The Setting

We work in a compact, riemanian, orientable, smooth Manifold (M, g) of dimension m together with a Morse-Smale funktion

$$f : M \to \mathbb{R} \tag{1}$$

We assume that q is a critical point of index $k+1$ and p is a critical point of index k. we define $f(q) = b$ and $f(p) = a$ and assume that there is no kritical point in $f^{-1}(a, b) \subseteq M$. We define the sub and superlevelsets

$$M^t := \{x \in M | f(x) \leq t\} \quad \text{and} \quad M_t := \{x \in M | f(x) \geq t\}, \tag{2}$$

and the constants

$$c \in (a, b) \quad , \quad \varepsilon > 0 \text{ small} \quad , \quad T > 0 \text{ big} . \tag{3}$$

With this we define the sets:

$$N_q := \{x \in M_c | f(\varphi_{-T}(x)) \leq b + \varepsilon\}, \tag{4}$$
$$L_q := \{x \in N_q | f(x) = c\}, \tag{5}$$
$$N_p := \{x \in M^c | f(\varphi_T(x)) \geq a - \varepsilon\}, \tag{6}$$
$$L_p := \{x \in N_p | f(\varphi_T(x)) = a - \varepsilon\}, \tag{7}$$

and finally:

$$C := N_p \cup N_q , \tag{8}$$
$$B := N_p \cup L_q , \tag{9}$$
$$A := L_p \cup (L_q \overset{\circ}{-} N_p) . \tag{10}$$

Lemma 0.0.1. *We claim that*

1. (N_q, L_q) *is a regular index pair for* q .
2. (C, B) *is an index pair for* q .
3. (N_p, L_p) *is a regular index pair for* p .
4. (B, A) *is an index pair for* p .
5. N_p *is a tubular neighbourhood of* $W(\to p) \cap M^c$.

Definition 0.0.2 (Jacobian of the Gradient)**.** The gradient is a section into the tangent bundle, $\text{grad}(f) : M \to TM$. Let $\psi : M \supseteq U \to V \subseteq \mathbb{R}^m$ be a chart. The coordinate map q_ψ on TU assigns to a vector $v \in T_pM$ (where $p \in U$ with $\psi(p) = x$) its components with respect to the basis $\left(\frac{\partial}{\partial x^i}\Big|_p\right)$, i.e., $q_\psi(v) = (v^1, \dots, v^m)$ if $v = \sum_{i=1}^m v^i \frac{\partial}{\partial x^i}\Big|_p$. We define the Jacobian of the gradient with respect to the chart ψ as the Jacobian matrix of the coordinate representation of the gradient in this chart:

$$J(\mathrm{grad}(f))_\psi(x) \coloneqq J\left(q_\psi \circ \mathrm{grad}(f) \circ \psi^{-1}\right)(x) = \left(\frac{\partial (q_\psi \circ \mathrm{grad}(f) \circ \psi^{-1})_i}{\partial x_j}(x)\right)_{ij}$$

Lemma 0.0.3. *Let $t \in \mathbb{R}$ be small enough and $\varphi_t : M \to M$ be the flow map corresponding to the negative gradient. Assume that U is the domain of a chart ψ and $p \in U$ such that $\varphi_t(p) \in U$. Then the linear map $\mathrm{d}\varphi_t\big|_p : T_pM \to T_{\varphi_t(p)}M$ has a local representation in the coordinates induced by ψ given by:*

$$q_{\psi(\varphi_t(p))} \circ \mathrm{d}\varphi_t\big|_p \circ q^{-1}_{\psi(p)} = \exp\left(-J(\mathrm{grad}(f))_\psi(\psi(p)) \cdot t\right)$$

where $J(\mathrm{grad}(f))_\psi(\psi(p))$ is the Jacobian matrix of the gradient evaluated at the coordinates of p.

Proof. We will inspect what differebtial equation the map $q_{\psi(\varphi_t(p))} \circ \mathrm{d}\varphi_t\big|_p \circ q^{-1}_{\psi(p)}$ solves and hence we differentiate:

$$\frac{\partial}{\partial t} q_{\psi(\varphi_t(p))} \circ \mathrm{d}\varphi_t\big|_p \circ q^{-1}_{\psi(p)}$$

□

Lemma 0.0.4. *Let ψ be a coordinate system arround a critical point p. I.e. $\psi : p \in U \to V$ with $\psi(p) = 0$ and let $\left(\frac{\partial}{\partial x_1}\Big|_x, \dots, \frac{\partial}{\partial x_m}\Big|_x\right)$ be the induced basis of T_xM Then the jacobean of the gradient*

$$J(\mathrm{grad}(f))_\psi(0) = \sum_k g^{ki}(0) \frac{\partial^2 f \circ \psi^{-1}}{\partial x_j \partial x_k} \tag{11}$$

where $(g^{ki}(x))$ denotes the matrix corresponding to the gradient in T_xM with respect to the basis $\left(\frac{\partial}{\partial x_1}\Big|_x, \dots, \frac{\partial}{\partial x_m}\Big|_x\right)$.

Proof. Let q_ψ denote the koordinate funktion $TU \to R^m$ induced from the basis $\left(\frac{\partial}{\partial x_1}\Big|_x, \dots, \frac{\partial}{\partial x_m}\Big|_x\right)$. then the gradient in ψ reads:

$$q_\psi \circ \mathrm{grad}(f) \circ \psi^{-1} = \left(\sum_k g^{k1} \frac{\partial f \circ \psi^{-1}}{\partial x_k}, \dots, \sum_k g^{km} \frac{\partial f \circ \psi^{-1}}{\partial x_k}\right)$$

Hence, we can calculate Hence, we can differentiate:

$$\frac{\partial}{\partial x} Q \circ \operatorname{grad}(f) \circ \psi^{-1} = \left(\frac{\partial}{\partial x_j} \sum_k g^{ki} \frac{\partial f \circ \psi^{-1}}{\partial x_k} \right)_{ij}$$
$$= \left(\sum_k \left[\frac{\partial g^{ki}}{\partial x_j} \frac{\partial f \circ \psi^{-1}}{\partial x_k} + g^{ki} \frac{\partial^2 f \circ \psi}{\partial x_j \, \partial x_k} \right] \right)_{ij} .$$

and now in $p = \psi{-}1(0)$ we have

$$\frac{\partial}{\partial x} Q \circ \operatorname{grad}(f) \circ \psi^{-1} \Big|_0 = \left(\sum_k \left[g^{ki}(0) \frac{\partial^2 f \circ \psi^{-1}}{\partial x_j \, \partial x_k} \right] \right)_{ij} .$$

□

Theorem 0.0.5 (Sylvesters Law of Inertia). *Let $A \in \operatorname{Mat}(n, \mathbb{R})$ be symmetric and let $T, T' \in \operatorname{GL}(n, \mathbb{R})$ and $k, k', l, l' \in \mathbb{N}$ such that*

$$T^t \circ A \circ T = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & -1_l & 0 \\ 0 & 0 & 0_{n-k-l} \end{pmatrix} \text{ and } T'^T \circ A \circ T' = \begin{pmatrix} 1_{k'} & 0 & 0 \\ 0 & -1_{l'} & 0 \\ 0 & 0 & 0_{n-k'-l'} \end{pmatrix}$$

then $k = k'$, $l = l'$ and $\operatorname{rank}(A) = k + l$.

Proof. Since T, T' are invertible we have that

$$k + l = \operatorname{rank}(T^t \circ A \circ T) = \operatorname{rank}(A) = \operatorname{rank}(T'^T \circ A \circ T') = k' + l' . \tag{12}$$

Hence it suffices to show that $k = k'$. Which we will do by proofing the claim:

$$k = \max \left\{ \dim(U) | U \subseteq \mathbb{R}^n \text{subspace such that } x^t A x > 0 \ \forall x \in U \setminus \{0\} \right\}$$

So we start by showing „$\leq$“: Denote the first k colomus of T with $x_1, ..., x_k$. They form a basis of $\mathbb{R}^n$ and with $0 \neq x = \sum_{i=1}^k \lambda i x^i$ we have that by bilinearity

$$x^T A x = \sum_{i=1}^k \lambda_i x_i^T A x = \sum_{i,j=1}^k \lambda_i \lambda_j x_i^T A x_j = \sum_{i=1}^k (\lambda_i)^2 \geq 0 . \tag{13}$$

This conculdes the first inequality.

Now let U be any k−dimensional subspace such that for all non zero $x \in U$ $x^T A x > 0$. By a calculation analog to the one above we have that for any $x \in W := \operatorname{span}(x_{k+1}, \dots x_n)$ the number $x^t A x$ is less or equal to zero. Hence, $W \cap U = \{0\}$ and with this we conclude:

$$\dim(U) = \dim(U + W) - \dim(W) + \dim(U \cap W) \leq (k + (n - k)) - (n - k) + 0 = k . \tag{14}$$

This is the last inequality proving the statement. □

Lemma 0.0.6. *Assume that $\psi : U \to V$ is a chart and $L : R^m \to R^m$ a linear function. Then the diagramm*

$$\begin{tikzcd} T_pM \arrow[r, "q_\psi|_p"] \arrow[dr, "q_{L\circ\psi}|_p"'] & \mathbb{R}^m \arrow[d, "L"] \\ & \mathbb{R}^m \end{tikzcd}$$

commutes for all $p \in U$.

Proof. We show this by showing the inverse: For any $v \in \mathbb{R}^m$ we have the two $(q_\psi\big|_p)^{-1} \circ L^{-1}(v)$ and $(q_{\psi\circ L}\big|p)^{-1}$ and claim that they are the same. Assume that $L^{-1} = (\lambda_{ij})_{ij}$, $\psi(p) = x_0$, $L(x_0) = \tilde{x_0}$, $(v^1, \dots v^m) = v \in \mathbb{R}^m$ and denote the basis of T_pM coming from the chart ψ by $\left(\frac{\partial}{\partial x^i}\big|_p\right)_i$ and the one coming from $L^{-1} \circ \psi$ by $\left(\frac{\partial}{\partial \tilde{x}^i}\big|_p\right)_i$. We now calculate for $f_p \in \mathcal{E}_pM$:

$$\begin{aligned}
\left[(q_{\psi\circ L}\big|_p)^{-1}(v)\right](f_p) &= \sum_i v^i \frac{\partial}{\partial \tilde{x}^i}\Big|_p (f_p) \\
&= \sum_i v^i \frac{\partial}{\partial x^i}\Big|_{\tilde{x}_0} (f \circ \psi^{-1} \circ L^{-1}) \\
&= \sum_i v^i \sum_j \frac{\partial f \circ \psi^{-1}}{\partial x^j}\Big|_{x_0} \cdot \frac{\partial L_j^{-1}}{\partial x^i}\Big|_{\tilde{x_0}} \\
&= \sum_{i,j} v^i \frac{\partial}{\partial x^j}\Big|_p (f_p) \cdot \lambda_{ji} \\
&= \sum_{i,j} v^i q_\psi\big|_p^{-1}(e_j)(f_p) \cdot \lambda_{ji} \\
&= \left[q_\psi\big|_p^{-1}\Big(\sum_{i,j} v^i \lambda_{ji} e^j\Big)\right](f_p) \\
&= \left[q_\psi\big|_p^{-1}(L^{-1}(v))\right](f_p)
\end{aligned}$$

This concludes the proof. ∎

Theorem 0.0.7 (Simultaneous Diagonalisation of Quadratic Forms)**.** *Let p be a critical point of a Morse function f on a manifold M and let $g(p)$ be the Riemannian metric on T_pM. Then there exists a Morse chart ψ arround p such that the representation of $g(p)$ in the basis induced by the Morse chart is given by a a diagonal matrix $diag(\mu_1, \dots \mu_m)$ with $\mu_i > 0$ for all i .*

Proof. The quadratic form of $f \circ \psi^{-1} - f(p)$ in the coordinates of any Morse chart ψ is $q_H(v) = v^T H v$. The quadratic form induced by the Riemannian metric $g(p)$ is $q_G(v) = v^T G v$, where $v \in \mathbb{R}^m$ are the coordinate vectors with respect to the Morse chart. To be precice this means for $v, w \in \mathbb{R}^m$:

$$g \circ (q_\psi\big|_p \oplus q_\psi\big|_p)(v, w) = v^T G w \text{ and } f \circ \psi^{-1}(v) = f(p) + v^T H v\,.$$

We now aim to manipulate ψ such that G becomes diagonal: Since $g(p)$ is positive definite, G is also positive definite, and H is symmetric.

Consider the generalized eigenvalue problem $Hv = \lambda G v$. Since H and G are real symmetric matrices and G is positive definite, this problem has m real eigenvalues $\lambda_1, \ldots, \lambda_m$ and corresponding eigenvectors $v_1, \ldots, v_m$, which can be chosen to be orthogonal with respect to the bilinear form defined by G, such that $v_i^T G v_j = \delta_{ij}$, since if v_i and v_j are different eigenvectors with respect to different eigenvalues, we can calculate:

$$v_j^T H v_i = (Hv_j)^T v_i = \lambda_j (Gv_j)^T v_i = \lambda_j v_j^T G(v_i) \quad \text{and} \quad v_j^T H v_i = v_j^T \lambda_i G(v_i) = \lambda_i v_j^T G(v_i)\,.$$

Hence we have the equality

$$\lambda_j (Gv_j)^T v_i = \lambda_i v_j^T G(v_i) \Leftrightarrow \underbrace{(\lambda_j - \lambda_i)}_{\neq 0} v_j^T G(v_i) = 0$$

Hence all eigenspaces are orthogonal and we can use Gram Schmitt to make the bases of the eigenspaces orthogonal and by rescaling orthonormal.

Let $L = [v_1| \ldots |v_m]$ be the matrix whose columns are these G-orthonormal eigenvectors. The linear change of coordinates $y = Lz$ leads to new coordinates z. In these new coordinates, the quadratic forms transform as follows:

$$q_G(y) = y^T G y = (Lz)^T G(Lz) = z^T L^T G L z = z^T I z = \sum_{i=1}^m z_i^2$$
$$q_H(y) = y^T H y = (Lz)^T H(Lz) = z^T L^T H L z$$

Since $Hv_i = \lambda_i G v_i$, we have $L^T H L = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$. The eigenvalues λ_i are real and have the same signature as the eigenvalues of H (l negative, $m - l$ positive).This is true to sysvesters law of inertia. By a further scaling of the v_i and a reordering the matrix $L^T H L$ can be brought into the form $\operatorname{diag}(-1, \ldots, -1, 1, \ldots, 1)$. (by doing this however, the matrix $L^T G L$ becomes $L^T G L = \operatorname{diag}(\mu_1, \ldots \mu_m)$ with $\mu_j > 0\ \forall j$). If ψ is the Morse chart we started with, then $L^{-1} \circ \psi$ is the new Morse chart:

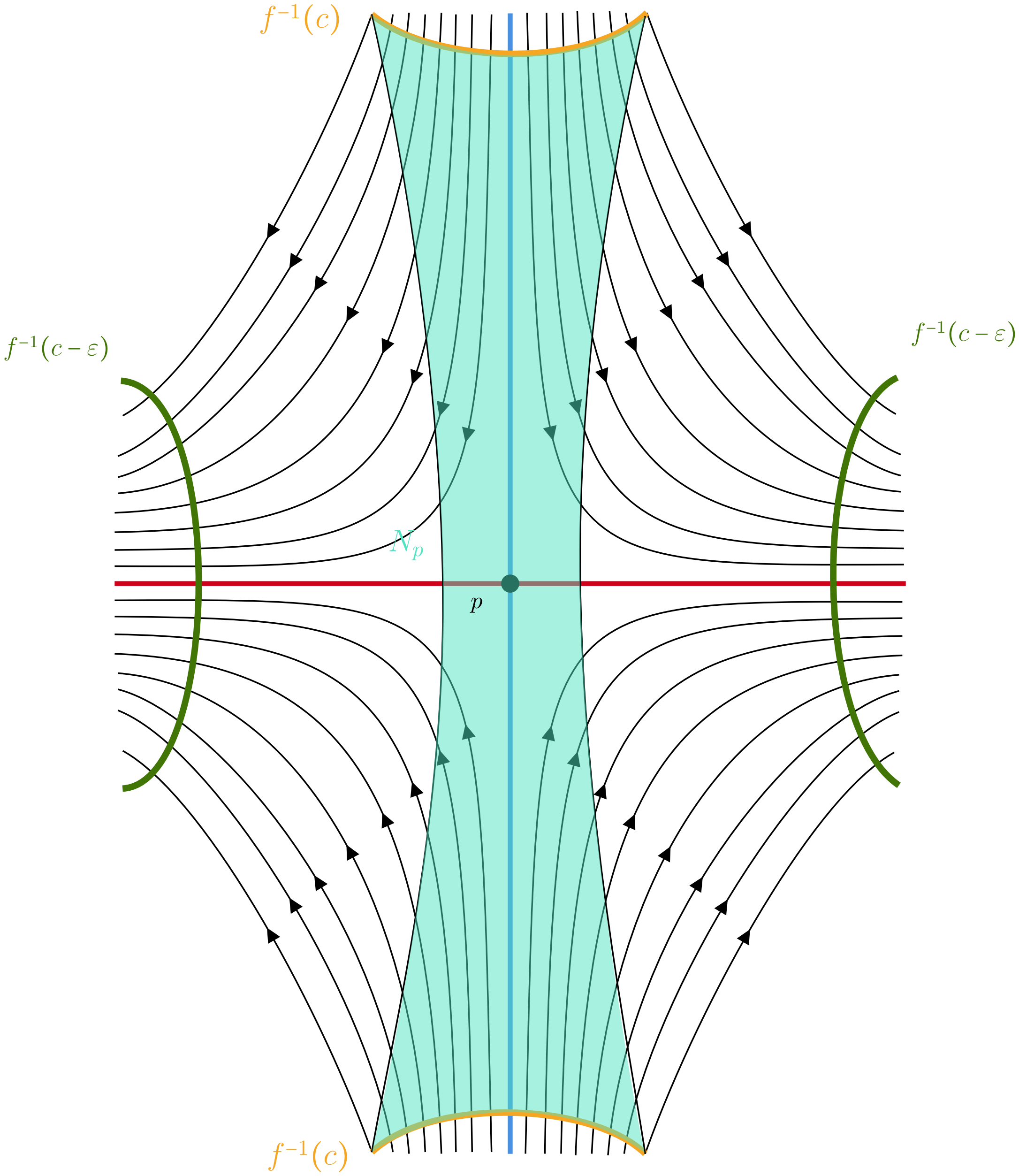
$$f \circ (\psi^{-1} \circ L)(y) = f \circ \psi^{-1}(L(y)) = q_H(L(y) = \sum_{i=1}^{l} -y_i^2 + \sum_{i=l+1}^{m} y_i^1$$

Furthermore, we want to inspect the metric $g\big|_p : T_p M^2 \to \mathbb{R}$ with respect to the chart $L^{-1} \circ \psi$-induced basis $\left(\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^m}\big|_p\right)$. By the lemma 0.0.6

$$g(q_{L^{-1} \circ \psi}\big|_p^{-1}, q_{L^{-1} \circ \psi}\big|_p^{-1})(v, w) = g(q_\psi\big|_p^{-1}, q_\psi\big|_p^{-1})(Lv, Lw) = (Lv)^T G L W = v^T L^T G L w$$

$\square$

Proof. The statements 1-4 are easy to proof by the definitions. The regularity can be derived from a general fact for pairs (X, Y) in metric spaces: If Y is closed in X and there is a neighbourhood of Y that is open in X such that Y is a strong deformation retract of U. So the only thing left to show is, that N_p is a tubular neighbourhood. in [**MorseTheorySalmbon**] in the proof of lemma 3.2 Salamon claims this to be true without a proof (page 119 in the attached source). Similar, in [**banyaga2004lectures**] Banyaga claims this (also without any argument). I find this difficult to proof, since we cannot use the flow for the map from the normal bundle, as points leave N_p along the flow: The Set N_p is sketched in the figure below.



Idea: Show it locally in a morse chart arround p. Hopefully we can archive, that in such a chart the property $\varphi_T(x) \geq a - \varepsilon$ translates to $x = \psi(x_s, x_u)$ where $\|x_u\| \leq T_x$. Then assume T to be big enough such that for any $x \in N_p$ there is a chart U arround a point $x_p \in W(\to p)$ and a t_0 such that $\varphi_{t_0}(U)$ lives in said morse chart. Finaly check if the property required arround said x_p can be furmulated such that U has a tubular structure. So lets start this procedure! Let

$$\psi : p \in V \to U \subseteq \mathbb{R}^m$$

be a morse chart. Here the function f is of the form

$$f \circ \psi^{-1}(x_s, x_u) = a + x^{s2} - x^{u2}\,.$$

where x^{s2} and x^{u2} denotes the sum of all the squares from the morse lemma. Without restrictions we call $x^s = (x^1, \dots x^l$ and $x^u : (x^{l+1}, \dots x^m)$. Now we inspect the set

$$\psi^{-1}(N_p \cap V) = \left\{(x^s, x^u) \in U \mid f \circ \varphi_T(\psi^{-1}(x^s, x^u)) \geq a - \varepsilon\right\}\,. \tag{15}$$

First we inspect the gradient in those local coordinats, i.e. $x = \psi^{-1}(u)$:

$$\mathrm{grad}(f)(x) = g^{ik}\,\frac{\partial f}{x^k}\frac{\partial}{x^i}$$

For now assume that $g_{ik} = \mathrm{diag}(1, \dots 1)$, i.e. that we work with the euklidean metric. Then the gradient is:

$$\sum_{i,k} \frac{\partial f \circ \psi^{-1}}{\partial x^k}\frac{\partial}{\partial x^i} = 2\frac{\partial}{x^s} - 2\frac{\partial}{x^u}\,.$$

Hence, the flow corresponding to $\psi_*(-\mathrm{grad}(f))$ is of the form

$$t \mapsto \varphi_t(x) = (e^{-2t}x^s, e^{2t}x^u)\,.$$

Assume that ε is small enough, such that all $x \in \psi(N_p \cap V) \smallsetminus W(\to p)$ flow through $f^{-1}(a - \varepsilon)$ inside of U, i.e. we can formulate the property:

$$f \circ \varphi_T(\psi^{-1}(x^s, x^u)) \geq a - \varepsilon \quad \Leftrightarrow \quad f\left(\psi^{-1}(e^{-2T}x^s, e^{2T}x^u)\right) = a + \left(e^{-2T}x^s\right)^2 - (e^{2T}x^u)^2 \geq a - \varepsilon \tag{16}$$

This however reads:

$$a + \left(e^{-2T}x^s\right)^2 - (e^{2T}x^u)^2 \geq a - \varepsilon \quad \Leftrightarrow \quad (e^{2T}x^u)^2 \leq \varepsilon + \left(e^{-2T}x^s\right)^2$$

And hence we have that:

$$\psi(N_p \cap V) = \left\{(x^s, x^u) \in U \;\middle|\; \|x^u\| \leq R(x^s)\right\}$$

where $R(x^s)$ is a smooth function. Now let g be a general metric, and ψ a Morse chart such that $g(p)$ is a diagonal quadratic form wit respect to that morse chart. We now want to interlinear approximate $\psi \circ \varphi_t \circ \psi^{-1} : U \to U$ which leads to the need of its jacobean: For this we consider the function $(t, x) \mapsto \psi\varphi_t(\psi^{-1}(x)) : \mathbb{R} \times U \to U$ For fixed x we can calculate

$$q_\psi \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(\psi^{-1}(x)) = -q_\psi \mathrm{grad}(f)(\varphi_t(\psi^{-1}(x)))$$

and by changing the order of integration we have:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}\psi\varphi_t(\psi^{-1}(x))$$

$\square$