Entanglement
Notebook

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Entanglement

The following is a summary of this thesis and this pdf.

If two systems interacted in the past it is, in general, not possible to assign a single state vector to either of the two subsystems.

Entanglement appears as the consequence of the combination of two of the quantum postulates:

- Postulate 1: The state of a quantum system is described by a vector in a complex Hilbert space.
- Postulate 2: The Hilbert space of a composite system is the tensor product of the two local spaces
- As a consequence there is a superposition of pure states that cannot be written as the tensor product of pure states in each local space

Antipodean to entangled states are the separable states, i.e., a state is entangled iff it is not separable.

Whether a given state is entangled or just classically correlated is easy to determine for pure states. However, for arbitrary mixed states it is a hard problem. This is known as the *separability problem.*

Note that there are two more postulates dealing with evolution of quantu states via unitary operators.

> **Definition of Entanglement**: (from wiki): A general state in $H_A \otimes H_B$ is $|\psi_{AB}\rangle = \sum_{jk} c_{jk} |j_A\rangle \otimes |k_B\rangle$. It is entangled if any pair $c_j^A c_k^B \neq c_{jk}$, such that $|\psi_{AB}\rangle \neq \sum_j c_j^A |j_A\rangle \otimes \sum_k c_k^B |k_B\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, for any basis.

Computational Basis

A qubit is a quantum system in which states 0 and 1 are represented by a prescribed pair of normalised and mutually orthogonal quantum states labeled as $\{|0\rangle, |1\rangle\} = \{\binom{1}{0}, \binom{0}{1}\}$. These two states form a a **computational basis**, and any other (pure) stateof the qubit can be written as a superposition $a|0\rangle + b|1\rangle$, where normalisation implies $|a|^2 + |b|^2 = 1$.

Tensor Product

By the postulate of quantum mechanics, the Hilbert space of a composite system is the *tensor product* of the two local spaces. Let $|\psi_1\rangle = a|0\rangle + b|1\rangle$ and $|\psi_2\rangle = \alpha|0\rangle + \beta|1\rangle$ be states in Hilbert spaces H_1 and H_2. Then the joint state is given by

$$\begin{aligned} |\psi\ \rangle &= |\psi_1\rangle \otimes\ |\psi_2\rangle \\ &= (a|0\rangle + b|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) \\ &= a\alpha|00\rangle + a\beta|01\rangle + b\alpha|10\rangle + b\beta|11\rangle. \end{aligned}$$

One is free to order the states in the joint basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, but this particular order is common in quantum information, known as the **computational basis**. Then $|\psi\rangle$ corresponds to a column vector

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a\alpha \\ a\beta \\ b\alpha \\ b\beta \end{pmatrix}.$$

The inner products in tensor-product space is straightforward. Let $|v_i\rangle$ and $|w_i\rangle$ be elements of Hilbert spaces V and W respectively. Then an inner product on $V \otimes W$ will in general work like this:

$$\left(\sum_i a_i |v_i\rangle \otimes |w_i\rangle, \sum_j a_j |v'_j\rangle \otimes |w'_j\rangle \right) \equiv \sum_{ij} a_i^* b_j \langle v_i|v'_j\rangle\langle w_i|w'_j\rangle.$$

For the states mentioned earlier, we have

$$\langle |\psi\rangle, |\psi\rangle\rangle = \langle\psi|\psi\rangle = \langle (|\psi_1\rangle \otimes |\psi_2\rangle), (|\psi_1\rangle \otimes |\psi_2\rangle)\rangle = \langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle.$$

Example

Introduce a notation where X_2 is the Pauli operator σ_x acting on the second qubit. Show that the observable X_1Z_2 for a two qubit system measureed in the state $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is zero.

We know that the pauli operators σ_x and σ_z have eigenvalues $\{-1, 1\}$, with

$$\begin{aligned} \sigma_x|\psi_1\rangle &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \end{pmatrix} + a \begin{pmatrix} 1 \\ 0 \end{pmatrix} = b|0\rangle + a|1\rangle \\ \sigma_z|\psi_2\rangle &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha|0\rangle - \beta|1\rangle \end{aligned}$$

Measuring the observable X_1Z_2 with $a = b = \alpha = \beta = 2^{1/4}$ should then give

$$
\begin{aligned}
\langle\psi|X_1Z_2|\psi\rangle &= \langle\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)|X_1Z_2|\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\rangle \\
&= \frac{1}{2}\langle(\langle 00| + \langle 11|)|(|10\rangle - |01\rangle)\rangle \qquad (= 0 \text{ orthonormal basis in the joint space}) \\
&= \frac{1}{2}(\langle 00|10\rangle + \langle 00|01\rangle + \langle 11|01\rangle + \langle 11|10\rangle) \\
&= 0 \qquad \text{(because of orthogonality in the subspaces)}
\end{aligned}
$$

Pure States

The Schmidt theorem, or Schmidt decomposition, is useful for the deciding separability. Let's define some concepts first.

Definition of Pure state (or ensemble): In literature the words "state" and "ensemble" are often used interchangeably. By **pure state**, (or pure ensemble in Sakurai), we mean a collection of identically prepared physical systems, where all members can be described by the same ket $|\psi\rangle$. For instance, silver atoms coming out of a Stern-Gerlach filtering apparatus, where all spins come out pointing in some definite direction $\hat{n}$.

Pure states and entanglement: A **pure state** $|\psi_{AB}\rangle \in H_A \otimes H_B$ is **entangled** iff it is not **separable**, i.e., $|\psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$. Equivalently, if it were separable, it would not be entangled.

Definition of a Bipartite system: A **pure state** of a composite system $A \cup B$ is called **bipartite** and can be written as $|\psi_{AB}\rangle = \sum_{jk} c_{jk}|j_A\rangle \otimes |k_B\rangle$, where $\{|j_A\rangle\}$ and $\{|k_B\rangle\}$ are bases in Hilbert spaces H_A and H_B respectively.

Schmidt Theorem: Suppose $|\psi_{AB}\rangle$ is a pure state of a composite system AB. Then there are orthonormal states $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$ in H_A and H_B respectively, such that

$$
|\psi_{AB}\rangle = \sum_i^n c_i|i_A\rangle \otimes |i_B\rangle
$$

where c_i are non-negative real numbers called **Schmidt coefficients**, satisfying $\sum_i^n c_i^2 = 1$, and $n = \min(\dim H_A, \dim H_B)$. Furthermore, $\lambda_i = c_i^2$ are the eigenvalues of the two partial traces of ρ_{AB}, i.e., ρ_A and ρ_B.

Proof: As per definition, let $|\psi_{AB}\rangle = \sum_{jk} c_{jk}|j_A\rangle \otimes |k_B\rangle = \sum_{jk} c_{jk}|jk\rangle$ be a bipartite system with some matrix $\mathbf{c} = c_{jk}$. By singular value decomposition we can write $\mathbf{c} = \mathbf{udv}$ where $\mathbf{d}$ is a nonnegative diagonal matrix, and $\mathbf{u}$ and $\mathbf{v}$ are unitary matrices:

$$
|\psi_{AB}\rangle = \sum_{jik} u_{ji}d_{ii}v_{ik}|j_A\rangle \otimes |k_B\rangle
$$

Define $|i_A\rangle = \sum_j u_{ji}|j_A\rangle$ and $|i_B\rangle = \sum_k v_{ik}|k_B\rangle$. Then by identifying $c_i = d_{ii}$ we get the desired result.

Definition of the Schmidt rank: The Schmidt rank is the number of non-vanishing Schmidt coefficients. Equivalently, it is defined as the number of non-zero eigenvalues of ρ_A or ρ_B.

For pure states we can then say:

- $|\psi_{AB}\rangle$ is **entangled** or **non-separable** if it has Schmidt rank > 1.
- $|\psi_{AB}\rangle$ is **separable** *iff* there is only one non-zero Schmidt coefficient.
- If all Schmidt coefficients are non-zero and equal, the state is **maximally entangled**.

Mixed States

A completely random state and a pure state can be regarded as extremes of what is known as a mixed state (or mixed ensemble). For a mixed ensemble we introduce the concept of **fractional population** or probability weight $p_i \geq 0$ with $\sum_{i=1}^{N} p_i = 1$, which as the name suggests tells us the fraction of the populations described by each state ket.

A general density matrix for a mixed state has the form

$$\rho = \sum_i p_i [\sum_j c_{ij} (|\alpha_{ij}\rangle \otimes |\beta_{ij}\rangle)] \otimes [\sum_j c_{ij} (\langle\alpha_{ij}| \otimes \langle\beta_{ij}|)]$$

A mixed state ρ is entangled iff it is not separable, i.e.,

$$\rho_{AB} \neq \sum_{i=1}^{N} p_i [|\psi_A^i\rangle\langle\psi_A^i| \otimes |\psi_B^i\rangle\langle\psi_B^i|] = \sum_{i=1}^{N} p_i \rho_A^i \otimes \rho_B^i$$

where N is the number of populations.

Do not confuse p_i with c_i:

- p_i is a weight as encountered in classical probability theory. If $p_{\text{male}} = p_{\text{female}} = 0.5$ in a school, there is a 50% probability of picking a male student at random. This does not mean that the student is in a linear superposition of male and female.
- c_i is a *probability amplitude* or phase relation in a linear superposition of state kets. For instance, in $\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ we have $c_1 = c_2 = \frac{1}{\sqrt{2}}$ and their squares represent the probability of measuring up or down respectively.

Explanation from Sakurai

In a mixed state (or *mixed ensemble* in Sakurai), we have a mixture of pure states. For instance, a fraction — say 70% — of the members may be characterized by state ket $|\psi_A\rangle$, and the remaining 30% by state ket $|\psi_B\rangle$. Note that $|\psi_A\rangle$ and $|\psi_B\rangle$ do not need to be orthogonal!

In a mixed state (or ensemble) there are many different ways to decide if it is entangled, i.e. there are many different entanglement measures.

Separable States

Separable quantum states are states without quantum entanglement.

Let H_1 and H_2 be quantum systems with basis states $\{|a_i\rangle\}_{i=1}^{n}$ and $\{|b_j\rangle\}_{j=1}^{n}$. By a postulate of quantum mechanics, the composite system is given by the tensor product

$$H_1 \otimes H_2$$

with basis $\{|a_i\rangle \ \otimes |b_j\rangle\} = \{|a_i b_j\rangle\}$.

If a pure state $|\psi\rangle \in H_1 \otimes H_2$ can be written in the form $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, where $|\psi_i\rangle$ is a pure state in the ith subsystem, it is said to be **separable**. Otherwise it is **entangled**.

For separable states the joint state of two systems can be expressed with the density

$$\rho = \sum_k p_k \rho_A^k \otimes \rho_B^k. \tag{1}$$

Product States

A product state is a *simply separable* state, in which there are neither classical nor quantum correlations. The joint state of systems A and B can be espressed as

$$\rho = \rho_A \otimes \rho_B \tag{2}$$

The Density Operator

For pure states: The density operator for a pure state can be written as $\rho = |\psi\rangle\langle\psi|$, without summation. For instance, in the conventional basis, a beam of particles polarized in the z-direction is described by state vector$|\psi\rangle = |\uparrow\rangle$, and then the density operator becomes

$$\rho = |\psi\rangle\langle\psi| = |\uparrow\rangle\langle\uparrow| = \begin{pmatrix}1\\0\end{pmatrix}(1\,0) = \begin{pmatrix}1\,0\\0\,0\end{pmatrix}$$

For mixed states: For a mixture of states $|\psi^i\rangle$ we define the density operator as

$$\rho = \sum_i p_i |\psi^i\rangle\langle\psi^i|,$$

where the fractional population satisfies $\sum_i p_i = 1$. Obviously, such a mixture is pure iff $p_i = 1$ for some $|\psi^i\rangle$, and zero for the remaining. A completely random ensemble has diagonal elements $\frac{1}{N}$, where N is the number of populations.

Averages The *expectation value* $\langle A\rangle$ (or *ensemble average* in Sakurai), for a mixed state is:

$$\langle A\rangle \equiv \sum_i p_i \langle i|A|i\rangle = \mathrm{Tr}(\rho A)$$

The Partial Trace (summary from StackExchange) If we're only interested in predictions for a subsystem A in a system composed of AB then A is described by a density matrix ρ_A calculable by "tracing over" the indices of the Hilbert space for B:

$$\rho_A = \mathrm{Tr}_B \rho_{AB}$$

Note that if the whole system AB is in a pure state we have $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$.

Separability Criteria

Summary from

> Mintert, F., Viviescas, C., & Buchleitner, A. (2009). Entanglement and Decoherence, 768, 61–86. https://doi.org/10.1007/978-3-540-88169-8 http://www.springer.com/cda/content/document/cda_downloaddocument/9783540881681-c2.pdf?SGWID=0-0-45-646111-p173849909

Checking separability can be much more involved than

Entanglement measures

(See also this pioneering paper) Here we address how can quantum entanglement be quantified and classified, but first, we need to defined what an entanglement measure *is*:

- Entanglement is nonnegative. It is zero iff the state is separable.
- Entanglement of independent systems is additive, i.e. $E(\psi^{\otimes n}) = nE(\psi)$.
- Entanglement is conserved under local unitary operations, i.e. $E(\psi) = E(U\psi)$. In other words, a local change of basis has no effect on E.

The von Neumann Entropy The entropy of entanglement defined as

$$S = -\mathrm{Tr}(\rho \ln \rho) = -\sum_i \lambda_i \ln \lambda_i,$$

is useful for both **pure** and **mixed** states. For a pure bipartite state we also have $E(\psi) = S(\rho_A) = S(\rho_B)$.

General properties of the von Neumann entropy:

1. $S(\rho) \geq 0$. If it is zero, it is a pure state.
2. $\dim H = d \Rightarrow S(\rho) \leq \log_2(d)$ and $S(\rho) = \log(d) \Leftrightarrow \rho = 1/d$
3. ρ_{AB} on $H_A \otimes H_B$ pure $\Rightarrow S(\rho_A) = S(\rho_B)$.
4. Let ρ_j be density operators with support on orthogonal subspaces. Then $S(\sum_j p_j \rho_j) = H(\{p_j\}) + \sum_j p_j S(\rho_j)$, where $H(\{p_j\}) = -\sum_j p_j \log_2 p_j$ is the **Shannon entropy**.
5. **Joint entropy theorem:** Let $\{|j_A\rangle\} \subset H_A$ orthonormal, ρ_j^B density operators on H_B. Then $S(\sum_j p_j |j_A\rangle\langle |j_A| \otimes \rho_j^B) = H(\{p_j\}) + \sum_j p_j S(\rho_j^B)$.
6. $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$
7. $S(\rho_A) = 0 \Leftrightarrow |\psi_{AB}\rangle$ separable, $S(\rho_A) > 0 \Leftrightarrow |\psi_{AB}\rangle$ entangled.
8. $S(\rho_A) = \log(\dim H_A)(= \text{maximal}) \Leftrightarrow |\psi_{AB}\rangle$ maximally entangled, assuming $\dim H_A \leq \dim H_B$

The rank of the Schmidt decomposition If A is a subset of n spins and B the rest of them, then the Schmidt decomposition for some partition of $A : B$ reads

$$|\psi_{AB}\rangle = \sum_{\alpha=1}^{\chi_A} \lambda_\alpha |\psi_\alpha^A\rangle \otimes |\psi_\alpha^B\rangle.$$

The rank χ_A of the reduced density matrix ρ_A measures the entanglement between spins in A and B.

Examples

In the following examples we have two spin-$\frac{1}{2}$ particles coming out of a Stern-Gerlach type apparatus which can be rotated. We use the basis where z is diagonal, i.e., $|\uparrow\rangle = (1,0)^T$ and $|\downarrow\rangle = (0,1)^T$.

Example 1 - Pure, separable, not entangled:

Particles A and B come out pointing in the z-direction. We then have a non-entangled and separable state:

$$|\psi_{AB}\rangle = |\uparrow_A\rangle \otimes |\uparrow_B\rangle = |\uparrow\uparrow\rangle.$$

The matrix c_{ij} then has elements $c_{00} = 1$, and all others are zero. The SVD decomposition tells us that $\mathbf{c} = \mathbf{udv}$ where $d_{ii}^2 = c_i^2$ are the eigenvalues λ_i of the matrix $\mathbf{c}^T\mathbf{c}$. Recall that c_i are the Schmidt coefficients! We get

$$\mathbf{c}^T\mathbf{c} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

with eigenvalues $\lambda_0 = 0, \lambda_1 = 1$.

Furthermore, the density operator becomes:

$$\rho_{AB} = |\psi\rangle\langle\psi| = (|\uparrow_A\rangle \otimes |\uparrow_B\rangle)(\langle\uparrow_A| \otimes \langle\uparrow_B|)$$

$$= |\uparrow_A\rangle\langle\uparrow_A| \otimes |\uparrow_B\rangle\langle\uparrow_B| = \rho_A \otimes \rho_B = 4 \times 4 \text{ matrix}$$

where $\rho_A = \mathrm{Tr}_B\rho_{AB} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\rho_B = \mathrm{Tr}_A\rho_{AB} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and thus both have eigenvalues $\lambda_0 = 0$ and $\lambda_1 = 1$.

In conclusion: - Since the state is separable, it is not entangled. - Since only one eigenvalue of ρ_A, ρ_B is non-zero, the state is separable, and not entangled. - Since 1 is one of the eigenvalues of ρ_A, ρ_B, the state is pure.

Example 2 - Pure, separable, not entangled:

Particles A and B come out in a linear superposition

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_A\rangle| \otimes |\uparrow_B\rangle + |\uparrow_A\rangle \otimes |\downarrow_B\rangle) = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle)$$

Then the coefficients c_{ij} become: $c_{00} = c_{01} = \frac{1}{\sqrt{2}}$, and all others zero. The SVD decomposition tells us that $\mathbf{c} = \mathbf{udv}$ where $d_{ii}^2 = c_i^2$ are the eigenvalues λ_i of the matrix $\mathbf{c}^T\mathbf{c}$. Recall that c_i are the Schmidt coefficients!

Thus we have $\mathbf{c}^T\mathbf{c} = \frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, with eigenvalues $\lambda_0 = 0, \lambda_1 = 1$.

Furthermore we have the density operator

$$
\begin{aligned}
\rho_{AB} &= |\psi_{AB}\rangle\langle\psi_{AB}| \\
&= \frac{1}{2}\left[|\uparrow_A\rangle\langle\uparrow_A| \otimes |\uparrow_B\rangle\langle\uparrow_B| + |\uparrow_A\rangle\langle\uparrow_A| \otimes |\uparrow_B\rangle\langle\downarrow_B| + |\uparrow_A\rangle\langle\uparrow_A| \otimes |\downarrow_B\rangle\langle\uparrow_B| + |\uparrow_A\rangle\langle\uparrow_A| \otimes |\downarrow_B\rangle\langle\downarrow_B|\right] \\
&= \frac{1}{2}\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}_A \otimes \begin{pmatrix}1&0\\0&0\end{pmatrix}_B + \begin{pmatrix}1&0\\0&0\end{pmatrix}_A \otimes \begin{pmatrix}0&1\\0&0\end{pmatrix}_B + \begin{pmatrix}1&0\\0&0\end{pmatrix}_A \otimes \begin{pmatrix}0&0\\1&0\end{pmatrix}_B + \begin{pmatrix}1&0\\0&0\end{pmatrix}_A \otimes \begin{pmatrix}0&0\\0&1\end{pmatrix}_B\right] \\
&= \frac{1}{2}\left[\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right] \\
&= \frac{1}{2}\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}
\end{aligned}
$$

which has eigenvalues $\lambda_0 = 0$ with multiplicity 3 and $\lambda_1 = 1$ with multiplicity 1.

We get eigenvectors $v^{(1)} = \frac{1}{\sqrt{2}}(-1, 1, 0, 0)^T$, $v^{(2)} = \frac{1}{\sqrt{2}}(1, -1, 0, 0)^T$, for $v^{(3)} = (0, 0, 1, 0)^T$, $v^{(4)} = (0, 0, 0, 1)^T$. Therefore, $\rho_{AB} = PDP^{-1}$ with $D = \left(\begin{smallmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{smallmatrix}\right)$, and $P = [v^1, v^2, v^3, v^4]$

Note that $\rho_A = \mathrm{Tr}_B\rho_{AB} = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $\rho_B = \mathrm{Tr}_A\rho_{AB} = \frac{1}{2}\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$. Simply trace the A and B parts into scalars, respectively, in the third line.

In conclusion:

- Since one eigenvalue of ρ_{AB} is 1, the state (or ensemble) is pure.
- Since $\mathrm{Tr}\rho_{AB}^2 = \sum_i \lambda_i^2 = 1$, the state (or ensemble) is pure.
- Since $\rho_{AB} = \rho_{AB}^2$, the state (or ensemble) is pure.
- Since we have only one non-zero Schmidt Coefficient, the state is separable and not entangled.

Example 3 - Pure non-separable, entangled:

The Bell state $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$ is maximally entangled. If we measure the spin of A we immediately know the spin of B. Let's repeat the steps of previous examples.

Particles A and B come out in a linear superposition $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_A\rangle| \otimes |\uparrow_B\rangle + |\downarrow_A\rangle \otimes |\downarrow_B\rangle)$ Then the coefficients c_{ij} become: $c_{00} = c_{11} = \frac{1}{\sqrt{2}}$, and all others zero. The SVD decomposition tells us that $\mathbf{c} = \mathbf{udv}$ where $d_{ii}^2 = c_i^2$ are the eigenvalues λ_i of the matrix $\mathbf{c}^T\mathbf{c}$. Recall that c_i are the Schmidt coefficients!

Thus we have $\mathbf{c}^T\mathbf{c} = \frac{1}{2}\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) = \frac{1}{2}\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, with eigenvalues $\lambda_0 = \lambda_1 = \frac{1}{2}$.

Furthermore we have the density operator

$$
\begin{aligned}
\rho_{AB} &= |\psi_{AB}\rangle\langle\psi_{AB}| = \\
&= \frac{1}{2}\left[|\uparrow_A\rangle\langle\uparrow_A| \otimes |\uparrow_B\rangle\langle\uparrow_B| + |\uparrow_A\rangle\langle\downarrow_A| \otimes |\uparrow_B\rangle\langle\downarrow_B| + |\downarrow_A\rangle\langle\uparrow_A| \otimes |\downarrow_B\rangle\langle\uparrow_B| + |\downarrow_A\rangle\langle\downarrow_A| \otimes |\downarrow_B\rangle\langle\downarrow_B|\right] \\
&= \frac{1}{2}\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}_A \otimes \begin{pmatrix}1&0\\0&0\end{pmatrix}_B + \begin{pmatrix}0&1\\0&0\end{pmatrix}_A \otimes \begin{pmatrix}0&1\\0&0\end{pmatrix}_B + \begin{pmatrix}0&0\\1&0\end{pmatrix}_A \otimes \begin{pmatrix}0&0\\1&0\end{pmatrix}_B + \begin{pmatrix}0&0\\0&1\end{pmatrix}_A \otimes \begin{pmatrix}0&0\\0&1\end{pmatrix}_B\right] \\
&= \frac{1}{2}\left[\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\right] \\
&= \frac{1}{2}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}
\end{aligned}
$$

which has eigenvalues $\lambda_0 = 0$ with multiplicity 3 and $\lambda_1 = 1$ with multiplicity 1.

We get eigenvectors $v^{(1)} = \frac{1}{\sqrt{2}}(-1,0,0,1)^T$,
for $v^{(2)} = (0,1,0,0)^T$, $v^{(3)} = (0,0,1,0)^T$ and $v^{(4)} = \frac{1}{\sqrt{2}}(1,0,0,-1)^T$. Therefore, $\rho_{AB} = PDP^{-1}$ with $D = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)$, and $P = [v^1, v^2, v^3, v^4]$

Note that $\rho_A = \mathrm{Tr}_B \rho_{AB} = \frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\rho_B = \mathrm{Tr}_A \rho_{AB} = \frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, each with eigenvalues $\lambda_1 = \lambda_2 = \frac{1}{2}$. Simply trace the A and B parts into scalars, respectively, in the third line above.

In conclusion:

- Since one eigenvalue of ρ_{AB} is 1, the state (or ensemble) is pure.
- Since $\mathrm{Tr}\rho_{AB}^2 = \sum_i \lambda_i^2 = 1$, the state (or ensemble) is pure.
- Since $\rho_{AB} = \rho_{AB}^2$, the state (or ensemble) is pure.
- Since there are two non-zero Schmidt coefficients, the state is entangled.
- Since all Schmidt coefficients are non-zero and equal, the state is maximally entangled.

Example 4 - A different approach

Let $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|\downarrow_A\rangle \otimes |\uparrow_B\rangle + |\uparrow_A\rangle \otimes |\downarrow_B\rangle) = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$.

This time we study this state in another way.

> Recall that a general state in $H_A \otimes H_B$ is $|\psi_{AB}\rangle = \sum_{jk} c_{jk} |j_A\rangle \otimes |k_B\rangle$, and that it is entangled if any pair $c_j^A c_k^B \neq c_{jk}$, such that $|\psi_{AB}\rangle \neq \sum_j c_j^A |j_A\rangle \otimes \sum_k c_k^B |k_B\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, for any basis.

The basis of $H_{A,B}$ is $\{|\uparrow\rangle, |\downarrow\rangle\}_{A,B}$. Then $c_{10} = -c_{01} = \frac{1}{\sqrt{2}}$. On the other hand, we have the the linear combination:

$$(c_0^A |\uparrow_A\rangle + c_1^A |\downarrow_A\rangle) \otimes (c_0^B |\uparrow_B\rangle + c_1^B |\downarrow_B\rangle) = c_0^A c_0^B |\uparrow\uparrow\rangle + c_0^A c_1^B |\uparrow\downarrow\rangle + c_1^A c_0^B |\downarrow\uparrow\rangle + c_1^A c_1^B |\downarrow\downarrow\rangle$$

Now, notice the **contradiction**: if $c_{10} = c_1^A c_0^B \neq 0$ and $c_{01} = c_0^A c_1^B \neq 0$, then all c_j^A and c_k^B are nonzero $\Rightarrow$ It is impossible to get rid of any terms! $\Rightarrow$ **Entangled state!**

By the same argument, it is also possible to see why Example 2 was not entangled: $c_1^A = 0$ and all others nonzero satisfies the condition without contradiction.