# AMA 2111 — HW2

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1、

$$y' = \begin{bmatrix} -1 & 4 & 2 \\ -2 & 5 & 2 \\ 1 & -2 & 0 \end{bmatrix} y$$

$$|A-\lambda I| = \begin{vmatrix} -1-\lambda & 4 & 2 \\ -2 & 5-\lambda & 2 \\ +1 & -2 & -\lambda \end{vmatrix} \xrightarrow{R_2=R_2+2R_3} \begin{vmatrix} -1-\lambda & 4 & 2 \\ 0 & 1-\lambda & 2-2\lambda \\ 1 & -2 & -\lambda \end{vmatrix} \xrightarrow{C_3=C_3-2C_2} \begin{vmatrix} -1-\lambda & 4 & -6 \\ 0 & 1-\lambda & 0 \\ 1 & -2 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -1-\lambda & -6 \\ 1 & 4-\lambda \end{vmatrix}$$

$$\therefore = (1-\lambda)\left[(-1-\lambda)(4-\lambda)+6\right] = -(\lambda-1)^2(\lambda-2) \Rightarrow \lambda_1=\lambda_2=1, \lambda_3=2$$

$\therefore \lambda_1=\lambda_2=1$:

$$\left[\begin{array}{ccc|c} -2 & 4 & 2 & 0 \\ -2 & 4 & 2 & 0 \\ 1 & -2 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$V = \begin{bmatrix} +2t+s \\ t \\ s \end{bmatrix} \Rightarrow V_1 = \begin{bmatrix} +2t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} s \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore \lambda_3 = 2$

$$\left[\begin{array}{ccc|c} -3 & 4 & 2 & 0 \\ -2 & 3 & 2 & 0 \\ 1 & -2 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore V = \begin{bmatrix} -2s \\ -2s \\ s \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = C_1 e^{x}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + C_2 e^{x}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C_3 e^{2x}\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

$$y(0) = \begin{bmatrix} 7 \\ 5 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 1 \\ C_3 = -2 \end{cases}$$

$$\therefore y = e^{x}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + e^{x}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2e^{2x}\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

2、

$m^2-2m+5=0$

$m=1\pm 2i$

$y_1=e^x\cos 2x,\ y_2=e^x\sin 2x$

$y_h=C_1y_1+C_2y_2=C_1e^x\cos(2x)+C_2e^x\sin(2x)$

$\because R(x)=10\cos(x)$

$\therefore y_p=A\cos x+B\sin x$

$\therefore \begin{cases} y_p'=-A\sin x+B\cos x \\ y_p''=-A\cos x-B\sin x \end{cases}$

$\therefore y_p''-2y_p'+5y_p=10\cos x$

$(-A\cos x-B\sin x)-2(-A\sin x+B\cos x)+5(A\cos x+B\sin x)=10\cos x$

$\therefore \begin{cases} 2A+4B=0 \\ 4A-2B=10 \end{cases}$

$\therefore \begin{cases} A=2 \\ B=-1 \end{cases}$

$\therefore y=C_1e^x\cos(2x)+C_2e^x\sin(2x)+2\cos(x)-\sin(x)$

3、$y'' - 3y' + 2y = 0$.

$y_h = C_1 y_1 + C_2 y_2$, $m^2 - 3m + 2 = 0 \Rightarrow m_1 = 1, m_2 = 2 \Rightarrow y_h = C_1 e^x + C_2 e^{2x}$

$R(x) \to y_p \to y'' + py' + qy = R(x)$

$y(0) = 1$, $y'(0) = 1$

let $y_{p_1} = A_1 x + A_0$

$y'_{p_1} = A_1$, $y''_{p_1} = 0$

$\Rightarrow 0 - 3A_1 + 2(A_1 x + A_0) = 4x$

$\begin{cases} A_1 = 2 \\ A_0 = 3 \end{cases}$

$y_{p_2} = A_0 e^{2x} \cdot x \Rightarrow y'_{p_2} = A_0 e^{2x} + 2A_0 x e^{2x}$

$y''_{p_2} = 2A_0 e^{2x} + 2A_0 e^{2x} + 4A_0 x e^{2x}$

$y''_{p_2} - 3y'_{p_2} + 2y_{p_2} \Rightarrow 3A_0 e^{2x} = e^{2x}$

$\Rightarrow A_0 = \frac{1}{3}$

$\Rightarrow \begin{cases} y_{p_1} = 2x + 3 \\ y_{p_2} = \frac{1}{3} x e^{2x} \end{cases}$

$\therefore y = C_1 e^x + C_2 e^{2x} + 2x + 3 + \frac{1}{3} x e^{2x}$

4、

$y''+y'=\frac{1}{e^{x}+e^{-x}}$

$m^2+m=0 \Rightarrow m=0,-1 \Rightarrow y_1=C_1,\ y_2=C_2e^{-x}$

$y_h=C_1+C_2e^{-x}$

$\begin{cases} y_1V_1'+y_2V_2'=0 \\ y_1'V_1'+y_2'V_2'=R(x) \end{cases}$

$y_p=V_1+V_2e^{-x}$

$\Downarrow$

$\begin{cases} V_1'+e^{-x}V_2'=0 \\ -e^{-x}V_2'=\frac{1}{e^{x}+e^{-x}} \end{cases} \Rightarrow \begin{cases} V_2'=-\left(\frac{e^{x}}{(e^{x}+e^{-x})}\right) \\ V_1'=\frac{1}{e^{x}+e^{-x}} \end{cases} \Rightarrow \begin{cases} V_2=-\frac{1}{2}(x+\ln(e^{x}+e^{-x})) \\ V_1=\arctan(e^{x}) \end{cases}$

$y_p=\arctan(e^{x})-\frac{1}{2}e^{-x}(x+\ln(e^{x}+e^{-x}))$

$y=C_1+C_2e^{-x}+y_p$

5、

$m^2-4m+4=0 \Rightarrow m=2\ (\alpha=0,\beta=2) \Rightarrow y_1=e^{2x},\ y_2=xe^{2x}$

$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} * \begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = \begin{bmatrix} 0 \\ R(x) \end{bmatrix}$

$\begin{cases} y_1V_1'+y_2V_2'=0 \\ y_1'V_1'+y_2'V_2'=R(x) \end{cases} \Rightarrow \begin{cases} e^{2x}\cdot V_1'+xe^{2x}V_2'=0 \\ 2e^{2x}\cdot V_1'+e^{2x}(1+2x)V_2'=\frac{e^{2x}}{(1+x)^2} \end{cases}$

$\Rightarrow V_2'(1+2x-2x)e^{2x}=\frac{e^{2x}}{(1+x)^2} \Rightarrow V_2'=\frac{1}{(1+x)^2} \Rightarrow V_2=-(1+x)^{-1}$

$\Rightarrow V_1'=-\frac{x}{(1+x)^2} \Rightarrow V_1=-\ln|1+x|-\frac{1}{1+x}$

$y=y_h+y_p=C_1e^{2x}+C_2xe^{2x}+\left(-\frac{1}{1+x}e^{2x}\right)-\left(\ln|1+x|+\frac{1}{1+x}\right)xe^{2x}$

$\Rightarrow y=C_1e^{2x}+C_2xe^{2x}-e^{2x}-\ln|1+x|\,xe^{2x}$

$\because y(0)=1 \Rightarrow C_1+(-1)*=1 \Rightarrow C_1=2$

$y'(0)=0 \Rightarrow 2C_1+C_2-2=0 \Rightarrow C_2=-2$

$\therefore y=e^{2x}-2xe^{2x}-\ln|1+x|\cdot x\cdot e^{2x}$

6、 $y''-2y'+y=t^2e^t+2\sin\frac{t}{2}\cos\frac{t}{2}, y(0)=1, y'(0)=0$

$s^2Y-s-2(sY-1)+Y=\frac{2}{(s-1)^3}+\frac{1}{s^2+1}$

$\Rightarrow Y=\frac{2}{(s-1)^5}+\frac{1}{(s^2+1)(s-1)^2}+\frac{s-2}{(s-1)^2}$

$\mathcal{L}\{\frac{2}{(s-1)^5}\}=\frac{1}{12}t^4e^t$

$\frac{1}{(s^2+1)(s-1)^2}=\frac{\frac{1}{2}s}{(s^2+1)}+\frac{-\frac{1}{2}}{(s-1)}+\frac{\frac{1}{2}}{(s-1)^2}$

$\mathcal{L}\{\frac{1}{(s^2+1)(s-1)}\}=\frac{1}{2}\cos t-\frac{1}{2}e^t+\frac{1}{2}te^t$

$\frac{s-2}{(s-1)^2}=\frac{1}{s-1}-\frac{1}{(s-1)^2}$ $\quad\mathcal{L}\{\frac{s-2}{(s-1)^2}\}=e^t-te^t$

$\therefore y=\frac{1}{12}t^4e^t+\frac{1}{2}\cos t+\frac{1}{2}e^t-\frac{1}{2}te^t$

7、

$F(s)=\int_1^3 2te^{-st}dt+\int_3^{+\infty}-te^{-st}dt$

$=(\frac{2}{s}+\frac{2}{s^2})e^{-s}-(\frac{9}{s}+\frac{3}{s^2})e^{-3s}$

$s^2Y+Y=(\frac{2}{s}+\frac{2}{s^2})e^{-s}-(\frac{9}{s}+\frac{3}{s^2})e^{-3s}$

$\begin{cases}\frac{1}{s(s^2+1)}=\frac{1}{s}-\frac{s}{s^2+1}\\ \frac{1}{s(s^2+1)}=\frac{1}{s^2}-\frac{1}{s^2+1}\end{cases}$

$\mathcal{L}\{\frac{1}{s(s^2+1)}\}=1-\cos t$ $\quad\mathcal{L}\{\frac{1}{s^2(s^2+1)}\}=t-\sin t$

$y=[2t-2\cos(t-1)-2\sin(t-1)]H(t-1)-[3t-9\cos(t-3)-3\sin(t-3)]H(t-3)$

8、 $z_x(x,y),\ z_y(x,y),\ z_{xy}(x,y)$

$x^2z - xy^2 = \sin(z)$

$\frac{\partial(x^2z-xy^2)}{\partial x} = \frac{\partial(\sin z)}{\partial x}$

$2xz + x^2 z_x - y^2 = \cos z \cdot z_x$

$z_x = \frac{y^2-2xz}{x^2-\cos z}$

$\frac{\partial(x^2z-xy^2)}{\partial y} = \frac{\partial(\sin z)}{\partial x}$

$x^2 z_y - 2xy = \cos z \cdot z_y$

$z_y = \cancel{\frac{\cos z+2xy}{x^2}}\ \frac{2xy}{x^2-\cos z}$

$z_{xy} = \frac{\partial(z_x)}{\partial y}$

$= \frac{-2x^2y - 2y\cos z - \frac{2xy^3\sin z - 4x^2y^2\sin z}{x^2-\cos z}}{(x^2-\cos z)^2}$

$= \frac{-2x^2y - 2y\cos z}{(x^2-\cos z)^2} - \frac{2xy^3\sin z - 4x^2y^2\sin z}{(x^2-\cos z)^3}$

9、

i) let $w=f(k)$


$k=e^x+2xy$

$w_y = \frac{dw}{dk}\cdot k_x = (e^x+2y)w'$

$(e^x+2y)\cdot 2xw' - 2x(e^x+2y)\cdot w' = 0$

∴ satisfies

ii) $f(a)=\cos(a)$


$a=e^x+2xy$

$w(x,y)=\cos(e^x+2xy)$

$w_x = -\sin(e^x+2xy)\cdot(e^x+2y)$

$w_{xy} = -\cos(e^x+2xy)\cdot 2x\cdot(e^x+2y) - 2\sin(e^x+2xy)$

$\therefore w_{xy} = -(2xe^x+4xy)\cos(e^x+2xy) - 2\sin(e^x+2xy)$

10、

$A(a,b)=\pi ab$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\left|\frac{\Delta a}{a}\right|\leq 1.5\%$

$\left|\frac{\Delta b}{b}\right|\leq 0.5\%$

$|\Delta A|\approx|A_a\Delta a+A_b\Delta b|$

$=\left|\pi ab\frac{\Delta a}{a}+\pi ab\frac{\Delta b}{b}\right|\leq\pi ab\left(\left|\frac{\Delta a}{a}\right|+\left|\frac{\Delta b}{b}\right|\right)$

$4x^2+9y^2=36$

$\frac{x^2}{9}+\frac{y^2}{4}=1$

$\begin{cases}a=3\\b=2\end{cases}\Rightarrow|\Delta A|=6\pi(1.2\%+0.4\%)$

$|\Delta A|\leq 0.096\pi\approx 0.3016$