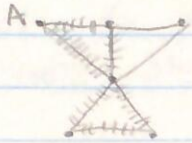
26/02/16

Math 239 - Lecture #21

Recall walks & paths.

Definition: A closed walk is a walk that starts and ends at the same vertex.



- Starts & ends at the same vertex 'A'

Recall: If the is a u,v-walk, then there is a u,v-path.

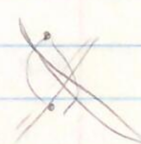
Corollary: If a u,v-path and a v,w-path exist, then a u,w-path also exists.

Proof - Join the given u,v-path and v,w-path to get a u,w-path. By the above theorem, there is a u,w-path.

The relation of "u,w is in a path" is a transitive property. This is also reflexive "u,u is in a path" and symmetric "if u,v is in a path, then v,u is in a path"; so this is an equivalence relation!

Cycles: A cycle is a non-trivial closed walk with no repeated vertices (except the start/end vertex).

etc.

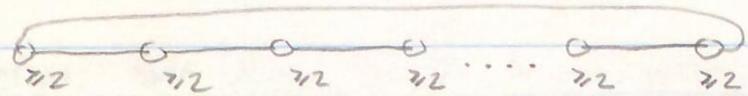


In a cycle, the length is the # of edges which is the same as the # of vertices. Length must be atleast 3.

A cycle is 2-regular. Hence, in any graph, if a vertex is part of a cycle, it must have degree at least 2.

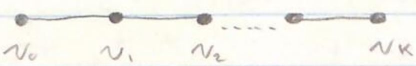
Theorem: If every vertex has degree atleast 2, then G contains a cycle.

Idea:



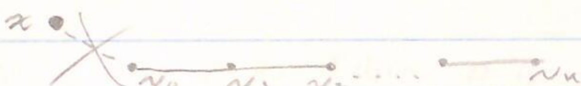
we have to go back!

Proof: Let $v_0, v_1, v_2, \ldots, v_k$ be a path of longest length in G. Such a path exists since a path cannot be longer than $|V(G)|$



Consider v_0: It has v_1 as one neighbour.

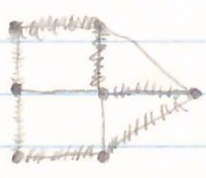
Since v_0 has degree atleast 2, it must have another neighbour x. This x cannot be outside of the path, for otherwise $x, v_0, v_1, v_2, \ldots, v_k$ is a path longer than the longest path.



So x is on the path, say $x = v_i$ for some $i \geq 2$. Then $v_0, v_1, \ldots, v_i, v_0$ is a cycle in G. □

Definition: A <u>Hamilton cycle</u> is a cycle that contains all vertices in the graph.

Ex:



Peterson

Can't be a Ham!

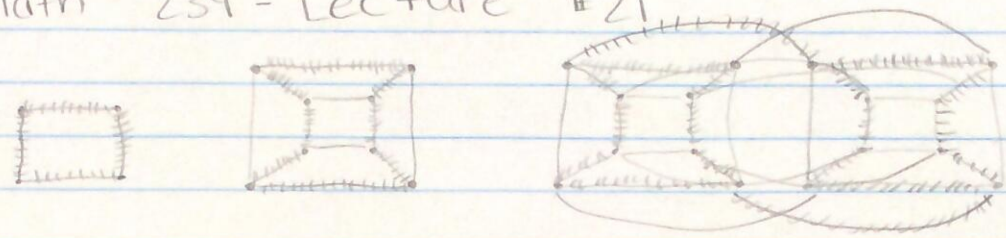
-Ham cycle

Theorem: The n-cube has a Hamilton cycle.

26/02/16

Math 239 - Lecture #21

N-cube:

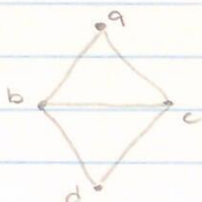


Proof-sketch: Suppose we have a cycle on the (n-1)-cube. Duplicate the cycle on the n-cube. Remove a corresponding edge from both cycles. Join using the edges joining corresponding vertices. $\square$ kind of

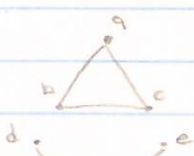
Connectedness

A graph G is <u>connected</u> if there is a u,v-path for every pair $u,v \in V(G)$

Ex -



This is connected since there is an a,b-path, an a,c-path, an a,d-path, a b,c-path, a b,d-path, and a c,d-path.



No a,e-path exists, so it is not connected.

Theorem: If there exists a vertex u in G such that a u,v-path exists for all $v \in V(G)$, then G is connected.