The Hong Kong Polytechnic University
Department of Applied Mathematics

Subject Code: AMA2111 Session: Semester 1, 2022/2023
Date: 23 Oct 2022 Time: HKT 13:30-15:00

This question paper has **2** pages (including this page).
Instruction to Candidates:
Attempt ALL questions (6 in total).
Provide detailed solutions to the questions.

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1. Consider the linear system

$$\begin{bmatrix} 1 & 3 & 2 & -1 \\ 2 & 5 & 2 & -3 \\ 1 & 0 & -a & -5 \\ 3 & 7 & 2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -b \\ 2 \end{bmatrix}.$$

(a) Find the conditions satisfied by a and b such that the system has

i. no solutions;

ii. infinitely many solutions;

iii. a unique solution.

Also solve the system when it has infinitely many solutions.

(b) Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 0 \\ 7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ -a \\ 2 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} -1 \\ -3 \\ -5 \\ a \end{bmatrix}$ and $\mathbf{v}_5 = \begin{bmatrix} 1 \\ 0 \\ -b \\ 2 \end{bmatrix}$ in $\mathbb{R}_4$.

i. Find the conditions satisfied by a and b such that the span of $\mathbf{v}_1$,$\mathbf{v}_2$,$\mathbf{v}_4$ and $\mathbf{v}_5$ is $\mathbb{R}_4$.

ii. Find the conditions satisfied by a and b such that $\mathbf{v}_3$,$\mathbf{v}_4$ and $\mathbf{v}_5$ are linearly dependent and write $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_4$ and $\mathbf{v}_5$, if possible.

[30 marks]

$-2-(-3)$
$=$
$-3-a$
$6-2-a$
$=4-a$
$-3-4=-7$
$6-1-b$
$=5-b$
$2-(-(3+a))$
$=2+3+a$
$6-1-b$
$=1$
$2-(-(3+a))$
$=5+a$
$(5-b)(5+a)-3$
$=25+5a-5b-ab-3$
$=22+5a-5b-ab$
$-1+3$
$-a+(-6)$
$=-a$
$a-2b$

a

$6-2\lambda-3+\lambda$
$=3-\lambda$

$10-6\lambda-5\lambda+3\lambda^2-(6-2\lambda-6\lambda+2\lambda^2)$
$=10-11\lambda+3\lambda^2-6+8\lambda-2\lambda^2$
$=\lambda^2-3\lambda+4$

2. Let $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -4 & 5 \end{bmatrix}$.

$=$

$2+1-\lambda=3-$

$4-2\lambda+2$
$=2-2\lambda$

$2-\lambda+1=$ $\quad 4-2\lambda+2$
$=6-2\lambda$ $\quad -2+5-\lambda$
$=3-\lambda$ $\quad \cancel{4+2\lambda-2}= \cancel{2-2\lambda}$

(a) Find a nonsingular matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$2-2\lambda+3-\lambda$
$=5-3\lambda$

(b) Find an orthogonal matrix such that $A = QDQ^T$ if possible, otherwise prove it is impossible.

$x-2t+5=0$

$2-\lambda+1$
$=3-\lambda$

(c) Find the value of the determinant $|A^7|$.

[20 marks]

3. Find all the 5th roots of $w = (\sqrt{3} - i)^4(-1 - i)^{10}$ and plot them in the complex plane.

$-(\frac{2}{3}+\frac{5}{2})=-\frac{4+15}{6}=-\frac{19}{6}$

[10 marks]

4. Find the explicit solution $y(x)$ of the initial value problem

$$y' + \frac{2}{x}y = \frac{e^x}{x^2}, \quad y(1) = e.$$

[15 marks]

5. Find the general explicit solution $y(x)$ of the differential equation

$$y' + xy = xy^3.$$

[15 marks]

6. Find the general explicit solution $y(x)$ of the differential equation

$$y' = \frac{y^3 + 2xy^2 + x^2y + x^3}{x(y + x)^2}.$$

[10 marks]

End