Pg #1

08/02/16

Math 239 - Lecture #16

Recall: Char poly of roots $r_1, \dots, r_k$ with multiplicities $e_1, \dots, e_k$
$\Rightarrow a_n = P_1(n) r_1^n + \dots + P_k(n) r_k^n$
- $P_i(n)$ is a poly in n of deg $e_i - 1$

So if $\{a_n\}$ satisfies $a_n + c_1 a_{n-1} + \dots + c_k a_{n-k} = 0$
Char poly is $x^k + c_1 x^{k-1} + \dots + c_k$

Example: $\{a_n\}$ satisfies $a_0 = 1$, $a_1 = 2$, $a_2 = 1$,
for $n \geq 3$, $a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$

*won't ask for degree 2 or more on exam

- Char poly is $x^3 - 3x^2 + 3x - 1 = (x-1)^3$

Roots $x = 1$ with multiplicities 3.
So $a_n = \underbrace{(A + Bn + Cn^2)}_{\text{Poly of deg } 3-1} \cdot 1^n$ for some constants A, B, C.

Use initial conditions to solve for A, B, C.

$$\left.\begin{array}{lll} a_0 = 1 = A + (B \cdot 0) + (C \cdot 0) = A & & n=0 \\ a_1 = 2 = A + B + C & & n=1 \\ a_2 = 1 = A + 2B + 4C & & n=2 \end{array}\right\} \text{Solve System}$$

$\therefore A = 1$, $B = 2$, $C = -1$, $\therefore a_n = 1 + 2n - n^2$

Example: $\{a_n\}$ satisfies $a_0 = 7$, $a_1 = 10$, $a_2 = 13$
for $n \geq 3$, $a_n - 7a_{n-1} + 15a_{n-2} - 9a_{n-3} = 0$

- Char poly is $x^3 - 7x^2 + 15x - 9 = (x-3)^2(x-1)$

Roots are $x = 3$ with mult 2, $x = 1$ with mult 1.
So $a_n = (A + Bn)3^n + C \cdot 1^n$ for some constants A, B, C

$$\left.\begin{array}{lll} a_0 = 7 = A + C & & n=0 \\ a_1 = 10 = 3A + 3B + C & & n=1 \\ a_2 = 13 = 9A + 18B + C & & n=2 \end{array}\right\} \Rightarrow A = 3, B = -1, C = 4$$

So $a_n = (3-n)3^n + 4$

Homogeneous Recurrences Continued

Example: (Fib seq) $\{f_n\}$ $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

$f_n - f_{n-1} - f_{n-2} = 0$, so Char Poly $= x^2 - x - 1$

So $x = \dfrac{1 \pm \sqrt{5}}{2}$ by Quadratic Formula

So $f_n = A \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + B \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$ for some constants A, B.

$$\left.\begin{array}{l} f_0 = 0 = A + B \\ f_1 = 1 = \left(\frac{1+\sqrt{5}}{2}\right)A + \left(\frac{1-\sqrt{5}}{2}\right)B \end{array}\right) \Rightarrow A = \frac{1}{\sqrt{5}},\ B = \frac{-1}{\sqrt{5}}$$

Then $f_n = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$

- Believe it or not, for $\forall n \in \mathbb{N}$, this is an integer!

$f_n \to \sim \left(\frac{1+\sqrt{5}}{2}\right)^n$ as $n \to \infty$

Solving non-homogeneous recurrences

Example: $\{a_n\}$ satisfies $a_0 = 11$, $a_1 = 42$,

for $n \geq 2$, $a_n - 3a_{n-1} + 2a_{n-2} = 10 \cdot 3^{n-1}$

$(a_2 = 3a_1 - 2a_0 + 10 \cdot 3^1 = 134)$

A specific solution b_n is a sequence that satisfies the recurrence, but not necessarily the initial conditions.

Want b_n where $b_n - 3b_{n-1} + 2b_{n-2} = 10 \cdot 3^{n-1}$

"Guess" $b_n = \alpha \cdot 3^n$ for some constant α

Then $b_n - 3b_{n-1} + 2b_{n-2} = \alpha \cdot 3^n - 3 \cdot \alpha \cdot 3^{n-1} + 2 \cdot \alpha \cdot 3^{n-2}$

$$= \alpha \cdot 3^{n-1}\left(3 - 3 + 2/3\right) = \frac{2}{3}\alpha \cdot 3^{n-1}$$

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Ex Cont: This equals $10 \cdot 3^{n-1}$. So $\frac{2}{3}\alpha = 10$, $\alpha = 15$
So $b_n = 15 \cdot 3^n$ is a specific solution.

So ① - ②: $(a_n - b_n) - 3(a_{n-1} - b_{n-1}) + 2(a_{n-2} - b_{n-2}) = 0$ for $n \geq 2$.
$a_n - b_n$ satisfies a homogeneous recurrence!

Char poly is $x^2 - 3x + 2 = (x-2)(x-1)$ Roots $x = 2, 1$
So $a_n - b_n = A \cdot 2^n + B \cdot 1^n$ for some const A, B
So $a_n = A \cdot 2^n + B + 15 \cdot 3^n$
Now use initial conditions to solve!

$$\left.\begin{array}{l} a_0 = 11 = A + B + 15 \\ a_1 = 42 = 2A + B + 45 \end{array}\right) \Rightarrow A = 1,\ B = -5$$

So $a_n = 2^n - 5 + (15 \cdot 3^n)$.

In general, $a_n = \boxed{\text{general sol'n to homogeneous recur}} + \boxed{\text{specific sol'n } b_n}$

Example: $\{a_n\}$ satisfies $a_0 = 2$, $a_1 = 3$,
For $n \geq 2$, $a_n - a_{n-1} - a_{n-2} = -4n + 16$

Let b_n be a specific solution.
"Guess" $b_n = \alpha n + \beta$.

Then $b_n - b_{n-1} - b_{n-2} = (\alpha n + \beta) - (\alpha(n-1) + \beta) - 2(\alpha(n-2) + \beta)$

$= (\alpha n - \alpha n - 2\alpha n) + (\beta + \alpha - \beta + 4\alpha - 2\beta)$

$= -2\alpha n + (5\alpha - 2\beta)$. This equals $-4n + 16$,

So $-2\alpha = -4 \Rightarrow \alpha = 2$
$5\alpha - 2\beta = 16 \Rightarrow -2\beta = 6 \Rightarrow \beta = -3$

Cont: A specific solution is $b_n = 2n-3$

Char poly is $x^2 - x - 2 = (x-2)(x+1)$, roots $= 2, -1$

So $a_n = A \cdot 2^n + B \cdot (-1)^n + 2n - 3$

$$\left.\begin{array}{l} a_0 = 2 = A + B - 3 \\ a_1 = 3 = 2A - B - 1 \end{array}\right) \Rightarrow A = 3,\ B = 2$$

So $a_n = 3 \cdot 2^n + 2 \cdot (-1)^n + 2n - 3$.