 15/01/16

Math 239 - Lecture #6

Coefficient Rule

$[x^n]\, x^k A(x)$ $=$ $\begin{cases} [x^{n-k}]A(x) & \text{if } n \geq k \\ 0 & \text{if } n < k \end{cases}$

$A(x) = 1 + 3x + 7x^2$ $\Rightarrow x^3 A(x) = x^3 + 3x^4 + 7x^5$ — coeff x = coeff x^4

Example: Let $A(x) = 1 + 3x^2$, $B(x) = 1 + 2x + 4x^2 + 8x^3 + \dots = \sum_{i \geq 0} 2^i x^i$

$[x^n]A(x)B(x) = [x^n](1+3x^2)B(x)$

$= [x^n]B(x) + [x^n]3x^2 B(x)$

$= 2^n + 3[x^{n-2}]B(x) = 2^n + 3 \cdot 2^{n-2}$ for $n \geq 2$.

Calculate $n = 0, 1$ cases separately. $1 + 2x$

$$A(x)B(x) = 1 + 2x + \sum_{n \geq 2} \left(2^n + 3 \cdot 2^{n-2}\right) x^n$$

Inverse: The inverse of a power series $A(x)$ is another power series $B(x)$ where: $A(x)B(x) = 1$.

Example: $A(x) = 1 - x$. Suppose $B(x)$ is the inverse of $A(x)$.

Let $B(x) = \sum_{n \geq 0} b_n x^n$. $B(x)(1-x) = 1$

$= B(x) - xB(x)$

$= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \dots$
$\quad - b_0 x - b_1 x - b_2 x^2 - \dots$ $= b_0 + \sum_{n \geq 1} (b_n - b_{n-1}) x^n$

* This is equal to 1
* They have the same coefficients, so compare coefficients on both sides.

$n=0$ constant: $b_0 = 1$

$n=1$ x^1: $b_1 - b_0 = 0$ $\therefore b_1 = b_0 = 1$.

$n=2$ x^2: $b_2 - b_1 = 0$ $\therefore b_2 = b_1 = b_0 = 1$.

$\vdots$

$b_n = 1$ for all $n \geq 0$

i.e $b_0 = 1$, $b_n - b_{n-1} = 0$ for $n \geq 1$ $\Rightarrow b_n = b_{n-1}$ for all $n \geq 1$

$\Rightarrow B(x) = 1 + x + x^2 + x^3 + \dots = \dfrac{1}{1-x}$ (inv of $A(x)$)

"Geometric Series"

Example: $A(x) = x$. Suppose $B(x)$ is its inverse,

$B(x) = \sum_{n \geq 0} b_n x^n$ so $B(x) \cdot x = 1$

$B(x) \cdot x = b_0 x + b_1 x^2 + b_2 x^3 + \ldots = 1$

- Constant term tells us $0 = 1$, but this is not possible. Hence, x has no inverse.

So when $A(x)$ has a constant term 0, then $B(x)A(x)$ has constant term 0, which cannot equal 1.

Theorem: $A(x)$ has an inverse if and only if the constant term of $A(x)$ is not 0.

Finding coefficients of $\frac{A(x)}{B(x)}$ through a recurrence.

Example: Let $A(x) = \dfrac{1+x}{1-2x-3x^2}$ Suppose $A(x) = \sum_{n \geq 0} a_n x^n$

Multiply both sides by $(1-2x-3x^2)$: $A(x)(1-2x+3x^2) = \underline{1+x}$

$$A(x)(1-2x-3x^2) = A(x) - 2x A(x) - 3x^2 A(x)$$
$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$
$$- 2a_0 x - 2a_1 x^2 - 2a_2 x^3 - \ldots$$
$$- 3a_0 x^2 - 3a_1 x^3 - \ldots$$
$$= a_0 + (a_1 - 2a_0)x + \sum_{n \geq 2} (a_n - 2a_{n-1} - 3a_{n-2}) x^n$$

This equals $\underline{1+x}$ from before; compare coeff.

$n=0$ constant: $a_0 = 1$

$n=1$ x^1: $a_1 - 2a_0 = 1$, $\therefore a_1 = 3$] initial conditions

$n \geq 2$ x^n: $a_n - 2a_{n-1} - 3a_{n-2} = 0 \Rightarrow a_n = 2a_{n-1} - 3a_{n-2}$] recurrence

i.e $a_2 = 6 + 3 = 9$, $a_3 = 2a_2 + 3a_1 = 27, \ldots$

$A(x) = 1 + 3x + 9x^2 + 27x^3 + \ldots$

Use the recurrence to generate the series!