Pg #1

22/01/16

Ch. 2 start

# Math 239 - Lecture #9

**Integer Composition:** A $k$-tuple $(a_1, \ldots, a_k)$ of positive integers is a composition of $n$ if $a_1 + \ldots + a_k = n$

Composition has $k$ parts.

**Example:** Compositions of 5 include: $(1,3,1)$, $(2,3)$, $(3,2)$, $(1,1,1,1)$, $(5)$, ....

Notes
1) Every part is positive, cannot be zero.
2) Order of the parts does matter.
3) The number of compositions of 0 is 1, which is $()$.

**Example:** How many compositions of $n$ have exactly $k$ parts?

$n=4$, $k=3$
$(1,1,2)$, $(1,2,1)$, $(2,1,1)$ 3 of them

Define a set $S$ to be all compositions with exactly $k$ parts, ignoring $n$ for now.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ (positive integers)

Then $S = \mathbb{N}^k = \{(a_1, \ldots, a_k) \mid a_i \in \mathbb{N}\}$

Define weight function $w$ on $S$ where $w(a_1, \ldots, a_k) = a_1 + \ldots + a_k$
For each $\mathbb{N}$, we define weight function $\alpha$ where $\alpha(a) = a$. Then $w(a_1, \ldots, a_k) = \alpha(a_1) + \ldots + \alpha(a_k)$, so the product lemma applies.

The generating series for $\mathbb{N}$ w.r.t $\alpha$ is

$$\Phi_{\mathbb{N}}(x) = x + x^2 + x^3 + x^4 + \ldots = \frac{x}{1-x}$$

$\therefore$ By the product lemma, *

$$\Phi_S(x) = \left(\Phi_{\mathbb{N}}(x)\right)^k = \frac{x^k}{(1-x)^k}$$

Then $[x^n] \frac{x^k}{(1-x)^k}$ is the # of $k$-tuples with weight $n$, i.e the # of compositions of $n$ with $k$ parts.

Cont. $[x^{n-k}] \frac{1}{(1-x)^k}$ for $n \geq k$

$$= \binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1} \qquad \square$$

Combinatorial "Proof": Imagine this is putting $n$ balls into $k$ boxes, where each box contains atleast 1 ball

$$ {}^{\times}O \downarrow O \downarrow O \downarrow O \downarrow \ldots \downarrow O^{\times} $$

$n$ O's
$k-1$ dividers (or 1's)

no divider at the ends!

- Cannot have 2 dividers in the same slot
- There are then $n-1$ slots! Boom, $\binom{n-1}{k-1}$.

Example: How many compositions of $n$ are there? (Regardless of Parts)

$n=1$ (1)
$n=2$ (2) (1,1) • Clearly $2^{n-1}$
$n=3$ (3) (1,2) (2,1) (1,1,1)
$n=4$ (4) (1,3) (2,2) (3,1) (1,1,2) (1,2,1) (2,1,1) (1,1,1,1)

Let $S$ be the set of all possible compositions, regardless of $n$.

Partition $S$ according to the number of parts. The set with $k$ parts is $\mathbb{N}^k$. So
$S = \mathbb{N}^0 \cup \mathbb{N}^1 \cup \mathbb{N}^2 \cup \ldots = \bigcup_{k \geq 0} \mathbb{N}^k$.

This is a disjoint union, so the sum lemma applies. The weight of a composition is the sum of its parts.

We had $\Phi_{\mathbb{N}^k}(x) = \left(\frac{x}{1-x}\right)^k$ before (prev example)
$\therefore$ Using sum lemma,

$$\Phi_S(x) = \sum_{k \geq 0} \Phi_{\mathbb{N}^k}(x) = \sum_{k \geq 0} \left(\frac{x}{1-x}\right)^k$$

# Math 239 Lecture #9 - Cont

Ex: Continued

$$= \frac{1}{1-\frac{x}{1-x}}$$

· Constant term of the previous sum is clearly 0 ($x \cdot$ stuff).

$$= \frac{1-x}{1-2x}$$

So the number of compositions of $n$ is:

$$[x^n]\frac{1-x}{1-2x} = [x^n]\frac{1}{1-2x} - [x^n]\frac{x}{1-2x}$$

$$= 2^n - [x^{n-1}]\frac{1}{1-2x} = 2^n - 2^{n-1} = 2^{n-1}$$

when $n \geq 1$

* $2^{n-1}$ is half of $2^n$

$$= \begin{cases} 2^{n-1} & \text{when } n \geq 1 \\ 1 & \text{when } n = 0 \end{cases}$$

$\square$