08/01/16

Math 239 - Lecture #3

A <u>combinatorial proof</u> is any proof that involves counting arguments. We prove an equation by counting a set of objects in two different ways.

<u>Example</u>: Binomial Theorem, $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ - see lesson 1

set $x=1$: $2^n = \sum_{k=0}^{n} \binom{n}{k} = \underbrace{\binom{n}{0}}_{\text{subsets of size 0}} + \underbrace{\binom{n}{1}}_{\text{size 1}} + \underbrace{\binom{n}{2}}_{\text{size 2}} + \dots + \underbrace{\binom{n}{n}}_{\text{size n}}$

- Together gives all subsets up to size n.

<u>Combinat. Proof</u>: Let S be the set of all subsets of $\{1,\dots,n\}$. From last class, there is a bijection between 'S' and the set of binary strings of length n. So $|S| = 2^n$.

For $k=0,\dots,n$, define S_k to be the set of all subsets of $\{1,\dots,n\}$ of size k.

So $S = S_0 \cup S_1 \cup S_2 \cup \dots \cup S_n$.

Since every subset has one possible size, they must be disjoint (such that $S_i \cap S_j = \emptyset$).

$\therefore |S| = |S_0| + |S_1| + \dots + |S_n|$

But $|S_k| = \binom{n}{k}$ so this gives us $2^n = \sum_{k=0}^{n} \binom{n}{k}$. □

bin theorem

<u>Pascal's Triangle</u>:

n=0			1				
n=1		1	1				
n=2	1	2	1				
n=3	1	3	3	1			← $\binom{3}{0},\binom{3}{1},\binom{3}{2},\binom{3}{3}$
n=4	1	4	6	4	1		← $\binom{4}{0},\binom{4}{1},\binom{4}{2},\binom{4}{3},\binom{4}{4}$
n=5	1	5	10	10	5	1	

etc.

Example: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$$\begin{tikzcd}[column sep=small] \text{row } n-1 & \boxed{\binom{n-1}{k-1}} \arrow[dr] & & \boxed{\binom{n-1}{k}} \arrow[dl] \\ \text{row } n & & \boxed{\binom{n}{k}} & \end{tikzcd}$$

See Pascal tri

Combinat. Proof

Let S be the set of all subsets of $\{1,\dots,n\}$ of size k. Then $|S| = \binom{n}{k}$.
Let S_1 be subsets of $\{1,\dots,n\}$ of size k which includes element n. Let S_2 " ... " which does not include element n.

ex

$n=5,\ k=3$ Subsets of $\{1,2,3,4,5\}$ of size 3

$S_1 = \{ \{1,2,5\}, \{1,3,5\}, \{1,4,5\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\} \}$

$S_2 = \{ \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\} \}$

Then $S = S_1 \cup S_2$ is clearly a disjoint union, since no subset can both have and not have n.
So $|S| = |S_1| + |S_2|$

Each element of S_2 is a subset of $\{1,\dots,n\}$ of size k. So $|S_2| = \binom{n-1}{k}$. Each element of S_1 consists of $\{n\}$ union with a subset of $\{1,\dots,n-1\}$ of size $k-1$. So $|S_1| = \binom{n-1}{k-1}$.

$\therefore \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Consider each part from a Stat 230 POV.
In one we're skipping 5, in the other we're choosing it every time.

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Math 239 - Lecture #3 Cont.

Example:

$$\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1} = \binom{n-1}{n-1} + \binom{n}{n-1} + \dots + \binom{n+k-1}{n-1}$$

Combinat. Proof: Let S be the set of all subsets of $\{1, \dots, n+k\}$ of size n. Then $|S| = \binom{n+k}{n}$.

- Choosing $n-1$ every time
- Numerator increasing each time by 1

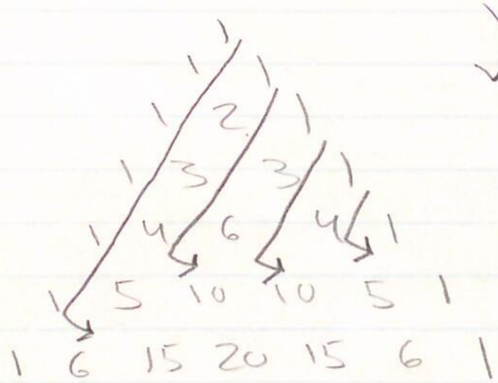
For $i = 0, \dots, k$, let S_i be the set of all subsets of $\{1, \dots, n+k\}$ of size n whose largest element is $n+i$.

$\{\_ \ \_ \ \_\}$ size n
- largest element is max $n+k$, and min n.

Then $S = S_0 \cup S_1 \cup \dots \cup S_k$ is a disjoint union.
So $|S| = \sum_{i=0}^{k} |S_i|$

Each element of S_i consists of $\{n+i\}$ union with a subset of $\{1, \dots, n+i-1\}$ of size $n-1$, since all such elements must be $< n+i$.

So $|S_i| = \binom{n+i-1}{n-1}$, $\therefore \binom{n+k}{n} = \sum_{n=0}^{k} \binom{n+i-1}{n-1}$. $\square$



that proof on the triangle

HOCKEY STICKS