Monday OH
3:30-4:20

13/01/16

# Math 239 - Lecture #5

Recall on generating series: $\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{n \geq 0} a_n x^n$

**Last Ex:** $S$ = all binary Strings; $w$ of a string is it's length
$\Phi_S(x) = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \ldots$ a power series!

**Notes on G. Series:**

1. Given a counting problem, define appropriate set and weight function so that the answers are encoded in the coefficients of the generating series.
2. "$x$" is a literal, we almost never put in values for $x$. Treat it as a variable we don't touch!
3. We will face problems where it is "easy" to find the generating series first, and then possibly determine its coefficients.

**Ex:** How many binstr of length $n$ have no "111"? (3 consec)
Using tools we learn, we will see that the answer is $[x^n] \dfrac{1+x+x^2}{1-x-x^2-x^3}$ set $S$ = all binstr with no 111, $w(\sigma)$ is the length.

**Formal Power Series:** Definition - Let $(a_0, a_1, a_2, \ldots)$ be a sequence of numbers. Then the formal power series associated with $\{a_n\}$ is
$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{k \geq 0} a_k x^k$
where $a_i$'s are well defined.
We say $a_k$ is the coefficient of $x^k$, denoted $[x^k] A(x)$.

Let $A(x) = \sum_{i \geq 0} a_i x^i$, $B(x) = \sum_{j \geq 0} b_j x^j$

**Equality:** $A(x) = B(x)$ if and only if $a_i = b_i$ for all $i \geq 0$. (coeff are equal)

Addition: $A(x)+B(x) = \sum_{i\geq 0} (a_i+b_i)x^i$

i.e $[x^i]A(x) + [x^i]B(x)$

- The addition of 2 power series is a power series, "closed under addition"

Multiplication: Ex - $(1+2x+3x^2)(1-3x+5x^2)$

Constant term: $1\cdot 1 = 1$

$x^1$ term: $(1\cdot -3x)+(1\cdot 2x) = -x$

$x^2$ term: $(1\cdot 3x^2)+(1\cdot 5x^2) = 8x^2$

$x^3$ term: $(-3x\cdot 3x^2)+(2x\cdot 5x^2) = x^3$

$x^4$ term: $(3x^2\cdot 5x^2) = 15x^4$

$$A(x)B(x) = \left(\sum_{i\geq 0} a_i x^i\right)\left(\sum_{j\geq 0} b_j x^j\right) = \sum_{i\geq 0}\sum_{j\geq 0} (a_i x^i)(b_j x^j)$$

$$= \sum_{i\geq 0}\sum_{j\geq 0} (a_i b_j) x^{i+j}$$

collecting similar terms

want: $(i+j)$ s.t $i+j=n$

$(0,n)\ (1,n-1)\dots(n,0)$

$$= \sum_{n\geq 0}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)x^n$$

$0\leq i\leq n,\quad j=n-i$

$(a_0+a_1x+a_2x^2+\dots)(b_0+b_1x+b_2x^2+\dots b_{n-1}x^{n-1}+b_nx^n)$

etc

$x^n$: Coeff is $a_0b_n + a_1b_{n-1}+\dots+a_nb_0$

Example:

$A(x) = 1+x+x^2+x^3+\dots = \sum_{i\geq 0} 1\cdot x^i$

$B(x) = 1+2x+3x^2+4x^3+\dots = \sum_{j\geq 0}(j+1)x^j$

$$[x^n]A(x)B(x) = \sum_{i=0}^{n}\left([x^i]A(x)\right)\left([x^{n-i}]B(x)\right)$$

$$= \sum_{i=0}^{n}(1)(n-i+1) = (n+1)+n+(n-1)+\dots+(2)+1$$

$$= \frac{(n+1)(n+2)}{2} = \binom{n+2}{2}$$

- The coeff of $x^n$ in $A(x)B(x)$ is a sum of $n+1$ finite numbers, which is finite.

$$A(x)B(x) = \sum_{n\geq 0}\binom{n+2}{2}x^n$$

closed under mult