

$$\begin{aligned}
\langle H^2 \rangle &= \left\langle \left(\sum_{i=0}^L \nabla_i^x \nabla_{i+1}^x + \frac{B}{2} (\nabla_i^2 + \nabla_{i+1}^2) \right)^2 \right\rangle \\
&= \left\langle \left(\underbrace{\sum_{i \text{ even}} \nabla_i^x \nabla_{i+1}^x + \frac{B}{2} (\nabla_i^2 + \nabla_{i+1}^2)}_{h_i} + \underbrace{\sum_{j \text{ odd}} \nabla_j \nabla_{j+1} + \frac{B}{2} (\nabla_j^2 + \nabla_{j+1}^2)}_{h_j} \right)^2 \right\rangle \\
&= \left\langle \left(\sum_{i \text{ even}} h_i + \sum_{j \text{ odd}} h_j \right)^2 \right\rangle = \left\langle \left(\sum_i h_i + \sum_j h_j \right) \left(\sum_{i'} h_{i'} + \sum_{j'} h_{j'} \right) \right\rangle \\
&= \left\langle \left(\sum_i h_i \right) \left(\sum_{i'} h_{i'} \right) \right\rangle + \left\langle \left(\sum_{j'} h_{j'} \right) \left(\sum_{j'} h_{j'} \right) \right\rangle \\
&\quad + \left\langle \left(\sum_i h_i \right) \left(\sum_{j'} h_{j'} \right) \right\rangle + \left\langle \left(\sum_{j'} h_{j'} \right) \left(\sum_i h_i \right) \right\rangle \\
&= \textcircled{\text{I}} + \textcircled{\text{II}} + \textcircled{\text{III}} + \textcircled{\text{IV}}
\end{aligned}$$

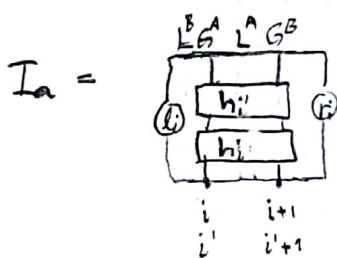
$$* \textcircled{\text{I}} = \left\langle \sum_{i=i'} h_i h_{i'} \right\rangle + \left\langle \sum_{i-i' \geq 2} h_i h_{i'} \right\rangle + \left\langle \sum_{i'-i \geq 2} h_i h_{i'} \right\rangle = I_a + I_b + I_c$$

$$* \textcircled{\text{II}} = \left\langle \sum_{j=j'} h_j h_{j'} \right\rangle + \left\langle \sum_{j-j' \geq 2} h_j h_{j'} \right\rangle + \left\langle \sum_{j'-j \geq 2} h_j h_{j'} \right\rangle = II_a + II_b + II_c$$

$$* \textcircled{\text{III}} = \left\langle \sum_{i+1=j'} h_i h_{j'} \right\rangle + \left\langle \sum_{i=j'+1} h_i h_{j'} \right\rangle + \left\langle \sum_{|i-j'| > 1} h_i h_{j'} \right\rangle = III_a + III_b + III_c$$

$$* \textcircled{\text{IV}} = \left\langle \sum_{j+1=i'} h_j h_{i'} \right\rangle + \left\langle \sum_{j=i'+1} h_j h_{i'} \right\rangle + \left\langle \sum_{|j-i'| > 1} h_j h_{i'} \right\rangle = IV_a + IV_b + IV_c$$

Diagrams :



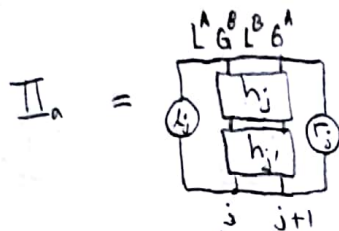
$$I_b = \textcircled{\text{I}} \left(\sum_{n=0}^{\infty} \left(\textcircled{\text{II}} \right)^n \right) \textcircled{\text{V}}$$

\uparrow
 E_i

$i' = i - 2n \quad (a)$

$i' = i + 2n \quad (b)$

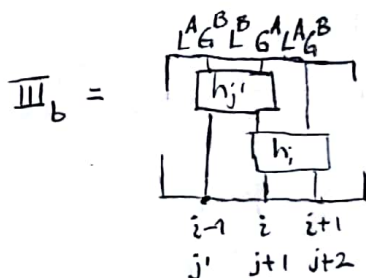
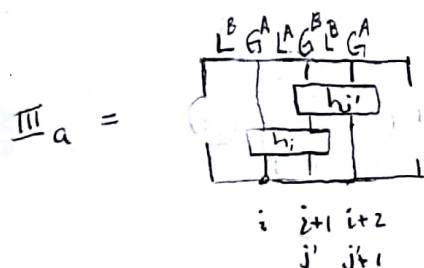
$$I_c = I_b$$



$\Pi_b =$

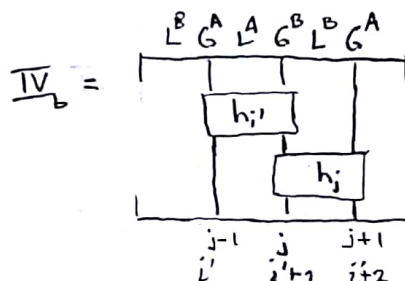
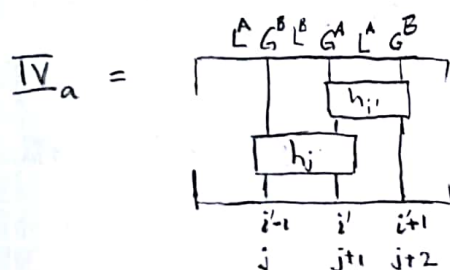


$\Pi_c = \Pi_b$



$\Pi_c = \langle \sum_{|i-j'| \geq 1} h_i h_{j'} \rangle = \langle \sum_{i-j' \geq 1} h_i h_{j'} \rangle + \langle \sum_{j'-i \geq 1} h_i h_{j'} \rangle$

$=$



$\Pi_c = \langle \sum_{|j-i'| \geq 1} h_j h_{i'} \rangle = \langle \sum_{j-i' \geq 1} h_j h_{i'} \rangle + \langle \sum_{i'-j \geq 1} h_j h_{i'} \rangle$

$=$