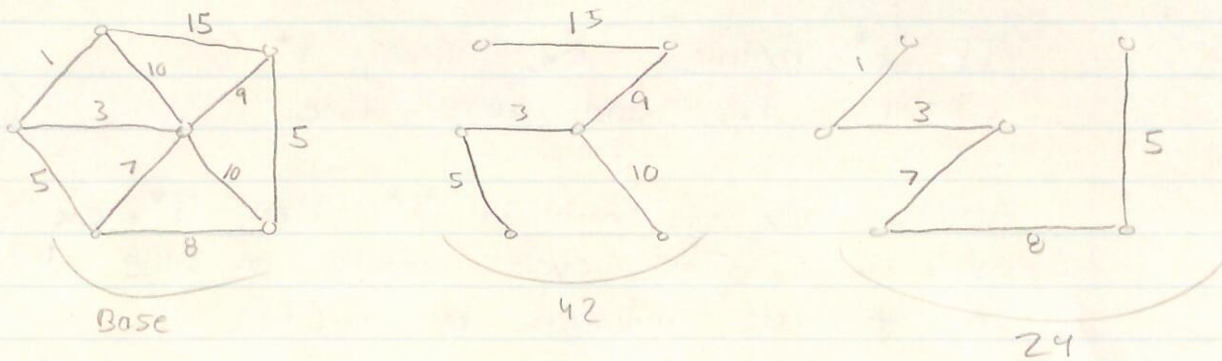
11/03/16

Math 239 - Lecture #27

MST (Minimum spanning Tree)

Problem - Given a connected graph G, and a weight function $w: E(G) \to \mathbb{R}$, Find a spanning tree whose total edge weight is minimized.



Prim's Algorithm:

1) Let v be any vertex in G, let T be the tree that consists of only v.
2) While T is not a spanning tree..
   a) Consider all edges in the cut induced by $V(T)$
   b) Let $e = uv$ be an edge of minimum weight in the cut, say $u \in V(T)$, $v \notin V(T)$
   c) Add v to $V(T)$, add e to $E(T)$.

Theorem: Prim's Algorithm always produces a MST.

Proof: Let $T_1, T_2, \ldots, T_n$ be the trees produced by the algorithm at each step, where the order of edges we add is $e_1, \ldots, e_{n-1}$. (we add e_i to T_i to get T_{i+1}).

We will prove by induction that T_k is a subgraph of some MST of G, for each $k = 1, \ldots, n$

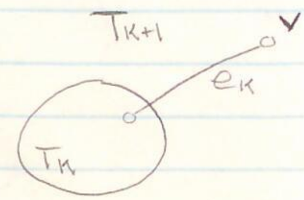
By doing so, we will have proved that T_n is contained in a MST, hence T_n is a MST of G.

Proof: Cont

Base: When $n=1$, T_1 is just one vertex, and every MST contains T_1.

Hyp: Assume there exists a MST T^* that contains T_K.

Conc: Consider T_{K+1}.



If T^* contains e_K, then T^* contains T_{K+1} and we're done.

Assume e_K is not in T^*. Then $T^* + e_K$ is a cycle in C. This cycle contains an edge e' that is in the cut induced by $v(T_K)$.

In Prim's algorithm, we picked e_K as an edge of minimum weight in the cut induced by $V(T_K)$. So $w(e_K) \leq w(e')$. Also, we see that $T^* + e_K - e'$ is a spanning tree.

So if $w(e_K) < w(e')$, then $w(T^* + e_K - e') = w(T^*) + w(e_K) - w(e') < w(T^*)$. But T^* is a MST and $T^* + e_K - e'$ has lower weight than T^*, contradiction.

So $w(e_K) = w(e')$, thus $w(T^* + e_K - e') = w(T^*)$, so $T^* + e_K - e'$ is a MST. (That contains e_K !!!)

Moreover, T^* contains T_{K+1}, so T_{K+1} is a MST. □