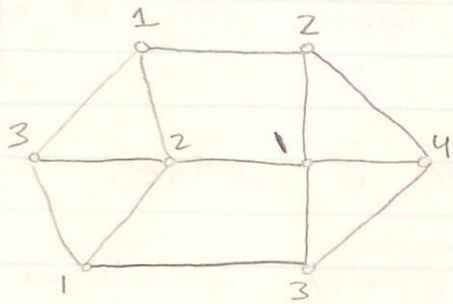
21/03/16

Math 239 - Lecture #31

Notes on Kuratowski's:

(1) Other than the main vertices of the K_5 or $K_{3,3}$, we cannot repeat vertices or edges.
(2) Usually we can find a $K_{3,3}$ edge subdivision in a non-planar graph.
(3) An algorithm for finding these subdivisions exists but is far beyond the level of this course.

Colouring:



4-colouring

A K-colouring of G is an assignment of a colour to each vertex of G using at most K colours such that adjacent vertices recieve different colours.

- If a graph has a K-colouring, then it is said to be K-colourable.
- If a graph is K-colourable, then it is also (K+1)-colourable.

Q: Given a graph, what is the minimum number of colours you need?

Theorem: K_n is n-colourable, but not (n-1)-colourable.

Theorem: A graph is 2-colourable if and only if it is a bipartite graph.

Anything between these boundaries is much more difficult to deal with.

Colouring Planar Graphs

*Theorem: Every planar graph is 6-colourable

Theorem: Every planar graph has a vertex of degree at most 5.
Proof: Let G be a planar graph with n vertices.
Suppose BWOC, that every vertex has degree at least 6. By HSL, $|E(G)| = 3n$.
But a planar graph with n vertices has at most $3n-6$ edges. Contradiction.

Proof: By induction on the number of vertices n.
(of *).

Base: When $n=1$, it is clearly 6-colourable.
Hyp: Assume any planar graph on $n-1$ vertices is 6-colourable.
Conc: Let G be a planar graph on n vertices; let v be a vertex of degree at most 5.

Let $G-v$ be the graph obtained by removing v and it's incident edges. Then $G-v$ is planar with $n-1$ vertices, and by ind hyp, $G-v$ is 6-colourable.

Keep the same colouring for G, and colour v with one not used by it's neighbours. This is possible since v has at most 5 neighbours, and there are 6 colours we can use.

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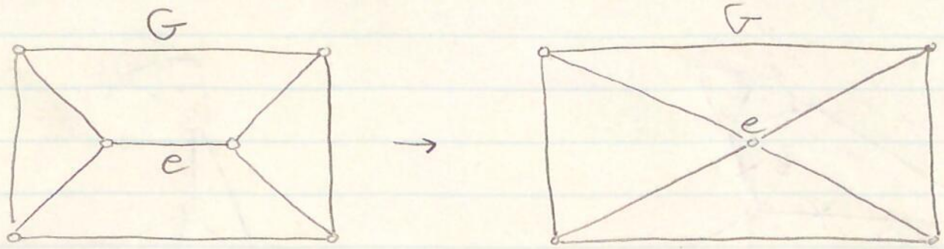
21/03/16

Math 239 - Lecture #31 Cont.

☆

Theorem: Every planar graph is 5-colourable. (...)

Contraction:



(merge two ends into one)

If G is planar, then G/e is also planar.

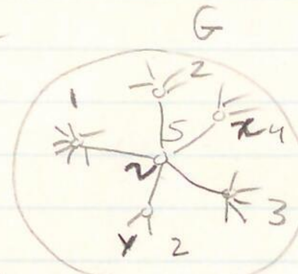
Proof of: ☆ By strong induction on n.

Base: Any $n \leq 5$ is 5-colourable.

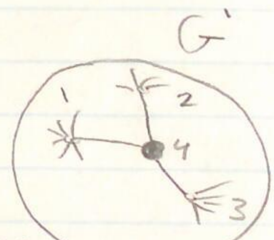
Hyp: Any planar graph with less than n vertices is 5-colourable.

Conc: Suppose G is planar with n-vertices. Let v be a vertex of $\deg \leq 5$. If v has $\deg \leq 4$, then we can apply the argument in the 6-colour theorem (one colour left).

Otherwise, assume $\deg(v) \geq 5$. At least 2 neighbours x, y are not adjacent, because otherwise we have K_5.



Contract xv and yv to get G'. By ind hyp, G' is 5-colourable.



We will keep the same colouring for G except assign the colour of the merge vertex to x, y (not adjacent). For v, there are ≤ 4 colours used in its 5 neighbours, so we have at least 1 colour available for v.

$\square$