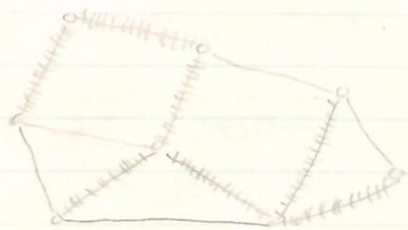
Math 239 - Lecture #26

Recall trees & spanning trees.

Definition: T is a <u>spanning tree</u> of G if T is a spanning subgraph of G that is a tree.

G



= Spanning Tree

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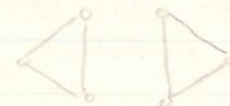


No spanning tree!

Theorem: G is connected if and only if G has a spanning tree.

Proof: ($\Leftarrow$) Suppose T is a spanning tree of G. Since T is a tree, there exists a u,v-path in T for all $u,v \in V(T)$. This is also a path in G since T is a subgraph of G. So G is connected.

($\Rightarrow$) Suppose G is connected. Prove by induction on the # of cycles, q.

<u>Base</u>: $q=0$, G is already a spanning tree since it is connected w/ 0 cycles.

<u>Hyp</u>: Assume any connected graph with $< q$ cycles has a spanning tree.

<u>Conc</u>: Let G be a connected graph with q cycles. Let e be an edge on a cycle in G. So e is not a bridge.

Then $G-e$ is connected with $< q$ cycles. By inductive hypothesis, $G-e$ has a spanning tree, which is also a spanning tree for G. $\square$

Corollary: If G is connected with n vertices and $n-1$ edges, then G is a tree.

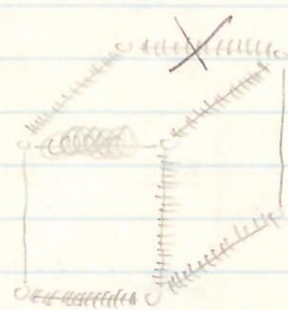
Proof: Since G is connected, by our previous theorem, G has a spanning tree T. Since T is a tree, T has $n-1$ edges. Since G has $n-1$ edges, T and G have the same set of edges. So $G=T$, and G is a tree. □

Theorem: Let G be a graph with n vertices. If any 2 of the 3 conditions below are satisfied, then G is a tree.

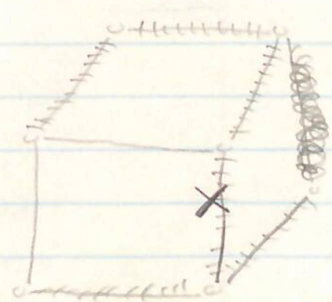
1. G is connected
2. G has no cycles
3. G has $n-1$ edges.

(2)(3) ⇒ Tree ; (2) ⇒ a forest $n-k$ edges, $k=$ # comp
G has $n-1$ edges ⇒ $k=1$ ⇒ connected

Theorem: If T is a spanning tree of G and e is an edge in G but not T, then $T+e$ contains exactly one cycle C. Moreover, if e' is an edge in C, then $T+e-e'$ is also a spanning tree of G.



Theorem: If T is a spanning tree of G and e is an edge in T, then $T-e$ has exactly 2 components. If e' is an edge in the cut induced by the vertices of one such component, then $T-e+e'$ is also a spanning tree of G.



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Math 239 - Lecture #26 Cont.

Bipartite Characterization

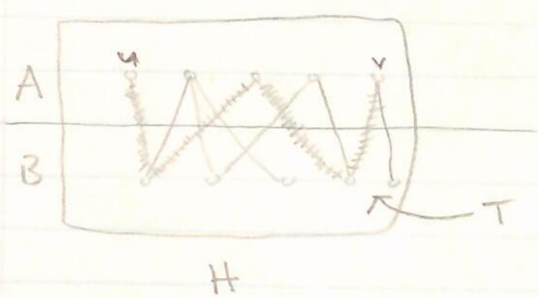
Theorem: G is bipartite if and only if G has no odd cycles.

Proof:

($\Rightarrow$) Suppose G is bipartite with bipartition (A,B). Let $C = v_1, v_2, \ldots, v_k, v_1$ be any cycle. Suppose WLOG $v_1 \in A$, then $v_2 \in B$, $v_3 \in A$, $v_4 \in B$, etc.

So then $v_i \in A$ if and only if i is odd. Since $v_k v_1$ is an edge, $v_k \in B$. So k is even, hence C has even length.

Suppose G is not bipartite. Then atleast one component H of G is not bipartite. Since H is connected, it has a spanning tree T. Since T is bipartite, it has a bipartition (A,B).



Since H is not bipartite, there exists an edge uv in H where u,v are in the same bipartition, WLOG say $u,v \in A$.

In T, there is a unique u,v-path P, which must have even length since it starts and ends in A.

Then $P + uv$ is a cycle in G. $\square$