4/01/16

Math 239 - Lecture #1

Assign #1 due next Friday
Office hours: M, W, Th 4:30-5:30
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Counting: sets.

① <u>Cartesian products</u>: If A, B are sets, then
$A \times B = \{(a,b) \mid a \in A, b \in B\}$

<u>Example:</u> $A = \{1,2\}$ $B = \{1,2,3\}$
$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$
$A^k = \{(a_1, \ldots, a_n) \mid a_i \in A\} = \underbrace{A \times A \times A \cdots \times A}$ → <u>k times</u>

* $|S|$ = size of S

If A, B are finite, then $|A \times B| = |A| \cdot |B|$ <u>product</u>
And it follows, $|A^k| = |A|^k$

<u>Example:</u> Throw two 6-sided dice. The set that enumerates all possible results, $S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$
$= \{(a,b) \mid a, b \in \{1, \ldots, 6\}\}$
$|S| = |\{1, \ldots, 6\}|^2 = 36$

<u>Example:</u> Binary strings of length n can be represented by
$S = \{0,1\}^n$ e.g. $(1,0,1) \in \{0,1\}^3$
$|\{0,1\}^n| = 2^n$

② <u>Disjoint Union</u>: Let $S = S_1 \cup S_2$ where $S_1 \cap S_2 = \emptyset$
Then $|S| = |S_1| + |S_2|$

<u>Example:</u> Let E be elements of $\{1, \ldots, 6\}^2$ ← 2 elems! where the sum of the 2 parts is even.
- Both odd
- Both even

$E = E_1 \cup E_2$
$E_1 = \{(a,b) \in \{1, \ldots, 6\}^2 \mid a, b \text{ are even}\}$
$E_2 = \{(a,b) \in \{1, \ldots, 6\}^2 \mid a, b \text{ are odd}\}$

So since $E_1 \cap E_2 = \emptyset$, $|E| = |E_1| + |E_2|$
$E_1 = \{2,4,6\}^2$ $E_2 = \{1,3,5\}^2$
$|E| = 9 + 9 = \underline{18}$ □

What if $S = S_1 \cup S_2$ but $S_1 \cap S_2$ might not be $\varnothing$.
$|S| = |S_1| + |S_2| - |S_1 \cap S_2|$ ← subtract intersect once

Example: Let F be elements of $S = \{1, \dots, 6\}^2$ where atleast 1 of 2 numbers is even.
Partition $F = F_1 \cup F_2$ where $F_1 = \{(a,b) \in S \mid a \text{ is even}\}$
$F_2 = \{(a,b) \in S \mid b \text{ is even}\}$
Then $F_1 \cap F_2 = \{(a,b) \in S \mid a,b \text{ are even}\} \neq \varnothing$
$\therefore |F| = |F_1| + |F_2| - |F_1 \cap F_2| = (3\cdot 6) + (6 \cdot 3) - (3 \cdot 3) = \underline{27}$. □

Binomial: Coefficient $\binom{n}{k}$ is the number of ways of selecting a set of 'k' objects out of 'n' objects.
- These could be subsets of $\{1, \dots, n\}$ of size k.

$\binom{n}{k}$: First pick k objects in order. There are $n \cdot (n-1) \cdot (n-2) \dots \underbrace{(n-k+1)}_{k^{th} \text{ object}}$ ways to do this.

Each set of k objects can be selected $k!$ times in order. So the # of sets is $\dfrac{n(n-1)\cdots(n-k+1)}{k!} = \dfrac{n!}{k!(n-k)!}$

Binomial: Theorem $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ x^k appears $\binom{n}{k}$ times

Expansion of $(1+x)^n = (1+x)(1+x)\dots(1+x)$ n times
Each term in the expansion has the form
$a_1 a_2 \cdots a_n$ where each $a_i \in \{1, x\}$

$(1+x)(1+x)(1+x)$
$= 1 \cdot 1 \cdot 1$
$+ 1 \cdot 1 \cdot x$
$+ 1 \cdot x \cdot x$
$\vdots$

How many times can we get x^2?
3 times!
$1 \cdot x \cdot x, \quad x \cdot 1 \cdot x, \quad x \cdot x \cdot 1$.

In order for $a_1 \dots a_n$ to be x^k, we need exactly k of the n a_is to be x_is. There are then $\binom{n}{k}$ ways to do so, so coeff x^k is $\binom{n}{k}$. □