Pg #1

29/02/16

Leap Year!

Math 239 - Lecture #22

Theorem: If there exists $u \in V(G)$ such that a u,v-path exists for all $v \in V(G)$, then G is connected.

Proof: Let $x, y \in V(G)$. By asscumption, there exists an x,u-path and a u,y-path. Joining them together yields an x,y-walk. By a previous theorem, there exists an x,y-path. So G is connected. $\square$

Theorem: The n-cube is connected.

Proof: Let v_0 be the string of n 0's, let x be any string of length n. Suppose x has k 1's, located at positions $i_1, \dots, i_k$

$n = 5$

$v_0 =$ 00000
00001
01001
01101

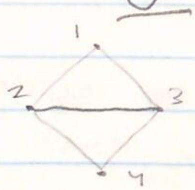
how to achieve ~~any~~ vertex! 1-bit at a time

Produce k strings $v_1, \dots, v_k$ where v_j is the string with exactly j 1's, located at positions $i_1, \dots, i_j$. Notice that v_j and v_{j+1} differ in exactly one bit at position i_{j+1}, so $v_j v_{j+1}$ is an edge. So $v_0, v_1, \dots, v_k = x$ is a v_0, x-path. So the n-cube is connected. $\square$

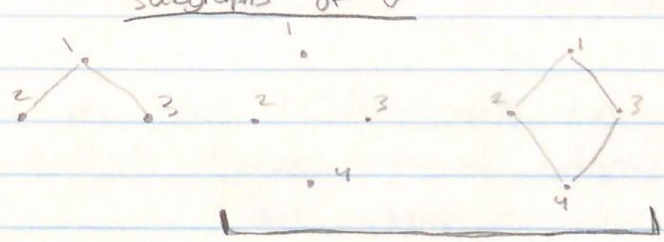
Components & Cuts: H is a subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and every edge in $E(H)$ joins two vertices in $V(H)$.

Ex:

G



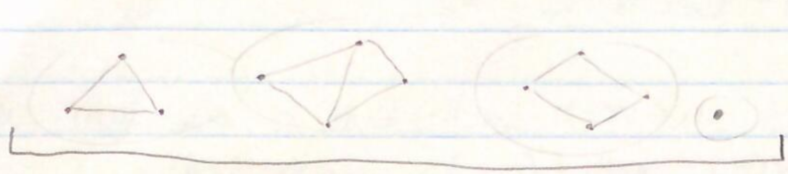
subgraphs of G



A subgraph is spanning if $V(H) = V(G)$.

FIVE STAR.

○ = Comp onent



A component of G is a maximally connected nonempty subgraph of G.

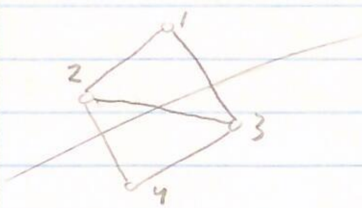
Maximally means there is no other connected nonempty subgraph that contains it.

A connected graph has one component; disconnected graphs have at least two components.

In a component, there is no edge joining a vertex in the component with a vertex outside of a component, due to maximality.

Def$^{\underline{n}}$: Let $x \subseteq V(G)$. The cut induced by x is the set of all edges with exactly one end in x.

Ex:



$x = \{1, 2\}$, the cut induced by x is $\{13, 23, 24\}$.

$Y = \{1, 2, 3, 4\}$ Cut is $\emptyset$

$Z = \{1, 4\}$ Cut is $\{12, 13, 24, 34\}$

A component induces an empty cut, i.e the set of vertices of a component induces an empty cut.

Theorem: G is not connected if and only if there exists a non-empty proper subset x of $V(G)$ that induces an empty cut.

Pg #2

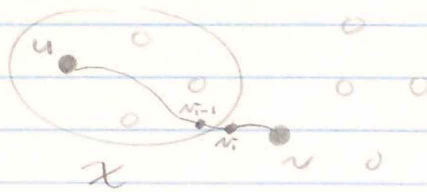
29/02/16

Math 239 - Lecture #22 Cont.

Proof: ($\Rightarrow$) Suppose G is not connected. Then G contains at least 2 components, let H be one of them. Then $V(H)$ is a nonempty proper subset of $V(G)$. That is non-empty by def^n, proper because another component exists.

Due to maximality, $V(H)$ induces an empty cut.

($\Leftarrow$) Suppose X is a nonempty proper subset of $V(G)$ that induces an empty cut. Let $u \in X$, $v \notin X$.



contradiction illustration

Suppose there exists a u,v-path $v_0, v_1, \ldots, v_k$. Since v_0 is in X & v_k is not in X, we can pick the smallest index i such that $v_i \notin X$. Then $v_{i-1} \in X$, and $v_{i-1}v_i$ is an edge in the cut induced by X. Contradiction.

So no u,v-path exists, and G is not connected. $\square$