Contents

1 Differentiable structures on topological manifolds

Definition 1.0.1 (Topological Manifold)**.** A second countable Hausdorffspace M is called **topological manifold** of dimension $m \in \mathbb{N}$, if it is locally homeomorphic to $\mathbb{R}^m$. To be precise, if for all $p \in M$ there exists an open neighborhood $U \subseteq M$ of p, an open set $V \subseteq \mathbb{R}^m$ and a map $\varphi : W \to V$ that is a homeomorphism. We call the map $\varphi : U \to V$ a **chart around p on M** and φ^{-1} a **local coordinate system around p on M**.

Definition 1.0.2 (Differentiable Manifold)**.** Let M be a topological manifold of dimension M.

1. A **differentiable atlas of class** $r \in \mathbb{N} \cup \{\infty\}$ is a family of charts $\mathfrak{A} = (\varphi_i : U_i \to V_i)_{i \in I}$ such that
   a) $\bigcup_{i \in I} U_i = M$, meaning that (U_i) is an open covering of M.
   b) For every pair $(i, j) \in I^2$ the **transition function**:
   $$\begin{aligned} \varphi_{ij} : \varphi_j(U_i \cap U_j) &\to \varphi_i(U_i \cap U_j) \\ x &\mapsto \left(\varphi_i \circ \varphi_j^{-1}\right)(x) \end{aligned}$$
   is differentiable of class r.

   We call such an atlas a C^r-atlas.
2. Two C^r-atlases $\mathfrak{A}$ and $\mathfrak{B}$ are called **equivalent** if the family $\mathfrak{A} + \mathfrak{B} = \left(\varphi_i, \varphi_j\right)_{ij}$ is a C^r-atlas.

A **differentiable structure of class** r on M is an equivalence class c of C^r-atlases. For $r = \infty$ we call the pair (M, c) a smooth manifold.

Corollary 1.0.3. Every transition functions φ_{ij} $i, j \in I^2$ of a differentiable atlas $\mathfrak{A} = (\varphi_i)_{i \in I}$ is not just a homeomorphism but also a diffeomorphism due to $\varphi_{ji} = \varphi_{ij}^{-1}$

Definition 1.0.4. Let (M, c) be a differentiable manifold of class r and $U \subseteq M$ open. We call a continuos function

$$f : U \to \mathbb{R}$$

differentiable of class r, if for any one (and hence for all) $(\varphi_i)_{i \in I} = \mathfrak{A} \in c$ the compositions $f \circ \varphi_i^{-1}$ are differentiable of class r. For $r = \infty$ we define:

$$\mathcal{E}(U) = \{f \in U \to \mathbb{R} \text{ continouos} \mid f \text{ is differentiable of class } \infty\}.$$

Corollary 1.0.5. Let (M, c) be a smooth manifold of dimension m and $U \subseteq M$ be a open subset. With pointwise defined operations, the set $(\mathcal{E}(U), +, \cdot, \circ)$ becomes an $\mathbb{R}$–algebra. Furthermore, $\mathcal{E}$ becomes a sheaf of $\mathbb{R}$-algebras.

Proof. There is not really a need for a proof.However, it might help to work through the definition of a sheaf as a reminder. First, $\mathcal{E}$ is a presheaf, where the restriction in the domain of a function gives the needed restriction homomorphism:

$$\begin{aligned} \mathrm{res}^U_V : \mathcal{E}(U) &\to \mathcal{E}(V) \\ f &\mapsto f\Big|_V . \end{aligned}$$

The required properties of a presheaf are trivial. Furthermore, this gives a sheaf as the requirement of locality is trivial for functions and the property of gluing is also trivial for functions, since differentiability is a local property. □

Definition 1.0.6. If $p \in M$ is fix, $f \in \mathcal{E}(U)$ and $g \in \mathcal{E}(U')$ such that $p \in U \cap U'$ we say that f and g have the same **germ in** p, if there is another open neighborhood $W \subseteq U \cap U'$ of p such that $f|_w = g|_W$. This defines an equivalence relation $\sim_p$. An equivalence class s of local functions around p is called a **germ in** p. We write $s = f_p$, if $s = [f]$ with $f \in \mathcal{E}(U)$. We write

$$\mathcal{E}_p(M) = \left(\sum_{U \text{open}, p \in U} \mathcal{E}(U) \right) \Big/ \sim_p \; .$$

For the set of germs and call it the **stalk in** p. Here $\sum$ denotes the co-product (also called sum) in $\top$ and hence the disjoint union.

Corollary 1.0.7. For a smooth manifold (M, c) the set $\mathcal{E}_p(M)$ inherits an $\mathbb{R}$–algebra structure from the $\mathcal{E}(U)$. Furthermore, it carries a natural (evaluation-)homomorphism:

$$\begin{aligned} \mathrm{eval}_p : \mathcal{E}_p(M) &\to \mathbb{R} \\ f_p &\mapsto f(p) =: f_p(p) \end{aligned}$$

The stalks are also local rings with maximal ideal $\mathfrak{m}_p = \ker(\mathrm{eval}_p)$. Hence, the pair $(M, \mathcal{E})$ gives us a locally ringed space.

Proof. Here, we only need to prove the statement about the locality of the stalks. This follows from $f_p \in \mathcal{E}_p(M)$ being invertible if and only if $f(p) \neq 0$ which is the same as $f_p \notin \ker(\mathrm{eval}_p)$. □

Definition 1.0.8. Let (M, c) be a smooth manifold of dimension m and $p \in M$. We call an $\mathbb{R}$-linear map $\delta : \mathcal{E}_p(M) \to \mathbb{R}$ a **derivation**, if it satisfies the Leibnitz-rule:

$$\delta(f_p \cdot g_p) = \delta(f_p) g_p(p) + f_p(p) \delta(g_p) \quad \text{for all } f, g \in \mathcal{E}_p(M) \, .$$

We call $\mathrm{Der}_{\mathbb{R}}(\mathcal{E}_p(M), \mathbb{R})$ the set of derivations and give it a $\mathbb{R}$ vector space structure by pointwise operations. We define the **tangent space of** M **at** p to be the vector space

$$TM_p := \mathrm{Der}_{\mathbb{R}}(\mathcal{E}_p(M), \mathbb{R}) \, .$$

Corollary 1.0.9. Let (M, c) be a smooth manifold and $\varphi : U \to V$ be a chart around p with $x_0 = \varphi(p)$ $(\varphi \in \mathfrak{A} \in c)$. Then

$$\xi = \frac{\partial}{\partial x^j}\bigg|_p : \mathcal{E}_p(M) \to \mathbb{R}$$
$$f_p \mapsto \xi(f_p) = \frac{\partial}{\partial x^j}\bigg|_{x_0} (f \circ \psi^{-1})$$

is well-defined and a tangent vector. In fact, the family

$$\left(\frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^m}\bigg|_p \right)$$

defines a basis of TM_p. Hence, the dimension of TM_p is m.

Definition 1.0.10. For a smooth manifold (M, c) the sum $\sum_p TM_p$ comes with a natural projection

$$\pi : TM \to M$$
$$\xi \mapsto p \text{ where } \xi \in TM_p$$

Furthermore, the **local vector fields** with respect to a chart $\varphi : U \to V$

$$\frac{\partial}{\partial x^j} : U \to \pi^{-1}(U)$$
$$p \mapsto \frac{\partial}{\partial x^j}\bigg|_p$$

induce a local trivialization:

$$\pi^{-1}(U) \cong U \times \mathbb{R}^m .$$

We can induce a topology on TM such that all those trivializations are continuos. This then gives an atlas for TM such that we have a $2m$-dimensional manifold. To be precise, the atlas is given by the maps $\pi^{-1}(U_i) \to R^m \times R^m ; x \mapsto (\pi(x), q_{\varphi \circ \pi(x)}(x))$ where q_p denotes the coordinate map corresponding to the basis $(\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^m}\big|_p)$ that depends on the chart φ_i. In fact, this yields a smooth manifold and a (smooth) vector bundle of dimension m. We call TM the **tangent bundle**.

Definition 1.0.11 (The Derivative)**.** Let (M, c) and (M', c') be smooth manifolds (from now on we suppress the differentiable structure in our notation). We call a continuous function $f : M \to M'$ **smooth**, if for all $\varphi \in \mathfrak{A} \in c$ and $\varphi' \in \mathfrak{A}' \in c'$ the maps

$$\varphi' \circ f \circ \varphi^{-1} : V \to V'$$

are smooth. A given smooth function induces a smooth function between the Tangent bundles as follows:

$$Df : TM \to TM' \ , Df_p(\xi)(g_p) = \xi((g \circ f)_p)$$

Here, $\xi \in T_pM$, $g_p \in E_p(M')$.

Corollary 1.0.12. Let $f : M \to \mathbb{R}$ be smooth and $\psi : U \to V$ be a chart. Then we can interprete $\mathrm{d}f$ as a one-form. To be prezice asume q to be the coordinate funktion $T\mathbb{R} \to \mathbb{R}$ from the basis induced by the identity as a chart. $v \in \Gamma TM$ we have

$$q \circ \mathrm{d}f(v) = v \cdot f$$

Proof. We proof this by showing it for the section $\frac{\partial}{\partial x^i}$ and thereby for any, since those sections form a basis of the space of sections as a $C^\infty(M, \mathbb{R})$ vectorspace. Now let $g_p \in \mathcal{E}_p(\mathbb{R})$ and $\psi(p) = c_0$. Then

$$\mathrm{d}f_p(\frac{\partial}{\partial x^i}\Big|_p)(g_p) = \frac{\partial}{\partial x^i}\Big|_p (g \circ f)_p = \frac{\partial}{\partial x^i}\Big|_{x_0}(g \circ f \circ \psi^{-1}) = \frac{\partial}{\partial x}\Big|_p(g_p) \cdot \frac{\partial}{\partial x^i}\Big|_p(f)$$

Hence, $\mathrm{d}f(v) = v(f)\frac{\partial}{\partial x}$ letting us conclude the statement. □