25/01/16

Math 239 - Lecture #10

Recall Int. Compositions: $n=7$: $(1,3,3), (4,3), \ldots$ Comp of n with k parts.
$S = \mathbb{N}^k$, $w(a_1, \ldots, a_k) = a_1 + \ldots + a_k$, $\Phi_s(x) = \left(\frac{x}{1-x}\right)^k$

Example: How many compositions of n are there with $2k$ parts, where the first k parts are atleast 5, and the last k parts are multiples of 3?

Let $A = \{5,6,7,8,\ldots\}$, $B = \{3,6,9,12,\ldots\}$ *can't start with 0!

The set of all compositions we consider is
$S = A^k \times B^k$ (which is $2k$ parts!)
The weight of a composition is the sum of its parts.

Then $\Phi_A(x) = x^5 + x^6 + x^7 + x^8 + \ldots$ $\frac{x^5}{1-x}$
$\Phi_B(x) = x^3 + x^6 + x^9 + x^{12} + \ldots$ $\frac{x^3}{1-x^3}$

By product lemma, $\Phi_s(x) = \left(\Phi_A(x)\right)^k \cdot \left(\Phi_B(x)\right)^k$

$$= \frac{x^{5k}}{(1-x)^k} \cdot \frac{x^{3k}}{(1-x^3)^k} = \frac{x^{8k}}{(1-x)^k(1-x^3)^k}$$

Answer: $[x^n] \frac{x^{8k}}{(1-x)^k(1-x^3)^k}$ *satisfactory for assignments

$$= [x^{n-8k}]\left(\frac{1}{(1-x)^k} \cdot \frac{1}{(1-x^3)^k}\right)$$

$$= [x^{n-8k}]\left(\sum_{i\geq 0}\binom{i+k-1}{k-1}x^i\right)\left(\sum_{j\geq 0}\binom{j+k-1}{k-1}x^{3j}\right)$$

$$= [x^{n-8k}]\sum_{i\geq 0}\sum_{j\geq 0}\binom{i+k-1}{k-1}\binom{j+k-1}{k-1}x^{i+3j}$$

Need $i+3j = n-8k$
So for $\forall j$, $\exists i$ exists

$j \geq 0$, $i \geq 0 \Rightarrow n-8k-3j \geq 0$ so $\frac{n-8k}{3} \geq j$

floor

$$= \sum_{j=0}^{\left\lfloor \frac{n-8k}{3} \right\rfloor}\binom{n-8k-3j+k-1}{k-1}\binom{j+k-1}{k-1}$$

Summary: How many compositions of n have certain properties?

1. Define set S of all compositions with these properties
2. Define weight of a composition to be the sum of its parts
3. Use Sum/Product lemmas to find $\Phi_S(x)$
4. Answer is $[x^n]\,\Phi_S(x)$

Example: How many compositions of n are there where every part is an odd number?

$n=5$

$(5), (1,1,3), (1,3,1), (3,1,1), (1,1,1,1,1)$

Let $\mathbb{N}_{odd} = \{1,3,5,7,9,11,\ldots\}$

The set of all compositions we consider is

$$S = \underbrace{\bigcup_{k\geq 0}}_{\text{any \# of parts}} \underbrace{\mathbb{N}_{odd}^{k}}_{k \text{ parts}} \qquad * \mathbb{N}_{odd}^{0} = \{()\}$$

Use the usual weight function.

Then $\Phi_{\mathbb{N}_{odd}}(x) = x + x^3 + x^5 + \ldots = \frac{x}{1-x^2}$

Using the sum & product lemmas,

$$\Phi_S(x) = \sum_{k\geq 0} \Phi_{\mathbb{N}_{odd}^{k}}(x) = \sum_{k\geq 0} \left(\Phi_{\mathbb{N}_{odd}}(x)\right)^k$$

$$= \sum_{k\geq 0} \left(\frac{x}{1-x^2}\right)^k \quad \text{geometric series, constant term is zero}$$

$$= \frac{1}{1-\frac{x}{1-x^2}} = \frac{1-x^2}{1-x-x^2} \qquad \left[\begin{array}{l} 1+x+x^2+\ldots = \frac{1}{1-x} \\ x \to \frac{x}{1-x^2} \end{array}\right.$$

Answer is $[x^n]\,\frac{1-x^2}{1-x-x^2}$

25/01/16

Math 239 - Lecture #10 Cont.

Cont:

Find a recurrence!

$(1-x-x^2)A(x) = 1-x^2$ $* \, A(x) = \sum_{n\geq 0} a_n x^n = \frac{1-x^2}{1-x-x^2}$

$(1-x-x^2)(a_0 + a_1 x + a_2 x^2 + \dots)$

$= a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + \sum_{n\geq 3}(a_n - a_{n-1} - a_{n-2})x^n$

Compare coeff

$a_0 = 1$

$a_1 - a_0 = 0 \Rightarrow a_1 = 1$

$a_2 - a_1 - a_0 = -1 \Rightarrow a_2 = 1$

For $n \geq 3$, $a_n - a_{n-1} - a_{n-2} = 0$

$\therefore a_n = a_{n-1} - a_{n-2}$ for $n \geq 3$

$a_3 = 2,\ a_4 = 3,\ a_5 = 5,$

$a_6 = 8,\ a_7 = 13,\ a_8 = 21, \dots$

So $a_n = f_n$ where f_n is the fibonacci sequence $n \geq 1$

$\left[f_n \quad \begin{matrix} 0, 1, 1, 2, 3, 5, 8, \dots \\ f_0, f_1, f_2, f_3, \dots \end{matrix} \right]$

a_0 is an exception

Bijection Thing:

Let C_n be the set of all compositions of n, where every part is odd.

So $|C_n| = |C_{n-1}| + |C_{n-2}|$. Find a bijection between C_n and $C_{n-1} \cup C_{n-2}$