Pg #1

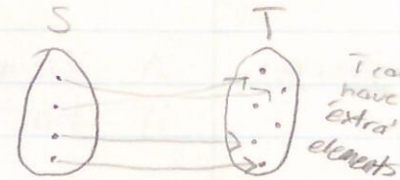
06/01/16

Math 239 - Lecture #2

Bijection: Function $f: \underset{\text{domain}}{S} \to \underset{\text{codomain}}{T}$ For any $x \in S$, $f(x) = y$ where $y \in T$

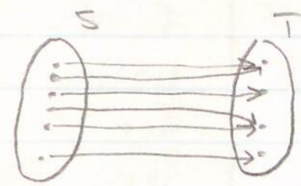
1. f is 1-1 (injective) if no two elements of S are mapped to the same element, in T.
   $[f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$
2. f is onto (surjective) if every element of T is mapped to by some element in S.
3. f is a bijection (or 1-1 correspondance) if f is 1-1 and onto.

Continued: Suppose $f: S \to T$ exists where:



1. f is 1-1

Then $|T| \geq |S|$, since each element in S is mapped to a distinct element in T.



2. f is onto

Then $|S| \geq |T|$, since each element in T has a distince pre-image in S.

3. f is a bijection

Then $|S| = |T|$ (since $|S| \geq |T|$ and $|S| \leq |T|$)

Thus, elements in S can be 'paired up' with elements in T.

Example: $A = \{a, b, c\}$ $B = \{1, 2, 3\}$.
Defince $f: A \to B$ by $f(a) = 1$, $f(b) = 2$, $f(c) = 3$.
So we have a bijection, $|A| = |B|$

Example: Let S be the set of all subsets $\{1, \ldots, n\}$ of size K.
Let T be the set of all subsets $\{1, \ldots, n\}$ of size $n-K$.

$n = 3$ $K = 1$

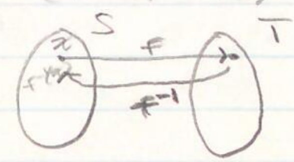
$S = \{\{1\}, \{2\}, \{3\}\}$ (size 1 subsets)

$T = \{\{1,2\}, \{1,3\}, \{2,3\}\}$ (size 2 subsets)

all subsets of $\{1,2,3\}$

In general, define $f: S \to T$ where for each $x \in S$, $f(x) = \{1, \dots, n\} \setminus X$.
Check: Is $f(x) \in T$? Yes, since $|x| = k$ and $x \subseteq \{1, \dots, n\}$, therefor, $|f(x)| = |\{1, \dots, n\} \setminus X| = n-k$

Definition: An inverse of $f: S \to T$ is a function $f^{-1}: T \to S$ such that for all $x \in S$, $f^{-1}(f(x)) = x$, and for $\forall y \in T$, $f(f^{-1}(y)) = y$.



i.e $x = f^{-1}(f(x))$

Theorem: A function is a bijection if and only if it has an inverse.

Previous Example Cont: $f(x) = \{1, \dots, n\} \setminus X$.
Define inverse $f^{-1}: T \to S$ by $f^{-1}(Y) = \{1, \dots, n\} \setminus Y$ for all $y \in T$.
Then $f(\{1,2\}) \to \{3\}$, $f(\{1,3\}) \to \{2\}$, $f(\{2,3\}) \to \{1\}$

Check: For any $X \in S$, $f^{-1}f(x) = f^{-1}(\{1, \dots, n\} \setminus X)$
$= \{1, \dots, n\} \setminus (\{1, \dots, n\} \setminus X) = X$
similar for any $Y \in T$. So f is a bijection and it follows that $|S| = |T|$.

$|S| = \binom{n}{k}$, $|T| = \binom{n}{n-k}$

By our proof (combinatorical), we know that
$\binom{n}{k} = \binom{n}{n-k}$

06/01/16

Math 239 - Lecture #2 Cont.

Example: Let S be the set of all subsets of $\{1,\dots,n\}$
Let T be the set of all binary strings of length n.

$n=3$

$S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

$T = \{000, 001, 010, 011, 100, 101, 110, 111\}$

Define a binary string '$a_3a_2a_1$' and any elements presence represent certain a_k,
i.e $\{3\} = \underline{1}00$

my poor explanation

Define $f: S \to T$ where for each $X \in S$,

$$f(x) = a_n a_{n-1} \dots a_1 \quad \text{where} \quad a_i = \begin{cases} 1 & \text{if } i \in x \\ 0 & \text{if } i \notin x \end{cases}$$

The inverse is $f^{-1}: T \to S$ where for each $b_n, \dots, b_1 \in T$, $f^{-1}(b_n \cdots b_1) = \{i \in \{1,\dots,n\} \mid b_i = 1\}$

So f is a bijection, and $|S| = |T| = 2^n$

To prove a bijection:

1. Clearly define $f: S \to T$
2. Check $f(x) \in T$ for all $x \in S$. (properly defined)
3. Give the inverse $f^{-1}: T \to S$ that is "intuitively clear" from my definition.