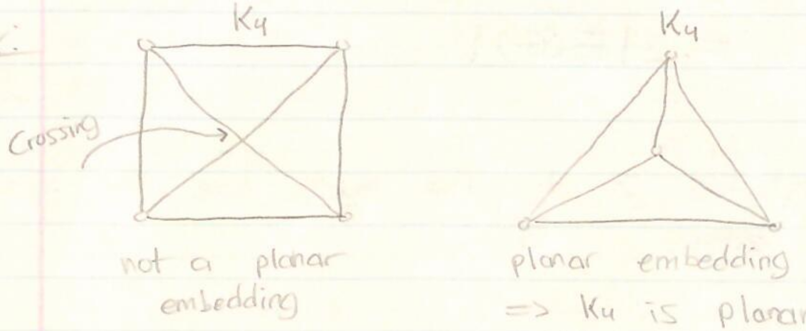
14/03/16
Pi day!

Math 239 - Lecture #28

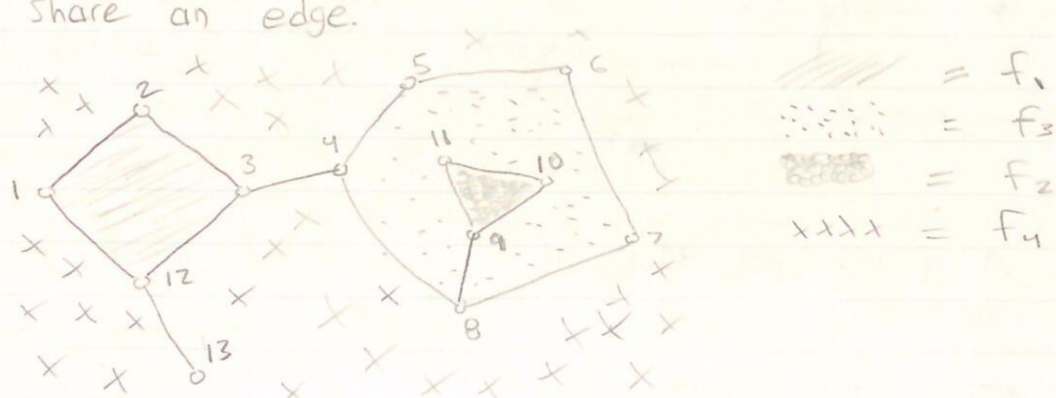
Definition: A planar embedding of G is a drawing of G on the plane such that vertices are at different points and edges intersect only at their common vertices, so no edges cross. A graph that has a planar embedding is called a planar graph.

Ex:



Definition: A face of a planer embedding is a connected region on the plane. Two faces are adjacent if they share an edge.

Ex:



Definition: For a connected planar embedding, the boundary walk for a face is a closed walk around its boundary once.

Ex:

f_1 : 1, 2, 3, 12, 1   *You can repeat edges!
f_3 : 4, 5, 6, 7, 8, 9, 10, 11, 9, 8, 4
f_2 : 9, 11, 10, 9
f_4 : 1, 2, 3, 4, 5, 6, 7, 8, 4, 3, 12, 13, 12, 1

Note how f_3 is 'inside' the pentagon, while f_4 is the entire outer shape.

Definition: The <u>degree</u> of a face is the length of its boundary walk.

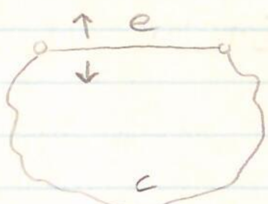
- For previous example, $\deg(f_1) = 4$, $\deg(f_2) = 3$, $\deg(f_3) = 10$, $\deg(f_4) = 13$

Theorem: Handshaking lemma for faces - let G be a planar graph with an embedding where F is the set of all faces.

Then: $$\sum_{f \in F} \deg(f) = 2|E(G)|$$

Proof: Each edge contributes 2 to the sum, one for each side of the edge. $\square$

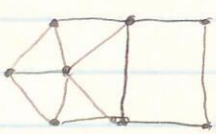
- An edge has 2 different faces on both sides if and only if the edge is <u>not</u> a bridge.



Jordan curve theorem: A simple closed curve separates the plane into 2 regions; one inside, one outside.

$\Rightarrow$ Not true for non-plane surfaces

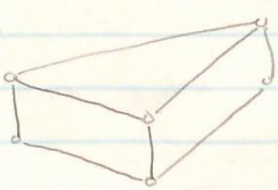
Euler's Formula: For a planar embedding of a connected graph G with n vertices, m edges, s faces, $\underline{n - m + s = 2}$



$n = 8$
$m = 13$
$s = 7$

$\underline{8 - 13 + 7 = 2}$

Happy Pi day!



$n = 6$
$m = 8$
$s = 4$

$\underline{6 - 8 + 4 = 2}$

Pg #2

14/03/16

Math 239 - Lecture #28 - Cont

Proof of: Euler's Formula

Fix n (# of vertices). Prove by induction on m (# of edges).

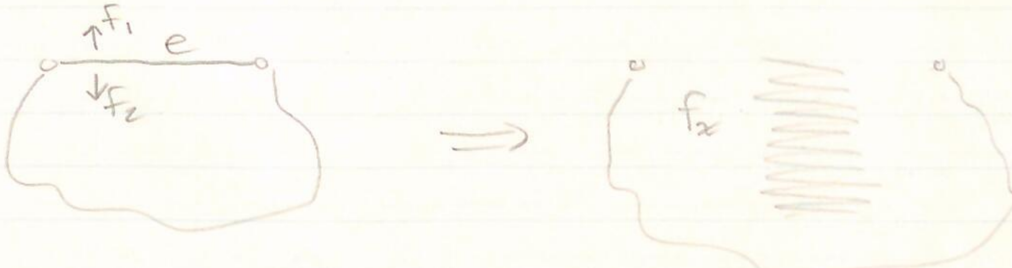
Base: $m = n-1$ (min # of edges in a connected graph, a tree)
A tree has n vertices, $n-1$ edges, 1 face.
$n-(n-1)+1 = 2$.

Hyp: Assume E.F. holds for any connected planar embedding with n vertices and $m-1$ edges.

Conc: Suppose G has a connected embedding with n vertices, m edges, s faces.

Since G is not a tree, it contains a cycle. Let e be any edge on a cycle. Then $G-e$ is still connected, and still planar. So we then have $m-1$ edges. So by inductive hypothesis, E.F. holds for $G-e$.

In $G-e$, the two faces on two sides of e merge into one, so $G-e$ has $s-1$ faces.
Using E.F., $n-(n-1)+(s-1)=2$
So $n-m+s=2$, and E.F. holds for G. $\square$



$\triangle$

$E \leq \frac{3}{2}n - 3$