Taller 03

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Matemáticas
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Índice

1. Punto 1: tabla de verdad y función

1.1. $p = \phi$

p
F
T

$$H_\phi(\mathtt{T}) = \mathtt{T}$$
$$H_\phi(\mathtt{F}) = \mathtt{F}$$

1.2. $(p \equiv r) = \phi$

p	r	$(p \equiv r)$
F	F	T
F	T	F
T	F	F
T	T	T

$$H_\phi(\mathtt{F},\mathtt{T}) = H_\phi(\mathtt{T},\mathtt{F}) = \mathtt{F}$$
$$H_\phi(\mathtt{F},\mathtt{F}) = H_\phi(\mathtt{T},\mathtt{T}) = \mathtt{F}$$

1.3. $((p \to (\neg q)) \to r) = \phi$

p	q	r	$(\neg q)$	$(p \to (\neg q))$	$((p \to (\neg q)) \to r)$
F	F	F	T	T	F
F	F	T	T	T	T
F	T	F	F	T	F
F	T	T	F	T	T
T	F	F	T	T	F
T	F	T	T	T	T
T	T	F	F	F	T
T	T	T	F	F	T

$$H_\phi(\mathtt{F},\mathtt{F},\mathtt{T}) = H_\phi(\mathtt{F},\mathtt{T},\mathtt{T}) = H_\phi(\mathtt{T},\mathtt{F},\mathtt{T}) = H_\phi(\mathtt{T},\mathtt{T},\mathtt{F}) = H_\phi(\mathtt{T},\mathtt{T},\mathtt{T})$$
$$H_\phi(\mathtt{F},\mathtt{F},\mathtt{F}) = H_\phi(\mathtt{F},\mathtt{T},\mathtt{F}) = H_\phi(\mathtt{T},\mathtt{F},\mathtt{F})$$

1.4. $((p \to q) \vee ((\neg p) \to (\neg q))) = \phi$

p	q	$(\neg p)$	$(\neg q)$	$(p \to q)$	$((\neg p) \to (\neg q))$	$((p \to q) \vee ((\neg p) \to (\neg q)))$
F	F	T	T	T	T	T
F	T	T	F	T	F	T
T	F	F	T	F	T	T
T	T	F	F	T	T	T

$$H_\phi(\mathtt{F},\mathtt{F}) = H_\phi(\mathtt{F},\mathtt{T}) = H_\phi(\mathtt{T},\mathtt{F}) = H_\phi(\mathtt{T},\mathtt{T}) = \mathtt{T}$$

1.5. $(p \to (q \to p)) = \phi$

p	q	$(q \to p)$	$(p \to (q \to p))$
F	F	T	T
F	T	F	T
T	F	T	T
T	T	T	T

$$H_\phi(\mathtt{F},\mathtt{F}) = H_\phi(\mathtt{F},\mathtt{T}) = H_\phi(\mathtt{T},\mathtt{F}) = H_\phi(\mathtt{T},\mathtt{T}) = \mathtt{T}$$

1.6. $((p \lor r) \land (p \to q)) = \phi$

p	q	r	$(p \lor r)$	$(p \to q)$	$((p \lor r) \land (p \to q))$
F	F	F	F	T	F
F	F	T	T	T	T
F	T	F	F	T	F
F	T	T	T	T	T
T	F	F	T	F	F
T	F	T	T	F	F
T	T	F	T	T	T
T	T	T	T	T	T

$$H_\phi(\mathtt{F},\mathtt{F},\mathtt{F}) = H_\phi(\mathtt{F},\mathtt{T},\mathtt{F}) = H_\phi(\mathtt{T},\mathtt{F},\mathtt{F}) = H_\phi(\mathtt{T},\mathtt{F},\mathtt{T}) = \mathtt{F}$$
$$H_\phi(\mathtt{F},\mathtt{F},\mathtt{T}) = H_\phi(\mathtt{F},\mathtt{T},\mathtt{T}) = H_\phi(\mathtt{T},\mathtt{T},\mathtt{F}) = H_\phi(\mathtt{T},\mathtt{T},\mathtt{T}) = \mathtt{T}$$

1.7. $(\neg((r \to (r \land (p \lor s))) \equiv (\neg((p \to q) \lor (r \land (\neg r)))))) = \phi$

Debido a la longitud de la proposición, decidí añadir subíndices a los conectores, para así no tener que hacer mención de las variables que hacen parte del mismo:

$$(\neg_0((r \to_2 (r \land_3 (p \lor_4 s))) \equiv_1 (\neg_2((p \to_4 q) \lor_3 (r \land_4 (\neg_5 r))))))$$

Posición	p	q	r	s
0	F	F	F	F
1	F	F	F	T
2	F	F	T	F
3	F	F	T	T
4	F	T	F	F
5	F	T	F	T
6	F	T	T	F
7	F	T	T	T
8	T	F	F	F
9	T	F	F	T
10	T	F	T	F
11	T	F	T	T
12	T	T	F	F
13	T	T	F	T
14	T	T	T	F
15	T	T	T	T

Posición	$\neg_5$	$\vee_4$	$\rightarrow_4$	$\wedge_4$	$\wedge_3$	$\vee_3$	$\rightarrow_2$	$\neg_2$	$\equiv_1$	$\neg_0$
0	T	F	T	F	F	T	T	F	F	T
1	F	T	T	F	F	T	T	F	F	T
2	T	F	T	F	F	T	F	F	T	F
3	F	T	T	F	T	T	T	F	F	T
4	T	F	T	F	F	T	T	F	F	T
5	F	T	T	F	F	T	T	F	F	T
6	T	F	T	F	F	T	F	F	T	F
7	F	T	T	F	T	T	T	F	F	T
8	T	T	F	F	F	F	T	T	T	F
9	F	T	T	F	F	F	T	T	T	F
10	T	T	T	F	T	F	T	T	T	F
11	F	T	F	F	T	F	T	T	T	F
12	T	T	T	F	F	T	T	F	F	T
13	F	T	T	F	F	T	T	F	F	T
14	T	T	T	F	T	T	T	F	F	T
15	F	T	T	F	T	T	T	F	F	T

$$\begin{aligned}H_\phi(\mathtt{F},\mathtt{F},\mathtt{F},\mathtt{F}) &= H_\phi(\mathtt{F},\mathtt{F},\mathtt{F},\mathtt{T}) = H_\phi(\mathtt{F},\mathtt{F},\mathtt{T},\mathtt{F}) = H_\phi(\mathtt{F},\mathtt{T},\mathtt{F},\mathtt{F}) = H_\phi(\mathtt{F},\mathtt{T},\mathtt{F},\mathtt{T}) = H_\phi(\mathtt{F},\mathtt{T},\mathtt{T},\mathtt{T}) \\ &= H_\phi(\mathtt{T},\mathtt{T},\mathtt{F},\mathtt{F}) = H_\phi(\mathtt{T},\mathtt{T},\mathtt{F},\mathtt{T}) = H_\phi(\mathtt{T},\mathtt{T},\mathtt{T},\mathtt{F}) = H_\phi(\mathtt{T},\mathtt{T},\mathtt{T},\mathtt{T}) = \mathtt{T}\end{aligned}$$

$$H_\phi(\mathtt{F},\mathtt{F},\mathtt{T},\mathtt{F}) = H_\phi(\mathtt{T},\mathtt{F},\mathtt{F},\mathtt{F}) = H_\phi(\mathtt{T},\mathtt{T},\mathtt{F},\mathtt{T}) = H_\phi(\mathtt{T},\mathtt{F},\mathtt{T},\mathtt{F}) = H_\phi(\mathtt{T},\mathtt{F},\mathtt{T},\mathtt{T}) = \mathtt{F}$$

2. Punto 2: Tabla de verdad

$((p \vee (q \vee r)) \equiv ((p \vee q) \vee r))$

p	q	r	$(q \vee r)$	$(p \vee q)$	$(p \vee (q \vee r))$	$((p \vee q) \vee r)$	$((p \vee (q \vee r)) \equiv ((p \vee q) \vee r))$
F	F	F	F	F	F	F	T
F	F	T	T	F	T	T	T
F	T	F	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	T	T	T
T	F	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	T	T	T	T	T	T	T

3. Punto 3: Justificar que la implicación no es asociativa

$$\boxed{\begin{array}{l}(\forall \phi, \psi, \tau \in \mathbb{B} : ((\phi \to \psi) \to \tau) \equiv (\phi \to (\psi \to \tau))) \\ (\exists \phi, \psi, \tau \in \mathbb{B} : \mathbf{v}[((\phi \to \psi) \to \tau)] \neq \mathbf{v}[(\phi \to (\psi \to \tau))])\end{array}}$$

propiedad a refutar

negación de la propiedad

$$\mathbf{v}[((\phi \to \psi) \to \tau)] = \mathtt{F}$$

suposición

$$\equiv$$

$$(\mathbf{v}[(\phi \to \psi)] = \mathtt{T}) \wedge (\mathbf{v}[\tau] = \mathtt{F})$$

Metateorema 2.23

$$\equiv$$

$$((\mathbf{v}[\phi] = \mathtt{F}) \vee (\mathbf{v}[\psi] = \mathtt{T})) \wedge (\mathbf{v}[\tau] = \mathtt{F})$$

Tomando $\mathbf{v} = \{\phi \mapsto \mathtt{F}, \psi \mapsto \mathtt{T}, \tau \mapsto \mathtt{F}\}$

$$\begin{aligned}\mathbf{v}[(\phi \to (\psi \to \tau))] &= H_{\to}[\mathtt{F}, \mathbf{v}[(\psi \to \tau)]] \\ &= H_{\to}[\mathtt{F}, H_{\to}(\mathtt{T}, \mathtt{F})] \\ &= H_{\to}[\mathtt{F}, \mathtt{F}] \\ &= \mathtt{T}\end{aligned}$$

$$\therefore (\exists \phi, \psi, \tau \in \mathbb{B} : \mathbf{v}[((\phi \to \psi) \to \tau)] \neq \mathbf{v}[(\phi \to (\psi \to \tau))])$$

por suposición

4. Punto 4: Tabla de verdad

$(((p \to q) \wedge (\neg true)) \equiv (r \vee q))$

p	q	r	$true$	$(\neg true)$	$(p \to q)$	$(r \vee q)$	$((p \to q) \wedge (\neg true))$	$(((p \to q) \wedge (\neg true)) \equiv (r \vee q))$
F	F	F	T	F	T	F	F	T
F	F	T	T	F	T	T	F	F
F	T	F	T	F	T	T	F	F
F	T	T	T	F	T	T	F	F
T	F	F	T	F	F	F	F	T
T	F	T	T	F	F	T	F	F
T	T	F	T	F	T	T	F	F
T	T	T	T	F	T	T	F	F

5. Punto 5: Nuevo conector

5.1. Definir $*$ en conectores lógicos clásicos

$$\frac{H_*(\phi, \psi) = H_\neg[H_\vee(\phi, \psi)]}{(\phi * \psi) \equiv (\neg(\phi \vee \psi))}$$

5.2. Hallar una proposición equivalente a $(\neg p)$ usando únicamente $\{p, *\}$

$$\begin{array}{ll} (\phi * \psi) \equiv (\neg(\phi \vee \psi)) & \text{sub-punto 1} \\ (p \vee p) \equiv p & \text{tabla de verdad} \\ \hline (p * p) \equiv (\neg p) & \end{array}$$

5.3. Hallar una proposición equivalente a $(p \wedge q)$ usando únicamente $\{p, q, *\}$

$$\begin{array}{ll} (\phi * \psi) \equiv (\neg(\phi \vee \psi)) & \text{sub-punto 1} \\ (p * p) \equiv (\neg p) & \text{sub-punto 2} \\ (\neg(\phi \vee \psi)) \equiv ((\neg\phi) \wedge (\neg\psi)) & \text{tabla de verdad} \\ \hline ((p * p) * (q * q)) \equiv (p \wedge q) & \end{array}$$

5.4. Justificar o refutar propiedades de $*$

Metateorema 2.23 para $*$

$\mathbf{v}[(\phi * \psi)] = \mathtt{T}$ si y solo si $\mathbf{v}[\phi] = \mathtt{F}$ y $\mathbf{v}[\psi] = \mathtt{F}$; de lo contrario $\mathbf{v}[(\phi * \psi)] = \mathtt{F}$

5.4.1. Conmutativa

$$\begin{array}{ll} (\forall \phi, \psi \in \mathbb{B} : \mathbf{v}[(\phi * \psi)] = \mathbf{v}[(\psi * \phi)]) & \text{def. propiedad conmutativa} \\ \mathbf{v}[(\phi * \psi)] = \mathtt{T} & \text{suposición} \\ (\mathbf{v}[\phi] = \mathtt{F}) \wedge (\mathbf{v}[\psi] = \mathtt{F}) & \\ (\mathbf{v}[\psi] = \mathtt{F}) \wedge (\mathbf{v}[\phi] = \mathtt{F}) & \text{Propiedad conmutativa de } \wedge \\ \mathbf{v}[(\psi * \phi)] = \mathtt{T} & \text{Metateorema 2.23} \\ \hline (\forall \phi, \psi \in \mathbb{B} : \mathbf{v}[(\phi * \psi)] = \mathbf{v}[(\psi * \phi)]) & \text{Es conmutativa} \end{array}$$

Se puede ver que cualquier otra valuación cumple la propiedad (negar el metatorema y aplicar propiedad conmutativa sobre $\vee$)

5.4.2. Asociativa

$$\begin{array}{ll} (\forall \phi, \psi, \tau \in \mathbb{B} : \mathbf{v}[(\phi * \psi) * \tau] = \mathbf{v}[(\phi * (\psi * \tau))]) & \text{def. propiedad asociativa} \\ \mathbf{v}[(\phi * \psi) * \tau] = \mathtt{T} & \text{suposición} \\ (\mathbf{v}[(\phi * \psi)] = \mathtt{F}) \wedge (\mathbf{v}[\tau] = \mathtt{F}) & \\ (\mathbf{v}[\phi] = \mathtt{T} \vee \mathbf{v}[\psi] = \mathtt{T}) \wedge (\mathbf{v}[\tau] = \mathtt{F}) & \end{array}$$

Tomando $\mathbf{v} = \{\phi \mapsto \mathtt{T}, \psi \mapsto \mathtt{T}, \tau \mapsto \mathtt{F}\}$

$$\begin{aligned}\mathbf{v}[(\phi * (\psi * \tau))] &= H_*[\phi, (\psi * \tau)]\\ &= H_*[\mathtt{T}, H_*(\mathtt{T}, \mathtt{F})]\\ &= H_*[\mathtt{T}, \mathtt{F}]\\ &= \mathtt{F}\end{aligned}$$

$$\therefore (\exists \phi, \psi, \tau \in \mathbb{B} : \mathbf{v}[(\phi * \psi) * \tau] \neq \mathbf{v}[(\phi * (\psi * \tau))]) \qquad \text{No es asociativa}$$

6. Punto 6: Nuevo conector

6.1. Definir $*$ en conectores lógicos clásicos

$$H_*(\phi, \psi) = H_\neg[H_\leftarrow(\phi, \psi)]$$

$$(\phi * \psi) \equiv (\neg(\phi \leftarrow \psi))$$

6.2. Justificar o refutar propiedades de $*$

Metateorema 2.23 para $*$

$\mathbf{v}[(\phi * \psi) = \mathtt{T}$ si y solo si $\mathbf{v}[\phi] = \mathtt{F}$ y $\mathbf{v}[\psi] = \mathtt{T}$; de lo contrario $\mathbf{v}[(\phi * \psi)] = \mathtt{F}]$

6.2.1. Asociativa

$$\begin{aligned}&(\forall \phi, \psi, \tau \in \mathbb{B} : \mathbf{v}[(\phi * \psi) * \tau] = \mathbf{v}[(\phi * (\psi * \tau))]) && \text{def. propiedad asociativa}\\ &\mathbf{v}[(\phi * \psi) * \tau] = \mathtt{T} && \text{suposición}\\ &\quad (\mathbf{v}[\phi] = \mathtt{F}) \wedge (\mathbf{v}[\psi] = \mathtt{T}) &&\\ &\quad (\mathbf{v}[\psi] = \mathtt{T}) \wedge (\mathbf{v}[\phi] = \mathtt{F}) && \text{Propiedad conmutativa de } \wedge\\ &\quad \mathbf{v}[(\psi * \phi)] = \mathtt{F} &&\end{aligned}$$

$$\therefore (\exists \phi, \psi, \tau \in \mathbb{B} : \mathbf{v}[(\phi * \psi) * \tau] \neq \mathbf{v}[(\phi * (\psi * \tau))]) \qquad \text{No es asociativa}$$

6.2.2. Transitiva

$$\boxed{\begin{array}{l}(\forall \phi, \psi, \tau \in \mathbb{B} : ((\mathbf{v}[(\phi * \psi)] = \mathtt{T}) \wedge (\mathbf{v}[(\psi * \tau)] = \mathtt{T})) \Rightarrow (\mathbf{v}[(\phi * \tau)] = \mathtt{T})) \\ \qquad\qquad\qquad \wedge \\ (\forall \phi, \psi, \tau \in \mathbb{B} : ((\mathbf{v}[(\phi * \psi)] = \mathtt{F}) \wedge (\mathbf{v}[(\psi * \tau)] = \mathtt{F})) \Rightarrow (\mathbf{v}[(\phi * \tau)] = \mathtt{F}))\end{array}} \quad \text{def. propiedad transitiva}$$

$$\begin{array}{cr} (\mathbf{v}[(\phi * \psi)] = \mathtt{T}) \wedge (\mathbf{v}[(\psi * \tau)] = \mathtt{T}) & \text{suposición} \\ ((\mathbf{v}[\phi] = \mathtt{F}) \wedge (\mathbf{v}[\psi] = \mathtt{T})) \wedge ((\mathbf{v}[\psi] = \mathtt{F}) \wedge (\mathbf{v}[\tau] = \mathtt{T})) & \\ (\mathbf{v}[\phi] = \mathtt{F}) \wedge \mathit{false} \wedge (\mathbf{v}[\tau] = \mathtt{T}) & \text{Propiedad asociativa de } \wedge \\ \mathit{false} & \end{array}$$

$$\boxed{\begin{array}{l}(\exists \phi, \psi, \tau \in \mathbb{B} : ((\mathbf{v}[(\phi * \psi)] = \mathtt{T}) \wedge (\mathbf{v}[(\psi * \tau)] = \mathtt{T})) \nRightarrow (\mathbf{v}[(\phi * \tau)] = \mathtt{T})) \\ \qquad\qquad\qquad \vee \\ (\exists \phi, \psi, \tau \in \mathbb{B} : ((\mathbf{v}[(\phi * \psi)] = \mathtt{F}) \wedge (\mathbf{v}[(\psi * \tau)] = \mathtt{F})) \nRightarrow (\mathbf{v}[(\phi * \tau)] = \mathtt{F}))\end{array}} \quad \text{No es transitiva}$$

7. Punto 7: considere las valuaciones v y w ...

$$\mathbf{v} = \{p \mapsto \mathtt{T},\ q \mapsto \mathtt{F},\ r \mapsto \mathtt{F},\ \ldots\}$$
$$\mathbf{w} = \{p \mapsto \mathtt{T},\ q \mapsto \mathtt{F},\ r \mapsto \mathtt{T},\ \ldots\}$$

Demostrar que $\mathbf{v}[(p \equiv (\neg q))] = \mathbf{w}[(p \equiv (\neg q))]$

Tomando **v** ...

$$\begin{aligned}\mathbf{v}[(p \equiv (\neg q))] &= H_{\equiv}[p, (\neg q)] \\ &= H_{\equiv}[\mathbf{v}[p], H_{\neg}(\mathbf{v}[q])] \\ &= H_{\equiv}[\mathbf{w}[p], H_{\neg}(\mathbf{w}[q])] \\ &= \mathbf{w}[(p \equiv (\neg q))]\end{aligned}$$

8. Punto 8: Demuestre que $\mathbf{v}[\phi] \neq \mathbf{v}[(\neg\phi)]$ para cualquier valuación v

$$\begin{array}{lr} \mathbf{v}[(\phi \equiv (\neg\phi))] = \mathtt{F} & \text{Enunciado/ proposición a probar} \\ \underline{\qquad\qquad \mathbf{v}[\phi] \neq \mathbf{v}[(\neg\phi)]} & \text{negación de Metateorema 2.23} \\ \therefore \mathbf{v}[(\phi \equiv (\neg\phi))] = \mathtt{F} & \end{array}$$