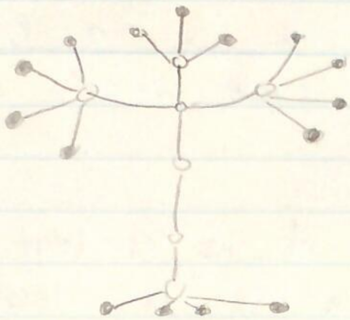
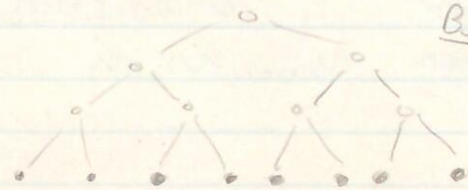
07/03/16

Math 239 - Lecture #25

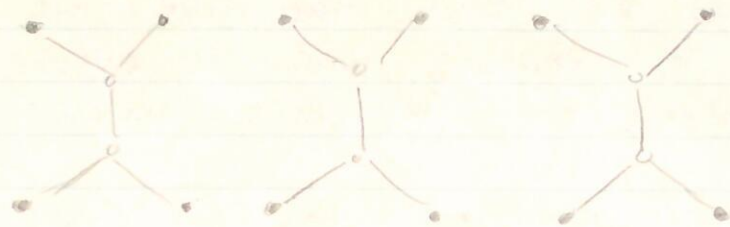
Trees: A <u>tree</u> is a connected graph with no cycles.

Ex:





Forests: A <u>forest</u> is a graph with no cycles.



<u>Lemma</u> - Every edge of a tree/forest is a bridge.

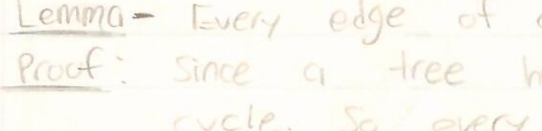
<u>Proof</u>: Since a tree has no cycles, no edge is in a cycle. So every edge is a bridge. □

- Thus removing any edge from a tree disconnects it. We can say that a tree is minimally-connected.

Definition: A <u>leaf</u> in a tree/forest is a vertex of deg 1.
- See filled in vertices above.

<u>Theorem</u> - Every tree with at least 2 vertices has at least two leaves.

<u>Proof</u>: Let $P = v_0, v_1, \ldots, v_k$ be a longest path in a tree T.



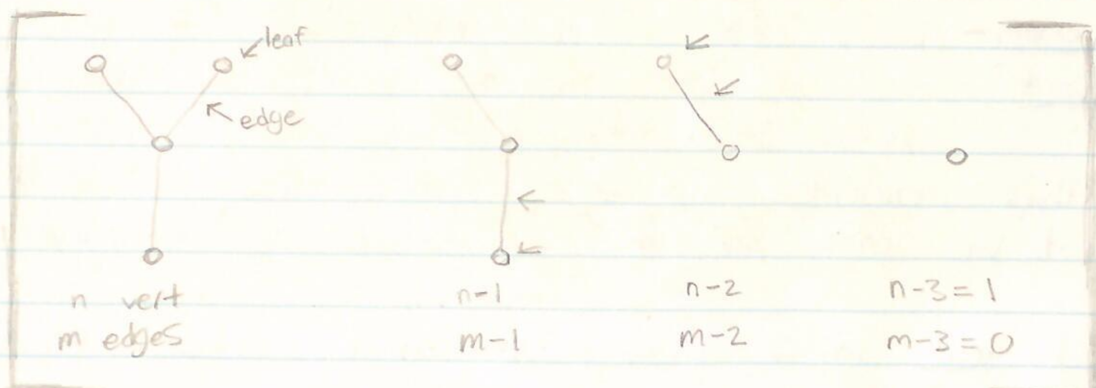
Since T has at least 2 vertices, P has length at least 1.

Proof Cont: Consider v_0. It has one neighbour v_1. Suppose v_0 has another neighbour x in T. If <u>x is not in</u> P, then $x, v_0, v_1, \ldots, v_k$ is a path longer than path P, contradiction. If <u>x is in P</u>, say $x = v_i$, then $v_0, \ldots, v_i, v_0$ is a cycle, which cannot exist in T.

So v_0 has degree 1, hence it is a leaf. A similar argument gives us that v_k is a leaf, so T has at least 2 leaves.

Look back at the first two trees we drew: The first had 21 vertices, and 20 edges. The BST had 15 vertices, and 14 edges. Hmm...

Theorem: A tree with n-vertices has n-1 edges.



Proof by induction on n.

Base: $n=1$, this tree has no edges, which is $n-1$ edges.
Hyp: Assume any tree with $n-1$ vertices has $n-2$ edges.
Step: Let T be any tree with n vertices. Let v be a leaf, and let e be the edge incident with v. When we remove e, there are 2 components, one consists of only v. So removing e and v from T results in a tree T', with $n-1$ vertices. By ind hyp, T' has $n-2$ edges. Hence T has $n-1$ edges. $\square$

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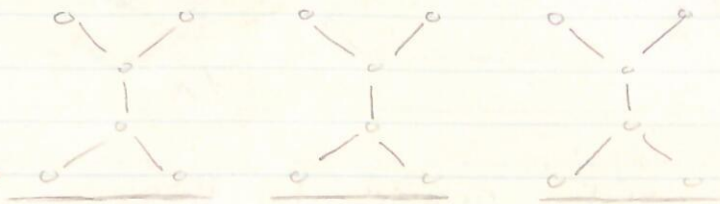
Math 239 - Lecture #25 Cont.

More on Forests

How many edges are in a forest with n vertices and K components?



$K=3$
18 vert, 15 edge.

$|V| = |E| + 1$ " " " "

- Each component has one fewer edge than vertices. There are K components, together there are K fewer edges than vertices; $\underline{n-K}$.

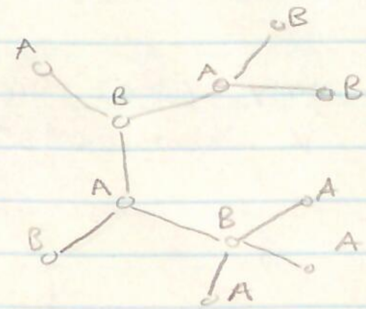
Theorem: There is a unique path between every pair of vertices in a tree.

Proof: Suppose there are 2 distinct u,v-paths P_1, P_2 in a tree T.

Then there is an edge xy in exactly one of P_1, P_2, say xy is in P_2, x is closer to u.
Then we can obtain an x,y walk in $T - xy$ by using P_2 to go from x to u, using P_1 to go from u to v, and then using P_2 to go from v to u.

So x,y are in the same component of $T - xy$, contradicting the fact that xy is a bridge. So we cannot have 2 u,v-paths. $\square$

Theorem: A tree is bipartite.



Proof by induction! :n

(do this myself)

Damn you Pei!

Can't believe I got around to this. 4/10/16

Induction on the number of vertices

Base - A tree of 1, 2 vertices is trivially bipartite.

Hyp - Assume a tree with n vertices is a bipartite graph, for some $n > 2$.

Conc - Prove for $n+1$.

So a tree with $n+1$ vertices; we can pick some arbitrary vertex of degree one (exists by previous theorem), let this vertex be v_0.

Since v_0 has degree one, we can put it in the opposite partition as the sole vertex its connected to.

Since $|G - v_0| = n+1-1 = n$, and by our hypothesis, $G + v_0$ is bipartite.

By POMI, any tree is bipartite.