Spectral Graph Theory and High Performance Computing
Eigenvalue theory

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1 Theory

The filter W is built on top of the kernel matrix K measuring the similarity between each pixel. The most popular kernel functions are the *Bilateral filter* [1] and the *Non-local Mean filter* [2]. In general, the kernel functions create a symmetric positive semi-definite (PSD) matrix K with $k_{ij} \geq 0$. For these specific functions, we can even define $0 \leq k_{ij} \leq 1$.

We shall use the re-normalised [3] Laplacian, which will result in a normalisation-free filter [4]. We define the Laplacian operator as

$$\mathcal{L} = \alpha(D - K),$$

with $alpha = \prime(\bar{d}^{-1})$ and $\bar{d} = mean(d_j)$.

For this definition, we know that $\mathcal{L}$ is symmetric and its eigenvalues $0 \leq \mu_i \leq 1$, meaning that $\mathcal{L}$ is symmetric positive definite (SPD).

The filter is defined as $W = I - \mathcal{L}$. The identity I is obviously SPD, so the filter is also SPD. We know from [5] that the eigenvalues of W are defined as $0 \leq \lambda_i^W \leq 1$ and the largest eigenvalue $\lambda_1^W = 1$.

The image processing algorithm is computing the eigendecomposition of the submatrix W_A. From the properties of SPD matrices, since W_A is a principal submatrix of W, is it also SPD. Furthermore, we can say that the eigenvalues $0 \leq \lambda_i^{W_A} \leq 1$ and $\lambda_1^{W_A} \leq 1$.

Proof Let A be a symmetric matrix, λ_{max}^A be the largest eigenvalue of A and u the associated eigenvector.

$$\lambda_{max}^A = max(\frac{A(u,u)}{(u,u)}).$$

Let R be the restriction operator, such as $Ru = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ 0 \end{pmatrix}$ for example.

So $\lambda_{max}^{RAR^T} \leq \lambda_{max}^{A}$ and we can say that $\lambda_{max}^{RAR^T} = max(\frac{A(R^T v, R^T v)}{(v,v)})$.
So $(v, v) = (u, u)$. (??)

From the definition of the filter $W = I - \mathcal{L}$, we have the submatrix $W_A = I - \mathcal{L}_A$, with I being the identity of appropriate order. For the algorithm, we need to compute the largest eigenvalues of W_A.

Theorem Computing the largest eigenvalues of W_A is equivalent to computing the smallest eigenvalues of $\mathcal{L}_A$.

Proof

$$W_A x = \lambda x \Leftrightarrow (I - \mathcal{L}_A)x = \lambda x$$

References

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