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Math 239 - Lecture #29

Definition: A connected planar graph is <u>platonic</u> if every vertex has an embedding where every vertex has the same degree (≥ 3) and every face has the same degree (≥ 3).

We can turn a planar embedding into a polyhedron by drawing it on a sphere and cut off the faces.

Suppose a platonic graph has vertex deg $d_v \geq 3$ and face deg $d_f \geq 3$, n vertices, m edges, s faces.

① Handshaking Lemma: $n \cdot d_v = 2m \Rightarrow n = \frac{2m}{d_v}$
② HSL for faces: $s \cdot d_f = 2m \Rightarrow s = \frac{2m}{d_f}$

③ Euler's Formula: $n - m + s = 2$

$\Rightarrow \frac{2m}{d_v} - m + \frac{2m}{d_f} = 2$, mult by $d_v d_f$

$\Rightarrow 2md_f - md_vd_f + 2md_v = 2d_vd_f > 0$, divide out m

$\Rightarrow 2d_f - d_vd_f + 2d_v > 0$

$\Rightarrow 2d_f - d_vd_f + 2d_v + 4 - 4 > 0$ factor

$\Rightarrow -(d_v - 2)(d_f - 2) + 4 > 0$

$\Rightarrow (d_v - 2)(d_f - 2) < 4$ $\quad * (d_v, d_f \geq 3)$

Possible (d_v, d_f) pairs

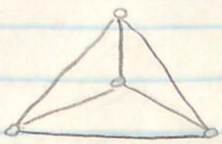
d_v	3	4	5
d_f	3,4,5	3	3
	$4 > d_f - 2$	$2 > d_f - 2$	$4/3 > d_f - 2$

These are all the possible pairs for platonic graphs

Cont.

Platonic Graphs:

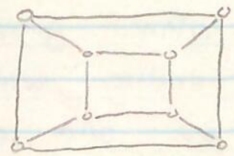
(1) $d_v = 3$, $d_f = 3$



$n=4$
$m=6$
$f=4$

Tetrahedron
(d4)

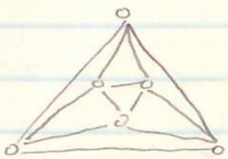
(2) $d_v = 3$, $d_f = 4$



$n=8$
$m=12$
$f=6$

cube/Hexahedron
(d6)

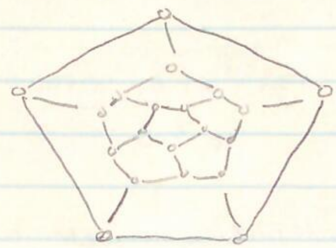
(3) $d_v = 4$, $d_f = 3$



$n=6$
$m=12$
$s=8$

Octahedron
(d8)

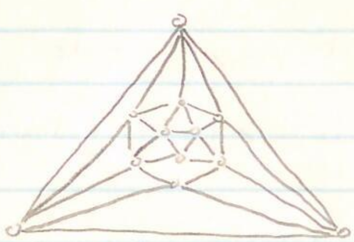
(4) $d_v = 3$, $d_f = 5$



$n=20$
$m=30$
$s=12$

Dodecthedron
(d12)

(5) $d_v = 5$, $d_f = 3$



$n=12$
$m=30$
$s=20$

isosahedron
(d20)

Non-Planar Graphs

Theorem: When $n \geq 3$, a connected planar graph on n vertices has at most $3n-6$ edges.

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Proof: Consider a connected embedding with n vertices, m edges, s faces. We claim that every face has deg at least 3.

If G does not have a cycle, then G is a tree, which has only 1 face. Since $n \geq 3$, G has at least 2 edges, so this face has deg at least 4.

If G contains a cycle, then every face must contain a cycle on its boundary (to seperate it from the other faces). So each face has deg ≥ 3.

Using HSLFF,

$$2m = \sum_{f \in F} \deg(f) \geq 3s$$
$$= 3(2-n+3) \quad \text{by E.F } (n-m+s=2)$$
$$= 6-3n+3m$$

$\therefore m \leq 3n-6$. $\square$