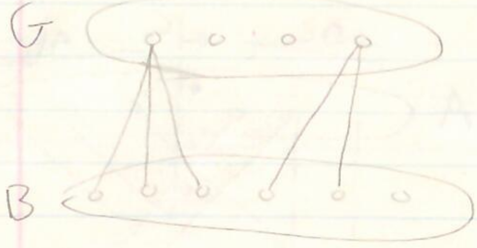
30/03/16

Math 239 - Lecture #35

Hall's Theorem: Given a bipartite graph with bipartition (A,B), is there a matching that saturates every vertex in A?



Are there ways we cannot marry off all the girls? Yes, there are many.

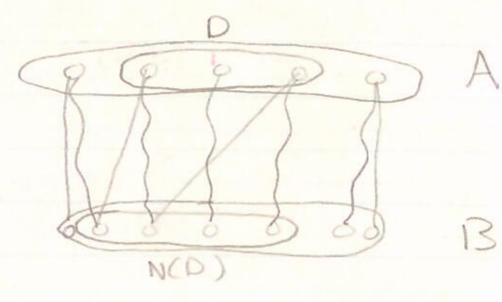
If there is a subset X of A for which the set of all neighbours is less than $|A|$, then we cannot find a matching saturating A.

Definition: Let $D \subseteq V(G)$. The neighbour set $N(D)$ is the set of all vertices adjacent to at least one vertex in D.

<u>Hall's Theorem</u> formally - Let G be bipartite with bipartition (A,B). Then G has a matching saturating every vertex in A if and only if every subset $D \subseteq A$ satisfies $|N(D)| \geq |D|$ (Hall's condition).

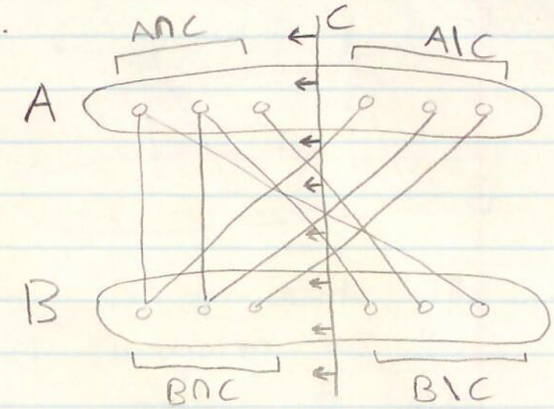
Proof: $(\Rightarrow)$ Let M be a matching that saturates A.

Let $D \subseteq A$. The matching edges in M with one end in D must have the other end in $N(D)$, and they are distinct neighbours.



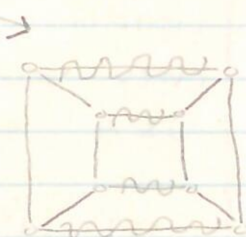
So $|D| \leq |N(D)|$

($\Leftarrow$) Suppose no matching saturates A (need to find a subset that violates Hall's condition).
Let M be a maximum matching. So $|M| < |A|$.
By Konig's theorem, there exists a cover C where $|C| = |M|$. So $|C| < |A|$.

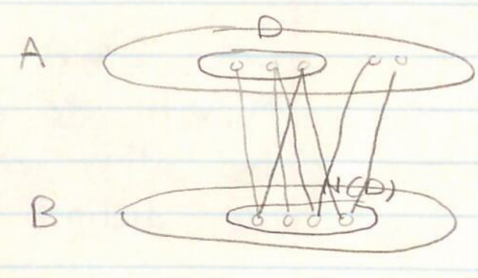


Since C is a cover, there are no edges between $A \setminus C$ and $B \setminus C$.
So $N(A \setminus C) \subseteq B \cap C$.
$|C| = |A \cap C| + |B \cap C|$.
$|A| = |A \cap C| + |A \setminus C|$.
Since $|C| < |A|$, $|B \cap C| < |A \setminus C|$.
Since $|N(A \setminus C)| \leq |B \cap C|$, we get $|N(A \setminus C)| \leq |A \setminus C|$.
So $A \setminus C$ violates Hall's condition. $\square$

Corollary: If G is a K-regular bipartite graph with $K \geq 1$, then G has a perfect matching.



Proof: Suppose G has bipartition (A, B). Let $D \subseteq A$. Every edge with one end in D has the other end in $N(D)$.



So $$\sum_{v \in D} \deg(v) \leq \sum_{v \in N(D)} \deg(v)$$

Since G is K-regular, $K|D| \leq K|N(D)|$. Since $K \geq 1$, we can divide by K to get $|D| \leq |N(D)|$. By Hall's Theorem, there is a matching that saturates A.
This also saturates B since $|A| = |B|$ (A6).
So then we have a perfect matching. $\square$