TSRT08 Optimal Control
Homework 3 - Implementation of algorithms

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December 14, 2023

Exercises

Analytical Solution

a) The Hamiltonian of our problem is

$$H = -x(t_f) + \lambda_1(w\cos(\theta) + y) + \lambda_2(w\sin(\theta))$$

and the adjoint equations are therefore

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x} = 0$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial y} = -\lambda_1$$

If we solve the adjoint equations for λ_1 and λ_2, we get

$$\lambda_1 = C_1$$

and

$$\lambda_2 = -C_1 t + C_2$$

We get the final value conditions by solving

$$\lambda_i(t_f) = \frac{\partial J}{\partial x_i(t_f)} = -\frac{\partial x(tf)}{\partial x_i(t_f)}$$

This gives us

$$\lambda_1(t_f) = -1$$

and

$$\lambda_2(t_f) = 0$$

Using this information to get C_1 and C_2, we get

$$C_1 = -1$$

and

$$C_2 = -t_f$$

Finally, this gives us

$$\lambda_1 = -1$$

and

$$\lambda_2 = t - t_f$$

b) The partial derivative of H with respect to θ is

$$\frac{\partial H}{\partial \theta} = -\lambda_1 w\sin(\theta) + \lambda_2 w\cos(\theta)$$

c) We can now set the derivative equal to zero in order to find the extreme point. We get

$$\frac{\lambda_2}{\lambda_1} = \tan(\theta)$$

using our previously computed values of λ_1 and λ_2, we get

$$t_f - t = \tan(\theta)$$

and thus

$$\theta = \arctan(t_f - t)$$

We now want to determine of this is a minimum or maximum point. The second derivative of H with respect to θ is

$$\frac{\partial^2 H}{\partial \theta^2} = -\lambda_1 w \cos(\theta) - \lambda_2 w \sin(\theta)$$

If we evaluate this at $\theta = \arctan(t_f - t)$ and plug in λ_1 and λ_2, we get

$$\frac{\partial^2 H}{\partial \theta^2} = w \cos(\arctan(t_f - t)) - (t - t_f) w \sin(\arctan(t_f - t))$$

Since $t_f \geq t$ and arctan ≥ 0 for arguments greater than or equal to zero combined with the fact that arctan approaches $\frac{\pi}{2}$ for positive arguments, we get

$$0 \leq \arctan(t_f - t) \leq \frac{\pi}{2}$$

Both cos and sin are non-negative for these possible values of $\arctan(t_f - t)$. We also know that $w \geq 0$ since it signifies speed. This means that

$$\frac{\partial^2 H}{\partial \theta^2} \geq 0$$

and that $\theta = \arctan(t_f - t)$ is a minimum solution to our problem.

d) When we plot the optimal trajectory with $t_f = w = 1$ in Matlab, we get the result in Figure 1.

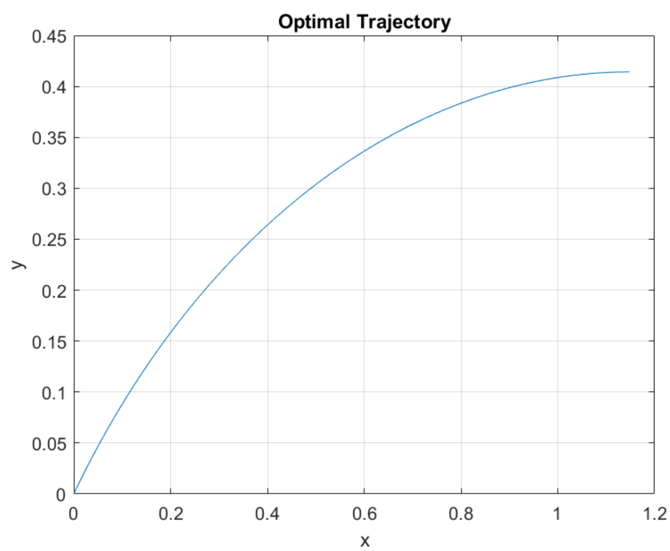

Figure 1: Optimal trajectory.

The numerical optimal objective value is $-x(t_f) = -1.148$.

Discretization Method Solution

a) If we discretize the system dynamics with sample time T, $w = 1$ and $v(y) = y$, we get

$$\begin{bmatrix} x[k+1] \\ y[k+1] \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x[k] \\ y[k] \end{bmatrix} + \begin{bmatrix} T\cos(\theta[k]) \\ T\sin(\theta[k]) \end{bmatrix}$$

b) A script and three functions that solves the discrete-time version of the problem with the Matlab function fmincon was implemented.

c) The numerical solution is very similar to the analytical solution, see Figure 2

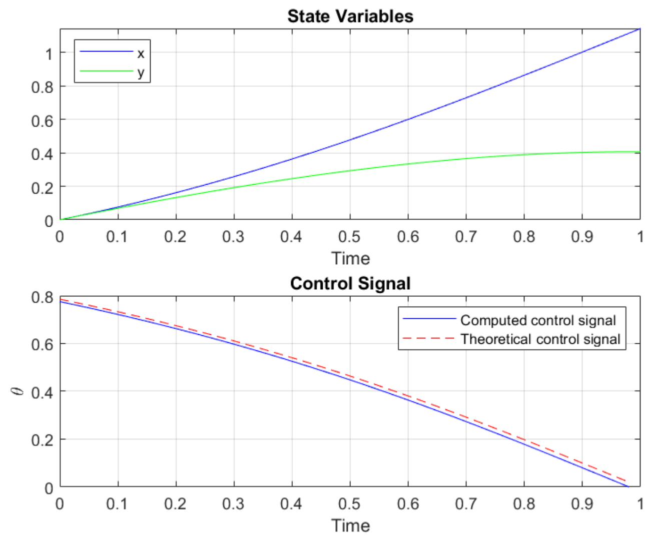

Figure 2: Comparison between numerical and analytical solution.

The numerical value of the cost was -1.143675 and the computational time was 0.265472 seconds.

d) When GradObj and GradConstr was set to 'off', the results in Figure 3 was obtained. The numerical value of the cost was -1.143675 and the computational time was 2.135054 seconds. We see very similar results but with a much longer computational time. This is because the program now has to search the entire space for an optimal solution instead of using the gradient to search in the right direction right away from the initial guess.

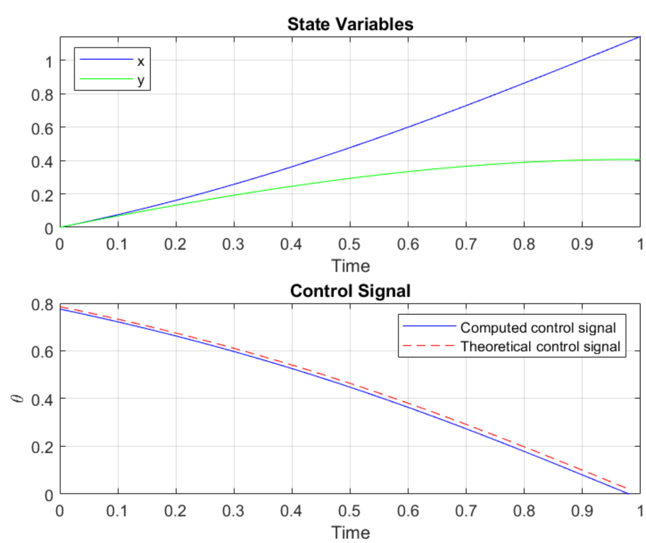

Figure 3: Comparison between numerical and analytical solution without gradient.

Gradient Method

a) A gradient search approach was implemented in Matlab to solve the problem.

b) We got the results in Figure 4. The numerical value of the cost was -1.147798 and the computational time was 0.055753 seconds. We prefer the gradient method compared to the discretization method since it gives us a better value of the cost function and is faster to compute.

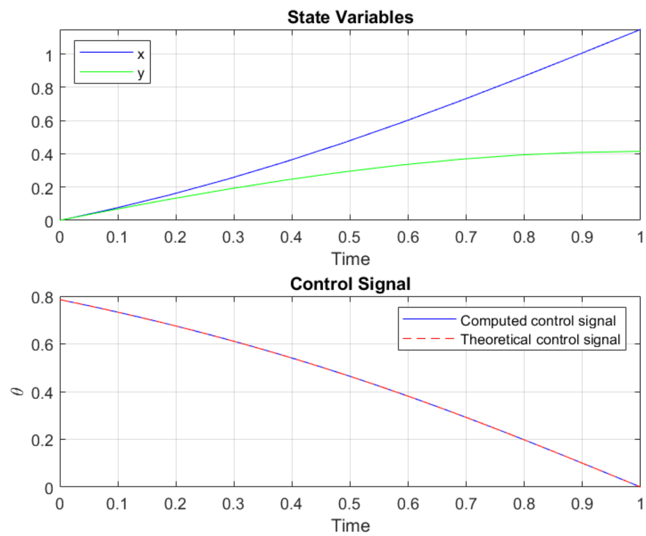

Figure 4: Comparison between numerical and analytical solution

Boundary Condition Iteration (Shooting)

a) Let $x = (r, u, v)^T$ and $\lambda = (\lambda_r, \lambda_u, \lambda_v)^T$.

The Hamiltonian is

$$H = -r(t_f) + \lambda_r u + \lambda_u\Big(\frac{v^2}{r} - \frac{1}{r^2} + a\sin(\theta)\Big) + \lambda_v\Big(-\frac{uv}{r} + a\cos(\theta)\Big)$$

and the adjoint equations are

$$\begin{aligned}
\dot{\lambda}_r &= \frac{\lambda_u v^2}{r^2} - \frac{2\lambda_u}{r^3} - \frac{\lambda_v uv}{r^2} \\
\dot{\lambda}_u &= -\lambda_r + \frac{\lambda_v v}{r} \\
\dot{\lambda}_v &= -\frac{2\lambda_u v}{r} + \frac{\lambda_v u}{r}
\end{aligned}$$

b) Let

$$G(x) = \begin{bmatrix} u \\ v - \frac{1}{\sqrt{r}} \end{bmatrix}$$

This means that

$$G_x(x(t_f))^T = \begin{bmatrix} 0 & \frac{1}{2r(t_f)^{3/2}} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Also, since

$$\phi(x) = -r(t_f)$$

we get

$$\nabla\phi(x(t_f)) = (-1, 0, 0)^T$$

Now, we can show that a terminal constraint can be obtained from (15) in the homework assignment.

$$\begin{aligned}
&\lambda(t_f) = \nabla\phi(x(t_f)) + G_x(x(t_f))^T \nu \\
\iff &\begin{bmatrix}\lambda_r(t_f)\\ \lambda_u(t_f)\\ \lambda_v(t_f)\end{bmatrix} = \begin{bmatrix}-1\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}0 & \frac{1}{2r(t_f)^{3/2}}\\ 1 & 0\\ 0 & 1\end{bmatrix}\nu \\
\implies &\lambda_r(t_f) = -1 + \frac{\nu(2)}{2r(t_f)^{3/2}} \\
\implies &\lambda_r(t_f) + 1 - \frac{\nu(2)}{2r(t_f)^{3/2}} = 0
\end{aligned}$$

From the last row once we have expanded $\lambda(t_f)$, we also get

$$\lambda_v(t_f) = \nu(2)$$

If we plug this into the expression for $\lambda_r(t_f)$, we get

$$\lambda_r(t_f) + 1 - \frac{\lambda_v(t_f)}{2r(t_f)^{3/2}} = 0$$

c) We begin by differentiating the Hamiltonian with respect to θ and setting it equal to zero.

$$\frac{\partial H}{\partial \theta} = \lambda_u a\cos(\theta) - \lambda_v a\sin(\theta) = 0$$

We can now rearrange the terms as

$$\lambda_u a\cos(\theta) = \lambda_v a\sin(\theta)$$

If we divide both sides by a and put both trigonometric functions on the same side, we get

$$\frac{\lambda_u}{\lambda_v} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$$

d) The algorithm is implemented in Matlab with $\lambda(0) = (-1, -1, -1)$. The results can be seen in Figure 5. The numerical value of the optimal cost is 1.526881 and the computational time was 0.298505 seconds.

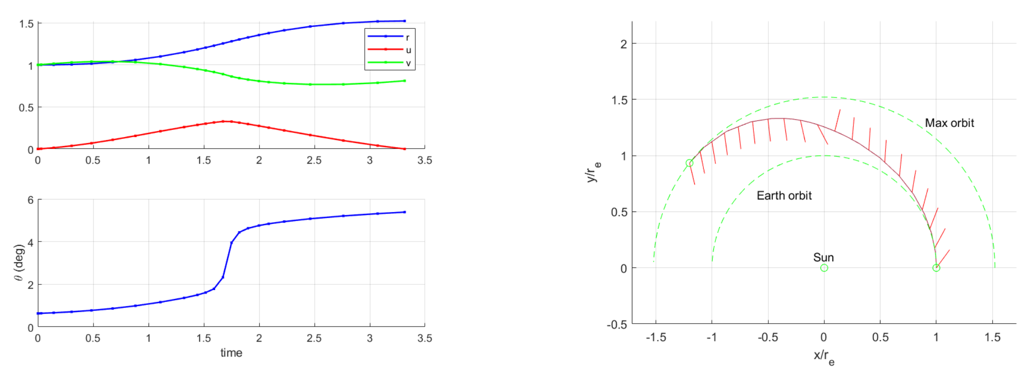

Figure 5: Results for task 2.1 (d).

If we instead use $\lambda(0) = (69, 420, 1337)$, we see the worse results in Figure 6. Now, the optimal cost is 1.145864 and the computational time was 4.241660 seconds.

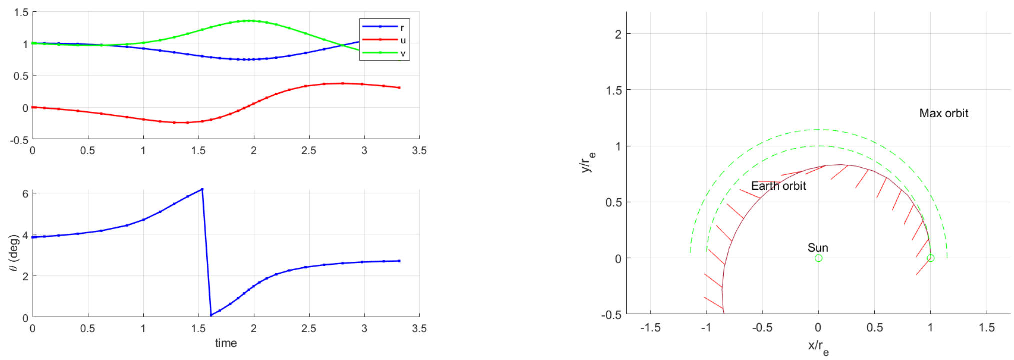

Figure 6: Worse results for task 2.1 (d).

CasADi

When we use CasADi to solve the problem in Matlab, the optimal cost is -1.523641 and the computational time was 4.147390 seconds. See Figure 7.

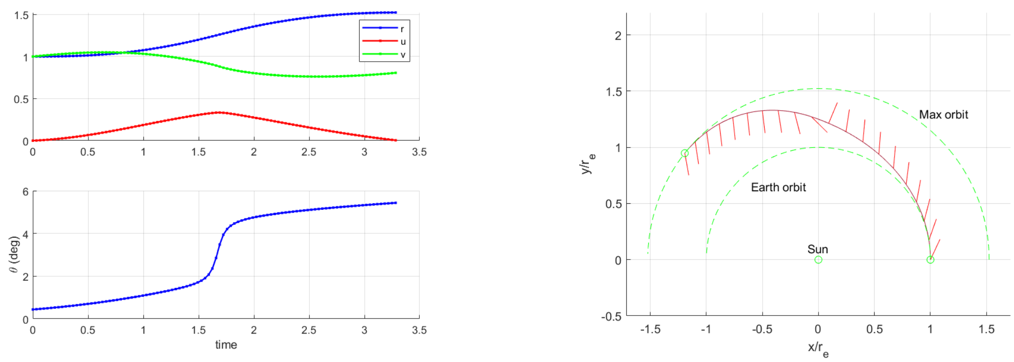

Figure 7: Result using CasADi.