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Numerical solutions of the generalized Kuramoto–Sivashinsky equation using B-spline functions

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ABSTRACT

A numerical technique based on the finite difference and collocation methods is presented for the solution of generalized Kuramoto–Sivashinsky (GKS) equation. The derivative matrices between any two families of B-spline functions are presented and are utilized to reduce the solution of GKS equation to the solution of linear algebraic equations. Numerical simulations for five test examples have been demonstrated to validate the technique proposed in the current paper. It is found that the simulating results are in good agreement with the exact solutions.

1. Introduction

As is said in [1] the generalized Kuramoto–Sivashinsky (GKS) equation is a model of nonlinear partial differential equation (NLPDE) frequently encountered in the study of continuous media which exhibits a chaotic behavior form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^4 u}{\partial x^4} = 0, \tag{1.1}$$

where α, β and γ are nonzero real constants.

For $\beta = 0$, Eq. (1.1) is called the Kuramoto–Sivashinsky (KS) equation which is a canonical nonlinear evolution equation arising in a variety of physical contexts, e.g. long waves on thin films, unstable drift waves in plasmas, reaction diffusion systems [2], etc. This equation was originally derived in the context of plasma instabilities, flame front propagation, and phase turbulence in reaction–diffusion system [3]. It appears in context of long waves on the interface between two viscous fluids [4], unstable drift waves in plasmas, and flame front instability [5]. The Kuramoto–Sivashinsky equation models the fluctuations of the position of a flame front, the motion of a fluid going down a vertical wall, or a spatially uniform oscillating chemical reaction in a homogeneous medium [6]. This equation is useful to model solitary pulses in a falling thin film [7]. As is mentioned in [8] this equation can be precisely recovered from a model in the continuity equation by adding an appropriate amending-function.

For $\alpha = \gamma = 1$ and $\beta = 0$ it represents models of pattern formation on unstable flame fronts and thin hydrodynamic films [5], Eq. (1.1) has thus been studied extensively [9,10].

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In the past several years ago, various methods have been proposed to solve this equation. Authors of [1] presented the Chebyshev spectral collocation scheme to solve this equation. This equation is solved by lattice Boltzmann technique [8]. The method of radial basis functions [11,12] developed in [13] to find the approximate solution of this equation. Authors of [14] proposed the local discontinuous Galerkin methods to solve this equation. The tanh function method is developed in [15] and author of [16] investigated this equation and proposed a technique based on the variational iteration method [17]. More investigations can be found in [18,19].

It is worth to point out that, recently various methods have been proposed to construct exact solutions of Kuramoto–Sivashinsky equation. For more details we refer the interested reader to [20–23]. Also many methods were developed for finding exact solutions of some nonlinear evolution equations [24–28].

We would like to mention that this equation is also called KdV–Burgers–Kuramoto (KBK) equation. Thus the interested reader can see [16,29,30].

The approach in the current paper is different. A numerical technique based on the finite difference [31,32] and collocation methods [33,34] is proposed for the solution of generalized Kuramoto–Sivashinsky (GKS) equation. In this work we reduce the given problem to a set of algebraic equations by expanding the unknown function as fifth order B-spline functions specially constructed on bounded intervals, with unknown coefficients. The derivative matrices between any two families of B-spline functions are given. These matrices together with the B-spline functions are then utilized to evaluate the unknown coefficients.

This paper is organized as follows: In Section 2, we describe the formulation of the B-spline functions on $[a,b]$, and the derivative matrices required for our subsequent development. In Section 3 the proposed method is used to approximate the solution of the problem in interval $[a,b]$. As the results a set of algebraic equations is formed and a solution of the considered problem is introduced. In Section 4 the theoretical stability of the linearized scheme is presented. In Section 5, we report our computational results and demonstrate the accuracy of the proposed approximate scheme by presenting numerical examples. Section 6 completes this paper with a brief conclusion. Note that we have computed the numerical results by Maple programming.

2. B-spline functions on $[a,b]$

The mth-order cardinal B-spline $N_m(t)$ has the knot sequence $\{\ldots,-1,0,1,\ldots\}$ and consists of polynomials of order m (degree $m-1$) between the knots. Let $N_1(t)=\chi_{[0,1]}(t)$ be the characteristic function of $[0,1]$. Then for each integer $m \geqslant 2$, the mth-order cardinal B-spline is defined, inductively by [35,36]

$$N_m(t) = (N_{m-1} * N_1)(t) = \int_{-\infty}^{\infty} N_{m-1}(t-x)N_1(x)dx = \int_0^1 N_{m-1}(t-x)dx. \tag{2.1}$$

It can be shown [37] that $N_m(t)$ for $m \geqslant 2$ can be achieved by using the following formula

$$N_m(t) = \frac{t}{m-1}N_{m-1}(t) + \frac{m-t}{m-1}N_{m-1}(t-1),$$

recursively, and $\mathrm{supp}[N_m(t)] = [0,m]$.

It can be seen [36] that a finite-energy sequence $g[m,k]$ exists such that the relation

$$N_m(t) = \sum_k g[m,k]N_m(2t-k), \tag{2.2}$$

where

$$g[m,k] = \begin{cases} 2^{-m+1}\binom{m}{k}, & 0 \leqslant k \leqslant m, \\ 0, & elsewhere, \end{cases}$$

is satisfied. The explicit expressions of $N_2(t)$, $N_3(t)$, $N_4(t)$ and $N_5(t)$ are [35–37]:

$$N_2(t) = \begin{cases} t, & t \in [0,1], \\ 2-t, & t \in [1,2], \\ 0, & \text{elsewhere}, \end{cases} \tag{2.3}$$

$$N_3(t) = \frac{1}{2}\begin{cases} t^2, & t \in [0,1], \\ -2t^2+6t-3, & t \in [1,2], \\ t^2-6t+9, & t \in [2,3], \\ 0, & \text{elsewhere}, \end{cases} \tag{2.4}$$

$$N_4(t) = \frac{1}{6}\begin{cases} t^3, & t \in [0,1], \\ 4 - 12t + 12t^2 - 3t^3, & t \in [1,2], \\ -44 + 60t - 24t^2 + 3t^3, & t \in [2,3], \\ 64 - 48t + 12t^2 - t^3, & t \in [3,4], \\ 0, & \text{elsewhere}, \end{cases} \tag{2.5}$$

$$N_5(t) = \frac{1}{24}\begin{cases} t^4, & t \in [0,1], \\ -5 + 20t - 30t^2 + 20t^3 - 4t^4, & t \in [1,2], \\ 155 - 300t + 210t^2 - 60t^3 + 6t^4, & t \in [2,3], \\ -655 + 780t - 330t^2 + 60t^3 - 4t^4, & t \in [3,4], \\ 625 - 500t + 150t^2 - 20t^3 + t^4, & t \in [4,5], \\ 0, & \text{elsewhere}. \end{cases} \tag{2.6}$$

Suppose $N_{i,j,k}(t) = N_i(2^j t - k),\ i = 1,2,3,4,5,\ j,k \in \mathbb{R}$ and $B_{i,j,k} = \text{supp}[N_{i,j,k}(t)] = clos\{t : N_{i,j,k}(t) \neq 0\}$. It is easy to see that

$$B_{i,j,k} = [2^{-j}k, 2^{-j}(i+k)], \quad i = 1,\ldots,5\ j,k \in \mathbb{R}.$$

Define the set of indices

$$S_{i,j} = \{k : B_{i,j,k} \cap (a,b) \neq \emptyset\}, \quad i = 1,\ldots,5,\ j \in \mathbb{R},$$

and suppose $\mathfrak{m}_{i,j} = min\{S_{i,j}\}$ and $\mathfrak{M}_{i,j} = max\{S_{i,j}\},\ i = 1,\ldots,5,\ j \in \mathbb{R}$.

We need that these functions intrinsically defined on $[a,b]$ so we put

$$N^i_{j,k}(t) = N_{i,j,k}(t)\chi_{[a,b]}(t), \quad i = 1,\ldots,5,\ j \in \mathbb{R},\ k \in S_{i,j}. \tag{2.7}$$

2.1. The function approximation

Suppose

$$\Phi_{i,j}(t) = \left[N^i_{j,\mathfrak{m}_{i,j}}(t), N^i_{j,\mathfrak{m}_{i,j}+1}(t), \ldots, N^i_{j,\mathfrak{M}_{i,j}}(t)\right]^T, \quad i = 1,\ldots,5,\ j \in \mathbb{R}. \tag{2.8}$$

For a fixed $j = J$, a function $f(t)$ defined over $[a,b]$ may be represented by the fifth order B-spline functions as

$$f(t) = \sum_{k=\mathfrak{m}_{5,J}}^{\mathfrak{M}_{5,J}} s_k N^5_{J,k}(t) = S^T \Phi_{5,J}(t), \tag{2.9}$$

where

$$S = [s_{\mathfrak{m}_{5,J}}, s_{\mathfrak{m}_{5,J}+1}, \ldots, s_{\mathfrak{M}_{5,J}}]^T, \tag{2.10}$$

where $s_k,\ k = \mathfrak{m}_{5,J}, \ldots, \mathfrak{M}_{5,J}$, can be found using the method presented in [38].

2.2. The derivative matrices

Using Eq. (2.1) we get [36]

$$N_m(t)' = \frac{d}{dt}N_m(t) = N_{m-1}(t) - N_{m-1}(t-1). \tag{2.11}$$

Now using Eqs. (2.8) and (2.11) we obtain

$$\Phi'_{i,J}(t) = D_{i,J}\Phi_{i-1,J}(t), \quad i = 2,3,4,5, \tag{2.12}$$

where

$$D_{i,J} = 2^J \times \begin{bmatrix} -1 & & & \\ 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}, \quad i = 2,3,4,5,$$

and $D_{i,J}$ is a $(\mathfrak{M}_{i,J} - \mathfrak{m}_{i,J}) \times (\mathfrak{M}_{i,J} - \mathfrak{m}_{i,J} - 1)$ matrix. Because $\text{supp}[N_m(t)] = [0,m]$, it can be found that $\mathfrak{M}_{i,J} - \mathfrak{m}_{i,J} - 1 = \mathfrak{M}_{i-1,J} - \mathfrak{m}_{i-1,J}$.

3. Description of the numerical method

Consider the GKS equation (1.1) with initial value

$$u(x,0) = h(x), \quad x \in [a,b], \tag{3.1}$$

and boundary conditions

$$\begin{cases} u(a,t) = f_1(t), \\ u(b,t) = f_2(t), \\ u_x(a,t) = g_1(t), \\ u_x(b,t) = g_2(t), \end{cases} \tag{3.2}$$

where $h(x)$, $f_1(t)$, $f_2(t)$, $g_1(t)$ and $g_2(t)$ are known functions.

Here we introduce a numerical scheme to solve the problem.

3.1. *Scheme based on Crank–Nicolson time stepping*

By integrating form Eq. (1.1) with respect to "t" in the interval $[t, t+\delta t]$ we get

$$u(x,t+\delta t) - u(x,t) + \int_t^{t+\delta t} u(x,t)u_x(x,t)dt + \alpha \int_t^{t+\delta t} u_{xx}(x,t)dt + \beta \int_t^{t+\delta t} u_{xxx}(x,t)dt + \gamma \int_t^{t+\delta t} u_{xxxx}(x,t)dt = 0. \tag{3.3}$$

Using the trapezoid method we can approximate Eq. (3.3) as

$$\begin{aligned} &u(x,t+\delta t) - u(x,t) + \frac{\delta t}{2}\{u(x,t+\delta t)u_x(x,t+\delta t) + u(x,t)u_x(x,t)\} + \alpha\frac{\delta t}{2}\{u_{xx}(x,t+\delta t) + u_{xx}(x,t)\} + \beta\frac{\delta t}{2}\{u_{xxx}(x,t+\delta t) \\ &\quad + u_{xxx}(x,t)\} + \gamma\frac{\delta t}{2}\{u_{xxxx}(x,t+\delta t) + u_{xxxx}(x,t)\} = 0, \end{aligned} \tag{3.4}$$

where δt is the time step size. To linearize the non-linear term $u(x,t+\delta t)u_x(x,t+\delta t)$ we use the linearization form given by Rubin and Graves [39]

$$u(x,t+\delta t)u_x(x,t+\delta t) \approx u(x,t+\delta t)u_x(x,t) + u(x,t)u_x(x,t+\delta t) - u(x,t)u_x(x,t). \tag{3.5}$$

Replacing Eq. (3.5) in Eq. (3.4), then rearranging it, and using the notation $u^n = u(x,t^n)$ where $t^n = t^{n-1} + \delta t$, we obtain

$$u^{n+1} - u^n + \frac{\delta t}{2}\left\{u^{n+1}\nabla u^n + u^n\nabla u^{n+1} + \alpha\left(\nabla^2 u^{n+1} + \nabla^2 u^n\right) + \beta\left(\nabla^3 u^{n+1} + \nabla^3 u^n\right) + \gamma\left(\nabla^4 u^{n+1} + \nabla^4 u^n\right)\right\} = 0, \tag{3.6}$$

where ∇ is the gradient operator.

Using Eq. (2.9), the function $u(x,t^n)$ can be approximated as

$$u^n(x) = \sum_{k=\mathfrak{m}_{5,J}}^{\mathfrak{M}_{5,J}} \mathrm{u}_k^n N_{J,k}^5(x) = U_n^T \Phi_{5,J}(x), \tag{3.7}$$

where $U_n = \left[\mathrm{u}^n_{\mathfrak{m}_{5,J}}, \mathrm{u}^n_{\mathfrak{m}_{5,J}+1}, \ldots, \mathrm{u}^n_{\mathfrak{M}_{5,J}}\right]^T$, and $\Phi_{5,J}(x)$ is defined as (2.8).

Also using (2.12) we can write

$$\nabla u^n = U_n^T \frac{d}{dx}\Phi_{5,J}(x) = U_n^T D_{5,J}\Phi_{4,J}(x), \tag{3.8}$$

$$\nabla^2 u^n = U_n^T \frac{d^2}{dx^2}\Phi_{5,J}(x) = U_n^T D_{5,J}\frac{d}{dx}\Phi_{4,J}(x) = U_n^T D_{5,J}D_{4,J}\Phi_{3,J}(x), \tag{3.9}$$

$$\nabla^3 u^n = U_n^T \frac{d^3}{dx^3}\Phi_{5,J}(x) = U_n^T D_{5,J}D_{4,J}D_{3,J}\Phi_{2,J}(x), \tag{3.10}$$

$$\nabla^4 u^n = U_n^T \frac{d^4}{dx^4}\Phi_{5,J}(x) = U_n^T D_{5,J}D_{4,J}D_{3,J}D_{2,J}\Phi_{1,J}(x). \tag{3.11}$$

Replacing Eqs. (3.7)–(3.11) in Eq. (3.6) we get

$$\begin{aligned} &\left\{\Phi_{5,J}^T(x)\left[1 + \frac{\delta t}{2}\left(\Phi_{4,J}^T(x)D_{5,J}^T U_n + U_n\Phi_{4,J}^T(x)D_{5,J}^T\right)\right] + \frac{\delta t}{2}\left(\alpha\Phi_{3,J}^T(x)D_{4,J}^T D_{5,J}^T + \beta\Phi_{2,J}^T(x)D_{3,J}^T D_{4,J}^T D_{5,J}^T + \gamma\Phi_{1,J}^T(x)D_{2,J}^T D_{3,J}^T D_{4,J}^T D_{5,J}^T\right)\right\}U_{n+1} \\ &\quad = \left\{\Phi_{5,J}^T(x) - \frac{\delta t}{2}\left(\alpha\Phi_{3,J}^T(x)D_{4,J}^T D_{5,J}^T + \beta\Phi_{2,J}^T(x)D_{3,J}^T D_{4,J}^T D_{5,J}^T + \gamma\Phi_{1,J}^T(x)D_{2,J}^T D_{3,J}^T D_{4,J}^T D_{5,J}^T\right)\right\}U_n. \end{aligned} \tag{3.12}$$

Using Eqs. (3.7) and (3.8) in (3.2) we have

$$\begin{aligned} \Phi_{5,J}^T(a)U_{n+1} &= f_1(t^{n+1}),\\ \Phi_{5,J}^T(b)U_{n+1} &= f_2(t^{n+1}),\\ \Phi_{4,J}^T(a)D_{5,J}^T U_{n+1} &= g_1(t^{n+1}),\\ \Phi_{4,J}^T(b)D_{5,J}^T U_{n+1} &= g_2(t^{n+1}). \end{aligned} \tag{3.13}$$

Collocating Eq. (3.12) in $\ell = \mathfrak{M}_{i,J} - \mathfrak{m}_{i,J} - 4$ points $x_i = i(b-a)/(\ell+1)+a$, $i = 1,2,\ldots,\ell$ we get

$$\begin{aligned} &\left\{\Phi_{5,J}^T(x_i)\left[1+\frac{\delta t}{2}\left(\Phi_{4,J}^T(x_i)D_{5,J}^T U_n + U_n\Phi_{4,J}^T(x_i)D_{5,J}^T\right)\right] + \frac{\delta t}{2}\left(\alpha\Phi_{3,J}^T(x_i)D_{4,J}^T D_{5,J}^T + \beta\Phi_{2,J}^T(x_i)D_{3,J}^T D_{4,J}^T D_{5,J}^T + \gamma\Phi_{1,J}^T(x_i)D_{2,J}^T D_{3,J}^T D_{4,J}^T D_{5,J}^T\right)\right\}U_{n+1}\\ &\quad = \left\{\Phi_{5,J}^T(x_i) - \frac{\delta t}{2}\left(\alpha\Phi_{3,J}^T(x_i)D_{4,J}^T D_{5,J}^T + \beta\Phi_{2,J}^T(x_i)D_{3,J}^T D_{4,J}^T D_{5,J}^T + \gamma\Phi_{1,J}^T(x_i)D_{2,J}^T D_{3,J}^T D_{4,J}^T D_{5,J}^T\right)\right\}U_n. \end{aligned} \tag{3.14}$$

Writing (3.14) together with (3.13) in a matrix form we have

$$A_n U_{n+1} = b_n, \quad n = 1,2,\ldots, \tag{3.15}$$

where A_n is a $(\ell+3)\times(\ell+3)$ matrix as

$$A_n := \begin{bmatrix} \Phi_{5,J}^T(a)\\ \Phi_{4,J}^T(a)D_{5,J}^T\\ \xi_1\\ \vdots\\ \xi_\ell\\ \Phi_{5,J}^T(b)\\ \Phi_{4,J}^T(b)D_{5,J}^T \end{bmatrix},$$

and

$$b_n := \begin{bmatrix} f_1(t^{n+1})\\ g_1(t^{n+1})\\ \zeta_1\\ \vdots\\ \zeta_\ell\\ f_2(t^{n+1})\\ g_2(t^{n+1}) \end{bmatrix},$$

with

$$\begin{aligned} \xi_i &= \Phi_{5,J}^T(x_i)\left[1+\frac{\delta t}{2}\left(\Phi_{4,J}^T(x_i)D_{5,J}^T U_n + U_n\Phi_{4,J}^T(x_i)D_{5,J}^T\right)\right]\\ &\quad + \frac{\delta t}{2}\left(\alpha\Phi_{3,J}^T(x_i)D_{4,J}^T D_{5,J}^T + \beta\Phi_{2,J}^T(x_i)D_{3,J}^T D_{4,J}^T D_{5,J}^T + \gamma\Phi_{1,J}^T(x_i)D_{2,J}^T D_{3,J}^T D_{4,J}^T D_{5,J}^T\right), \quad i = 1,2,\ldots,\ell, \end{aligned}$$

and

$$\zeta_i = \left\{\Phi_{5,J}^T(x_i) - \frac{\delta t}{2}\left(\alpha\Phi_{3,J}^T(x_i)D_{4,J}^T D_{5,J}^T + \beta\Phi_{2,J}^T(x_i)D_{3,J}^T D_{4,J}^T D_{5,J}^T + \gamma\Phi_{1,J}^T(x_i)D_{2,J}^T D_{3,J}^T D_{4,J}^T D_{5,J}^T\right)\right\}U_n, \quad i = 1,2,\ldots,\ell.$$

Eq. (3.15) using Eq. (3.1) as the starting points, gives a linear system of equations with $\ell+4$ unknowns and equations, which can be solved to find U_{n+1} in any step $n = 1,\ldots$. So the unknown functions $u(x,t^n)$ in any time $t = t^n$, $n = 1,2,\ldots$ can be found.

4. Theoretical stability of the linearized scheme

We use the notation of asymptotic stability of a numerical method as it is defined in [40] for a discrete problem of the form

$$\frac{dU}{dt} = L_J U,$$

where L_J is assumed to be a diagonalizable matrix.

Definition 1. The region of absolute stability of a numerical method is defined for the scalar model problem

$$\frac{dU}{dt} = \lambda U,$$

to be the set of all $\lambda\delta t$ such that $\|U_n\|$ is bounded as $t \to \infty$. We say that a numerical method is asymptotically stable for a particular problem if for sufficiently small $\delta t > 0$, the product of the δt times every eigenvalue of L lies within the region of absolute stability [41].

The Crank–Nicolson scheme: This method is absolutely stable in the entire left-half plane.

4.1. Stability for linearized GKS equation

We consider the linearized GKS equation

$$\frac{\partial u}{\partial t} + \mu\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^4 u}{\partial x^4} = 0, \tag{4.1}$$

where the linearization coefficient μ stands for values of u. We assume that $|\mu| \leqslant M$, where M is an upper bound of u. Here $L = \Gamma^{-1}\Lambda$, where Γ and Λ are matrices in the form $\Gamma = (V_1, V_2, \ldots, V_{\ell+4})$ and $\Lambda = (C_1, C_2, \ldots, C_{\ell+4})$ where V_i and C_i, $i = 1, 2, \ldots, \ell + 4$ are the vectors in the forms

$$V_i = -\Phi_{5,J}(x_i), \quad i = 1, 2, \ldots, \ell + 4,$$

and

$$C_i = -\left\{\mu D_{5,J}\Phi_{4,J}(x_i) + \alpha D_{5,J}D_{4,J}\Phi_{3,J}(x_i) + \beta D_{5,J}D_{4,J}D_{3,J}\Phi_{2,J}(x_i) + \gamma D_{5,J}D_{4,J}D_{3,J}D_{2,J}\Phi_{1,J}(x_i)\right\}, \quad i = 1, 2, \ldots, \ell + 4,$$

respectively, so that the discretized linearized GKS equation becomes

$$\frac{dU}{dt} = LU.$$

The Crank–Nicolson scheme is A-stable. The region of absolute stability for the Crank–Nicolson time stepping scheme is plotted in Fig. 1, for Examples 1 and 2 presented in this paper.

5. Numerical examples

In this section we give some computational results of numerical experiments with the method based on Section 3, to support our theoretical discussion.

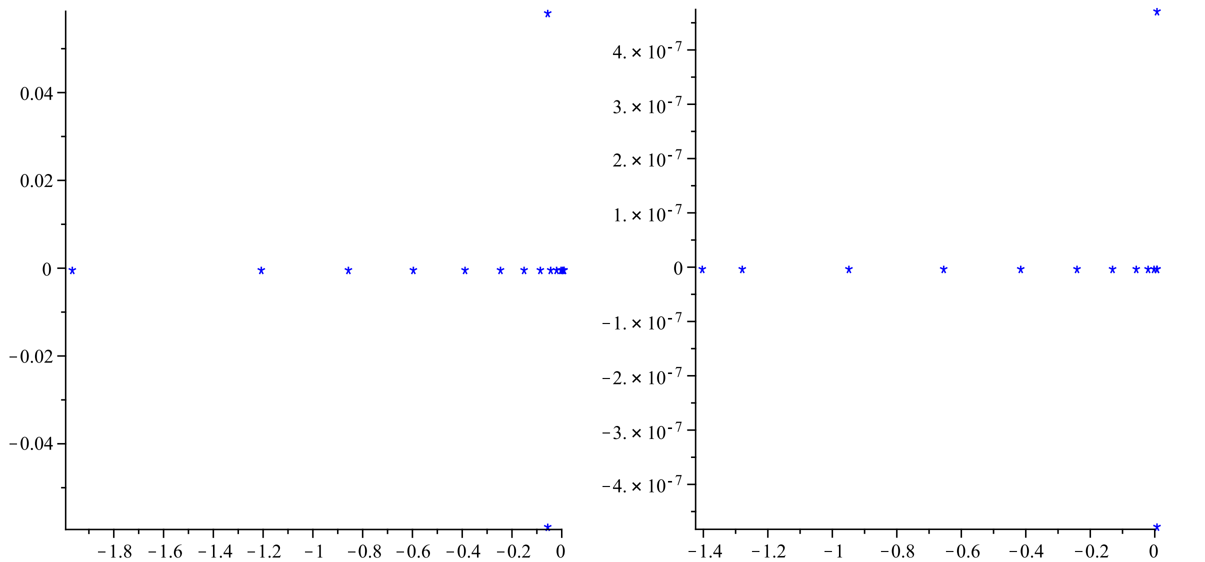

Fig. 1. $\delta t = 10^{-3}$ times the eigenvalues of L_1 for Example 1 (left) and Example 2 (right).

Table 1
$Ł_{\infty}$ and L_2 errors for $u(t)$ using presented method.

δt	$Ł_{\infty}$-error $J = 4$	L_2-error $J = 4$	$Ł_{\infty}$-error $J = 5$	L_2-error $J = 5$
0.1	7.3×10^{-3}	6.5×10^{-3}	4.4×10^{-3}	3.7×10^{-3}
0.01	6.1×10^{-3}	5.7×10^{-3}	3.3×10^{-5}	2.6×10^{-5}
0.001	1.0×10^{-4}	8.4×10^{-5}	2.8×10^{-6}	1.5×10^{-6}

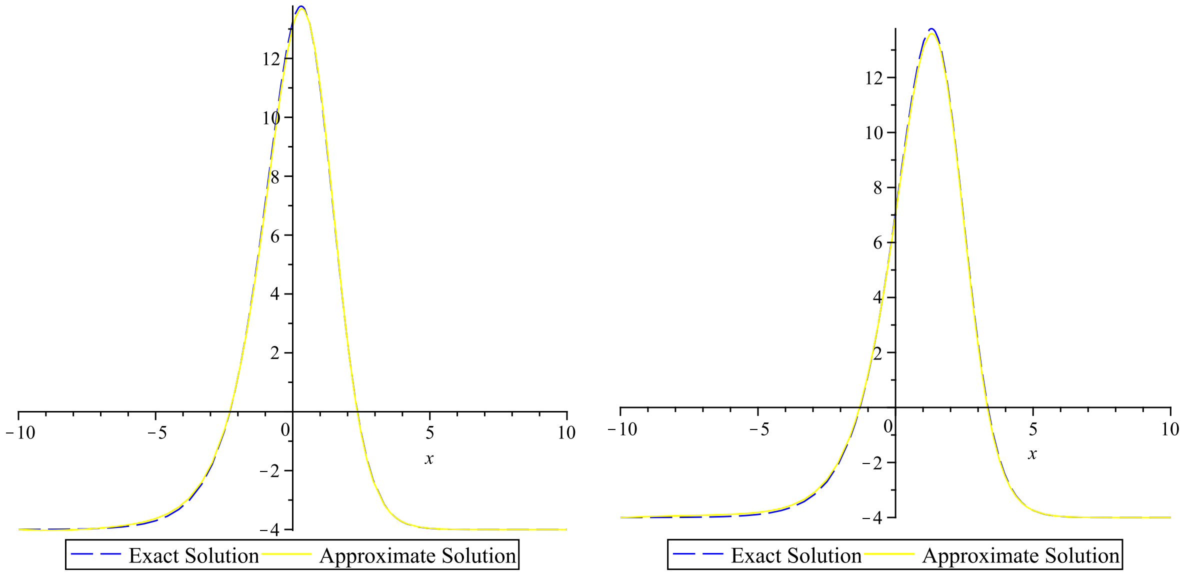

Fig. 2. Plot of exact and approximate solutions for $J = 1$, $\delta t = 0.01$, $T = 0.5$ (left) and $T = 1$ (right) for Example 1.

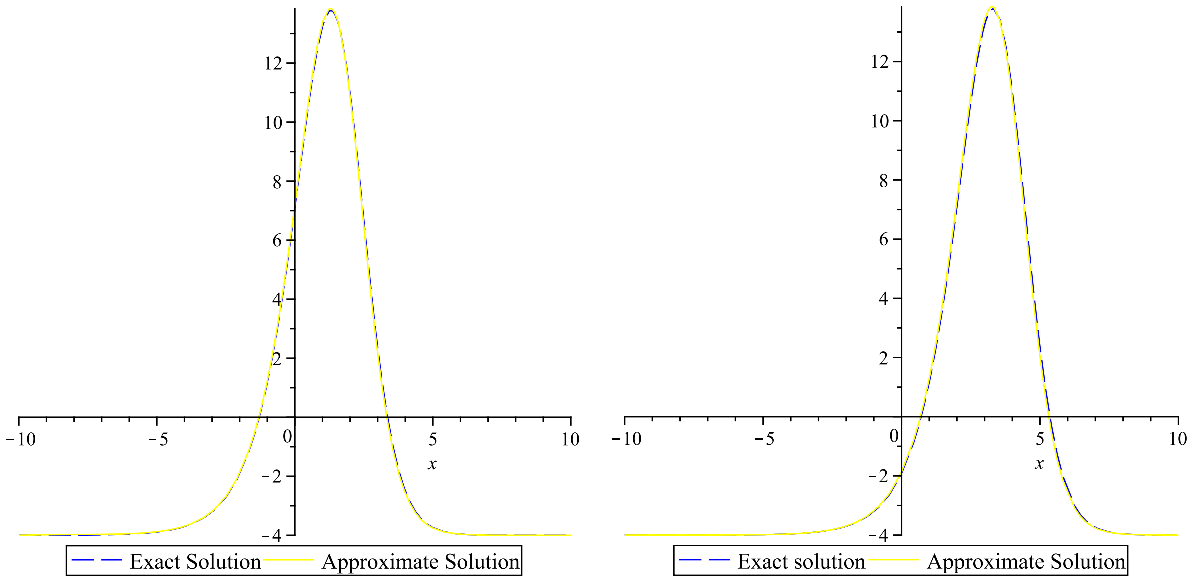

Fig. 3. Plot of exact and approximate solutions for $J = 1$, $\delta t = 0.1$, $T = 1$ (left) and $T = 2$ (right) for Example 1.

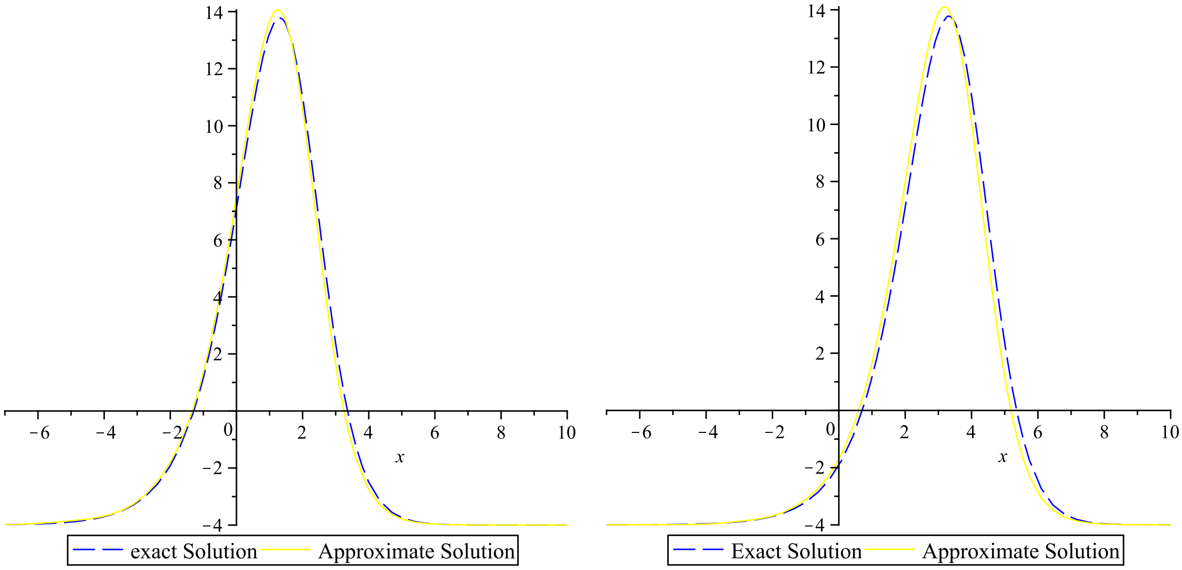

Fig. 4. Plot of exact and approximate solutions for $J = 1$, $\delta t = 0.2$, $T = 1$ (left) and $T = 2$ (right) for Example 1.

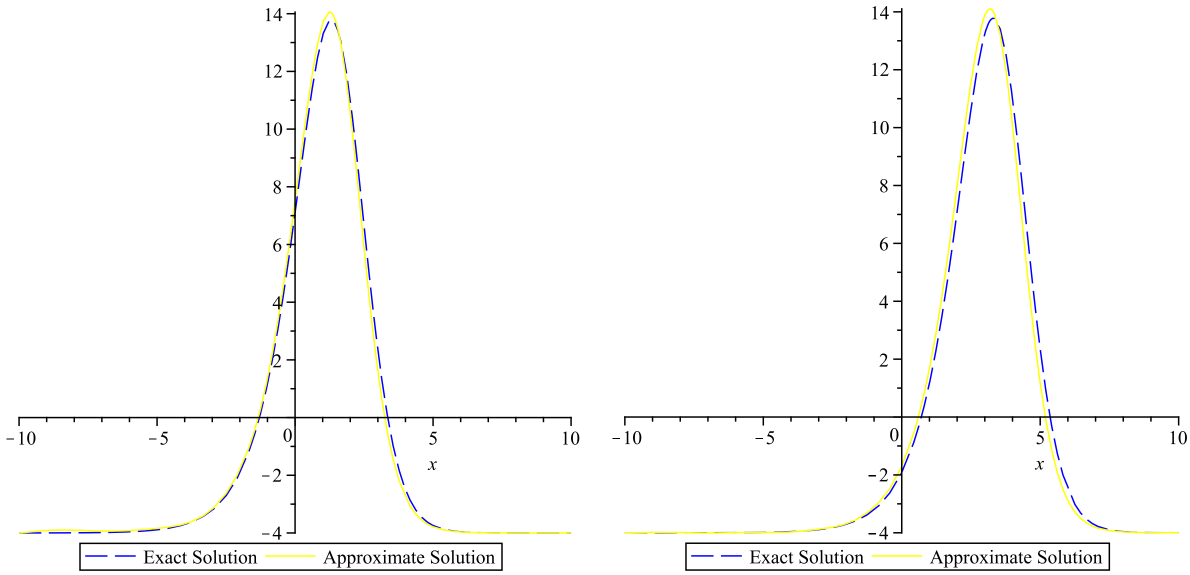

Fig. 5. Plot of exact and approximate solutions for $J = 2$, $\delta t = 0.2$, $T = 1$ (left) and $T = 2$ (right) for Example 1.

Example 1. In this example, we consider the GKS equation, represented by $\alpha = \gamma = 1$ and $\beta = 4$. The exact solution is [1]

$$u(x,t) = 11 + 15\,tanh\,\theta - 15\,tanh^2\,\theta - 15\,tanh^3\,\theta,$$

with $\theta = -\frac{1}{2}x + t$. We will use this solution, evaluated at $t = 0$, as the initial condition, and the boundary functions from the exact solution on the interval $[-1,1]$. The $Ł_\infty$ and L_2 errors are obtained in Table 1 for the presented method in time $t = 1$ for different values of δt and J.

Figs. 2–6 show the chaotic solutions for different values of J, T, δt on the interval $[-10,10]$.

Example 2. Consider Eq. (1.1) with $\alpha = 2$, $\gamma = 1$ and $\beta = 0$. The exact solution is [1]

$$u(x,t) = -\frac{1}{\kappa} + \frac{60}{19}\kappa(-38\gamma\kappa^2 + \alpha)\,tanh\,\theta + 120\gamma\kappa^3\,tanh^3\,\theta,$$

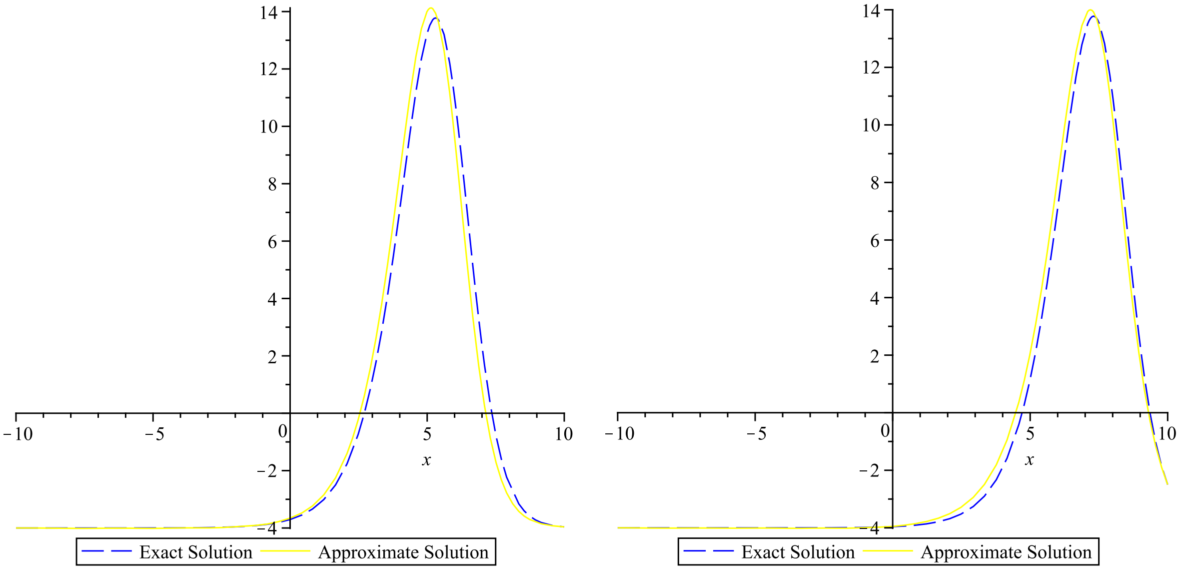

Fig. 6. Plot of exact and approximate solutions for $J = 2$, $\delta t = 0.2$, $T = 3$ (left) and $T = 4$ (right) for Example 1.

Table 2
$Ł_\infty$ and L_2 errors for $u(t)$ using the presented method.

δt	$Ł_\infty$-error $J = 4$	L_2-error $J = 4$	$Ł_\infty$-error $J = 5$	L_2-error $J = 5$
0.1	1.3×10^{-3}	1.2×10^{-3}	1.7×10^{-3}	1.5×10^{-3}
0.01	4.2×10^{-4}	3.7×10^{-4}	9.0×10^{-5}	8.0×10^{-5}
0.001	9.7×10^{-5}	9.1×10^{-5}	6.2×10^{-6}	5.5×10^{-6}

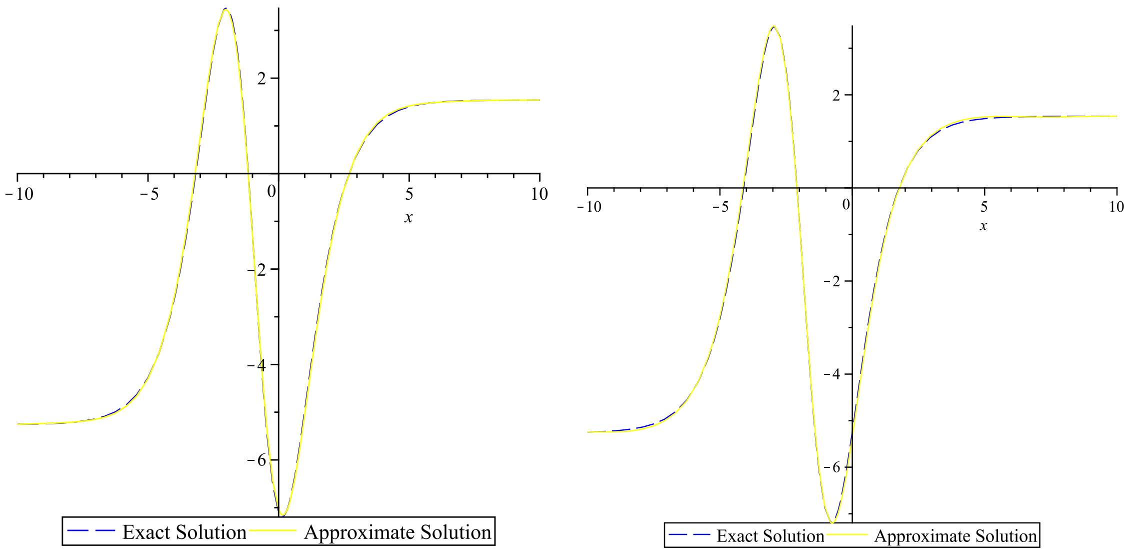

Fig. 7. Plot of exact and approximate solutions for $J = 1$, $\delta t = 0.01$, $T = 0.5$ (left) and $T = 1$ (right) for Example 2.

where $\theta = \kappa x + t$ and $\kappa = (1/2)\sqrt{11\alpha/19\gamma}$. Similar to the previous example, we extract the required boundary functions from the exact solution on the interval $[-1,1]$. The $Ł_\infty$ and L_2 errors are obtained in Table 2 for the presented method in time $t = 1$ for different values of δt and J. Figs. 7 and 8 show the chaotic solutions for different values of J, T, δt on the interval $[-10,10]$.

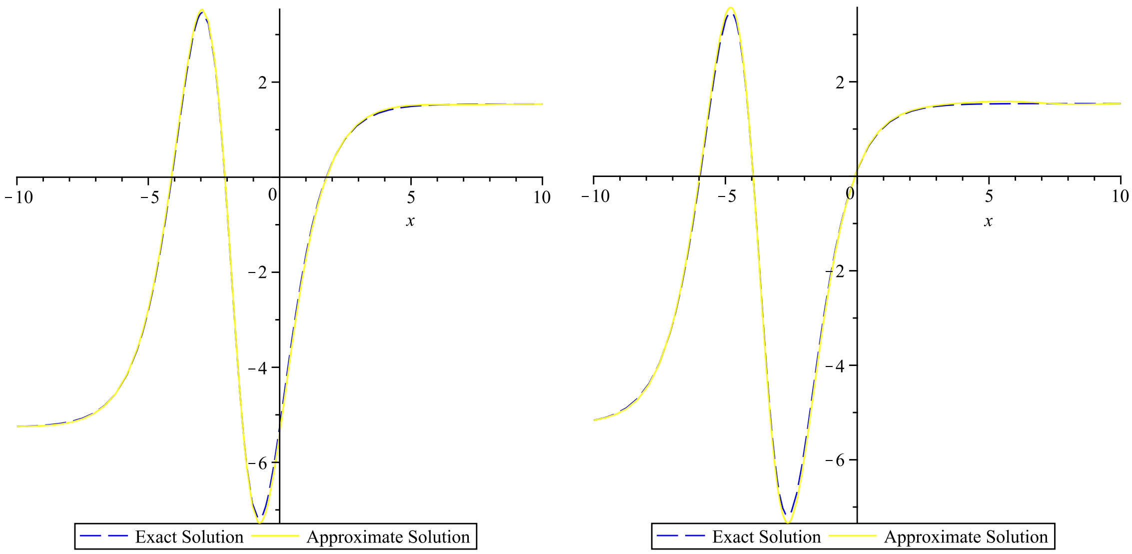

Fig. 8. Plot of exact and approximate solutions for $J = 1$, $\delta t = 0.1$, $T = 1$ (left) and $T = 2$ (right) for Example 2.

Table 3
$Ł_\infty$ and L_2 errors for $u(t)$ using the presented method.

δt	$Ł_\infty$-error $J = 3$	L_2-error $J = 3$	$Ł_\infty$-error $J = 4$	L_2-error $J = 4$
0.1	1.9×10^{-2}	1.7×10^{-2}	9.9×10^{-3}	8.9×10^{-3}
0.01	4.8×10^{-3}	4.4×10^{-3}	2.1×10^{-3}	1.6×10^{-3}

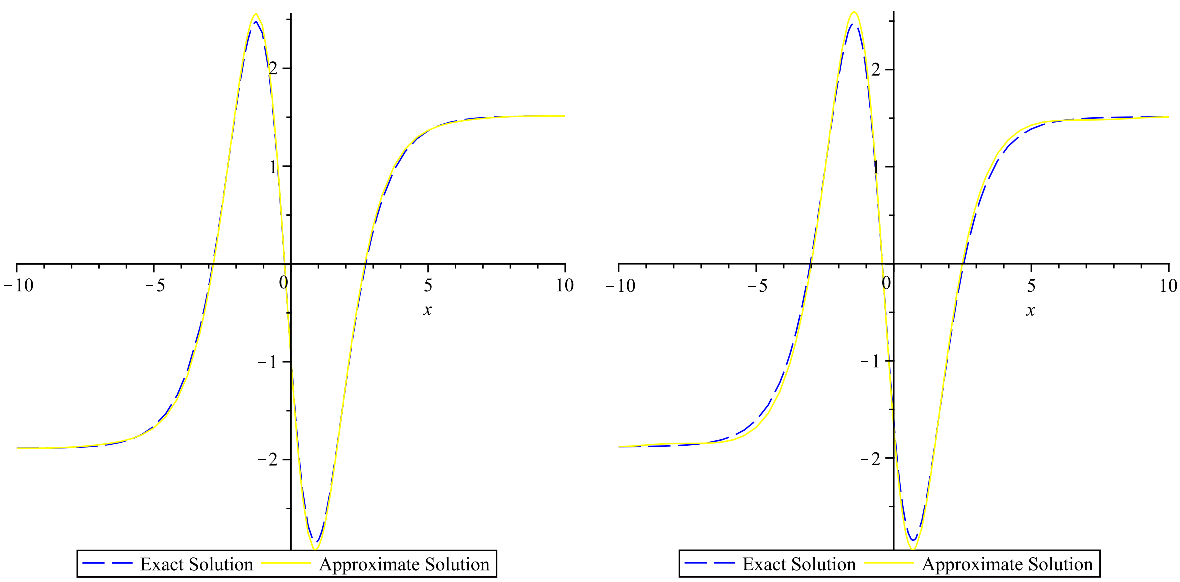

Fig. 9. Plot of exact and approximate solutions for $J = 2$, $\delta t = 0.1$, $T = 1$ (left), and $T = 2$ (right), for Example 3.

Example 3. Consider Eq. (1.1) with $\alpha = 1$, $\gamma = 0.5$ and $\beta = 0$. The exact solution is [1]

$$u(x,t) = -\frac{0.1}{\kappa} + \frac{60}{19}\kappa(-38\gamma\kappa^2 + \alpha)\, tanh\,\theta + 120\gamma\kappa^3\, tanh^3\,\theta,$$

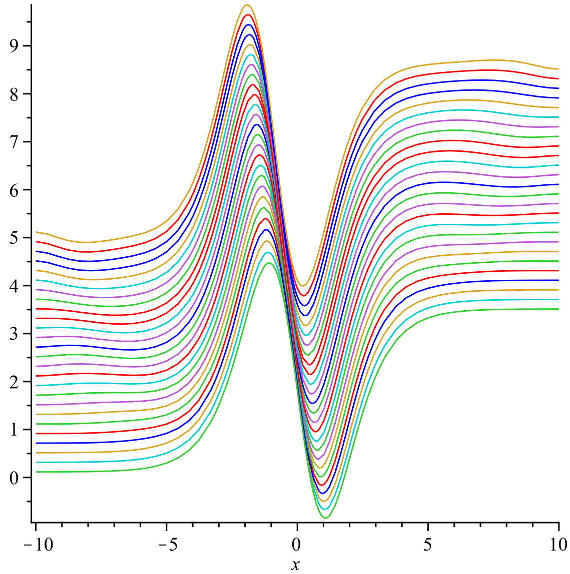

Fig. 10. The chaotic solution of the GKS equation for $J = 1$, $\delta t = 0.1$, $t \in [0,5]$, for Example 3.

Table 4
$\L_\infty$ error for $u(t)$ using CN method.

c	CN method	Method [1]
0.1	7.7×10^{-7}	2.6×10^{-4}
0.01	1.8×10^{-6}	3.2×10^{-5}
0.001	1.6×10^{-6}	3.2×10^{-5}

where $\theta = \kappa x + 0.1t$ and $\kappa = (1/2)\sqrt{11\alpha/19\gamma}$. Again, we extract the required boundary functions from the exact solution on the interval $[-1,1]$. The $\L_\infty$ and L_2 errors are obtained in Table 3 for the presented method in time $t = 1$ for different values of δt and J. Fig. 9 shows the chaotic solutions for $J = 2$, $\delta t = 0.1$ and $T = 1, 2$ on the interval $[-10,10]$. The chaotic solutions are shown in Fig. 10 for $J = 1$, $\delta t = 0.1$ and $t \in [0,5]$.

Example 4. Consider Eq. (1.1) with $\alpha = 1, \gamma = 1$ and $\beta = 4$. The exact solution is [1]

$$u(x,t) = 9 + 2c + 15\,tanh\,\theta - 15\,tanh^2\,\theta - 15\,tanh^3\,\theta,$$

where $\theta = -1/2x + ct$. Similar to the previous examples, we extract the required boundary functions from the exact solution on the interval $[-1,1]$. The $\L_\infty$ errors for different values of c are obtained in Table 4 for the presented method in time $t = 1$ for $\delta t = 0.05$ and $J = 5$, also we compare the results with the method proposed in [1].

Example 5. In this example, we consider another type of Kuramoto–Sivashinsky equation

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0,$$

where is the simplest nonlinear partial differential exhibiting the chaotic behavior over a finite spatial domain. Here we take the Gaussian initial condition [13,42]

$$u(x,0) = \exp(-x^2),$$

with the boundary conditions

$$u(a,t) = 0, \quad u(b,t) = 0, \quad u_x(a,t) = 0, \quad u_x(b,t) = 0.$$

The numerical results are presented in Fig. 11 for $a = -30$ and $b = 30$. We can observe that the numerical results are convergent for the very chaotic nature.

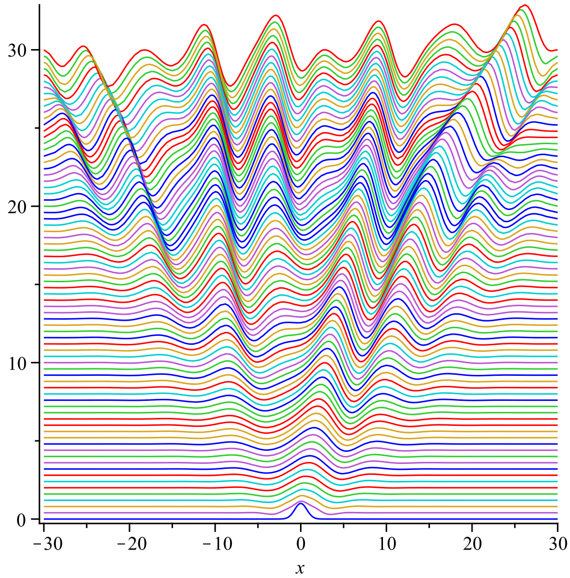

Fig. 11. The chaotic solution of the KS equation for $J = 1$, $\delta t = 0.1$, $t \in [0, 30]$, for Example 5.

6. Conclusion

In this paper we presented a numerical scheme for solving the generalized Kuramoto–Sivashinsky equation. The method employed to find the solution of this equation is based on the B-spline functions. The new method was applied on several test problems from the literature. The computational results are found to be in good agreement with the exact solutions. The algorithms proposed in the current paper can be employed to solve a large class of linear and nonlinear time-dependent partial differential equations.

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