AMA 2111 HW1

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1、 $\frac{(\sqrt{3}-i)^{10}}{(-1+\sqrt{3}i)^7}$

$\sqrt{3}-i = 2e^{i(-\frac{\pi}{6})}$

$-1+\sqrt{3}i = 2e^{i(\frac{2\pi}{3})}$

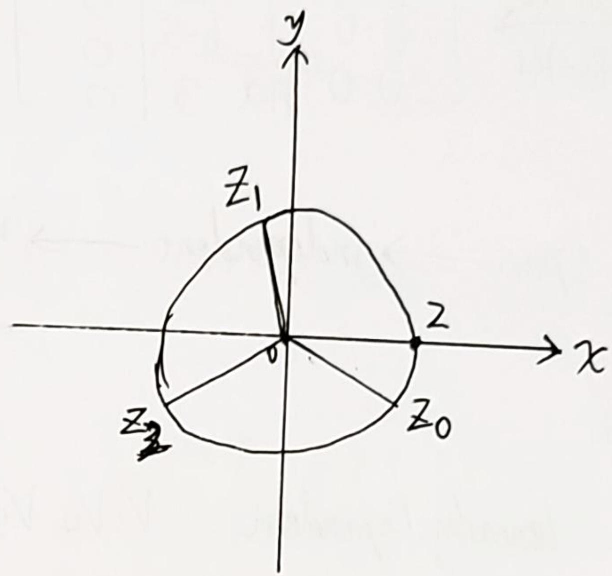
$z^3 = w = \frac{2^{10}e^{i(-\frac{5\pi}{3})}}{2^7 e^{i(\frac{14\pi}{3})}} = 2^3 e^{i(-\frac{19\pi}{3})}$

$z = 2e^{i\frac{(-\frac{19\pi}{3})+2k\pi}{3}} = 2e^{i\frac{-19+6k}{9}\pi}, K=0,1,2$

$z_0 = 2e^{i(\frac{-19}{9})\pi}$

$z_1 = 2e^{i(\frac{-13}{9})\pi}$

$z_2 = 2e^{i(\frac{-7}{9})\pi}$



2、

(a) $\left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 1 \\ -2 & -5 & -2 & 3 & 0 \\ -1 & 0 & a & 5 & b \\ 3 & 7 & 2 & a & 2 \end{array}\right] \xrightarrow[R_2=2R_1+R_2]{R_3=R_1+R_3} \left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 3 & 2+a & 4 & b+1 \\ 3 & 7 & 2 & a & 2 \end{array}\right] \xrightarrow{R_4=3R_1-R_3} \left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 3 & 2+a & 4 & b+1 \\ 0 & 2 & 4 & -3-a & 1 \end{array}\right]$

$\xrightarrow[R_3=3R_2-R_3]{R_4=2R_2-R_4} \left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & a-4 & 1 & b-5 \\ 0 & 0 & 0 & a+5 & 3 \end{array}\right]$

i、no solution: ~~a≠-5~~

$a+5=0$

$a=-5$

ii、∞ solution:

$a+5\neq0$ & $a-4=0 \Rightarrow$

$a\neq-5$ & $a=4$ & $b=\frac{16}{3}$

iii、unique solution:

$a-4\neq0$ & $a+5=0$

$a\neq4$ & $a\neq-5$

if $a-4=0$: ~~a≠4~~

$\left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & b-5 \\ 0 & 0 & 0 & a+5 & 3 \end{array}\right]$

$a=4$

$\therefore x_4=b-5$

$(a+5)x_4=3$

$x_4=\frac{1}{3}$

$\frac{1}{3}=b-5$

$b=\frac{16}{3}$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4t-\frac{11}{3} \\ \frac{5}{3}-2t \\ t \\ \frac{1}{3} \end{bmatrix}$

$= t\begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{11}{3} \\ \frac{5}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}$

(b)、

i)、

$$[V_1V_2V_4V_5] \Rightarrow \begin{bmatrix} 1 & 3 & -1 & 1 \\ -2 & -5 & 3 & 0 \\ -1 & 0 & 5 & b \\ 3 & 7 & a & 2 \end{bmatrix} \xrightarrow[R_4=R_4+R_2]{R_3=R_1+R_3} \left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 0 \\ -2 & -5 & 3 & 0 & 0 \\ 0 & 3 & 4 & b+1 & 0 \\ 1 & 2 & 3+a & 2 & 0 \end{array}\right] \xrightarrow[R_4=R_1-R_4]{R_2=2R_1+R_2} \left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 4 & b+1 & 0 \\ 0 & 1 & -4-a & -1 & 0 \end{array}\right]$$

$$\xrightarrow[R_4=R_2-R_4]{R_3=R_3-3R_2} \left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & b-5 & 0 \\ 0 & 0 & 5+a & 3 & 0 \end{array}\right] \xrightarrow{R_4=R_4-(5+a)R_3} \left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & b-5 & 0 \\ 0 & 0 & 0 & 3-(5+a)(b-5) & 0 \end{array}\right]$$

span $\longrightarrow$ independent $\longrightarrow$ unique solution $\longrightarrow 3-(5+a)(b-5) \neq 0$

ii)、

$\because$ Linearly Dependent : $V_3 V_4 V_5$

$$\therefore [V_3V_4V_5] \Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ -2 & 3 & 0 & 0 \\ a & 5 & b & 0 \\ 2 & a & 2 & 0 \end{array}\right] \xrightarrow[R_4=R_4-R_1]{R_2=R_1+R_2} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ a & 5 & b & 0 \\ 0 & a+1 & 1 & 0 \end{array}\right] \xrightarrow{\frac{R_1}{2}} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 2 & 1 & 0 \\ a & 5 & b & 0 \\ 0 & a+1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[\frac{R_2}{2}]{R_3=R_3-aR_1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 5+\frac{a}{2} & b-\frac{a}{2} & 0 \\ 0 & a+1 & 1 & 0 \end{array}\right] \xrightarrow{R_3=(5+\frac{a}{2})R_2-R_3} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{(5+\frac{a}{2})}{2}-b+\frac{a}{2} & 0 \\ 0 & a+1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_4=(a+1)R_2-R_4} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{(5+\frac{a}{2})}{2}-b+\frac{a}{2} & 0 \\ 0 & 0 & \frac{(a+1)}{2}-1 & 0 \end{array}\right]$$

dependent $\longrightarrow \infty$ solutions $\Longrightarrow \begin{cases} \frac{(5+\frac{a}{2})}{2}-b+\frac{a}{2} = \frac{(a+1)}{2}-1 \\ (\frac{5+\frac{a}{2}}{2})-b+\frac{a}{2}=0 \\ \frac{(a+1)}{2}-1=0 \end{cases}$

$$\frac{a+1}{2}-1=0 \qquad a+1=2 \qquad a=1$$

$$(\frac{5+\frac{1}{2}}{2})-b+\frac{1}{2}=0 \qquad b=\frac{13}{4}$$

$$\therefore \begin{cases} a=1 \\ b=\frac{13}{4} \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow t=\begin{bmatrix} -3s \\ -2s \\ 4s \end{bmatrix}$$

$$-3sV_3 - 2sV_4 + 4sV_5 = 0$$

$$\Downarrow$$

$$V_3 = \frac{-2sV_4+4sV_5}{3s} = \frac{-2V_4+4V_5}{3}$$

3、

$$A=\begin{bmatrix}2&2&1\\1&1&-1\\-2&4&5\end{bmatrix}$$

① $|A-\lambda|=\begin{vmatrix}2-\lambda&2&1\\1&1-\lambda&-1\\-2&4&5-\lambda\end{vmatrix}\xrightarrow{C_3=C_3+C_1}\begin{vmatrix}2-\lambda&2&3-\lambda\\1&1-\lambda&0\\-2&4&3-\lambda\end{vmatrix}\xrightarrow{R_3=R_3-R_1}\begin{vmatrix}2-\lambda&2&3-\lambda\\1&1-\lambda&0\\\lambda-4&2&0\end{vmatrix}$

② $(3-\lambda)\begin{vmatrix}1&1-\lambda\\\lambda-4&2\end{vmatrix}=(3-\lambda)(1\times2-(1-\lambda)(\lambda-4))=\cancel{(3-\lambda)(\lambda^2-\lambda+2)}\,(3-\lambda)(x^2-5x+6)$

$\therefore\ \lambda_1=\lambda_2=3\quad\lambda_3=2$

$\therefore\ \lambda_1=\lambda_2=3$:

$$\left[\begin{array}{ccc|c}2-3&2&1&0\\1&1-3&-1&0\\-2&4&5-3&0\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}-1&2&1&0\\1&-2&-1&0\\-2&4&2&0\end{array}\right]\xrightarrow{R_2=R_1+R_2}\left[\begin{array}{ccc|c}-1&2&1&0\\0&0&0&0\\-2&4&2&0\end{array}\right]$$

$$\xrightarrow[R_1\times(-1)]{R_2\leftrightarrow R_3}\left[\begin{array}{ccc|c}1&-2&-1&0\\-2&4&2&0\\0&0&0&0\end{array}\right]\xrightarrow{R_2=R_2+2R_1}\left[\begin{array}{ccc|c}1&-2&-1&0\\0&0&0&0\\0&0&0&0\end{array}\right]\Rightarrow V=\begin{bmatrix}2s+t\\s\\t\end{bmatrix}$$

$$V_1=\begin{bmatrix}2s\\s\\0\end{bmatrix}=\begin{bmatrix}2\\1\\0\end{bmatrix}\quad V_2=\begin{bmatrix}t\\0\\t\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$\therefore\ \lambda_3=2$:

$$\left[\begin{array}{ccc|c}0&2&1&0\\1&-1&-1&0\\-2&4&3&0\end{array}\right]\xrightarrow{R_1\leftrightarrow R_2}\left[\begin{array}{ccc|c}1&-1&-1&0\\0&2&1&0\\-2&4&3&0\end{array}\right]\xrightarrow{R_3=R_3+2R_1}\left[\begin{array}{ccc|c}1&-1&-1&0\\0&2&1&0\\0&2&1&0\end{array}\right]$$

$$\xrightarrow{R_3=R_2-R_3}\left[\begin{array}{ccc|c}1&-1&-1&0\\0&2&1&0\\0&0&0&0\end{array}\right]\Rightarrow V_3=\begin{bmatrix}t\\-t\\2t\end{bmatrix}=\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

$\therefore\ P=\begin{bmatrix}2&1&1\\1&0&-1\\0&1&2\end{bmatrix}\qquad\cancel{A=PDP^{-1}}$

$\therefore\ D=\begin{bmatrix}3&0&0\\0&3&0\\0&0&2\end{bmatrix}$

4.

(a) $A=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$

$$|A-\lambda|=\begin{vmatrix}1-\lambda&1&1\\1&1-\lambda&1\\1&1&1-\lambda\end{vmatrix}\xrightarrow{R_3=R_3-R_2}\begin{vmatrix}1-\lambda&1&1\\1&1-\lambda&1\\0&\lambda&-\lambda\end{vmatrix}\xrightarrow{C_3=C_3+C_2}\begin{vmatrix}1-\lambda&1&2\\1&1-\lambda&2-\lambda\\0&\lambda&0\end{vmatrix}$$

$$\Rightarrow(-\lambda)\cdot\begin{vmatrix}1-\lambda&2\\1&2-\lambda\end{vmatrix}=(-\lambda)\cdot\left[(1-\lambda)\cdot(2-\lambda)-2\right]$$

$$\Rightarrow\lambda_1=\lambda_2=0,\ \lambda_3=3$$

$\lambda_1=\lambda_2=0$:

$$\left[\begin{array}{ccc|c}1&1&1&0\\1&1&1&0\\1&1&1&0\end{array}\right]\xrightarrow{R_3=R_2-R_3}\left[\begin{array}{ccc|c}1&1&1&0\\1&1&1&0\\0&0&0&0\end{array}\right]\xrightarrow{R_2=R_1-R_2}\left[\begin{array}{ccc|c}1&1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right]\Rightarrow V=\begin{bmatrix}-s-t\\s\\t\end{bmatrix}$$

$V_1=\begin{bmatrix}-s\\s\\0\end{bmatrix}$ $V_2=\begin{bmatrix}-t\\0\\t\end{bmatrix}$

$V_1=\begin{bmatrix}-1\\1\\0\end{bmatrix}$ $V_2=\begin{bmatrix}-1\\0\\1\end{bmatrix}$

$\lambda_3=3$:

$$\left[\begin{array}{ccc|c}1-3&1&1&0\\1&1-3&1&0\\1&1&1-3&0\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}-2&1&1&0\\1&-2&1&0\\1&1&-2&0\end{array}\right]\xrightarrow{R_2=R_1+2R_2}\left[\begin{array}{ccc|c}-2&1&1&0\\0&-3&3&0\\1&1&-2&0\end{array}\right]\xrightarrow[\frac{R_2}{-3}]{R_1\leftrightarrow R_3}\left[\begin{array}{ccc|c}1&1&-2&0\\0&1&-1&0\\-2&1&1&0\end{array}\right]$$

$$\xrightarrow{R_3=R_3+2R_1}\left[\begin{array}{ccc|c}1&1&-2&0\\0&1&-1&0\\0&3&-3&0\end{array}\right]\xrightarrow{R_3=3R_2-R_3}\left[\begin{array}{ccc|c}1&1&-2&0\\0&1&-1&0\\0&0&0&0\end{array}\right]\Rightarrow V_3\begin{bmatrix}t\\t\\t\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

(b) $\langle V_1,V_2\rangle=1$, $\langle V_1,V_3\rangle=0$, $\langle V_2,V_3\rangle=0$

$\because$ $\langle V_1,V_2\rangle=1$ need change to $\langle V_1,U_2\rangle=0$

$$\therefore U_2=V_2-Proj_{V_1}V_2=V_2-\frac{\langle V_2,V_1\rangle}{\|V_1\|^2}V_1=\begin{bmatrix}-1\\0\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}$$

$$\|V_1\|=\sqrt{\langle V_1,V_1\rangle}=\sqrt{(-1)^2+1^2+0^2}=\sqrt{2}$$

$$\|U_2\|=\sqrt{\langle U_2,U_2\rangle}=\sqrt{(-\tfrac{1}{2})^2+(-\tfrac{1}{2})^2+1^2}=\frac{\sqrt{6}}{2}$$

$$\|V_3\|=\sqrt{\langle V_3,V_3\rangle}=\sqrt{1^2+1^2+1^2}=\sqrt{3}$$

$$\therefore Q=\begin{bmatrix}-\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{3}}\\0&\frac{2}{\sqrt{6}}&\frac{1}{\sqrt{3}}\end{bmatrix},\quad D=\begin{bmatrix}0&0&0\\0&0&0\\0&0&3\end{bmatrix}$$

5、

$$y' \cos x + \sin^2 y = \cos^2 y$$

$$y' = \frac{\cos^2 y - \sin^2 y}{\cos x}$$

$$y' = \frac{\cos(2y)}{\cos(x)}$$

$$\frac{dy}{dx} = \cos(2y) \times \frac{1}{\cos(x)}$$

$$\frac{dy}{\cos(2y)} = \frac{dx}{\cos(x)}$$

$$\int \frac{1}{\cos(2y)} dy = \int \frac{1}{\cos(x)} dx$$

$$\frac{1}{2} \ln(\sec(2y) + \tan(2y)) = \ln(\sec(x) + \tan(x)) + C$$

6、

$$-xy' + 2y = \ln x$$

$$\frac{dy}{dx} = -\frac{\ln x - 2y}{x}$$

$$\frac{dy}{dx} = -\frac{\ln x}{x} + \frac{2y}{x}$$

$$\frac{dy}{dx} - \frac{2y}{x} = -\frac{\ln x}{x}$$

$\because \frac{dy}{dx} + p(x)y = q(x) \Longleftarrow$ 1st order linear ODE

$\therefore p(x) = \frac{-2}{x}, \ q(x) = -\frac{\ln(x)}{x}$

$$\frac{du}{dx} = \mu p \qquad \frac{du}{dx} = \frac{2}{x}\mu \Rightarrow \int \frac{1}{\mu} d\mu = \int \frac{1}{\mu} du = \int -\frac{2}{x} dx$$

$$\ln \mu = -2\ln x \Rightarrow \ln \mu = \ln x^{-2} \Rightarrow \mu = x^{-2} = \frac{1}{x^2}$$

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)q(x)$$

Let $\frac{du}{dx} = \mu(x)(p(x))$

$$\mu(x)\frac{dy}{dx} + \frac{du}{dx}y = \mu(x)q(x) = \mu(x) \cdot -\frac{\ln(x)}{x} = -\frac{\mu(x)\ln(x)}{x}$$

$$\mu(x)y = \int \mu(x)q(x)dx + C \Rightarrow \cancel{y = \frac{\mu(x)}{\mu(x)}} \ \mu(x)y = \int -\frac{\mu(x)\ln(x)}{x} dx + C$$

$$y = \frac{\int -\frac{\mu(x)\ln(x)}{x} dx + C}{\mu(x)} \Rightarrow y = \frac{\int -\frac{\frac{1}{x^2}\ln(x)}{x} dx + C}{\frac{1}{x^2}}$$

$$y = \frac{\frac{\ln(x)}{2x^2} + \frac{1}{4x^2} + C}{\frac{1}{x^2}} = \frac{\ln(x)}{2} + \frac{1}{4} + C \cdot x^2$$

7、

$$y' - \frac{y}{x} = \frac{(x+y)^2}{x^2} \quad (x>0),\ y(1) = -2$$

$$y' - \frac{1}{x}y = \frac{x^2+y^2+2xy}{x^2}$$

$$y' - \frac{1}{x}y = 1 + \frac{y^2}{x^2} + \frac{2y}{x}$$

$$\frac{dy}{dx} = 1 + \frac{y^2}{x^2} + \frac{3y}{x}$$

let $v = \frac{y}{x}$

$\therefore y = vx$

$$\frac{dy}{dx} = v'x + v$$

$$x\frac{dv}{dx} + v = 1 + v^2 + 2v$$

$$x\frac{dv}{dx} = (v+1)^2$$

① $$\frac{dv}{(v+1)^2} = \frac{dx}{x}$$

讨 $(v+1)^2 \neq 0$

$$\int \frac{1}{(v+1)^2}\,dv = \int \frac{1}{x}\,dx$$

$\cancel{\ln|(v+1)^2| = \ln|x| + C}$

$\cancel{(v+1)^2 = e^{(\ln|x|+C)}}$

$\cancel{(v+1)^2 = e^{\ln|x|} \cdot e^{C}}$

$\cancel{(\frac{y}{x}+1)^2 = \pm x \cdot C = x}$

$\cancel{\because y(1) = -2}$

$\cancel{\therefore y = -2}$

$\cancel{x = 1}$

$$-\frac{1}{v+1} = \ln|x| + C$$

$\because y(1) = -2$

$\therefore x = 1$

$y = -2$

$\therefore v = -2$

$$-\frac{1}{-2+1} = \ln|1| + C$$

$$C = 1$$

$$\therefore \frac{-1}{v+1} = \ln|x| + 1$$

$$\frac{y}{x} = \frac{-1}{\ln|x|+1} - 1$$

$$y = \frac{-x}{\ln|x|+1} - x$$

$\because x > 0$

$$\therefore y = \frac{-x}{\ln x + 1} - x,\ x>0$$

② 讨 $(v+1)^2 = 0$

$v + 1 = 0$

$v = -1$

$-1 = \frac{y}{x}$

$y = -x$

$\because y(+1) = -2$

$y = -1 \neq y(1) = -2$

$\therefore$ is not a solution