JIANG Guanlin (21093962D)

AMA10071- Calculus and Linear Algebra
Assignment 1
Due: 10 March 2021 (Monday) at 5pm

1. Solve the following equations for x.

   (a) $0.5^x = 8$

   (b) $\log_3(x+5) = 2\log_3(x-1)$

1. (a) $(\frac{1}{2})^x = 8$
$-x = 3$
$x = -3$

(b) $x+5 = (x-1)^2$
$x+5 = x^2-2x+1$
$x^2-3x-4=0$
$x=4, x=-1$ (×)
$\therefore x = 4$

2. Find inverse functions for the following functions (Only need to find the formula of f^{-1}).

   (a) $f(x) = (2x+1)^3$

   (b) $f(x) = \sqrt[2]{5x+8}$

   (c) $f(x) = \dfrac{1+9x}{4-x}$

(a) $f(x) = (2x+1)^3$
$x = (2y+1)^3$
$\sqrt[3]{x} = 2y+1$
$y = \frac{\sqrt[3]{x}-1}{2}$
$f^{-1} = \frac{\sqrt[3]{x}-1}{2}$

(b) $f(x) = \sqrt[2]{5x+8}$
$x = \sqrt[2]{5y+8}$
$\sqrt[2]{x} = 5y+8$
$y = \frac{\sqrt[2]{x}-8}{5}$
$f^{-1} = \frac{\sqrt[2]{x}-8}{5}$

(c) $f(x) = \frac{1+9x}{4-x}$
$x = \frac{1+9y}{4-y}$
$y = \frac{4x-1}{x+9}$
$f^{-1} = \frac{4x-1}{x+9}$

3. Find the following limits. If the limit does not exist, write "This limit does not exist." For infinite limit results, please indicate $+\infty$ or $-\infty$.

   (a) $\lim\limits_{x\to1}\dfrac{x-1}{x^3-1}$

   (b) $\lim\limits_{x\to\infty}(\ln(1+2x)-\ln(x))$

   (c) $\lim\limits_{t\to0}\sqrt[3]{t}\sin(\dfrac{1}{t^2})$

   (d) $\lim\limits_{x\to-\infty} x\cos(1/x)$

   (e) $\lim\limits_{x\to3}\sqrt{\dfrac{x^2-2x-3}{\sin(x-3)}}$

   (f) $\lim\limits_{x\to\infty}(3x-\sqrt{9x^2-12x+1})$

   (g) $\lim\limits_{x\to0}\dfrac{\ln(1+\sin(x))}{x}$

(a) $= \lim_{x\to1}\frac{x-1}{(x-1)(x^2+x+1)}$
$= \frac{1}{3}$

(b) $= \lim_{x\to\infty}(\ln(\frac{1+2x}{x}))$
$= \lim_{x\to\infty}\ln(\frac{1}{x}+2)$
$= \ln(2)$

(c) $= \lim_{t\to0}\sqrt[3]{t}\sin(\frac{1}{t^2})$
$= 0$

(d) $= \lim_{x\to-\infty} x\cos(\frac{1}{x})$
$= -\infty$

(e) $= \sqrt{\lim_{x\to3}\frac{x^2-2x-3}{\sin(x-3)}}$
$\because (\frac{0}{0})$
$\therefore$ use L'Hopital's Rule
$\therefore = \sqrt{\lim_{x\to3}\frac{2x-2}{\cos(x-3)}}$
$= \sqrt{\frac{6-2}{1}}$
$= 2$

(f) $\lim_{x\to\infty}\frac{(3x-\sqrt{9x^2-12x+1})(3x+\sqrt{9x^2-12x+1})}{(3x+\sqrt{9x^2-12x+1})}$
$= \lim_{x\to\infty}\frac{12x-1}{3x+\sqrt{9x^2-12x+1}}$
$= \frac{12}{3+\sqrt{9}} = \frac{12}{6} = 2$

(g) $\because (\frac{0}{0})$
$\therefore$ use L'Hopital's Rule
$\therefore = \lim_{x\to0}\frac{\frac{1}{1+\sin(x)}\cdot\cos(x)}{1}$
$= 1$

4. Find $\dfrac{df}{dx}$ for each of the following $f(x)$.

   (a) $f(x) = \sin(x) + \ln(2x)$

   (b) $f(x) = x^3\sin(x) - \frac{1}{2}\ln(x)$

   (c) $f(x) = \tan(2x)$

   (d) $f(x) = 2\sin(3x+x^2)$

   (e) $f(x) = \dfrac{\sqrt{x}+2x}{7x-4x^2}$

4. (a) $f'(x) = \cos(x) + \frac{1}{2x}\times2$
$f'(x) = \cos(x) + \frac{1}{x}$

(b) $f'(x) = x^3\cos(x) + 3x^2\sin(x) - \frac{1}{2}\times\frac{1}{x}$
$f'(x) = x^3\cos(x) + 3x^2\sin(x) - \frac{1}{2x}$

(c) $f'(x) = \sec^2(2x)\times2 = 2\sec^2(2x)$

(d) $f'(x) = 2\cos(3x+x^2)\times(3+2x)$
$f'(x) = 6\cos(3x+x^2) + 4x\cos(3x+x^2)$

(e) $f'(x) = \frac{(\frac{1}{2\sqrt{x}}+2)(7x-4x^2)-(7-8x)(\sqrt{x}+2x)}{(7x-4x^2)^2}$
$f'(x) = \frac{16x^2+16x\sqrt{x}-7\sqrt{x}-4x^{\frac{3}{2}}}{2(7x-4x^2)^2}$

5. $\sin y + e^x - xy = 1$

$y = \frac{e^x-1}{x-1}$

$y' = \frac{e^x(x-1)-(e^x-1)}{(x-1)^2}$

$y' = \frac{xe^x-2e^x+1}{(x-1)^2}$

$y' = \frac{xe^x-2e^x+1}{(x-1)^2}\Big|_{x=0}$

$y' = -1$

$K = \frac{y_1-y}{x_1-x} \Rightarrow -1 = \frac{y_1-0}{x_1-0}$

$y = -x$

5. Find the equation of the tangent line to $\sin y + e^x - xy = 1$ at $x = 0$.

6. Find y' for each of the following function y.

(a) Let y be a function of x defined implicitly by $x - y + \frac{1}{2}\sin y = 0$

(b) Let $y = x^{\sin x}$, $x > 0$

(c) $y = \arcsin\sqrt{\frac{1-x}{1+x}}$

(d) $y = \ln(\sin(x) - \cot(x))$

(e) $y = \cos(x^2e^x)$

(f) $y = (e^{-6x} + \sin(2-x))^3$

6. (a) $1 - y' + \frac{\cos(y)\cdot y'}{2} = 0$

$\cos(y)\cdot y' = 2y' - 2$

$y' = \frac{-2}{\cos(y)-2}$

(b) $y = e^{\sin x \ln x}$

$y' = e^{\sin x\ln x}\cdot\left(\cos x\ln x + \frac{\sin x}{x}\right)$

$y' = x^{\sin x}\cdot\left(\cos x\ln x + \frac{\sin x}{x}\right)$

(c) $y' = \left(\frac{1}{\sqrt{1-\frac{1-x}{1+x}}}\right)\frac{d}{dx}\left(\sqrt{\frac{1-x}{1+x}}\right)$

$y' = \left(\frac{1}{\sqrt{1-\frac{1-x}{1+x}}}\right)\cdot -\frac{1}{\sqrt{1-x}\,(x+1)^{\frac{3}{2}}}$

(d) $y' = \frac{1}{\sin(x)-\cot(x)}\cdot(\cos(x)+\csc^2(x))$

$y' = \frac{\cos(x)+\csc^2(x)}{\sin(x)-\cot(x)}$

(e) $y' = -\sin(x^2e^x)\cdot(2xe^x + x^2e^x)$

(f) $y' = 3(e^{-6x}+\sin(2-x))^2\cdot(e^{-6x}\cdot(-6)-\cos(2-x))$

$y' = 3(e^{-6x}+\sin(2-x))^2\cdot(-6e^{-6x}-\cos(2-x))$

7. Find the local maximum and local minimum values of the following functions.

(a) $f(x) = 4 + \frac{1}{3}x - \frac{1}{2}x^2$

(b) $f(x) = |x^2 - 3x + 2|$, $\quad x \in [-3, 4]$

(a) $f'(x) = \frac{1}{3} - x$

$\frac{1}{3} - x = 0$

$x = \frac{1}{3}$

$f(\frac{1}{3}) = 4 + \frac{1}{3}\times\frac{1}{3} - \frac{1}{2}\times(\frac{1}{3})^2$

$f(\frac{1}{3}) = \frac{73}{18}$

∴ The local max is $\frac{73}{18}$ at $x = \frac{1}{3}$

(b) $f'(x) = \frac{(x^2-3x+2)(2x-3)}{|x^2-3x+2|} = 0$, $x \in [-3,4]$, $x \neq 2, 1$

$(x^2-3x+2)(2x-3) = 0$

$x_1 = \frac{3}{2}$, $\cancel{x_2 = 2}$, $\cancel{x_3 = 1}$

∴ $x = \frac{3}{2}$

∴ $f(\frac{3}{2}) = |(\frac{3}{2})^2 - 3\times\frac{3}{2} + 2| = \frac{1}{4}$

∴ The min is $\frac{1}{4}$ at $\frac{3}{2}$

8. Find the global extrema of f on the given interval.

(a) $f(x) = x|x-2|$, $0 \le x \le 3$

(b) $f(x) = xe^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$

8. (a)

$f(x) = x|x-2|$

$f'(x) = |x-2| + x\cdot\frac{d}{dx}(|x-2|)$

$f'(x) = |x-2| + x\cdot\frac{x-2}{|x-2|}$

$f'(x) = \frac{2x^2-6x+4}{|x-2|}$

$\frac{2x^2-6x+4}{|x-2|} = 0$

$x - 2 \neq 0$

$x \neq 2$

$x = 1$, $\cancel{x=2}$

$f''(x) = \frac{2(x-2)}{|x-2|}$

$\frac{2(x-2)}{|x-2|} = 0$

$x \neq 2$

$\cancel{x=2}$

global min: $f(0) = 0$

global max: $f(3) = 3$

(b) $f'(x) = e^{-\frac{x^2}{2}} + (-e^{-\frac{x^2}{2}}\cdot x^2)$

$f''(x) = e^{-\frac{x^2}{2}}x^3 - 3e^{-\frac{x^2}{2}}\cdot x$

$f'(x) = 0$

$x = \pm 1$

$f''(x) = 0$, $x \in \mathbb{R}$

$x = 0$, $\cancel{x = \sqrt{3}}$, $\cancel{x = -\sqrt{3}}$

∴ global min: $f(-1) = -e^{-\frac{1}{2}}$

global max: $f(+1) = e^{-\frac{1}{2}}$

9. Sketch the graph of the following functions. Label all intercepts, local extrema, and inflection points.

(a) $f(x) = x^3 - 3x + 1$

(b) $f(x) = \frac{x}{x^2-1}$

9. (a) $f'(x) = 3x^2 - 3$

$f''(x) = 6x$

$3x^2 - 3 = 0$

$x = \pm 1$

$6x = 0$

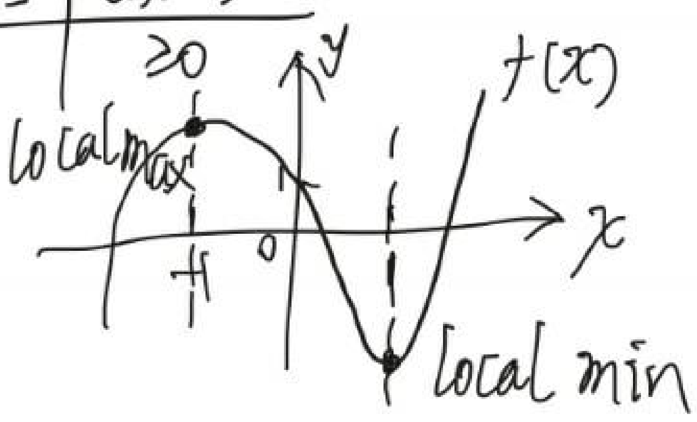
$x = 0$

① $f'(x) = 3x^2 - 3$

x	$(-\infty,-1]$	$[-1,1]$	$[1,+\infty)$
$f'(x)$	≥ 0	≤ 0	≥ 0

② $f''(x) = 6x$

x	$(-\infty,0]$	$(0,+\infty)$
$f''(x)$	≤ 0	≥ 0



(b) $f'(x) = \frac{(x^2-1) - x(2x)}{(x^2-1)^2} = \frac{-x^2-1}{(x^2-1)^2}$

$f''(x) = -\frac{2x(-x^2-3)}{(x^2-1)^3}$

$\frac{-x^2-1}{(x^2-1)^2}$

$-\frac{2x(-x^2-3)}{(x^2-1)^3}$

x	$(-\infty,-1)$	$(-1,1)$	$(1,+\infty)$
$f'(x)$	≤ 0	≤ -1	≤ 0

x	$(-\infty,-1)$	$(-1,1)$	$(1,+\infty)$
$f''(x)$	≤ 0		≥ 0