Math 239 - Lecture #8

Recall: Product lemma - sets A, B, weight functions α, β respectively. $A \times B$ weight function $\omega(a,b) = \alpha(a) + \alpha(b)$, then $\Phi_{A\times B}(x) = \Phi_A(x) \cdot \Phi_B(x)$.

Proof: $\Phi_A(x) \cdot \Phi_B(x) = \left(\sum_{a\in A} x^{\alpha(a)}\right)\left(\sum_{b\in B} x^{\beta(b)}\right) = \underbrace{\sum_{a\in A}\sum_{b\in B}}_{\text{sum over } A\times B} x^{\alpha(a)} x^{\beta(b)}$ exp are added

$$= \sum_{(a,b)\in A\times B} x^{\alpha(a)+\beta(b)} = \sum_{(a,b)\in A\times B} x^{\omega(a,b)} = \Phi_{A\times B}(x). \qquad \square$$

Example: Red, blue dice. Red die is doubled in the sum of the two dice.
Define $R = B = \{1, \ldots, 6\}$. Consider $S = R \times B$.
$\omega(r,b) = 2r + b$. Weight function for R is $\alpha(r) = 2r$, for B is $\beta(b) = b$.
Product lemma applies since $\omega(r,b) = \alpha(r) + \beta(b)$.

$$\Phi_R(x) = x^2 + x^4 + x^6 + x^8 + x^{10} + x^{12}$$

$$\Phi_B(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$$

So $\Phi_{R\times B}(x) = (x^2 + x^4 + \ldots + x^{12})(x + x^2 + \ldots + x^6)$ Product Lemma

# of ways to get a sum of n is $[x^n]\,\Phi_{R\times B}(x)$.

Example: How many ways can a sequence of k non-negative integers add up to n?

$k=2,\ n=3$: $(0,3), (1,2), (2,1), (3,0)$

Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$

The sequence of k non-negative integers is
$\mathbb{N}_0^k = \{(a_1, a_2, \ldots, a_k) \mid a_i \in \mathbb{N}_0\}$

Cont. Define $\omega(a_1, \dots, a_k) = a_1 + \dots + a_k$

For each $\mathbb{N}_0$, define $\sigma(a) = a$.

Generating series for $\mathbb{N}_0$: $\Phi_{\mathbb{N}_0}(x) = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$

By product lemma, $\Phi_{\mathbb{N}_0^k}(x) = \frac{1}{(1-x)^k}$

Then answer to our question is: $[x^n] \frac{1}{(1-x)^k}$

Before we said this was equal to the thing over there! $\binom{n+k-1}{k-1}$

Proof:

$n = 10$ $\quad x_1 + x_2 + x_3 + x_4 = 10$

$k = 4$ $\quad x_i \geq 0$

my thoughts during example:

i.e distributing 10 identical balls into 4 boxes

but how many ways can we do this?

- 3 barriers to seperate the balls into 4 boxes
- Hence $10 + 4 - 1 = 13$ numerator
- Then we arrange by choosing the $k-1$ dividers

Hence $\binom{n+k-1}{k-1}$

Actual Proof:

Let S be the set of all sequences of k non-neg integers that sum to n.

Let T be the set of all binary strings of length $n+k-1$, with $k-1$ 1's and n 0's.

Define $f: S \to T$ by $f(a_1, \dots, a_k) = 0^{a_1} 1 0^{a_2} 1 \dots 1 0^{a_k}$

- 0^b is a sequence of b consecutive 0's.
- There are $k-1$ 1's (they divide into k boxes), and $a_1 + \dots + a_k = n$ 0's. So $f(a_1, \dots, a_k)$ is in T.

This is invertible. Any binstr in T has the form $0^{a_1} 1 0^{a_2} 1 \dots 1 0^{a_k}$ where $a_i \geq 0$.

$f^{-1}(0^{a_1} 1 0^{a_2} 1 \dots 1 0^{a_k}) = (a_1, \dots, a_k)$ is the inverse.

So $|S| = |T| = \binom{n+k-1}{k-1}$ (bin str of length $n+k-1$, choose $k-1$ spots for 1's)

$|S| = \binom{n+k-1}{k-1}$. $\square$