Linear scalar instabilites in BTZ and global AdS$_3$

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Response under linear scalar perturbations

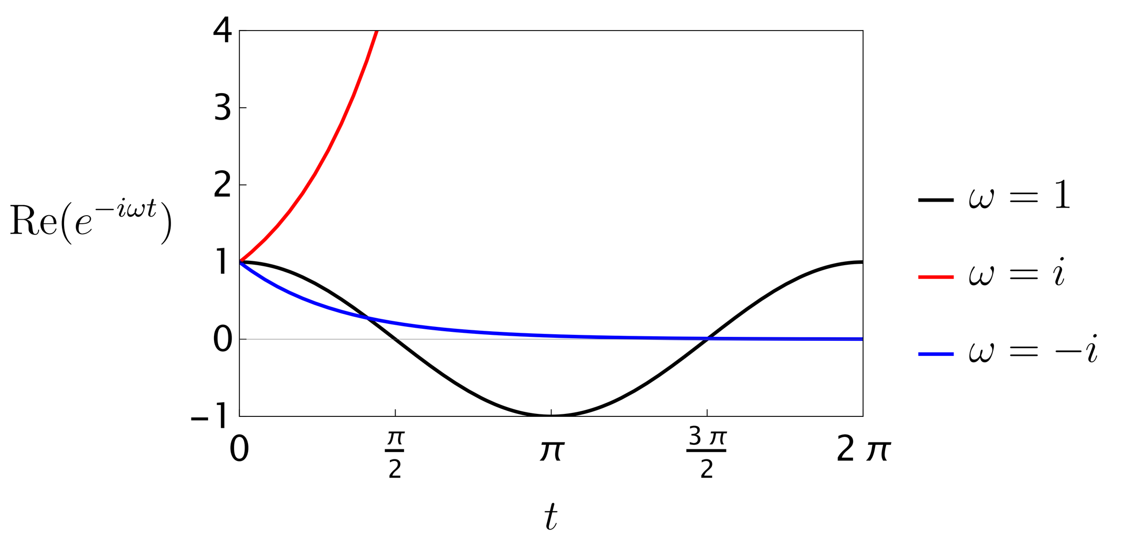
The linear response of a fixed background under a mass μ, neutral, scalar field perturbation Ψ, is found by solving

$$(\nabla_\alpha \nabla^\alpha - \mu^2)\Psi = 0, \qquad (1)$$

and imposing boundary conditions (BCs) which depend on asymptotics [1].

In stationary, axisymmetric backgrounds (as BTZ and global AdS$_3$), we write $\Psi \sim e^{-i\omega t + im\phi}\psi$, with ω the frequency and m the azimuthal number.

If $\begin{cases} \mathrm{Im}(\omega) > 0 \\ \mathrm{Im}(\omega) = 0 \\ \mathrm{Im}(\omega) < 0 \end{cases}$ the solution mode is $\begin{cases} \text{unstable} \\ \text{normal} \\ \text{quasinormal} \end{cases}$



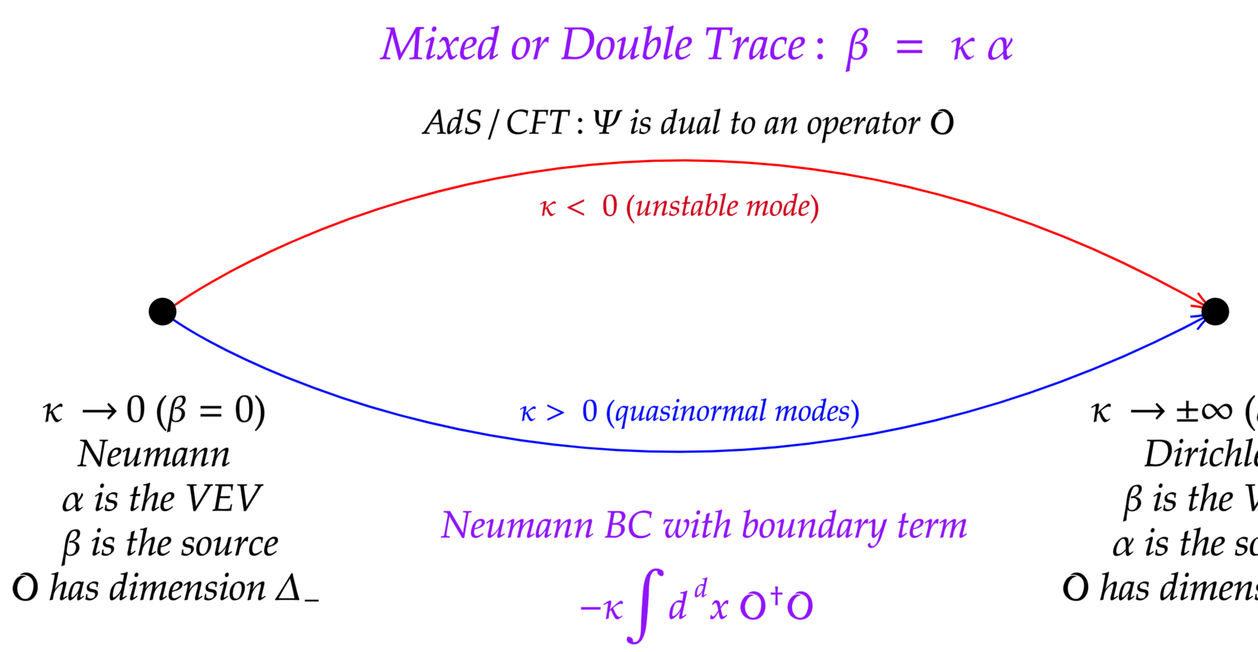
Mixed BCs in asymptotically AdS backgrounds

Asymptotically AdS backgrounds admit different choices of BCs for the scalar Ψ at the timelike boundary, depending on its mass μ.

In Fefferman-Graham coordinates $\{z, x^i\}$, near the timelike boundary ($z = 0$) of AdS$_{d+1}$ with radius L, we have [2]

$$\psi(z,x)|_{z=0} \sim \alpha(x) z^{\Delta_-} + \dots + \beta(x) z^{\Delta_+} + \dots, \quad \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + \mu^2 L^2}.$$

For $-d^2/4L^2 < \mu^2 < -d^2/4L^2 + 1/L^2$, both modes give a finite action (normalizable) [3]. We impose **mixed boundary conditions**: $\beta = \kappa\alpha$.



For $\kappa < 0$, spontaneous symmetry breaking causes instabilities [4].

Mixed BCs in BTZ and global AdS$_3$

In BTZ and global AdS$_3$, (1) with Dirichlet/Neumann BCs can be solved **analytically**. These cases are **stable**: $\mathrm{Im}(\omega) \le 0$ [1].

Global AdS$_3$ + Dirichlet/Neumann BCs $\to$ normal modes ($\mathrm{Im}(\omega) = 0$).
BTZ + Dirichlet/Neumann BCs $\to$ quasinormal modes ($\mathrm{Im}(\omega) < 0$).

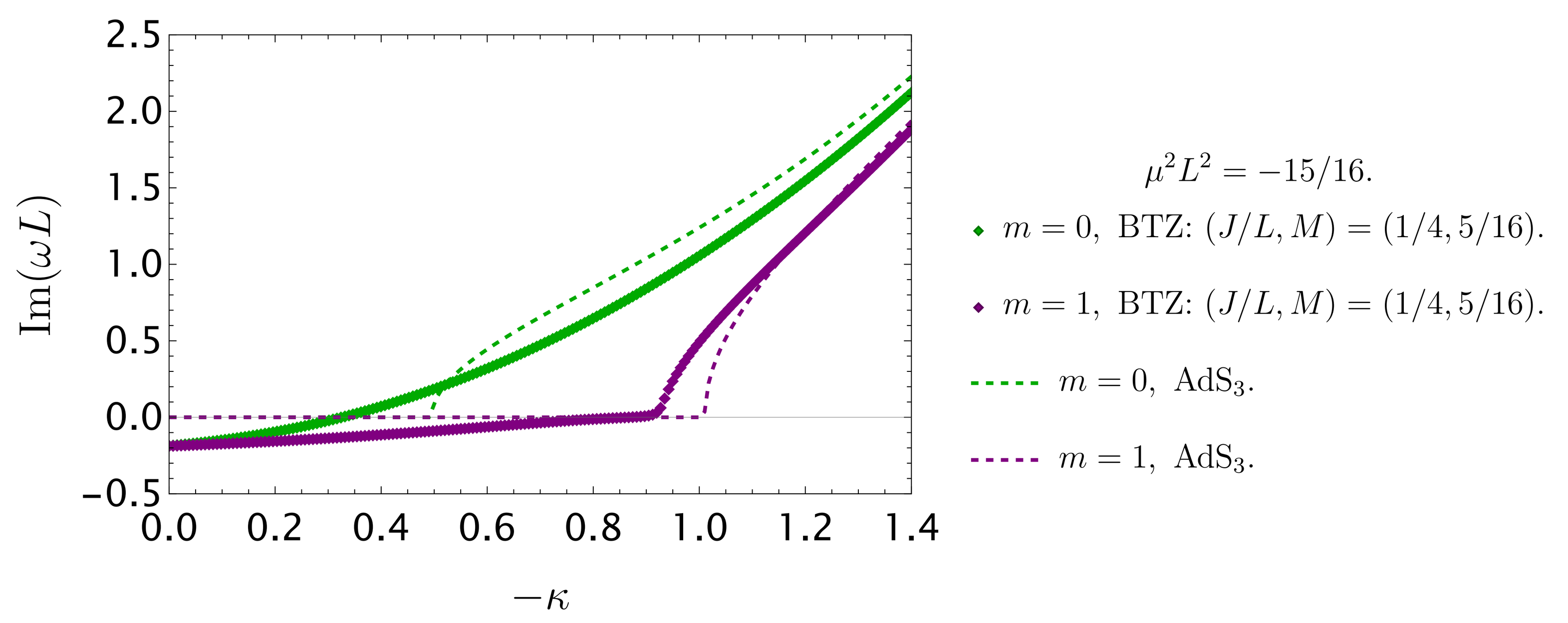
For modes with mixed BCs, we need to find an ω such that

$$\frac{\Gamma(A_1)}{\Gamma(A_2(\omega))\Gamma(A_3(\omega))} - \frac{\kappa}{A_4}\frac{\Gamma(A_5)}{\Gamma(A_6(\omega))\Gamma(A_7(\omega))} = 0, \qquad (2)$$

with A_i also depending on the scalar and background parameters. Hence, to compute these modes, either we solve (1) or (2) **numerically**.

The nature of $\kappa < 0$ instabilities in global AdS$_3$ and BTZ

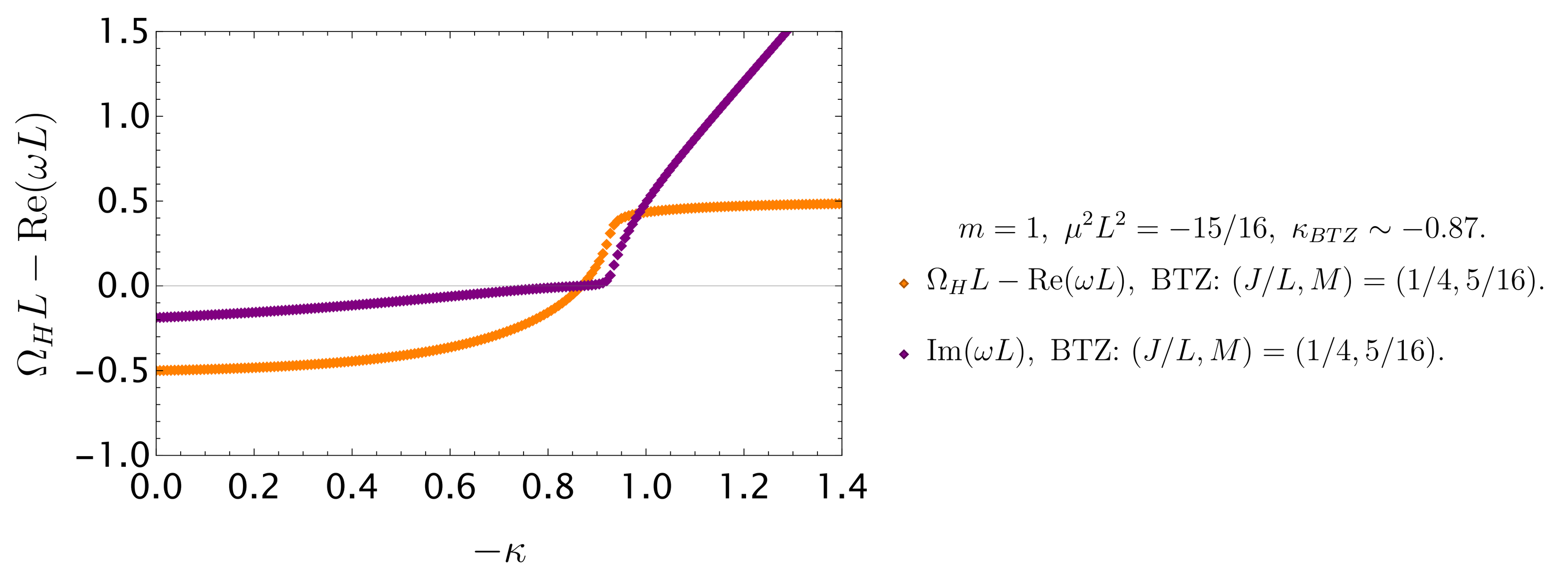
For some $\kappa < 0$, global AdS$_3$ is stable but the dominant mode of BTZ is unstable [5]. Then, as κ decreases, both backgrounds develop instabilities.



Setting $\omega = 0$ in (2) for global AdS$_3$, the onset occurs for

$$\kappa(m,\mu)_{\mathrm{AdS}_3} = \frac{\Gamma\left(-\sqrt{1+\mu^2L^2}\right)\Gamma\left(\frac{1}{2}\left(m+1+\sqrt{1+\mu^2L^2}\right)\right)^2}{\Gamma\left(\sqrt{1+\mu^2L^2}\right)\Gamma\left(\frac{1}{2}\left(m+1-\sqrt{1+\mu^2L^2}\right)\right)^2}. \qquad (3)$$

In BTZ, the onset of the instability occurs when $\omega = m\Omega_H$ [5].

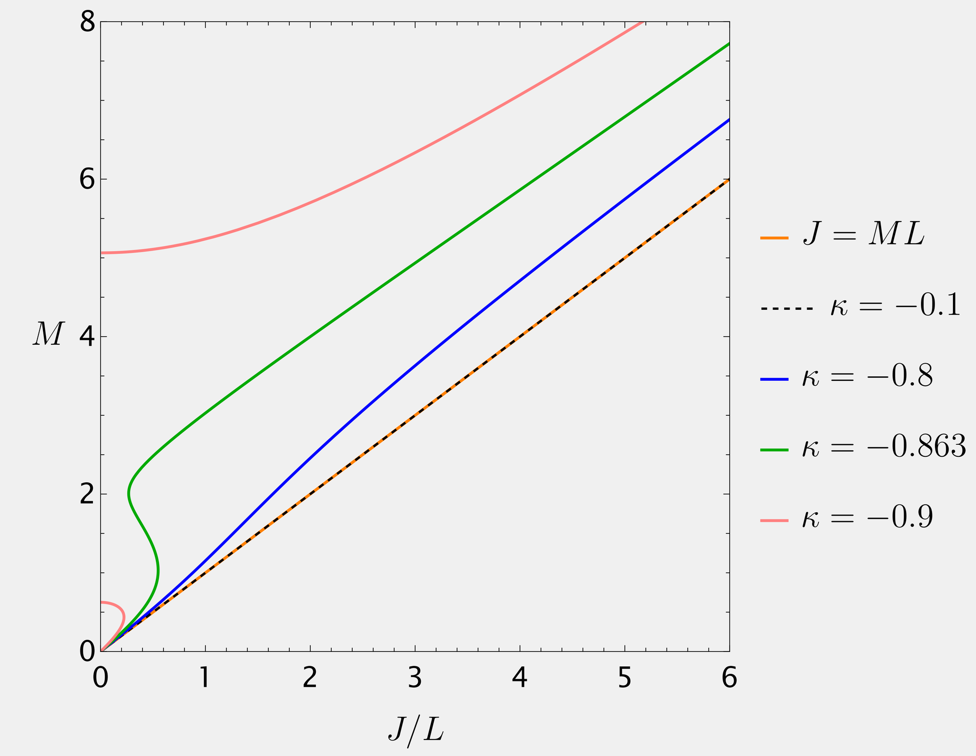


For $m > 0$, the onset of the instability coincides with the onset of **superradiance**: a wave-like analogue of the Penrose process.

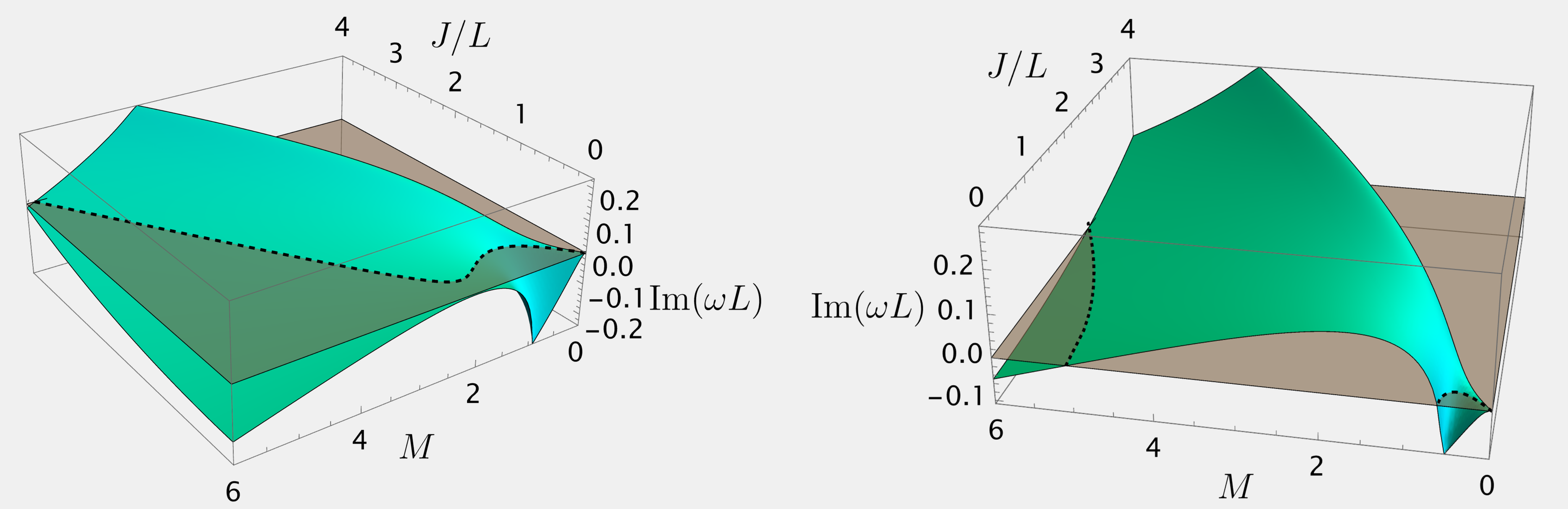
Instability regions for BTZ in (J, M)-space

Fixing κ in (2), we can find the (J, M)-line where the BTZ solution becomes unstable. We consider $\kappa \in (\kappa_{\mathrm{AdS}_3}(m,\mu), 0)$, where AdS$_3$ is stable.

For $m = 1$, $\mu^2 L^2 = -15/16$, $\kappa_{\mathrm{AdS}_3} \sim -1.01$ and the onset contours are



Regions bounded by the $J = M\,L$ line and a fixed κ contour are unstable under the least damped mode. For $\kappa = -0.863$ (left) and $\kappa = -0.9$ (right)



Conclusions and outlook

For $m = 1$ and $\kappa \in (\kappa_{\mathrm{AdS}_3}, 0)$, there is a region where BTZ is unstable (onset with superradiance) and global AdS$_3$ is stable. We believe the story extends qualitatively for $m > 1$. We need to understand the $m = 0$ instability.

Hairy BTZ black holes with $\kappa < 0$ exist [6, 7]. Our next goal is to study the **non-linear problem**, with full control of κ and the thermodynamics.

References

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