AdS Euclidean Wormholes

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Abstract

AdS Euclidean wormholes are an essential piece in the Factorization Problem puzzle. This essay aims to provide a literature review of Euclidean wormholes, describe such puzzle and explore our current progress in tackling it.

Firstly, with the Part III courses on General Relativity, Black Holes and Quantum Field Theory as a starting point, this essay provides an overall view of the terminology required to understand the Euclidean wormholes literature. Building on the Einstein-Rosen bridge, we begin by defining Euclidean wormholes and explicitly working with an example provided by Hawking. Then we define AdS Euclidean wormholes, argue how these are more restricted and review the maximally symmetric example from [1, 2]. Finally, we introduce the notion of *baby universes* and explain how these arise from wormholes.

After all the terminology has been laid out, we present the conflict of the second part: the Factorization Problem. As we will see, wormholes and the associated change of the topology of spacetime lead to inconsistencies when trying to establish a theory of Quantum Gravity. Coleman accounted for the case of theories with low energies and low dimensions and we review his approach. Nonetheless, the solution is far from clear in the intricate context of String Theory. In the final sections, we review the methodology and results from [1, 2], summarise the main ideas of the essay and suggest directions for future research.

Contents

1 Introduction

In 1935, Einstein and Rosen published the first historically known paper about wormholes [3]. By establishing a new coordinate system in Schwarschild spacetime, they discovered the existence of a bridge between two regions: the Einstein-Rosen bridge (ER-bridge). Ever since, human beings have been inspired by these wonderful astronomical objects and have wondered about the physical and philosophical issues that wormholes have brought to light.

In a moment of crisis for Theoretical Physics, the motivation behind Einstein and Rosen's article was to construct an atomistic theory of matter and electricity by means of General Relativity (GR). Referring to the ER-bridge, their paper triumphantly closes with

In any case here is a possibility for a general relativistic theory of matter which is logically completely satisfying and which contains no new hypothetical elements [4].

Nowadays, we know that Einstein and Rosen were wrong. Not only was the bridge originally constructed with ill-defined coordinates [3], but also the approach of carelessly using GR to consolidate Quantum Mechanics (QM) leads to unmanageable divergencies, as we will argue in section 5.2.3.

On the other hand, wormholes do exist as solutions to the Einstein equations and paradoxically, they play a fundamental role in QM. This is because every well-posed theory of Quantum Gravity must allow all possible spacetime topologies [1, 5]. Thus, we must grasp how wormholes behave and their impact on our computations.

In particular, we want to understand wormholes in Euclidean spacetime, i.e. Euclidean wormholes. The current widespread formalism to compute path integrals opts for performing a Wick rotation. This consists in a change of coordinates $\tau = it$, analytically continuing to imaginary time as a result [6, 7]. Starting with a scalar field ϕ and an action S, performing a Wick rotation yields

$$\int \mathcal{D}\phi e^{iS[\phi]} \xrightarrow{\tau=it} \int \mathcal{D}\phi e^{-S[\phi]}.$$

Thus the oscillatory behaviour of the exponent has been removed and the integral has a better chance to converge [6, 7]. When it comes to gravitation, this seemingly innocent change turns the signature of the Minkowski space from Lorentzian $(-,+,\dots,+,+)$ to Euclidean $(+,+,\dots,+,+)$. Hence, we will focus on grasping Euclidean wormholes and the consequences of their existence as solutions to Einstein equations. One of the most striking implications was addressed by Hawking:

It is an amusing thought that our Universe could be a rather large wormhole in an asymptotically flat space [5].

Clasically, the Einstein equations preserve the topology of spacetime under evolution [8], but in the Quantum Gravity approach, the topology of spacetime can change [9]. When the metric becomes part of an action for a theory of Quantum Gravity, it is treated as a quantum field, which can fluctuate. As a result, little closed universes of the Plack scale can extend beyond their original universe or join on [5]. We will see how wormholes can *branch off* our spacetime and the definition of *baby universes* in section 4, which formalises the intuition of having a newborn universe from a wormhole.

Having little universes that can leave or join your own leads to major trouble when you are trying to keep track of the quantum information. Imagine you have a particle with a known state, which then enters a region of spacetime that extends and closes away from your universe. As the particle is now in a separate and uncommunicated region, from your point of view you would need to sum over all possible quantum states for the closed universe [5]. You knew the state of the particle at the beginning and all of a sudden, not anymore. Back in the 80's this was known as the *loss of quantum coherence*.

For theories in low energies and dimensions, this problem was solved by Coleman, as we will cover in section 5.1. Nonetheless, in the framework of complete UV-energies and high dimensions of String Theory, this is an issue that remains unsolved. One of the main tools to perform computations in String Theory is the Anti de-Sitter (AdS) and Conformal Field Theory (CFT) correspondence, or AdS/CFT correspondence. This establishes a duality between a theory of gravitation and a field theory. The conflict, however, is that while from the field theory perspective the correlation functions across a theory with two boundaries should factorize, from the gravitation perspective, the existence of wormholes tells us it should not [1]. This puzzle, known as the *the Factorization Problem* is our main motivation to study AdS Euclidean wormholes.

This essay consists of two parts and its main goal is to provide a comprehensive review of the main notions and results, which are scattered in the literature of Euclidean wormholes.

The first half aims at laying out the groundwork to understand such literature. We take the Part III courses of GR, Black Holes and Quantum Field Theory (QFT) as a solid foundation, and we progressively construct definitions and examples. By initially leaning on the example of the ER-bridge, we define and exemplify what is a wormhole, then we cover the protagonists of the paper: AdS Euclidean wormholes, and finally we explain how baby universes are related to these. Along the way we will discover various instabilities present in wormholes and consequences of Wick rotations. These are what make the AdS Euclidean wormholes so hard to come by.

In the second part, we dive into the Factorization Problem. Once we have described the issue itself, we introduce the ensemble approach for low-dimensional and low-energy theories. For the UV-energy spectrum and higher dimensions, we require some notions from String Theory, which we briefly introduce. Finally, we review the impact of AdS Euclidean wormholes in computations for high-energy and high-dimensional models.

1.1 Conventions

For the sake of clarity, we introduce some conventions employed throughout the essay.

1. We work with natural units by setting $c = \hbar = 1$. For simplicity we have also set $G = 1$.
2. Einstein summation as in [10] is employed throughout this essay.

3. The term *spacetime* refers to a manifold equipped with a smooth metric. Unlike in [10], such metric is not limited to have Lorentzian signature.

4. We denote the Euclidean metric in n-dimensions by

$$\delta = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n \implies \delta_{ij} = [\text{diag}(1, 1, \ldots, 1)]_{ij}.$$

5. Let $n \geq 3$, denote by x^i the standard coordinates in $\mathbb{R}^n$ and $r = \sqrt{\delta_{ij} x^i x^j}$. The n-dimensional spherical coordinates are given by [11]

$$\begin{aligned} &x_1 = r\cos(\theta_1),\ x_j = r\cos(\theta_j)\prod_{k=1}^{j-1}\sin(\theta_k),\ \text{with } 2 \leq j \leq n-2 \text{ and } \theta_j \in (0, \pi), \\ &x_{n-1} = r\sin(\phi)\prod_{k=1}^{n-2}\sin(\theta_k) \text{ and } x_n = r\cos(\phi)\prod_{k=1}^{n-2}\sin(\theta_k),\ \text{with } \phi \sim \phi + 2\pi. \end{aligned} \tag{1}$$

By pulling back the Euclidean metric in these coordinates with $r = 1$ we generalise the round metric from [10] to the round metric for the $n-1$-sphere

$$d\Omega_{n-1}^2 = d\theta_1^2 + \sin^2(\theta_1)d\theta_2^2 + \sin^2(\theta_1)\sin^2(\theta_2)d\theta_3^2 + \cdots + \prod_{j=1}^{n-2}\sin^2(\theta_j)d\phi^2.$$

2 Euclidean wormholes

We begin by introducing the core definition and proving the ER-bridge is an Euclidean wormhole.

Definition 2.1 (Wormhole)**.** ***Wormholes*** *are connected geometries whose boundaries have more than one compact connected component [2].*

Given an arbitrary smooth curve parametrised along a spacetime, the ***boundary*** *refers to the set of points that lie an infinite proper distance away from any given point in such spacetime. This is also known as the conformal boundary.*

We have more than one compact connected component if the proper distance from one point on the boundary to another point on the boundary is infinitely large. We will refer to each of these compact components as one boundary in our wormhole.

If the metric g describes a wormhole, its signature tells us the type of wormhole we are dealing with.

Definition 2.2. *We distinguish two types of wormholes [12].*

1. *A wormhole is a* ***Lorentzian wormhole*** *if g is pseudo-Riemannian, i.e. everywhere non-degenerate, smooth and symmetric.*

2. *A wormhole is an* ***Euclidean wormhole*** *if g is Riemannian, i.e. everywhere positive-definite, smooth and symmetric.*

As explained in the introduction, we will focus on Euclidean wormholes. Although the number of boundaries is unlimited [13], we will restrict to examples of wormholes with two boundaries. As advertised, we prove that the ER-bridge is an Euclidean wormhole.

Example 2.3. ***The ER-bridge****. In isotropic coordinates with $\rho \in (0, \infty)$, the metric for the ER-bridge is given by [14]*

$$ds^2_{ER} = \left(1 + \frac{M}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\Omega_2^2). \tag{2}$$

I claim that this geometry has two compact connected components at the boundary: $\rho \to \infty$ and $\rho \to 0$, and thus we have a wormhole [14]. Consider a curve along the ρ direction going from ρ_1 to ρ_2. The proper distance along such curve is given by

$$L(\rho_1, \rho_2) = \int_{\rho_1}^{\rho_2} \sqrt{g_{\rho\rho}} d\rho = \int_{\rho_1}^{\rho_2} \left(1 + \frac{M}{2\rho}\right)^2 d\rho = \rho + \ln|\rho| - \frac{M^2}{2\rho}\Bigg|_{\rho_1}^{\rho_2}.$$

If either $\rho_2 \to \infty$ or $\rho_1 \to 0$, then $L \to \infty$. By definition 2.1, each of these limits for ρ is at the boundary on our spacetime. In addition, since taking both $\rho_2 \to \infty$ and $\rho_1 \to 0$ also yields $L \to \infty$, we must have two compact connected components in our boundary. Therefore, the ER-bridge is an Euclidean wormhole.

As $\rho \to \infty$ we can see that $M/\rho \to 0$ and thus (2) looks like the spatial part of Minkowski metric in $\mathbb{R}^3$ spherical polar coordinates. Intuitively, the boundary "seems to be flat". To formalise this intuition, I will use the following definition.

Definition 2.4 (Asymptotically flat end)**.** *Let (M, h) be an n-dimensional Riemannian manifold. Then (M, h) is* ***an asymptotically flat end*** *if the following hold [14, 15].*

1. *M is diffeomorphic to $\mathbb{R}^n \setminus B$, where B is a closed ball centred at the origin in $\mathbb{R}^n$.*

2. *Pull-back the $\mathbb{R}^n$ coordinates to define coordinates x^i on M and let $r = \sqrt{\delta_{ij} x^i x^j}$. For $r \to \infty$ we have*

$$h_{ij} = \delta_{ij} + \mathcal{O}(r^{-1}) \text{ and } h_{ij,k} = \mathcal{O}(r^{-2}).$$

3. *Derivatives of the latter expression above also hold, e.g. $h_{ij,kl} = \mathcal{O}(r^{-3})$.*

4. *If matter fields are present, these should also decay at a suitable rate for large r.*

A Riemannian manifold (M, h) is ***asymptotically flat with N ends*** *if it is the union of a compact set with N asymptotically flat ends.*

Remark. *Contrary to the definition in [10], we do not even mention the extrinsic curvature tensor here. The metrics we work with are Riemannian and do not necessarily belong to a larger Lorentzian spacetime, so we make Riemannian manifolds the protagonists in our definition.*

Nonetheless, this is just a change of perspective. If the Riemannian spacetime does embed into a larger Lorenztian spacetime, the conditions on the extrinsic curvature tensor will be implied by the conditions on the derivative of h from the above definition.

In the GR course we showed that the ER-bridge is asymptotically flat with two ends: the surfaces $\Sigma_1 = \{\rho > M/2\}$ and $\Sigma_2 = \{0 < \rho < M/2\}$. These are joined together by the compact bifurcation sphere at $\rho = M/2$ [10].

To consider an unseen example, we prove how the geometry Hawking proposes in [5] is an Euclidean wormhole with 2 asymptotically flat ends.

Example 2.5. ***The Hawking Wormhole***. *Consider the geometry (H, h) described by the Euclidean line element* [1]

$$ds_H^2 = \left(1 + \frac{b^2}{x^2}\right)^2 (d(x^1)^2 + d(x^2)^2 + d(x^3)^2 + d(x^4)^2) = \left(1 + \frac{b^2}{x^2}\right)^2 \delta_{ij} dx^i dx^j, \tag{3}$$

where $x^i \in \mathbb{R}$ are the usual standard coordinates in $\mathbb{R}^4$ and $x^2 = \sqrt{\delta_{ij}x^i x^j}$. Hawking claims this has two boundaries: as $x \to 0$ and $x \to \infty$ [5]. What follows are my own computations to prove Hawking's claims. Aiming at (2), we set $n = 4$ in (1) so that $x_1 = r\cos(\theta_1)$, $x_2 = r\sin(\theta_1)\cos(\theta_2)$, $x_3 = r\sin(\theta_1)\sin(\theta_2)\cos(\phi)$ and $x_4 = r\sin(\theta_1)\sin(\theta_2)\sin(\phi)$. In these coordinates (3) becomes

$$ds_H^2 = \left(1 + \frac{b^2}{r^2}\right)^2 [dr^2 + r^2(d\theta_1^2 + \sin^2(\theta_1)(d\Omega_2^2)] = \left(1 + \frac{b^2}{r^2}\right)^2 (dr^2 + r^2 d\Omega_3^2). \tag{4}$$

The proper length of a curve going from r_1 to r_2 along the radial coordinate r is given by

$$L(r_1, r_2) = \int_{r_1}^{r_2} \sqrt{h_{rr}} dr = \int_{r_1}^{r_2} \left(1 + \frac{b^2}{r^2}\right) dr = r - \frac{b^2}{r}\Bigg|_{r_1}^{r_2}.$$

Letting either $r_1 \to 0$ or $r_2 \to \infty$ causes this integral to diverge. Thus we have an Euclidean wormhole with two compact connected components at the boundary: $r \to 0$ and $r \to \infty$. Notice that the transformation $r \to b^2/r$ is an isotropy since

$$dr \to \frac{-b^2}{r^2} dr \implies dr^2 \to \frac{b^4}{r^4} dr^2 \implies ds_H^2 \to \left(1 + \frac{r^2}{b^2}\right)^2 \left[\frac{b^4}{r^4} dr^2 + \frac{b^4}{r^2} d\Omega_3^2\right] =$$
$$= \left(1 + \frac{b^2}{r^2}\right)^2 \frac{r^4}{b^4} \left[\frac{b^4}{r^4} dr^2 + \frac{b^4}{r^2} d\Omega_3^2\right] = \left(1 + \frac{b^2}{r^2}\right)^2 (dr^2 + r^2 d\Omega_3^2) = ds_H^2.$$

[1] I have set $x_0 = 0$ for simplicity, since having $x_0 \neq 0$ as in [5] only shifts the $x \to 0$ boundary to $x \to x_0$.

With this in mind, remains to show we have two asymptotically flat ends which are joined by a compact hypersurface. Let Σ_b be the hypersurface $r = b$, a 3-sphere with radius $2b$. Then we can recover the entire Hawking wormhole by considering $\{0 < r < b\} \cup \Sigma_b \cup \{r > b\}$. By the isometry, it suffices to show that $\{r > b\}$ is an asymptotically flat end. We can see that

$$\{r > b\} = \{(r, \theta_1, \theta_2, \phi) \mid b < r < \infty,\ 0 < \theta_1 < \pi,\ 0 < \theta_2 < \pi,\ \phi \sim \phi + 2\pi\} = \mathbb{R}^4 \setminus B_b(0),$$

where $B_b(0)$ is the closed ball of radius $r = b$ centred at the origin. As $r \to \infty$ we can expand the conformal factor to obtain

$$\left(1 + \frac{b^2}{r^2}\right)^2 \sim 1 + \frac{2b^2}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right) \implies h_{ij} = \left(1 + \frac{b^2}{r^2}\right)^2 \delta_{ij} \sim \delta_{ij} + \mathcal{O}\left(\frac{1}{r^2}\right),$$

and similarly for higher derivatives.

These introductory examples have served to illustrate how Euclidean wormholes arise in spacetimes with asymptotically flat ends. Nonetheless, the title of the essay is Anti de-Sitter (AdS) Euclidean wormholes. What are these? And why would we care about such spacetimes specifically?

3 AdS Euclidean Wormholes

3.1 AdS Euclidean wormholes in the AdS/CFT conjecture

As we will cover more rigorously in section 3.2, the AdS Euclidean wormholes are Euclidean wormholes that, instead of having asymptotically flat ends, present the geometry of AdS in their boundaries.

In the introduction of this essay, we argued that wormholes are a key piece of the Quantum Gravity puzzle and that they must be taken into account when computing the path integrals. But why do we care about AdS Euclidean wormholes in particular?

In 1997, Juan Maldacena published a paper that shook the world of physics [2]. In his paper he conjectured what we know as the AdS/CFT correspondence or gauge/gravity duality, establishing an unexpected connection between two previously unrelated fields of Theoretical Physics.

The *AdS* part of the duality comes from the Anti de-Sitter spacetimes that we met in our Black Holes course. The *CFT* part of the duality stands for Conformal Field Theory. A CFT is a quantum field theory that is invariant under transformations from the conformal group, which includes some rescaling transformations [17]. For some $N > n$, the AdS/CFT correspondence conjectures that there is a mapping between a N-dimensional theory of Quantum Gravity in AdS, what String Theory is pursuing, and an ordinary n-dimensional non-gravitational CFT [16, 18].

[2]Reportedly, at the Strings '98 conference all participants danced the "Maldacena", a modified version of the popular Spanish dance "la Macarena" [16].

This duality between theories has lead to many insights and can be used both ways. When the CFT is strongly-coupled our perturbative methods fail, but we can write the problem in the String Theory setting where we have a weakly-coupled theory and perform our computations there. When the computations and findings from the String Theory approach seem obscure, we can reinterpret these in the CFT language, which is more tractable [16]. So far, the AdS/CFT correspondence has withstood all the tests we have put it under, has been cited over 20.000 times and given birth to loads of open questions and wonderful papers in other areas as superfluid mechanics or quantum information. See e.g. [19, 20].

The importance of AdS spacetimes for String Theory computations cannot be overstated. Thanks to the AdS/CFT duality it plays a major role in our search for Quantum Gravity. Therefore, we must grasp any topology that arises in these and in particular, the AdS Euclidean wormholes.

3.2 Definition of an AdS Euclidean Wormholes

Just as we had a notion to define when the boundary seems to be flat, we have a formal notion of the boundary behaving as AdS spacetime.

Definition 3.1 (Asymptotically locally AdS spacetimes)**.** *A spacetime (M, g) is **asymptotically locally AdS** [3] if there exists a spacetime $(\bar{M}, \bar{g})$ and [13]*

1. *There exists a function $\Omega > 0$ on M such that $(\bar{M}, \bar{g})$ is an extension of $(M, \Omega^2 g)$. Hence if we regard M as a subset of $\bar{M}$, then $\bar{g} = \Omega^2 g$ on M.*

2. *Within $\bar{M}$, M can be extended to obtain a manifold-with-boundary $M \cup \partial M$.*

3. *Ω can be extended to a function on $\bar{M}$ such that $\Omega = 0$ and $d\Omega \neq 0$ on ∂M.*

4. *The spacetime (M, g) solves the Einstein equations*

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{5}$$

for some $\Lambda < 0$ and with $\Omega^2 T_{\mu\nu}$ admitting a continuous limit to ∂M.

Remark. *These conditions ensure that the spacetime metric g will asymptotically approach the geometry of AdS under a suitable choice of coordinates [13].*

Definition 3.2 (AdS Euclidean wormholes)**.** *An **AdS Euclidean wormhole** is an Euclidean wormhole that is asymptotically locally AdS.*

Now that we know the definition of AdS Euclidean wormhole, a warning: these are not easy to find, not at all. Counterintuitively, the first example that naively comes to mind will never be the ambient space for a wormhole.

[3]If we impose the additional condition of the metric at the conformal boundary being that of $\mathbb{R} \times S^{n-1}$, we will have an asymptotically AdS spacetime, without the "locally". Confusingly, these two terms are used interchangeably in the literature, as in [2, 13]. Throughout this essay we will not impose any topology at the boundary and focus on asymptotically locally AdS spacetimes.

3.3 Why does AdS not have a wormhole itself?

In $n+1$ dimensions, the line element of the AdS_{n+1} spacetime can be written as [13]

$$ds^2_{AdS_{n+1}} = \frac{l^2}{\cos^2(\alpha)}(-dT^2 + d\alpha^2 + \sin^2(\alpha)d\Omega_n^2).$$

Here $\alpha \in (0, \pi/2)$, l is the radius of curvature of AdS (thinking of it as arising from a hyperboloid [13]) and $T \in \mathbb{R}$. Taking a constant T slice, we have the Euclidean metric

$$ds^2_{\Sigma_T} = \frac{l^2}{\cos^2(\alpha)}(d\alpha^2 + \sin^2(\alpha)d\Omega_n^2).$$

Compare this expression to our previous examples of Euclidean wormholes (2) and (3). The conformal factor blows up with $\alpha \to \frac{\pi}{2}$ and AdS spacetime is asymptotically locally AdS by definition. These facts point at the existence of an AdS Euclidean wormhole with two S^n boundaries linked through the bulk. However, notice that the proper length of a curve along α going from α_1 to α_2 is given by

$$L(\alpha_1, \alpha_2) = l\int_{\alpha_1}^{\alpha_2} \frac{d\alpha}{\cos(\alpha)} = \ln(|\tan(\alpha) + \sec(\alpha)|)|_{\alpha_1}^{\alpha_2}.$$

Taking $\alpha_1 \to 0$ and $\alpha_2 < \frac{\pi}{2}$ yields a finite result, whereas for $\alpha_2 \to \frac{\pi}{2}$ the integral will always diverge. We only have one compact connected component at the boundary $\alpha \to \infty$, which can be shown to be S^n [2]. Thus we cannot have a wormhole.

This impossibility of having a wormhole does not only occur for AdS spacetime, it also applies to a broad category of spacetimes in vacuum, with negative scalar curvature and with positively curved boundaries. We introduce some terminology before quoting the general result.

Definition 3.3. *A pseudo-Riemannian manifold (M, g) is **Einstein** if there exists a real constant β such that $R_{\mu\nu} = \beta g_{\mu\nu}$ [21].*

Einstein manifolds are solutions of (5) with $T_{\mu\nu} = 0$ and $\Lambda \neq 0$ for $n \geq 2$.

Proposition 3.4. *Let (M, g) be a pseudo-Riemannian metric. Then (M, g) is Einstein if and only if for some constant Λ it is a solution to the vacuum Einstein equation*

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0.$$

Proof. Fix (M, g) an n-dimensional pseudo-Riemannian manifold.

$\Longrightarrow$ direction. Suppose that (M, g) is an Einstein manifold, with $R_{\mu\nu} = \beta g_{\mu\nu}$ so that $R = \beta g^{\mu\nu} g_{\mu\nu} = n\beta$. Then choosing $\Lambda = \beta(n-2)/2$ we have

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \beta g_{\mu\nu} - \frac{n}{2}\beta g_{\mu\nu} + \beta\left(\frac{n-2}{2}\right)g_{\mu\nu} = 0,$$

so the vacuum Einstein equations hold.

$\Longleftarrow$ direction. Suppose that we have

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \implies R_{\mu\nu} = \left(\frac{1}{2}R - \Lambda\right) g_{\mu\nu}.$$

Thus (M, g) is Einstein and we are done. $\square$

We finally quote the Witten-Yau theorem, which explains why we cannot have an Euclidean wormhole in vacuum AdS spacetime and in other more general settings.

Theorem 3.5 (Witten-Yau)**.** *Let M be a complete Einstein manifold of negative curvature and with Penrose compactification such that at least one component of the conformal boundary of M has positive scalar curvature. Then the conformal boundary of M^{n+1} is connected [22].*

AdS spacetime is an Einstein manifold with negative curvature and S^n at the boundary. Therefore, we will only have one compact connected component at the boundary, excluding the existence of wormholes. So when do we have an AdS Euclidean wormhole?

3.4 Maximally symmetric AdS Euclidean wormhole in vacuum

In [1] and [2] the geometry of a maximally symmetric AdS Euclidean wormhole is stated. In this section we review the metric describing such geometry and prove how it obeys our definition of AdS Euclidean wormhole.

Example 3.6 (Santos, Marolf, Maldacena and Liat's wormhole)**.** *Let Σ_n describe a maximally symmetric Euclidean geometry. This means that either Σ_n is a sphere S^n, an Euclidean plane $\mathbb{R}^n$ or a hyperbolic plane $\mathbb{H}^n$. An AdS Euclidean wormhole whose boundary contains two copies of Σ_n and that preserves this symmetry through the bulk is given by*

$$ds^2_{SM^3} = d\tau^2 + a^2(\tau)d\Sigma_n^2, \tag{6}$$

where $a(\tau) > 0$ is the scale factor [23] and $a(\tau) \sim e^{|\tau|}$ as $\tau \to \pm\infty$ [1, 2]. This metric depends on the Euclidean time given by τ, but we can recover the Lorentzian signature metric by replacing $\tau = it$ [4]. We now fill in the gaps from both papers and show how such geometry describes an AdS Euclidean wormhole.

The proper distance computed for a curve going from τ_1 to τ_2 along the τ coordinate will clearly diverge as either $\tau_1 \to \infty$ or $\tau_2 \to -\infty$. Hence we have an Euclidean wormhole with two boundaries at $\tau \to \pm\infty$.

In the $\tau \to \pm\infty$ domain we have that $a(\tau) \sim e^{|\tau|}$ and the rest of the metric is invariant under $\tau \to -\tau$. By symmetry, it suffices to check that the metric is asymptotically locally AdS as $\tau \to \infty$. In this limit, the line element (6) can be expressed as

$$ds^2_{SM^3} = d\tau^2 + e^{2\tau}d\Sigma_n^2.$$

[4] A more detailed discussion of the method to analytically continue the metric instead, which also applies to non-stationary metrics, can be found in [24].

We want to append the hypersurface $\tau \to \infty$ to this spacetime. To achieve so, let $x = e^{-\tau}$, so that $x \to 0$ as $\tau \to \infty$. Then $dx = -xd\tau \implies dx^2/x^2 = \tau$ and the metric becomes

$$ds^2_{SM^3} = \frac{dx^2}{x^2} + \frac{1}{x^2} d\Sigma_n^2.$$

Choosing the conformal factor $\Omega = x \implies d\Omega = dx \neq 0$, we have that

$$d\bar{s}^2_{SM^3} = x^2 ds^2_{SM^3} = dx^2 + d\Sigma_n^2.$$

This can now be extended through $x = 0$ as we wanted. The hypersurface $x = 0$ is described by the variables contained in $d\Sigma_n^2$ and thus we have either $\mathbb{R}^n$, S^n or $\mathbb{H}^n$ at the boundaries. The check of the Einstein equations for each choice of boundary is done in Mathematica in the appendix B. Therefore, the spacetime described by (6) is an AdS Euclidean wormhole.

Notice that throughout the computation we have allowed $d\Sigma_n^2$ to describe the metric of $\mathbb{R}^n$, S^n or $\mathbb{H}^n$, which yields a curvature k equal to 0, 1 or -1 respectively. It seems that we can have $k = 1$ at the boundary. How does this not contradict theorem 3.5?

3.5 Friedmann equations to check the Witten-Yau theorem

To put the theorem to a test and highlight the meaning of its conditions, we employ Einstein's equations to obtain a condition for the curvature k depending on whether the universe has matter or not. If we consider an isotropic and homogeneous universe filled with some matter, using (5) and (6) we obtain the Friedman equation [2]

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{16\pi}{n(n-1)}\rho + \frac{1}{l^2} + \frac{k}{a^2}, \tag{7}$$

where $\dot{a} = \frac{da}{d\tau}$, $1/l^2 = -\Lambda/n(n-1)$ and $\rho = -T_{\tau\tau}$ with a minus sign due to the Euclidean signature of the metric [2]. This equation determines the evolution of $a(\tau)$ within a geometry described by the metric (6). To obtain an asymptotically AdS spacetime we require that $a(\tau) \to \infty$ for $\tau \to \pm\infty$. Santos and Marolf claim that this forces $a(\tau)$ to have a minimum and I will show this.

Proof. As $a(\tau)$ is a real function, having both limits diverging means that there exist τ_1, τ_2 such that $a(\tau) > a(\tau_2)$ for all $\tau > \tau_2$ and also that $a(\tau) > a(\tau_1)$ for all $\tau < \tau_1$. Considering the closed interval $[\tau_1, \tau_2]$, by the Extreme Value Theorem a must attain a minimum there, say a_0 for $\tau_0 \in [\tau_1, \tau_2]$. Hence, we must have that $a(\tau) \geq a_0$, for all $\tau \in \mathbb{R}$. □

Evaluating (7) at τ_0 and $\rho = 0$, we have $\dot{a} = 0$ as this is a minimum and then

$$0 = \frac{1}{l^2} + \frac{k}{a_0^2}.$$

To obtain a solution we must have $k = -1$. In addition, there is a marginal case where we can have $k = 0$. If instead of looking at the minimum, we multiply both sides of (7) by a^2 and set $\rho = 0$ we find

$$\dot{a}^2 = \frac{a^2}{l^2} + k.$$

Setting $k = 0$ here gives the solution $a(\tau) = le^{\frac{|\tau|}{l}}$[5]. The conformal factor in front of the spatial part of the metric shapes the profile curve that, when revolved around, determines the geometry of the wormhole after embedding it into Euclidean spacetime (see appendix A for an example). If we glue together two copies of solutions with such $a(\tau)$, which yields exponentials as profile curves, the resulting geometry will describe a degenerate case of Euclidean wormhole where the neck of the wormhole has become infinitely large and thin [2].

If $\rho = 0$, for a wormhole to exist we cannot have a positive curvature at the boundary. Thus $k \leq 0$, as established by theorem 3.5. Now let $\rho \neq 0$. Witten's theorem no longer applies as our geometry is not Einstein anymore. What are the restrictions on k?

Evaluating (7) with $\rho \neq 0$ and at $\tau = \tau_0$ yields the condition

$$0 = -\frac{16\pi}{n(n-1)}\rho + \frac{1}{l^2} + \frac{k}{a_0^2},$$

which can be solved for any $k \in \{-1, 0, 1\}$ if we take $\rho > 0$ large enough [2]. For $k = 0$, as we have seen above $\rho \sim 0$ suffices and $k = -1$ clearly has a solution. For $k = 1$, we instead require ρ to exceed a critical threshold. We expect $k = 0$ wormholes to have negligible matter sources on the boundaries whilst $k = 1$ wormholes will appear for ρ values large enough [2].

The addition of matter fields to wormholes does give more freedom of choice for k, but this does not come for free. When matter comes into play, the change of coordinates $\tau = it$ yields instabilities in the model.

3.6 Consequences of Wick rotations

The trick of Wick rotations produces two major changes in the dynamics of the wormhole.

1. If $\rho > 0$, then **gravitational attraction of matter is inverted**. Matter that is gravitationally attractive in Lorentz signature will become gravitationally repulsive in the Euclidean metric. Performing a Wick rotation $t \to it = \tau$ yields

$$\rho = T_{tt} = \frac{\partial(it)}{\partial t}\frac{\partial(it)}{\partial t}T_{\tau\tau} = -T_{\tau\tau}.$$

so effectively "mass changes its sign". Gravitationally repulsive matter contributes to $\ddot{a} > 0$ at $a = a_0$, which is required to have asymptotically AdS geometries [2]. This tells us that we do want to choose this type of matter, as especially in the $k = 1$ case we have a large ρ.

[5]There is a typo in Santos and Marolf's paper, where they write $e^{\frac{\tau}{l}}$ instead of $e^{\frac{|\tau|}{l}}$ [2].

2. **Kinetic energy may flip its sign**. If the kinetic energy in the Lorenztian signature comes from d^2/dt^2, then for any smooth $f(t)$

$$\frac{df}{dt} = \frac{d\tau}{dt}\frac{df}{d\tau} = i\frac{df}{d\tau} \implies \left(\frac{df}{dt}\right)^2 = -\left(\frac{df}{d\tau}\right)^2.$$

This tells us that in a QFT we would have imaginary first derivatives and a negative term in the Euclidean action. This negative contributions yield what we call **negative modes**, are problematic in any field theory admiring a Lagrangian description. These cause instabilities that we review and explain how to avoid in section 5.2.1.

Overall, we have shown that there are many more restrictions to construct the geometry of an AdS Euclidean wormhole. Obeying the properties required to have an Euclidean wormhole and the asymptotically locally AdS characterization are necessary but not sufficient conditions for a stable AdS Euclidean wormhole.

For a wormhole in vacuum, in the light of Witten-Yau theorem we cannot have positive curvature at the conformal boundary. This is allowed in the $\rho > 0$ case, but the existence of matter does force us to make choices to obtain consistent and sensible dynamics. Two major problems arise as a consequence of the Wick rotation. Firstly, we have to deal with the negative modes arising from a negative kinetic energy term in our action. Secondly, and especially for $k = 1$, we want to choose gravitationally attractive matter in Lorentz signature, to end up with gravitationally repulsive matter.

We have now formally established what we mean by AdS Euclidean wormholes, covered a general example and their limitations. How to visualise Euclidean wormholes via embeddings in Euclidean space and an example of how wormholes can open up and then pinch off is covered in appendix A. In the next section, we will tackle how wormholes can *branch off* from spacetimes and how this leads to newborn universes called *baby universes*.

4 Baby universes

In any theory of Quantum Gravity, the topology of spacetime is no longer preserved [5, 25]. As soon as we write an action including terms related to the metric, we are giving the metric the same footing as a quantum field. For scales of the order of Planck length, the metric and the topology of spacetime can thus fluctuate [25]. In particular, Euclidean wormholes can use their dynamic behaviour to join on or extend away from an ambient universe, i.e. branch off. This is the case for the Hawking wormhole when b is of the order of the Planck length [5], and also of AdS Euclidean wormholes [13].

When a wormhole has branched off, we can focus on a particular compact region or slice and study what happens to it. Before doing so, we must establish the quantum machinery required to formally slice spacetimes.

4.1 Slicing and establishing a wormhole QFT

In the QFT framework we compute the contributions from a wormhole on the set of matter fields Φ in the asymptotic regions. The approach is the same for asymptotically flat [5] or asymptotically AdS regions [26]. Here we will restrict to matter fields that are conformally invariant and wormholes of the Planck size so that all the effects are small [5].

Definition 4.1 (Path integral in a wormhole)**.** *Consider a spacetime where the asymptotic boundary has s compact connected components. The path integral over such boundary in a theory with a set of fields Φ, an action $S[\Phi]$ and associated boundary conditions J_i, which describe the behaviour of fields near the boundary, i.e. $\Phi \sim J = \{J_1, \dots, J_s\}$, is given by*

$$\langle Z[J_1] \dots Z[J_s] \rangle := \int_{\Phi \sim J} \mathcal{D}\Phi e^{-S[\Phi]}. \tag{8}$$

Each J_i induces an n-dimensional metric on the boundaries, say g_i [26].

Imagine that we want to compute the interactions at a hypersurface from our wormhole. If $|\xi\rangle$ encodes the state of such hypersurface, to find the interactions at the cross-section of interest we can factorize the path integral in two terms

$$\langle Z[J_1] \dots Z[J_n] \rangle = \langle 0| Z[J_1] \dots Z[J_n] |0\rangle = \langle 0| Z[J_1] \dots Z[J_k] |\xi\rangle \langle \xi| Z[J_{k+1}] \dots Z[J_n] |0\rangle ,$$

where $1 \leq k \leq s$ and $|0\rangle$ is the usual vacuum state with all $\phi_i \to 0$ [5]. This factorization of the path integral into two different pieces is what is generally called *slicing the path integral* in the literature. To be able to formally slice the path integral, we need to define how we evaluate the inner-products involving $|\xi\rangle$, which requires us to construct a Hilbert space $\mathcal{H}$ for the wormhole.

Nonetheless, there is a crucial difference to the usual QFT approach. We normally define the field-dependent wave functions as solutions to our Schrödinger equation [27]. This implies that we have a time derivative of the states, so at some point we have acknowledged a general time definition for all the observers. But in GR there is no absolute notion of time, it is a coordinate that we can define in several ways and will vary depending on where the observer is located.

For this reason, to develop a theory of Quantum Gravity we consider wave functions that obey the Wheeler-DeWitt equation instead. Such equation depends on the metric but not on the background or the coordinates we choose, and can be obtained for both asymptotically flat spacetimes [5, 28] and asymptotically locally AdS spacetimes [29]. Contrary to the Schrödinger equation that relies on a single spacetime, the Wheeler-DeWitt equation does allow for a superposition of spacetimes [30].

With this essential distinction clarified, we now provide a definition of an inner-product in the wormhole space.

Definition 4.2. *Let Σ be a compact hypersurface on a wormhole and let g_Σ and Φ_Σ denote the metric and the values of the matter fields from Φ on such hypersurface Σ. For two* *solutions of the Wheeler-DeWitt equation* Ψ_1, Ψ_2 *on* Σ*, with associated quantum states* $|\psi_1\rangle$*,* $|\psi_2\rangle$*, we define the inner-product [5, 31]*

$$\langle\psi_1|\psi_2\rangle = \int_\Sigma \sqrt{\det(g_\Sigma)}\Psi_1^*(\Phi_\Sigma, g_\Sigma)\Psi_2(\Phi_\Sigma, g_\Sigma).$$

Combining all the solutions to the Wheeler-DeWitt equation that correspond to wormholes, we obtain the Hilbert space $\mathcal{H}$ we were looking for. If we consider a basis $|\xi_i\rangle$ for $\mathcal{H}$, we can write the path integral computation as [5]

$$\langle 0| Z[J_1]\dots Z[J_s]|0\rangle = \sum_i \langle 0| Z[J_1]\dots Z[J_q]|\xi_i\rangle\langle\xi_i| Z[J_{q+1}]\dots Z[J_s]|0\rangle\,,$$

which is what we mathematically mean by *slicing the path integral* into a sum of bra-kets.

4.2 The baby universe sector

To slice the path integral we had to construct the Hilbert space of the theory, and we are free to choose at which compact hypersurface of the wormhole geometry we slice through. In particular, we can consider Σ to be an S^2 in the ER-bridge or S^3 in the Hawking wormhole. Slicing the path integral gives a new sector that lies infinite affine parameter distance away from any of the asymptotically flat or AdS boundaries, so it does not interact with these [5, 26]. Moreover, this new sector is instead associated with spatially compact universes.

As explained previously, a wormhole can branch off from an asymptotically locally AdS or asymptotically flat spacetime. We can now slice through the region that has branched off and treat such slice as having been emitted by the asymptotic flat or AdS universe, see Fig. 1. This is where the name of *baby universe* comes from, as we think of it as a closed universe that was emitted by the parent universe at its boundary.

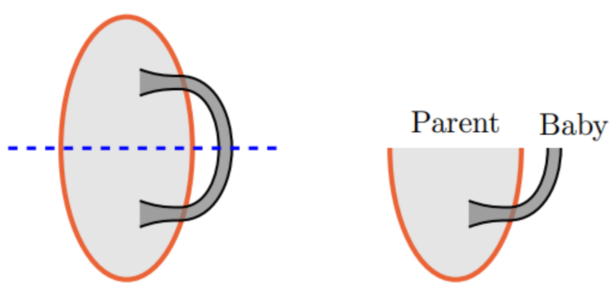

Figure 1: Pictorial representation of slicing a wormhole that branches off and rejoins a spacetime. The slice is represented by the blue dotted line and the boundary of the spacetime in orange. On the disconnected geometry of the slice we have a baby universe, which was emitted from the parent asymptotic universe and does not intersect the boundary at the slice [26].

Definition 4.3 (The baby universe sector)**.** *Consider the setup as in definition 4.1 and pick a compact hypersurface Σ on the wormhole that does not intersect any of its conformal boundaries. Label one of the boundaries "past" and the other one "future". Then the Hilbert space from definition 4.2 is the **baby universe** Hilbert space $\mathcal{H}_{BU}$, which represents the space of closed universes in the theory. Within this Hilbert space, the **baby universe state***

$$|Z[J_1]\dots Z[J_s]\rangle \in \mathcal{H}_{BU}$$

is defined by the boundary conditions $J_1, \dots, J_s$ for the path integral and encapsulates all closed universes with all possible spatial metrics and field configurations on Σ. Notice that these states are unchanged by permutations of the J_i's [26].

Remark. *In the definition above I have made two simplifications for the sake of clarity.*

1. The requirement of not intersecting any of the conformal boundaries is not necessary. This definition can be generalized to slices through the conformal boundaries [26].

2. Explicitly computing the wave-function instead of giving the associated state in $\mathcal{H}_{BU}$ can be done, but is rather complicated. As we argued for the Wheeler-DeWitt equation, there is no absolute notion of time in GR, so we do not know where to cut and establish the past and future. This forces us to impose the constraints from the Wheeler-DeWitt in such wave-functions, which is quite challenging when we try to account for universes splitting and joining. For an in-depth explanation check [25, 28].

Now that we have understood how baby universes can arise from slicing wormholes, in the second part of the essay we describe how allowing for tiny universes to join on or branch off spacetimes is rather problematic for Quantum Mechanics.

5 The Factorization Problem

Up until the arrival of wormholes, the AdS/CFT correspondence had withstood all the tests and computations not only in String Theory, but also in condensed matter, nuclear and particle physics; proving to be a rather robust conjecture [16]. However, when AdS Euclidean wormholes come into play, we have an unresolved puzzle. Its more general statement is thought to be the following

The sum over all geometries with fixed boundary conditions is the same as the partition function of a conformal field theory living on the boundary [1].

Given a geometry with a single compact boundary and conditions specified by J_1, we expect to have the path integral given by $\langle Z[J_1]\rangle$. Similarly, if we have two disconnected boundaries with the same conditions specified by J_1, due to the tensor product structure of the boundary (two parts that are disconnected from each other), we would expect to obtain the path integral from the product $\langle Z[J_1]^2\rangle = \langle Z[J_1]Z[J_1]\rangle = \langle Z[J_1]\rangle^2$ [32]. This is not the case if we take Euclidean wormholes into account. Since we have to include all the geometries, we must take into account the contribution from wormholes connecting these through the bulk. This means that

$$\langle Z[J_1]^2\rangle = \langle Z[J_1]\rangle^2 + \{\text{contribution from wormholes}\} \neq \langle Z[J_1]\rangle\langle Z[J_1]\rangle.$$

The path integral does not factorize. This is what we call the **Factorization Problem**.

Predating the existence of the AdS/CFT theory, Hawking argued about the Quantum Mechanical version of this problem: the *loss of quantum coherence*. Consider we have a Black Hole or a baby universe as described in previous section, both examples of closed universes, and place an observer outside. Imagine now that a particle with a known pure quantum state falls into the closed universe. There is no way to measure the state of the closed universe, so the observer outside the closed universe needs to sum up over all possible quantum states for the closed universe. Hence, we end up with a mixed state for the same particle, despite having started from a pure quantum state. This leads to a loss of quantum coherence [5].

This issue was addressed and solved by Coleman by means of the baby universes [33]. His key insight was to treat each $\langle Z[J_i]\rangle$ as not dual to one, but to an ensemble of theories. The difference between what we expect and what we have can then be interpreted as fluctuations: $\delta Z[J_1]^2 := \langle Z[J_1]^2\rangle - \langle Z[J_1]\rangle^2$, which allow our partition function to differ from the ensemble mean $\langle Z[J_1]\rangle$. These can be interpreted as fluctuations in the constants of nature, which we do not know precisely [33]. Computations following this philosphy have been explicitly carried out for the 2D Jackiw-Teitelboim gravity [32] and pure gravity in AdS_3 [34] among many other examples. In section 5.1 we will develop this formalism by constructing the α-states and recovering factorization [26].

Unfortunately, it is unclear how to apply the ensemble perspective on familiar AdS/CFT theories [1, 2]. To inspect the problem closely, we will cover the saddle-point approximation method in section 5.2, which allows us to approximate the non-trivial gravitational path integrals in high dimensions. Then we review the results from the low dimensional Einstein $U(1)^3$ theory. Finally, we will move onto higher dimensions, address the additional issue of brane instabilities [1] when dealing with String Theories and comment on the results for the factorization problem for Einstein-$U(1)^2$ wormholes in 11-dimensions.

5.1 The ensemble approach

The aim of this section is to review the construction of α-states as a means to restore the factorization property. Be wary: we are assuming that (8) is well-defined and that it gives exact results. This is only the case for very simple gravitational models, as for most of the examples we have divergences and we need to consider asymptotic expansions [26].

5.1.1 Completion of the baby universe Hilbert space

We focus on the big picture, avoiding the issue of gauge equivalent states which lead to dramatical physical consequences. For an in-depth explanation, check [26].

To begin with, we need some extra ingredients. We pick up from definition 4.2 and consider the specific case of no boundary condition in the past.

Definition 5.1 (The Hartle-Hawking state)**.** *Consider the case where we have no boundary conditions of a state in $\mathcal{H}_{BU}$, i.e. set $s = 0$. This defines the **Hartle-Hawking state** $|HH\rangle$, which represents a state of the full collection of an indefinite number of baby universes [26].*

Remark. *The diffeomorphism invariance of General Relativity gives us some gauge freedoms that we need to fix in order to obtain a consistent theory. The $|HH\rangle$ state is something we choose for the theory, and thus fixes the gravitational constraints required for our path integral. Once we have fixed our gravitational constraints, we can then compute the inner products.*

Additionally, notice that we have chosen to use the first-quantized path integrals to fix our gravitational constraints [35], instead of using first-quantized path integrals to compute Green functions as in [7].

Once we have fixed the $|HH\rangle$ state, we rewrite the inner-product in $\mathcal{H}_{BU}$ using the $Z[J_i]$ notation. This is the same as in 4.2, but here we impose some additional properties.

Definition 5.2. *Consider the framework of $\mathcal{H}_{BU}$. Given a future state with boundary conditions $\tilde{J}_1, \dots, \tilde{J}_r$ and a past state with boundary conditions $J_1, \dots, J_s$, their inner-product is given by [26]*

$$\langle Z[\tilde{J}_1] \dots Z[\tilde{J}_r] | Z[J_1] \dots Z[J_s] \rangle := \langle Z[\tilde{J}_1^*] \dots Z[\tilde{J}_r^*] Z[J_1] \dots Z[J_s] \rangle, \tag{9}$$

where $*$ is the Charge-Parity-Time (CPT) reversal operation [36] and the R.H.S. corresponds to the path integral (8). This inner-product is chosen to obey [13]

*1. **Hermiticity**, so we have $\langle Z[\tilde{J}_1^*] \dots Z[\tilde{J}_r^*] \rangle = \langle Z[\tilde{J}_1] \dots Z[\tilde{J}_r] \rangle^*$*

*2. **Positive-definiteness**, i.e. $||\Theta||^2 := \langle \Theta | \Theta \rangle \geq 0$ for all $|\Theta\rangle = \sum_{i=1}^{N} c_i \, |Z[J_i, 1] \dots Z[J_{i,s_i}]\rangle$. Here J_{i,s_i} corresponds to the case with i disconnected boundaries, and we are selecting the conditions for each boundary from a set with s_i possibilities.*

We are now set to construct the states that solve the Factorization Problem.

5.1.2 Construction of operators and α-states

The essence of what follows is similar to the development of the quantum harmonic oscillator, in the sense that we want to end up with annihilation and creation operators and recover orthonormality as in [27].

Definition 5.3 (Operator from functional)**.** *For each boundary functional $Z[J]$, we define the operator $\widehat{Z[J]}$ to act on $\mathcal{H}_{BU}$ as [26]*

$$\widehat{Z[J]} \, |Z[J_1] \dots Z[J_k]\rangle = |Z[J] Z[J_1] \dots Z[J_k]\rangle \, .$$

In analogy to the quantum harmonic oscillator, this means that $\widehat{Z[J]}$ can be regarded as a *creation operator* for boundary conditions, that allows us to obtain all the states by acting on the $|HH\rangle$ no-boundary state

$$|Z[J_1]\dots Z[J_k]\rangle = \widehat{Z[J_1]}\dots\widehat{Z[J_k]}\,|HH\rangle\,.$$

Using this definition and the properties we have reviewed, we have the following.

Proposition 5.4. *The Hermitian conjugate* $\widehat{Z[J]}^{\dagger} = \widehat{Z[J^*]}$.

Proof. The proof is not done in [26] and it takes a few lines. Using the definition of hermitian conjugate

$$\begin{aligned}
&\langle Z[\tilde{J}_1]\dots Z[\tilde{J}_r]|\,\widehat{Z[J]}^{\dagger}\,|Z[J_1]\dots Z[J_k]\rangle = \langle Z[J_1]\dots Z[J_k]|\,\widehat{Z[J]}\,|Z[\tilde{J}_1]\dots Z[\tilde{J}_r]\rangle^{*}\\
&= \langle Z[J_1]\dots Z[J_k]|\,|Z[J]Z[\tilde{J}_1]\dots Z[\tilde{J}_r]\rangle^{*}\,,\ \text{by definition of the operator.}\\
&= \langle Z[\tilde{J}_1]\dots Z[\tilde{J}_r]Z[J]|\,|Z[J_1]\dots Z[J_k]\rangle\,,\ \text{using sesquilinearity and rearranging.}\\
&= \langle Z[\tilde{J}_1]\dots Z[\tilde{J}_r]|\,|Z[J^*]Z[J_1]\dots Z[J_k]\rangle\,,\text{by (9).}\\
&= \langle Z[\tilde{J}_1]\dots Z[\tilde{J}_r]|\,\widehat{Z[J^*]}\,|Z[J_1]\dots Z[J_k]\rangle\,.
\end{aligned}$$

□

As the states are not modified by permutting the J_i's, for $J \neq J'$ we deduce that

$$[\widehat{Z[J]},\widehat{Z[J']}] = 0 \implies [\widehat{Z[J]},\widehat{Z[J]}^{\dagger}] = [\widehat{Z[J]},\widehat{Z[J^*]}] = 0,$$

after using the proposition 5.4. Hence, it follows from the spectral theorem that we have a complete basis of orthonormal states.

Theorem 5.5. *The baby universe Hilbert space $\mathcal{H}_{BU}$ has a basis of orthonormal states $|\alpha\rangle$ which are simultaneous eigenvectors for all $\widehat{Z[J]}$ operators*

$$\widehat{Z[J]}\,|\alpha\rangle = Z_\alpha[J]\,|\alpha\rangle\,,\ \textit{for all } J.$$

These eigenvectors are called ***α-states*** *and we have $\langle\alpha'|\alpha\rangle = \delta_{\alpha',\alpha}$ for the case of a discrete spectrum and the relevant delta function and normalization if the spectrum is continuous. Thus the set $\widehat{Z[J]}$ constitutes a complete set of commuting observables and we have a probabilistic interpretation of the α-states [26, 33].*

So how can we think about these α-states? The α's labels the theories in the ensemble, and the eingenvalues $Z_\alpha[J]$ give definite and precise values for observables in the theory associated with each particular label of α. Each state $|\alpha\rangle$ corresponds to one member of the ensemble from $\mathcal{H}_{BU}$ [26].

We argued that the Factorization Problem, from the ensemble perspective, arises from the fluctuations of the constant in nature. Notice, on the other hand, that the eigenvalues of $|\alpha\rangle$ states are precise and also that we are free to choose the $|HH\rangle$ state. If we fix $|HH\rangle = |\alpha\rangle$, the eigenvalues of $|\alpha\rangle$ turn out to be a unique choice of fixed constants of nature [33] and we can explicitly check that

$$\langle Z[J_1]Z[J_1]\rangle = \langle\alpha|\,\widehat{Z[J_1]}^{\dagger}\widehat{Z[J_1]}\,|\alpha\rangle = ||Z[J_{1,\alpha}]||^2 = \langle\alpha|\,Z[J_1]\,|\alpha\rangle\,\langle\alpha|\,Z[J_1]\,|\alpha\rangle = \langle Z[J_1]\rangle\langle Z[J_1]\rangle.$$

This is because after fixing the choice of constants of nature, the additional term for the fluctuations vanishes. Hence the ensemble perspective explicitly solves the Factorization Problem for well-behaved theories with low energies and low dimensions.

5.2 Models beyond the ensemble approach

In a general gravitational model, chances are that (8) is ill-defined and full of divergences. Moreover, computations get even more complicated when dealing with higher dimensions in String Theory. Instead, we perform an asymptotic expansion of our path integral. In this section we introduce the methodology to estimate the wormhole contributions and compare these to the disconnected spaces solution to see whether this additional term should be taken into account or not.

5.2.1 Saddle-point approximation and negative Euclidean modes

Suppose we have a general Euclidean partition function $Z[J]$ associated to an Euclidean action $S_E[\Phi]$, where J imposes the boundary condition of a set of Einstein-Scalar fields Φ. Expand $\Phi = \Phi^0 + \delta\Phi$, with Φ^0 containing the solutions of the classical equations of motion so that $\delta S_E[\Phi^0] = 0$. In the saddle-point approximation we have [2]

$$\langle Z[J]\rangle = e^{-S_E[\Phi^0]}\int_{\delta\Phi\sim J} D(\delta\Phi)e^{-S_E^{(2)}[\delta\Phi]} + \dots,$$

where we have kept terms up to $\mathcal{O}(\delta\Phi^2)$ and $S_E^{(2)}[\delta\Phi]$ represents the second functional derivative of the action evaluated at $\delta\Phi$. This is the approximation we will employ in the coming computations.

Notice that the saddle-point approximation relies on two facts: the negative sign of the exponent and Φ^0 being a minimum of the action. If Φ^0 is not a minimum of the action, then $S_E^{(2)}[\Phi^0] \leq 0$ and we can reduce it even further. Therefore, nearby configurations of the integral along the real Euclidean contour can dominate over this choice.

As we commented on section 3.6, this problem arises in any effective field theory admiring a Lagrangian description. The kinetic energy term will have a minus sign at the front and thus we can always lower the action. This is known as the conformal factor problem of Euclidean quantum gravity.

When we have $\rho = 0$, it suffices to perform another Wick rotation to end up with a convergent Gaussian integral and solve the issue in both linear and non-linear levels [2]. On the other hand, when matter is coupled to gravity it is not so clear how to perform the Wick rotation. Einstein equations relate the ρ and our metric. This means that perturbations of the trace of the metric will couple to matter perturbations, and also that the trace-free part of the metric may couple with the trace itself [2]. There is an outline of the method in [37] and it is employed in [38] for the Reisnner-Nordström black hole.

To avoid dealing with these issues and restrict ourselves to real fields, in this essay we choose to take all τ derivatives to vanish at a_0. Assuming that the scalar sources are identical on both boundaries, this is equivalent to imposing that the wormhole that is symmetric under reflection along its narrowest cross-section, i.e. invariant under $\mathbb{Z}_2$ symmetry [2]. We now review a simple low-dimensional solution with $\rho > 0$ due to spatial (but not τ-dependent) gradients of the matter fields.

5.2.2 Wormhole in AdS_4 Einstein-Maxwell theory

Here we review an example in low energies and low dimensions that illustrates the main techniques we have covered. We will focus on the geometrical aspect of the problem, for the details of the action and fields setup check [2] (section 4). Consider an AdS_4 spacetime with S^3's at the boundaries. Within such spacetime, we let

$$\sigma_1 = -\sin(\eta)d\theta + \cos(\eta)\sin(\theta)d\varphi,\ \ \sigma_2 = \cos(\eta)d\theta + \sin(\eta)\sin(\theta)d\varphi,\ \text{ and } \sigma_3 = d\eta + \cos(\theta)d\varphi,$$

with $\eta \in (0, 4\pi)$, $\theta \in (0, \pi)$ and $\varphi \in (0, 2\pi)$. Then choose three distinct Maxwell fields to be given by

$$A_i = L\frac{\sigma_i}{2}\Phi(r),$$

where $\Phi(r)$ is the source for these fields. These are chosen to break the symmetries of the solution into the symmetries of the S^3, which is achieved by our choice of σ_i's since $d\Omega_3^2 = (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)/4$ [2]. The solutions we want are of the form [2]

$$ds^2 = \frac{dr^2}{U(r)} + g_{S^3}(r)d\Omega_3^2,$$

with g_{S^3} giving us some gauge freedom to consider a solution where we have a wormhole and another solution where we do not.

No-wormhole solution For the no-wormhole solution we want to have $g_{S^3}(r) = r^2$ with $r \in (0, \infty)$. If we imagine $r = 0$ as the origin where the S^3 sphere is compressed to a dot, then for smoothness of the metric and Maxwell fields at the origin we must have [2]

$$\left.\frac{d\Phi(r)}{dr}\right|_{r=0} = 0 \text{ and } U(0) = 1.$$

These boundary conditions yield $U(r) = 1 + r^2/l^2$ as we had in section 3.3, and [2]

$$\Phi(r) = \Phi_\infty \frac{\sqrt{l^2 + r^2} - l}{\sqrt{l^2 + r^2} + l},$$

with Φ_∞ telling us how the electromagnetic sources behave at the boundary. This allowed Santos and Marolf to compute the action

$$S_{\text{no-wormhole}} = 8\pi^2 l^2 (1 + 3\Phi_\infty^2).$$

In this simple case everything works out analytically and we obtain a finite number for the action. We now introduce the wormhole solution to compare values.

Wormhole solutions If we want to have a wormhole with S^3 at each boundary and invariant under the $\mathbb{Z}^2$ symmetry, for some $r_0 > 0$ we consider $g_{S^3} = r^2 + r_0^2$. The metric now becomes

$$ds^2 = \frac{dr^2}{U(r)} + (r^2 + r_0^2) d\Omega_3^2.$$

Here $r \in \mathbb{R}$, we have two asymptotic boundaries as $r \to \pm\infty$ and the constant r_0 represents the minimum radius at the neck of the wormhole. As we now have extended the domain of r, after fixing some constants to avoid having a singularity at $r = 0$, using the equations of motion we find [2]

$$U(r) = \frac{l^2 + r^2 + 2r_0^2}{l^2}.$$

Finally, imposing the $\mathbb{Z}^2$ symmetry so that $d\Phi/dr(0) = 0$, expanding for large r gives [2]

$$\Phi(r) = \Phi(0) \cosh\left[\frac{2}{b} F\left(\arctan\left(\frac{r}{La}\right) \Big| 1 - \frac{a^2}{b^2}\right)\right] \xrightarrow{r\to\infty} \Phi_\infty = \Phi(0) \left[\frac{2}{b} K\left(1 - \frac{a^2}{b^2}\right)\right]. \quad (10)$$

Here $a = (1 + 16\Phi(0))^{1/4}$, $b = \sqrt{(a^2-1)/2} = r_0/l$, $F(\phi|m)$ is the elliptic integral of the first kind and $K(m)$ is the complete elliptic integral of the first kind (check for instance section 19 of [39]).

Despite having the A_i in terms of Φ_∞, we write the solutions in terms of $\Phi(0)$. This is because for each Φ_∞ value we can have multiple solutions, but these are uniquely determined by their value of $\Phi(0)$. We can see in plot Fig. 2 that wormhole solutions exist if $\Phi_\infty \geq 3.563349$ and that for each value of $\Phi(0)$ we indeed have the two solutions. What do these solutions physically mean?

Since $r_0 = bl$ and b depends on $\Phi(0)$, inverting the relation (10) we can express r_0 as a function of Φ_∞. Each value of Φ_∞ gives two solutions, so we will obtain two different wormholes: a large wormhole with a bigger r_0 and a smaller one. When $\Phi_\infty = 3.563349 \implies r_0 = 1.251462l$ and both branches merge.

We finally arrive at the most important result. The difference between the action of two disconnected universes and the wormhole solution is given by

$$\Delta S_{U(1)^3} = 2 * S_{no-wormhole} - S_{wormhole},$$

whose plot can be seen in Fig. 2. If $\Delta S_{U(1)^3} > 0$, the action coming from the wormhole is lower with the same value of Φ_∞, and thus will dominate when it comes to evaluating the

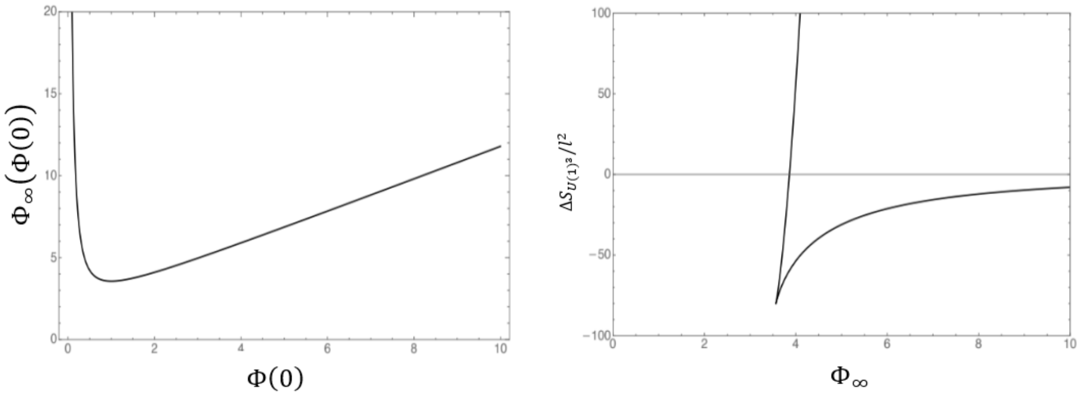

Figure 2: Plots corresponding to the wormhole in AdS_4 Einstein-Maxwell theory. **Left panel:** graph of the source for Maxwell fields A_i as a function of $\Phi(0)$. We have a minimum at $\Phi_\infty \approx 3.563349$. **Right panel:** graph of the difference in action between the wormhole and no-wormhole solution. At any Φ_∞, the larger value of $\Delta S_{U(1)^3}$ corresponds to the large wormhole and the smaller value to the small wormhole [2].

path integral. For $\Phi_\infty > 3.859673$, we have that the large wormhole solutions are dominant, and subdominant elsewhere. The small wormholes are always subdominant. With a careful treatment of perturbations, it can be shown that the large wormholes are free of negative modes, so that they are stable under these perturbations and dominate over the disconnected solution for r_0 large enough [2]. In this theory, the Factorization Problem remains unsolved.

For the final example, we will deal with the complete UV-spectrum, thus requiring a change of approach. In the following section, we motivate and introduce some String Theory definitions.

5.2.3 A brief introduction to String Theory terminology

String Theory is a theory that tries to unify both General Relativity and QM, which is a rather intricate task. One of the main issues is that the coupling of Einstein gravity to matter renders non-renormalizable theories: QFT's which carry divergences we do not know how to reabsorb with a finite number of counter-terms in our Feynman expansion [17].

Consider the case of two particles interchanging a graviton. It can be shown that the amplitude is proportional to E^2/M_p^2, where $M_p \sim 10^{19}$ GeV is the Planck mass. For $E \gg M_p$, the gravitational action is small, but when $E \ll M_p$ these interactions become strongly relevant [40]. In particular, for the exchange of two gravitons, the amplitude for the Feynman diagram is badly divergent as it scales as

$$\frac{1}{M_p^4}\int^\infty d^4k.$$

To deal with these theories, we introduce a cut-off, but we will not be able to describe what happens at the cut-off range or beyond.

As a solution, String Theory proposes to substitute the 0-dimensional fundamental notion of particle in our QFT, by 1-dimensional strings which have additional degrees of freedom. These have a fixed length of approximately 10^{-33} cm and can be open or closed, depending on the type of particle they are replacing. Consequently worldlines in spacetime become worldsheets and Feynman diagrams become surfaces [17].

To define branes, we focus on open strings. What characterizes these is that they have two different endpoints. Think of the open string as a curve parametrised by σ in n-dimensions and with its position being determined by X^μ. Then to be able to quantise the string and perform physics with it, we need to set up boundary conditions on both ends. We have two choices:

- Neumann boundary conditions. The end of the string does not oscillate in a certain direction, i.e. $\partial_\sigma X^\mu = 0$ for some values of μ.
- Dirichlet boundary conditions. The end of the string is fixed in certain coordinates at a value which we specify, i.e. $X^\mu = c^\mu$, for some values of μ and c^μ all constants.

If we specify $(p+1)$ Neumann conditions and $n-(p+1)$ Dirichlet conditions, the endpoints of the string are fixed to be in a $(p+1)$-hypersurface in spacetime. This hypersurface is what we commonly refer to the D-brane (or simply brane in the literature), the hypersurface in spacetime where the end-points of the string are constrained to be after we impose the boundary conditions. If we want to explicitly write the dimension, we write np-brane. As common examples we have the $n0$-brane, which is a particle; or the $n1$-brane which is a string itself. In this sense, branes should be regarded as a new dynamical object [17].

Now that we have an intuition on the meaning of a brane, we review the brane instability issue that arises when considering embeddings of wormholes in String Theories.

5.2.4 Brane instabilities

Inserting a brane in our theory will yield a term that modifies the overall value of the action [1]. If such term has a negative sign, this means that we can find a lower-action solution and thus we have an instability for the theory, just as with negative Euclidean modes. To test whether a theory has brane instabilities or not, we have to compute the change in the action after inserting a brane and check its sign [2].

Consider a theory with a background field strength described by a p-form F. This is similar to the setup in section 5.2.2. Suppose its geometry is described by [1]

$$ds_M^2 = d\rho^2 + \cosh^2(\rho) ds_{\Sigma_d}^2, \tag{11}$$

with $\rho \in \mathbb{R}$ and with two negatively curved compact boundaries at $\rho \to \pm\infty$. Now imagine we have an Euclidean $(n-1)$-brane around the n-dimensional hyperbolic slices of the metric (11). We are concerned about branes that partially screen F, so we think of these as being charged by the same potential $(n+1)$-form A, with $F = dA$. With this setup, in AdS spacetime with $l = 1$, the action of this theory can be written as [1]

$$S \sim (\cosh(\rho))^n - n \int^{\rho} d\rho' (\cosh(\rho'))^n \sim \frac{-2n}{2^n(n-2)} e^{(n-2)\rho} + \dots$$

For large ρ and $n > 2$, the contribution to the action coming from the branes is negative. This means that we can create a brane-antibrane pair and move one to one of the boundaries at $\rho \to \pm\infty$ to decrease the action [1]. Such instability always occurs in negatively curved spacetime and leads to an effective negative mass term for the scalar, which causes an instability.

This instability comes from the topology of the boundaries [1]. This was already hinted by the Witten-Yau theorem, whose origin is in the String Theory framework. As Witten and Liat explain in their paper, the CFT's arising in the AdS/CFT correspondence are only well-behaved when the scalar curvature of the boundary is non-negative [22]. In fact, the argument to prove such theorem consists in computing the action for a brane in the spacetime and showing that it is unbounded below if the curvature at the boundary is negative.

Note, however, that is not only restricted to the case where we have negative curvature. In our final example, we will see that brane instabilities may arise in some cases for a wormhole with S^3's at the boundaries. This issue did not arise in the AdS_4 wormhole in section 5.2.2 because we were in a low-dimensional and low energy setting, just as with all previous wormholes.

When we are dealing with UV energies and higher dimensions, we must keep the brane instabilities in mind for a truthful evaluation of our final results.

5.2.5 Einstein-$U(1)^2$ wormholes in 11-dimensional supergravity

As a final example, we finally review an AdS Euclidean wormhole in the framework of String Theory. The mathematics to deeply understand this example and its computations lie beyond the scope of this essay. Therefore, we focus on reviewing and commenting on the results from [2].

Our aim here is to review the wormhole contribution to the path integral in an UV-complete theory as it is String Theory. Santos and Marolf begin with a theory of supergravity in 11 dimensions, i.e. gravity with restrictions imposed by the theory of supersymmetry, and then they truncate it to a theory in 4 dimensions similar to the one in section 5.2.2, but with two Maxwell fields instead of three. Then any solution of the equations of motion of the truncated action can be uplifted to the 11 dimensions by means of supersymmetry. The details on the setup can be found in section 6 of [2].

Focusing on the metric, we seek for a solution with S^3's at the boundaries, imposing the $SO(4)$ symmetry only at the the boundaries themselves and relaxing this condition to an $U(2)$ symmetry on the bulk. As the structure is similar to the theory from section 5.2.2, we have the metric and Maxwell fields [2]

$$ds^2 = \frac{dr^2}{U(r)} + \frac{g(r)}{4}[h(r)\sigma_3^2 + \sigma_1^2 + \sigma_2^2],\ A_1 = \frac{L}{2}\Phi(r)\sigma_1 \text{ and } A_2 = \frac{L}{2}\Phi(r)\sigma_2.$$

We seek for solutions where the operator Hodge dual to A_i has a non-vanishing source, i.e. such that $\Phi_\infty = \lim_{r\to\infty}\Phi(r)$ is not zero.

No-wormhole solution. The solution can be found analytically. The stress-energy tensor $F_i = dA_i = 0$ for $i \in \{1,2\}$ and we recover the same solution as in section 5.2.2 with $h(r) = 1$, $g(r) = r^2$, $U(r) = r^2/l^2 + 1$ and [2]

$$\Phi(r) = \Phi_\infty \frac{\sqrt{l^2 + r^2} - l}{\sqrt{l^2 + r^2} + l}.$$

For solutions with $h \neq 1$ the boundary metrics are not S^3's and this will lead to negative Euclidean modes [2]. The action is given by $S_{\text{no-wormhole}} = 8\pi^2 l^2(1 + 2\Phi_\infty^2)$.

Wormhole solution. An analytic solution could not be found in this case. Instead, we take an ansatz of the form [2]

$$ds^2 = \frac{l^2}{(1-y)^2}\left[f(y)\frac{dy^2}{y(2-y)} + \frac{y_0^2}{4}(g(y)\sigma_3^2 + \sigma_1^2 + \sigma_2^2)\right],$$

where we want to find f and g numerically in terms of $y \in (0,1)$. Here y_0 represents the minimum radius of the wormhole and $y = 0$ is our wormhole plane of symmetry for the $\mathbb{Z}_2$ reflection. In the numerical solution, we find a one-parameter family of solutions with wormholes having S^3's at the two ends. This one-parameter family of solutions is labelled by y_0 and allows us to compute all the remaining quantities of interest, including Φ_∞ and the action for the wormhole.

Similar to previous example, we can see that in Fig. 3 for $\Phi_\infty > 4.0162(5)$ we find a small and a large wormhole solution. For the case of the large wormhole we have no negative Euclidean modes, although the small wormhole does have one negative Euclidean mode in some specific sector [2]. For the case of the large AdS Euclidean wormholes, we have that $\Delta S_{U(1)^2} = 2 * S_{no-wormhole} - S_{wormhole}$ is less than zero for $\Phi_\infty > 4.352(8)$, which tells us that the wormhole solution dominates in this regime.

To determine whether the wormhole is susceptible to brane instabilities, we can compute the contribution of a brane wrapped around the S^3 to the action. Denoted by S_{M2}, as it is the action of a 2-dimensional brane in a String Theory called *M-theory*, if $S_{M2} < 0$ then brane instabilities can arise. As we can observe in Fig. 3, all large wormholes which dominate over the no-wormhole solution do have a negative $S_{U(1)^2}$ and thus are unstable under brane nucleation. In the regime where the wormholes do not present instabilities, we have $\Delta S_{M2} > 0$, so they do not dominate over the no-wormhole solution [2].

What happens when $S_{M2} > 0$? If we explore solutions without S^3 at the boundaries, i.e. $g(1) \neq 1$, we can approximate [2]

$$S_{M2}(y \sim 1) = \frac{\pi^2 (4 - g(1)) \sqrt{g(1)} l^3 y_0}{1 - y} + \mathcal{O}(1).$$

This means that for $0 < g(y) < 4$ (and $g(1) \neq 1$), then S_{M2} will be positive. Unfortunately, within this regime the wormhole solution does not dominate over the no-wormhole solution, although wormholes do not have negative modes either. For this example, no dominant contributions coming from stable wormholes were found. The values of the action are either subdominant or unstable [2].

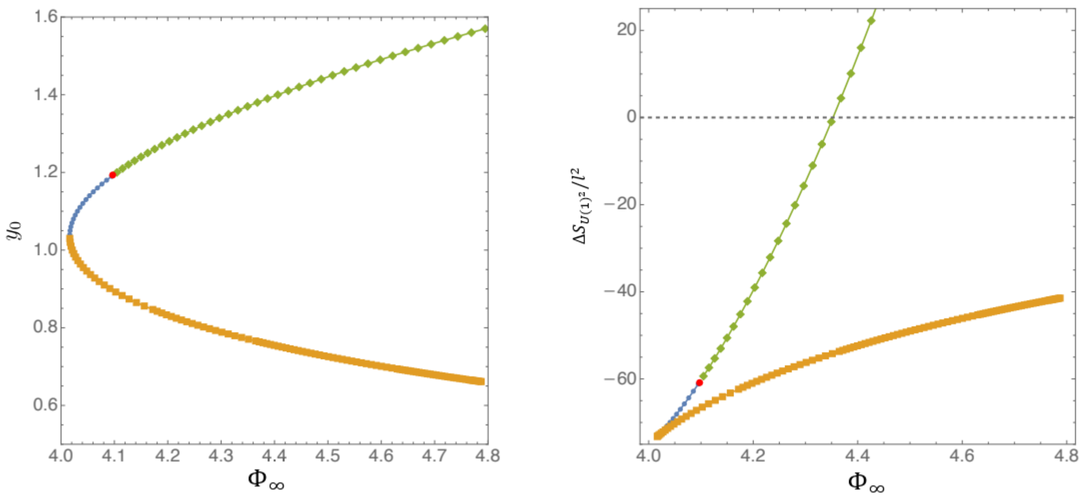

Figure 3: Plots corresponding to Einstein-$U(1)^2$ wormholes in 11-dimensional supergravity. **Left panel:** plot of the radius of the wormhole solutions as a function of the source $\Phi(0)$. Wormholes only exist for $\Phi(0) > 4.0162(5)$. **Right panel:** plot of the difference in action between the wormhole and no-wormhole solution as a function of $\Phi(0)$. The wormhole solutions have a lower action for $\Phi_\infty > 4.0162(5)$. In both plots orange squares correspond to small wormholes, blue disks correspond to large wormholes where the M2 Euclidean action is positive and green diamonds correspond to large wormholes where the Euclidean action for M2 is not positive definite. The red disk indicates the value of $\Phi(0)$ at which the Euclidean action changes to not positive [2].

6 Final discussion

For any newcomer in the wonderful real of Euclidean wormholes, the amount of keywords and distinct notations and approaches will seem, as it did to me, daunting. My goal with this essay is to briefly introduce the literature by constructing a clear path that progressively introduces and connects all the core ideas, and to provide some intuition on the Factorization Problem. I now summarise the key ideas and point towards possible further research and readings.

At the beginning of the essay, we decided to Wick rotate time and focus on Euclidean metrics, arguing that in this framework the path integral has a better chance to converge [5, 6, 7]. Taking as a starting point the ER-bridge, we formally defined what we mean by a wormhole: a connected geometry with more than on compact connected components which we call boundaries. We showed that the Hawking wormhole ticks all the boxes and characterised its boundary by showing it is an asymptotically flat spacetime with 2 ends.

After having laid out the core definitions, we tackled the AdS Euclidean wormholes. Understanding everything that is related to AdS spacetimes is essential for the AdS/CFT duality, which embodies the cornerstone of String Theory computations. Nevertheless, this task is far from trivial.

Introducing locally AdS asymptotics restricts the kinds of wormholes that we can find and renders unstable QFT's under many choices. To begin with, intuition will take you down the wrong road: the vacuum AdS spacetime cannot contain wormholes itself because of its S^n boundary. Moreover, whenever we have $\rho = 0$, by the Witten-Yau's theorem we must have that $k \leq 0$ at the boundaries.

For $\rho > 0$, we can have any kind of curvature at the boundary, but in general we must fine tune the matter we introduce in our theory. The Wick rotation we performed to ease the convergence of the path integral comes right back to bite us, forcing us to choose matter that is gravitationally attractive in Lorentz signature and making our theories unstable through the negative Euclidean modes. For instance, the case with $k > 0$ and some hypothetical spinless and uncharged bosons with low mass known as *axions* is treated in [41], where they concluded that the instabilities arising from these would break the wormholes. Thus, following Marolf and Santos' approach, we committed to studying $\mathbb{Z}_2$ invariant wormholes, where the τ derivatives vanish and these issues are no longer present. If you were to take a guess for the metric of an AdS Euclidean wormhole, chances are that your guess would be unstable or plain wrong. The large amount of restrictions required to have stable AdS Euclidean wormholes make them really hard to come by.

Despite how limited these are, they have been a source of inspiration and issues for the pursuit of Quantum Gravity. If we treat the metrics describing the geometry of spacetime as quantum fields, these can fluctuate for scales of the order of Planck length. This means that Euclidean wormholes can branch off from asymptotically flat or locally asymptotically AdS spacetimes. If we slice through these new regions of spacetime that have branched off we will obtain closed universes which we call *baby universes.*

There is a conflict between the gravitational and field theory perspectives here that underlies the Factorization Problem, initially addressed as addressed as the *loss of quantum coherence.* We have two different regions that are connected through the bulk, but should not interact as these lie infinitely apart. Computing the path integral over all topologies for two boundaries with conditions J_1 and J_2, the result will not factorize into the path integral from J_1 times the path integral from J_2.

For the case of well-behaved, low energy and low dimensional theories, the loss of quantum coherence was sorted by Coleman. The construction of the α-states, a complete set of orthonormal states which fully describes the baby universe Hilbert space $\mathcal{H}_{BU}$, allowed us to recover the factorization property of the path integrals after we had chosen our no-boundary state to be one of these α states. In the ensemble perspective, the fluctuations of constants in nature are the cause of the Factorization Problem. Choosing a fixed value of constants by fixing our no-boundary state removes all ambiguity and thus we recover factorization.

But what happens in the context of AdS/CFT? For the ensemble approach, we assume that the path integrals are convergent, well-behaved and we looked at low-energies, so all these assumptions break down. Furthermore, the most familiar cases of AdS/CFT theories have a set of identifications imposed by supersymmetries. The restrictions, which must be present in all the members of the ensemble, may even limit us to a single physically distinct theory in some cases (see section 5 of [1]). It is unclear how to extend the ensemble theory for most of AdS/CFT examples.

With these issues in mind, we tackled the problem using a different approach: the saddle-point approximation of the path integral. The goal is to approximate the values of the path integrals and compare the wormhole contributions to the no-wormhole contributions.

When dealing with String Theories, a new kind of instability arises: the *brane instabilities*. After introducing some terminology, we reviewed how negatively curved boundaries in high dimensions allow us to create a brane and move it all the way to a boundary to decrease the action. If the boundary has negative curvature, the theory will present brane instabilities [1]. On contrast, if the boundary has positive or zero curvature, it may be stable or unstable depending on further details [2, 22]. For instance, the wormhole from section 5.2.5 does present brane instabilities, despite having S^3's at the boundaries.

The examples studied for the AdS/CFT correspondence are well-behaved only when the scalar curvature of the boundary is non-negative [22]. To have the best chance to avoid brane instabilities, we want to have a theory with positively curved boundaries. Nonetheless, if we want to include wormholes in the theory, Witten-Yau's theorem tells us that we must have matter fields. These conditions narrow down the possibility of having stable wormholes in String Theories to theories with positively curved boundaries and with $\rho \neq 0$.

In the final sections, we review the study of AdS Euclidean wormholes in two examples: the first one in low-energy and the second one in the complete UV-spectrum. For the case of the low-energy in AdS_4 Einstein-Maxwell theory, we have two families of solutions, corresponding to a small and a large wormhole. Marolf and Santos found that the large two-boundary Euclidean wormholes are stable and dominate over disconnected solutions for large enough sources [2]. Thus we cannot ignore their in the path integral.

For the String Theory model, again large wormhole and small wormhole solution were found. The large wormhole solution dominates for large enough sources and have no negative modes. The distinction however, is that brane instabilities are present whenever the wormholes are large enough. If a wormhole dominates, it will not be stable [2].

What does this mean for the AdS/CFT Factorization Problem? For the case of complete UV-theories, no example of stable wormhole that dominates the computation of the partition function was found in [2]. This occurs even when the parameters that we would expect to promote the stability of solutions are tuned to large values [2]. However, some subdominant and stable large wormholes were found for the case of the String Theory model. These must be taken into account when computing the contributions from wormholes. Unless these contributions conspire to cancel themselves in a fashion that we are not able to check, the Factorization Problem remains far from being resolved [1, 2].

On the one hand, there is scope for further research along the lines of AdS Euclidean wormholes. The no-wormhole solution of the model in section 5 of [1] has not been found for comparison, and the regime where the sources become exponentially large and wormholes could turn into the dominant contribution was not probed in [2].

In addition to these unexplored topics, we could follow Coleman's perspective of α-states and establish a duality between AdS gravity in UV-complete theories and an ensemble of quantum theories [1]. Setting aside the problematic Yang-Mills theory with 4-dimensional boundaries where the ensemble interpretation breaks down [1, 2], the wormholes would have associated a large number of constants of nature. This number would be given by the dimension of the baby universe Hilbert space coming from a compact slice of the wormhole. On contrast, given a field theory from [1] corresponding to a choice of constants of nature, Maldacena and Liat claim these are hard to deform to obtain other values for the constants of nature. Chances are that just as the field theory on the boundary fixes such constants, the wavefunction of the baby universes is also fixed by the field theory. Thus we should carefully study the field theory and check whether it contains information on baby universes [1].

On the other hand, we may be not be seeing the solution because we are looking in the wrong direction. Hawking claimed that *quantum gravity has to be formulated in the Euclidean domain*, arguing that real, non-singular Lorentzian metrics do not allow for topology change of spacetime [5]. The Quantum Gravity literature I cited argues that chances of convergence are greater and that Einstein equations in Lorentzian spacetimes with a source with non-negative energy density enforce the conservation of topology [8]. However, examples of non-singular Lorentzian theories with topology change have been found [42].

A different approach would consist in establishing fundamental definitions of the path integral in Lorentzian spacetime, with real Lorentz-signature metrics defining the contour of integration. This would allow us to avoid the conformal factor problem. If possible, the idea is to deform the contour into the complex plane to recover the corresponding Euclidean path integral that we would have in the previous framework [43]. Even though oscillatory integrals are not expected to be well-behaved, they often converge when treated as distributions instead, e.g. the ubiquitous case of the δ-function $2\pi\delta(x) = \int dk e^{ikx}$.

Promisingly, recent work from Marolf, Santos and Mahajan following the publication of [2] has brought to light the double cone spacetime. Following the Lorentzian perspective, which leans on the thermodynamical perspective from [43], the double cone spacetime does not present brane instabilities at the saddle point level and is dominant over the no-wormhole contributions. If its stability under negative field theory modes is confirmed and we cannot obtain further corrections from other components in the bulk, it would violate boundary factorization and force us to have duality not to one well-defined CFT as the AdS/CFT correspondence predicts, but to an ensemble of theories as it was suggested by Coleman.

Not only does God play dice, but... he sometimes throws them where they cannot be seen [44]. We live in fascinating times to study Theoretical Physics.

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A How to visualise a wormhole

Both in the scientific and entertainment realm, wormholes are visualised as a tube connecting two wider ends. We now go over how to arrive at such visualisation and provide a precise definition of some terms that repeatedly arise in the literature.

Definition A.1. *In cylindrical polar coordinates, the Euclidean metric for $\mathbb{R}^{n+1}$ is given by*

$$ds^2_{\mathbb{R}^{n+1}} = dR^2 + dz^2 + R^2 d\Omega^2_{n-1},$$

with $z \in (-\infty, \infty)$ the usual cartesian coordinate and $d\Omega^2_{n-1}$ the round metric for S^{n-1}, with $d\Omega^2_0 = d\phi$ and $\phi \sim \phi + 2\pi$. Here R is the area-radius function, given by $R(p) = \sqrt{A(p)/4\pi}$ where $A(p)$ is the area of the S^{n-1} orbit through the point p, so we must have $n-1$ spherical symmetry to properly define R [14].

To be able to visualize an n-sphere, we embed it into $\mathbb{R}^{n+1}$ Euclidean space. For a n-dimensional Euclidean wormhole, provided it has S^{n-1} spherical symmetry, we can embed it into $\mathbb{R}^{n+1}$ Euclidean space in cylindrical polar coordinates. Without such condition, we will certainly have an embedding into Euclidean space by the Nash embedding theorem [45], but we will require more than $n+1$ dimensions. In this essay, we will restrict to wormholes with S^{n-1} spherical symmetry. Generalising the method from [46], we have the line element

$$ds^2_{\mathbb{R}^{n+1}} = dR^2 + dz^2 + R^2 d\Omega^2_{n-1} = \left(\left(\frac{dR}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2\right) d\sigma^2 + R^2(\sigma) d\Omega^2_{n-1}, \tag{12}$$

for σ the radial coordinate from our wormhole metric. We compare (12) to our expression for the wormhole geometry to find $z(\sigma)$ and $R(\sigma)$. Then we omit all but one angular variable to produce the plots.

Definition A.2 (Throat and proper radius)**.** *Consider a n-dimensional Euclidean wormhole with S^{n-1} spherical symmetry and embed it into $\mathbb{R}^{n+1}$. Then*

- *the **proper radius** of the wormhole is $R(\sigma)$.*
- *the **throat** of the wormhole is the hypersurface that minimizes $R(\sigma)$ [12].*

We now explicitly embed the Hawking's wormhole, which is not done on his paper.

Example A.3 (Embedding of the Hawking wormhole)**.** *Looking at the geometry*

$$ds^2_{\Sigma_t} = \left(1 + \frac{b^2}{r^2}\right)^2 (dr^2 + r^2 d\Omega^2_3),$$

we have $n = 4$ with S^3 spherical symmetry, so we need to embed it into Euclidean $\mathbb{R}^5$. Setting $\sigma = r$, comparison to (12) yields the equations

$$\left(\frac{dz}{dr}\right)^2 + \left(\frac{dR}{dr}\right)^2 = \left(1 + \frac{b^2}{r^2}\right)^2, \tag{13}$$

$$R^2 = r^2 \left(1 + \frac{b^2}{r^2}\right)^2. \tag{14}$$

As $R > 0$ we can only take the positive root in (14). Hence

$$R = r\left(1 + \frac{b^2}{r^2}\right) \implies \frac{dR}{dr} = 1 - \frac{b^2}{r^2} \tag{15}$$

We can identify the minimum at $r = b$ with $R(b) = 2b$, as Hawking claims in [5]. We substitute (15) in (13) to obtain

$$\left(\frac{dz}{dr}\right)^2 = \left(1 + \frac{b^2}{r^2}\right)^2 - \left(\frac{dR}{dr}\right)^2 = \left(1 + \frac{b^2}{r^2}\right)^2 - \left(1 - \frac{b^2}{r^2}\right)^2 = \frac{4b^2}{r^2}.$$

Any root and choice of constant will solve (13). We choose the positive root and set to 0 the integration constant so that

$$\frac{dz}{dr} = \frac{2b}{r} \implies z(r) = 2b\ln(r).$$

Thus we have found

$$R(r) = r\left(1 + \frac{b^2}{r^2}\right) \quad \text{and} \quad z(r) = 2b\ln(r).$$

In Fig. 4 we can see how plotting in $\mathbb{R}^5$ after suppressing the θ_1 and θ_2 coordinates yields the expected tube, with each of the constant z slices representing S^3 as we have omitted two angular coordinates.

Both of these examples provide us with single snapshot of wormholes through the bulk. Yet wormholes also have a dynamical side to them that we are missing through these snapshots: these can open up, extend and then pinch off. This happens in the often disregarded dynamical evolution of the ER-bridge.

Example A.4 (Various snapshots of the Einstein-Rosen bridge)**.** *We focus on the line element in Kruskal coordinates (u, v, θ, ϕ)*

$$ds^2 = \frac{32}{r(u,v)} e^{\frac{r}{2}}(-dv^2 + du^2) + r^2(u,v)d\Omega_2^2,$$

where $u = t - r_*$ and $v = t + r_*$ with r_* the tortoise coordinate [14]. We also have the relation

$$u^2 - v^2 = \left(\frac{r}{2} - 1\right) e^{\frac{r}{2}}. \tag{16}$$

which allows us to express r as a function of u and v [47]. Just as before we took constant t slices, we now focus on constant v slices. Letting $v = v_0$ we find

$$ds^2 = \frac{32}{r(u,v_0)} e^{\frac{r}{2}} du^2 + r^2(u,v_0)d\Omega_2^2,$$

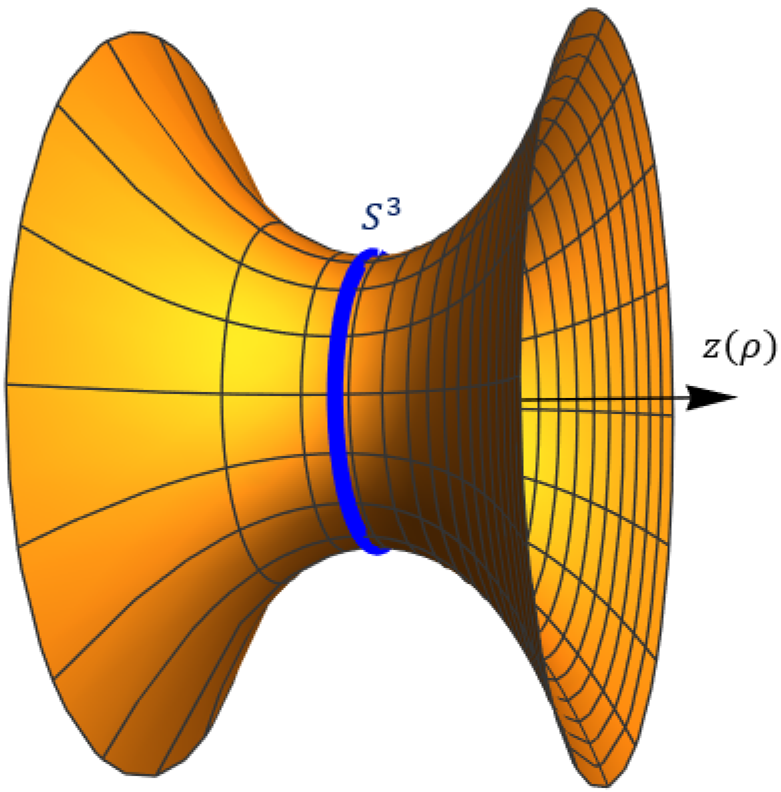

Figure 4: Mathematica plot of the Hawking wormhole embedded in $\mathbb{R}^5$ after suppresing the θ_1 and θ_2 coordinates. Each of the cross-sections of the wormhole as the one through the throat in blue correspond to an S^3.

and we end up with an Euclidean signature metric. We have $n = 3$, so we need to embed this in $\mathbb{R}^4$ Euclidean space. Following the methods from previous example we find [47]

$$R(u) = r(u, v_0) \implies \frac{dR}{du} = \frac{8ue^{\frac{-r}{2}}}{r} \text{ and } z(u) = \int_{r(0)}^{r(u)} \left[\frac{2(e^{\frac{r}{2}} - v_0^2)}{re^{\frac{r}{2}} - 2(e^{\frac{r}{2}} - v_0^2)}\right]^{\frac{1}{2}} dr, \tag{17}$$

where we are restricted to $v_0 \in [-1, 1]$. To see why, notice that from (17) we have

$$\left(\frac{dz}{dr}\right)^2 = \frac{2(e^{\frac{r}{2}} - v_0^2)}{re^{\frac{r}{2}} - 2(e^{\frac{r}{2}} - v_0^2)}.$$

The L.H.S. is positive, so the R.H.S. must be positive. The sign of the R.H.S. is determined by the numerator [47], so we must have that $r \geq 2\ln(v_0^2)$, and equation (16) yields

$$u^2 = v_0^2 + \left(\frac{r}{2} - 1\right) e^{\frac{r}{2}} \geq v_0^2 \ln(v_0^2) = u_0^2.$$

Hence the embedding requires $|v_0| \leq 1 \implies v_0 \in [-1, 1]$*, which also prevents us from touching the singularity* $r = 0$ *described by the hyperbola* $u^2 - v^2 = -1$*. In Fig. 5, we can see that the embedding with* $v_0 = 0$ *gives the well-known ER-bridge. In addition, as either* $v_0 \to -1$ *or* $v_0 \to 1$*, the throat of the wormhole shrinks until almost pinching off at* $v_0 = \pm 1$*. Further values of* u *and* v *can be considered if we impose instead that* $v(u)$ *is a linear function of* $|u|$ *[47], and we would see that the wormhole throat closes.*

As we argued in the introduction, the Einstein equations preserve the topology of spacetime and this example is no exception. It is the geometry of a constant t hypersurface within Kruskal spacetime that is changing, but Kruskal spacetime remains being Kruskal itself. This dynamical aspect of wormholes is essential to understand how, in the Quantum Gravity framework, these allow the creation of separate and closed universes known as *baby universes*.

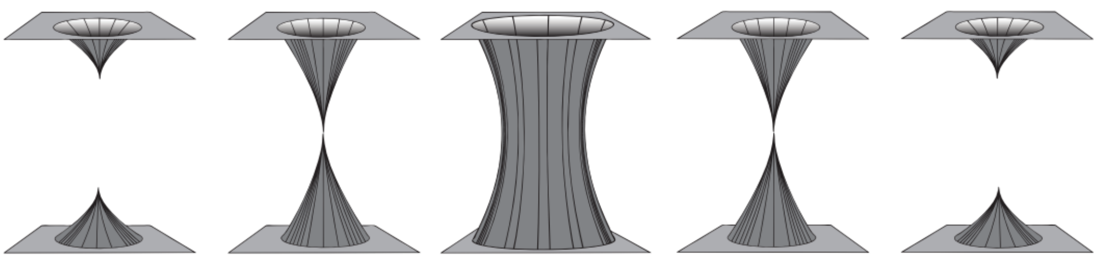

Figure 5: Snapshots of a constant t hypersurface for different fixed values of v. Left to right: Kruskal time $v < -1$, before the bridge opens up; formation of the bridge at $v = -1$; what is normally presented as the ER-bridge at $v = 0$; pinching off at $v = 1$; post separation at Kruskal time $v > 1$ [47].

B Mathematica code

We check that the metric from (6) obeys the Einstein equations for $k \in \{-1, 0, 1\}$. Let

$$S_k(\chi) = \begin{cases} \sin(\chi) & \text{if } k > 0 \\ \chi & \text{if } k = 0 \\ \sinh(\chi) & \text{if } k < 0 \end{cases}$$

With this in mind, following the approach for the FLRW metric from [23], we write (6) as

$$ds^2_{SM^3} = d\tau^2 + \frac{1}{x^2}[d\chi^2 + S_k^2(\chi)d\Omega^2_{n-1}].$$

This is the setup I used for the code in Mathematica, attached in the following pages. As we can observe, for $k = 0$ the metric obeys the vacuum Einstein equations and for $k \neq 0$ we have a non-zero $T_{\mu\nu}$ tensor. In all the cases, $x^2 T_{\mu\nu}$ can easily be extended through $x = 0$ as it presents no divergences here.

Check of the Einstein equations for Santos, Marolf, Liat and Maldacena's wormhole.

Setup of the differential geometry toolkit

In[ ]:=
```
ClearAll;
InverseMetric[g_] := Simplify[Inverse[g]]
ChristoffelSymbol[g_, xx_] := Block[{n, ig, res}, n = 4;
  ig = InverseMetric[g];
  res = Table[(1/2) * Sum[ig[[i, s]] *
       (-D[g[[j, k]], xx[[s]]] + D[g[[j, s]], xx[[k]]] + D[g[[s, k]], xx[[j]]]),
      {s, 1, n}], {i, 1, n}, {j, 1, n}, {k, 1, n}];
  Simplify[res]]
RiemannTensor[g_, xx_] := Block[{n, Chr, res}, n = 4;
  Chr = ChristoffelSymbol[g, xx];
  res = Table[D[Chr[[i, k, m]], xx[[l]]] - D[Chr[[i, k, l]], xx[[m]]] +
     Sum[Chr[[i, s, l]] * Chr[[s, k, m]], {s, 1, n}] - Sum[Chr[[i, s, m]] * Chr[[s, k, l]],
      {s, 1, n}], {i, 1, n}, {k, 1, n}, {l, 1, n}, {m, 1, n}];
  Simplify[res]]
RicciTensor[g_, xx_] := Block[{Rie, res, n}, n = 4;
  Rie = RiemannTensor[g, xx];
  res = Table[Sum[Rie[[s, i, s, j]], {s, 1, n}], {i, 1, n}, {j, 1, n}];
  Simplify[res]]
RicciScalar[g_, xx_] := Block[{Ricc, ig, res, n}, n = 4;
  Ricc = RicciTensor[g, xx];
  ig = InverseMetric[g];
  res = Sum[ig[[s, i]] × Ricc[[s, i]], {s, 1, n}, {i, 1, n}];
  Simplify[res]]
```

Code by "Sjoerd C. de Vries" and "Aries" users from StackExchange: https://mathematica.stackexchange.com/questions/8895/how-to-calculate-scalar-curvature-ricci-tensor-and-christoffel-symbols-in-mathe. Here we will focus on the case of 4 dimensions for simplicity. We define the variables

In[ ]:=
```
xx = {τ, χ, θ, ϕ};
```

0. Check of the AdS spacetime.

First we check that the code above works to check the Einstein equations for the 4-dimensional AdS spacetime metric with l^2 =1. We need Λ = -3.

In[ ]:=
```
gAdS = {{-(1 + χ^2), 0, 0, 0},
   {0, 1/(1 + χ^2), 0, 0}, {0, 0, χ^2, 0}, {0, 0, 0, (Sin[θ])^2 χ^2}};
Simplify[RicciTensor[gAdS, xx] - 1/2 * gAdS * RicciScalar[gAdS, xx] - 3 * gAdS] //
 MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1. Case of k = 0.

We now solve the case with zero curvature, where we simply have the Euclidean metric in spherical polar coordinates. We need Λ = -3.

In[∘]:=
```
g0 = {{1, 0, 0, 0}, {0, (Exp[τ])^2, 0, 0},
   {0, 0, (Exp[τ])^2 χ^2, 0}, {0, 0, 0, (Exp[τ])^2 χ^2 (Sin[θ])^2}};
RicciTensor[g0, xx] - 1/2 * g0 * RicciScalar[g0, xx] - 3 * g0 // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. Case of k =1.

If k=1, then we have Sin[χ]^2 instead of χ^2 itself. By trial and error I found Λ = -3.

In[∘]:=
```
g1 = {{1, 0, 0, 0}, {0, (Exp[τ])^2, 0, 0},
   {0, 0, (Exp[τ])^2 (Sin[χ])^2, 0}, {0, 0, 0, (Exp[τ])^2 (Sin[χ])^2 (Sin[θ])^2}};
Simplify[RicciTensor[g1, xx] - 1/2 * g1 * RicciScalar[g1, xx] - 3 * g1] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -3\, e^{-2\tau} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\mathrm{Sin}[\chi]^2 & 0 \\ 0 & 0 & 0 & -\mathrm{Sin}[\theta]^2\, \mathrm{Sin}[\chi]^2 \end{pmatrix}$$

Thus we obtain a solution of the Einstein equations if we take our stress-energy tensor to be

In[∘]:=
```
-8 π * Simplify[RicciTensor[g1, xx] - 1/2 * g1 * RicciScalar[g1, xx] - 3 * g1] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 24\, e^{-2\tau}\, \pi & 0 & 0 & 0 \\ 0 & 8\pi & 0 & 0 \\ 0 & 0 & 8\pi\, \mathrm{Sin}[\chi]^2 & 0 \\ 0 & 0 & 0 & 8\pi\, \mathrm{Sin}[\theta]^2\, \mathrm{Sin}[\chi]^2 \end{pmatrix}$$

3. Case of k = -1.

If k=-1, then we have Sinh[χ]^2 instead of χ^2 itself. By trial and error I found Λ = -3.

In[∘]:=
```
g2 = {{1, 0, 0, 0}, {0, (Exp[τ])^2, 0, 0},
   {0, 0, (Exp[τ])^2 (Sinh[χ])^2, 0}, {0, 0, 0, (Exp[τ])^2 (Sinh[χ])^2 (Sin[θ])^2}};
Simplify[RicciTensor[g2, xx] - 1/2 * g2 * RicciScalar[g2, xx] - 3 * g2] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 3\, e^{-2\tau} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{Sinh}[\chi]^2 & 0 \\ 0 & 0 & 0 & \mathrm{Sin}[\theta]^2\, \mathrm{Sinh}[\chi]^2 \end{pmatrix}$$

Thus we obtain a solution of the Einstein equations if we take our stress-energy tensor to be

In[ ]:=
```
-8 π * Simplify[RicciTensor[g2, xx] - 1/2 * g2 * RicciScalar[g2, xx] - 3 * g2] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -24\, e^{-2\tau}\pi & 0 & 0 & 0 \\ 0 & -8\pi & 0 & 0 \\ 0 & 0 & -8\pi\, \mathrm{Sinh}[\chi]^2 & 0 \\ 0 & 0 & 0 & -8\pi\, \mathrm{Sin}[\theta]^2\, \mathrm{Sinh}[\chi]^2 \end{pmatrix}$$