Pg #1

04/03/16

Math 239 - Lecture #24

Theorem: Let G be connected. Then G has a Eularian circuit if and only if every vertex has even degree.

Proof: ($\Rightarrow$) A Eularian circuit contributes 2 to the degree of a vertex for each visit. So every vertex has even degree.

($\Leftarrow$) Prove by induction on the number of edges (in G)

Base: When $m=0$ (no edges), G is a single vertex, which has a trivial E.C.

Ind Hyp: Assume that any connected graph with all even degrees and less than m edges has an E.C.

Ind Step: Suppose G has m edges, connected, all of even degree.

→ Find a non-trivial closed walk w with no repeated edges. Such a walk exists since G has a cycle due to all $\deg \geq 2$.

1) If w contains all edges, then w is an E.C

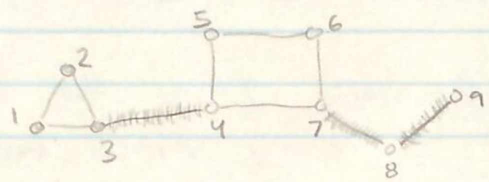
→ Otherwise, we remove edges of w from G to get G'. Then G' consists of components where every vertex has even degree. (This is because a vertex has even deg in both G and w).

2). So by I.H, each component of G' has an E.C. Since G is connected, each component shares at least a vertex with W. We can obtain an E.C. by attaching the E.C of these components to W.

By 1) and 2), $\square$.

Bridges: An edge e is a <u>bridge</u> in G if removing e from G increases the # of components.
$G - e$ (read G minus e)

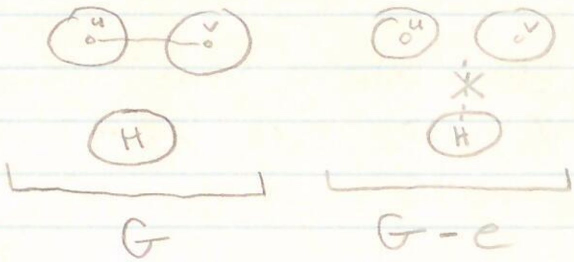
Ex:



= bridge / cut-edge / isthmus

Theorem: If $e = uv$ is a bridge of a connected graph G, then $G - e$ has exactly 2 components. Moreover, u, v are in different components of $G - e$.

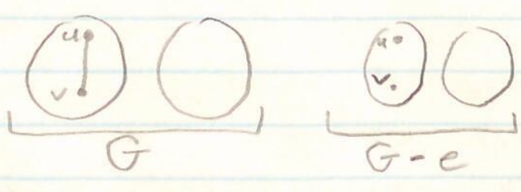
Proof: Suppose $G - e$ has 3 components BWOC.
Let H be a component not containing u nor v. The cut induced by $V(H)$ is empty in $G - e$.



Since u, v are not in H, u, v is not in the cut induced by $V(H)$ in G. So this cut is empty in G.

So G is disconnected, contradiction.
$\therefore$ $G - e$ has 2 components.

Suppose u, v are in the same component of $G - e$.



Let J be the component not containing u, v. Then uv is not in the cut induced by $V(J)$ in G. So $V(J)$ induces an empty cut in G, contradiction.

So u, v are in diff components of $G - e$. $\square$

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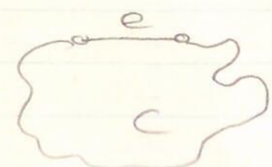
Math 239 - Lecture #24 Cont.

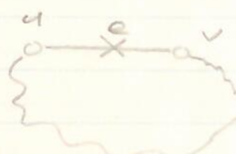
Theorem: An edge e is a bridge in G if and only if e is not in any cycle of G.

Contrapositive - e is in a cycle of G if and only if e is not a bridge.

($\Rightarrow$) Suppose $e=uv$ is in a cycle C. Then in $G-e$, we have $C-e$ is a u,v-path. So u,v are in the same component in $G-e$. By previous theorem, e is not a bridge.



($\Leftarrow$) Suppose $e=uv$ is not a bridge. By bridge definition, u,v are in the same component of $G-e$. So there exists a u,v-path P which does not contain e. The $P+e$ is a cycle in G containing e.



$\square$