# Contents

# 1 Differentiable structures on topological manifolds

**Definition 1.0.1** (Topological Manifold)**.** A second countable Hausdorffspace $M$ is called **topological manifold** of dimension $m \in \mathbb{N}$, if it is locally homeomorphic to $\mathbb{R}^m$. To be precise, if for all $p \in M$ there exists an open neighborhood $U \subseteq M$ of $p$, an open set $V \subseteq \mathbb{R}^m$ and a map $\varphi : W \to V$ that is a homeomorphism. We call the map $\varphi : U \to V$ a **chart around** $p$ **on** $M$ and $\varphi^{-1}$ a **local coordinate system around** $p$ **on** $M$.

**Definition 1.0.2** (Differentiable Manifold)**.** Let $M$ be a topological manifold of dimension $M$.

1. A **differentiable atlas of class** $r \in \mathbb{N} \cup \{\infty\}$ is a family of charts $\mathfrak{A} = (\varphi_i : U_i \to V_i)_{i \in I}$ such that
   a) $\bigcup_{i \in I} U_i = M$, meaning that $(U_i)$ is an open covering of $M$.
   b) For every pair $(i, j) \in I^2$ the **transition function**:
   $$\begin{aligned} \varphi_{ij} : \varphi_j(U_i \cap U_j) &\to \varphi_i(U_i \cap U_j) \\ x &\mapsto \left(\varphi_i \circ \varphi_j^{-1}\right)(x) \end{aligned}$$
   is differentiable of class $r$.

   We call such an atlas a $C^r$-atlas.
2. Two $C^r$-atlases $\mathfrak{A}$ and $\mathfrak{B}$ are called **equivalent** if the family $\mathfrak{A} + \mathfrak{B} = \left(\varphi_i, \varphi_j\right)_{ij}$ is a $C^r$-atlas.

A **differentiable structure of class** $r$ on $M$ is an equivalence class $c$ of $C^r$-atlases. For $r = \infty$ we call the pair $(M, c)$ a smooth manifold.

**Corollary 1.0.3.** Every transition functions $\varphi_{ij}$ $i, j \in I^2$ of a differentiable atlas $\mathfrak{A} = (\varphi_i)_{i \in I}$ is not just a homeomorphism but also a diffeomorphism due to $\varphi_{ji} = \varphi_{ij}^{-1}$

**Definition 1.0.4.** Let $(M, c)$ be a differentiable manifold of class $r$ and $U \subseteq M$ open. We call a continuos function

$$f : U \to \mathbb{R}$$

**differentiable of class** $r$, if for any one (and hence for all) $(\varphi_i)_{i \in I} = \mathfrak{A} \in c$ the compositions $f \circ \varphi_i^{-1}$ are differentiable of class $r$. For $r = \infty$ we define:

$$\mathcal{E}(U) = \{f \in U \to \mathbb{R} \text{ continouos} \mid f \text{ is differentiable of class } \infty\}.$$

**Corollary 1.0.5.** Let $(M, c)$ be a smooth manifold of dimension $m$ and $U \subseteq M$ be a open subset. With pointwise defined operations, the set $(\mathcal{E}(U), +, \cdot, \circ)$ becomes an $\mathbb{R}$–algebra. Furthermore, $\mathcal{E}$ becomes a sheaf of $\mathbb{R}$-algebras.

*Proof.* There is not really a need for a proof.However, it might help to work through the definition of a sheaf as a reminder. First, $\mathcal{E}$ is a presheaf, where the restriction in the domain of a function gives the needed restriction homomorphism:

$$\begin{aligned} \mathrm{res}^U_V : \mathcal{E}(U) &\to \mathcal{E}(V) \\ f &\mapsto f\Big|_V . \end{aligned}$$

The required properties of a presheaf are trivial. Furthermore, this gives a sheaf as the requirement of locality is trivial for functions and the property of gluing is also trivial for functions, since differentiability is a local property. □

**Definition 1.0.6.** If $p \in M$ is fix, $f \in \mathcal{E}(U)$ and $g \in \mathcal{E}(U')$ such that $p \in U \cap U'$ we say that $f$ and $g$ have the same **germ in** $p$, if there is another open neighborhood $W \subseteq U \cap U'$ of $p$ such that $f|_w = g|_W$. This defines an equivalence relation $\sim_p$. An equivalence class $s$ of local functions around $p$ is called a **germ in** $p$. We write $s = f_p$, if $s = [f]$ with $f \in \mathcal{E}(U)$. We write

$$\mathcal{E}_p(M) = \left( \sum_{U \text{open}, p \in U} \mathcal{E}(U) \right) \Big/ \sim_p \; .$$

For the set of germs and call it the **stalk in** $p$. Here $\sum$ denotes the co-product (also called sum) in $\top$ and hence the disjoint union.

**Corollary 1.0.7.** For a smooth manifold $(M, c)$ the set $\mathcal{E}_p(M)$ inherits an $\mathbb{R}$–algebra structure from the $\mathcal{E}(U)$. Furthermore, it carries a natural (evaluation-)homomorphism:

$$\begin{aligned} \mathrm{eval}_p : \mathcal{E}_p(M) &\to \mathbb{R} \\ f_p &\mapsto f(p) =: f_p(p) \end{aligned}$$

The stalks are also local rings with maximal ideal $\mathfrak{m}_p = \ker(\mathrm{eval}_p)$. Hence, the pair $(M, \mathcal{E})$ gives us a locally ringed space.

*Proof.* Here, we only need to prove the statement about the locality of the stalks. This follows from $f_p \in \mathcal{E}_p(M)$ being invertible if and only if $f(p) \neq 0$ which is the same as $f_p \notin \ker(\mathrm{eval}_p)$. □

**Definition 1.0.8.** Let $(M, c)$ be a smooth manifold of dimension $m$ and $p \in M$. We call an $\mathbb{R}$-linear map $\delta : \mathcal{E}_p(M) \to \mathbb{R}$ a **derivation**, if it satisfies the Leibnitz-rule:

$$\delta(f_p \cdot g_p) = \delta(f_p) g_p(p) + f_p(p) \delta(g_p) \quad \text{for all } f, g \in \mathcal{E}_p(M) .$$

We call $\mathrm{Der}_{\mathbb{R}}(\mathcal{E}_p(M), \mathbb{R})$ the set of derivations and give it a $\mathbb{R}$ vector space structure by pointwise operations. We define the **tangent space of** $M$ **at** $p$ to be the vector space

$$TM_p := \mathrm{Der}_{\mathbb{R}}(\mathcal{E}_p(M), \mathbb{R}) .$$

**Corollary 1.0.9.** Let $(M,c)$ be a smooth manifold and $\varphi : U \to V$ be a chart around $p$ with $x_0 = \varphi(p)$ $(\varphi \in \mathfrak{A} \in c)$. Then

$$\xi = \frac{\partial}{\partial x^j}\bigg|_p : \mathcal{E}_p(M) \to \mathbb{R}$$
$$f_p \mapsto \xi(f_p) = \frac{\partial}{\partial x^j}\bigg|_{x_0}(f \circ \varphi^{-1})$$

is well-defined and a tangent vector. In fact, the family

$$\left(\frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^m}\bigg|_p\right)$$

defines a basis of $TM_p$. Hence, the dimension of $TM_p$ is $m$.

**Definition 1.0.10.** For a smooth manifold $(M,c)$ the sum $\sum_p TM_p$ comes with a natural projection

$$\pi : TM \to M$$
$$\xi \mapsto p \text{ where } \xi \in TM_p$$

Furthermore, the **local vector fields** with respect to a chart $\varphi : U \to V$

$$\frac{\partial}{\partial x^j} : U \to \pi^{-1}(U)$$
$$p \mapsto \frac{\partial}{\partial x^j}\bigg|_p$$

induce a local trivialization:

$$\pi^{-1}(U) \cong U \times \mathbb{R}^m .$$

We can induce a topology on $TM$ such that all those trivializations are continuos. This then gives an atlas for $TM$ such that we have a $2m$-dimensional manifold. To be precise, the atlas is given by the maps $\pi^{-1}(U_i) \to R^m \times R^m; x \mapsto (\pi(x), q_{\varphi \circ \pi(x)}(x))$ where $q_p$ denotes the coordinate map corresponding to the basis $(\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^m}\big|_p)$ that depends on the chart $\varphi_i$. In fact, this yields a smooth manifold and a (smooth) vector bundle of dimension $m$. We call $TM$ the **tangent bundle**.

**Definition 1.0.11** (The Derivative)**.** Let $(M,c)$ and $(M',c')$ be smooth manifolds (from now on we suppress the differentiable structure in our notation). We call a continuous function $f : M \to M'$ **smooth**, if for all $\varphi \in \mathfrak{A} \in c$ and $\varphi' \in \mathfrak{A}' \in c'$ the maps

$$\varphi' \circ f \circ \varphi^{-1} : V \to V'$$

are smooth. A given smooth function induces a smooth function between the Tangent bundles as follows:

$$Df : TM \to TM' \ , Df_p(\xi)(g_p) = \xi((g \circ f)_p)$$

Here, $\xi \in T_pM$, $g_p \in E_p(M')$.

**Corollary 1.0.12.** Let $f : M \to \mathbb{R}$ be smooth and $\varphi : U \to V$ be a chart. Then we can interprete $\mathrm{d}f$ as a one-form. To be prezice asume $q$ to be the coordinate funktion $T\mathbb{R} \to \mathbb{R}$ from the basis induced by the identity as a chart. $v \in \Gamma TM$ we have

$$q \circ \mathrm{d}f(v) = v \cdot f$$

Here on the righd side, $f$ denotes a map $p \mapsto f_p$ such that $v(f)(p) \coloneqq v_p(f_p)$ is well defined. We keep this notations so vector fields can take funktions as an input.

*Proof.* We proof this by showing it for the section $\frac{\partial}{\partial x^i}$ and thereby for any, since those sections form a basis of the space of sections as a $C^\infty(M, \mathbb{R})$ vectorspace. Now let $g_p \in \mathcal{E}_p(\mathbb{R})$ and $\varphi(p) = x_0$. Then

$$\mathrm{d}f(\frac{\partial}{\partial x^i}\Big|_p)(g_p) = \frac{\partial}{\partial x^i}\Big|_p(g \circ f)_p = \frac{\partial}{\partial x^i}\Big|_{x_0}(g \circ f \circ \varphi^{-1}) = \frac{\partial}{\partial x}\Big|_p(g_p) \cdot \frac{\partial}{\partial x^i}\Big|_p(f_p)$$

Hence, $\mathrm{d}f(v) = v(f)\frac{\partial}{\partial x}$ letting us conclude the statement. □

**Definition 1.0.13** (A Metric)**.** Let $M$ be a smooth manifold. A section $g \in \Gamma(TM^* \otimes TM^*)$ into the tensor product of the dual Tangend bundle with itself is a **riemannian metric** if the following are satisfied for all $p \in M$:

1. $g$ is **non degenerate**, meaning for a $v_p \in T_pM$ $g_p(v_p, w_p) = 0$ $\forall w_p \in T_pM$ then $v_p = 0$.
2. $g$ is **symmetric**, meaning $g_p(v_p, w_p) = g_p(w_p, v_p)$ $\forall v_p, w_p \in T_pM$ .
3. $g$ is **positiv definite**, meaning that $g_p(v_p, v_p) \geq 0 \forall v_p$ and vanishes only for $v_p = 0$.

We cal a touple $(M, g)$ a **riemannian manifold**.

**Definition 1.0.14** (The Gradient)**.** Assume that $(M, g)$ is a smooth manifold together with a riemannian metric. Let $f : M \to \mathbb{R}$ be a smooth function. We define its **gradient** to be a vector field $\mathrm{grad}(f) \in \Gamma(TM)$ such that for any vecor field $V$:

$$q \circ \mathrm{d}f(V) = g(\mathrm{grad}(f), V)\,.$$

Here $q$ denotes the canonical coordinate funtion $TR \to \mathbb{R}$.

**Definition 1.0.15.** Let $(M, g)$ be a riemannian manifold and $\varphi : U \to V$ be a chart. We define the smooth functions $g_{ij} : U \to \mathbb{R}$ to be :

$$g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\,.$$

Then if two vectorfields $v, w$ over $U$ are given with $v = \sum v^i \frac{\partial}{\partial x^i}$ and $w = \sum_i w^i \frac{\partial}{\partial x^i}$ we can calculate the metric locally:

$$g(v, w) = \sum_{ij} g_{ij} v^i w^j \ : U \to \mathbb{R}$$

Furthermore, we can invert the matricies $(g_{ij}(p))_{ij}$ for all $p$ (which is a smooth procedure by the theorem of cayley hamilton) to get smooth functinos

$$\begin{aligned} g^{ij} : U &\to \mathbb{R} \\ p &\mapsto g^{ij}(p) = \left((g_{kl}(p))^{-1}_{kl}\right)_{ij} \end{aligned}$$

**Lemma 1.0.16.** *Let $\varphi : U \to V$ be a chart. Then the gradient has the local form:*

$$\operatorname{grad}(f) = \sum_{i,j} g^{ij} \left(\frac{\partial}{\partial x^i}(f)\right) \frac{\partial}{\partial x^j} =: \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} .$$

*Here, $g^{ij}$ denotes the smooth functions given by the coordinates of the function $x \mapsto (g_{ij}(x))$ that defines the coordinate representation of the gradient in $x$.*

*Proof.* Assume that $v, w \in \Gamma(TM)$ such that $v = \sum_i v^i \frac{\partial}{\partial x^i}$ and $w = \sum_i w^i \frac{\partial}{\partial x^i}$. Then $g(v, w) = \sum_{i,j} g_{ij} v^i w^i$. Suppose that $\operatorname{grad}(f) = \sum_j G^j \frac{\partial}{\partial x^j}$ and $q$ is the coordinate map $T\mathbb{R} \to \mathbb{R}$ in each tangend space. Then by definition of the gradient we have:

$$\frac{\partial}{\partial x^j}(f) = q \circ \mathrm{d}f(\frac{\partial}{\partial x^j}) = q \circ \frac{\partial f}{\partial x^j} = g(\operatorname{grad}(f), \frac{\partial}{\partial x^j}) = \sum_{ij} g_{ij} G^i$$

Hence,

$$\begin{aligned} (G^1, \cdots G^m)(g_{ij}) &= \left(\frac{\partial}{\partial x^1}(f), \cdots, \frac{\partial}{\partial x^m}(f)\right) \\ \Leftrightarrow (G^1, \cdots G^m) &= \left(\frac{\partial}{\partial x^1}(f), \cdots, \frac{\partial}{\partial x^m}(f)\right)(g^{ij}) . \end{aligned}$$

But then we can conclude that $G^j = \sum_i g^{ij} \frac{\partial}{\partial x^i}(f)$.

□