The Setting

We work in a compact, riemanian, orientable, smooth Manifold (M, g) of dimension m together with a Morse-Smale funktion

$$f : M \to \mathbb{R} \tag{1}$$

We assume that q is a critical point of index $k+1$ and p is a critical point of index k. we define $f(q) = b$ and $f(p) = a$ and assume that there is no kritical point in $f^{-1}(a,b) \subseteq M$. We define the sub and superlevelsets

$$M^t := \{x \in M | f(x) \leq t\} \quad \text{and} \quad M_t := \{x \in M | f(x) \geq t\}\,, \tag{2}$$

and the constants

$$c \in (a,b) \quad , \quad \varepsilon > 0 \text{ small} \quad , \quad T > 0 \text{ big}\,. \tag{3}$$

With this we define the sets:

$$\begin{align}
N_q &:= \{x \in M_c | f(\varphi_{-T}(x)) \leq b + \varepsilon\}\,, \tag{4}\\
L_q &:= \{x \in N_q | f(x) = c\}\,, \tag{5}\\
N_p &:= \{x \in M^c | f(\varphi_T(x)) \geq a - \varepsilon\}\,, \tag{6}\\
L_p &:= \{x \in N_p | f(\varphi_T(x)) = a - \varepsilon\}\,, \tag{7}
\end{align}$$

and finally:

$$\begin{align}
C &:= N_p \cup N_q\,, \tag{8}\\
B &:= N_p \cup L_q\,, \tag{9}\\
A &:= L_p \cup (L_q \overset{\circ}{-} N_p)\,. \tag{10}
\end{align}$$

Lemma 0.0.1. *We claim that*

1. (N_q, L_q) *is a regular index pair for* q .
2. (C, B) *is an index pair for* q .
3. (N_p, L_p) *is a regular index pair for* p .
4. (B, A) *is an index pair for* p .
5. N_p *is a tubular neighbourhood of* $W(\to p) \cap M^c$.

Definition 0.0.2 (Jacobian of the Gradient)**.** The gradient is a section into the tangent bundle, $\mathrm{grad}(f) : M \to TM$. Let $\psi : M \supseteq U \to V \subseteq \mathbb{R}^m$ be a chart. The coordinate map q_ψ on TU assigns to a vector $v \in T_pM$ (where $p \in U$ with $\psi(p) = x$) its components with respect to the basis $\left(\frac{\partial}{\partial x^i}\Big|_p\right)$, i.e., $q_\psi(v) = (v^1, \dots, v^m)$ if $v = \sum_{i=1}^m v^i \frac{\partial}{\partial x^i}\Big|_p$. We define the Jacobian of the gradient with respect to the chart ψ as the Jacobian matrix of the coordinate representation of the gradient in this chart:

$$J(\mathrm{grad}(f))_\psi(x) := J\left(q_\psi \circ \mathrm{grad}(f) \circ \psi^{-1}\right)(x) = \left(\frac{\partial (q_\psi \circ \mathrm{grad}(f) \circ \psi^{-1})_i}{\partial x_j}(x)\right)_{ij}$$

Lemma 0.0.3. *Let $t \in \mathbb{R}$ be small enough and $\varphi_t : M \to M$ be the flow map corresponding to the negative gradient. Assume that U is the domain of a chart ψ and $p \in U$ such that $\varphi_t(p) \in U$. Then the linear map $\mathrm{d}\varphi_t\big|_p : T_pM \to T_{\varphi_t(p)}M$ has a local representation in the coordinates induced by ψ given by:*

$$q_{\psi(\varphi_t(p))} \circ \mathrm{d}\varphi_t\big|_p \circ q_{\psi(p)}^{-1} = \exp\left(-J(\mathrm{grad}(f))_\psi(\psi(p)) \cdot t\right)$$

where $J(\mathrm{grad}(f))_\psi(\psi(p))$ is the Jacobian matrix of the gradient evaluated at the coordinates of p.

Lemma 0.0.4. *Let ψ be a coordinate system arround a critical point p. I.e. $\psi : p \in U \to V$ with $\psi(p) = 0$ and let $\left(\frac{\partial}{\partial x_1}\Big|_x, \dots, \frac{\partial}{\partial x_m}\Big|_x\right)$ be the induced basis of T_xM Then the jacobean of the gradient*

$$J(\mathrm{grad}(f))_\psi(0) = \sum_k g^{ki}(0) \frac{\partial^2 f \circ \psi^{-1}}{\partial x_j \partial x_k} \tag{11}$$

where $(g^{ki}(x))$ denotes the matrix corresponding to the gradient in T_xM with respect to the basis $\left(\frac{\partial}{\partial x_1}\Big|_x, \dots, \frac{\partial}{\partial x_m}\Big|_x\right)$.

Proof. Let q_ψ denote the koordinate funktion $TU \to R^m$ induced from the basis $\left(\frac{\partial}{\partial x_1}\Big|_x, \dots, \frac{\partial}{\partial x_m}\Big|_x\right)$. then the gradient in ψ reads:

$$q_\psi \circ \mathrm{grad}(f) \circ \psi^{-1} = \left(\sum_k g^{k1} \frac{\partial f \circ \psi^{-1}}{\partial x_k}, \dots, \sum_k g^{km} \frac{\partial f \circ \psi^{-1}}{\partial x_k}\right)$$

Hence, we can calculate Hence, we can differentiate:

$$\begin{aligned}\frac{\partial}{\partial x} Q \circ \mathrm{grad}(f) \circ \psi^{-1} &= \left(\frac{\partial}{\partial x_j} \sum_k g^{ki} \frac{\partial f \circ \psi^{-1}}{\partial x_k}\right)_{ij} \\ &= \left(\sum_k \left[\frac{\partial g^{ki}}{\partial x_j} \frac{\partial f \circ \psi^{-1}}{\partial x_k} + g^{ki} \frac{\partial^2 f \circ \psi}{\partial x_j\, \partial x_k}\right]\right)_{ij}.\end{aligned}$$

and now in $p = \psi{-}1(0)$ we have

$$\frac{\partial}{\partial x} Q \circ \mathrm{grad}(f) \circ \psi^{-1}\Big|_0 = \left(\sum_k \left[g^{ki}(0) \frac{\partial^2 f \circ \psi^{-1}}{\partial x_j \, \partial x_k} \right] \right)_{ij} .$$

□

Theorem 0.0.5 (Sylvesters Law of Inertia)**.** *Let $A \in \mathrm{Mat}(n,\mathbb{R})$ be symmetric and let $T, T' \in \mathrm{GL}(n,\mathbb{R})$ and $k, k', l, l' \in \mathbb{N}$ such that*

$$T^t \circ A \circ T = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & -1_l & 0 \\ 0 & 0 & 0_{n-k-l} \end{pmatrix} \text{ and } T'^T \circ A \circ T' = \begin{pmatrix} 1_{k'} & 0 & 0 \\ 0 & -1_{l'} & 0 \\ 0 & 0 & 0_{n-k'-l'} \end{pmatrix}$$

then $k = k'$, $l = l'$ and $\mathrm{rank}(A) = k + l$.

Proof. Since T, T' are invertible we have that

$$k + l = \mathrm{rank}(T^t \circ A \circ T) = \mathrm{rank}(A) = \mathrm{rank}(T'^T \circ A \circ T') = k' + l' . \tag{12}$$

Hence it suffices to show that $k = k'$. Which we will do by proofing the claim:

$$k = \max\left\{ \dim(U) | U \subseteq \mathbb{R}^n \text{subspace such that } x^t A x > 0 \ \forall x \in U \setminus \{0\} \right\}$$

So we start by showing „$\leq$“: Denote the first k colomus of T with $x_1, ..., x_k$. They form a basis of $\mathbb{R}^n$ and with $0 \neq x = \sum_{i=1}^k \lambda i x^i$ we have that by bilinearity

$$x^T A x = \sum_{i=1}^k \lambda_i x_i^T A x = \sum_{i,j=1}^k \lambda_i \lambda_j x_i^T A x_j = \sum_{i=1}^k (\lambda_i)^2 \geq 0 . \tag{13}$$

This conculdes the first inequality.

Now let U be any $k-$dimensional subspace such that for all non zero $x \in U$ $x^T A x > 0$. By a calculation analog to the one above we have that for any $x \in W := \mathrm{span}(x_{k+1}, \ldots x_n)$ the number $x^t A x$ is less or equal to zero. Hence, $W \cap U = \{0\}$ and with this we conclude:

$$\dim(U) = \dim(U + W) - \dim(W) + \dim(U \cap W) \leq (k + (n-k)) - (n-k) + 0 = k . \tag{14}$$

This is the last inequality proving the statement. □

Corollary 0.0.6 (Simultaneous Diagonalisation of Quadratic Forms)**.** Let p be a critical point of a Morse function f on a manifold M and let $g(p)$ be the Riemannian metric on T_pM. Then there exists a Morse chart ψ arround p such that the representation of $g(p)$ in the basis induced by the Morse chart is given by a a diagonal matrix $diag(\mu_1, \ldots \mu_m)$ with $\mu_i > 0$ for all i .

Proof. The quadratic form of $f \circ \psi^{-1} - f(p)$ in the coordinates of the Morse chart is $q_H(v) = v^T H v$. The quadratic form induced by the Riemannian metric $g(p)$ is $q_G(v) = v^T G v$, where $v \in \mathbb{R}^m$ are the coordinate vectors. Since $g(p)$ is positive definite, G is also positive definite, and H is symmetric.

Consider the generalized eigenvalue problem $Hv = \lambda G v$. Since H and G are real symmetric matrices and G is positive definite, this problem has m real eigenvalues $\lambda_1, \ldots, \lambda_m$ and corresponding eigenvectors $v_1, \ldots, v_m$, which can be chosen to be orthogonal with respect to the bilinear form defined by G, such that $v_i^T G v_j = \delta_{ij}$, since if v_i and v_j are different eigenvectors with respect to different eigenvalues, we can calculate:

$$v_j^T H v_i = (H v_j)^T v_i = \lambda_j (G v_j)^T v_i = \lambda_j v_j^T G(v_i) \quad \text{and} \quad v_j^T H v_i = v_j^T \lambda_i G(v_i) = \lambda_i v_j^T G(v_i)\,.$$

Hence we have the equality

$$\lambda_j (G v_j)^T v_i = \lambda_i v_j^T G(v_i) \Leftrightarrow \underbrace{(\lambda_j - \lambda_i)}_{\neq 0} v_j^T G(v_i) = 0$$

Hence all eigenspaces are orthogonal and we can use Gram Schmitt to make the bases of the eigenspaces orthogonal and by rescaling orthonormal.

Let $L = [v_1|\ldots|v_m]$ be the matrix whose columns are these G-orthonormal eigenvectors. The linear change of coordinates $y = Lz$ leads to new coordinates z. In these new coordinates, the quadratic forms transform as follows:

$$q_G(y) = y^T G y = (Lz)^T G(Lz) = z^T L^T G L z = z^T I z = \sum_{i=1}^{m} z_i^2$$
$$q_H(y) = y^T H y = (Lz)^T H(Lz) = z^T L^T H L z$$

Since $H v_i = \lambda_i G v_i$, we have $L^T H L = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$. The eigenvalues λ_i are real and have the same signature as the eigenvalues of H (l negative, $m-l$ positive).This is true to sysvesters law of inertia. By a further scaling of the coordinates z and reordering, the matrix $L^T H L$ can be brought into the form $\operatorname{diag}(-1, \ldots, -1, 1, \ldots, 1)$. (by doing this however, the matrix G becomes $G = \operatorname{diag}(\mu_1, \ldots \mu_m)$ with $\mu_j > 0 \ \forall j$). If ψ is the Morse chart we started with, then $L^{-1} \circ \psi$ is the new morse chart. Here we have:

$$f \circ (\psi^{-1} \circ L)(y) = f \circ \psi^{-1}(L(y)) = q_H(L(y) = \sum_{i=1}^{l} -y_i^2 + \sum_{i=l+1}^{m} y_i^1$$

Furthermore, we want to inspect the metric $g\big|_p : T_p M^2 \to \mathbb{R}$ with respect to the chart $\psi \circ L^{-1}$-induced basis $\left(\frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_m}\Big|_p\right)$ We calculate $g(p)\left(\frac{\partial}{\partial x_i}\Big|_p, \frac{\partial}{\partial x_j}\Big|_p\right)$. With the coordinate map we have that $g(p)\left(q^{-1}_{\psi \circ L^{-1}}, q^{-1}_{\psi \circ L^{-1}}\right)$ is a quadratic form on R^n and given by a matrix $\tilde{G}$. □

Check how $\tilde{G}$ and G are connected, by analysing how $q_{\psi \circ L^{-1}}$ and q_ψ are connected

Corollary 0.0.7. Assume that ψ is a Morse chart and$(l_{ij}) \in \mathrm{GS}(l,m)$. Then $f \circ \psi^{-1} \circ L^{-1}$ reads:

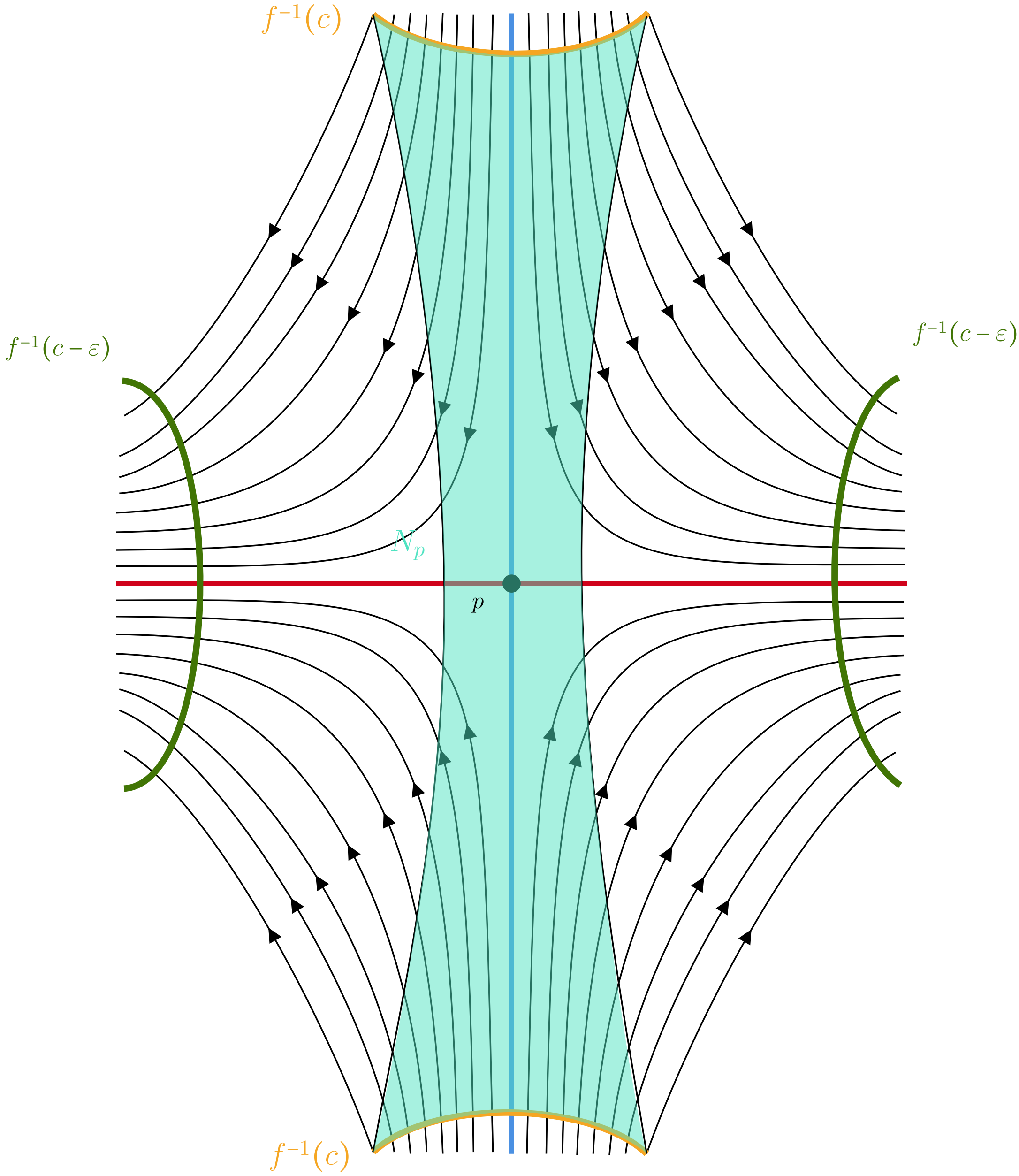
$$f \circ \psi^{-1} \circ L \circ L^{-1}(x^1, \ldots x^m) =$$

Lemma 0.0.8. *If* $(x_1, \dots, x_m)$ *is a local coordinate system arround a critical point* $p \in M$ *such that* $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right)$ *is a orthonormal basis for the tangent space at* p *(with respect to* g_p*, then for any* $t \in \mathbb{R}$ *the matrix for the differential of* φ_t *at* p *is the exponential of the negative matrix for the Hessian at* p*, i.e.:*

$$\frac{\partial}{\partial x}\varphi_t\bigg|_p = \exp(-\operatorname{Hess}(f)_p \cdot t\,.)$$

Proof. Compare Lemma 4.5 in [**banyaga2004lectures**]. □

Proof. The statements 1-4 are easy to proof by the definitions. The regularity can be derived from a general fact for pairs (X, Y) in metric spaces: If Y is closed in X and there is a neighbourhood of Y that is open in X such that Y is a strong deformation retract of U. So the only thing left to show is, that N_p is a tubular neighbourhood. in [**MorseTheorySalmbon**] in the proof of lemma 3.2 Salamon claims this to be true without a proof (page 119 in the attached source). Similar, in [**banyaga2004lectures**] Banyaga claims this (also without any argument). I find this difficult to proof, since we cannot use the flow for the map from the normal bundle, as points leave N_p along the flow: The Set N_p is sketched in the figure below.



Idea: Show it locally in a morse chart arround p. Hopefully we can archive, that in such a chart the property $\varphi_T(x) \geq a - \varepsilon$ translates to $x = \psi(x_s, x_u)$ where $\|x_u\| \leq T_x$. Then assume T to be big enough such that for any $x \in N_p$ there is a chart U arround a point $x_p \in W(\to p)$ and a t_0 such that $\varphi_{t_0}(U)$ lives in said morse chart. Finaly check if the property required arround said x_p can be furmulated such that U has a tubular structure. So lets start this procedure! Let

$$\psi : p \in V \to U \subseteq \mathbb{R}^m$$

be a morse chart. Here the function f is of the form

$$f \circ \psi^{-1}(x_s, x_u) = a + x^{s2} - x^{u2} .$$

where x^{s2} and x^{u2} denotes the sum of all the squares from the morse lemma. Without restrictions we call $x^s = (x^1, \dots x^l$ and $x^u : (x^{l+1}, \dots x^m)$. Now we inspect the set

$$\psi^{-1}(N_p \cap V) = \left\{ (x^s, x^u) \in U \mid f \circ \varphi_T(\psi^{-1}(x^s, x^u)) \geq a - \varepsilon \right\} . \tag{15}$$

First we inspect the gradient in those local coordinats, i.e. $x = \psi^{-1}(u)$:

$$\mathrm{grad}(f)(x) = g^{ik} \frac{\partial f}{x^k} \frac{\partial}{x^i}$$

For now assume that $g_{ik} = \mathrm{diag}(1, \dots 1)$, i.e. that we work with the euklidean metric. Then the gradient is:

$$\sum_{i,k} \frac{\partial f \circ \psi^{-1}}{\partial x^k} \frac{\partial}{\partial x^i} = 2 \frac{\partial}{x^s} - 2 \frac{\partial}{x^u} .$$

Hence, the flow corresponding to $\psi_*(-\mathrm{grad}(f))$ is of the form

$$t \mapsto \varphi_t(x) = (e^{-2t} x^s, e^{2t} x^u) .$$

Assume that ε is small enough, such that all $x \in \psi(N_p \cap V) \setminus W(\to p)$ flow through $f^{-1}(a - \varepsilon)$ inside of U, i.e. we can formulate the property:

$$f \circ \varphi_T(\psi^{-1}(x^s, x^u)) \geq a - \varepsilon \quad \Leftrightarrow \quad f\left(\psi^{-1}(e^{-2T} x^s, e^{2T} x^u)\right) = a + \left(e^{-2T} x^s\right)^2 - (e^{2T} x^u)^2 \geq a - \varepsilon \tag{16}$$

This however reads:

$$a + \left(e^{-2T} x^s\right)^2 - (e^{2T} x^u)^2 \geq a - \varepsilon \quad \Leftrightarrow \quad (e^{2T} x^u)^2 \leq \varepsilon + \left(e^{-2T} x^s\right)^2$$

And hence we have that:

$$\psi(N_p \cap V) = \left\{ (x^s, x^u) \in U \;\middle|\; \|x^u\| \leq R(x^s) \right\}$$

where $R(x^s)$ is a smooth function. Now let g be a general metric, that is a section into the bundle of two forms with certain properties. (non-degeneret, symmetric and positive definite) We want to Taylor expand the gradient locally. So define $V^i(x) := \sum_{j=1}^m g^{ij}(x) \frac{\partial f \circ \psi^{-1}}{x^j}(x)$. Now we taylor this component in p:

$$V^i(x) = V^i(p) + \sum_{k=1}^m \frac{\partial V^i}{x^k}(p)(x^k - x_0^k) + \mathcal{O}(\|x\|^2)$$

and

$$
\begin{aligned}
\frac{\partial V^i}{x^k}(0) &= \frac{\partial}{x^k}(0)\sum_{j=1}^m g^{ij}(0)\frac{\partial f}{x^j}(0) \\
&= \sum_{j=1}^m \left(\frac{\partial g^{ij}}{\partial x^k}(0)\frac{\partial (f\circ\psi)}{\partial x^j}(0) + g^{i,j}\frac{\partial^2 (f\circ\psi)}{\partial x^k \partial x^i}(0)\right) \\
&= \sum_{j=1}^m \left(g^{i,j}\frac{\partial^2 (f\circ\psi)}{\partial x^k \partial x^i}(0)\right)
\end{aligned}
$$

$$
\sum_{i,k} g^{ik}\frac{\partial f\circ\psi^{-1}}{\partial x^k}\frac{\partial}{\partial x^i} = \sum_{k=1}^{l} 2g^{kk}\frac{\partial}{x^k} - \sum_{k=l+1}^{m} 2g^{kk}\frac{\partial}{x^k}
$$

where $g^{ik} \in C^\infty(M,\mathbb{R})$. Hence we can tayler all the g^{ik} in p to get them to be of the form $\mathrm{d}g^{ik}(p) + r^{ik}(x)$ □

Lemma 0.0.9. *Assume that $\psi : U \to V$ is a chart and $L : R^m \to R^m$ a linear function. Then the diagramm*

$$
\begin{tikzcd}
T_pM \arrow[r, "q_\psi"] \arrow[dr, "q_{\psi\circ L}"] & \mathbb{R}^m \arrow[d, "L"] \\
& \mathbb{R}^m
\end{tikzcd}
$$

commutes for all $p \in U$.

Proof. We show this by showing the inverse: For any $v \in \mathbb{R}^m$ we have the two $(q_\psi\big|_p)^{-1}L^{-1}(v)$ and $(q_{\psi\circ L}|p)^{-1}$ and claim that they are the same. So let $f_p \in \mathcal{E}_pM$. Assume that $L^{-1} = (\lambda_{ij})_{ij}$, $\psi(p) = x_0$ and $L(x_0) = \tilde{x_0}$. We now calculate

$$
\left[(q_\psi\big|_p)^{-1}L^{-1}(v)\right](f_p) = \left[(q_\psi\big|_p)^{-1}\sum_{i,j} e_i\lambda_{ij}v_j\right](f_p) = \left[\sum_{i,j}\lambda_{ij}v_j(q_\psi\big|_p)^{-1}(e_i)\right](f_p)
$$

and

$$
\begin{aligned}
\left[(q_{\psi\circ L}\big|_p)^{-1}(v)\right](f_p) &= \sum_i v_i\frac{\partial}{\partial x_i}\Big|_p(f_p) \\
&= \sum_i v_i\frac{\partial}{\partial x_i}\Big|_{x_1}(f\circ\psi^{-1}\circ L^{-1}) \\
&= \sum_{i,j} v_i\frac{\partial}{\partial x_i}\Big|_{x_0}(f\circ\psi^{-1})\lambda_{ij}
\end{aligned}
$$

□