Review session dec 8

Question 1

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 4 & 2 \\ 3 & 4 & 2 \end{bmatrix}$$

a- Find null space of A.

$$null(A) = \{ X \in \mathbb{R}^3 : AX = 0 \}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 4 & 2 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & -8 & -4 & 0 \end{array}\right] R_3 \to R_3 - 2R_2 \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-4y - 2z = 0 \qquad z = \frac{-4}{2} y = -2y$$

$$x + 4y + 2z = 0 \qquad x = -4y + 4y = 0$$

$$null(A) = \{ x \in \mathbb{R}^3 : X = \begin{bmatrix} 0 \\ y \\ -2y \end{bmatrix} \; y \in \mathbb{R} \}$$

b- Find a basis of null (A) Call it S_1

$\{v_1, \dots, v_n\}$ are basis of V iff they span V & they're lin independent

$X \in null(A)$ iff $X = \begin{bmatrix} 0 \\ y \\ -2y \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ $\quad \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ spans null (A)

$a \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad a = 0$ so lin ind

$S_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$ is a basis of null(A).

c- deduce the nullity and rank(A)

nullity(A) = dim (null(A)) $\quad$ rank(A) = dim(rowspace (A)) = dim(columnspace(A))

nullity (A) + rank(A) = nbr of columns in A.

nullity (A) = 1 $\quad$ rank(A) = 3 - 1 = 2 d - Complete S_1 into a basis of $\mathbb{R}^3$. Call it S.

$S_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$ $\dim(\mathbb{R}^3) = 3$ (so 3 lin ind vectors are needed)

$\left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

e - verify that $T = \left\{ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

3 vectors in $\mathbb{R}^3$; it's enough to prove that they're lin. indep

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1 \times 1 \times 3 = 3 \neq 0$$. so T is a basis of $\mathbb{R}^3$.

f) find $[v]_T$ for $v = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ $\left(T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\} \right)$

$[v]_T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $v = a_1 u_1 + a_2 u_2 + a_3 u_3$ (lin comb of the vectors in T)

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$a_1 = 1 - 2a_2 + a_3 = 2$

$a_2 = 0$

$3a_3 = 3$ $a_3 = 1$

$$[v]_T = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

g) Find the transition matrix from T to S and deduce $[v]_S$

$$S = \left\{ \underset{s_1}{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}}; \underset{s_2}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}; \underset{s_3}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \right\} \qquad T = \left\{ \underset{u_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}, \underset{u_3}{\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}} \right\}$$

$$P_{S \leftarrow T} = \left[[u_1]_S \; [u_2]_S \; [u_3]_S \right] \qquad [v]_S = P_{S \leftarrow T} [v]_T$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 3 \end{array}\right] R_1 \to R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ -2 & 0 & 0 & 0 & 0 & 3 \end{array}\right] R_3 \to R_3 + 2R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & 0 & 2 & 3 \end{array}\right] R_3 \to \frac{R_3}{2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 & 1 & 3/2 \end{array}\right] R_1 \to R_1 - R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 & 1 & 3/2 \end{array}\right] \qquad \mathcal{P}_{S \leftarrow T} = \begin{bmatrix} 0 & 0 & -3/2 \\ 1 & 2 & -1 \\ 0 & 1 & 3/2 \end{bmatrix}$$

$$[v]_S = \begin{bmatrix} 0 & 0 & -3/2 \\ 1 & 2 & -1 \\ 0 & 1 & 3/2 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1 \\ 3/2 \end{bmatrix}$$

h - transform T into an orthonormal basis W and find $[v]_W$.

$$T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\} \qquad v = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\quad u_1 \qquad u_2 \qquad u_3$$

$$v_1 = u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$\langle u_2, v_1 \rangle = 2 \qquad \langle v_1, v_1 \rangle = 1^2 = 1$$

$$v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = u_3 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$\langle u_3, v_2 \rangle = 0 \qquad \langle v_2, v_2 \rangle = 1$$

$$\langle u_3, v_1 \rangle = -1$$

$$v_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$ is an orthog basis of $\mathbb{R}^3$

$$\langle v_3, v_3 \rangle = 9 \qquad \frac{v_3}{\|v_3\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = W$$

$$[v]_W = \begin{bmatrix} \langle v, w_1 \rangle \\ \langle v, w_2 \rangle \\ \langle v, w_3 \rangle \end{bmatrix}$$

$[v]_W = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ since W is the natural basis

Question 2 $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^3$ s.t. $\mathcal{L}\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_1 + 3u_2 \\ u_1 - u_2 \end{bmatrix}$

a. Prove that $\mathcal{L}$ is a linear transformation

$u, v \in \mathbb{R}^2$ $\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v)$; $\mathcal{L}(\lambda u) = \lambda \mathcal{L}(u)$ $\forall \lambda \in \mathbb{R}$

$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$

$$\mathcal{L}(u+v) = \begin{pmatrix} u_1 + v_1 \\ u_1 + v_1 + 3(u_2 + v_2) \\ u_1 + v_1 - (u_2 + v_2) \end{pmatrix} = \begin{bmatrix} u_1 \\ u_1 + 3u_2 \\ u_1 - u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_1 + 3v_2 \\ v_1 - v_2 \end{bmatrix} = \mathcal{L}(u) + \mathcal{L}(v)$$

$$\mathcal{L}(\lambda u) = \mathcal{L}\left(\begin{bmatrix} \lambda u_1 \\ \lambda u_2 \end{bmatrix}\right) = \begin{bmatrix} \lambda u_1 \\ \lambda u_1 + 3\lambda u_2 \\ \lambda u_1 - \lambda u_2 \end{bmatrix} = \lambda \begin{bmatrix} u_1 \\ u_1 + u_2 \\ u_1 - u_2 \end{bmatrix} = \lambda \mathcal{L}(u)$$

$\mathcal{L}$ is a linear transformation

b. Find $Ker(\mathcal{L})$ and check whether $\mathcal{L}$ is one-to-one. $\mathcal{L}: V \longrightarrow W$

$$\mathcal{L}\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_1 + 3u_2 \\ u_1 - u_2 \end{bmatrix}$$

$$\mathcal{L}\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_1 + 3u_2 \\ u_1 - u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} u_1 = 0 & u_1 = 0 \\ u_1 + 3u_2 = 0 & \\ u_1 - u_2 = 0 & u_2 = u_1 = 0 \end{cases}$$

$u \in Ker(\mathcal{L})$ iff $u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

so $\mathcal{L}$ is one-to-one

$Ker(\mathcal{L}) = \{ u \in V : \mathcal{L}(u) = 0_W \}$

$\mathcal{L}$ is one-to-one iff $Ker(\mathcal{L}) = \{0_V\}$

$Range(\mathcal{L}) = \{ w \in W : \exists\, v \in V \;\; \mathcal{L}(v) = w \}$

$\mathcal{L}$ is onto if $Range(\mathcal{L}) = W$

$\dim(Ker(\mathcal{L})) + \dim(Range(\mathcal{L})) = \dim(V)$

$\mathcal{L}$ is rep by $A_{m \times n}$ $\quad m = \dim(W)$, $n = \dim(V)$

$S = \{v_1, \dots, v_n\}$ basis of V & $T = \{w_1, \dots, w_m\}$ basis of W

$$A = \left[[\mathcal{L}(v_1)]_T \; [\mathcal{L}(v_2)]_T \cdots [\mathcal{L}(v_n)]_T \right]$$

$$[\mathcal{L}(v)]_T = A\,[v]_S$$

$\mathcal{L}$ is inv iff $\mathcal{L}$ is one-to-one and onto

C- Find the range ($\mathcal{L}$) and check whether $\mathcal{L}$ is onto.

Rmk deduce if $\mathcal{L}$ is onto you have to use the part b

$\dim(\ker(\mathcal{L})) = 0$

$\dim(\text{range}(\mathcal{L})) = \dim(\mathbb{R}^2) = 2$

so range $(\mathcal{L}) \neq \mathbb{R}^3$ so $\mathcal{L}$ is not onto

$$\text{range}(\mathcal{L}) = \left\{ w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{R}^3 : \exists\, u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2 : \mathcal{L}(u) = w \right\}$$

$$\mathcal{L}\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) = \begin{bmatrix} u_1 \\ u_1 + 3u_2 \\ u_1 - u_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 1 & 0 & w_1 \\ 1 & 3 & w_2 \\ 1 & -1 & w_3 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & w_1 \\ 0 & 3 & w_2 - w_1 \\ 0 & -1 & w_3 - w_1 \end{array}\right] R_3 \to R_3 + \tfrac{1}{3} R_2 \left[\begin{array}{cc|c} 1 & 0 & w_1 \\ 0 & 3 & w_2 - w_1 \\ 0 & 0 & w_3 - w_1 + \tfrac{1}{3} w_2 - \tfrac{1}{3} w_1 \end{array}\right]$$

$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \text{Range}(\mathcal{L})$ iff $w_3 - \frac{4}{3} w_1 + \frac{1}{3} w_2 = 0$ so Range $(\mathcal{L}) \neq \mathbb{R}^3$ and $\mathcal{L}$ is not onto

Question 3 Consider the symmetric matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

a- find a diag matrix D and an orthogonal matrix P s.t. $A = PDP^T$

orthog matrix $P^{-1} = P^T$ (the columns are an orthonormal set)

eigenvectors associated to $\neq$ eigenvalues are $\perp$

step1 eigenvectors $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{vmatrix} 3-\lambda & 1 & -1 \\ -2+\lambda & 2-\lambda & 0 \\ -1 & -1 & 5-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 3-\lambda & 1 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 5-\lambda \end{vmatrix}$$

$$\xrightarrow{C_2 \to C_2 + C_1} (2-\lambda) \begin{vmatrix} 3-\lambda & 4-\lambda & -1 \\ -1 & 0 & 0 \\ -1 & -2 & 5-\lambda \end{vmatrix} = (2-\lambda)(-1)(-1)^3 \begin{vmatrix} 4-\lambda & -1 \\ -2 & 5-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(4-\lambda)(5-\lambda) - 2\right] = (2-\lambda)\left[\lambda^2 - 9\lambda + 18\right]$$

$\Delta = 81 - 72 = 9$

$\lambda' = \frac{9-3}{2} = 3$

$\lambda'' = \frac{9+3}{2} = 6$

$|A - \lambda I| = (2-\lambda)(\lambda-3)(\lambda-6)$ so $\lambda_1 = 2$; $\lambda_2 = 3$ and $\lambda_3 = 6$ are the eigenvalues of A.

$A_{n\times n}$ if we have $n \neq$ eigenvalues then A is diag

$$(A-\lambda I) = \begin{bmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{bmatrix}$$

since A has $3 \neq$ eigenv A is diag and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

Step 2 in order to find P we have to find the eigenvectors.

$\lambda_1 = 2 \quad (A-2I)X = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 3 & 0 \end{array}\right] \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{matrix} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

$x_1 = -x_2 + x_3 = -x_2$

$x_3 = 0$

$$X = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_2 = 3 \quad (A-3I)X = 0$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] R_1 \to R_2 \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] R_3 \to R_3 + R_1 \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] R_3 \to R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]. \quad \begin{matrix} x_1 = x_3 \\ x_2 = x_3 \end{matrix}$$

$$X = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda_3 = 6 \quad (A - 6I)X = 0$

$$\left[\begin{array}{ccc|c} -3 & 1 & -1 & 0 \\ 1 & -3 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \; R_1 \to R_2 \; \left[\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ -3 & 1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \; \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ 0 & -8 & -4 & 0 \\ 0 & -4 & -2 & 0 \end{array}\right] \; R_3 \to R_3 - \tfrac{1}{2}R_2 \; \left[\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ 0 & -8 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 3x_2 + x_3 = x_2$

$-8x_2 = 4x_3 \qquad x_3 = -2x_2$

$$X = \begin{bmatrix} x_2 \\ x_2 \\ -2x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ they're orthog since they're associated to $\neq$ eigenvalues

$\|u_1\| = \sqrt{1+1+0} = \sqrt{2} \quad \|u_2\| = \sqrt{1+1+1} = \sqrt{3} \quad \|u_3\| = \sqrt{6}$

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

b- Compute A^{100}

$$A^{100} = P\, D^{100}\, P^T$$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2^{100} & 0 & 0 \\ 0 & 3^{100} & 0 \\ 0 & 0 & 6^{100} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}$$

$= \ldots$