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Math 239 - Lecture #7

Common Series:

$$\frac{1}{1-x} = 1+x+x^2+x^3+\dots \quad \sum_{i\geq 0} x^i$$ geometric series

$$\frac{1-x^{k+1}}{1-x} = 1+x+x^2+\dots+x^k$$ Partial geometric series

$$\left(\frac{1}{1-x}\right)^k = \frac{1}{(1-x)^k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n$$

Composition of Series:

Let $G(x) = \frac{1}{1-x} = 1+x+x^2+\dots$

$$G(2x) = \frac{1}{1-2x} = 1+2x+(2x)^2+(2x)^3+\dots = \sum_{n\geq 0} 2^n x^n$$

$$G(x^3) = \frac{1}{1-x^3} = 1+x^3+(x^3)^2+(x^3)^3+\dots = \sum_{n\geq 0} x^{3n}$$

$$[x^n]\frac{1}{1-x^3} = \begin{cases} 1 & \text{if } 3\mid n \\ 0 & \text{otherwise} \end{cases}$$

$$G(3x^2) = \sum_{n\geq 0}(3x^2)^n = \sum_{n\geq 0} 3^n x^{2n} = \frac{1}{1-3x^2}$$

$$[x^m]\frac{1}{1-x^3} = \begin{cases} 3^{\frac{m}{2}} & \text{if } 2\mid m \\ 0 & \text{if } 2\nmid m \end{cases}$$ need $2n=m$, so $n=\frac{m}{2}$

Example:

$$A(x) = \frac{x}{1-x} = x+x^2+x^3+\dots \qquad G(A(x)) = 1+A(x)+A(x)^2+\dots$$

$$= \frac{1}{1-A(x)} = \frac{1}{1-\frac{x}{1-x}} = \frac{1-x}{1-2x}$$

$$[x^n]\,G(A(x)) = [x^n]\frac{1-x}{1-2x} = [x^n]\frac{1}{1-2x} - [x^n]\frac{x}{1-2x}$$

$$= 2^n - [x^{n-1}]\frac{1}{1-2x} = 2^n - 2^{n-1} \qquad n\geq 1$$

Example:

$$\left(\frac{1}{1-3x^2}\right)^{30} = \sum_{n\geq 0}\binom{n+30-1}{30-1}(3x^2)^n = \sum_{n\geq 0}\binom{n+29}{29}3^n x^{2n}$$

$$[x^m]\left(\frac{1}{1-3x^2}\right)^{30} = \begin{cases} \binom{\frac{m}{2}+29}{29} 3^{\frac{m}{2}} & 2\mid m \\ 0 & 2\nmid m \end{cases}$$ * $2n=m$

This jazz does not always work.

$G(1+x^2) = 1 + (1+x^2) + (1+x^2)^2 + (1+x^2)^3 + \dots$

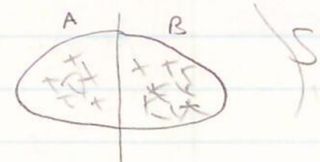
- We end up with an infinite amount of $+1+\dots+1+\dots$
- Constant term of $(1+x^2)^k$ is 1 for any k, so constant is ∞, thus not a power series.

In $G(\frac{x}{1-x})$, $(\frac{x}{1-x})^k$ has min term x^k.
So $[x^n]\, G(\frac{x}{1-x})$ can be non-zero in $1, \frac{x}{1-x}, \dots, (\frac{x}{1-x})^k$, which is finite.

Theorem: If constant term of $B(x)$ is 0, then $A(B(x))$ is always a power series.
(If const $\neq 0$, $A(B(x))$ may or may not be a power series)

Sum Lemma: Recall: set S, weight w, $\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)}$

So let $S = A \cup B$ where $A \cap B = \emptyset$ (disjoint). Let w be a weight function on S.
Then $\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$



Proof: $\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{\sigma \in A} x^{w(\sigma)} + \sum_{\sigma \in B} x^{w(\sigma)}$

$= \Phi_A(x) + \Phi_B(x)$ $\square$

Example: $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ $w(\sigma) = 2a$

$\Phi_{\mathbb{N}_0}(x) = 1 + x^2 + x^4 + x^6 + \dots = \frac{1}{1-x^2}$

Let $E = \{0, 2, 4, 6, \dots\}$ $O = \{1, 3, 5, 7, \dots\}$

$\Phi_E(x) = 1 + x^4 + x^8 + x^{12} = \frac{1}{1-x^4}$ $\Phi_O(x) = 1 + x^2 + x^6 + \dots$ $\frac{x^2}{1-x^4}$

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Cont

$$\Phi_E(x) + \Phi_o(x) = \frac{1}{1-x^4} + \frac{x^2}{1-x^4} = \frac{1}{1-x^2} = \Phi_{\mathbb{N}_0}(x).$$

The sum lemma can be extended to a disjoint union of any number of sets.

Product Lemma:

Theorem - Let A, B be sets with weight functions α, β, respectively. Suppose $A \times B$ has weight function $w(a,b) = \alpha(a) + \beta(b)$. Then

$$\underset{\substack{\uparrow \\ w}}{\Phi_{A\times B}}(x) = \underset{\substack{\uparrow \\ \alpha}}{\Phi_A}(x) \cdot \underset{\substack{\uparrow \\ \beta}}{\Phi_B}(x)$$

Example:

Throw 2 dice, $A = B = \{1,2,3,4,5,6\}$

Results are $A \times B$, $w(a,b) = a+b$

Define $\alpha(a) = a$ for A, $\beta(b) = b$ for B.

Then $w(a,b) = \alpha(a) + \beta(b)$.

$\Phi_A(x) = x + x^2 + \dots + x^6$, $\Phi_B(x) = x + x^2 + \dots + x^6$

By product lemma, $\Phi_{A\times B}(x) = \Phi_A(x) \cdot \Phi_B(x) = (x + x^2 + \dots + x^6)^2$.

$$\frac{2x}{1-x^3} \cdot \frac{x^4}{(1-x^6)}$$

$$\frac{2x - 2x^7 + x^4 - x^7}{1 - x^6 - x^3 + x^9} = \frac{2x + x^4 - 3x^7}{1 - x^3 - x^6 + x^9}$$