

Spectral Graph Theory and High Performance Computing

Eigenvalue theory

David Wobrock `david.wobrock@inria.fr`

January 22, 2018

1 Theory

The filter W is built on top of the kernel matrix K measuring the similarity between each pixel. The most popular kernel functions are the *Bilateral filter* [1] and the *Non-local Mean filter* [2] to measure these similarities. The kernel functions create a symmetric positive semi-definite (PSD) matrix K with $K_{ij} \geq 0$.

In our case, we shall use the re-normalised Laplacian [3], which will result in a normalisation-free filter [4]. We define the Laplacian operator as

$$\mathcal{L} = \alpha(D - K),$$

with $\alpha = \mathcal{O}(\bar{d}^{-1})$, $\bar{d} = \text{mean}(d_i)$, $d_i = \sum_{j=1}^n K_{ij}$ and $D = \text{diag}(d_i)$. From its definition, we know that \mathcal{L} is symmetric and its eigenvalues $0 \leq \mu_i \leq 1$, meaning that \mathcal{L} is symmetric positive definite (SPD).

The filter is defined as $W = I - \mathcal{L}$. The identity I is obviously SPD, so the filter is also SPD. We know from [5] that the eigenvalues of W are defined as $0 \leq \lambda_i^W \leq 1$ and the largest eigenvalue $\lambda_1^W = 1$.

The image processing algorithm contains the computation of the eigendecomposition of the submatrix W_A . From the properties of SPD matrices, since W_A is a principal submatrix of W , is it also SPD. Furthermore, we can say that the eigenvalues $0 \leq \lambda_i^{W_A} \leq 1$ and $\lambda_1^{W_A} \leq 1$.

Proof Let A be a symmetric matrix of size n , λ_n^A be the largest eigenvalue of A and λ_1^A the smallest one. Let R be the restriction operator, such as, with u a

non-zero vector, $Ru = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ for example. This defines RAR^T a $s \times s$ principal

submatrix of A with $s \in [1; n]$. Suppose the remaining rows and columns of A in RAR^T are indexed by S of size s .

Let $\mathcal{U} \in \mathbb{R}^s$ and $u \in \mathbb{R}^n$ with $\begin{cases} u_i = \mathcal{U}_i & \text{if } i \in S \\ u_i = 0 & \text{if } i \notin S \end{cases}$. Given a $k \in [1; s]$, the

Courant-Fischer theorem, involving the Rayleigh-Ritz quotient, implies that,

$$\max \left(\frac{\langle Au, u \rangle}{\langle u, u \rangle} \right) = \max \left(\frac{\langle RAR^T \mathcal{U}, \mathcal{U} \rangle}{\langle \mathcal{U}, \mathcal{U} \rangle} \right) \leq \lambda_k^A.$$

So $\lambda_k^{RAR^T} \leq \lambda_k^A$, thus $\lambda_k^{RAR^T} \leq \lambda_n^A$ and finally $\lambda_s^{RAR^T} \leq \lambda_n^A$. This concludes the proof, showing that the eigenvalues of the submatrix are bounded by the eigenvalues of the original matrix.

From the definition of the filter $W = I - \mathcal{L}$, we have the submatrix $W_A = I - \mathcal{L}_A$, with I being the identity of appropriate order. For the algorithm, we need to compute the largest eigenvalues of W_A .

Theorem Computing the largest eigenvalues of W_A is equivalent to computing the smallest eigenvalues of \mathcal{L}_A .

Proof

$$\begin{aligned} W_A x = \lambda x &\Leftrightarrow (I - \mathcal{L}_A)x = \lambda x \\ &\Leftrightarrow x - \mathcal{L}_A x = \lambda x \\ &\Leftrightarrow \mathcal{L}_A x = x - \lambda x \\ &\Leftrightarrow \mathcal{L}_A x = (1 - \lambda)x \end{aligned} \tag{1}$$

So the eigenvalues of the Laplacian submatrix $\mu = 1 - \lambda$. We know that $\mu \geq 0$, so $1 - \lambda \geq 0$.

We can thus get the greatest eigenvalues of W_A by computing the smallest eigenvalues of \mathcal{L}_A .

Speed of convergence For both these problems, finding the greatest and smallest eigenvalues, the most famous methods are, respectively, the power method and inverse power method¹.

¹Those are also called power iteration and inverse iteration. The inverse method has a variant called inverse subspace iteration, to find the associated subspace to the eigenvalues.

For the power iteration, the convergence rate is $|\frac{\lambda_2}{\lambda_1}|$, with λ_2 being the second largest eigenvalue. We know that $\lambda_2^{W_A} \leq \lambda_1^{W_A} \leq 1$ and thus $\frac{\lambda_2^{W_A}}{\lambda_1^{W_A}} \leq \frac{\lambda_1^{W_A}}{\lambda_1^{W_A}} = 1$. The convergence rate is lower than 1.

The inverse iteration has a speed of convergence of $|\frac{\mu_1}{\mu_2}|$, with μ_2 the second smallest eigenvalue. Again, we know that $0 \leq \mu_1^{L_A} \leq \mu_2^{L_A}$. So the convergence speed is also lower than 1. (erreur quelque part? je pensais que la mthode de la puissance inverse tait plus rapide dans notre cas)

Inverse iterations implies either to compute the inverse of the matrix $x_{k+1} = A^{-1}x_k$, or to solve a system of linear equations $Ax_{k+1} = x_k$. Since the image processing context suggests having dense matrices, we want to explore the performances of Krylov methods and domain decomposition methods (e.g. the Additive Schwarz method) on such dense matrices.

References

- [1] C. Tomasi and R. Manduchi. “Bilateral filtering for gray and color images”. In: *Sixth International Conference on Computer Vision (IEEE Cat. No.98CH36271)*. Jan. 1998, pp. 839–846. DOI: 10 . 1109 / ICCV . 1998 . 710815.
- [2] C. Kervrann and J. Boulanger. “Optimal Spatial Adaptation for Patch-Based Image Denoising”. In: *IEEE Transactions on Image Processing* 15.10 (Oct. 2006), pp. 2866–2878. ISSN: 1057-7149. DOI: 10 . 1109 / TIP . 2006 . 877529.
- [3] Peyman Milanfar. *Non-conformist Image Processing with Graph Laplacian Operator*. 2016. URL: <https://www.pathlms.com/siam/courses/2426/sections/3234>.
- [4] Peyman Milanfar and Hossein Talebi. “A new class of image filters without normalization”. In: *Image Processing (ICIP), 2016 IEEE International Conference on*. IEEE, 2016, pp. 3294–3298.
- [5] Hossein Talebi and Peyman Milanfar. “Global Image Denoising”. In: *IEEE Transactions on Image Processing* 23.2 (Feb. 2014), pp. 755–768. ISSN: 1057-7149, 1941-0042. DOI: 10 . 1109 / TIP . 2013 . 2293425. URL: <http://ieeexplore.ieee.org/document/6678291/> (visited on 10/03/2017).