AMA2111 Mathematics I

Homework 1 Solution

1. Find all the cubic roots of $\dfrac{(\sqrt{3}-i)^{10}}{(-1+\sqrt{3}i)^7}$ and plot them in the complex plane.

   Solution: Since
   $$\sqrt{3}-i = 2e^{-i\frac{\pi}{6}}, \quad -1+\sqrt{3}i = 2e^{i\frac{2}{3}\pi},$$
   we have
   $$\frac{(\sqrt{3}-i)^{10}}{(-1+\sqrt{3}i)^7} = \frac{2^{10}e^{-i\frac{10\pi}{6}}}{2^7 e^{i\frac{14}{3}\pi}} = 2^3 e^{-i\frac{19}{3}\pi} = 2^3 e^{-i\frac{1}{3}\pi}.$$
   The cubic roots are
   $$z_k = 2e^{-i\frac{1}{9}\pi + i\frac{6k\pi}{9}}, \quad k = 0, 1, 2.$$
   That is
   $$z_0 = 2e^{-i\frac{1}{9}\pi}, \;\; z_1 = 2e^{i\frac{5}{9}\pi}, \;\; z_2 = 2e^{i\frac{11}{9}\pi}.$$

2. Consider the linear system

$$\begin{bmatrix} 1 & 3 & 2 & -1 \\ -2 & -5 & -2 & 3 \\ -1 & 0 & a & 5 \\ 3 & 7 & 2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ b \\ 2 \end{bmatrix}.$$

(a) Find the conditions satisfied by a and b such that the system has

i. no solutions;

ii. infinitely many solutions;

iii. a unique solution.

Also solve the system when it has infinitely many solutions.

(b) Consider the vectors $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 3 \\ -5 \\ 0 \\ 7 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 2 \\ -2 \\ a \\ 2 \end{bmatrix}$, $\boldsymbol{v}_4 = \begin{bmatrix} -1 \\ 3 \\ 5 \\ a \end{bmatrix}$ and $\boldsymbol{v}_5 = \begin{bmatrix} 1 \\ 0 \\ b \\ 2 \end{bmatrix}$ in $\mathbb{R}_4$.

i. Find the conditions satisfied by a and b such that the span of $\boldsymbol{v}_1$,$\boldsymbol{v}_2$,$\boldsymbol{v}_4$ and $\boldsymbol{v}_5$ is $\mathbb{R}_4$.

ii. Find the conditions satisfied by a and b such that $\boldsymbol{v}_3$,$\boldsymbol{v}_4$ and $\boldsymbol{v}_5$ are linearly dependent and write $\boldsymbol{v}_3$ as a linear combination of $\boldsymbol{v}_4$ and $\boldsymbol{v}_5$, if possible.

Solution. (a) Reduce the augmented matrix to the row-echelon form:

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 1 \\ -2 & -5 & -2 & 3 & 0 \\ -1 & 0 & a & 5 & b \\ 3 & 7 & 2 & a & 2 \end{array}\right] \xrightarrow[r_4-3r_1\to r_4]{\substack{r_2+2r_1\to r_2 \\ r_3+r_1\to r_3}} \left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 3 & a+2 & 4 & b+1 \\ 0 & -2 & -4 & a+3 & -1 \end{array}\right] \xrightarrow[r_4+2r_2\to r_4]{r_3-3r_2\to r_3} \left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & a-4 & 1 & b-5 \\ 0 & 0 & 0 & a+5 & 3 \end{array}\right].$$

When $a \neq 4$ and $a \neq -5$, the system has a unique solution.

When $a = -5$, the last row of the row-echelon form gives $0 = 3$, so the system has no solution.

When $a = 4$, we conduct the row operation once more:

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & b-5 \\ 0 & 0 & 0 & 9 & 3 \end{array}\right] \xrightarrow{r_4-9r_3\to r_4} \left[\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & b-5 \\ 0 & 0 & 0 & 0 & 48-9b \end{array}\right].$$

When $48 - 9b \neq 0$, namely, $b \neq \frac{16}{3}$, the system has no solution. Otherwise, $b = \frac{16}{3}$, the system has infinitely many solutions. In summary,

- when $a \neq 4$ and $a \neq -5$, the system has a unique solution;
- when $a = -5$ or $\left\{ \begin{array}{l} a = 4 \\ b \neq \frac{16}{3} \end{array} \right.$, the system has no solution;
- when $a = 4$ and $b = \frac{16}{3}$, the system has infinitely many solutions. Now, the system is equivalent to

$$\left\{ \begin{aligned} x_1 + 3x_2 + 2x_3 - x_4 &= 1 \\ x_2 + 2x_3 + x_4 &= 2 \\ x_4 &= \frac{1}{3} \end{aligned} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} x_1 = -\dfrac{11}{3} + 4s \\ x_2 = \dfrac{5}{3} - 2s \\ x_3 = s \\ x_4 = \dfrac{1}{3} \end{array} \right. .$$

(b) (i) According to the row operations in (a), we have

$$[\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_4\ \boldsymbol{v}_5] \to \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & b-5 \\ 0 & 0 & a+5 & 3 \end{bmatrix} \xrightarrow{r_4-(a+5)r_3 \to r_4} \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & b-5 \\ 0 & 0 & 0 & 3-(a+5)(b-5) \end{bmatrix}.$$

When $(a+5)(b-5) \neq 3$, the vectors $\boldsymbol{v}_1$,$\boldsymbol{v}_2$,$\boldsymbol{v}_4$ and $\boldsymbol{v}_5$ are linearly independent, and thus, their spanned space is $\mathbb{R}_4$.

(ii)

$$[\boldsymbol{v}_3\ \boldsymbol{v}_4\ \boldsymbol{v}_5] = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & 0 \\ a & 5 & b \\ 2 & a & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & b - \frac{3a}{4} - \frac{5}{2} \\ 0 & 0 & 1-a \end{bmatrix}.$$

To make $\boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5$ be linearly dependent, we need $1-a=0$ and $b - \frac{3a}{4} - \frac{5}{2} = 0$, namely, $a = 1$ and $b = \frac{13}{4}$.

Therefore, when $a = 1$ and $b = \frac{13}{4}$, the vectors $\boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5$ are linearly dependent. Now, the equation $t_3\boldsymbol{v}_3 + t_4\boldsymbol{v}_4 + t_5\boldsymbol{v}_5 = \mathbf{0}$ is equivalent to

$$\left\{ \begin{aligned} t_3 - \frac{1}{2}t_4 + \frac{1}{2}t_5 &= 0 \\ t_4 + \frac{1}{2}t_5 &= 0 \end{aligned} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} t_3 = -\dfrac{3}{4}s \\ t_4 = -\dfrac{1}{2}s \\ t_5 = s \end{array} \right. .$$

Then, we have

$$-\frac{3}{4}\boldsymbol{v}_3 - \frac{1}{2}\boldsymbol{v}_4 + \boldsymbol{v}_5 = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{v}_3 = -\frac{2}{3}\boldsymbol{v}_4 + \frac{4}{3}\boldsymbol{v}_5.$$

3. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & -1 \\ -2 & 4 & 5 \end{bmatrix}$.

Find a nonsingular matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution.

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 1-\lambda & -1 \\ -2 & 4 & 5-\lambda \end{vmatrix} \overset{r_2+r_1\to r_2}{=\!=\!=\!=} \begin{vmatrix} 2-\lambda & 2 & 1 \\ 3-\lambda & 3-\lambda & 0 \\ -2 & 4 & 5-\lambda \end{vmatrix}$$
$$\overset{c_2-c_1\to c_2}{=\!=\!=\!=} \begin{vmatrix} 2-\lambda & \lambda & 1 \\ 3-\lambda & 0 & 0 \\ -2 & 6 & 5-\lambda \end{vmatrix} \overset{\text{expand along } c_1}{=\!=\!=\!=\!=} (-1)^{2+1}(3-\lambda) \begin{vmatrix} \lambda & 1 \\ 6 & 5-\lambda \end{vmatrix}$$
$$= -(\lambda-3)(\lambda^2-5\lambda+6) = -(\lambda-2)(\lambda-3)^2.$$

Therefore, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = 3$.

For $\lambda_1 = 2$,

$$A - 2I = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & -1 \\ -2 & 4 & 3 \end{bmatrix} \xrightarrow[r_3+2r_1\to r_3]{r_1\leftrightarrow r_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A linearly independent eigenvector corresponding to $\lambda_1 = 2$ is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

For $\lambda_2 = \lambda_3 = 3$,

$$A - 3I = \begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & -1 \\ -2 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Two linearly independent eigenvectors corresponding to $\lambda_2 = \lambda_3 = 3$ are given by

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Let

$$P = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

4. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^T$.

Solution:

(a)

Step 1 (The eigenvalues): The characteristic polynomial is given by

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \xlongequal{r_1 - r_2 \to r_1} \begin{vmatrix} -\lambda & \lambda & 0 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

$$\xlongequal{c_1 + c_2 \to c_2} \begin{vmatrix} -\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ 1 & 2 & 1-\lambda \end{vmatrix} = -\lambda[(2-\lambda)(1-\lambda) - 2] = -\lambda^2(\lambda - 3).$$

Hence, equation $|A - \lambda I| = 0$ has roots

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = 3,$$

which are eigenvalues of matrix A.

Step 2 (The eigenvectors): The eigenvectors $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ with respect to eigenvalues $\lambda_1 = \lambda_2 = 0$ satisfy $A\mathbf{v} = 0\mathbf{v} = 0$. This yields linear equation

$$v_1 + v_2 + v_3 = 0.$$

Hence, the degree of freedom is 2. We have $\boldsymbol{v} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix}$, for any $st \neq 0$. We have two linearly independent eigenvectors w.r.t. eigenvalue 0:

$$\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ with respect to eigenvalues $\lambda_3 = 3$ satisfy $A\mathbf{v} = 3\mathbf{v}$. Solving this system of linear equations, we have $\boldsymbol{v} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$ for any $t \neq 0$. Hence, we have eigenvector w.r.t. eigenvalue 3: $\boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b)

Step 3 (Gram-Schmidt process): Notice that

$$\langle \boldsymbol{v}_1, \boldsymbol{v}_3 \rangle = \langle \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle = 0.$$

We normalize $\boldsymbol{v}_2,\ \boldsymbol{v}_3$ and obtain

$$\mathbf{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \quad \mathbf{q}_3 = \frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|} = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}.$$

For $\boldsymbol{v}_1$, we first get the orthogonal part w.r.t. $\mathbf{q}_2,\ \mathbf{q}_3$:

$$\mathbf{w}_1 = \boldsymbol{v}_1 - \langle \boldsymbol{v}_1, \mathbf{q}_2 \rangle \mathbf{q}_2 = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}.$$

Normalize $\mathbf{w}_1$ and we have

$$\mathbf{q}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{bmatrix} -\sqrt{6}/6 \\ \sqrt{6}/3 \\ -\sqrt{6}/6 \end{bmatrix}.$$

Hence, the orthogonal matrix is given by $Q = \begin{bmatrix} -\sqrt{6}/6 & -\sqrt{2}/2 & \sqrt{3}/3 \\ \sqrt{6}/3 & 0 & \sqrt{3}/3 \\ -\sqrt{6}/6 & \sqrt{2}/2 & \sqrt{3}/3 \end{bmatrix}$ and the diagonal matrix is given by $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

5. Solve the differential equation $y' \cos x + \sin^2 y = \cos^2 y$.

Solution: The equation is equivalent to

$$\cos x \frac{dy}{dx} = \cos^2 y - \sin^2 y = \cos(2y)$$

If $\cos(2y) \neq 0$

$$\int \sec(2y) dy = \int \sec x dx$$

$$\frac{1}{2} \ln|\sec(2y) + \tan(2y)| = \ln|\sec(x) + \tan(x)| + C$$

From $\cos(2y) = 0$, we get solutions $y = \frac{k\pi}{2} + \frac{\pi}{4}$, $k \in \mathbb{Z}$.

Therefore $\cos(2y) = C[1 + \sin(2y)] \left(\frac{\cos x}{1 + \sin x} \right)^2$. (No need to write.)

6. Solve the differential equation $-xy' + 2y = \ln x$.

Solution: This is a linear equation.

$$y' - \frac{2}{x}y = -\frac{\ln x}{x}$$

Then, we have

$$\ln \mu = \int -\frac{2}{x}dx = \ln x^{-2} \quad \Rightarrow \quad \mu = \frac{1}{x^2}$$

$$\mu y = \int -\frac{\ln x}{x}\mu dx$$

$$\frac{1}{x^2}y = \int -\frac{\ln x}{x}\frac{1}{x^2}dx = \frac{1}{2}x^{-2}\ln x - \frac{1}{2}\int x^{-3}dx = \frac{1}{2}x^{-2}\ln x + \frac{1}{4}x^{-2} + C$$

$$y = \frac{1}{2}\ln x + \frac{1}{4} + Cx^2$$

7. Find the explicit $y(x)$ of the initial value problem $y' - \dfrac{y}{x} = \dfrac{(x+y)^2}{x^2}$ $(x > 0)$ and $y(1) = -2$.

Solution: The equation is equivalent to

$$\frac{dy}{dx} - \frac{y}{x} = \left(1 + \frac{y}{x}\right)^2.$$

Let $v = \dfrac{y}{x}$.

$$x\frac{dv}{dx} + v - v = (1+v)^2.$$

As $v(1) = -2$, separate variables and we have

$$\int \frac{1}{(v+1)^2} dv = \int \frac{1}{x} dx$$

which gives

$$\int \frac{1}{(v+1)^2} dv = -\frac{1}{v+1} = \ln x + C$$
$$x = 1,\ v = -2 \quad \Rightarrow \quad C = 1$$
$$v + 1 = -\frac{1}{\ln x + 1}$$
$$\frac{y}{x} = -1 - \frac{1}{\ln x + 1}$$

Hence,

$$y = -x - \frac{x}{\ln x + 1} = -x\frac{\ln x + 2}{\ln x + 1}.$$