18/03/16

Math 239 - Lecture #30

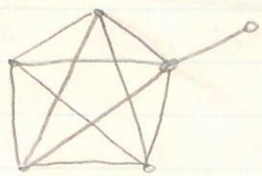
Recall: If $n \geq 3$, then any connected planar graph has at most $3n-6$ edges.

(☆) The converse of this is not true. A graph with $\leq 3n-6$ edges might not be planar. (Example: Peterson)

Corollary: K_5 is not planar.

Proof: K_5 has 5 vertices, and any planar graph on 5 vertices has at most $(3 \cdot 5)-6=9$ edges. But K_5 has 10 edges, so K_5 is not planar. □

(☆) Counterexample:



6 vertices
11 edges $\leq 3n-6$
$6 \cdot 3-6=12$; not planar

$K_{3,3}$

6 vertices
9 edges $\leq 3n-6=12$

Theorem: When $n \geq 3$, a connected planar bipartite graph on n vertices has at most $2n-4$ edges.

Proof: Consider a planar embedding of a bipartite graph G with n vertices, m edges, s faces. We claim that every face has deg ≥ 4. We proved the tree case in previous lesson.

If there is a cycle, then all face boundaries contain cycles of length ≥ 4 (since no △ exist in a bipartite graph). So every face has deg ≥ 4.

By HSLFF, $2m = \sum_{f \in F} \deg(f) \geq 4s$

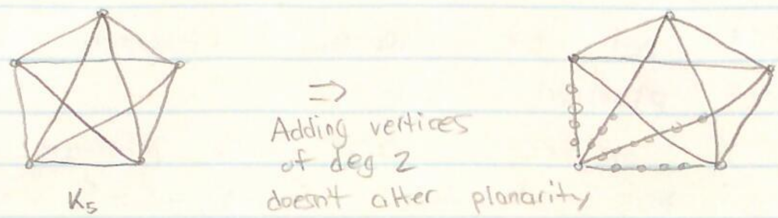
$= 4(2-n+m)$ by EF $(n-m+s=2)$
$= 8-4n+4m$

So $2n-4 \geq m$. □

Corollary: $K_{3,3}$ is not planar.

Proof: $K_{3,3}$ has 6 vertices. Any planar bipartite graph with 6 vertices has at most $(2 \cdot 6) - 4 = 8$ edges. But $K_{3,3}$ has 9 edges, so it is not planar. $\square$

Kuratowski's Theorem



Def$^{\underline{n}}$: An edge subdivision of G is obtained by introducing some new vertices of degree 2 to the edges of G.

- Could be 0 new vertices
- We are replacing each edge with a new path

*See handout

A graph is not planar if and only if any edge subdivision of the graph is not planar.

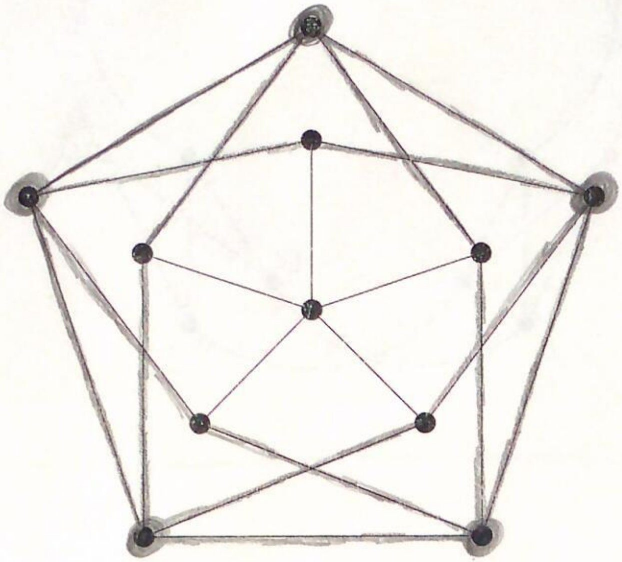
Kuratowski's theorem says that a graph is planar if and only if it does not contain an edge subdivision of K_5 or $K_{3,3}$ as a subgraph.

Planarity Practice

Find an edge-subdivision of K_5 and/or $K_{3,3}$ in each of the following graphs. This proves that each graph is not planar.

No repeated vertices!!

Grötzsch graph

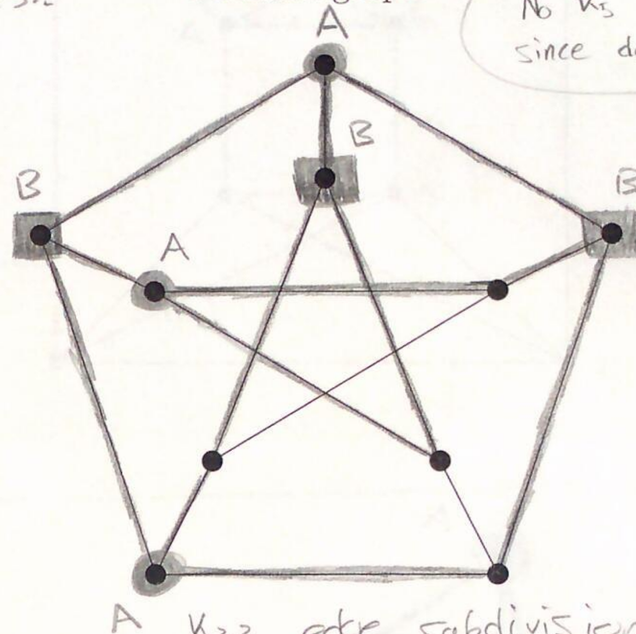


K_5 edge subdivision

$\Rightarrow$ Not planar

Petersen graph

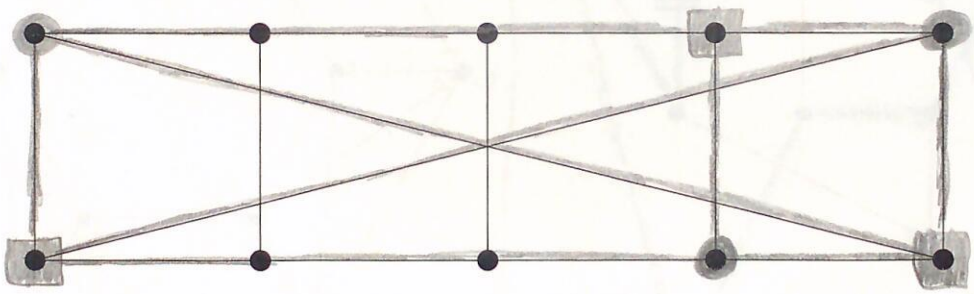
No K_5 ES since deg < 4



$K_{3,3}$ edge subdivision

$\Rightarrow$ Not planar

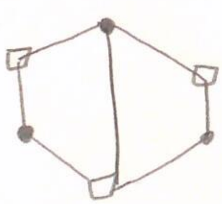
M_5



$K_{3,3}$ edge substitution

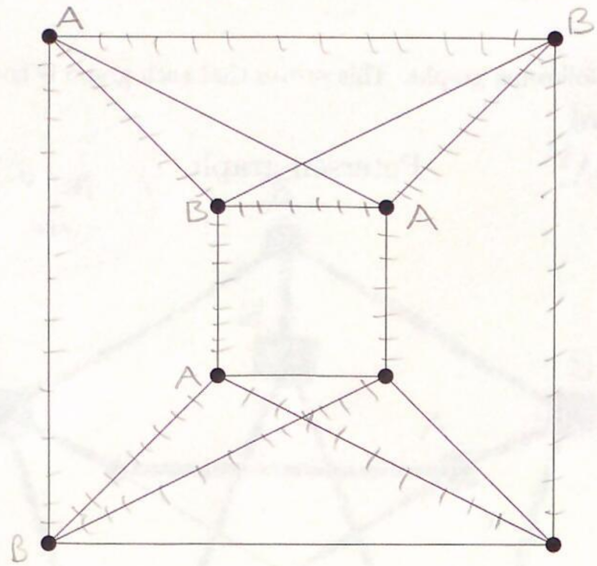
$\Rightarrow$ Not planar

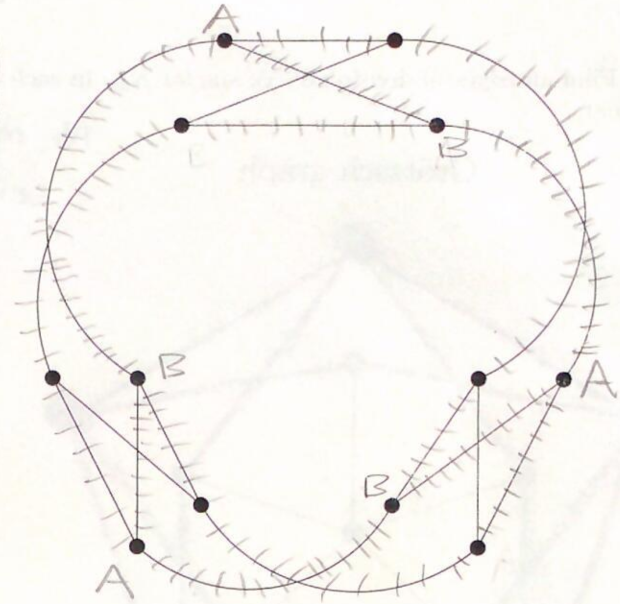
*

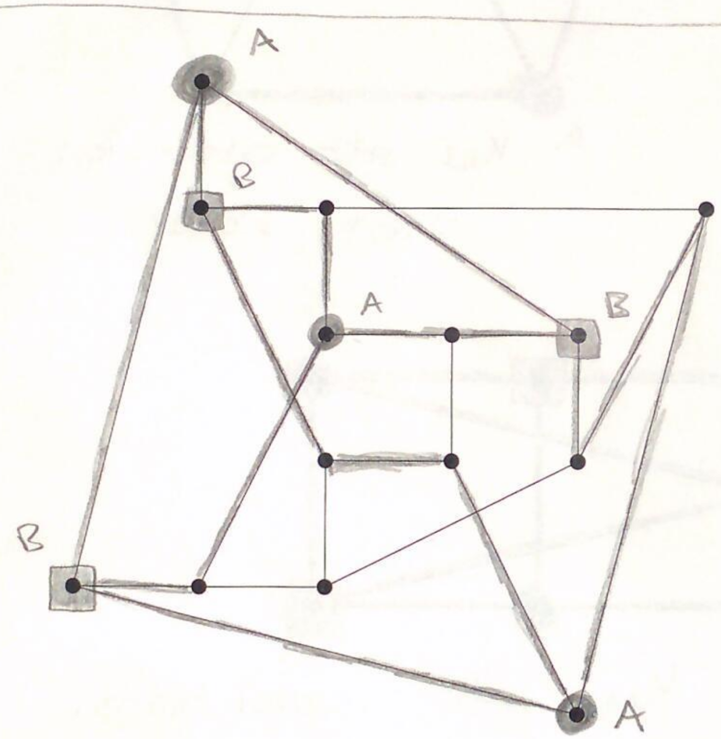


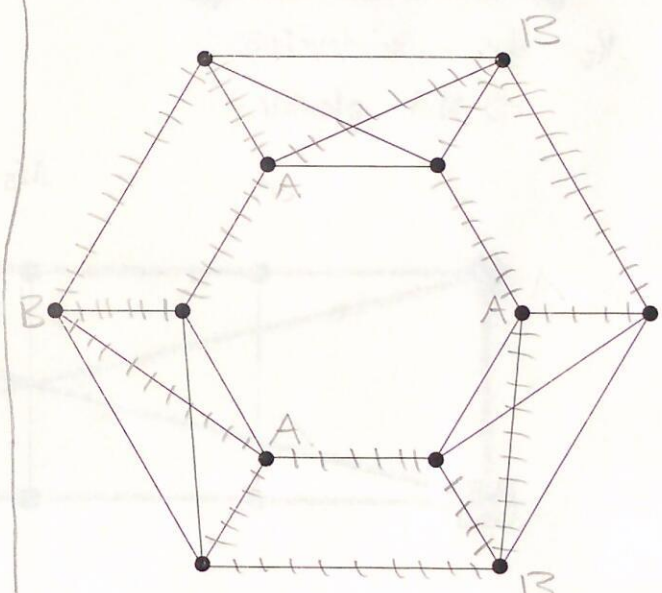
$K_{3,3}$ also!

Additional practice.









All deg 3, not gonna have K_5

$K_{3,3}$ edge substitution

$\Rightarrow$ Not Planar