \in \tilde{N} \setminus \tilde{L}$ by implication (10). And since $\tilde{L}$ is the exit set this implies the missing second inclusion.
„$\Leftarrow$" For the first inclusion notice that $x = \varphi_0(x) \in N$ by definition and $\varphi_{[t,5t]}(x) \subseteq \tilde{N} \setminus \tilde{L}$ implies $\varphi_{[2t,4t]}(x) \subseteq N \setminus L$ by implication (7). Using the exit set property we conclude that $\varphi_{[0,4t]} \subseteq N \setminus L$. Finally, $\varphi_{[t,5t]}(x) \subseteq \tilde{N} \setminus \tilde{L}$ implies that $\varphi_{[2t,4t]}(x) \subseteq N' \setminus L'$ by (9).

Together with $\varphi_{[4t,6t]}(x) \subseteq N' \setminus L$ this lets us conclude that $\varphi_{[2t,6t]}(x) \subseteq N' \setminus L'$. The additive property of the flow does the rest letting us conclude that we are in the first case of h and h' if and only if the stated property is satisfied. □

Theorem 6.12 (Homotopy equivalence of index pairs)**.** *If* S *is an isolated compact invariant set and* (N, L) *and* $(\tilde{N}, \tilde{L})$ *are two index pairs of* S. *Then* N/L *and* $\tilde{N}/\tilde{L}$ *are homotopy equivalent as pointed spaces.*

Proof. Let $h_t : N/L \to \tilde{N}/\tilde{L}$ and $g_t : \tilde{N}/\tilde{L} \to N/L$ be the continuous family of maps from lemma 6.10. By lemma 6.11 we get that

$$g_t \circ h_t([x]) = \begin{cases} [\varphi_{6t}(x)] & \text{if } \varphi_{[0,4t]} \subseteq N \setminus L \text{ and } \varphi_{[2t,6t]}(x) \subseteq N \setminus L \\ [L] & \text{otherwise.} \end{cases}$$

Furthermore, we use lemma 6.10 for $T = 0$. And now we can explicitly write down a homotopy from the identity to $g_t \circ h_t([x])$ as follows:

$$H : N/L \times [0,1] \to N/L$$
$$([x], t') \mapsto \begin{cases} [\varphi_{6t't}(x)] & \text{if } \varphi_{[0,4t't]} \subseteq N \setminus L \text{ and } \varphi_{[2t't,6t't]}(x) \subseteq N \setminus L \\ [L] & \text{otherwise.} \end{cases}$$

This homotopy is continuous by lemma 6.10 and $H(\cdot,0) = \mathrm{id}$ and $H(\cdot,1) = g_t \circ h_t$. Similar we show that $h_t \circ g_t$ is homotopic to the identity. With this we have that

$$N/L \simeq \tilde{N}/\tilde{L}.$$

□

Definition 6.13 (The homotopy Conley index)**.** The **Conley index** of an invariant set S is defined as $\pi_1(N, L)$ where (N, L) is a regular index pair. This definition is invariant of the chosen pair and is an invariant of invariant sets.

7 A Morse Theoretic Filtration

In this subchapter we will finally prove the Morse homology theorem. We start by defining some index sets that we will later use in the prove. Again, this prove is due to Solomon and found in [**MorseTheorySalmbon**].

Example 7.1 (Critical points as isolated compact invariant subsets)**.** Let $f : M \to \mathbb{R}$ be a Morse function on a smooth Riemannian manifold (M, g). Let $q \in \mathrm{Crit}_k(f)$. Around q we have coordinate charts $\varphi : U \to T_qM$ (after identifying q_i and ∂q_i) where $\varphi(W(q \to) \cap U) \subseteq T_q^uM$ and $\varphi(W(\to q) \cap U) \subseteq T_q^sM$. With this we can define the balls:

$$\begin{aligned} D_\varepsilon^s &= \{v \in T_q^sM \mid \|v\| \le \varepsilon\}, \\ D_\varepsilon^u &= \{v \in T_q^uM \mid \|v\| \le \varepsilon\}. \end{aligned}$$

They give rise to the index pair $N_q := \varphi^{-1}(D^s \times D^u)$ and $L_q = \varphi^{-1}(D^s \times \partial D^u)$. This index pair is in fact regular, since $(N_q, L_q) \cong (D^s \times D^u, D^s \times \partial D^u) \cong (D_\varepsilon^u, \partial D_\varepsilon^u)$. Now we can make the first step towards singular homology, since we know the homology of such a tuple:

$$H_i^{\mathrm{sing}}(N_q, L_q) = H_i^{\mathrm{sing}}(D^k, \partial D^k) = \begin{cases} \mathbb{Z} & \text{if } i = k, \\ 0 & \text{else.} \end{cases} \tag{11}$$

Notice that for $k = 0$ we need to consider $\partial D^k = \emptyset$, to get an index pair. That's why we let ∂ denote the manifold boundary insted of the topological boundary. However, now we can identify for all k:

$$C_k(M, f) = \bigoplus_{\mathrm{Crit}_k(f)} \mathbb{Z} \cong \bigoplus_{q \in \mathrm{Crit}_k(f)} H_k(N_q, L_q; \mathbb{Z}). \tag{12}$$

Furthermore, this isomorphism can be made canonically, by using orientations: Notice for this that (N_q/L_q) is homotopic equivalent to $W(q \to)/(W(q \to) \setminus \{q\})$. Here we get a generator of the k−homology group induced by an orientation on $T^u_p M$ that then canonically maps to the k−th homology group of (N_q, L_q).

Example 7.2 (A map of homology groups)**.** Let $f : M \to \mathbb{R}$ be a Morse Smale function on a smooth compact Riemannian manifold (M, g). Let $q \in \mathrm{Crit}_k(f)$ and $p \in \mathrm{Crit}_{k-1}(f)$. Assume for a moment we already have an isolated regular compact index pair (N_2, N_0) of $S = W(p \to q) \cup \{p, q\}$. For a $c \in (f(p), f(q))$ we set $N_1 = N_0 \cup (N_2 \cap M^c)$ depicted in figure 4, where M^c is the sublevel set. Clearly now (N_2, N_1) is an index pair for q and (N_1, N_0) is a pair for p. Assuming we have regular index pairs $(N_q.L_q)$ and (N_p, L_p) for p, q we can now define a map:

$$\Delta_k(q \to p) : H_k^{\mathrm{sing}}(N_q, L_q) \to H_{k-1}^{\mathrm{sing}}(N_p, L_p)$$

as the composition:

$$H_k(N_q, L_q; \mathbb{Z}) \xrightarrow{\cong} H_k(N_2, N_1) \xrightarrow{\delta_*} H_{k-1}(N_1, N_0) \xrightarrow{\cong} H_{k-1}(N_p, L_p)$$

where δ_* is the connecting homomorphism, and the other two maps are induced by the homotopic equivalence of two index pairs corresponding to the same isolated compact invariant subset. Putting all together we can define a morphism:

$$\Delta_k : \bigoplus_{q \in \mathrm{Crit}_k(f)} H_k^{\mathrm{sing}}(N_q, L_q) \to \bigoplus_{p \in \mathrm{Crit}_{k-1}(f)} H_{k-1}^{\mathrm{sing}}(N_p, L_p). \tag{13}$$

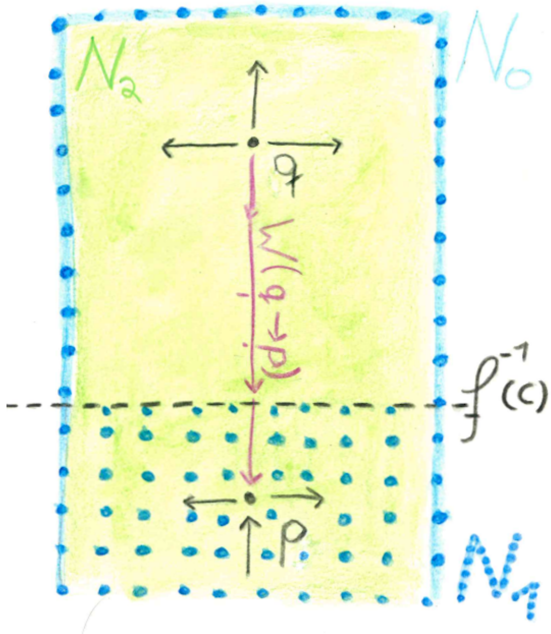


Figure 4: Index pairs for gradient flow lines.

Definition 7.3 (Regular index pairs and a filtration on M)**.** As always let $f : M \to \mathbb{R}$ be a Morse-Smale function on a compact smooth Riemannian manifold (M, g) of dimension m. Let $\varphi_t : M \to M$ be the flow given by $-\mathrm{grad} f$. For $0 \le j \le k \le m$ define:

$$W(k, j) = \bigcup_{j \le \lambda_p \le \lambda_q \le k} W(q \to p).$$

We know that this space is compact. Assume that N is a compact neighbourhood of $W(k, j)$, such that $\mathrm{Crit}(f) \cap N = \mathrm{Crit}(f) \cap W(k, j)$, meaning that N does not contain any new critical points. Then the biggest invariant subspace from definition 6.5 is reference

$$I(N) := \{x \in N | \varphi_t(x) \in N \forall t \in \mathbb{R}\} = W(j, k).$$

This follows from every gradient flow line starting and ending in a critical point. Therefore, $W(k, j)$ is an isolated compact invariant set. By a corollary of the lambda lemma we know that reference

$$\begin{aligned} W_j^s &:= \bigcup_{j \le \lambda_p} W(\to p) \\ W_j^u &:= \bigcup_{\lambda_p \le j} W(p \to) \end{aligned}$$

for all $j = 0, \ldots, m$ are compact as they are finite unions of compact sets. Now we define $N_m := M$ and choose a cofibered compact neighbourhood N_{m-1} of $W_{m-1}^{p \to}$ that is positively invariant and satisfies $N_{m-1} \cap W_m^{\to p} = \varnothing$ One could for example define:

$$N_{m-1} := N_m \setminus \Big(\bigcup_{q \in \mathrm{Crit}_m(f)} \mathring{N}_q \Big),$$

where N_q is taken from example 7.1. Once we check that N_{m-1} is an exit set with respect to N_m we can conclude that (N_m, N_{m-1}) is a regular index pair for $\mathrm{Crit}_m(f)$. The first statement is clear, since every gradient line starting in $N_m \setminus N_{m-1}$ has to pass through N_{m-1}, as they go towards other critical points. This tells us that they have to leave $N_m \setminus N_{m-1}$, which is enclosed by N_{m-1}. The second thing to check can be done by showing that there is a neighbourhood $U \subset N_M$ of N_{m-1} that deforms to N_{m-1}. For this notice that each N_q with $\lambda_q = m$ is homeomorphic to D^m. Now choose a open neighbourhood U_q in the disc of its boundary. By definition this retracts to the boundary with a retraction induced by the flow. And finally define $U := N_{m-1} \cap_{q \in \mathrm{Crit}_m(f)} U_q$. This retracts to N_{m-1} telling us that our index pair is indeed regular.

Now we want to inductively define a filtration: For this we choose a compact cofibered neighborhood N_{m-2} of W_{m-2}^u that is positifely invariant in N_{m-1} and has an empty intersection with $W_{m-1}^{\to p}$. This can be done similar to the construction of N_{m-1} but instead of cutting out neighbourhoods if $q \in \mathrm{Crit}_m(f)$, we can cut out tubular neighborhoods of W_{m-1}^s. This is depicted in figure 5 for the torus. By iterating this process we get a filtration:

$$\varnothing =: N_{-1} \subset N_0 \subset N_1 \subset \cdots \subset N_{m-1} \subset N_m = M, \tag{14}$$

such that (N_k, N_{j-1}) is a regular index pair for $W(k, j)$ for all $0 \le j \le k \le m$.

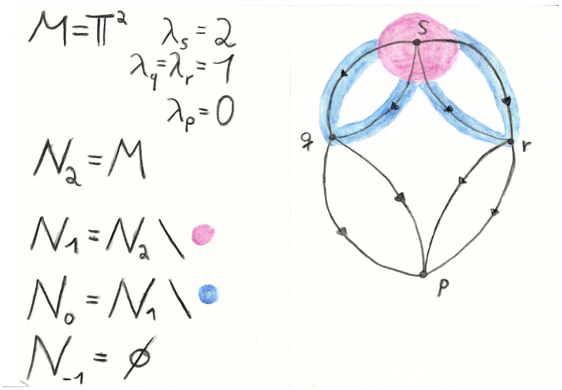


Figure 5: The filtration of the torus.
The picture shows the phase diagram of the torus and schematically depicts the filtration.

8 Orientations with K-theorie

Before we start the hole section, we establish a little reminder on how the K-Groups of the spheres look like: First of all the n-th K-group of a k-Sphere is by definition the reduced K-group of a $n+k$ Sphere. with this we have the general rule:

> If the dimension is **odd**, the K-group matters **not**!

Definition 8.1 (Orientations on Vector Spaces)**.** Let's start with a recall of what an orientation is. For this consideration we start with a vector space $V \cong \mathbb{R}^m$. An orientation of V is the equivalence class of an **ordered basis** $(v_1, \dots v_m)$, where the relation is as follows:
Two basis $v = (v_1, \dots, v_m)$ and $w = (w_1, \dots w_m)$ are equivalent, if the base-change matrix M_w^v has positive determinant. By this relation we get two orientations on $\mathbb{R}^m$. If $m = 0$, we define the orientation to be either 1 or -1. A vector space V together with an orientation $\theta = \overline{v}$ is called an oriented vector space and an ordered basis w of V is called **positive**, if $w \in \theta$, or in other words, if $w \sim v$.

Remark 8.2 (From Real Space to the Sphere)**.** We now want to construct a canonical homeomorphism between S^m and $\overline{\mathbb{R}^m}$. First we use the stereographic projection from the South Pole:

$$\begin{aligned} \pi_S : S^m \setminus \{S\} \subseteq \mathbb{R}^{m+1} &\to \mathbb{R}^m \\ (x_1, \dots, x_m, x_{m+1}) &\mapsto \left(\frac{x_1}{1 + x_{m+1}}, \dots, \frac{x_m}{1 + x_{m+1}} \right) \end{aligned}$$

with its inverse:

$$\begin{aligned} \pi_S^{-1} : \mathbb{R}^m &\to S^m \setminus \{S\} \subseteq \mathbb{R}^{m+1} \\ (y_1, \dots y_m) &\mapsto \left(\frac{2y_1}{1 + \|y\|^2}, \dots, \frac{2y_m}{1 + \|y\|^2}, \frac{1 - \|y\|}{1 + \|y\|} \right) \end{aligned}$$

Those are inverse maps and they give rise to maps between their one-point-compactifications. Hence $\overline{\mathbb{R}^m} \cong S^m$. Lets calculate $\pi_S^{-1} \circ \pi_s$, since we will later need a little bit of insight in those calculations:

$$\begin{aligned} \pi_S^{-1} \circ \pi_s(x_1, \dots, x_{m+1}) &= \pi_S^{-1} \left(\frac{x_1}{1 + x_{m+1}}, \dots, \frac{x_m}{1 + x_{m+1}} \right) \\ &= \left(\frac{2 \frac{x_1}{1+x_{m+1}}}{1 + \sum_{i=1}^m (\frac{x_i^2}{(1+x_{m+1})^2})}, \dots, \frac{2 \frac{x_m}{1+x_{m+1}}}{1 + \sum_{i=1}^m (\frac{x_i^2}{(1+x_{m+1})^2})}, \frac{1 - \sum_{i=1}^m (\frac{x_i^2}{(1+x_{m+1})^2})}{1 + \sum_{i=1}^m (\frac{x_i^2}{(1+x_{m+1})^2})} \right) \\ &= \end{aligned}$$

Before we keep calculating, we notice the denominator can be rewritten as:

$$1 + \sum_{i=1}^m \left(\frac{x_i^2}{(1 + x_{m+1})^2} \right) = \frac{(1 + x_{m+1})^2 + \sum_{i=1}^m x_i^2}{(1 + x_{m+1})^2}$$

And hence:

$$
\begin{aligned}
\frac{2\frac{x_j}{1+x_{m+1}}}{1+\sum_{i=1}^m\left(\frac{x_i^2}{(1+x_{m+1})^2}\right)} &= \frac{2\frac{x_j}{1+x_{m+1}}}{\frac{(1+x_{m+1})^2+\sum_{i=1}^m x_i^2}{(1+x_{m+1})^2}} && (15)\\
&= \frac{2x_j(1+x_{m+1})}{(1+x_{m+1})^2+\sum_{i=1}^m x_i^2} && (16)\\
&= \frac{2x_j(1+x_{m+1})}{(1+x_{m+1})^2+\sum_{i=1}^m x_i^2} && (17)\\
&= \frac{2x_j(1+x_{m+1})}{1+2x_{m+1}+\underbrace{\sum_{i=1}^{m+1} x_i^2}_{=1}} && (18)\\
&= x_j && (19)
\end{aligned}
$$

And the calculation of the last term is:

$$
\begin{aligned}
\frac{1-\sum_{i=1}^m\left(\frac{x_i^2}{(1+x_{m+1})^2}\right)}{1+\sum_{i=1}^m\left(\frac{x_i^2}{(1+x_{m+1})^2}\right)} &= \frac{\frac{(1+x_{m+1})^2-\sum_{i=1}^m x_i^2}{(1+x_{m+1})^2}}{\frac{(1+x_{m+1})^2+\sum_{i=1}^m x_i^2}{(1+x_{m+1})^2}}\\
&= \frac{(1+x_{m+1})^2-\sum_{i=1}^m x_i^2}{(1+x_{m+1})^2+\sum_{i=1}^m x_i^2}\\
&= \frac{1+2x_{m+1}+x_{m+1}^2-\sum_{i=1}^m x_i^2}{1+2x_{m+1}+\underbrace{x_{m+1}^2+\sum_{i=1}^m x_i^2}_{=1}}\\
&= \frac{1+2x_{m+1}+2x_{m+1}^2-\sum_{i=1}^{m+1} x_i^2}{2+2x_{m+1}}\\
&= \frac{x_{m+1}(2+2x_{m+1})}{2+2x_{m+1}} = x_{m+1}
\end{aligned}
$$

This concludes the calculation of $\pi_S^{-1} \circ \pi_s = \mathrm{id}$ the other order is similar.

Definition 8.3 (K-theoretic orientations)**.** We define a **K-theoretic orientation** of a vector space V to be a generator of the group $K^m(\overline{V})$. As a sainity check we can confirm: since $\overline{V} \cong S^m$ we are looking at a generator of a group isomorphic to $K(S^{2m}) \cong \mathbb{Z}$, which has two possible generators.

Now V is congruent to $\mathbb{R}^m$ via the coordinate map B_v and that depends on an ordered basis v of v. So the idea is as follows: We have a canonical generator of $K^m(S^m)$ and a map:

$$K^n(S^n) \xrightarrow{(\pi_S^{-1})^*} K^n(\overline{\mathbb{R}^m}) \xrightarrow{B_v^*} K^m(\overline{V})$$

This induces a generator in $K^m(\overline{V})$ that depends on the ordered basis.

Remark 8.4. The K−theoretic orientaion only depends on the orientation.

replace B_v^* mit B_v^-1, extension und so wohl unterscheiden. Vllt den beweis funktoriell aufziehen und faktorisiert über hTop verwenden

Proof. Let v and v' be two equivalent orientations, meanin $\det(M_{v'}^v) > 0$. This means that $M_{v'}^v \simeq \mathrm{id}$, since $\mathrm{GL}(m, \mathbb{R})$ has two path-connected components determined by the determinant. Alternative we can give an homotopy explicit by

$$\begin{aligned} H : \overline{\mathbb{R}^m} \times I &\to \overline{\mathbb{R}^m} \\ (x, t) &\mapsto (1-t)M_{v'}^v + t(\mathrm{id}) \\ (\infty, t) &\mapsto \infty \end{aligned}$$

This is continuos for all t, since it is continuous for all $x \neq \infty$. Furthermore, it is invertible for all t and $x \neq \infty$, because for $x \neq \infty$ we can view H as a continuous map $H(\cdot, t) : I \to \mathrm{GL}(m, \mathbb{R})$ and since I is connected it maps into one connected component and that is $\mathrm{GL}^+(n, \mathbb{R})$. Hence, the composition maps into $\det(H(\cdot, t)) \in \mathbb{R}_{>0}$ for all t. Finally, we have that the restricted map where the points at infinity are excluded is proper and thereby our extended map H is continuos, giving us a continuous homotopy. With this we can conclude, that the diagramm commutes up to homotopy:

$$\begin{tikzcd} \overline{V} \arrow[r, "\mathrm{id}"] \arrow[d, "B_v"'] & \overline{V} \arrow[d, "B_{v'}"] \\ \overline{\mathbb{R}^m} \arrow[r, bend left, "M_{v'}^v"] \arrow[r, bend right, "\mathrm{id}"'] & \overline{\mathbb{R}^m} \end{tikzcd}$$

Hence, the diagram commutes,

$$\begin{tikzcd} K^m(S^m) \arrow[r, "(\pi_S^{-1})^*"] & K^n(\overline{\mathbb{R}^m}) \arrow[r, "B_v^*"] \arrow[d, "\mathrm{id}"] & K^m(\overline{V}) \arrow[d, "\mathrm{id}"] \\ K^m(S^m) \arrow[r, "(\pi_S^{-1})^*"] & K^n(\overline{\mathbb{R}^m}) \arrow[r, "B_{v'}^*"] & K^m(\overline{V}) \end{tikzcd}$$

Thereby the induced generators agree. □

Lemma 8.5 (K-Theoretic Orientations Detect Orientations)**.** *Assume, that v' and v are non-equivalent orientations. Hence, $\det(M_{v'}^v) < 0$. Then they induce different generators.*

Proof. With a similar argument to above we can conclude, that $M_{v'}^v \simeq \Big((x_1, \dots x_m) \mapsto (x_1, \dots x_{m-1}, -x_m)\Big)$ and again there extensions to the one point compactifications are homotypic. Now we define two maps:

$$\begin{aligned} T : S^m &\to S^m \\ (x_1, \dots, x_{m+1}) &\mapsto (-x_1, \dots, x_{m+1}) \end{aligned}$$

and

$$\begin{aligned} S : \overline{R^m} &\to \overline{R^m} \\ (x_1, \dots x_m) &\mapsto (-x_1, \dots x_m) \end{aligned}$$

With those maps we have the commutative diagram:

$$\begin{tikzcd}
S^m & \overline{\mathbb{R}^m} \arrow[l, "\pi_s^{-1}"'] \\
S^m \arrow[u, "T"] & \overline{\mathbb{R}^m} \arrow[l, "\pi_s^{-1}"'] \arrow[u, "S"']
\end{tikzcd}$$

And hence since $S \simeq \overline{M_{v'}^{v}}$ we get the diagram, that is commutative up to homotopie:

$$\begin{tikzcd}
S^m & \overline{\mathbb{R}^m} \arrow[l, "\pi_s^{-1}"'] & \overline{V} \arrow[l, "B_{v'}^{-1}"'] \\
S^m \arrow[u, "T"] & \overline{\mathbb{R}^m} \arrow[l, "\pi_s^{-1}"'] \arrow[u, "S"] & \overline{V} \arrow[l, "B_v^{-1}"'] \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

This induces the commutative diagramm

$$\begin{tikzcd}
K^m(S^m) \arrow[r, "(\pi_S^{-1})^*"] \arrow[d, "T^*"'] & K^m(\overline{\mathbb{R}^m}) \arrow[r, "(B_v^{-1})^*"] \arrow[d, "S^*"] & K^m(\overline{V}) \arrow[d, "\mathrm{id}"] \\
K^m(S^m) \arrow[r, "(\pi_S^{-1})^*"'] & K^m(\overline{\mathbb{R}^m}) \arrow[r, "(B_{v'}^{-1})^*"'] & K^m(\overline{V})
\end{tikzcd} \tag{20}$$

Now we want to see that $T^* = -\mathrm{id}$. For this, we start by looking at $K^m(S^m) = K(S^{2m})$. In this case we define

$$\begin{aligned}
T' : S^m \wedge S^m = S^{2m} &\to S^{2m} \\
(x_1, \ldots x_m, y_1, \ldots, y_m) &\mapsto (x_1, \ldots x_m, -y_1, \ldots, y_m)
\end{aligned}$$

Now by definition $T^* = (T')^*$ Now we have the homotopy between T' and

$$\begin{aligned}
R \wedge \mathrm{id} : S^{2m} &\to S^{2m} \\
(x_1, \ldots x_{2m}) &\mapsto (-x_1, \ldots x_{2m})
\end{aligned}$$

because the change of order of bases elements is homotopic to the identity. (This can be seen by reusing the proof of lemma 8.4, since the needed base-change matrix has determinant 1). Now the way we defined $S^{2m} \subseteq \mathbb{R}^{2m+1}$ the map $R \wedge \mathrm{id}$ really is the wedge product of the identity on S^{2m-1} with the map R, that exchanges the South and North Pole. Hence by the lemma we had before , this agrees with the multiplication with -1, and hence the map T^* is the multiplication with -1. Finally, we can compare the induced generators: for this we denote $\beta_v := (B_v^{-1})^* \circ (\pi_S^{-1})^*(\beta)$ to be the generator given by the oriented basis v and $\beta_{v'} := (B_{v'}^{-1})^* \circ (\pi_S^{-1})^*(\beta)$ given by v'. Now by the

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commutativity of the diagram 20 we have that

$$\begin{aligned}
-\beta_{v'} &= -(B_{v'}^{-1})^* \circ (\pi_S^{-1})^*(\beta) \\
&= (B_{v'}^{-1})^* \circ (\pi_S^{-1})^*(-\beta) \\
&= -\beta_{v'} = -(B_{v'}^{-1})^* \circ (\pi_S^{-1})^*(\beta) \\
&= \underbrace{(B_{v'}^{-1})^* \circ (\pi_S^{-1})^* \circ T^*}_{=(B_v^{-1})^* \circ (\pi_S^{-1})^*}(\beta)) \\
&= \beta_v
\end{aligned}$$

□

Definition 8.6 (Oriented Vector Bundles)**.** For a vector bundle $\pi : E \to B$, an orientation is a family $\{\omega_x\}_{x \in B}$ of orientations of the fibers E_x such that there is an atlass Φ with the following propertie:
if $\varphi\Big|_U \to U \times \mathbb{R}^n$ is in Φ, then

$$\varphi_x : (E_x, \omega_x) \to (\mathbb{R}^n, \omega^n)$$

is orientation preserving. ω^n denotes the standars orientation of $\mathbb{R}^n$. If such an orientation exists, we call the bundle **orientable**.

Definition 8.7 (Orientable Manifold)**.** We call a manifold orientable, if its tangent bundle TM is orientable.

Corollary 8.8. A orientable manifold has a atlass, such that for all charts the jacobian of the transition maps is orientation preserving.

Proof. Let $(U_\alpha, \varphi_\alpha)$ be an atlas nof M. Each chart $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$ induces a basis of the Tangend space $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m})$ for all points in U_α. Let Φ be an oriented atlas of the tangend bundle $\pi : TM \to M$. Hence, we get □

9 What are Spheres

In this section we want to inspect representations of homotopical Spheres. To give a little overview we will define the sphere to come with an orientation. To be more specific we have the definition:

Definition 9.1 (The k−-Sphere)**.** Let $\mathbb{R}^k$ be the k-dimensional vector space given as the span of $e_1, \dots e_k$ together with the eucliden metric $\|\cdot\|_2$. Then the set

$$S^k := \{x \in \mathbb{R}^k | \, \|x\|_2 = 1\}$$

is called the k−**sphere**.

This sphere will be our save haven where we will always come back to. In this definition there is no room for "diffrent orientations on a sphere". In our reality we will detect the two "different orientations" of a space A homotopic to a sphere that arise from two constructions as follows: assume that α and β are the maps $A \to S^k$. then there is an endomorphism $f : S^k \to S^k$ such that our diagramm commutes:

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ S^k \arrow[rr, "f"] & & S^k \end{tikzcd}$$

Now $f^* : K^n(S^k) \to K^n(S^k)$ is an isomorphism and hence either the identity or minus the identity.

Now we have a lot of spaces that are homotopic or even homeomorph to the sphere. Just to name a view constructions:

1. The disc in $\mathbb{R}^k$ modulo its boundary $D^k/\partial D^k$.
2. The cone over the boundary of a D^k disc.
3. The resduced and unreduced susopension of a $k-1$ sphere.
4. The one-point compactification of a vector space together with an oriented basis.

We already saw how we can map the one-point compactification of a vector space with an ordered basis to a sphere wia the stereographic projections from the south pole. So lets inspect the other construction

Corollary 9.2. Given the disk $D^k \subseteq \mathbb{R}^k$ we have a homeomorphism

$$\overline{D^k}/\partial D^k \to S^k$$

Proof. We will construct a proper map from the open disk to R^k and by this we get a homeomorphism between their one-point-compaktifications.

$$\begin{aligned} s : D^k &\to \mathbb{R}^k \\ x &\mapsto x \cdot \frac{1}{1 - \|x\|_2} \end{aligned}$$

This is a homeomorphism and proper. Hence, we get a map between the onepoint compactifications

$$\overline{D^k}/\partial D^k \to \mathbb{R}^k \cup \{\infty\}$$

and the latter is homeomorphic to the Sphere. □

Corollary 9.3. Let $\overline{D^k} \cup C\partial D^k$ be the cone over the boundary of the disk. This is homeomorphic to the k–sphere.

Proof. Again we will use the language of onepoint compactifications. First we give a homeomorphism from the "open cone" over the boundary to the open disc. To do this we include our space into R^k as follows:

$$f : \overline{D^k} \cup \partial D^k \times [0,1) \to R^k$$
$$x \mapsto \begin{cases} x & \text{if } x \in \overline{D^k}\,, \\ (1+t)x & \text{if } (x,t) \in \partial D^k \times [0,1)\,. \end{cases}$$

This is a homeomorphism (if we restrict the image to $\{x \in R^k | \, \|x\|_2 < 2\}$). This can be seen as follows: Obviously it is bijective and the contiuouty can be derived from the gluing lemma. To see the contiuoity of the invese we can explicitly describe it:

$$f^{-1} : \{x \in R^k | \, \|x\|_2 < 2\} \to \overline{D^k} \cup \partial D^k \times [0,1)$$
$$x \mapsto \begin{cases} x & \text{if } x \in \overline{D^k}\,, \\ (\frac{x}{\|x\|_2}, (1 - \|x\|_2)) & \text{if } \|x\|_2 \geq 1\,, \end{cases}$$

Again, gluing lemma gives continouity, and a calculating gives us that they are inverse to each other. To see that the map is proper, we check if preimages of compact sets are compact. So let $K \subset \{x \in R^k | \, \|x\|_2 < 2\}$ be compact. Then $K = A \cup B$ where $A = K \cap \overline{D^k}$ and $B = K \cap \{x \in R^k | 1 \leq \|x\|_2 < 2\}$. Now A and B are again compact. (A is the intersection of two compact Hausdorff spaces and B can also be constructed as such a intersection) Since $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ we have to check if f^{-1} is a closed map restricted to $\overline{D^k}$ and $\{x \in R^k | 1 \leq \|x\|_2 < 2\}$. But those restrictions give rise to homeomorphisms and hence they are closed. Now we have a proper homeomorphism which lets us conclude, that the map induced on their onepoint compactifications are also homeomorphisms. After the rescaling:

$$r : \{x \in R^k | \, \|x\|_2 < 2\} \to D^k$$
$$x \mapsto \frac{x}{2}$$

we have a proper homeomorphism $\overline{D^k} \cup \partial D^k \times [0,1) \to D^k$. This lets us deduce the statment, as the onepoint compactification of the disk can be mapped to the sphere as seen in the corollary above. □

We will later use similar maps in our main prove, but now we want to talk about spheres in K−theorie.

Corollary 9.4. We want to understand homöomorphisms between boukettes of spheres. Or to beginn with let:

$$f : S^k \to \bigvee S^k$$

Now how does $f*$ look like? for this we

Proof. L □

10 Morsetheoretic Atiyah Hirzebruch Spectralsequeze

Definition 10.1. Let M be a smooth manifold with morse data and

$$\varnothing =: N_{-1} \subset N_0 \subset N_1 \subset \cdots \subset N_{m-1} \subset N_m = M,$$

be a filtrations constructed in 7.3. Then we define the Groups

$$E_r^{p,q}(M) := \frac{\ker\left[K^{p+q}(N_{p+r-1}, N_{p-r}) \to K^{p+q}(N_{p-1}, N_{p-r})\right]}{\ker\left[K^{p+q}(N_{p+r-1}, N_{p-r}) \to K^{p+q}(N_p, N_{p-r})\right]}.$$

To define a boundary operator, we proceed as follows: The following diagram commutes (by naturality of the boundary in the triple sequence).

$$\begin{tikzcd}
K^{p+q}(N_{p+r-1}, N_{p-r}) \arrow[r, "\alpha"] \arrow[d, "\delta^*_{triple}"] & K^{p+q}(N_p, N_{p-r}) \arrow[d, "\delta^*_{triple}"] & \\
K^{p+q+1}(N_{p+2r-1}, N_{p+r-1}) \arrow[r, "\beta"] & K^{p+q+1}(N_{p+2r-1}, N_p) \arrow[r] & K^{p+q+1}(N_{p+r-1}, N_p)
\end{tikzcd}$$

Hence, $\beta \circ \delta^*_{triple}$ factors over $K^{p+q}(N_{p+r-1}, N_{p-r})/\ker\alpha$ and because the bottom row is exact (by the triple sequence) we have get a map

$$K^{p+q}(N_{p+r-1}, N_{p-r})/\ker\alpha \to \ker\left[K_{p+q+1}(N_{p+2r-1}, N_p) \to K^{p+q+1}(N_{p+r-1}, N_p)\right]$$

Now since $K^{p+q}(N_{p+r-1}, N_{p-r}) \supseteq \ker\left[K^{p+q}(N_{p+r-1}, N_{p-r}) \to K^{p+q}(N_{p-1}, N_{p-r})\right]$ we can restrict the domain and furthermore compose with the quotient map to get a well-defined map

$$\begin{aligned} d_r^{p,q}:\ E_r^{p,q} &\to E_r^{p+r,q-r+1} \\ [x] &\mapsto [\beta \circ \delta^*_{triple}(x)] \end{aligned}$$

Corollary 10.2 (The first page)**.** We start by analysing the first page

$$\begin{aligned} E_1^{p,q}(M) := \frac{\ker\left[K^{p+q}(N_p, N_{p-1}) \to K^{p+q}(N_{p-1}, N_{p-1})\right]}{\ker\left[\underbrace{K^{p+q}(N_p, N_{p-1}) \to K^{p+q}(N_p, N_{p-1})}_{\mathrm{id}}\right]} &\cong K^{p+q}(N_p, N_{p-1}) \\ &= \tilde{K}^{p+q}(N_p/N_{p-1}) \end{aligned}$$

Now since (N_p, N_{p-1}) is a regular index pair for $\mathrm{Crit}_p(f)$ we have a flow induced homotopiy equivalentce:

$$N_p/N_{p-1} \simeq \Big(\bigcup_{w \in Crit_p(f)} D^u_\varepsilon(w)\Big)/\Big(\bigcup_{w \in Crit_p(f)} \partial D^u_\varepsilon(w)\Big) \cong \bigvee_{w \in \mathrm{Crit}_p(f)} S^p$$

Here, we use the definitions from above

$$D^u_\varepsilon(w) \ := \ \varphi_w^{-1}\big(\{v \in T^u_w M \mid \|v\| \le \varepsilon\}\big),$$

and the last isomorphism is induced by orientations in the unstable manifolds. We can continue our inspection:

$$\begin{aligned} E_1^{p,q} &\cong \tilde{K}^{p+q}(N_p/N_{p-1}) \\ &\cong \bigoplus_{w\in \mathrm{Crit}_p(f)} \tilde{K}^q(pt) = C^k(M,f,g,\mathfrak{o};\tilde{K}^q(pt)) \end{aligned}$$

In sum we have the chain of horizontal isomorphisms:

$$\begin{tikzcd} E_1^{p,q} \arrow[r, "\alpha"] \arrow[d, "d_1^{p,q}"] & \tilde{K}^{p+q}(N_p/N_{p-1}) \arrow[r, "\beta"] \arrow[d] & \oplus_{w\in \mathrm{Crit}_p(f)} \tilde{K}^q(pt) \arrow[d] & C^p(M,\mathfrak{A},\tilde{K}^q(pt)) \arrow[l] \arrow[d] \\ E_1^{p+1,q} \arrow[r, "\alpha'"] & \tilde{K}^{p+1+q}(N_{p+1}/N_p) \arrow[r, "\beta'"] & \oplus_{w\in \mathrm{Crit}_{p+1}(f)} \tilde{K}^q(pt) & C^{p+1}(M,\mathfrak{A},\tilde{K}^q(pt)) \arrow[l] \end{tikzcd}$$

By naturality of the triple boundary operator, that the second vertical map is the boundary operator. Now on the next rug we can induce two maps. From the left to make the diagram commute and from the right. Do they agree? Or asked differently, is the map induced from the d_1 map the boundary operator of morse homology?

Definition 10.3. Let $q \in \mathrm{Crit}_{k+1}(f)$ and $p \in \mathrm{Crit}_p(f)$. Assume for the moment, that we have a regular index pair (N_2, N_1) for $S := W(q \to p) \cup \{p,q\}$. For $c \in (f(p), f(q))$ we define $N_1 = N_0 \cup (N_2 \cup M^c)$, where M^c denotes the sublevelset. Now (N_2, N_1) is a index pair for q and (N_1, N_0) is one for p. Assume that we are givben any two index pairs (N_q, L_q) and (N_p, L_p) of q and p. Then we define a map

$$\Delta_k(q \to p) : K^k(N_p, L_p) \to K^k(N_q, L_q)$$

as the composition:

$$K^k(N_p, L_p) \xrightarrow{\cong} K^k(N_1, L_0) \xrightarrow{\delta_{triple}} K^k(N_2, L_1) \xrightarrow{\cong} K^k(N_q, L_q)$$

Putting all those together we have the definition of a map:

$$\Delta_k : \bigoplus_{p\in \mathrm{Crit}_k(f)} K^k(N_p, L_p) \to \bigoplus_{q\in \mathrm{Crit}_{k+1}(f)} K^k(N_q, L_q)$$

If we denote the inclusion $i_q : K^k(N_q, L_q) \to \oplus_{q\in \mathrm{Crit}_{k+1}(f)} K^k(N_q, L_q)$ we have the above map given by:

$$\Delta_k = \sum_{q\in \mathrm{Crit}_{k+1}(f)} i_q \circ \Big(\bigoplus_{p\in \mathrm{Crit}_f(q)} \Delta_k(q \to p) \Big)$$

all depends on the choice of index pairs?

Lemma 10.4. *Let M be an oriented manifold and $q \in \mathrm{Crit}(f)$. Furthermore, let (N_q, L_q) be a regular index pair. Then an orientation in $T_q^u M$ induces a generator of $K^{\mathrm{ind}(q)}(N_p, L_p)$*

Proof. First, a generator of $T_q^u M$ induces a generator of $K^{\mathrm{ind}(q)}(\overline{T_q^u M})$. Now we need to find a natural map $(N_p/L_p) \to \overline{T_q^u M}$. We do this in steps. First, we define:

$$\begin{aligned} s : B_q^u(\varepsilon) := \{x \in T_q^u M \mid \|x\| \leq \varepsilon\} &\to \overline{T_q^u M} \\ x &\mapsto \begin{cases} \frac{x}{\|x\| - \varepsilon} & \text{if } \|x\| < \varepsilon \\ \infty & \text{else .} \end{cases} \end{aligned}$$

This map is can be made into a homomorphism if we make it bijective, meaning we identify all fibers of ∞ giving us the map

$$\begin{aligned} \overline{s} : B_q^u(\varepsilon)/\partial B_q^u(\varepsilon) &\to \overline{T_q^u M} \\ x &\mapsto s(x) \end{aligned}$$

now this map is again a homeomorphisms. Now we need to get a map from $N_p/L_p \to B_q^u(\varepsilon)/\partial B_q^u(\varepsilon)$. We proceed as follows: First we get a diffrent index pair as follows: Let $\varphi : U \to T^u M$ be a morse chart and $D_u^s(\varepsilon)$ be a closed ball of radius ε living in $T_q^s M$. Then $\left(\varphi^{-1}(B_q^u(\varepsilon) \times B_q^s M)\ ,\ \varphi^{-1}(\partial B_q^u(\varepsilon) \times B_q^s M)\right)$ is a regular index pair for q, hence there is a flow induced map

$$\psi :\ (N_p/L_p) \to \left(\varphi^{-1}(B_q^u(\varepsilon) \times B_q^s M) \Big/ \varphi^{-1}(\partial B_q^u(\varepsilon) \times B_q^s M)\right)$$

Furthermore we have the homöomorphism induced from φ:

$$\left(\varphi^{-1}(B_q^u(\varepsilon) \times B_q^s M) \Big/ \varphi^{-1}(\partial B_q^u(\varepsilon) \times B_q^s M)\right) \to \left(B_q^u(\varepsilon) \times B_q^s M \Big/ \partial B_q^u(\varepsilon) \times B_q^s M\right)$$

and the contraction

$$\left(B_q^u(\varepsilon) \times B_q^s M \Big/ \partial B_q^u(\varepsilon) \times B_q^s M\right) \ \to\ B_q^u(\varepsilon)/B_q^s(\varepsilon),$$

which concludes the proof.

The independence of the choice of the chart comes from M being oriented and hence diffrent charts induce the same orientation. Furthermore, in the definition of the map between regular index pairs, we choose a T, but diffrent suitable T give homotopic maps! □

show T and T' induced maps between regular index pairs are homotopic

Theorem 10.5. *Given the filtration $N_{-1} \subseteq N_0 \subseteq \cdots \subseteq N_{m-1} \subseteq N_m = M$ from 14. The following diagram commutes:*

$$\begin{tikzcd}
C^k(M, \mathfrak{A}, \tilde{K}^l(pt)) \arrow[r, "\partial^p"] \arrow[d] & C^{k+1}(M, \mathfrak{A}, \tilde{K}^l(pt)) \arrow[d] \\
\bigoplus\limits_{q \in \mathrm{Crit}_k(f)} \tilde{K}^{k+l}(N_q, L_q) \arrow[r, "\Delta_{k+l}"] \arrow[d] & \bigoplus\limits_{p \in \mathrm{Crit}_{k+1}(f)} \tilde{K}^{k+l+1}(N_p, L_p) \arrow[d] \\
K^{k+l}(N_k, N_{k-1}) \arrow[r, "\delta_{triple}"] & K^{p+l+1}(N_{k+1}, N_k)
\end{tikzcd}$$

Proof. The lower part should just be homological algebra. The top part is the interesting part! So we assume for the moment, that $q \in \mathrm{Crit}_{p+1}(f)$ and $p \in \mathrm{Crit}_p(f)$ are the only critical points in $f^{-1}([a,b])$, where $a := f^{-1}(p)$ and $b := f^{-1}(q)$. .

this can be archived by alternations of f

Now, we choose the index pairs wisely: First we define the notations:

$$M^t := \{x \in M | f(x) \leq t\} \ , \ M_t := \{x \in M | f(x) \geq t\}$$

and the constants:

$$c \in (a,b) \ , \ \varepsilon > 0 \text{ small enough } \ , \ T > 0 \text{ large enough}$$

Now we define the following sets:

$$\begin{aligned} N_q &:= \{x \in M_c | f(\varphi_{-T}(x)) \leq b + \varepsilon\} \\ L_q &:= \{x \in N_q | f(x) = c\} \\ N_p &:= \{x \in M^c | f(\varphi_T(x)) \geq a - \varepsilon\} \\ L_p &:= \{x \in N_p | f(\varphi_T(x)) = a - \varepsilon\} \end{aligned}$$

and with those the sets;

$$\begin{aligned} C &:= N_p \cup N_q \\ B &:= N_p \cup L_q \\ A &:= L_p \cup (L_q \overset{\circ}{-} N_p) \end{aligned}$$

With those we have the following list of facts:

1. (N_q, L_q) is a regular index pair for q.
2. (C, B) is an index pair for q.
3. (N_p, L_p) is a regular index pair for p.
4. (B, A) is an index pair for p

proof the list

We have a contraction

$$c : (N_q, L_q) \to (W(q \to) \cap M_c, W(q \to) \cap f^{-1}(c))$$

Furthermore we show that N_p is a tubular neighbourhood of $W(\to p) \cap M^c$ and hence after analysing L_p we get the isomorphisms to :

Muss noch gezeigt werden, dass das wirrklich deformiert!

$$(N_p, L_p) \cong (\underbrace{D^{k-1}}_{\dim W(p\to)} \times \underbrace{D^{m-k+1}}_{\dim W(\to p)}, \partial D^{k-1} \times D^{m-k+1})$$

Now by compactnes the space $N_p \cap (W(q \to) \cap f^{-1}(c))$ has finitly many connected components $V_1, \dots V_n$, and for each V_j there is a $x_j \in V_j \cap W(q \to p)$. Since N_p is a tubular neighbourhood we get the diffeomorphism

Hier fehlt auch noch ein Beweis

$$\psi_p : N_p \to \underbrace{D^{k-1}}_{\dim W(p\to)} \times \underbrace{D^{m-k+1}}_{\dim W(\to p)}$$

such that

1. $\psi(L_p) = \partial D^{k-1} \times D^{m-k+1}$,
2. $\psi_p(N_p \cap W(\to p) = \{0\} \times D^{m-k+1}$,
3. $\psi(V_j) = D^{k-1} \times \{\theta_j\}$ where $\theta_j \in \partial D^{m-k+1}$.

Using this map we get diffoemorphisms

$$\psi_j : V_j \to D^{k-1}$$
$$x \mapsto \pi_1 \circ \psi_p(x) \text{ where } \pi_1 \text{ is the projection onto the first factor.}$$

This map restricts to a diffeomorphism from $\partial V_j = V_j \cap L_p$ to ∂D^{k-1} Hence, we get the continouos maps.

das durch-denken, ob das klar ist

$$(N_p, L_p) \xrightarrow{\pi_1 \circ \psi_p} (D^{k-1}, \partial D^{k-1}) \xleftarrow{\psi_j} (V_j, \partial V_j)$$

and hence

$$K^{k-1}(N_p, L_p) \xleftarrow{(\pi_1 \circ \psi_p)^*} K^{k-1}(D^{k-1}, \partial D^{k-1}) \xrightarrow{(\psi_j)^*} K^{k-1}(V_j, \partial V_j)$$

An orientation of $T_p^u M$ induces a generator in the middle from the left (via $((\pi_1 \circ \psi_p)^*)^{-1}$). This generator can be mapped via $((\psi_j)^*)^{-1}$ to $K^{k-1}(V_j, \partial V_j)$. Another way to get a generator on the right side is from an orientation of $T_q^u M$ as follows: We notice, that

$$T_{x_j} V_j = (-\text{grad}(f)_{x_j})^\perp \cap T_{x_j} W(q \to)$$

Hence, a orthogonal oriented basis $(-\text{grad}(f)_{x_j}\ ,\ b_q^u)$ gives a basis (b_q^u) of $T_{x_j} V_j$, and hence a generator of $K^{k-1}()$

different idea

Using parallel transport, we get a basis of $T_{x_j} V_j$ in two ways: First from a basis of $T_p^u M$ via parrallel transport along the flow line of $\varphi_t(x_j)$ and secondly because

$$T_{x_j} V_j = (-\text{grad}(f)_{x_j})^\perp \cap T_{x_j} W(q \to) he$$

Hence, a orthogonal oriented basis $(-\text{grad}(f)_{x_j}\ ,\ b_q^u)$ gives a basis (b_q^u) of $T_{x_j} V_j$. We define $n_j = \pm 1$ depending on wheather those agree ore not. (This agrees with the sign in the Morse boundary operator associated to the orbit containing x_j) Now define $S(q \to) := W(q \to) \cap f^{-1}(c) = W^u(q) \cup L_q$. With this we have the commuting diagram

$$\begin{tikzcd}
\bigoplus_j K^k(S(q \to), \overline{S(q \to) \setminus V_j}) \arrow[d] & \\
K^k(S(q \to), \overline{S(q \to) \setminus \bigsqcup_j V_j}) \arrow[r, "\delta_{triple}"] \arrow[d] & K^{k+1}(W(q \to) \cap N_q, S(q \to)) \arrow[d] \\
K^k(B, A) \arrow[r, "\delta_{triple}"] & K^{k+1}(C, B)
\end{tikzcd}$$

Now inspect the map

$$\delta_j : K^k(S(q\to), \overline{S(q\to)\setminus V_j}) \to K^{k+1}(W(q\to)\cap N_q, S(q\to))$$

This map is induced by the follwing maps:

$$\begin{tikzcd}
(W(q\to)\cap N_q)\cup C_1S(q\to) \arrow[r] \arrow[d, hook] & W(q\to)\cap N_q)/(S(q \arrow[d] \\
\big((W(q\to)\cap N_q)\cup C_1S(q\to)\big)\cup C_2(W(q\to)\cap N_q) \arrow[r] \arrow[d, hook] & S\wedge(S(q\to)) \arrow[d] \\
\Big(\big((W(q\to)\cap N_q)\cup C_1S(q\to)\big)\cup C_2(W(q\to)\cap N_q)\Big)\cup C_3(C_1\overline{S(q\to)\setminus V_j}) \arrow[r] & S\wedge(S(q\to))/(S\wedge\overline{S(q -}
\end{tikzcd}$$

Let β be a generator of the Group

$$K^{k+1}(W(q\to)\cup N_q, W(q\to)\cup L_q)$$

Since The $V_j \subset W(q\to)\cup N_q$ are contractible, without restrictions we have that $\beta\big|_{V_j} = V_j \times \mathbb{C}^{k+1}$ with the basis (b_j, v_j) such that b_j is an orientation of $T_{x_j}V_j$ and (b_j, v_j) is an oriented basis of $T_{x_j}(W(q\to)\cup N_q)$. Now how does

$$\delta^j_{pair}(\beta) \in K^k$$

Notice how our map is from a $k+1-$ sphere $(W(q\to)\cap N_q)\big/S(q\to) \cong S^{k+1}$ to a $k+1-$Sphere $S\wedge(V_j\big/\partial V_j) \cong S^{k+1}$. Hence, we want to figure out, if it is orientation reversing or not. For this we need to really focus on the sphere-isomorphisms. And ask, how the map looks if the diagramm commutes:

$$\begin{tikzcd}
(W(q\to)\cap N_q)\cup C_1S(q\to) \arrow[r] \arrow[d, hook] & S^{k+1} \arrow[d] \\
\Big(\big((W(q\to)\cap N_q)\cup C_1S(q\to)\big)\cup C_2(W(q\to)\cap N_q)\Big)\cup C_3(C_1\overline{S(q\to)\setminus V_j}) \arrow[r] & S^{k+1}
\end{tikzcd}$$

Hence we need to specify the horizontal (and vertical) maps. So we start with the top one:

$$\big(W(q\to)\cap N_q \cup C_1S(q\to)\big) \to S^{k+1}$$

First we have a map $E^u : T^u_qM \to W(q\to)$ from theorem 5.2 which is a diffeomorphism onto its range. Hence, using its inverse we have a homeopmorphism $W(q\to)\to T^u_qM \cong$

$\mathbb{R}^{k+1}$ where the last congruence is given by an orientation. We can restrict this homeomorphism to $W(q \to) \cap N_q = W(q \to) \cap M_c$ to get a mapping onto a closed subspace of R^{k+1} and after a rescaling this is a closed Disk in R^{k+1}. This follows from $f(E^u(x))$ strictly decreasing if $\|x\|$ decreses, which again follows from the definiton of E^u, which comes from „flowing along the flow downwards "Hence the condition in the image of f being greater or equal to c translates to a condition in the domain of having norm less or equal to someting. w

In fact this is a closed Disc, since the map $E^u(x) = E^u_n(x) = \varphi^n \circ \chi^{-1} \circ \tilde{h}^n(x)$ for a suitible n. (χ is a chart in this context and we can assume that its image is a disc of radius 1) Careful, we swiched back to our discrete setting defining $\varphi := \varphi_1$. Compare the proof of theorem 5.2 to the definitinos. Hence, the inverse is given by $\tilde{h}^{-n}(x) \circ \chi \circ \varphi^{-n}$. Since there is no critical point above c we can assume, that all points in $W(q \to)$ flow below c in finite time. Let $U \subset W(q \to) \cap M_c$ be the domain of χ and ∂U be its boundary. Now for each $x \in D^{k+1}$ we have $\chi^{-1}(\frac{x}{\|x\|}) \in \partial U$ and for each x we define t_x such that $\varphi_{t_x}(\chi^{-1}(\frac{x}{\|x\|})) \in f^{-1}(x)$. With this we define the rescaling

$$\begin{aligned} \chi\big(\varphi^{-n}(W(q \to) \cap M_c)\big) &\to \overline{D^{k+1}} \\ x &\mapsto t_x x \end{aligned}$$

Define t_x such that $f(\varphi_{t_x}(x)) = c$. Now let n be the smallest natural number such $\varphi^{-n}(W(q \to) \cap M_c)$ is contained in the domain of χ. Now $\varphi^{-n}(W(q \to) \cup f^{-1}(c))$

Lets restrict the chart $\chi^0 =: \chi : T^u M \supset U \to W(q \to)$ such that $U = \overline{D^{k+1}}$ is a closed Disc. Now for each

$x \in U \setminus q$ there is a number $t_{0,x} \geq 0$ such that $\varphi_{t_{0,x}}(\chi(x)) \in \chi(\partial U)$. Furthermore for each $x \in U$, there is a number$t_{1,x}$ such that $\varphi_{t_{1,x}}(\chi(x)) \in f^{-1}(c)$ Those numbers all smoothly depend on x.

Now define the map

$$\begin{aligned} \Psi : \overline{D^{k+1}} &\to W(q \to) \cap M_c \\ x &\mapsto \varphi_{(t_{1,x} - t_{0,x})}\chi((x)) \end{aligned}$$

This map is a homeomorphism, since χ is one, and the flow as a map $\varphi : \mathbb{R} \times M \to M$ is continouos. Furthermore there is an inverse given as follows: For each $x \in W(q \to) \cap M_c$ there is a real number $s_{0,x}$ such that $\varphi_{s_{0,x}}(x) \in \chi(\partial U)$, and a real number $s_{1,x}$ such that $\varphi_{s_{1,x}}(x) \in f^{-1}(c)$. Then the map

$$\begin{aligned} \Phi : W(q \to) \cap M_c &\to \overline{D^{k+1}} \\ x &\mapsto \chi^{-1}(\varphi_{s_{0,x} - s_{1,x}})(x) \end{aligned}$$

is the inverse of χ: To see this notice that for $y = \chi(x)$ we have $t_{0,x} = s_{0,y}$, and $t_{1,x} = s_{1,y}$ by definition.

Now call $A := \varphi_{(t_{1,x} - t_{0,x})}\chi(x) = \varphi_{(s_{1,\chi(x)} - s_{0,\chi(y)})}\chi(x)$, then we have:

- $s_{1,A} = s_{0,\chi(x)}$ and,

- $s_{0,A} = -s_{1,\chi(x)} + 2s_{0,\chi(x)}$.

This is because:

$$\varphi_{s_{0,\chi(x)}} \circ \varphi_{(t_{1,x}-t_{0,x})}\chi(x) = \varphi_{s_{0,\chi(x)}} \circ \varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x) = \varphi_{(s_{1,\chi(x)})}\chi((x)) \in f^{-1}(c)\,,$$

and

$$\varphi_{-s_{1,\chi(x)}+2s_{0,\chi(x)}} \circ \varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x) = \varphi_{(s_{0,\chi(x)})}\chi((x)) \in f(\partial U)\,.$$

Hence

$$\begin{aligned}\Phi \circ \Psi(x) &= \Phi\Big(\varphi_{(t_{1,x}-t_{0,x})}\chi((x))\Big)\\ &= \Phi\Big(\underbrace{\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x)}_{=A}\Big)\\ &= \chi^{-1} \circ \varphi_{s_{0,A}-s_{1,A}}(\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x))\\ &= \chi^{-1} \circ \underbrace{\varphi_{-s_{1,\chi(x)}+2s_{0,\chi(x)}-s_{0,\chi(x)}}(\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}}_{=\mathrm{id}}\chi(x))\\ &= x\end{aligned}$$

And the other way round we have the relations for $B := \chi^{-1}(\varphi_{s_{0,x}-s_{1,x}}(x))$:

$$\begin{aligned}t_{0,B} &= s_{1,x}\\ t_{1,B} &= -s_{0,x} + 2s_{1,x}\,,\end{aligned}$$

since we can calculate:

$$\begin{aligned}\varphi_{s_{1,x}}(\chi(B)) &= \varphi_{s_{1,x}}(\varphi_{s_{0,x}-s_{1,x}}) = \varphi_{s_{0,x}}(x) \in f(\partial U)\\ \varphi_{-s_{0,x}+2s_{1,x}}(\chi(B)) &= \varphi_{-s_{0,x}+2s_{1,x}}(\varphi_{s_{0,x}-s_{1,x}}) = \varphi_{s_{1,x}}(x) \in f^{-1}(c)\end{aligned}$$

$$\begin{aligned}\Psi \circ \Phi(x) &= \Psi\big(\chi^{-1}(\varphi_{s_{0,x}-s_{1,x}})(x)\big)\\ &= \varphi_{t_{1,B}-t_{0,B}}\chi \circ \chi^{-1}(\varphi_{s_{0,x}-s_{1,x}})(x)\\ &= \varphi_{-s_{0,x}+2s_{1,x}-s_{1,x}}\varphi_{s_{0,x}-s_{1,x}})(x)\\ &= x\end{aligned}$$

So now we have a diffeomorpism $\Psi : W(q \to) \cap N_q \to \overline{D^{k+1}}$. We extend this to a homeomorphism

$$\begin{aligned}(W(q \to) \cap N_q) \cup C_1 S(q \to) &\to \overline{D^{k+1}} \cup C(\partial D^{k+1}) \subseteq (\mathbb{R}^{k+1} \times I)\big/\mathbb{R}^{k+1} \times \{1\}\\ (x,t) &\mapsto (\Phi(x), t)\end{aligned}$$

This map is well defined, since $S(q\rightarrow) = W(q\rightarrow) \cap f^{-1}(c)$, and if $x \in S(q\rightarrow)$, then $s_{1,x} = 0$ and hence:

$$\Phi(x) = \chi^{-1}(\underbrace{\varphi_{s_{0,x}}(x)}_{\in\chi(\partial U)}) \in \partial(U) = \partial(\overline{D^{k+1}})$$

Now we want to construct the map

$$\left(\Big((W(q\rightarrow)\cap N_q)\cup C_1S(q\rightarrow)\Big)\cup C_2(W(q\rightarrow)\cap N_q)\right)\cup C_3(C_1\overline{S(q\rightarrow)\setminus V_j}) \rightarrow S^{k+1} \quad (21)$$

First we contract all unnecesarry parts:

$$\begin{aligned}
&\left(\Big((W(q\rightarrow)\cap N_q)\cup C_1S(q\rightarrow)\Big)\cup C_2(W(q\rightarrow)\cap N_q)\right)\cup C_3(C_1\overline{S(q\rightarrow)\setminus V_j})\\
&\rightarrow \left(\Big((W(q\rightarrow)\cap N_q)\cup C_1S(q\rightarrow)\Big)\Big/(W(q\rightarrow)\cap N_q)\right)\Big/(C_1\overline{S(q\rightarrow)\setminus V_j})\\
&= \Big((W(q\rightarrow)\cap N_q)\cup C_1S(q\rightarrow)\Big)\Big/\Big((W(q\rightarrow)\cap N_q)\cup(C_1\overline{S(q\rightarrow)\setminus V_j})\Big)\\
&= \Big(C_1S(q\rightarrow)\Big)\Big/\Big(S(q\rightarrow)\cup(C_1\overline{S(q\rightarrow)\setminus V_j})\Big)\\
&= \big(C_1V_j\big)\big/\big(V_j\cap C_1(\partial V_j)\big)
\end{aligned}$$

To summarize, we get a continouos map Θ:

$$\begin{aligned}
&\left(\Big((W(q\rightarrow)\cap N_q)\cup C_1S(q\rightarrow)\Big)\cup C_2(W(q\rightarrow)\cap N_q)\right)\cup C_3(C_1\overline{S(q\rightarrow)\setminus V_j})\\
&\rightarrow \big(C_1V_j\big)\big/\big(V_j\cap C_1(\partial V_j)\big)
\end{aligned}$$

Careful! This map does not neet to be a homotopy equivalence and hence does not give us a continuous inverse up to homotopie. It is a quotiet map induced by collapsing contractible subsets, and hence induces an isomorphism in the K-groups. via ψ we get a map $V_j \rightarrow D^k$, which lets us build a map

$$\begin{aligned}
\tilde{\psi} : \big(C_1V_j\big)\big/\big(V_j\cap C_1(\partial V_j)\big) &\rightarrow D^k\times I\big/\big(\partial D^k\times I\cup D^k\times\{0,1\}\big)\\
(x,t) &\mapsto \psi(x,t)
\end{aligned}$$

Hence we get the map

Inhalt...

□

Theorem 10.6. *Given the filtration $N_{-1}\subseteq N_0\subseteq\cdots\subseteq N_{m-1}\subseteq N_m = M$ from 14. The following diagram commutes:*

$$\begin{tikzcd}
C^k(M,\mathfrak{A},\tilde{K}^l(pt)) \arrow[r, "\partial^p"] \arrow[d] & C^{k+1}(M,\mathfrak{A},\tilde{K}^l(pt)) \arrow[d] \\
\bigoplus_{p\in \mathrm{Crit}_k(f)} \tilde{K}^{k+l}(N_p,L_p) \arrow[r, "\Delta_{k+l}"] \arrow[d] & \bigoplus_{q\in \mathrm{Crit}_{k+1}(f)} \tilde{K}^{k+l+1}(N_q,L_q) \arrow[d] \\
K^{k+l}(N_k,N_{k-1}) \arrow[r, "\delta_{triple}"] & K^{p+l+1}(N_{k+1},N_k)
\end{tikzcd}$$

Proof. So we assume for the moment, that $q \in \mathrm{Crit}_{k+1}(f)$ and $p \in \mathrm{Crit}_k(f)$ are the only critical points in $f^{-1}([a,b])$, where $a := f^{-1}(p)$ and $b := f^{-1}(q)$. Now, we choose the index pairs wisely: First we define the notations:

$$M^t := \{x \in M | f(x) \le t\}\ ,\ M_t := \{x \in M | f(x) \ge t\}$$

and the constants:

$$c \in (a,b)\ ,\ \varepsilon > 0 \text{ small enough }\ ,\ T > 0 \text{ large enough}$$

Now we define the following sets:

$$\begin{aligned}
N_q &:= \{x \in M_c | f(\varphi_{-T}(x)) \le b + \varepsilon\} \\
L_q &:= \{x \in N_q | f(x) = c\} \\
N_p &:= \{x \in M^c | f(\varphi_T(x)) \ge a - \varepsilon\} \\
L_p &:= \{x \in N_p | f(\varphi_T(x)) = a - \varepsilon\}
\end{aligned}$$

and with those the sets;

$$\begin{aligned}
C &:= N_p \cup N_q \\
B &:= N_p \cup L_q \\
A &:= L_p \cup (L_q \overset{\circ}{-} N_p)
\end{aligned}$$

With those we have the following list of facts:

1. (N_q, L_q) is a regular index pair for q.
2. (C, B) is an index pair for q.
3. (N_p, L_p) is a regular index pair for p.
4. (B, A) is an index pair for p

Since N_p is a tubular neighbourhood of the contractible $W(\to p) \cap M^c$ we get the diffeomorphism:

$$\psi_p : N_p \to \underbrace{\overline{D^{m-k}}}_{\dim W(\to p)} \times \overline{D^k} \subseteq T^u_p M$$

The image here is a subspace of the total space of the trivialized normal bundle. This map satisfies that

1. $\psi(L_p) = \overline{D^{m-k}} \partial D^k$,
2. $\psi_p(N_p \cap W(\to p) = \{0\} \times \overline{D^{m-k}}$,
3. $\psi(V_j) = \{\theta_j\} \times \overline{D^k}$ where $\theta_j \in \partial D^{m-k}$.

Using this map we get diffoemorphisms

$$\begin{aligned} \psi_j : V_j &\to \overline{D^k} \\ x &\mapsto \pi_1 \circ \psi_p(x) \text{ where } \pi_1 \text{ is the projection onto the first factor.} \end{aligned}$$

This map restricts to a diffeomorphism from $\partial V_j = V_j \cap L_p$ to ∂D^{k-1} Hence, an orientation of the unstable tangend space of p induces a map: $V_j \to \overline{D}^k$

das durch-denken, ob das klar ist

Now we want to figure out how the map between the spheres on the right looks if the diagramm commutes:

$$\begin{tikzcd} (W(q \to) \cap N_q) \cup C_1 S(q \to) \arrow[r, "\Phi"] \arrow[d, hook] & S^{k+1} \arrow[d] \\ \Big(\big((W(q \to) \cap N_q) \cup C_1 S(q \to) \big) \cup C_2 (W(q \to) \cap N_q) \Big) \cup C_3 (C_1 \overline{S(q \to) \setminus V_j}) \arrow[r, "\Psi"] & S^{k+1} \end{tikzcd}$$

Hence we need to specify the horizontal (and vertical) maps. So we start with the top one(or rqther its inverse): From the proof of theorem 5.2 we recycle a view maps. Notice how the orientation of $T_q^u M$ gives us a way to identify $T_q^u M$ with $\mathbb{R}^{k+1}$ (the coordinate map) and hence we can restrict the chart $\chi^0 =: \chi : T_q^u M \supset U \to W(q \to)$ such that $U = \overline{D^{k+1}}$ is a closed Disc in $\mathbb{R}^n$.. Now for each $x \in U \setminus q$ there is a number $t_{0,x} \geq 0$ such that $\varphi_{t_{0,x}}(\chi(x)) \in \chi(\partial U)$. Furthermore for each $x \in U$, there is a number$t_{1,x}$ such that $\varphi_{t_{1,x}}(\chi(x)) \in f^{-1}(c)$ Those numbers all smoothly depend on x.

Why is the left an inclusion?

careful, the map χ is just a chart and hence not natural in any sence. But since $W(q \to)$ is orientable, and hence we can make χ and ori-ented chart.

Now define the map

$$\begin{aligned} \tilde{\Phi} : \overline{D^{k+1}} &\to W(q \to) \cap M_c \\ x &\mapsto \varphi_{(t_{1,x} - t_{0,x})} \chi((x)) . \end{aligned}$$

This map is a homeomorphism, since χ is one, and the flow as a map $\varphi : \mathbb{R} \times M \to M$ is continouos. Furthermore there is an inverse given as follows: For each $x \in W(q \to) \cap M_c$ there is a real number $s_{0,x}$ such that $\varphi_{s_{0,x}}(x) \in \chi(\partial U)$, and a real number $s_{1,x}$ such that $\varphi_{s_{1,x}}(x) \in f^{-1}(c)$. Then the map

$$\begin{aligned} \Phi : W(q \to) \cap M_c &\to \overline{D^{k+1}} \\ x &\mapsto \chi^{-1}(\varphi_{s_{0,x} - s_{1,x}})(x) \end{aligned}$$

is the inverse of $\tilde{\Phi}$: To see this notice that for $y = \chi(x)$ we have $t_{0,x} = s_{0,y}$, and $t_{1,x} = s_{1,y}$ by definition.

Now call $A := \varphi_{(t_{1,x} - t_{0,x})} \chi(x) = \varphi_{(s_{1,\chi(x)} - s_{0,\chi(y)})} \chi(x)$, then we have:

- $s_{1,A} = s_{0,\chi(x)}$ and,
- $s_{0,A} = -s_{1,\chi(x)} + 2s_{0,\chi(x)}$.

This is because:

$$\varphi_{s_{0,\chi(x)}} \circ \varphi_{(t_{1,x}-t_{0,x})}\chi(x) = \varphi_{s_{0,\chi(x)}} \circ \varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x) = \varphi_{(s_{1,\chi(x)})}\chi((x)) \in f^{-1}(c)\,,$$

and

$$\varphi_{-s_{1,\chi(x)}+2s_{0,\chi(x)}} \circ \varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x) = \varphi_{(s_{0,\chi(x)})}\chi((x)) \in f(\partial U)\,.$$

Hence

$$\begin{aligned}\Phi \circ \tilde{\Phi}(x) &= \Phi\Big(\varphi_{(t_{1,x}-t_{0,x})}\chi((x))\Big)\\ &= \Phi\Big(\underbrace{\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x)}_{=A}\Big)\\ &= \chi^{-1} \circ \varphi_{s_{0,A}-s_{1,A}}(\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}\chi(x))\\ &= \chi^{-1} \circ \underbrace{\varphi_{-s_{1,\chi(x)}+2s_{0,\chi(x)}-s_{0,\chi(x)}}(\varphi_{(s_{1,\chi(x)}-s_{0,\chi(y)})}}_{=\mathrm{id}}\,\chi(x))\\ &= x\end{aligned}$$

And the other way round we have the relations for $B := \chi^{-1}(\varphi_{s_{0,x}-s_{1,x}}(x))$:

$$\begin{aligned}t_{0,B} &= s_{1,x}\\ t_{1,B} &= -s_{0,x} + 2s_{1,x}\,,\end{aligned}$$

since we can calculate:

$$\begin{aligned}\varphi_{s_{1,x}}(\chi(B)) &= \varphi_{s_{1,x}}(\varphi_{s_{0,x}-s_{1,x}}) = \varphi_{s_{0,x}}(x) \in f(\partial U)\\ \varphi_{-s_{0,x}+2s_{1,x}}(\chi(B)) &= \varphi_{-s_{0,x}+2s_{1,x}}(\varphi_{s_{0,x}-s_{1,x}}) = \varphi_{s_{1,x}}(x) \in f^{-1}(c)\end{aligned}$$

$$\begin{aligned}\tilde{\Phi} \circ \Phi(x) &= \tilde{\Phi}\big(\chi^{-1}(\varphi_{s_{0,x}-s_{1,x}})(x)\big)\\ &= \varphi_{t_{1,B}-t_{0,B}}\chi \circ \chi^{-1}(\varphi_{s_{0,x}-s_{1,x}})(x)\\ &= \varphi_{-s_{0,x}+2s_{1,x}-s_{1,x}}\varphi_{s_{0,x}-s_{1,x}})(x)\\ &= x\end{aligned}$$

So now we have a diffeomorpism $\Phi : W(q \to) \cap N_q \to \overline{D^{k+1}}$. We extend this to a continuous map

$$(W(q \to) \cap N_q) \cup C_1 S(q \to) \to \overline{D^{k+1}}/\partial\overline{D^{k+1}}$$

by first contracting $C_1S(q\to)$: So now we have a diffeomorpism $\Phi : W(q\to)\cap N_q \to \overline{D^{k+1}}$. We extend this to a continuous map

$$(W(q\to)\cap N_q)\cup C_1S(q\to)\to \overline{D^{k+1}}\cup C_1\partial\overline{D^{k+1}}$$

This is neither a homeopmorphism nor a homotopie equivalence. But it is continouos and since $C_1S(q\to)$ was contractible, it induces an isomorphims in the K-groups. Now since Φ maps $S(q\to)$ to ∂D^{k+1} we can compose the contracting with Φ to get a continouos map that induces an isomorphism in the K-group, and which we will also call Φ:

$$\Phi : (W(q\to)\cap N_q)\cup C_1S(q\to)\to \overline{D^{k+1}}/\partial\overline{D^{k+1}}\,.$$

Now we want to construct the map

$$\Psi : \left(\Big((W(q\to)\cap N_q)\cup C_1S(q\to)\Big)\cup C_2(W(q\to)\cap N_q)\right)\cup C_3(C_1\overline{S(q\to)\setminus V_j})\to S^{k+1} \quad (22)$$

First we contract all unnecesarry parts:

$$\begin{aligned}
&\left(\Big((W(q\to)\cap N_q)\cup C_1S(q\to)\Big)\cup C_2(W(q\to)\cap N_q)\right)\cup C_3(C_1\overline{S(q\to)\setminus V_j})\\
&\to \left(\Big((W(q\to)\cap N_q)\cup C_1S(q\to)\Big)\Big/(W(q\to)\cap N_q)\right)\Big/(C_1\overline{S(q\to)\setminus V_j})\\
&= \Big((W(q\to)\cap N_q)\cup C_1S(q\to)\Big)\Big/\Big((W(q\to)\cap N_q)\cup (C_1\overline{S(q\to)\setminus V_j})\Big)\\
&= \Big(C_1S(q\to)\Big)\Big/\Big(S(q\to)\cup (C_1\overline{S(q\to)\setminus V_j})\Big)\\
&= \big(C_1V_j\big)\big/\big(V_j\cap C_1(\partial V_j)\big)
\end{aligned}$$

To summarize, we get a continouos map Θ:

$$\begin{aligned}
&\left(\Big((W(q\to)\cap N_q)\cup C_1S(q\to)\Big)\cup C_2(W(q\to)\cap N_q)\right)\cup C_3(C_1\overline{S(q\to)\setminus V_j})\\
&\to \big(C_1V_j\big)\big/\big(V_j\cap C_1(\partial V_j)\big)
\end{aligned}$$

Again this is continouos but not even a homotopie equivalence. However, it induces an isomorphism in the K-groups. via ψ we get a map $V_j\to\overline{D^k}$.

$$\begin{aligned}
\tilde{\psi} : \big(C_1V_j\big)\big/\big(V_j\cap C_1(\partial V_j)\big) &\to \overline{D^k}\times I\big/\big(\partial\overline{D^k}\times I\cup\overline{D^k}\times\{0,1\}\big)\\
(x,t)&\mapsto \psi(x,t)
\end{aligned}$$

Now by rescaling the last factor:

$$\tilde{r} : \overline{D^k}\times I\to\overline{D^{k+1}}$$

we get the homomorphism

$$r : \overline{D^k} \times I \big/ \big(\partial \overline{D^k} \times I \cup \overline{D^k} \times \{0,1\}\big) \to \overline{D^{k+1}} \big/ \partial \overline{D^{k+1}}$$
$$(x,t) \mapsto \overline{r(x,t)}\,.$$

To see the well definition, notice how for closed sets we have the equality $\partial(A \times B) = (\partial A \times B) \cup (A \times \partial B)$. In sum we get the map :

$$\Psi : \left(\Big((W(q \to) \cap N_q) \cup C_1 S(q \to) \Big) \cup C_2 (W(q \to) \cap N_q) \right) \cup C_3 (C_1 \overline{S(q \to) \smallsetminus V_j})$$
$$\to \overline{D^{k+1}} \big/ \partial \overline{D^{k+1}}$$

given by $\Psi : r \circ \tilde{\psi} \circ \Theta$ Now we want to ask, how the map

$$\Psi \circ i \circ \Phi^{-1} : \overline{D^{k+1}} \big/ \partial \overline{D^{k+1}} \to \overline{D^{k+1}} \big/ \partial \overline{D^{k+1}}$$

looks like. Our claim is, that this map is homotopic to the identity, if the orientation in $T_{x_j} V_j$ induced from the one in $T^u_q M$ and from $T^u_p M$ agree. To do this we start with the inclusion. (But we need the "Coordinates from Φ maby we can look at $\Phi(x,t)$ and compare it to $\Psi \circ i(x,t)$.

$$i : (W(q \to) \cap N_q) \cup C_1 S(q \to)$$
$$\to \left(\Big((W(q \to) \cap N_q) \cup C_1 S(q \to) \Big) \cup C_2 (W(q \to) \cap N_q) \right) \cup C_3 (C_1 \overline{S(q \to) \smallsetminus V_j})$$

So assume (t,x) lives in the domain. Then $(\Theta \circ i)(t,x) = \overline{(t,x)}$ where the equivalence is given by the collaps of everything but the interior of $C_1 V_j$. Now inspect $\psi(V_j)$. we can identify via a trivialitation of the normal bundle $\psi(V_j)$ with $T^u_p M$ and the orientation of the latter induces a map to D^{k-1}.

□

Corollary 10.7 (The Logic and To-Dos of my Proof)**.** We have the definitions above, of all sets. with those we first want to construct a map of triples

$$t : (A, B, C) \to (N_{p+1}, N_p, N_{p-1})$$

By naturality and definition we then have the commutative diagram:

$$\begin{array}{ccccc} K^{-p+1}(N_{p+1}, N_p)) & \xrightarrow{t^*} & K^{-p+1}(A,B) & \Longrightarrow & K^{-p+1}(A,B) \\ \delta_{triple}\uparrow & & \delta_{triple}\uparrow & & s^*\uparrow \\ K^{-p}(N_p, N_{p-1}) & \xrightarrow{t^*} & K^{-p}(B,C) & \Longrightarrow & K^{-p+1}(s \vee (B,C)) \end{array}$$

We now have to show that s^* is induced from a continuous map. Then we want to use the maps Φ and Ψ to induce a map $p \simeq \pm\mathrm{id}$ such that the diagram commutes up to homotopie:

$$\begin{tikzcd}
(A,B) \arrow[r, "\Phi"] \arrow[d, "s"] & \bigvee_{j=1}^{l} S^{k+1} \arrow[d, "\bigvee_j \delta_j \,\mathrm{id}"] \\
S \wedge (B,C) \arrow[r, "\Psi"] & S^{k+1}
\end{tikzcd}$$

where $\delta_j \in \{-1,1\}$(with $+\mathrm{id}$ we denote the identity and with $-id$ we denote a homeomorphism of degree -1, i.e. not homotopic to the identity.) and Ψ is a homotopie equivalence. Now since $K^{-p+1}(\bigvee_{j=1}^{l} S^{p+1}) \cong \mathbb{Z}^l$ we can define the following maps. Let β_q be a generator of $K^{-p+1}(A,B)$ and β_p of $K^{-p+1}(S \vee (B,C))$. Then define the koordinate maps

$$\begin{aligned}
q_q : K^{-p+1}(\bigvee_{j=1}^{l} S^{k+1}) \to \mathbb{Z}^l; \quad (\Phi^*)^{-1}(\beta_q) \mapsto \sum_j e_j \\
q_p : K^{-p+1}(S^{k+1}) \to \mathbb{Z}; \quad (\Psi^*)^{-1}(\beta_p) \mapsto 1\,.
\end{aligned}$$

With those we get the diagram:

$$\begin{tikzcd}
K^{-p+1}(A,B) & K^{-p+1}(\bigvee_{j=1}^{l} S_j^{k+1}) \arrow[l, "\Phi^*"'] \arrow[r, "q_q"] & \mathbb{Z}^l \\
K^{-p+1}(s \vee (B,C)) \arrow[u, "s^*"] & K^{-p+1}(S^{k+1}) \arrow[l, "\Psi^*"'] \arrow[r, "q_p"] \arrow[u, "g"] & \mathbb{Z} \arrow[u, "h"]
\end{tikzcd}$$

The map h is given by $h : \mathbb{Z} \to Z^l$; $e_i \mapsto \sum_{j=1}^{l} \delta_j$ with the δ_j from above. This all concludes in the final calculation:

$$\begin{aligned}
s^*(\beta_p) &= \Phi^* \circ g \circ (\Psi^*)^{-1}(\beta_p) \\
&= \Phi^* \circ q_q^{-1} \circ h \circ q_p \circ (\Psi^*)^{-1}(\beta_p) \\
&= \Phi^* \circ q_q^{-1} \circ h(\sum_j e_j) \\
&= \Phi^* \circ q_q^{-1}(\sum_j \delta_j) \\
&= \sum_j \delta_j \beta_q
\end{aligned}$$

Now the hope is that δ_j is the sign that I would get from inspecting the morse boundary operator along a certain flow line. The todo's are:

1. The map t.
2. The map Φ.
3. The map Ψ.
4. Is the map s^* induced?

5. The map h corresponds to a proceedure similar to the Morse boundary.

Once we have done all the above we want to connect the considerations with the boundary operator. To do this we do the proveedure to all pairs $(q,p) \in \mathrm{Crit}(f)_{k+1} \times \mathrm{Crit}(f)_k$. For this we enrich the notaion and call the triple coresponding to such a pair $(A^{q,p}, B^{q,p}, C^{q,p})$. All the maps and spaces are enriched in that way, by adding the pair (q,p) as a superscrit. For T big enough we assume that all $A^{q,p}$ are pairwise disjoint. Then we want a homomorphism (that is a homotopic eqiuvalence)

do wee needc continuity or rather something weaker?

$$\Omega : \bigsqcup_{(q,p)\in \mathrm{Crit}_{k+1}\times \mathrm{Crit}_k} (A^{q,p}, B^{q,p}, C^{q,p}) \to (N_{p+1}, N_p, N_{p-1})$$
$$x \mapsto t^{q,p}(x) \text{ for } x \in A^{q,p}$$

Now, together with the isomorphism

$$\bigoplus_{\mathrm{Crit}_{k+1}} \mathbb{Z} \to K^p(N_{-p+1}, N_p) \ , \ q \mapsto \beta_q \text{ which is a generator of } K^{-p+1}(A^{q,p}, B^{q,p})$$

Fuck das funktioniert alles nicht!

Lemma 10.8. *Lets say that $p \in -\mathbb{N}$ is negative for the moment. We have the three spaces*

$$A := N_q \cup N_p \quad , \quad B := L_q \cup N_p \quad , \quad C := L_p \cup (L_q \smallsetminus \mathring{N}_p)\,. \tag{23}$$

By definition we get the triple connecting morphism as the composition of the inclusion with the connecting homomorphism of the pair::

$$\begin{tikzcd}
 & K^p[N_q \cup N_p, L_q \cup N_p] \arrow[r, Rightarrow] & K^p[N_q \cup N_p \cup /(L_q \cup N_p)] \\
 & & K^p[N_q \cup N_p \cup C_1(L_q \cup N_p)] \arrow[u, "(m^*)^{-1}"] \\
K^{p-1}[L_q \cup N_p] \arrow[uur, "\delta_{pair}"] \arrow[r, Rightarrow] & K^p[S_1 \wedge (L_q \cup N_p)] \arrow[r, "\alpha^*"] & K^p[N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)] \arrow[u, "i^*_\alpha"] \\
K^{p-1}[L_q \cup N_p, L_p \cup (L_q \smallsetminus \mathring{N}_p)] \arrow[u, "i^*"] \arrow[r, Rightarrow] & K^p[S_1 \wedge L_q \cup N_p, S_1 \wedge L_p \cup (L_q \smallsetminus \mathring{N}_p)] \arrow[u, "i^*_\beta"] &
\end{tikzcd}$$

The bottom left diagramm commutes by definition. All maps with names $i^*_{something}$ are induced from obvious inclusions. The other maps are defined as follows, where p denotes the choosen point of our pointed spaces:

$$\alpha : N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p) \to S_1 \wedge (L_q \cup N_p)$$
$$x \mapsto \begin{cases} p & \text{if } x \in C_2(N_q \cup N_p) \\ x & \text{else} \end{cases}$$

This map is well defined and continouos, as every point in $S_1 \wedge (L_q \cup N_p)$ is of the form $[(t_1, x)]$ where $x \in L_q \cup N_p$. and those are the points in the domain of α that not get *collapsed to p We have the maps:*

$$c_\gamma : [N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)] \to \left[\frac{N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \Big/ \frac{C_1(L_p \cup \overline{(L_q \smallsetminus N_p)}) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \right]$$
$$x \mapsto \begin{cases} p & \text{if } x \in C_2(N_q \cup N_p) \quad \cup \quad C_1(L_p \cup \overline{(L_q \smallsetminus N_p)}) \\ x & \text{else} \end{cases}$$

and

$$c_\beta : [S_1 \wedge (L_q \cup N_p)] \to \left[S_1 \wedge L_q \cup N_p \Big/ S_1 \wedge L_p \cup (L_q \smallsetminus \mathring{N}_p) \right]$$
$$x \mapsto \begin{cases} p & \text{if } x \in S_1 \wedge L_p \cup (L_q \smallsetminus \mathring{N}_p) \\ x & \text{else} \end{cases}$$

This map is again well defined and continouos and is also a map of pairs. In fact, both maps α and β can be written as $x \mapsto \overline{x}$ and since the collapsed space is contractible, they induce isomorphism between the K-groups. The map γ is just the identity.

Furthermore, we have the commutative diagramm:

$$\begin{array}{ccc} K^p[S_1 \wedge (L_q \cup N_p)] & \xrightarrow{\quad\alpha^*\quad} & K^p[N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)] \\ \uparrow{\scriptstyle c_\beta^*} & & \uparrow{\scriptstyle c_\gamma^*} \\ K^p\left[S_1 \wedge L_q \cup N_p \Big/ S_1 \wedge L_p \cup (L_q \smallsetminus \mathring{N}_p) \right] & \xrightarrow{\gamma^*} & K^p\left[\frac{N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \Big/ \frac{C_1(L_p \cup \overline{(L_q \smallsetminus N_p)}) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \right] \end{array}$$

To see the commutativitie we have to show that

$$c_\beta \circ \alpha = \gamma \circ c_\gamma : N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p) \to \left[S_1 \wedge L_q \cup N_p \Big/ S_1 \wedge L_p \cup (L_q \smallsetminus \mathring{N}_p) \right]$$

since α and β are both of the form $x \mapsto \overline{x} = x$ or p we compare the case distinctions:

- $c_\beta \circ \alpha(x) = p \Leftrightarrow x \in C_1\left(L_p \cup (L_q \smallsetminus \mathring{N}_p)\right) \cup C_2(N_q \cup N_p)$
- $\gamma \circ c_\gamma(x) = p \Leftrightarrow x \in C_2(N_q \cup N_p) \cup C_1(L_p \cup \overline{(L_q \smallsetminus N_p)}) \cup C_2(N_q \cup N_p)$

Which are the same, since $L_q \smallsetminus \mathring{N}_p = \overline{L_q \smallsetminus N_p}$ due to L_q being closed. Now we consider the maps:

$$c : [N_q \cup N_p \cup C_1(L_q \cup N_p)] \to \left[\frac{N_q \cup N_p \cup C_1(L_q \cup N_p)}{\mathrm{cl}\left(N_q \cup N_p \cup C_1(L_q \cup N_p) \smallsetminus \bigcup_{j=1}^{l} C_1 V_j \right)} \right]$$
$$x \mapsto \begin{cases} p & \text{if } x \in \mathrm{cl}(N_q \cup N_p \cup C_1(L_q \cup N_p) \smallsetminus \bigcup_{j=1}^{l} C_1 V_j) \\ x & \text{else} \end{cases}$$

and

$$c_\iota : [N_q \cup N_p \cup C_1(L_q \cup N_p)] \to \left[\frac{N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \Big/ \frac{N_q \cup N_p \cup C_1\left(L_p \cup \mathrm{cl}\left(L_q \setminus \bigcup_{j=1}^{l} V_j\right)\right) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \right]$$

$$x \mapsto \begin{cases} p & \textit{if } x \in N_q \cup N_p \cup C_1\left(L_p \cup \mathrm{cl}\left(L_q \setminus \bigcup_{j=1}^{l} V_j\right)\right) \cup C_2(N_q \cup N_p) \\ x & \textit{else} \end{cases} .$$

We define V_j *to be the the finitly many components* $N_p \cap S(q \to)$ *and hence we can identify the sets* $\overline{L_q \setminus N_p} = \mathrm{cl}(L_q \setminus (L_q \cap N_q)) = \mathrm{cl}(L_q \setminus (\bigcup_{j=1}^{l}))$ *and with this the map* η *in the diagramm below is just the identity:*

$$\begin{tikzcd}
K^p\big[N_q \cup N_p \cup /(L_q \cup N_p)\big] & \\
K^p\left[N_q \cup N_p \cup C_1(L_q \cup N_p)\right] \arrow[u, "(m^*)^{-1}"] & K^p\left[\frac{N_q \cup N_p \cup C_1(L_q \cup N_p)}{\mathrm{cl}\left(N_q \cup N_p \cup C_1(L_q \cup N_p) \setminus \bigcup_{j=1}^{l} C_1 V_j\right)}\right] \arrow[l, "c^*"'] \\
K^p\left[N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)\right] \arrow[u, "i_\alpha^*"] & K^p\left[\frac{C_1(L_q \cup N_p)}{\mathrm{cl}\left(C_1(L_q \cup N_p) \setminus \bigcup_{j=1}^{l} C_1 V_j\right)}\right] \arrow[u, "i_\delta^*"] \\
 & K^p\left[\frac{C_1(L_q)}{\mathrm{cl}\left(C_1(L_q) \setminus \bigcup_{j=1}^{l} C_1 V_j\right)}\right] \arrow[u, "i_\eta^*"] \\
K^p\left[\frac{N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \Big/ \frac{C_1(L_p \cup \overline{(L_q \setminus N_p)}) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)}\right] \arrow[uu, "c_\gamma^*"] & K^p\left[\frac{N_q \cup N_p \cup C_1(L_q \cup N_p) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)} \Big/ \frac{N_q \cup N_p \cup C_1\left(L_p \cup \mathrm{cl}\left(L_q \setminus \bigcup_{j=1}^{l} V_j\right)\right) \cup C_2(N_q \cup N_p)}{C_2(N_q \cup N_p)}\right] \arrow[l, "\eta^*"'] \arrow[u, "c_\iota^*"]
\end{tikzcd}$$

Now the mamoth task is to show that this beast commutes: For this we preoceed with the same strategy since again all maps are inclusions and collases and hence the maps look like $x \mapsto \overline{x} = x$ *or* p*. For the consideration assume that* $x \neq p$ *in the domain. Then:*

$$\eta \circ c_\gamma \circ i_\alpha(X) = p \Leftrightarrow x \in C_1(L_p \cup \overline{(L_q \setminus N_p)}) \cup C_2(N_q \cup N_p)$$

On the other hand:

$$c_\iota \circ i_\eta \circ i_\delta \circ c(x) = p \Leftrightarrow c(x) = p \text{ or } c_\iota(x) = p$$

$$x \in \operatorname{cl}\left(N_q \cup N_p \cup C_1(L_q \cup N_p) \setminus \bigcup_{j=1}^{l} C_1 V_j\right) \cup N_q \cup N_p \cup C_1(L_p \cup \operatorname{cl}\left(L_q \setminus \bigcup_{j=1}^{l} V_j\right)) \cup C_2(N_q \cup N_p)$$

$$= C_1 \operatorname{cl}\left((L_q \cup N_p) \setminus \bigcup_{j=1}^{l} V_j\right) \cup C_1(L_p \cup \operatorname{cl}\left(L_q \setminus \bigcup_{j=1}^{l} V_j\right)) \cup C_2(N_q \cup N_p)$$

$$= C_1 \operatorname{cl}\left((L_q) \setminus \bigcup_{j=1}^{l} V_j\right) \cup C_1(L_p \cup \operatorname{cl}\left(L_q \setminus \bigcup_{j=1}^{l} V_j\right)) \cup C_2(N_q \cup N_p)$$

$$= C_1(L_p \cup \operatorname{cl}\left(L_q \setminus \bigcup_{j=1}^{l} V_j\right)) \cup C_2(N_q \cup N_p)$$

And now the conditions agree and hence the big diagramm commutes. Now notice how we have the homeomorphism

$$\Lambda : \left[\frac{C_1(L_q)}{\operatorname{cl}\left(C_1(L_q) \setminus \bigcup_{j=1}^{l} C_1 V_j\right)}\right] \to \bigvee_{j=1}^{l} C_1 V_j \big/ \partial C_1 V_j$$

that is the identity on all $C_1 \mathring{V}_j$. Furthermore we have the compression maps

$$c_j : \bigvee_{j=1}^{l} C_1 V_j \big/ \partial C_1 V_j \to C_1 V_j \big/ \partial C_1 V_j$$

Hence by the additivity of $K-$Theorie, the pair

$$\left(K^p\left[\frac{C_1(L_q)}{\operatorname{cl}\left(C_1(L_q) \setminus \bigcup_{j=1}^{l} C_1 V_j\right)}\right], \left((c_j \circ \Lambda)^*\right)_j\right)$$

satisfies the coproduct universal property.

So consider the maps

$$c^* \circ i_\delta^* \circ i_\eta^* \circ (c_j \circ \Lambda)^* : C_1 V_j \big/ \partial C_1 V_j \to K^p\left[N_q \cup N_p \cup C_1(L_q \cup N_p)\right]$$

This map is induced from

$$N_q \cup N_p \cup C_1(L_q \cup N_p) \to C_1 V_j \big/ \partial C_1 V_j$$
$$x \mapsto \begin{cases} x & \text{if } x \in C_1(\mathring{V}_j)\,, \\ p & \text{else }. \end{cases}$$

For all C_1V_j we get an homeomorphism to S^{k+1}: More on ψ

$$r : C_1V_j \big/ \partial C_1V_j \to I \times D^k \big/ \partial(I \times D^k)$$
$$(t, x) \mapsto (t, \psi(x))$$