Taller 15

Hecho por

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Punto 1

$limit(f, 0)$

Tomando el procedimiento 7.4.1

$$\left\{\epsilon > 0, n > \frac{1}{\epsilon}\right\} \vdash_{\mathrm{DS}(\mathcal{L})} |f(n) - 0| < \epsilon$$

$$\begin{aligned}
& |f(n) - 0| \\
\equiv \quad & \langle \text{ Def.}(f) \ \rangle \\
& \left|\frac{1}{n+1} - 0\right| \\
\equiv \quad & \langle \text{ Aritmética } \rangle \\
& \frac{1}{n+1} \\
\equiv \quad & \langle \text{ Álgebra } \rangle \\
& 0 < \frac{1}{n+1} < \frac{1}{n} \\
\equiv \quad & \langle\ n > m \wedge m = \frac{1}{\epsilon}\ \rangle \\
& 0 < \frac{1}{n+1} < \frac{1}{n} < \frac{1}{m} = \epsilon \\
\equiv \quad & \langle \text{ Transitividad}(<)\ \mathit{true}\ \rangle
\end{aligned}$$

Punto 2

$limit(f, 1)$

Tomando el procedimiento 7.4.1

$$\left\{\epsilon > 0, n > \frac{1}{\epsilon}\right\} \vdash_{\mathrm{DS}(\mathcal{L})} |f(n) - 1| < \epsilon$$

$$\boxed{f(n) = \frac{n}{n+1} = 1 - \frac{1}{n+1}}$$

$$\begin{aligned}
& \left|1 - \frac{1}{n+1} - 1\right| \\
\equiv \quad & \langle \text{ Aritmética } \rangle \\
& \frac{1}{n+1} \\
\equiv \quad & \langle \text{ Álgebra } \rangle \\
& 0 < \frac{1}{n+1} < \frac{1}{n} \\
\equiv \quad & \langle\ n > m \wedge m = \frac{1}{\epsilon}\ \rangle \\
& 0 < \frac{1}{n+1} < \frac{1}{n} < \frac{1}{m} = \epsilon \\
\equiv \quad & \langle \text{ Transitividad}(<) \rangle \\
& \mathit{true}
\end{aligned}$$

Punto 3

$limit(f, 0)$

Tomando el procedimiento 7.4.1

$$\left\{\epsilon > 0, n > \frac{1}{\epsilon^2}\right\} \vdash_{\mathrm{DS}(\mathcal{L})} |f(n) - 0| < \epsilon$$

$$\begin{aligned}
& \left|\frac{1}{\sqrt{n+1}} - 0\right| \\
\equiv \quad & \langle \text{ Aritmética } \rangle \\
& \frac{1}{\sqrt{n+1}} \\
\equiv \quad & \langle \text{ Álgebra } \rangle \\
& 0 < \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \\
\equiv \quad & \langle\ n > m \wedge m = \frac{1}{\epsilon^2}\ \rangle \\
& 0 < \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{m}} = \epsilon \\
\equiv \quad & \langle \text{ Transitividad}(<) \rangle \\
& \mathit{true}
\end{aligned}$$

Punto 4

limit$(f, 1)$

Tomando el procedimiento 7.4.1

$$\left\{\epsilon > 0, n > \frac{1}{\epsilon} + 1\right\} \vdash_{\mathrm{DS}(\mathcal{L})} |f(n) - 1| < \epsilon$$

$$\begin{aligned}
& \left|\frac{1}{n-1}\right| \\
\equiv \quad & \langle \text{ Aritmética } \rangle \\
& \frac{1}{n-1} \\
\equiv \quad & \langle \text{ Álgebra } \rangle \\
& 0 < \frac{1}{n-1} \\
\equiv \quad & \langle\, n > m \wedge m = \frac{1}{\epsilon} + 1 \,\rangle \\
& 0 < \frac{1}{n-1} < \frac{1}{m-1} = \epsilon \\
\equiv \quad & \langle \text{ Transitividad}(<) \rangle \\
& \mathit{true}
\end{aligned}$$

Punto 5

limit$(f, 0)$

Tomando el procedimiento 7.4.1

$$\left\{\epsilon > 0, n > e^{1/\epsilon} + 1\right\} \vdash_{\mathrm{DS}(\mathcal{L})} |f(n) - 0| < \epsilon$$

$$\begin{aligned}
& \left|\frac{1}{\ln(n)} - 0\right| \\
\equiv \quad & \langle \text{ Aritmética } \rangle \\
& \frac{1}{\ln(n)} \\
\equiv \quad & \langle \text{ Álgebra } \rangle \\
& 0 < \frac{1}{\ln(n)} \\
\equiv \quad & \langle\, n > m \wedge m = e^{1/\epsilon} \,\rangle \\
& 0 < \frac{1}{\ln(n)} < \frac{1}{\ln(m)} = \epsilon
\end{aligned}$$

Punto 6

$limit(f, 1)$

Tomando el procedimiento 7.4.1

$$\left\{\epsilon > 0, n > \frac{\epsilon}{3} + 1\right\} \vdash_{\mathrm{DS}(\mathcal{L})} |f(n) - 1| < \epsilon$$

$$\boxed{f(n) = \frac{n^2}{(n+1)^2} = 1 - \frac{2n+1}{(n+1)^2}}$$

$$\begin{aligned}
& \left|1 - \frac{2n+1}{(n+1)^2}\right| \\
\equiv \quad & \langle \text{ Aritmética } \rangle \\
& \frac{2n+1}{(n+1)^2} \\
\equiv \quad & \langle \text{ Álgebra } \rangle \\
& 0 < \frac{2n+1}{(n+1)^2} < \frac{2n+1}{n+1} < \frac{2n+1}{n} < 3n \\
\equiv \quad & \langle\ n > m \wedge m = \frac{\epsilon}{3}\ \rangle \\
& 0 < \frac{2n+1}{(n+1)^2} < \frac{2n+1}{n+1} < \frac{2n+1}{n} < 3n < 3m = \epsilon \\
\equiv \quad & \langle \text{ Transitividad}(<) \rangle \\
& \mathit{true}
\end{aligned}$$

Procedimientos

7.4.1

$$limit(f,0) \equiv (\forall \epsilon \in \mathbb{R} \mid \epsilon > 0 : (\exists m \in \mathbb{R} \mid m \geq 0 : (\forall n \in \mathbb{N} \mid n > m : |f(n) - 0| < \epsilon)))$$

Por metateorema 7.22, demostrar

$$\{\epsilon > 0\} \vdash_{\mathrm{DS}(\mathcal{L})} (\exists m \in \mathbb{R} \mid m \geq 0 : (\forall n \in \mathbb{N} \mid n > m|f(n) - 0| < \epsilon))$$

$$\begin{array}{ll}
& (\exists m \in \mathbb{R} \mid m \geq 0 : (\forall n \in \mathbb{N} \mid n > m : |f(n) - 0| < \epsilon)) \\
\equiv & \langle \text{ Azúcar sintáctico } \rangle \\
& \exists m \in \mathbb{R}(m \geq 0 \wedge (\forall n \in \mathbb{N} \mid n > m : |f(n) - 0| < \epsilon)) \\
\Leftarrow & \langle \text{ instanciación con testigo } a \rangle \\
& a \geq 0 \wedge (\forall n \in \mathbb{N} \mid n > a|f(n) - 0| < \epsilon) \\
\equiv & \langle\ a \geq 0 \equiv \mathit{true}, \text{ Identidad}(\wedge) \ \rangle \\
& (\forall n \in \mathbb{N} \mid n > a|f(n) - 0| < \epsilon)
\end{array}$$

Por metateorema 7.22, demostrar

$$\{\epsilon > 0, n > a\} \vdash_{\mathrm{DS}(\mathcal{L})} |f(n) - 0| < \epsilon$$