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Math 239 - Lecture #4

Generating Series

Example: How many subsets of $\{1,2,3\}$ have size k?
Let S be the set of all subsets of $\{1,2,3\}$
For each element σ in S, give a weight w
where $w(\sigma) = |\sigma|$.

Rephrase Q: "How many elements in S have weight k?"
For each element σ of weight k, we contribute x^k
to the "generating series" of S.

σ	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$w(\sigma)$	0	1	1	1	2	2	2	3
$x^{w(\sigma)}$	x^0	x^1	x^1	x^1	x^2	x^2	x^2	x^3
	$=1$	x	x	x	x^2	x^2	x^2	x^3

The generating series of S is the sum of all contributions
$\Phi_S(x) = 1 + 3x + 3x^2 + x^3 = (1+x)^3$

What does the coefficient of x^k represent?
The number of elements of S of weight k, subsets of size k in this case.

Rephrase Q: "What is the coeff of x^k in $\Phi_S(x)$?

Definition: Given a set S where each element $\sigma \in S$ is given a non-negative integer weight $w(\sigma)$, the <u>generating series</u> of S with respect to w is

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)}$$

We can rewrite this as: Let a_k be the number of elements of S of weight k. Then,

$$\Phi_S(x) = \sum_{k \geq 0} a_k x^k$$

i.e $[x^2](1+x)^3 = 3$

Notation for coefficients: $[x^k]\, \Phi_S(x)$ is the coeff of x^k in the series

Example: How many subsets of $\{1, \dots, n\}$ have size k?

Let S be the set of all subsets of $\{1, \dots, n\}$.
For each $\sigma \in S$, let $w(\sigma) = |\sigma|$.
The answer to our question is $\binom{n}{k}$, so the coefficient of x^k in $\Phi_S(x)$ with respect to w is $\binom{n}{k}$.

binomial theorem

$$\text{So } \Phi_S(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

Generating series answer to our question is $[x^k]\Phi_S(x)$.

Examples: How many ways can we throw two 6-sided dice to get a sum of k?

We can enumerate all possible ways of throwing 2 dice by the set $S = \{1, \dots, 6\} \times \{1, \dots, 6\}$.
For each $(a,b) \in S$, define $w(a,b) = a+b$.

In $\Phi_S(x)$ w.r.t w, the answer is $[x^k]\Phi_S(x)$.

- $w(3,5) = 8$
- $\Phi_S(x) = \underset{(1,1)}{1x^2} + \underset{\substack{(1,2)\\(2,1)}}{2x^3} + \underset{\substack{(1,3)\\(2,2)\\(3,1)}}{3x^4} + 4x^5 + 5x^6 + \underset{\substack{(1,6)\\(2,5)\\\vdots\\(6,1)}}{6x^7} + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + 1x^{12}$

Settlers of Catan!!

$$= (x + x^2 + x^3 + x^4 + x^5 + x^6)^2$$
$$= (x + x^2 + x^3 + x^4 + x^5 + x^6)(x + x^2 + x^3 + x^4 + x^5 + x^6)$$

$$x^3 \cdot x^5 = x^8 = x^{3+5} \qquad (3,5)$$

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Math 239 - Lecture #4 Cont.

Example: How many binary strings have length k?

Let S be the set of all binary strings.
For each $\sigma \in S$, define $w(\sigma)$ to be the length of σ.   • $w(00101) = 5$ • $w(11) = 2$

In $\Phi_S(x)$ w.r.t w, the coeff x^k is the # of binary strings of length k. This is 2^k.
So $\Phi_S(x) = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \dots = \dfrac{1}{1-2x}$

geometric series