CSC446 Project

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Overview

The Kuramoto-Sivashinsky equation is known as the simplest chaotic PDE [1].

$$u_t + u_{xxxx} + u_{xx} + \frac{1}{2}|u_x|^2 = 0, u = u(x,t)$$

In this project, we will try to solve 1-D K-S equation using different schemes of finite difference method and compare their stability in different time scale. Also we will investigate some properties of chaotic dynamical systems that presented in the solutions of K-S equation.

K-S Equation

First we can try to understand the K-S equation by its components. First consider the heat equation

$$u_t = u_{xx}$$

As we know the heat equation "smooths" out the peak value as the heat diffuses to the surrounding. In the K-S equation, we have the reverse of the heat equation

$$u_t = -u_{xx}$$

which amplifies or magnifies small regions of high temperature. This part takes account to the chaotic behaviour of the K-S equation. However, instead of growing completely chaotic,

$$u_t = -u_{xx} - u_{xxxx}$$

the 4th order derivative makes the sharp spikes in u tend to damped out exponentially [2]. Lastly, the nonlinear part

$$u_t = -u_{xx} - u_{xxxx} - \frac{1}{2}|u_x|^2$$

stabilizes the solution [3]. Notice that the solution to the K-S equation has the form

$$u(x,t) = -c_0^2 t + v(x,t)$$

where $c_0^2 \approx 1.2$ is a universal constant independent of the initial condition. $v(x,t)$ here is irregular but has properties of a quasi-periodic wave with a characteristic wave length $l_0 = 2\pi/\omega_0, \omega_0 = \sqrt{2}/2$ [4]. Then for the steady solution, we have

$$u(x,t) = -c_0^2 t + v(x)$$

which will be used in Leapfrog method.

Numerical solution

In this section, we will use the Euler, Leapfrog and Lax-Wendroff methods. Notice that all these methods are using the same setup of number of time steps N and number of grid m but using different total time for some of the methods due to the chaotic behaviour. The range of x is $[-5, 5]$ with initial condition as Gaussian with $\sigma = 2$ [5].

Firstly for the Euler's method, we use $T = 10^{-12}$.

m	N	e_h^k	e_{2h}^{2k}/e_h^k	p_h^k
10	25	2.60e-9	-	-
20	50	2.03e-9	0.36	0.36
40	100	2.00e-8	0.07	-3.30
80	200	1.68e-7	0.07	-3.07
160	400	1.33e-6	0.14	-2.98

For the Leapfrog, we use $T = 10^{-5}$. Notice that in Leapfrog, we need two initial conditions to continue the induction. So we use the steady solution to initialize the initial conditions.

m	N	e_h^k	e_{2h}^{2k}/e_h^k	p_h^k
10	25	4.89e-7	-	-
20	50	7.52e-9	64.99	6.02
40	100	9.40e-9	0.80	-0.32
80	200	3.71e-8	0.25	-1.98
160	400	1.48e-7	0.25	-1.99

For the Lax-Wendroff methods, we use $T = 10^{-5}$.

m	N	e_h^k	e_{2h}^{2k}/e_h^k	p_h^k
10	25	8.90e-9	-	-
20	50	7.28e-9	1.22	0.29
40	100	9.36e-9	0.78	-0.36
80	200	3.71e-8	0.25	-1.99
160	400	1.48e-7	0.25	-1.99

Overall, we can see that the error increases in all the method when we using smaller grid size and time steps. On the other hand, the Leapfrog and Lax-Wendroff methods are more stable than the Euler's method. Once the total time T larger than 10^{-12}, due to the chaotic behaviour of K-S equation, the error will explode for the last row of choice of m and N. Compare between the Leapfrog method and the Lax-Wendroff method, the latter shows a better stability than the other method. This matches our results from A3.

Chaotic Properties

For any chaotic system, the model should satisfy the following properties:

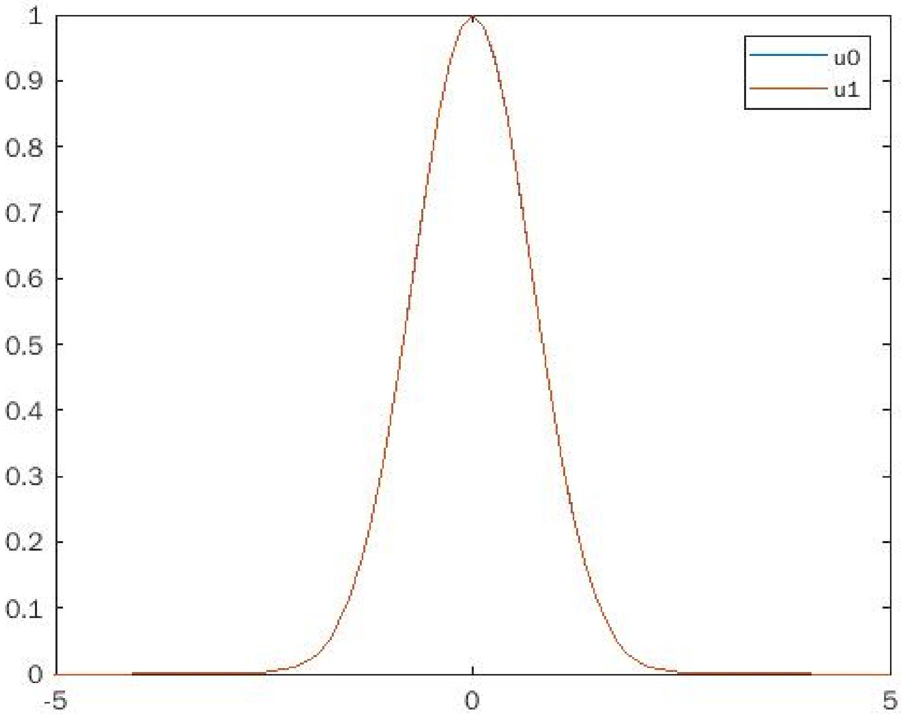
- For any two different individual initial conditions, the motion in this vector field will be completely different no matter how small the difference is. In another word, the future development is very sensitive to the initial conditions.
- For any large set of data, the over all solutions have some special trends. In another word, the chaotic system is statistically stable.

Here we have a setup that the range of x is $[-10, 10]$ with initial condition as Gaussian with $\sigma = 1$. Also we have number of grid $m = 150$, total time $T = 7.5 * 10^{-4}$ and number of time step $N = 50$. However we change make two plots with a small change in initial condition that $x0 = 0.001, x1 = 0.002$ with

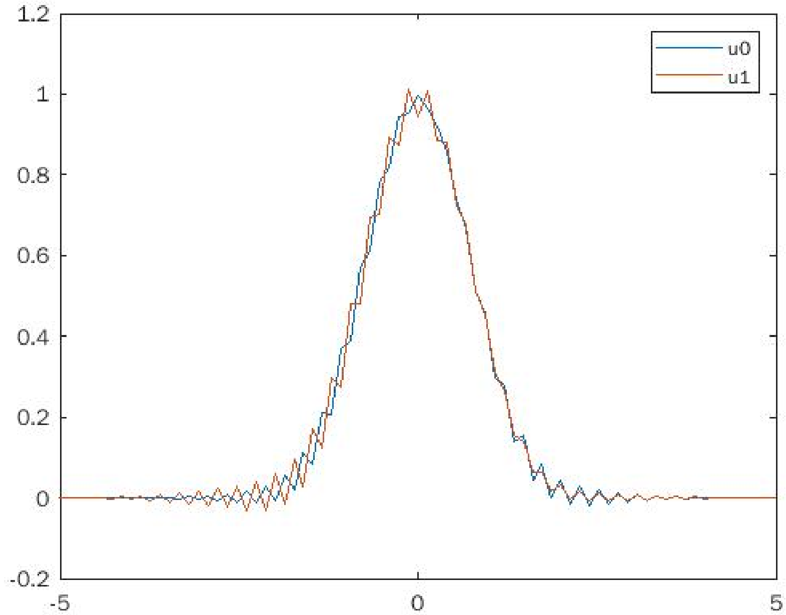
$$u_0 = e^{-(\frac{(x-x0)}{\sigma})^2}$$

$$u_1 = e^{-(\frac{(x-x1)}{\sigma})^2}$$

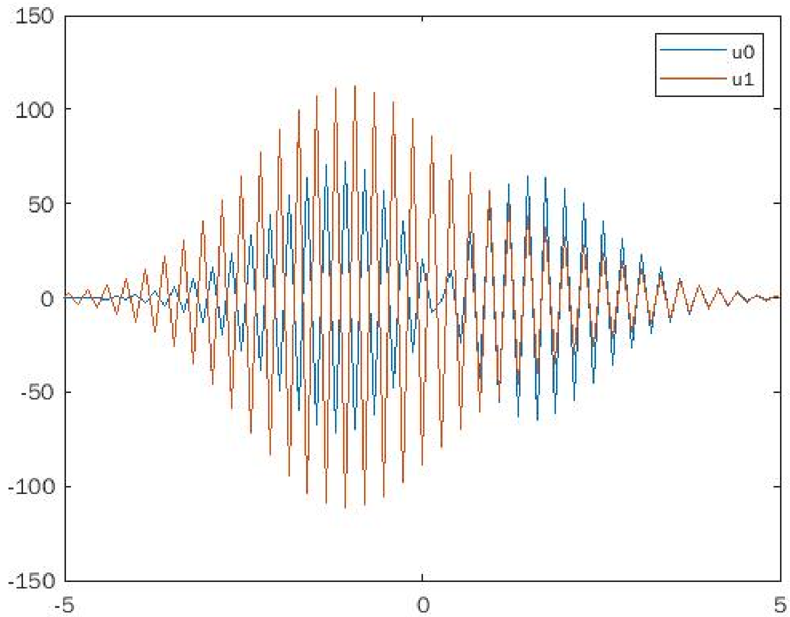
Here is a graph of our initial condition magnified to $x = [-5, 5]$.



Currently, it's hard to distinguish between two plots. Then after 38 time steps, we have the following graph,



As we can see, both plots start to fluctuate in different x value. As time goes on, the chaotic behaviour becomes more and more significant. At the end, we have the graph



Here, u_0 and u_1 develops into two complete different graph but we can see that the frequency of how they oscillating is same as we mention in the earlier part. We verified the sensitive dependence on initial condition in the above graph. Now we check the statistically stable.

Grid	% of points of u_0	% of points of u_1
1	5	5
2	9	7
3	12	11
4	16	15
5	24	28
6	85	85
7	88	89
8	91	92
9	95	95
10	100	100

Since they have different extreme value, we divide each plot of u_0 and u_1 into 10 equally spaced grid. Then we calculate the cumulative percentage of points in and above this grid. As we can see, although the plots are completely different, the percentage of points in each grid are the same. This result is consistent if we change the grid size. Therefore we have proven the statistically stable properties of K-S equation.

Reference

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