# TSRT08 Optimal Control
# Homework 1 - Discrete Dynamic Programming

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# Exercises

a)

$$\begin{aligned} \max \sum_{k=0}^{N-1} u_k & \\ \text{subject to } x_{k+1} &= x_k + \theta(x_k - u_k) \\ x_k &\geq 0,\ \forall k = \{0, 1, \ldots, N-1\} \\ 0 \leq u_k &\leq x_k \end{aligned} \tag{1}$$

Further, it is also assumed that $\theta > 0$ since it is necessary to increase the company profit the following year.

b) The Bellman equation $J(k, x)$ is written down, assuming the solution $J(k, x) = \alpha_k x$ and then simplified.

$$\begin{aligned} J(k, x) &= \max_{0 \leq u_k \leq x_k} \{u_k + J(k+1, x_k + \theta(x_k - u_k))\} \\ J(k, x) &= \alpha_k x \\ \alpha_k x &= \max_{0 \leq u_k \leq x_k} \{u_k + \alpha_{k+1}(x_k + \theta(x_k - u_k))\} \\ \alpha_k x &= \max_{0 \leq u_k \leq x_k} \{u_k + \alpha_{k+1} x_k + \alpha_{k+1}\theta x_k - \alpha_{k+1}\theta u_k\} \\ \alpha_k x &= \max_{0 \leq u_k \leq x_k} \{u_k(1 - \alpha_{k+1}\theta) + x_k(\alpha_{k+1} + \alpha_{k+1}\theta)\} \end{aligned}$$

Since the function we want to maximize for $u_k$ is linear, we only need to check the endpoints of the interval of the max operation. This yields the following cases:

$$\begin{cases} \alpha_k x_k = x_k(\alpha_{k+1} + \alpha_{k+1}\theta), & \text{if } u_k = 0 \\ \alpha_k x_k = x_k(\alpha_{k+1} + 1), & \text{if } u_k = x_k \end{cases}$$

Since we want to maximize $J(k, x) = \alpha_k x$ we get

$$\alpha_k x_k = \max \begin{cases} x_k(\alpha_{k+1} + \alpha_{k+1}\theta) \\ x_k(\alpha_{k+1} + 1) \end{cases}$$

which simplifies to

$$\alpha_k = \max \begin{cases} \alpha_{k+1} + \alpha_{k+1}\theta \\ \alpha_{k+1} + 1 \end{cases}$$

Since the term $\alpha_{k+1}$ is the same in both cases we only need to apply the max operation on the second terms, hence

$$\alpha_k = \alpha_{k+1} + \max \begin{cases} \alpha_{k+1}\theta \\ 1 \end{cases}$$

It is now possible to shift this in k to get the desired expression.

$$\alpha_{k-1} = \alpha_k + \max \begin{cases} \alpha_k \theta \\ 1 \end{cases}$$

c) Now the backwards recursion is solved for $\alpha_k$ to find the optimal control signal $u_k^*$. It is assumed that $\alpha_k \geq 0$. We begin at $k = N$.

$$\alpha_{N-1} = \alpha_N + \max \begin{cases} \alpha_N \theta \\ 1 \end{cases}$$

Since there is no distribution to shareholders in the last year, $J(N, x) = \alpha_N x_N = 0$. Since $x_N = 0$ would imply all $x_k = 0$ and there is no distribution to shareholders, we assume $x_N \neq 0$. Therefore, $\alpha_N = 0$. Using this, we get

$$\alpha_{N-1} = 1$$

Next, we look at $N - 2$

$$\alpha_{N-2} = \alpha_{N-1} + \max \begin{cases} \alpha_{N-1} \theta \\ 1 \end{cases}$$

Using the value for $\alpha_{N-1} = 1$, we get

$$\alpha_{N-2} = 1 + \max \begin{cases} \theta \\ 1 \end{cases}$$

Now, we get two cases. One where $\theta < 1$ and one where $\theta \geq 1$. More generally, with an unknown value for $\alpha_k$, we get the cases $\alpha_k \theta < 1$ and $\alpha_k \theta \geq 1$.

We begin by examining the case where $\alpha_k \theta \geq 1$.

$$\alpha_{N-2} = 1 + \theta$$

$$\alpha_{N-3} = \alpha_{N-2} + \max \begin{cases} \alpha_{N-2} \theta \\ 1 \end{cases} \quad = 1 + \theta + \max \begin{cases} (1+\theta)\theta \\ 1 \end{cases} \quad = 1 + 2\theta + \theta^2$$

$$\alpha_{N-4} = \alpha_{N-3} + \max \begin{cases} \alpha_{N-3} \theta \\ 1 \end{cases} \quad = 1 + 2\theta + \theta^2 + \theta + \max \begin{cases} (1 + 2\theta + \theta^2)\theta \\ 1 \end{cases} \quad = 1 + 3\theta + 3\theta^2 + \theta^3$$

If we extend the pattern looking like binomial expansion, we get

$$\alpha_k = (1 + \theta)^{N-k-1}$$

This expression for $\alpha_k$ is valid as long as $\alpha_k \theta \geq 1$. We now check for which $k$ this assumption breaks down.

$$
\begin{aligned}
\theta(1+\theta)^{N-k-1} &= 1 \\
(1+\theta)^{N-k-1} &= \frac{1}{\theta} \\
(N-k-1)\ln(1+\theta) &= -\ln(\theta) \\
N-k-1 &= \frac{-\ln(\theta)}{\ln(1+\theta)} \\
k &= N + \frac{\ln(\theta)}{\ln(1+\theta)} - 1
\end{aligned}
$$

For $k$ below this threshold, we instead get that $\alpha_k \theta < 1$.

$$
\alpha_{N-2} = \alpha_{N-1} + \max \begin{cases} \alpha_{N-1}\theta \\ 1 \end{cases} = 2
$$

$$
\alpha_{N-3} = \alpha_{N-2} + \max \begin{cases} \alpha_{N-2}\theta \\ 1 \end{cases} = 3
$$

If we extend the pattern, we get

$$
\alpha_k = N - k
$$

This expression for $\alpha_k$ holds for $\alpha_k \theta < 1$.

We can now compute the optimal control signal $u_k^*$.

$$
u_k^* = \underset{0 \le u_k \le x_k}{\arg\max} \{u_k + \alpha_{k+1}(x_k + \theta(x_k - u_k))\}
$$

If $k$ is above the threshold, we get

$$
u_k^* = \underset{0 \le u_k \le x_k}{\arg\max} \{u_k + (1+\theta)^{N-k}(x_k + \theta(x_k - u_k))\}
$$

$$
\begin{aligned}
u_k = 0: &\quad (1+\theta)^{N-k}(x_k + \theta x_k) = (1+\theta)^{N-k} x_k + (1+\theta)^{N-k}\theta x_k \\
u_k = x_k: &\quad x_k + (1+\theta)^{N-k} x_k = (1+\theta)^{N-k} x_k + x_k
\end{aligned}
$$

For these two, since $(1+\theta)^{N-k}\theta > 1$, $u_k = 0$ gives the maximum value. Therefore, $u_k^* = 0$.

If $k$ is bellow the threshold, we get

$$
u_k^* = \underset{0 \le u_k \le x_k}{\arg\max} \{u_k + (N-k+1)(x_k + \theta(x_k - u_k))\}
$$

$$
\begin{aligned}
u_k = 0: &\quad (N-k+1)(x_k + \theta x_k) = (N-k+1)x_k + (N-k+1)\theta x_k \\
u_k = x_k: &\quad x_k + (N-k+1) = (N-k+1)x_k + x_k
\end{aligned}
$$

Therefore, $u_k^* = x_k$ in this case.