

## The Setting

We work in a compact, riemanian, orientable, smooth Manifold  $(M, g)$  of dimension  $m$  together with a Morse-Smale funktion

$$f : M \rightarrow \mathbb{R} \quad (1)$$

We assume that  $q$  is a critical point of index  $k+1$  and  $p$  is a critical point of index  $k$ . we define  $f(q) = b$  and  $f(p) = a$  and assume that there is no kritikal point in  $f^{-1}(a, b) \subseteq M$ . We define the sub and superlevelsets

$$M^t := \{x \in M | f(x) \leq t\} \quad \text{and} \quad M_t := \{x \in M | f(x) \geq t\}, \quad (2)$$

and the constants

$$c \in (a, b) \quad , \quad \varepsilon > 0 \text{ small} \quad , \quad T > 0 \text{ big} . \quad (3)$$

With this we define the sets:

$$N_q := \{x \in M_c | f(\varphi_{-T}(x)) \leq b + \varepsilon\}, \quad (4)$$

$$L_q := \{x \in N_q | f(x) = c\}, \quad (5)$$

$$N_p := \{x \in M^c | f(\varphi_T(x)) \geq a - \varepsilon\}, \quad (6)$$

$$L_p := \{x \in N_p | f(\varphi_T(x)) = a - \varepsilon\}, \quad (7)$$

and finally:

$$C := N_p \cup N_q, \quad (8)$$

$$B := N_p \cup L_q, \quad (9)$$

$$A := L_p \cup (L_q - N_p). \quad (10)$$

**Lemma 0.0.1.** *We claim that*

1.  $(N_q, L_q)$  is a regular index pair for  $q$  .
2.  $(C, B)$  is an index pair for  $q$  .
3.  $(N_p, L_p)$  is a regular index pair for  $p$  .
4.  $(B, A)$  is an index pair for  $p$  .
5.  $N_p$  is a tubular neighbourhood of  $W(\rightarrow p) \cap M^c$  .

**Definition 0.0.2** (Jacobian of the Gradient). The gradient is a section into the tangent bundle,  $\text{grad}(f) : M \rightarrow TM$ . Let  $\psi : M \supseteq U \rightarrow V \subseteq \mathbb{R}^m$  be a chart. The coordinate

map  $q_\psi$  on  $TU$  assigns to a vector  $v \in T_p M$  (where  $p \in U$  with  $\psi(p) = x$ ) its components with respect to the basis  $\left(\frac{\partial}{\partial x^i}\Big|_p\right)$ , i.e.,  $q_\psi(v) = (v^1, \dots, v^m)$  if  $v = \sum_{i=1}^m v^i \frac{\partial}{\partial x^i}\Big|_p$ . We define the Jacobian of the gradient with respect to the chart  $\psi$  as the Jacobian matrix of the coordinate representation of the gradient in this chart:

$$J(\text{grad}(f))_\psi(x) := J\left(q_\psi \circ \text{grad}(f) \circ \psi^{-1}\right)(x) = \left(\frac{\partial(q_\psi \circ \text{grad}(f) \circ \psi^{-1})_i}{\partial x_j}(x)\right)_{ij}$$

**Lemma 0.0.3.** *Let  $t \in \mathbb{R}$  be small enough and  $\varphi_t : M \rightarrow M$  be the flow map corresponding to the negative gradient. Assume that  $U$  is the domain of a chart  $\psi$  and  $p \in U$  such that  $\varphi_t(p) \in U$ . Then the linear map  $d\varphi_t|_p : T_p M \rightarrow T_{\varphi_t(p)} M$  has a local representation in the coordinates induced by  $\psi$  given by:*

$$q_{\psi(\varphi_t(p))} \circ d\varphi_t|_p \circ q_{\psi(p)}^{-1} = \exp\left(-J(\text{grad}(f))_\psi(\psi(p)) \cdot t\right)$$

where  $J(\text{grad}(f))_\psi(\psi(p))$  is the Jacobian matrix of the gradient evaluated at the coordinates of  $p$ .

**Lemma 0.0.4.** *Let  $\psi$  be a coordinate system around a critical point  $p$ . I.e.  $\psi : p \in U \rightarrow V$  with  $\psi(p) = 0$  and let  $\left(\frac{\partial}{\partial x_1}\Big|_x, \dots, \frac{\partial}{\partial x_m}\Big|_x\right)$  be the induced basis of  $T_x M$ . Then the jacobean of the gradient*

$$J(\text{grad}(f))_\psi(0) = \sum_k g^{ki}(0) \frac{\partial^2 f \circ \psi^{-1}}{\partial x_j \partial x_k} \quad (11)$$

where  $(g^{ki}(x))$  denotes the matrix corresponding to the gradient in  $T_x M$  with respect to the basis  $\left(\frac{\partial}{\partial x_1}\Big|_x, \dots, \frac{\partial}{\partial x_m}\Big|_x\right)$ .

*Proof.* Let  $q_\psi$  denote the koordinate funktion  $TU \rightarrow \mathbb{R}^m$  induced from the basis  $\left(\frac{\partial}{\partial x_1}\Big|_x, \dots, \frac{\partial}{\partial x_m}\Big|_x\right)$ . then the gradient in  $\psi$  reads:

$$q_\psi \circ \text{grad}(f) \circ \psi^{-1} = \left(\sum_k g^{k1} \frac{\partial f \circ \psi^{-1}}{\partial x_k}, \dots, \sum_k g^{km} \frac{\partial f \circ \psi^{-1}}{\partial x_k}\right)$$

Hence, we can calculate Hence, we can differentiate:

$$\begin{aligned} \frac{\partial}{\partial x} Q \circ \text{grad}(f) \circ \psi^{-1} &= \left(\frac{\partial}{\partial x_j} \sum_k g^{ki} \frac{\partial f \circ \psi^{-1}}{\partial x_k}\right)_{ij} \\ &= \left(\sum_k \left[\frac{\partial g^{ki}}{\partial x_j} \frac{\partial f \circ \psi^{-1}}{\partial x_k} + g^{ki} \frac{\partial^2 f \circ \psi^{-1}}{\partial x_j \partial x_k}\right]\right)_{ij}. \end{aligned}$$

and now in  $p = \psi^{-1}(0)$  we have

$$\left. \frac{\partial}{\partial x} Q \circ \text{grad}(f) \circ \psi^{-1} \right|_0 = \left( \sum_k \left[ g^{ki}(0) \frac{\partial^2 f \circ \psi^{-1}}{\partial x_j \partial x_k} \right] \right)_{ij}.$$

□

**Theorem 0.0.5** (Sylvester's Law of Inertia). *Let  $A \in \text{Mat}(n, \mathbb{R})$  be symmetric and let  $T, T' \in \text{GL}(n, \mathbb{R})$  and  $k, k', l, l' \in \mathbb{N}$  such that*

$$T^t \circ A \circ T = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & -1_l & 0 \\ 0 & 0 & 0_{n-k-l} \end{pmatrix} \text{ and } T'^t \circ A \circ T' = \begin{pmatrix} 1_{k'} & 0 & 0 \\ 0 & -1_{l'} & 0 \\ 0 & 0 & 0_{n-k'-l'} \end{pmatrix}$$

*then  $k = k'$ ,  $l = l'$  and  $\text{rank}(A) = k + l$ .*

*Proof.* Since  $T, T'$  are invertible we have that

$$k + l = \text{rank}(T^t \circ A \circ T) = \text{rank}(A) = \text{rank}(T'^t \circ A \circ T') = k' + l'. \quad (12)$$

Hence it suffices to show that  $k = k'$ . Which we will do by proofing the claim:

$$k = \max \{ \dim(U) \mid U \subseteq \mathbb{R}^n \text{ subspace such that } x^t A x > 0 \ \forall x \in U \setminus \{0\} \}$$

So we start by showing „ $\leq$ “: Denote the first  $k$  columns of  $T$  with  $x_1, \dots, x_k$ . They form a basis of  $\mathbb{R}^n$  and with  $0 \neq x = \sum_{i=1}^k \lambda_i x_i$  we have that by bilinearity

$$x^t A x = \sum_{i=1}^k \lambda_i x_i^t A x = \sum_{i,j=1}^k \lambda_i \lambda_j x_i^t A x_j = \sum_{i=1}^k (\lambda_i)^2 \geq 0. \quad (13)$$

This concludes the first inequality.

Now let  $U$  be any  $k$ -dimensional subspace such that for all non zero  $x \in U$   $x^t A x > 0$ . By a calculation analog to the one above we have that for any  $x \in W := \text{span}(x_{k+1}, \dots, x_n)$  the number  $x^t A x$  is less or equal to zero. Hence,  $W \cap U = \{0\}$  and with this we conclude:

$$\dim(U) = \dim(U + W) - \dim(W) + \dim(U \cap W) \leq (k + (n - k)) - (n - k) + 0 = k. \quad (14)$$

This is the last inequality proving the statement. □

**Lemma 0.0.6.** *Assume that  $\psi : U \rightarrow V$  is a chart and  $L : \mathbb{R}^m \rightarrow \mathbb{R}^m$  a linear function. Then the diagram*

$$\begin{array}{ccc}
T_p M & \xrightarrow{q_\psi|_p} & \mathbb{R}^m \\
& \searrow q_{L \circ \psi}|_p & \downarrow L \\
& & \mathbb{R}^m
\end{array}$$

commutes for all  $p \in U$ .

*Proof.* We show this by showing the inverse: For any  $v \in \mathbb{R}^m$  we have the two  $(q_\psi|_p)^{-1} \circ L^{-1}(v)$  and  $(q_{\psi \circ L}|_p)^{-1}$  and claim that they are the same. Assume that  $L^{-1} = (\lambda_{ij})_{ij}$ ,  $\psi(p) = x_0$ ,  $L(x_0) = \tilde{x}_0$ ,  $(v^1, \dots, v^m) = v \in \mathbb{R}^m$  and denote the basis of  $T_p M$  coming from the chart  $\psi$  by  $\left(\frac{\partial}{\partial x^i}\bigg|_p\right)_i$  and the one coming from  $L^{-1} \circ \psi$  by  $\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_p\right)_i$ . We now calculate for  $f_p \in \mathcal{E}_p M$ :

$$\begin{aligned}
\left[(q_{\psi \circ L}|_p)^{-1}(v)\right](f_p) &= \sum_i v^i \frac{\partial}{\partial \tilde{x}^i}\bigg|_p(f_p) \\
&= \sum_i v^i \frac{\partial}{\partial x^i}\bigg|_{\tilde{x}_0}(f \circ \psi^{-1} \circ L^{-1}) \\
&= \sum_i v^i \sum_j \frac{\partial f \circ \psi^{-1}}{\partial x^j}\bigg|_{x_0} \cdot \frac{\partial L_j^{-1}}{\partial \tilde{x}^i}\bigg|_{\tilde{x}_0} \\
&= \sum_{i,j} v^i \frac{\partial}{\partial x^j}\bigg|_p(f_p) \cdot \lambda_{ji} \\
&= \sum_{i,j} v^i q_\psi|_p^{-1}(e_j)(f_p) \cdot \lambda_{ji} \\
&= \left[q_\psi|_p^{-1}\left(\sum_{i,j} v^i \lambda_{ji} e^j\right)\right](f_p) \\
&= \left[q_\psi|_p^{-1}(L^{-1}(v))\right](f_p)
\end{aligned}$$

This concludes the proof. □

**Theorem 0.0.7** (Simultaneous Diagonalisation of Quadratic Forms). *Let  $p$  be a critical point of a Morse function  $f$  on a manifold  $M$  and let  $g(p)$  be the Riemannian metric on  $T_p M$ . Then there exists a Morse chart  $\psi$  around  $p$  such that the representation of  $g(p)$  in the basis induced by the Morse chart is given by a diagonal matrix  $\text{diag}(\mu_1, \dots, \mu_m)$  with  $\mu_i > 0$  for all  $i$ .*

*Proof.* The quadratic form of  $f \circ \psi^{-1} - f(p)$  in the coordinates of any Morse chart  $\psi$  is  $q_H(v) = v^T H v$ . The quadratic form induced by the Riemannian metric  $g(p)$  is  $q_G(v) = v^T G v$ , where  $v \in \mathbb{R}^m$  are the coordinate vectors with respect to the Morse chart.

To be precise this means for  $v, w \in \mathbb{R}^m$ :

$$g \circ (q_\psi|_p \oplus q_\psi|_p)(v, w) = v^T G w \text{ and } f \circ \psi^{-1}(v) = f(p) + v^T H v.$$

We now aim to manipulate  $\psi$  such that  $G$  becomes diagonal: Since  $g(p)$  is positive definite,  $G$  is also positive definite, and  $H$  is symmetric.

Consider the generalized eigenvalue problem  $Hv = \lambda Gv$ . Since  $H$  and  $G$  are real symmetric matrices and  $G$  is positive definite, this problem has  $m$  real eigenvalues  $\lambda_1, \dots, \lambda_m$  and corresponding eigenvectors  $v_1, \dots, v_m$ , which can be chosen to be orthogonal with respect to the bilinear form defined by  $G$ , such that  $v_i^T G v_j = \delta_{ij}$ , since if  $v_i$  and  $v_j$  are different eigenvectors with respect to different eigenvalues, we can calculate:

$$v_j^T H v_i = (H v_j)^T v_i = \lambda_j (G v_j)^T v_i = \lambda_j v_j^T G(v_i) \quad \text{and} \quad v_j^T H v_i = v_j^T \lambda_i G(v_i) = \lambda_i v_j^T G(v_i).$$

Hence we have the equality

$$\lambda_j (G v_j)^T v_i = \lambda_i v_j^T G(v_i) \Leftrightarrow \underbrace{(\lambda_j - \lambda_i)}_{\neq 0} v_j^T G(v_i) = 0$$

Hence all eigenspaces are orthogonal and we can use Gram Schmitt to make the bases of the eigenspaces orthogonal and by rescaling orthonormal.

Let  $L = [v_1 | \dots | v_m]$  be the matrix whose columns are these  $G$ -orthonormal eigenvectors. The linear change of coordinates  $y = Lz$  leads to new coordinates  $z$ . In these new coordinates, the quadratic forms transform as follows:

$$\begin{aligned} q_G(y) &= y^T G y = (Lz)^T G (Lz) = z^T L^T G L z = z^T I z = \sum_{i=1}^m z_i^2 \\ q_H(y) &= y^T H y = (Lz)^T H (Lz) = z^T L^T H L z \end{aligned}$$

Since  $Hv_i = \lambda_i Gv_i$ , we have  $L^T H L = \text{diag}(\lambda_1, \dots, \lambda_m)$ . The eigenvalues  $\lambda_i$  are real and have the same signature as the eigenvalues of  $H$  ( $l$  negative,  $m - l$  positive). This is true to Sylvester's law of inertia. By a further scaling of the  $v_i$  and a reordering the matrix  $L^T H L$  can be brought into the form  $\text{diag}(-1, \dots, -1, 1, \dots, 1)$ . (by doing this however, the matrix  $L^T G L$  becomes  $L^T G L = \text{diag}(\mu_1, \dots, \mu_m)$  with  $\mu_j > 0 \ \forall j$ ). If  $\psi$  is the Morse chart we started with, then  $L^{-1} \circ \psi$  is the new Morse chart:

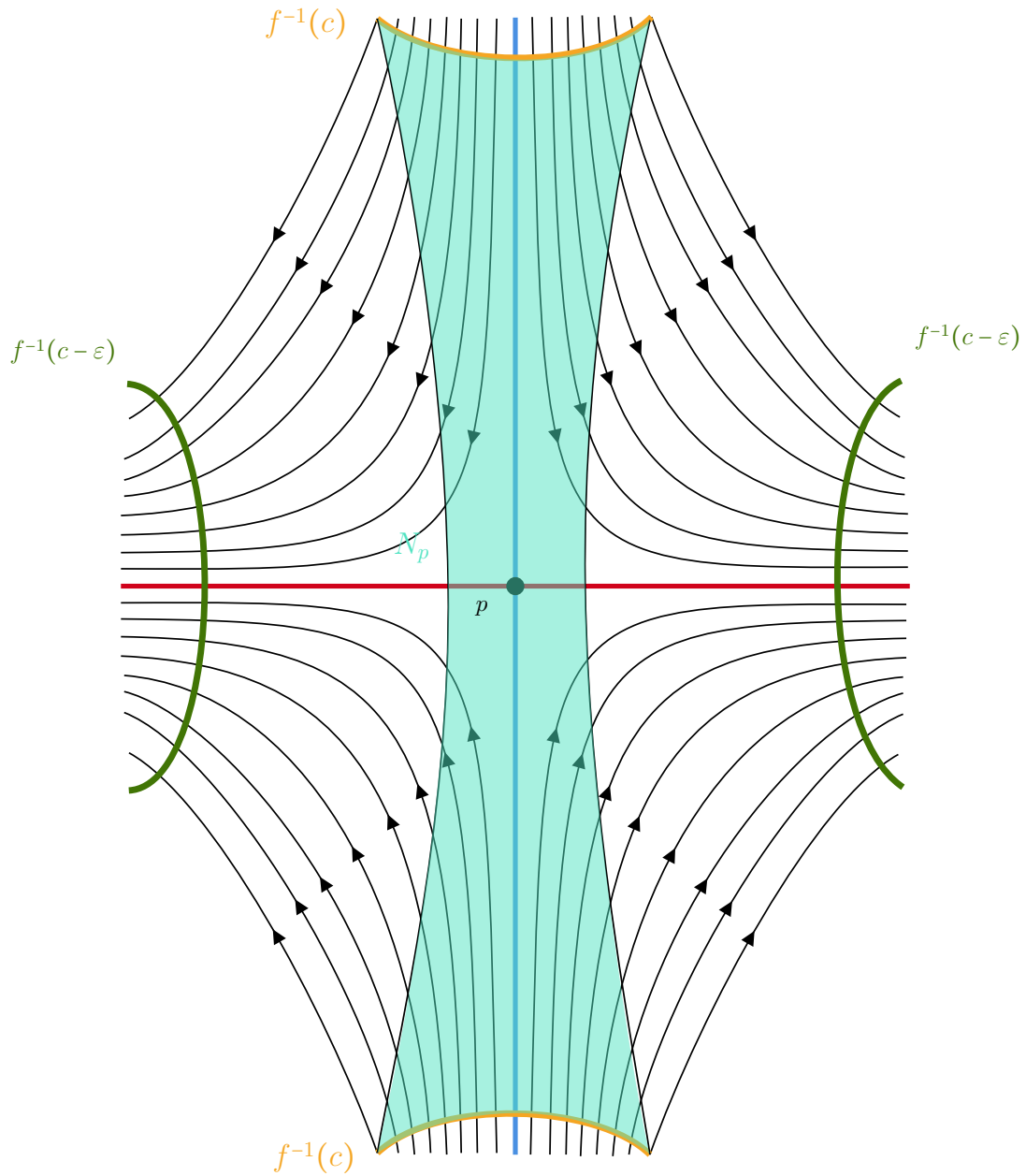
$$f \circ (\psi^{-1} \circ L)(y) = f \circ \psi^{-1}(L(y)) = q_H(L(y)) = \sum_{i=1}^l -y_i^2 + \sum_{i=l+1}^m y_i^2$$

Furthermore, we want to inspect the metric  $g|_p : T_p M^2 \rightarrow \mathbb{R}$  with respect to the chart  $L^{-1} \circ \psi$ -induced basis  $\left( \frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^m}|_p \right)$ . By the lemma 0.0.6

$$g(q_{L^{-1} \circ \psi}|_p^{-1}, q_{L^{-1} \circ \psi}|_p^{-1})(v, w) = g(q_\psi|_p^{-1}, q_\psi|_p^{-1})(Lv, Lw) = (Lv)^T G L w = v^T L^T G L w$$

□

*Proof.* The statements 1-4 are easy to proof by the definitions. The regularity can be derived from a general fact for pairs  $(X, Y)$  in metric spaces: If  $Y$  is closed in  $X$  and there is a neighbourhood of  $Y$  that is open in  $X$  such that  $Y$  is a strong deformation retract of  $U$ . So the only thing left to show is, that  $N_p$  is a tubular neighbourhood. in [**MorseTheorySalmbo**n] in the proof of lemma 3.2 Salamon claims this to be true without a proof (page 119 in the attached source). Similar, in [**banyaga2004lectures**] Banyaga claims this (also without any argument). I find this difficult to proof, since we cannot use the flow for the map from the normal bundle, as points leave  $N_p$  along the flow: The Set  $N_p$  is sketched in the figure below.



Idea: Show it locally in a morse chart around  $p$ . Hopefully we can archive, that in such a chart the property  $\varphi_T(x) \geq a - \epsilon$  translates to  $x = \psi(x_s, x_u)$  where  $\|x_u\| \leq T_x$ . Then assume  $T$  to be big enough such that for any  $x \in N_p$  there is a chart  $U$  around a point  $x_p \in W(\rightarrow p)$  and a  $t_0$  such that  $\varphi_{t_0}(U)$  lives in said morse chart. Finaly check if the property required around said  $x_p$  can be furmulated such that  $U$  has a tubular structure. So lets start this procedure! Let

$$\psi : p \in V \rightarrow U \subseteq \mathbb{R}^m$$

be a morse chart. Here the function  $f$  is of the form

$$f \circ \psi^{-1}(x_s, x_u) = a + x^{s^2} - x^{u^2}.$$

where  $x^{s^2}$  and  $x^{u^2}$  denotes the sum of all the squares from the morse lemma. Without restrictions we call  $x^s = (x^1, \dots, x^l)$  and  $x^u = (x^{l+1}, \dots, x^m)$ . Now we inspect the set

$$\psi^{-1}(N_p \cap V) = \left\{ (x^s, x^u) \in U \mid f \circ \varphi_T(\psi^{-1}(x^s, x^u)) \geq a - \varepsilon \right\}. \quad (15)$$

First we inspect the gradient in those local coordinats, i.e.  $x = \psi^{-1}(u)$ :

$$\text{grad}(f)(x) = g^{ik} \frac{\partial f}{\partial x^k} \frac{\partial}{\partial x^i}$$

For now assume that  $g_{ik} = \text{diag}(1, \dots, 1)$ , i.e. that we work with the euklidean metric. Then the gradient is:

$$\sum_{i,k} \frac{\partial f \circ \psi^{-1}}{\partial x^k} \frac{\partial}{\partial x^i} = 2 \frac{\partial}{\partial x^s} - 2 \frac{\partial}{\partial x^u}.$$

Hence, the flow corresponding to  $\psi_*(-\text{grad}(f))$  is of the form

$$t \mapsto \varphi_t(x) = (e^{-2t} x^s, e^{2t} x^u).$$

Assume that  $\varepsilon$  is small enough, such that all  $x \in \psi(N_p \cap V) \setminus W(\rightarrow p)$  flow through  $f^{-1}(a - \varepsilon)$  inside of  $U$ , i.e. we can formulate the property:

$$f \circ \varphi_T(\psi^{-1}(x^s, x^u)) \geq a - \varepsilon \quad \Leftrightarrow \quad f\left(\psi^{-1}(e^{-2T} x^s, e^{2T} x^u)\right) = a + (e^{-2T} x^s)^2 - (e^{2T} x^u)^2 \geq a - \varepsilon \quad (16)$$

This however reads:

$$a + (e^{-2T} x^s)^2 - (e^{2T} x^u)^2 \geq a - \varepsilon \quad \Leftrightarrow \quad (e^{2T} x^u)^2 \leq \varepsilon + (e^{-2T} x^s)^2$$

And hence we have that:

$$\psi(N_p \cap V) = \left\{ (x^s, x^u) \in U \mid \|x^u\| \leq R(x^s) \right\}$$

where  $R(x^s)$  is a smooth function. Now let  $g$  be a general metric, and  $\psi$  a Morse chart such that  $g(p)$  is a diagonal quadratic form wit respect to that morse chart.  $\square$