AMA2111 Mathematics I

Test Solution

1. (a) Reduce the augmented matrix to the row-echelon form:

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 1 \\ 2 & 5 & 2 & -3 & 0 \\ 1 & 0 & -a & -5 & -b \\ 3 & 7 & 2 & a & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & a-4 & 1 & b-5 \\ 0 & 0 & 0 & a+5 & 3 \end{array}\right].$$

When $a \neq 4$ and $a \neq -5$, the system has a unique solution.

When $a = -5$, the last row of the row-echelon form gives $0 = 3$, so the system has no solution.

When $a = 4$, we conduct the row operation once more:

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & b-5 \\ 0 & 0 & 0 & 9 & 3 \end{array}\right] \xrightarrow{r_4 - 9r_3 \to r_4} \left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & b-5 \\ 0 & 0 & 0 & 0 & 48-9b \end{array}\right].$$

When $48 - 9b \neq 0$, namely, $b \neq \frac{16}{3}$, the system has no solution. Otherwise, $b = \frac{16}{3}$, the system has infinitely many solutions. In summary,

- when $a \neq 4$ and $a \neq -5$, the system has a unique solution;
- when $a = -5$ or $\begin{cases} a = 4 \\ b \neq \frac{16}{3} \end{cases}$, the system has no solution;
- when $a = 4$ and $b = \frac{16}{3}$, the system has infinitely many solutions. Now, the system is equivalent to

$$\begin{cases} x_1 + 3x_2 + 2x_3 - x_4 = 1 \\ \qquad x_2 + 2x_3 + x_4 = 2 \\ \qquad\qquad x_4 = \dfrac{1}{3} \end{cases} \Rightarrow \begin{cases} x_1 = -\dfrac{11}{3} + 4s \\ x_2 = \dfrac{5}{3} - 2s \\ x_3 = s \\ x_4 = \dfrac{1}{3} \end{cases}.$$

(b) (i) According to the row operations in (a), we have

$$[\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_4\ \boldsymbol{v}_5] \to \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & b-5 \\ 0 & 0 & a+5 & 3 \end{bmatrix} \xrightarrow{r_4 - (a+5)r_3 \to r_4} \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & b-5 \\ 0 & 0 & 0 & 3-(a+5)(b-5) \end{bmatrix}.$$

When $(a+5)(b-5) \neq 3$, the vectors $\boldsymbol{v}_1$,$\boldsymbol{v}_2$,$\boldsymbol{v}_4$ and $\boldsymbol{v}_5$ are linearly independent, and thus, their spanned space is $\mathbb{R}_4$.

(ii)

$$[\boldsymbol{v}_3 \ \boldsymbol{v}_4 \ \boldsymbol{v}_5] = \begin{bmatrix} 2 & -1 & 1 \\ 2 & -3 & 0 \\ -a & -5 & -b \\ 2 & a & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & b-\frac{3a}{4}-\frac{5}{2} \\ 0 & 0 & 1-a \end{bmatrix}.$$

To make $\boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5$ be linearly dependent, we need $1-a=0$ and $b-\frac{3a}{4}-\frac{5}{2}=0$, namely, $a=1$ and $b=\frac{13}{4}$.

Therefore, when $a=1$ and $b=\frac{13}{4}$, the vectors $\boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5$ are linearly dependent. Now, the equation $t_3\boldsymbol{v}_3 + t_4\boldsymbol{v}_4 + t_5\boldsymbol{v}_5 = \mathbf{0}$ is equivalent to

$$\begin{cases} t_3 - \dfrac{1}{2}t_4 + \dfrac{1}{2}t_5 = 0 \\ \qquad\quad t_4 + \dfrac{1}{2}t_5 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} t_3 = -\dfrac{3}{4}s \\ t_4 = -\dfrac{1}{2}s \\ t_5 = s \end{cases}.$$

Then, we have

$$-\frac{3}{4}\boldsymbol{v}_3 - \frac{1}{2}\boldsymbol{v}_4 + \boldsymbol{v}_5 = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{v}_3 = -\frac{2}{3}\boldsymbol{v}_4 + \frac{4}{3}\boldsymbol{v}_5.$$

2. (a)

$$|A - \lambda I| = -(\lambda - 2)(\lambda - 3)^2 = 0.$$

Therefore, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = 3$.

A linearly independent eigenvector corresponding to $\lambda_1 = 2$ is given by

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Two linearly independent eigenvectors corresponding to $\lambda_2 = \lambda_3 = 3$ are given by

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Let

$$P = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} -1 & 2 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(b) Cannot find an orthogonal matrix such that $A = QDQ^T$.

If $A = QDQ^T$, $A^T = QD^TQ^T = QDQ^T = A$. However, $A^T \neq A$.

(c)

$$|A^7| = |PD^7P^{-1}| = |P||D|^7|P^{-1}| = |PP^{-1}||D|^7 = |D|^7 = (2 \times 3 \times 3)^7 = 18^7$$

3

$\sqrt{3} - i = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$

$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$\begin{aligned}
&\left(\sqrt{3} - i\right)^4 (1+i)^{10} \\
&= \left[2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)\right]^4 \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{10} \\
&= 2^9\left[\cos\left(4\times\frac{11\pi}{6} + 10\times\frac{\pi}{4}\right) + i\sin\left(4\times\frac{11\pi}{6} + 10\times\frac{\pi}{4}\right)\right] \\
&= 2^9\left(\cos\frac{59\pi}{6} + i\sin\frac{59\pi}{6}\right) = 2^9\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)
\end{aligned}$$

$$z = 2^{\frac{9}{5}} e^{i\left(\frac{11\pi}{6} + 2k\pi\right)\cdot\frac{1}{5}} = 2^{\frac{9}{5}} e^{i\frac{11+12k}{30}\pi}$$

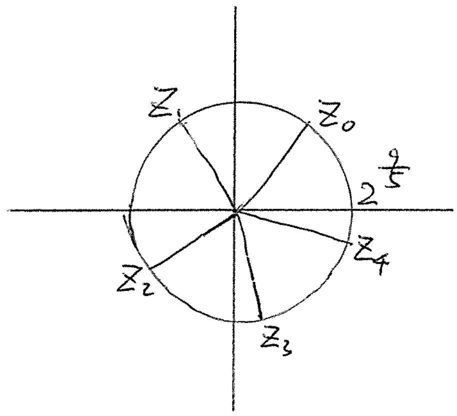
$$\therefore z_0 = 2^{\frac{9}{5}} e^{i\frac{11\pi}{30}}$$

$$z_1 = 2^{\frac{9}{5}} e^{i\frac{23\pi}{30}}$$

$$z_2 = 2^{\frac{9}{5}} e^{i\frac{35\pi}{30}}$$

$$z_3 = 2^{\frac{9}{5}} e^{i\frac{47\pi}{30}}$$

$$z_4 = 2^{\frac{9}{5}} e^{i\frac{59\pi}{30}}$$



4. Let $\mu(x) = e^{\int \frac{2}{x}dx} = x^2$, then

$$(\mu(x)y(x))' = \frac{\mu(x)e^x}{x^2} = e^x \quad \Rightarrow \quad \mu(x)y(x) = e^x + C,$$

so $y(x) = \dfrac{e^x + C}{x^2}$. Using the initial condition, we get $C = 0$. Thus,

$$y(x) = \frac{e^x}{x^2}.$$

5. This is a Bernoulli equation with $n = 3$. $y = 0$ is the trivial solution. If $y \neq 0$, divide the equation by y^3 to get

$$y^{-3}y' + xy^{-2} = x.$$

Let $v = y^{-2}$, then $v' = -2y^{-3}y'$, and thus

$$v' - 2xv = -2x.$$

Let

$$\mu(x) = e^{\int(-2x)dx} = e^{-x^2},$$

then

$$\int (-2x)\mu(x)dx = \int (-2x)e^{-x^2}dx = e^{-x^2},$$

and thus,

$$v(x) = \frac{e^{-x^2} + C}{e^{-x^2}} = 1 + Ce^{x^2}.$$

Therefore,

$$y(x) = \frac{\pm 1}{\sqrt{1 + Ce^{x^2}}}.$$

6. $y = ux; u'x + u = \dfrac{u^3 + 2u^2 + u + 1}{(u+1)^2} = \dfrac{u(u+1)^2 + 1}{(u+1)^2} = u + \dfrac{1}{(u+1)^2}; u'x = \dfrac{1}{(u+1)^2};$
$(u+1)^2u' = \dfrac{1}{x}; \dfrac{(u+1)^3}{3} = \ln|x| + c; (u+1)^3 = 3(\ln|x| + c); \left(\dfrac{y}{x} + 1\right)^3 = 3(\ln|x| + c);$
$(y + x)^3 = 3x^3(\ln|x| + c).$

$\therefore Y = X(3\ln|X| + C)^{\frac{1}{3}} - X$