Pg #1

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Math 239 - Lecture #15

Problem: Given $A(x) = \frac{P(x)}{q(x)}$, what is $[x^n]A(x)$?

Suppose $A(x) = \sum_{n\geq 0} a_n x^n$ where $A(x) = \frac{3-8x-2x^2}{1-7x+16x^2-12x^3}$

$A(x) = \frac{3-8x-2x^2}{(1-2x)^2(1-3x)}$ · Using partial fractions,

$$= \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1-3x}$$

for some constants A, B, C.

$$= \frac{A(1-2x)(1-3x) + B(1-3x) + C(1-2x)^2}{(1-2x)^2(1-3x)} \quad \vdash = 3-8x-2x^2$$

Solving yields $A=-1$, $B=3$, $C=1$.

So $A(x) = \frac{-1}{1-2x} + \frac{3}{(1-2x)^2} + \frac{1}{1-3x}$

$\therefore [x^n]A(x) = [x^n]\frac{-1}{1-2x} + [x^n]\frac{3}{(1-2x)^2} + [x^n]\frac{1}{1-3x}$

$$= -2^n + 3\binom{n+2-1}{2-1}2^n + 3^n$$

$$= -2^n + 3(n-1)2^n + 3^n \quad = (3n+2)\cdot 2^n + 3^n$$ book familiar

Def^n:

$$*\binom{n+k-1}{k-1} = \frac{(n+k-1)!}{(k-1)!\,(n)!} = \frac{(n+k-1)(n+k-2)\dots(n+1)}{k!}$$

Product of (k-1) terms, each having an 'n'

constant denom

· This is a polynomial in n of degree k-1.

Generalization: $A(x) = \frac{P(x)}{q(x)}$ where $\deg(P(x)) < \deg(q(x))$, and

$$q(x) = (1-r_1x)^{e_1}(1-r_2x)^{e_2}\dots(1-r_kx)^{e_k}$$

By partial fraction, there exists constants $C_{i,j}$ where

$$A(x) = \frac{C_{1,1}}{1-r_1x} + \frac{C_{1,2}}{(1-r_1x)^2} + \dots + \frac{C_{1,e_1}}{(1-r_1x)^{e_1}} + \dots$$

$$\frac{C_{k,1}}{1-r_kx} + \dots + \frac{C_{k,e_k}}{(1-r_kx)^{e_k}}$$

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$$\text{So } [x^n]A(x) = \Big(\overbrace{C_{1,1}\binom{n+1-1}{1-1} + \dots + C_{1,e_1}\binom{n+e_1-1}{e_1-1}}^{\text{Poly in } n \text{ of deg } e_1-1}\Big) r_1^n + \dots$$

$$\therefore [x^n]A(x) = P_1(n) r_1^n + \dots + P_k(n) r_k^n$$

where $P_i(n)$ is a polynomial in n of degree $e_i - 1$

Example: $A(x) = \dfrac{1-2x}{(1+3x)^2(1-5x)^3}$

Then $[x^n]A(x) = \underbrace{(An+B)}_{\text{deg } 1}(-3)^n + \underbrace{(Cn^2+Dn+E)}_{\text{deg } 2} 5^n$

for constants A, B, C, D, E

$q(x) = (1-r_1x)^{e_1} \dots (1-r_kx)^{e_k}$

$q^*(x) = (x-r_1)^{e_1} \dots (x-r_k)^{e_k}$ ← Characteristic Polynomial

The roots are $r_1, \dots, r_k$ with multiplicities $e_1, \dots, e_k$ respectively.

Example: $q(x) = 1 - 7x + 10x^2 = (1-2x)(1-5x)$

Char Poly: $q^*(x) = (x-2)(x-5) = x^2 - 7x + 10$ inverted coefficients!

Generally, the characteristic polynomial of $1 - c_1x - c_2x^2 - \dots - c_kx^k$ is $x^k - c_1x^{k-1} - c_2x^{k-2} - \dots - c_k$

This will help to solve homogeneous recurrences.

Example: Suppose $\{a_n\}$ satisfies $a_0 = 1$, $a_1 = 5$, $a_n - 5a_{n-1} + 6a_{n-2} = 0$ for $n \geq 2$.

$(a_2 = 5a_1 - 6a_0 = 19,\ a_3 = 5a_2 - 6a_1 = 65, \dots)$

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Cont.

$$\left[\frac{\sim\sim}{1-c_1x-c_2x^2} \longleftrightarrow a_n - c_1a_{n-1} - c_2a_{n-2} = 0\right]$$

$A(x) = \sum_{n\geq 0} a_n x^n$ has the form $\dfrac{P(x)}{1-5x+6x^2}$

- Char poly is $x^2-5x+6 = (x-2)(x-3)$
- Roots are $x=2,3$ with multiplicity 1 each.

So $a_n = A\cdot 2^n + B\cdot 3^n$ for some constants A,B.

Using the initial conditions (last page),

$n=0$: $a_0 = 1 = A\cdot 2^0 + B\cdot 3^0$

$n=1$: $a_1 = 5 = A\cdot 2^1 + B\cdot 3^1$

Solving: $A+B=1$, $2A+3B=5$

$A=-2$, $B=3$

Thus $a_n = -2\cdot 2^n + 3\cdot 3^n$

- This process can be even quicker!

Solve $\{a_n\} = a_n - c_1a_{n-1} - \dots - c_ka_{n-k} = 0$

Char poly is $x^k - c_1x^{k-1} - \dots - c_k$

Find roots & mults. Find the form of a_n.

Solve unknown constants using initial conditions.

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