The Setting

We work in a compact, riemanian, orientable, smooth Manifold (M, g) of dimension m together with a Morse-Smale funktion

$$f : M \to \mathbb{R} \tag{1}$$

We assume that q is a critical point of index $k+1$ and p is a critical point of index k. we define $f(q) = b$ and $f(p) = a$ and assume that there is no kritical point in $f^{-1}(a, b) \subseteq M$. We define the sub and superlevelsets

$$M^t := \{x \in M | f(x) \leq t\} \quad \text{and} \quad M_t := \{x \in M | f(x) \geq t\}\,, \tag{2}$$

and the constants

$$c \in (a, b) \quad , \quad \varepsilon > 0 \text{ small} \quad , \quad T > 0 \text{ big}\,. \tag{3}$$

With this we define the sets:

$$\begin{aligned} N_q &:= \{x \in M_c | f(\varphi_{-T}(x)) \leq b + \varepsilon\}\,, && (4)\\ L_q &:= \{x \in N_q | f(x) = c\}\,, && (5)\\ N_p &:= \{x \in M^c | f(\varphi_T(x)) \geq a - \varepsilon\}\,, && (6)\\ L_p &:= \{x \in N_p | f(\varphi_T(x)) = a - \varepsilon\}\,, && (7) \end{aligned}$$

and finally:

$$\begin{aligned} C &:= N_p \cup N_q\,, && (8)\\ B &:= N_p \cup L_q\,, && (9)\\ A &:= L_p \cup (L_q \overset{\circ}{-} N_p)\,. && (10) \end{aligned}$$

Lemma 0.0.1. *We claim that*

1. (N_q, L_q) *is a regular index pair for q* .
2. (C, B) *is an index pair for q* .
3. (N_p, L_p) *is a regular index pair for p* .
4. (B, A) *is an index pair for p* .
5. N_p *is a tubular neighbourhood of* $W(\to p) \cap M^c$.

The gradient is a section into the tangent bundle. Hence it is of the form:

$$\operatorname{grad}(f) : M \to TM$$

Lemma 0.0.2. *If $(x_1, ..., x_m)$ is a local coordinate system arround a critical point $p \in M$ and let $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right)$ be the induced basis of T_pM. Then the Matrix of the differential of the gradient* $\mathrm{grad}(f) : U \to \mathbb{R}^m$ *inj p is of the form:*

$$\frac{\partial}{\partial x}\mathrm{grad}(f)\Big|_p = \left(\sum_k g^{ki} \frac{\partial^2 f}{\partial x_j \partial x_k}\right)_{ij}$$

Proof. Compare lemma 4.4 in [**banyaga2004lectures**]: First we notice, that in the coordinates $x_1, ..., x_n$ respectifly $(\frac{\partial}{\partial x_1}, \ldots \frac{\partial}{\partial x_m})$(given by the chart $\psi : U \to V$)the $\mathrm{grad}(f) \circ \psi^{-1}$ reads:

$$Q \circ \mathrm{grad}(f) \circ \psi^{-1} = \left(\sum_k g^{k1} \frac{\partial f \circ \psi}{\partial x_k}, \sum_k g^{k2} \frac{\partial f \circ \psi}{\partial x_k}, \ldots, \sum_k g^{km} \frac{\partial f \circ \psi}{\partial x_k}\right)$$

Where $Q : TU \to \mathbb{R}^m$ denotes the koordinate map with respect to the chart on ψ. Hence, we can differentiate:

$$\begin{aligned}\frac{\partial}{\partial x} Q \circ \mathrm{grad}(f) \circ \psi^{-1} &= \left(\frac{\partial}{\partial x_j} \sum_k g^{ki} \frac{\partial f \circ \psi}{\partial x_k}\right)_{ij} \\ &= \left(\sum_k \left[\frac{\partial g^{ki}}{\partial x_j} \frac{\partial f \circ \psi}{\partial x_k} + g^{ki} \frac{\partial^2 f \circ \psi}{\partial x_j\, \partial x_k}\right]\right)_{ij} .\end{aligned}$$

Hence, in a critical point $p = \psi^{-1}(x_0)$ we have that:

$$\frac{\partial}{\partial x} Q \circ \mathrm{grad}(f) \circ \psi^{-1}\Big|_{x_0} = \left(\sum_k \left[g^{ki} \frac{\partial^2 f \circ \psi}{\partial x_j\, \partial x_k}\right]\right)_{ij} .$$

The last line is the precice notion of the lemma. □

Now assume that $(x_1, ..., x_m)$ is a morse chart, hence $f(x_1, ..., x_m) = \sum_{i=1}^{l} x_i^2 - \sum_{i=l+1}^{m} x_i^2$ and thereby in this chart:

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = \begin{cases} \delta_{ik} 2 & \text{if } i \le l\,, \\ -\delta_{ik} 2 & \text{else}\,. \end{cases} \tag{11}$$

This lets us calculate:

$$\frac{\partial}{\partial x}\mathrm{grad}(f)\bigg|_p = \begin{pmatrix} 2g^{11} & 2g^{12} & \dots & 2g^{1m} \\ \vdots & \vdots & \vdots & \vdots \\ 2g^{l1} & 2g^{l2} & \dots & 2g^{lm} \\ -2g^{l+11} & -2g^{l+12} & \dots & -2g^{l+1m} \\ \vdots & \vdots & \vdots & \vdots \\ -2g^{m1} & -2g^{m2} & \dots & -2g^{mm} \end{pmatrix}$$

Now we want to calculate $\frac{\partial}{\partial x}\varphi_t\Big|_p$ in such a Morse chart: For this we start by differentiating in $t - \mathit{direction}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x) = -\mathrm{grad}(f)(\varphi_t(x))$$

If we first differentiate this in $x-$direction and then in t dircetion we have:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial x}\varphi_t(x) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\varphi_t(x)\right) \\ &= \frac{\partial}{\partial x}\big(-\mathrm{grad}(f)(\varphi_t(x))\big) \\ &= -\frac{\partial}{\partial x}\mathrm{grad}(f)(\frac{\partial}{\partial x}\varphi_t(x))\,. \end{aligned}$$

In other words: $\Psi(t,x) := \frac{\partial}{\partial x}\varphi_t(x)$ is a solution of the system of linear ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Psi(t,x) &= -\frac{\partial}{\partial x}\mathrm{grad}(f)(\Psi(t,x))\,, \\ \Psi(0,x) &= 1_{m\times m}\,. \end{aligned}$$

By the uniqueness of such systems we have that

$$\Psi(t,x) = \exp\left(-\frac{\partial}{\partial x}\mathrm{grad}(f)\cdot t\right).$$

Now choose an orthonormal basis $B = (b_1, \dots b_m)$ of T_pM with resprect to the bilinear form $g(p)(\psi^{-1} \oplus \psi^{-1} =: \mathbb{R}^m \oplus \mathbb{R}^m \to \mathbb{R}$. Since for a chart that is centered in p the family

$$\left(\frac{\partial}{\partial x^i}\right)_i$$

forms a basis, there is a unique base change between the two bases, that can be realized as a linear change of coordinates in the following way: Define $\lambda^i_k \in \mathbb{R}$ such that $b_i =$

$\sum_k \lambda_i^k (\frac{\partial}{\partial x^k})$. With this we define the linear map $L : \mathbb{R}^m \to \mathbb{R}^m$ by $x \mapsto \left((\lambda_k^i)_{ik}\right)^{-1} x$. If we define $(L^{-1})_i$ to be the ith component of the inverse map, we have

$$\frac{\partial (L^{-1})_i}{\partial x^j} = \frac{\partial(\sum_k \lambda_k^i \cdot x^k)}{\partial x^j} = \lambda_j^i \,.$$

Hence, the basis

$$\left(\frac{\partial}{\partial \tilde{x}^i}\right)_i$$

corresponding to the new chart $\psi \circ L : U \to V \subseteq \mathbb{R}^m$ reads for any $f_p \in \mathcal{E}_p(M)$:

$$\begin{aligned}
\frac{\partial}{\partial \tilde{x}^j}(f_p) &= \frac{\partial}{\partial x^j}\bigg|_0 (f \circ \psi^{-1} \circ L^{-1}) \\
&= \sum_k \frac{\partial f \circ \psi^{-1}}{\partial x^k}(L^{-1}(0))\frac{\partial (L^{-1})_k}{\partial x^j}(0) \\
&= \sum_k \underbrace{\frac{\partial f \circ \psi^{-1}}{\partial x^k}(0)}_{=\frac{\partial}{\partial x^k}(f_p)} \underbrace{\frac{\partial (L^{-1})_k}{\partial x^j}(0)}_{=\lambda_j^k} \\
&= \left(\sum_k \lambda_j^k (\frac{\partial}{\partial x^k})\right)(f_p) = b_j(f_p)\,.
\end{aligned}$$

Hence, a linear change of koordinates lets us manipulate any chart such that the basis of the tangendspaces in the origin of the given chart is orthongonal (or even orthonormal) with resprect to a given metric.

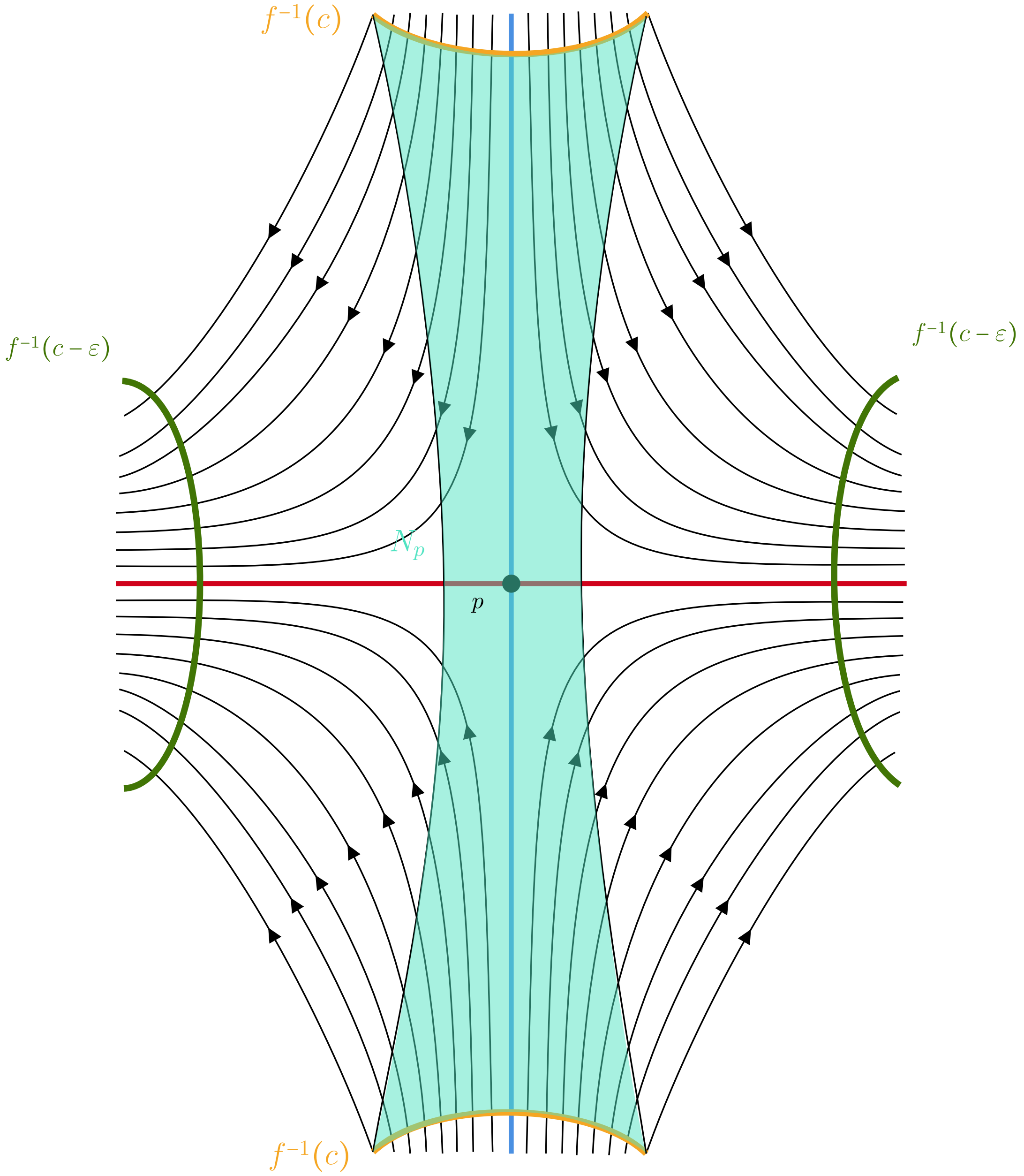
Now this is somewhat troublesome: We could now choose a basis, such that $\frac{\partial}{\partial x}\mathrm{grad}(f)$ is diagonal and hence the exponential is easy to calculate. But then we dont know how f looks like. But with a Morse chart (where we controle f) we have trouble calculating the exponential of $\frac{\partial}{\partial x}\mathrm{grad}(f)$. The hope ist that there is a change of basis that keeps the splitting.

Lemma 0.0.3. *If $(x_1, \ldots, x_m)$ is a local coordinate system arround a critical point $p \in M$ such that $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right)$ is a orthonormal basis for the tangent space at p (with respect to g_p, then for any $t \in \mathbb{R}$ the matrix for the differential of φ_t at p is the exponential of the negative matrix for the Hessian at p, i.e.:*

$$\frac{\partial}{\partial x}\varphi_t\bigg|_p = \exp(-\operatorname{Hess}(f)_p \cdot t\,.)$$

Proof. Compare Lemma 4.5 in [**banyaga2004lectures**]. ☐

Proof. The statements 1-4 are easy to proof by the definitions. The regularity can be derived from a general fact for pairs (X, Y) in metric spaces: If Y is closed in X and there is a neighbourhood of Y that is open in X such that Y is a strong deformation retract of U. So the only thing left to show is, that N_p is a tubular neighbourhood. in [**MorseTheorySalmbon**] in the proof of lemma 3.2 Salamon claims this to be true without a proof (page 119 in the attached source). Similar, in [**banyaga2004lectures**] Banyaga claims this (also without any argument). I find this difficult to proof, since we cannot use the flow for the map from the normal bundle, as points leave N_p along the flow: The Set N_p is sketched in the figure below.



Idea: Show it locally in a morse chart arround p. Hopefully we can archive, that in such a chart the property $\varphi_T(x) \geq a - \varepsilon$ translates to $x = \psi(x_s, x_u)$ where $\|x_u\| \leq T_x$. Then assume T to be big enough such that for any $x \in N_p$ there is a chart U arround a point $x_p \in W(\to p)$ and a t_0 such that $\varphi_{t_0}(U)$ lives in said morse chart. Finaly check if the property required arround said x_p can be furmulated such that U has a tubular structure. So lets start this procedure! Let

$$\psi : p \in V \to U \subseteq \mathbb{R}^m$$

be a morse chart. Here the function f is of the form

$$f \circ \psi^{-1}(x_s, x_u) = a + x^{s2} - x^{u2} .$$

where x^{s2} and x^{u2} denotes the sum of all the squares from the morse lemma. Without restrictions we call $x^s = (x^1, \dots x^l$ and $x^u : (x^{l+1}, \dots x^m)$. Now we inspect the set

$$\psi^{-1}(N_p \cap V) = \left\{ (x^s, x^u) \in U \mid f \circ \varphi_T(\psi^{-1}(x^s, x^u)) \geq a - \varepsilon \right\} . \tag{12}$$

First we inspect the gradient in those local coordinats, i.e. $x = \psi^{-1}(u)$:

$$\operatorname{grad}(f)(x) = g^{ik} \ \frac{\partial f}{x^k} \frac{\partial}{x^i}$$

For now assume that $g_{ik} = \operatorname{diag}(1, \dots 1)$, i.e. that we work with the euklidean metric. Then the gradient is:

$$\sum_{i,k} \frac{\partial f \circ \psi^{-1}}{\partial x^k} \frac{\partial}{\partial x^i} = 2 \frac{\partial}{x^s} - 2 \frac{\partial}{x^u} .$$

Hence, the flow corresponding to $\psi_*(-\operatorname{grad}(f))$ is of the form

$$t \mapsto \varphi_t(x) = (e^{-2t} x^s, e^{2t} x^u) .$$

Assume that ε is small enough, such that all $x \in \psi(N_p \cap V) \setminus W(\to p)$ flow through $f^{-1}(a - \varepsilon)$ inside of U, i.e. we can formulate the property:

$$f \circ \varphi_T(\psi^{-1}(x^s, x^u)) \geq a - \varepsilon \quad \Leftrightarrow \quad f\left(\psi^{-1}(e^{-2T} x^s, e^{2T} x^u)\right) = a + \left(e^{-2T} x^s\right)^2 - (e^{2T} x^u)^2 \geq a - \varepsilon \tag{13}$$

This however reads:

$$a + \left(e^{-2T} x^s\right)^2 - (e^{2T} x^u)^2 \geq a - \varepsilon \quad \Leftrightarrow \quad (e^{2T} x^u)^2 \leq \varepsilon + \left(e^{-2T} x^s\right)^2$$

And hence we have that:

$$\psi(N_p \cap V) = \left\{ (x^s, x^u) \in U \;\middle|\; \|x^u\| \leq R(x^s) \right\}$$

where $R(x^s)$ is a smooth function. Now let g be a general metric, that is a section into the bundle of two forms with certain properties. (non-degeneret, symmetric and positive definite) We want to Taylor expand the gradient locally. So define $V^i(x) := \sum_{j=1}^m g^{ij}(x)\frac{\partial f\circ\psi^{-1}}{x^j}(x)$. Now we taylor this component in p:

$$V^i(x) = V^i(p) + \sum_{k=1}^m \frac{\partial V^i}{x^k}(p)(x^k - x_0^k) + \mathcal{O}(\|x\|^2)$$

and

$$\begin{aligned}\frac{\partial V^i}{x^k}(0) &= \frac{\partial}{x^k}(0)\sum_{j=1}^m g^{ij}(0)\frac{\partial f}{x^j}(0)\\ &= \sum_{j=1}^m \left(\frac{\partial g^{ij}}{\partial x^k}(0)\frac{\partial(f\circ\psi)}{\partial x^j}(0) + g^{i,j}\frac{\partial^2(f\circ\psi)}{\partial x^k\partial x^i}(0)\right)\\ &= \sum_{j=1}^m \left(g^{i,j}\frac{\partial^2(f\circ\psi)}{\partial x^k\partial x^i}(0)\right)\end{aligned}$$

$$\sum_{i,k} g^{ik}\frac{\partial f\circ\psi^{-1}}{\partial x^k}\frac{\partial}{\partial x^i} = \sum_{k=1}^l 2g^{kk}\frac{\partial}{x^k} - \sum_{k=l+1}^m 2g^{kk}\frac{\partial}{x^k}$$

where $g^{ik}\in C^\infty(M,\mathbb{R})$. Hence we can tayler all the g^{ik} in p to get them to be of the form $\mathrm{d}g^{ik}(p) + r^{ik}(x)$ $\square$