Signals + Systems: A Fresh Look
Solutions Manual

Chapter 2

2.1 No. A could be one of two possible locations.

2.2 Yes. We draw a circle from every known point with the given distance as its radius. The intersection of the three circles yield A.

2.3 Yes. It is the ratio of the integer and integer 1.

2.4 Let $r_i = p_i / q_i$, $i = 1, 2$, be any two rational numbers, where p_i and q_i are integers. Its mid-point is

$$\frac{1}{2}(r_1 + r_2) = \frac{1}{2}\left(\frac{p_1}{q_1} + \frac{p_2}{q_2}\right) = \frac{p_1 q_2 + q_1 p_2}{2 q_1 q_2}$$

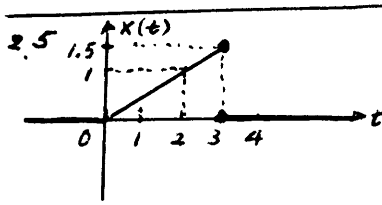
It is a rational number because both $(p_1 q_2 + q_1 p_2)$ and $2 q_1 q_2$ are integers.

Yes, we can. The first three rational numbers are

$$\frac{1}{2}\left(\frac{3}{5} + \frac{2}{3}\right) = \frac{1}{2} \cdot \frac{9+10}{15} = \frac{19}{30}$$

$$\frac{1}{2}\left(\frac{3}{5} + \frac{19}{30}\right) = \frac{1}{2} \cdot \frac{18+19}{30} = \frac{37}{60}$$

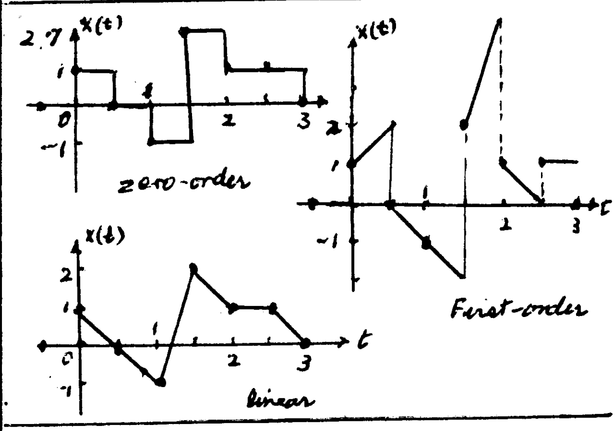
$$\frac{1}{2}\left(\frac{19}{30} + \frac{2}{3}\right) = \frac{1}{2} \cdot \frac{19+20}{30} = \frac{39}{60} = \frac{13}{20}$$

2.5 

It is not a signal because its value at $t=3$ is not unique. It becomes a function if we modify it as

$$x(t) = \begin{cases} 0.5t, & 0 \le t \le 3 \\ 0, & t < 0 \text{ and } t > 3 \end{cases}$$

2.6 $x(0)=0$, $x(0.5)=0.25$, $x(1)=0.5$, $x(1.5) = 0.75$, $x(2)=1$, $x(2.5)=1.25$, $x(3) = 1.5$, $x(3.5)=0$, $\cdots$

2.7 

2.8 $0.1\,\delta(t - t_c)$
$- 2\,\delta(t - t_0)$

2.9 1. $\int_{-\infty}^{\infty} \cos t \, \delta(t)\, dt = \cos t \Big|_{t=0} = 1$

2. $\int_{t=0}^{\pi/2} \sin t \, \delta(t - \frac{\pi}{2})\, dt = \sin t \Big|_{t=\pi/2} = 1$

3. $\int_{t=\pi}^{\infty} \sin t \cdot \delta(t - \pi/2)\, dt = 0$ because the impulse is located at $\pi/2$ and the integration interval does not cover nor touch it.

4. $\int_{t=0}^{\infty} \delta(t + \frac{\pi}{2}) \sin(t - \pi)\, dt = 0$
The impulse at $-\pi/2$ is outside $[0, \infty)$.

5. $\int_{-\infty}^{\infty} \delta(t)(t^3 - 2t^2 + 10t + 1)\, dt$
$= t^3 - 2t^2 + 10t + 1 \Big|_{t=0} = 1$

6. $\int_{0}^{0} \delta(t) e^{2t}\, dt = e^{2t}\Big|_{t=0} = e^0 = 1$

7. $\int_{0}^{0} 10^{10} e^{2t}\, dt = 0$

2.10 $\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-3)d\tau = x(\tau)\Big|_{t-\tau-3=0}$

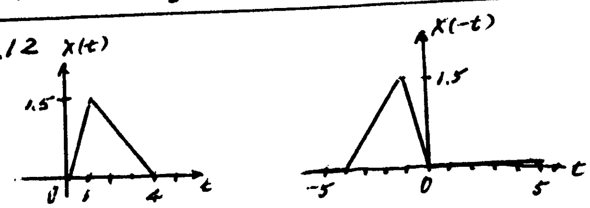
$= x(\tau)\Big|_{\tau=t-3} = x(t-3)$

$\int_{-\infty}^{\infty} x(t-\tau)\,\delta(\tau-t_0)d\tau = x(t-\tau)\Big|_{\tau-t_0=0}$

$= x(t-t_0)$

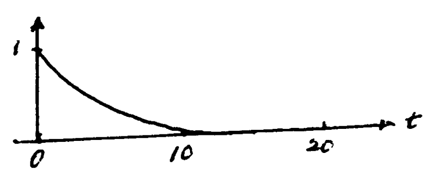
2.11 It is discontinuous and has infinitely many discontinuities in $(-\infty,\infty)$. However, in every finite time interval, it has only a finite number of discontinuities. Thus it is of bounded variation.

2.12

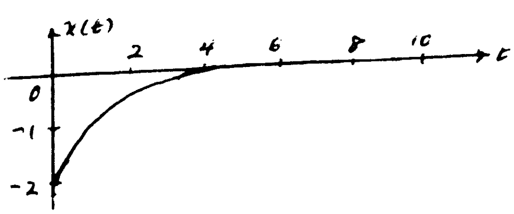


By direct substitution, we can plot $x(-t)$ as shown. Thus $x(-t)$ flips $x(t)$ to negative time with respect to $x=0$.

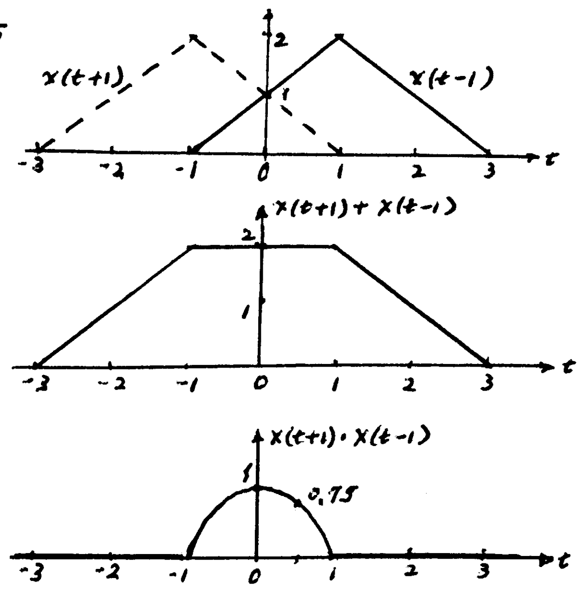
2.13 time constant $= \frac{1}{0.4} = 2.5$. Thus $x(t)$ takes $4 \times 2.5 = 10$ seconds to reach zero.



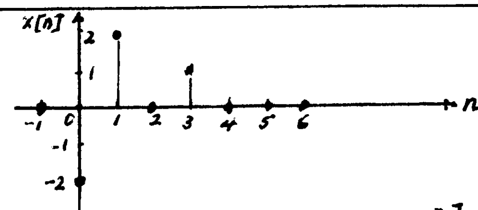
2.14 time constant $= \frac{1}{1.2} = 0.83$ and $x(t)$ takes $5 \times 0.83 = 4.15$ seconds to reach zero



2.15

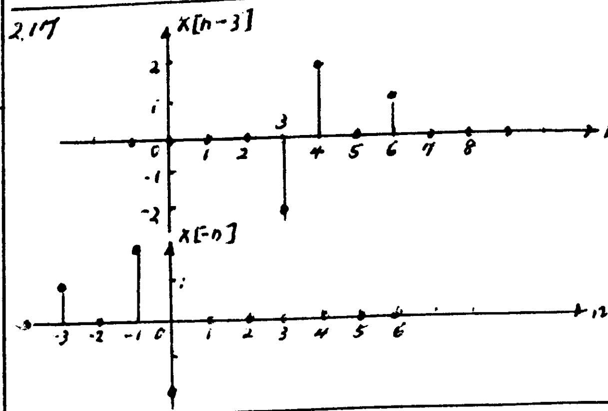


2.16



$x[n] = -2\delta_d[n] + 2\delta_d[n-1] + \delta_d[n-3]$

2.17



2.18 $x(nT) = e^{-0.4nT} = e^{-0.2n} = b^n$

where $b = e^{-0.2}$

Time const $= -1/\ln b = -1/(-0.2) = 5$ samples

$= 5 \times T = 0.5 \times 5 = 0.25$ seconds

2.19 $x(nT) = -2e^{-1.2nT} = -2e^{-0.24n} = -2b^n$

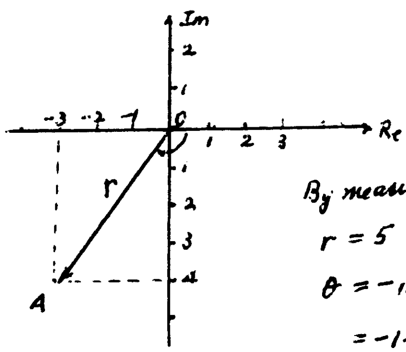
where $b = e^{-0.24}$

Time const $= -1/\ln b = -1/(-0.24) = 4.17$ samples

$= 4.17 \times 0.2 = 0.83$ seconds

Chapter 3

3.1



By measurement

$r = 5$

$\theta = -127^\circ$

$= -127 \times \frac{\pi}{180}$ rad

$= -2.2$ rad

$A = -3 - j4 = 5e^{-j2.2}$

3.2 $-3 = 3e^{j\pi}$, $\quad -1005 = 1005e^{j\pi}$

$1 - j1 = 1.4e^{-j\pi/4}$, $\quad 1 + j1 = 1.4e^{j\pi/4}$

$-10j = 10e^{-j\pi/2}$, $\quad 20j = 20e^{j\pi/2}$

$-1 - j1 = 1.4e^{-j3\pi/4}$, $\quad -1 + j1 = 1.4e^{j3\pi/4}$

$51 = 51e^{j0}$

3.3 $a = -ae^{j\pi}$

3.4

3.5 $e^{jn\pi} = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$

3.6 $e^{jnk2\pi} = 1$ for any integers n and k.

3.7 Let $A = \alpha + j\beta$, where α and β are real.

$A \cdot A = (\alpha + j\beta)(\alpha + j\beta) = \alpha^2 - \beta^2 + j2\alpha\beta$

$|A| = \sqrt{\alpha^2 + \beta^2}$, $\quad |A|^2 = \alpha^2 + \beta^2$

Clearly $A \cdot A \neq |A|^2$

If A is real, then $\beta = 0$ and $A \cdot A = |A|^2$

3.8 $x = \alpha_1 + j\beta_1$, $\quad y = \alpha_2 + j\beta_2$

$(xy)^* = (\alpha_1\alpha_2 - \beta_1\beta_2 + j(\alpha_1\beta_2 + \beta_1\alpha_2))^*$

$= \alpha_1\alpha_2 - \beta_1\beta_2 - j(\alpha_1\beta_2 + \beta_1\alpha_2)$

$x^* y^* = (\alpha_1 - j\beta_1)(\alpha_2 - j\beta_2)$

$= \alpha_1\alpha_2 - \beta_1\beta_2 - j(\alpha_1\beta_2 + \beta_1\alpha_2)$

Indeed we have $(xy)^* = x^* y^*$

$$\left(\int x(t) e^{j\omega t} dt\right)^* = \int [x(t) e^{j\omega t}]^* dt$$

$$= \int x^*(t) (e^{j\omega t})^* dt = \int x^*(t) e^{-j\omega t} dt$$

3.9 $I_3 M$ is not defined.

$$I_2 M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 4 & -1.5 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -1.5 & 0 \end{bmatrix} = M$$

$$M I_3 = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -1.5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -1.5 & 0 \end{bmatrix} = M$$

3.10 xy is not defined.

$$xy' = [1 \ 2 \ 3] \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} = 2 + 10 - 9 = 3$$

$$x'y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [2 \ 5 \ -3] = \begin{bmatrix} 2 & 5 & -3 \\ 4 & 10 & -6 \\ 6 & 15 & -9 \end{bmatrix}$$

3.11, 3.12 Direct verification.

3.13 Define $X(\omega) = \int_0^\infty x(t) e^{-j\omega t} dt$

Then $\int_0^\infty x(t) e^{j\omega t} dt = X(-\omega)$

and $\int_0^\infty x(t) e^{-st} dt = X(s/j)$

3.14 No. Because the summation may go to $-\infty$.

3.15 There is no difference.

3.16 We write

$1 - 1 + 1 - 1 + \cdots = (5-4) - (5-4) + (5-4)$

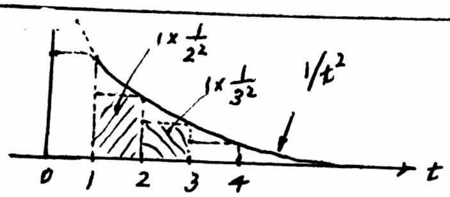
$-(5-4) + \cdots = 5 - (4-4) - (5-5)$

$-(4-4) - (5- \cdots$

$= 5 - 0 - 0 - 0 \cdots = 5$

3.17 Not true. The sequence in (3.15) is summable but does not approach zero as $n \to \infty$.

3.18



From the plot, we see that

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots < \int_1^\infty \frac{1}{t^2}\,dt = 1$$

Thus we conclude

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 2 < \infty$$

3.19 Direct verification. See page 304 of the text.

3.20 No. The sequence

$$x[n] = \frac{1}{n}, \quad \text{for } n \geq 1$$

is a counterexample as shown in the preceding two problems.

3.21 No. As Problem 3.19 shows. This is the reason, we substitute $|x[n]| < M_1$ only once in (3.22).

3.22 $\int_{-\infty}^{\infty} \delta_a(t-t_0)\,dt = 1$ for all a.

The total energy of $\delta_a(t-t_0)$ is

$$E = \int_{-\infty}^{\infty} \delta_a^2(t-t_0)\,dt = \int_{t_0}^{t_0+a} \frac{1}{a^2}\,dt$$

$$= \frac{1}{a^2}(t_0+a-t_0) = \frac{a}{a^2} = \frac{1}{a}$$

It approaches ∞ as $a \to 0$. Thus an impulse has infinite amount of energy and cannot be generated in practice. In general, absolutely integrable does not imply squared absolutely integrable.

3.23 $$\int_0^{1/n^3} [x(t)]^2\,dt = \int_0^{1/n^3} n^8 t^2\,dt = \frac{n^8}{3} t^3 \Big|_{t=0}^{1/n^3}$$

$$= \frac{n^8}{3}\left(\frac{1}{n^9} - 0\right) = \frac{1}{3n}$$

The triangle at $t=n$ in Fig 3.4 consists of two $x(t)$. Thus its energy is $4(1/3n) = 4/3n$. Consequently, the total energy of the signal in Fig. 3.4 is $\frac{4}{3}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) = \infty$.

3.24 $$\int_1^\infty \left|\frac{1}{t}\right| dt = \int_1^\infty \frac{1}{t}\,dt = \ln t \Big|_{t=1}^{\infty} = \infty$$

$$\int_1^\infty \frac{1}{t^2}\,dt = \frac{-1}{t}\Big|_{t=1}^{\infty} = (0-(-1)) = 1$$

Thus squared absolutely integrable does not imply absolutely integrable.

3.25 $p^2 = 2$ is even, but $p = \sqrt{2}$ is not. Thus we require the preamble.

3.26 n=0:0.5:2.5 means

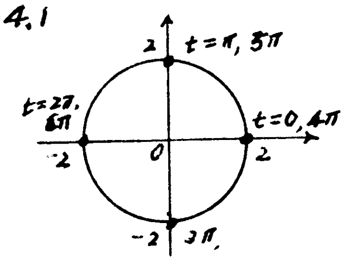
n=[0 0.5 1 1.5 2 2.5].

n=0:2.5 means [0 1 2].

3.31 change subplot(1,2,1) and subplot(1,2,2) to subplot(2,1,1) and subplot(2,1,2).

<u>Chapter 4</u>

4.1



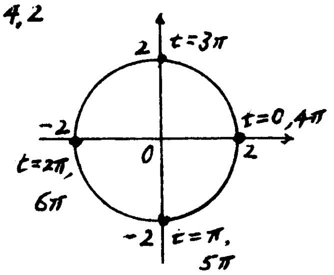
It rotates counter-clockwise.

$P = 4\pi$ s.

$f = \frac{1}{P} = \frac{1}{4\pi}$ Hz

$\omega = 2\pi f = \frac{2\pi}{4\pi} = 0.5$ rad/s

4.2



It rotates clockwise.

$P = 4\pi$ s.

$f = \frac{1}{P} = \frac{1}{4\pi}$ Hz

$\omega = \frac{2\pi}{4\pi} = 0.5$ rad/s

4.3 $\sin 3t + \sin \pi t$ is not periodic because $3/\pi$ is not a rational number.

4.4 $\sin 3t$ is periodic with (fundamental) period $2\pi/3$. 1.2 is periodic with any period. Thus they have the common period $2\pi/3$. The fundamental period of $1.2 + \sin 3t$ is $P_0 = 2\pi/3$. Its fundamental frequency is $\omega_0 = 2\pi/P_0 = 3$ rad/s

4.5 It is periodic because 1.2 and $\sin \pi t$ have common period $P_0 = 2\pi/\pi = 2$ seconds. Its fundamental period is $P_0 = 2$ and its fundamental frequency is $\omega_0 = 2\pi/P_0 = \pi$ rad/s.

4.6 The greatest common divisor (gcd) of 2.1 and 2.8 is 0.7. Thus the fundamental frequency is 0.7 rad/s and the fundamental period is $P_0 = 2\pi/0.7$

4.7
$$x(t) = -1.2 - 2\left(\frac{e^{j2.1t} - e^{-j2.1t}}{2j}\right) + 3\left(\frac{e^{j2.8t} + e^{-j2.8t}}{2}\right)$$
$$= 1.2e^{j\pi} + e^{j\pi/2}e^{j2.1t} + e^{-j\pi/2}e^{-j2.1t} + 1.5e^{j2.8t} + 1.5e^{-j2.8t}$$

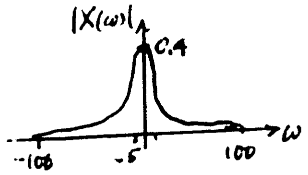
4.8
$$\int_{-\infty}^{\infty} |x(t)|\,dt = \int_0^{\infty} 3e^{-0.2t}\,dt = 3\cdot\frac{1}{-0.2}e^{-0.2t}\Big|_{t=0}^{\infty}$$
$$= \frac{-3}{0.2}(0-1) = 15 < \infty$$

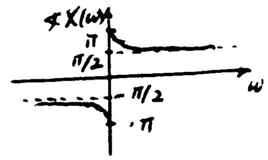
4.9 $x(t) = 2e^{5t} \to \infty$ as $t \to \infty$ and its spectrum is not defined.

4.10
$$X(\omega) = \int_0^{\infty} (-2e^{-5t})e^{-j\omega t}\,dt$$
$$= -2\int_0^{\infty} e^{(-5-j\omega)t}\,dt = \frac{-2}{-5-j\omega} \times e^{(-5-j\omega)t}\Big|_0^{\infty} = \frac{2}{5+j\omega}(0-1)$$
$$= \frac{-2}{j\omega+5} \quad \text{for all } \omega \text{ in } (-\infty, \infty).$$

$$X(0) = \frac{-2}{0+5} = -0.4 = 0.4e^{j\pi}$$
$$X(5) = \frac{-2}{j5+5} = \frac{2e^{j\pi}}{7.1e^{j\pi/4}} = 0.28e^{j(3\pi/4)}$$
$$X(-5) = \frac{-2}{-5j+5} = \frac{2e^{j\pi}}{7.1e^{-j\pi/4}} = 0.28e^{j(5\pi/4)}$$
$$= 0.28e^{-j3\pi/4}$$ (to express the angle in the principal range $(-\pi, \pi]$)

$$X(100) = \frac{-2}{j100+5} \approx \frac{-2}{j100} = \frac{2e^{j\pi}}{100e^{j\pi/2}} = 0.02e^{j\pi/2}$$
$$X(-100) = \frac{-2}{-j100+5} \approx \frac{-2}{-j100} = \frac{2e^{j\pi}}{100e^{-j\pi/2}} = 0.02e^{j3\pi/2} = 0.02e^{-j\pi/2}$$





4.11 $x(t) = \delta(t)$

$$X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega t}\Big|_{t=0} = 1$$

for all ω. Its total energy is

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega =$$

$$= \frac{1}{2\pi}\,\omega\Big|_{\omega=-\infty}^{\infty} = \frac{1}{2\pi}(\infty - (-\infty)) = \infty$$

An impulse cannot be generated in practice because it requires infinite amount of energy.

4.12

$$X_0(\omega) = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega(t-t_0+t_0)} dt$$

$$= e^{-j\omega t_0}\int_{t=-\infty}^{\infty} x(t-t_0) e^{-j(t-t_0)}$$

Define $\bar{t} = t - t_0$. Then we have

$$X_0(\omega) = e^{-j\omega t_0}\int_{\bar{t}=-\infty}^{\infty} x(\bar{t}) e^{-j\omega\bar{t}} d\bar{t} = e^{-j\omega t_0} X(\omega)$$

which implies

$$|X_0(\omega)| = |e^{-j\omega t_0}||X(\omega)| = |X(\omega)|$$

$$\measuredangle X_0(\omega) = \measuredangle e^{-j\omega t_0} + \measuredangle X(\omega)$$

$$= \measuredangle X(\omega) - \omega t_0$$

4.13 If $x(t)$ is real, then

$$\mathrm{Re}\, X(\omega) = \int_{-\infty}^{\infty} x(t) \cos\omega t\, dt$$

$$\mathrm{Im}\, X(\omega) = -\int_{-\infty}^{\infty} x(t) \sin\omega t\, dt$$

Because $\cos(-\omega)t = \cos\omega t$, we have

$$\mathrm{Re}\, X(-\omega) = \int_{-\infty}^{\infty} x(t)\cos(-\omega)t\, dt = \mathrm{Re}\, X(\omega) \text{ (even)}$$

Because $\sin(-\omega)t = -\sin\omega t$, we have

$$\mathrm{Im}\, X(-\omega) = \int_{-\infty}^{\infty} -x(t)\sin(-\omega)t\, dt = +\int_{-\infty}^{\infty} x(t)\sin\omega t\, dt = -\mathrm{Im}\, X(\omega)$$

$$|X(-\omega)| = \sqrt{[\mathrm{Re}\, X(-\omega)]^2 + [\mathrm{Im}(-\omega)]^2}$$

$$= \sqrt{[\mathrm{Re}\, X(\omega)]^2 + [-\mathrm{Im}\, X(\omega)]^2}$$

$$= \sqrt{[\mathrm{Re}\, X(\omega)]^2 + [\mathrm{Im}\, X(\omega)]^2} = |X(\omega)|$$

$$\measuredangle X(-\omega) = \tan^{-1}[\mathrm{Im}\, X(-\omega)]/[\mathrm{Re}\, X(-\omega)]$$

$$= \tan^{-1}[-\mathrm{Im}\, X(\omega)]/[\mathrm{Re}\, X(\omega)] = -\measuredangle X(\omega).$$

This establishes (4.29). This is slightly more complicated than the proof in the text.

4.14 If $x(t)$ is real and $x(-t) = x(t)$, then $x(t)\sin\omega t$ is odd (because $\sin(-\omega)t = -\sin\omega t$). Thus its integration over $(-\infty,\infty)$ is zero, that is

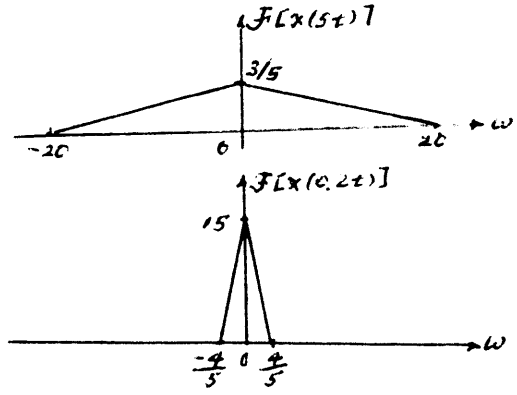
$\mathrm{Im}\, X(\omega) = 0$. Thus $X(\omega) = \mathrm{Re}\, X(\omega)$ is real and even.

If $x(t)$ is positive time, then it cannot be even. It is unlikely to find a $x(t)$ so that

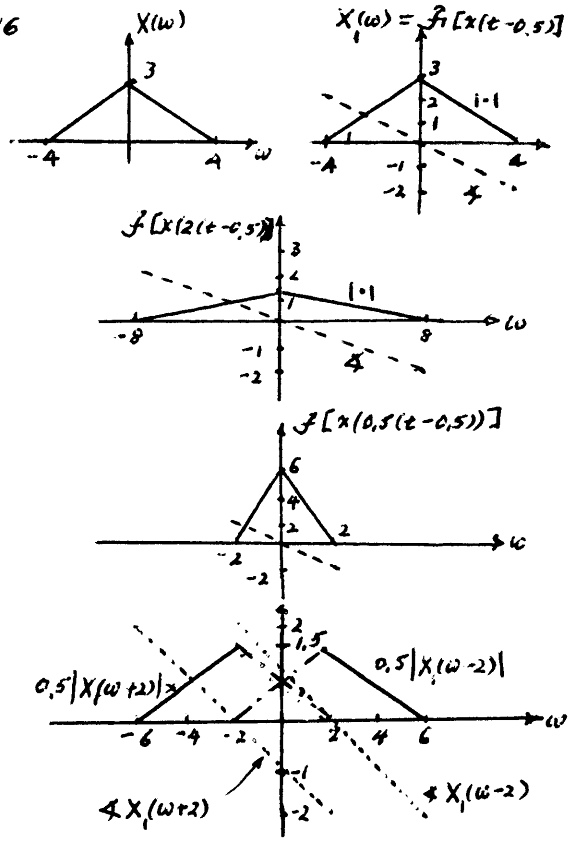
$$\int_0^{\infty} x(t)\sin\omega t\, dt = 0 \quad \text{for all } \omega$$

Thus we rarely, or never, encounter real-valued spectra in practice.

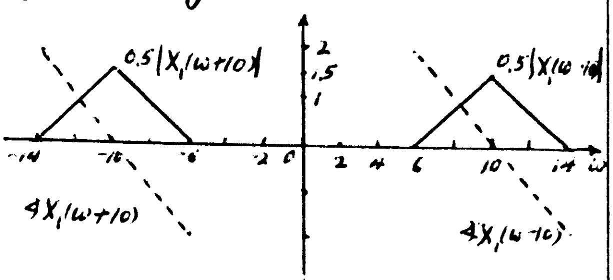
4.15



4.16



There are overlapping of $0.5X_1(\omega-2)$ and $0.5X_1(\omega+2)$ for ω in $[-2, 2]$. Its effect is complicated because it involves the addition of complex numbers and cannot be easily obtained graphically.



There is no overlapping of $0.5X_1(\omega-10)$ and $0.5X_1(\omega+10)$ and the signal $x(t-0.5)$ can be recovered from its modulated signal $x(t-0.5)\cos 10t$.

4.17 $P = 2P_0$, $\bar{\omega}_0 = 2\pi/P = \pi/P_0 = \omega_0/2$.

$$x(t) = \sum_{m=-\infty}^{\infty} \bar{c}_m e^{jm\bar{\omega}_0 t}$$

$$\bar{c}_m = \frac{1}{2P_0}\int_0^P x(t)e^{-jm\bar{\omega}_0 t}dt$$

$$= \frac{1}{2P_0}\left[\int_0^{P_0} x(t)e^{-jm\bar{\omega}_0 t}dt + \int_{P_0}^{2P_0} x(t)e^{-jm\bar{\omega}_0 t}dt\right]$$

Define $\bar{t} = t - P_0$. Then we have

$$\int_{t=P_0}^{2P_0} x(t)e^{-jm\bar{\omega}_0 t}dt = \int_{\bar{t}=0}^{P_0} x(\bar{t}+P_0)e^{-jm\bar{\omega}_0(\bar{t}+P_0)}d\bar{t}$$

$$= \int_{\bar{t}=0}^{P_0} x(\bar{t})e^{-jm\bar{\omega}_0\bar{t}}e^{-jm\bar{\omega}_0 P_0}d\bar{t}$$

Because $e^{-jm\bar{\omega}_0 P_0} = e^{-jm\pi} = \begin{cases} 1 & m: \text{even} \\ -1 & m: \text{odd} \end{cases}$

if m is odd, we have

$$\bar{c}_m = \frac{1}{2P_0}\left[\int_0^{P_0} x(t)e^{-jm\bar{\omega}_0 t}dt - \int_0^{P_0} x(t)e^{-jm\bar{\omega}_0 t}dt\right]$$

$$= 0$$

If m is even or $m = 2\bar{m}$, then

$$\bar{c}_m = \frac{1}{2P_0}\left[\int_0^{P_0} x(t)e^{-jm\bar{\omega}_0 t} + \int_0^{P_0} x(t)e^{-jm\bar{\omega}_0 t}dt\right]$$

$$= \frac{1}{P_0}\int_0^{P_0} x(t)e^{-j\bar{m}\omega_0 t}dt = c_{\bar{m}} = c_{m/2}$$

4.18 $x(t) = \sum_{m=-\infty}^{\infty} c_m e^{jm\omega_0 t}$

with $\omega_0 = 2\pi/P_0$ and $c_m = \frac{1}{P_0}\int_{-P_0/2}^{P_0/2} x(t)e^{-jm\omega_0 t}dt$

$$X(\omega) = \int_{-P_0/2}^{P_0/2} x(t)e^{-j\omega t}dt$$

Direct substitution yields $c_m = X(m\omega_0)/P_0$

$$X(\omega) = \int_{-P_0/2}^{P_0/2}\left(\sum c_m e^{jm\omega_0 t}\right)e^{-j\omega t}dt$$

$$= \sum c_m \int_{-P_0/2}^{P_0/2} e^{j(m\omega_0-\omega)t}dt$$

$$X(\omega) = \sum c_m \cdot \frac{1}{j(m\omega_c-\omega)} e^{j(m\omega_c-\omega)t}\Big|_{t=-P_c/2}^{P_c/2}$$

$$= \sum \frac{c_m}{j(m\omega_c-\omega)}\left[e^{j(m\omega_c-\omega)P_c/2} - e^{-j(m\omega_c-\omega)P_c/2}\right]$$

$$= \sum c_m \frac{2j\sin[(m\omega_c-\omega)P_c/2]}{j(m\omega_c-\omega)}$$

$$= \sum_{m=-\infty}^{\infty} c_m \cdot \frac{2\sin[(\omega-m\omega_0)P_c/2]}{\omega-m\omega_c}$$

4.19

$$X(\omega)=\int_0^L x(t)e^{-j\omega t}dt = \int_0^L \sum c_m e^{jm\omega_c t} e^{-j\omega t}dt$$

$$=\sum \frac{c_m}{j(m\omega_c-\omega)} e^{j(m\omega_c-\omega)t}\Big|_{t=0}^{L}$$

$$=\sum_{m=-\infty}^{\infty}\frac{c_m}{j(m\omega_0-\omega)}\left(e^{j(m\omega_c-\omega)L}-1\right)$$

4.20 Its total energy is

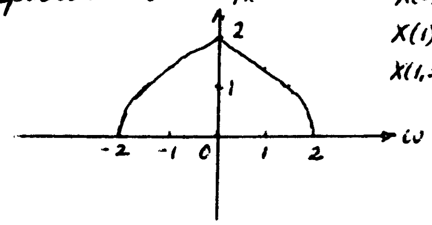
$$E=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}|X(\omega)|^2 d\omega = \frac{1}{2\pi}\cdot\frac{4\times4}{2}=\frac{4}{\pi}$$

Its spectrum is

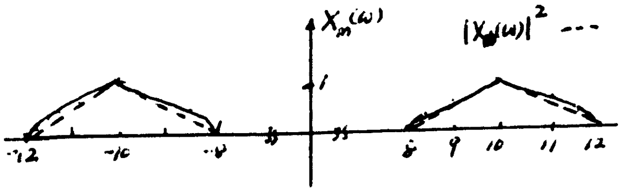
$X(0)=\sqrt{4}=2$

$X(1)=\sqrt{2}=1.4$

$X(1.5)=\sqrt{1}=1$



The spectrum of $x(t)\cos 10t$ is



$$E_m = \frac{1}{2\pi}\times2\times1\times4\times\frac{1}{2} = \frac{2}{\pi} = \frac{1}{2}E$$

The modulated signal has only half of the energy of the original signal.

4.21 $X_m(\omega) = 0.5[X(\omega-\omega_c) + X(\omega+\omega_c)]$

If $X(\omega-\omega_c)$ and $X(\omega+\omega_c)$ do not overlap, then

$$|X_m(\omega)| = 0.5[|X(\omega-\omega_c)| + |X(\omega+\omega_c)|]$$

and

$$|X_m(\omega)|^2 = 0.25[|X(\omega-\omega_c)|^2 + |X(\omega+\omega_c)|^2]$$

The total energy of $x_m(t)$ is

$$E_m = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X_m(\omega)|^2 d\omega = \frac{0.25}{2\pi}\left[\int_{-\infty}^{\infty}|X(\omega-\omega_c)|^2 d\omega + \int_{-\infty}^{\infty}|X(\omega+\omega_c)|^2 d\omega\right]$$

The total energy of $x(t)$ is

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega-\omega_c)|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega+\omega_c)|^2 d\omega$$

Thus we have

$$E_m = 0.25(E+E) = 0.5E.$$

4.22 If $T=1$, NFR $= (-\frac{\pi}{T}, \frac{\pi}{T}] = (-\pi, \pi]$.

$0 = 2\pi = 4\pi = 6\pi \pmod{2\pi}$

Thus the three $x_i(nT)$ has the same frequency 0 rad/s and their principal forms all equal $x(nT) = \cos(0\cdot nT) = 1$, for all n.

4.23 If $T = 0.5$, NFR $= (\frac{-\pi}{0.5}, \frac{\pi}{0.5}] = (-2\pi, 2\pi]$

Because $2\pi = 6\pi \pmod{4\pi}$, $x_1(nT)$ and $x_3(nT)$ have the same frequency 2π rad/s. $x_1(nT)$ is in principal form. The principal form of $x_3(nT)$ is $\cos 2\pi nT$.

Because $4\pi = 0 \pmod{4\pi}$, $x_2(nT)$ has frequency 0 and its principal form is $\cos(0\cdot nT) = 1$ for all n. There are two distinct sequences.

4.24 $T=0.1$, $NFR=(-10\pi, 10\pi]$.

$x_i(nT)$, $i=1,2,3$ are three distinct sequences and are in principal form.

4.25 $x(nT) = \sin 10nT$

If $T=1$, then $NFR=(-\pi, \pi]$
$=(-3.14, 3.14]$

Because 10 is outside the NFR, the frequency of $\sin 10nT$ is not 10 rad/s.

$$10 = 10-6.28 = 3.72-6.28 = -2.56 \pmod{6.28}$$

Thus $\sin 10nT$ with $T=1$ has frequency -2.56 rad/s and principal form $\sin(-2.56)nT$.

4.26 $T=0.5$ $NFR=(-2\pi, 2\pi]=(-6.28, 6.28]$

$10 = 10-12.56 = -2.56 \pmod{12.56}$

Thus $\sin 10nT$, with $T=0.5$, has freq. -2.56 and principal form $\sin(-2.56)nT$.

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$T=0.3$, $NFR=(\frac{-\pi}{T}, \frac{\pi}{T}]=(-10.47, 10.47]$

Because 10 lies inside the NFR, $\sin 10nT$, with $T=0.3$, has freq. 10 and is in principal form.

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$T=0.1$ $NFR=(-10\pi, 10\pi]=(-31.4, 31.4]$

$\sin 10nT$, with $T=0.1$, is in principal form and has freq. 10 rad/s.

4.27 $T=\pi/4$, $NFR=(-\frac{\pi}{T}, \frac{\pi}{T}]=(-4, 4]$

$10 = 10-8 = 2 \pmod 8$

$20 = 20-8 = 12-8 = 4 \pmod 8$

$x(nT) = 2-3\sin 10nT + 4\cos 20nT$

$= 2-3\sin 2nT + 4\cos 4nT$ (Principal form)

It has aliased freq. 2 and 4 rad/s and we cannot recover $x(t)$ from $x(nT)$ with $T=\pi/4$.

4.28 $T=\pi/5$. $NFR=(-\frac{\pi}{T}, \frac{\pi}{T}]=(-5, 5]$

$10 = 0 \pmod{10}$

$20 = 10 = 0 \pmod{10}$

$x(nT) = 2-3\sin 10nT + 4\cos 20nT$

$= 2-3\sin 0\cdot nT + 4\cos 0\cdot nT$

$= 2-3\times 0+4\times 1 = 6$ (Principal form)

It has aliased freq. 0 which coincides with the original freq. 0. We cannot recover $x(t)$ from $x(nT)$ with $T=\pi/5$.

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$T=\pi/10$, $NFR=(-10, 10]$

$20 = 0 \pmod{20}$

$x(nT) = 2-3\sin 10nT + 4\cos 20nT$

$= 2-3\sin 10nT + 4 = 6-3\sin 10nT$ (Principal form)

It has aliased freq. 0. We cannot recover $x(t)$ from $x(nT)$ with $T=\pi/10$.

- - - - - - - - - - - - - - - -

$T=\pi/25$, $NFR=(-25, 25]$

$x(nT) = 2-3\sin 10nT + 4\cos 20nT$ (Principal form)

It has no aliased freq. and we can recover $x(t)$ from $x(nT)$ with $T=\pi/25$.

4.29 $$(1-r)\sum_{n=0}^{N} r^n = \sum_{n=0}^{N}(r^n - r^{n+1}) = (r^0-r) + (r-r^2)+(r^2-r^3)+\cdots+(r^N-r^{N+1})$$
$$= 1-r^{N+1}$$

Thus we have $$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$

4.30

$N=0$: $\sum_{n=0}^{0} r^n = r^0 = 1 = \frac{1-r}{1-r} = 1$

$N=1$: $\sum_{n=0}^{1} r^n = 1+r = \frac{1-r^2}{1-r} = \frac{(1-r)(1+r)}{1-r}$
$= 1+r$.

Suppose (4.66) holds for N or
$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$

Now we consider $N+1$
$$\sum_{n=0}^{N+1} r^n = \sum_{n=0}^{N} r^n + r^{N+1}$$

$$= \frac{1-r^{N+1}}{1-r} + r^{N+1} = \frac{1-r^{N+1}+r^{N+1}-r^{N+2}}{1-r}$$

$$= \frac{1-r^{N+2}}{1-r}$$

So (4.66) is verified for any positive integer N.

4.31 $x(nT) = 3^n \to \infty$ as $n \to \infty$

Its spectrum diverges and is not defined.

4.32 $x(nT) = 0.3^n$, $n \geq 0$ and $T = 0.1$

$$X_d(\omega) = \sum_{n=0}^{\infty} (0.3)^n e^{-j\omega nT} = \sum_{n=0}^{\infty} (0.3e^{-j\omega T})^n$$

$$= \frac{1}{1-0.3e^{-j\omega T}}$$

Note that $|r| = |0.3e^{-j\omega T}| = 0.3 < 1$ and (4.67) can be applied.

4.33 $x(nT) = 1.2$, $n = 0, 2$ and $T = 0.2$

$$X_d(\omega) = 1.2 + 0 \times e^{-j\omega T} + 1.2e^{-j2\omega T}$$

$$= 1.2e^{-j\omega T}(e^{j\omega T} + e^{-j\omega T})$$

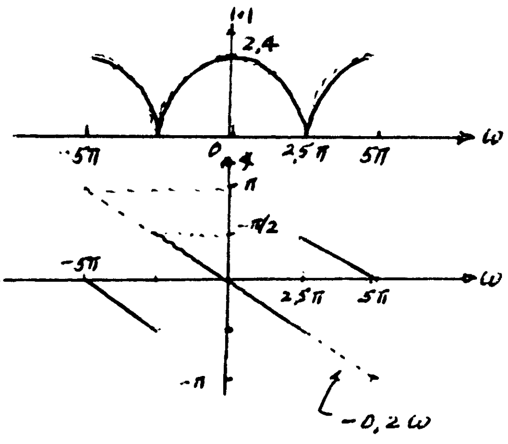
$$= 1.2e^{-j\omega T} \times 2\cos\omega T = 2.4e^{-j\omega T}\cos\omega T$$

$$|X_d(\omega)| = |2.4e^{-j\omega T}\cos\omega T| = 2.4|\cos\omega T|$$

$$\measuredangle X_d(\omega) = \measuredangle 2.4 + \measuredangle e^{-j\omega T} + \measuredangle \cos\omega T$$

$$= 0 - \omega T + \begin{cases} 0 & \text{if } \cos\omega T \geq 0 \\ \pi & \text{if } \cos\omega T < 0 \end{cases}$$

$T = 0.2$, $NFR = (\frac{-\pi}{T}, \frac{\pi}{T}] = (-5\pi, 5\pi]$



For ω in $[-2.5\pi, 2.5\pi]$, $\cos 0.2\omega$ is positive and $\measuredangle X_d(\omega) = -0.2\omega$.

For ω in $[2.5\pi, 5\pi]$, $\cos\omega T$ is negative and $\measuredangle X_d(\omega) = -0.2\omega + \pi$

For ω in $[-5\pi, -2.5\pi]$, $\cos 0.2$ is negative and $\measuredangle X_d(\omega) = -0.2\omega + \pi$. But the angle will be outside the principal range $(-\pi, \pi]$. Thus we substract 2π to yield

$$\measuredangle X_d(\omega) = -0.2\omega - \pi$$

4.34

$$X_d(\omega) = \sum_{n=0}^{\infty} 3 \cdot 0.98^n e^{-j\omega nT} = 3\sum_{n=0}^{\infty}(0.98e^{-j\omega T})^n$$

$$= 3 \cdot \frac{1}{1-0.98e^{-j\omega T}}$$

4.35 $T = 0.2$

$$X_d(\omega) = 1 - 2e^{-j\omega T} + e^{-j2\omega T}$$

$$= e^{-j\omega T}[e^{j\omega T} - 2 + e^{-j\omega T}]$$

$$= e^{-j\omega T}[2\cos\omega T - 2]$$

$$= 2e^{-j\omega T}(\cos\omega T - 1) = -2e^{-j\omega T}(1-\cos\omega T)$$

$$= 2e^{-j(\omega T - \pi)}(1-\cos\omega T)$$

Because $1 - \cos\omega T \geq 0$ for all ω, we have

$$|X_d(\omega)| = 2(1-\cos\omega T)$$

$\measuredangle X_d(\omega) = -\omega T + \pi$ (Note that the phase plot must be limited to $(-\pi, \pi]$.)

$$\omega = m(2\pi/NT) = m(2\pi/(3 \times 0.2)) = m(10\pi/3)$$

$$= 0,\ 10\pi/3,\ 20\pi/3,\quad m = 0:2$$

$|X_d(0)| = 0$, $\measuredangle X_d(0) = \pi$. Note that the computer yields $\measuredangle X_d(0) = 0$ which differs from the analytical result. We pay no attention to this discrepancy.

$$|X_d(10\pi/3)| = 2(1-\cos(\tfrac{10\pi}{3} \times 0.2)) = 2(1+0.5) = 3$$

$$\measuredangle X_d(10\pi/3) = -\frac{10\pi}{3} \times 0.2 + \pi = \frac{\pi}{3} = 1.05 \text{ rad}$$

$|X_d(20\pi/3)| = 2(1-\cos(4\pi/3)) = 2(1+0.5) = 3$

$\measuredangle X_d(20\pi/3) = -\frac{20\pi}{3}\times 0.2 + \pi = -\frac{4\pi}{3} + \pi$

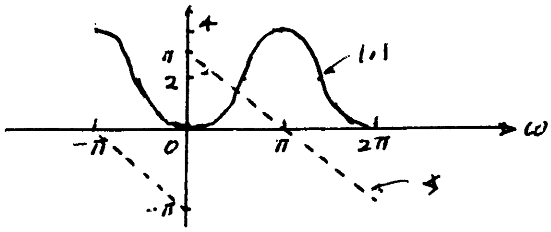
$= -\pi/3 = -1.05$ rad.

3.36 $T=1$, $NFR = (-\pi, \pi]$

$$X_d(\omega) = 2e^{-j(\omega T - \pi)}(1-\cos\omega T)$$
$$= 2e^{-j(\omega-\pi)}(1-\cos\omega)$$

$|X_d(\omega)| = 2(1-\cos\omega)$

$\measuredangle X_d(\omega) = -\omega + \pi$



$\omega = m(2\pi/NT) = m(2\pi/3)$

$= 0,\ 2\pi/3,\ 4\pi/3,\quad m = 0:2$

$|X_d(0)| = 0, \quad \measuredangle X_d(0) = \pi$

$|X_d(2\pi/3)| = 2(1-\cos(2\pi/3)) = 2(1+0.5) = 3$

$\measuredangle X_d(2\pi/3) = \frac{-2\pi}{3} + \pi = \frac{\pi}{3} = 1.05$

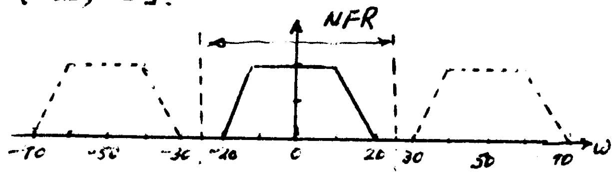
$|X_d(4\pi/3)| = 2(1-\cos(4\pi/3)) = 3$

$\measuredangle X_d(4\pi/3) = -\frac{4\pi}{3} + \pi = \frac{-\pi}{3} = -1.05$

They are the same as those computed in Problem 3.35.

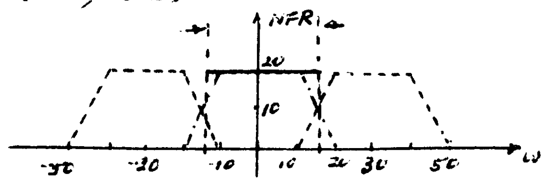
Chapter 5

5.1 $T = \pi/25$, $\omega_s = 2\pi/T = 50$, $NFR = (-25, 25]$.



The dotted curves are shifting of the solid curve to $\pm 50 = \pm\omega_s$ or folding with respect to $\pm\omega_s/2 = \pm 25$. There is no overlapping. Thus the spectrum of $Tx(nT)$ is as shown with the solid line. There is no frequency aliasing and we can recover $X(\omega)$ from $X_d(\omega)$ or recover $x(t)$ from $x(nT)$ using the ideal interpolator in (5.15).

5.2 $T = \pi/15$, $\omega_s = 2\pi/T = 30$, $NFR = (-15, 15]$.

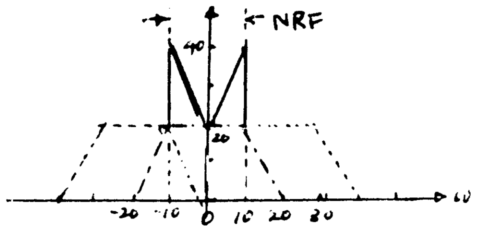


There are overlapping. The sum of those overlapping yields

$TX_d(\omega) = 20$ for all ω in $(-15, 15]$

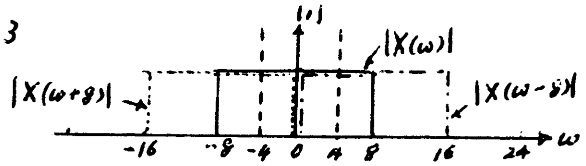
There is frequency aliasing and we can not recover $x(t)$ from $x(nT)$.

$T = \pi/10$, $\omega_s = 2\pi/T = 20$, $NFR = (-10, 10]$

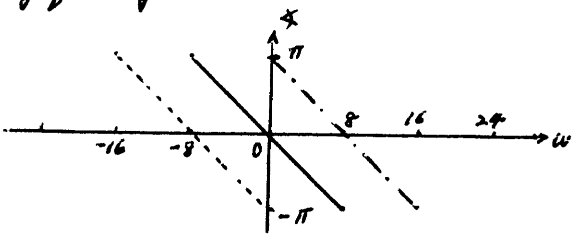


There is frequency aliasing and we cannot recover $x(t)$ from $x(nT)$.

5.3



$|X(\omega \pm 8)|$ can be obtained from $|X(\omega)|$ by folding with respect to ∓ 4.



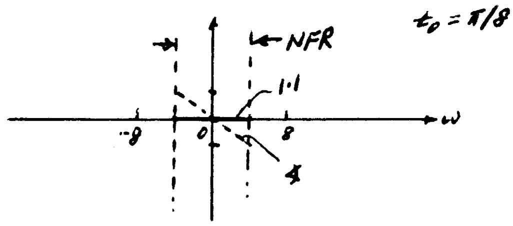
$\measuredangle X(\omega \pm 8)$ cannot be obtained from $\measuredangle X(\omega)$ by folding. They must be obtained by shifting to ∓ 8.

If $x(t)$ is sampled with $T = \pi/4$ or with $\omega_s = 2\pi/T = 8$, then we have

$$TX_d(\omega) = \sum_{m=-\infty}^{\infty} X(\omega + m\omega_s)$$

Because $X_d(\omega)$ is periodic with period ω_s, we need to plot $TX_d(\omega)$ for ω in the NFR $= (-4, 4)$ or just the positive NFR $[0, 4]$. From the preceding plots, we see that for ω in $[0, 4]$, we have

$$TX_d(\omega) = X(\omega) + X(\omega - 8)$$
$$= e^{-j\omega t_0} + e^{-j(\omega-8)t_0}$$
$$= e^{-j\omega t_0}\left(1 + e^{j8t_0}\right)$$
$$= e^{-j\omega t_0}\left(1 + e^{\pm j\pi}\right) = 0 \cdot e^{-j\omega t_0} \quad \text{if } t_0 = \pi/8$$



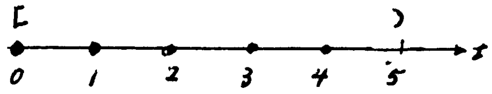
Because the magnitude is even and the phase is odd, once we have $TX_d(\omega)$ for ω in $[0, 4]$, we can obtain $TX_d(\omega)$ for all ω in $[-4, 4]$ as shown. For this simple example, we can obtain the magnitude and phase plot of $TX_d(\omega)$ from those of $X(\omega + m\omega_s)$. If t_0 is different from $\pi/8$, then $(1 + e^{j8t_0})$ can range from 0 to 2 in magnitude and from 0 to π in phase. In this case, to obtain $TX_d(\omega)$ graphically will be more complicated. In conclusion, the effect of frequency aliasing is generally very complicated. In practice, there is no need to be concerned with the effect so long as the effect is very small.

5.4 $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

If $X(\omega) = 0$ for $|\omega| > \omega_{max}$ and if $T < \pi/\omega_{max}$, then $X(\omega) = 0$ for $|\omega| > \pi/T > \omega_{max}$.

$$x(t) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} X(\omega) e^{j\omega t} d\omega$$
$$= \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega nT} e^{j\omega t} d\omega$$
$$= \frac{T}{2\pi}\sum_{n=-\infty}^{\infty} x(nT) \int_{-\pi/T}^{\pi/T} e^{j(t-nT)\omega} d\omega$$
$$= \frac{T}{2\pi}\sum_{n=-\infty}^{\infty} x(nT) \cdot \frac{1}{j(t-nT)} e^{j(t-nT)\omega}\Big|_{\omega=-\pi/T}^{\pi/T}$$
$$= \frac{T}{2\pi}\sum_{n=-\infty}^{\infty} \frac{e^{j(t-nT)\pi/T} - e^{-j(t-nT)\pi/T}}{j(t-nT)} x(nT)$$
$$= \sum_{n=-\infty}^{\infty} x(nT) \cdot \frac{T \cdot 2j \sin[(t-nT)\pi/T]}{2\pi j(t-nT)}$$
$$= \sum_{n=-\infty}^{\infty} x(nT) \cdot \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

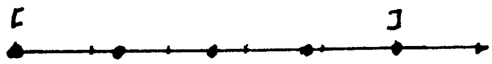
5.5



$x(0),\ x(1),\ x(2),\ x(3),\ x(4)$

$T = \frac{5}{5} = 1 \qquad L = 5 = NT = 5$

5.6



$x(0) \quad x(1.25) \quad x(2.5) \quad x(3.75) \quad x(5)$

$T = \frac{L}{N-1} = \frac{5}{5-1} = \frac{5}{4} = 1.25$

$T \cdot N = 1.25 \times 5 = 6.25 \neq L = 5$

5.7 $N=6$

$m = 0, 1, 2, 3, 4, 5$

Frequency range $m \cdot D = 0 \to \frac{5}{6} \cdot \frac{2\pi}{T}$

$[0, 2\pi/T)$

$mp = 0, 1, 2, 3$

$mp \cdot D = 0 \to \frac{3}{6} \cdot \frac{2\pi}{T} = \frac{\pi}{T}$

Frequency range $[0, \pi/T]$.

5.8 $N=5$

$m = 0, 1, 2, 3, 4$

$m \cdot D = 0 \to \frac{4}{5} \cdot \frac{2\pi}{T}$

Frequency range $[0, 2\pi/T)$

$mp = 0, 1, 2$

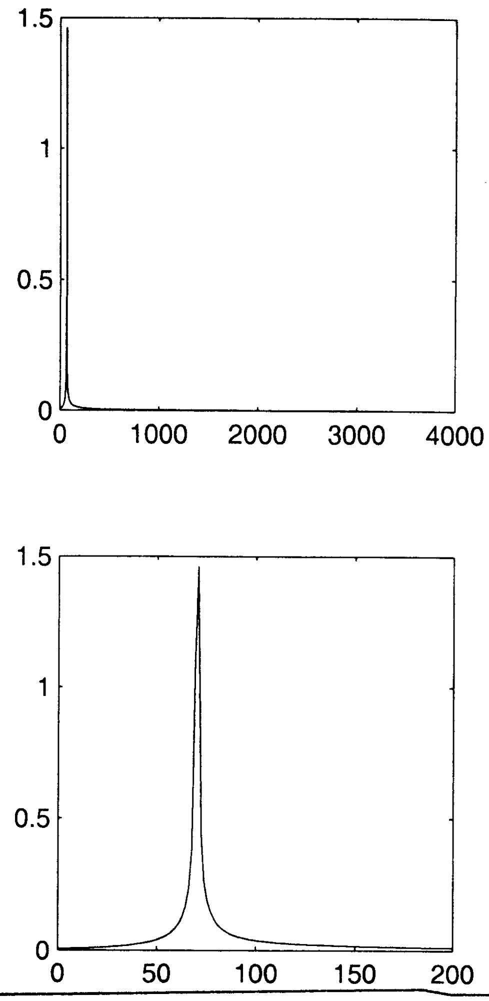
$mp \cdot D = 0 \to \frac{2}{5} \cdot \frac{2\pi}{T} = \frac{4}{5} \cdot \frac{\pi}{T}$

Frequency range $[0, \pi/T)$

5.9

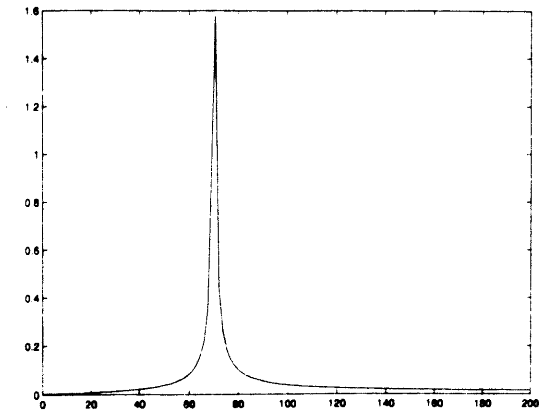
```
T=0.001;t=0:T:4;N=4/T;
x=cos(70*t);
X=T*fft(x);
mp=0:N/2;D=2*pi/(N*T);
subplot(2,1,1)
plot(mp*D,abs(X(mp+1))),axis square
subplot(2,1,2)
plot(mp*D,abs(X(mp+1))),axis square
axis([0 200 0 1.5])
```

For $T=0.001$, the positive NFR is $[0, \pi/T]$ $= [0\ 3140]$.



5.10 Yes

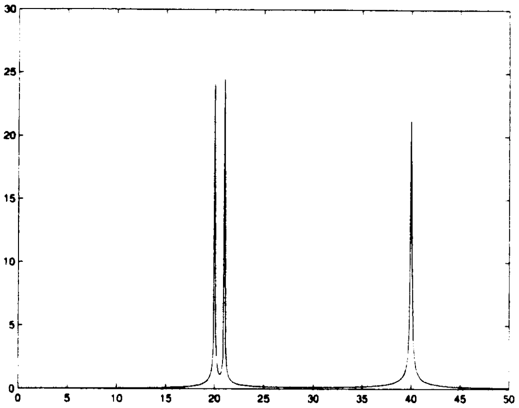
```
T=pi/200;t=0:T:4;N=4/T;
x=cos(70*t);
X=T*fft(x);
mp=0:N/2;D=2*pi/(N*T);
plot(mp*D,abs(X(mp+1)))
```



Yes, the results are roughly the same. However, this computation uses only $L/T = 4/\pi/200 = 800/\pi = 254$ samples. Whereas, the computation in Problem 5.4 uses 4000 samples, about 15 times more.

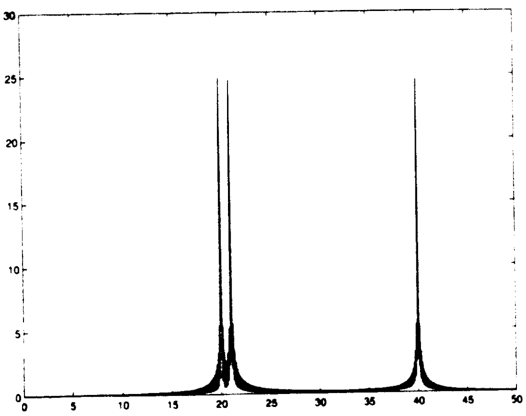
5.11

```
L=50;N=1000;T=L/N;n=0:N-1;
x=sin(20*n*T)+cos(21*n*T)-sin(40*n*T);
X=T*fft(x,N);
mp=0:N/2;D=2*pi/(N*T);
plot(mp*D,abs(X(mp+1)))
axis([0 50 0 30])
```



The peak magnitudes of the three spikes at $\omega=20, 21,$ and 40 are not the same even though the three sinusoids have amplitudes 1 or -1. The peak magnitudes remain different when we use $L=100$, $N=3000$ or some other values.

5.12



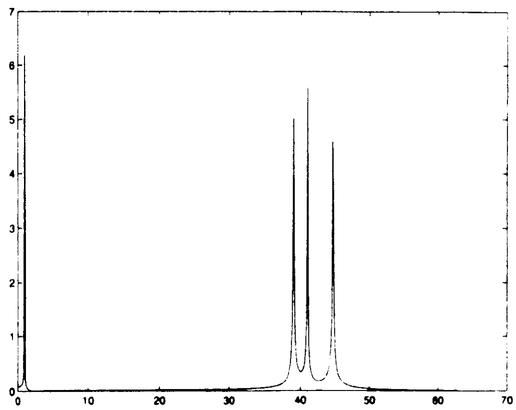
The plot is generated by

```
L=50;N=1000;T=L/N;n=0:N-1;
x=sin(20*n*T)+cos(21*n*T)-sin(40*n*T);
X=T*fft(x,5*N);
mp=0:5*N/2;D=2*pi/(5*N*T);
plot(mp*D,abs(X(mp+1)))
axis([0 50 0 30])
```

It uses 5N-point FFT using 1000 samples of $x(t)$ for t in $[0, 50)$ and 4000 trailing zeros. The peak magnitudes of the three spikes at $\omega=20, 21,$ and 40 are roughly the same. Both plots in Problems 5.11 and 5.12 are FFT computed spectra of $x(t)$ for t in $[0, 50)$, but the latter has a better frequency resolution and is closer to the exact spectrum of $x(t)$.

5.13

```
L=50;N=1000;T=L/N;n=0:N-1;
x=sin(20*n*T).*cos(21*n*T).*sin(40*n*T);
X=T*fft(x);
mp=0:N/2;D=2*pi/(N*T);
plot(mp*D,abs(X(mp+1)))
```

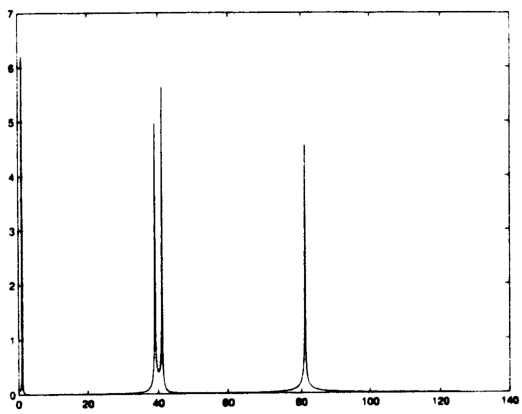


Using the formula, we have

$$x(t) = 0.25\left[\cos 41t - \cos 39t + \cos t - \cos 81t\right]$$

The magnitude spectrum shows spikes at $\omega = 1, 39, 41,$ and 45 rad/s. Because $\omega=81$ does not appear in the plot, there must be frequency aliasing. Indeed for $T = L/N = 0.05$ the NFR is $(-\pi/T, \pi/T] = (-62.8, 62.8)$. Thus the frequency 81 is shifted inside the range as $81 - 125.6 = -44.6$ or 44.6 because of the evenness of magnitude spectrum. Thus the spike at $\omega \approx 45$ is due to aliasing. Note that for this example, even though the computed spectrum is practically zero in the neighborhood of π/T, frequency aliasing still occur. Thus in practice, we shall start with the smallest possible T or largest possible N to find roughly ω_{max} as discussed in Subsection 5.4.1 and Problem 5.10. We can then use $T = \pi/\omega_{max}$.

If N in the program is changed to $N = 2000$, then it will yield



It has four spikes at $\omega = 1, 39, 41$ and 81 and frequency aliasing is probably not significant.

8.14 Direct application

Chapter 6

6.1 $y(t)$ depends on the future input $u(\tau)$ for $t < \tau < t+1$. Thus the system is not causal.

6.2 $y(t)$ does not depend on any future input. Thus the system is causal.

6.3 $y[n]$ depends on $u[n+1]$, a future input. Thus the system is not causal.

6.4 The current output $y[n]$ depends on current input $u[n]$ and past inputs $u[n-1]$ and $u[n-2]$. It does not depend on any future input. Thus the system is causal.

6.5 The system described by

$$y(t) = 1.5\,[u(t)]^2$$

is memoryless because $y(t)$ depends only on $u(t)$. It is time invariant because the coefficient 1.5 is independent of t. It is nonlinear because if $u = u_1 + u_2$, then

$$y(t) = 1.5\,(u_1(t) + u_2(t))^2 = 1.5\,u_1^2(t) + 3\,u_1(t)u_2(t) + 1.5\,u_2^2(t) = y_1(t) + y_2(t) + 3u_1(t)u_2(t) \neq y_1(t) + y_2(t).$$

6.6 The system described by

$$y(t) = (2t+1)\,u(t)$$

is memoryless. It is time varying because the coefficient $(2t+1)$ is a function of time. If $u = u_1 + u_2$, then

$$y(t) = (2t+1)(u_1(t) + u_2(t)) = y_1(t) + y_2(t)$$

and $(2t+1)\,\alpha u_1(t) = \alpha(2t+1)\,u_1(t) = \alpha y_1(t)$.

Thus the system is linear.

6.7 The modulating system

$$y(t) = (\cos \omega_c t)\, u(t)$$

is memoryless. It is time varying because the coefficient $\cos \omega_c t$ is a function of time. If $u = u_1 + u_2$, then

$$y(t) = (\cos \omega_c t)(u_1(t) + u_2(t)) = y_1(t) + y_2(t)$$

$$(\cos \omega_c t)\, \alpha u_1(t) = \alpha (\cos \omega_c t)\, u_1(t) = \alpha y_1(t)$$

Thus the system is linear.

6.8 The DT system described by

$$y[n] = 1.5\,(u[n])^2$$

is memoryless, time-invariant but nonlinear for the same reasons as in Problem 6.5.

6.9 The DT system described by

$$y[n] = (2n+1)\, u[n]$$

is memoryless, because $y[n]$ depends only on $u[n]$. It is time varying because the coefficient $(2n+1)$ is a function of the time index. It is linear as in Problem 6.6.

6.10 If $u = u_1 + u_2$, then

$$y = \alpha(u_1 + u_2) + 1 = \alpha u_1 + 1 + \alpha u_2 + 1 - 1$$

$$= y_1 + y_2 - 1 \neq y_1 + y_2$$

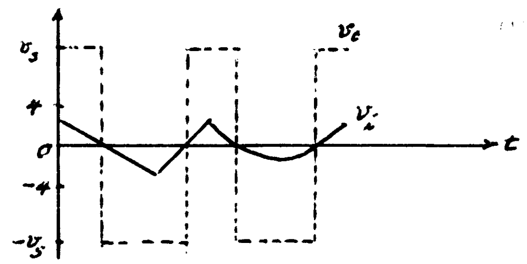
Thus the equation is not linear. If we define $\bar{y} = y - 1$ and $\bar{u} = u$, then $y = \alpha u + 1$ becomes $\bar{y} = \alpha \bar{u}$. This is a linear equation.

6.11 Define $\bar{u}(t) = u(t) - u_0$ and $\bar{y}(t) = y(t) - y_0$. For $\bar{u}(t)$ small, we have

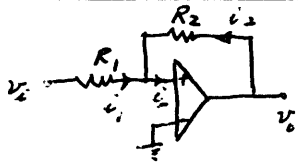
$$\bar{y}(t) = R\, \bar{u}(t)$$

This is a linear time-invariant equation.

6.12



6.13



$e_- = 0$

Because $e_+ = e_-$ we also have $e_+ = 0$

$$i_1 = \frac{v_i - e_+}{R_1} = \frac{v_i}{R_1}, \quad i_2 = \frac{v_o - e_+}{R_2} = \frac{v_o}{R_2}$$

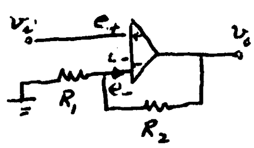
$$i_1 + i_2 - i_- = 0 \quad \text{Because } i_- = 0$$

we have $i_1 = -i_2$ or

$$\frac{v_i}{R_1} = -\frac{v_o}{R_2} \Rightarrow v_o(t) = -\frac{R_2}{R_1} v_i(t)$$

It is the same as (6.19).

6.14



Because $i_- = 0$

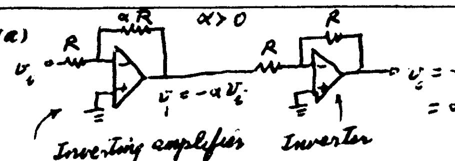
$$e_- = \frac{v_o}{R_1 + R_2} \cdot R_1$$

Because $e_- = e_+ = v_i$

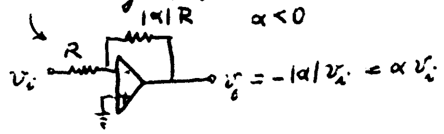
we have $v_i = \dfrac{v_o}{R_1 + R_2} \cdot R_1$ or

$$\frac{v_o}{v_i} = \frac{R_1 + R_2}{R_1} = 1 + \frac{R_2}{R_1}$$

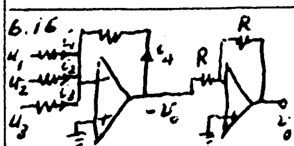
6.15 (a)



(b)



6.16



All resistors have resistance R. $i_k = u_k / R$, $k = 1, 2, 3$

$i_4 = -v_o / R$. $i_1 + i_2 + i_3 + i_4 = 0$

or $\dfrac{u_1}{R} + \dfrac{u_2}{R} + \dfrac{u_3}{R} + \dfrac{-v_o}{R} = 0$

Thus we have $v_o = u_1 + u_2 + u_3$

Chapter 7

7.1 It is not causal because $y[n]$ depends on $u[n+1]$. It is nonlinear because it contains the product of $u[n+1]$ and $u[n-2]$. It is time invariant because its coefficients 2 and 1 are independent of time.

7.2 (1) Causal, time invariant but nonlinear because it contains 10.

(2) Causal, linear and time invariant.

(3) Causal, linear and time invariant.

7.3 For $\beta u[n]$, $n \geq 0$, we have

$$\frac{(\beta u[n])^2}{\beta u[n-1]} = \beta \frac{u^2[n]}{u[n-1]} = \beta y[n]$$

Thus the equation meets the homogeneity property. If $u = u_1 + u_2$, then

$$\frac{(u_1[n]+u_2[n])^2}{u_1[n-1]+u_2[n-1]} \neq \frac{u_1^2[n]}{u_1[n-1]} + \frac{u_2^2[n]}{u_2[n-1]}$$

Thus the equation does not meet the additivity property.

7.4 We use $y = \mathcal{L}[u]$ to denote $u[n], n \geq 0 \rightarrow y[n], n \geq 0$. Let p be any positive integer. Applying the additivity property p times, we have

$$\mathcal{L}[pu] = p\mathcal{L}[u] = py$$

Let q be a positive integer. Define $u = qv$. Then $\mathcal{L}[u] = \mathcal{L}[qv] = q\mathcal{L}[v]$ which implies

$$\mathcal{L}[v] = \mathcal{L}[\tfrac{1}{q}u] = \tfrac{1}{q}\mathcal{L}[u]$$

Let $\alpha = p/q$ by any positive rational number. Then

$$\mathcal{L}[\alpha u] = \mathcal{L}[\tfrac{p}{q}u] = p\mathcal{L}[\tfrac{1}{q}u] = \tfrac{p}{q}\mathcal{L}[u]$$

$$= \alpha\mathcal{L}[u].$$

Because $\mathcal{L}[u+0] = \mathcal{L}[u] + \mathcal{L}[0] = \mathcal{L}[u]$, we have $\mathcal{L}[0] = 0$.

$$\mathcal{L}[\alpha u + (-\alpha u)] = \mathcal{L}[\alpha u] + \mathcal{L}[-\alpha u] = 0$$

which implies

$$\mathcal{L}[-\alpha u] = -\mathcal{L}[\alpha u] = -\alpha\mathcal{L}[u]$$

Thus additivity implies homogeneity for any positive or negative rational number.

7.5 $h[n] = 2\delta_d[n-1] - 4\delta_d[n-3]$

$h[n] = 0$ for all $n < 0$

$h[0] = 2\delta_d[-1] - 4\delta_d[-3] = 0$

$h[1] = 2\delta_d[0] - 4\delta_d[-2] = 2$

$h[2] = 2\delta_d[1] - 4\delta_d[-1] = 0$

$h[3] = 2\delta_d[2] - 4\delta_d[0] = -4$

$h[4] = 2\delta_d[3] - 4\delta_d[1] = 0$

$h[n] = 0$ for $n = 5, 6, \cdots$

Thus the impulse response of

$$y[n] = 2u[n-1] - 4u[n-3]$$

is $h[1] = 2$, $h[3] = -4$ and $h[n] = 0$ for all n other than 1 and 3. It is FIR.

7.6 The impulse response of $y[n] = 2.5u[n]$ is $h[n] = 2.5\delta_d[n] = \begin{cases} 2.5, & n = 0 \\ 0, & n \neq 0 \end{cases}$

It is FIR.

7.7 $y[n] = \frac{1}{4}\{u[n] + u[n-1] + u[n-2] + u[n-3]\}$

It is a 4-point moving average with impulse response

$$h[n] = \begin{cases} 1/4 = 0.25 & \text{for } n = 0:3, \\ 0 & \text{otherwise.} \end{cases}$$

It is FIR.

7.8 (1) We write $2y[n]-3y[n-1]=4u[n]$

as $y[n]=1.5y[n-1]+2u[n]$

Substituting $u[n]=\delta_d[n]$ and using $y[n]=0$, for $n<0$, we compute recursively

$n=0$: $y[0]=1.5y[-1]+2\delta_d[0]=2$

$n=1$: $y[1]=1.5y[0]+2\delta_d[1]=1.5\times 2$

$n=2$: $y[2]=1.5y[1]+2\delta_d[2]=2\times(1.5)^2$

$n=3$: $y[3]=1.5y[2]+2\delta_d[3]=2\times(1.5)^3$

$\vdots$

$y[n]=2\times(1.5)^n$

$\vdots$

Thus the impulse response is

$h[n]=2\times(1.5)^n$, for all $n\geq 0$

$h[n]=0$, for all $n<0$, because the system is causal. The system is IIR system.

(2) We write the equation as

$y[n]=-2y[n-1]-2\delta_d[n-1]-\delta_d[n-2]+6\delta_d[n-3]$

We compute recursively, using $y[n]=0$, for $n<0$,

$n=0$: $y[0]=-2y[-1]-2\delta_d[-1]-\delta_d[-2]+6\delta_d[-3]=0$

$n=1$: $y[1]=-2y[0]-2\delta_d[0]-\delta_d[-1]+6\delta_d[-2]=-2$

$n=2$: $y[2]=-2y[1]-2\delta_d[1]-\delta_d[0]+6\delta_d[-1]=(-2)\times(-2)-1=4-1=3$

$n=3$: $y[3]=-2y[2]-2\delta_d[2]-\delta_d[1]+6\delta_d[0]=(-2)\times 3+6=0$

$n=4$: $y[4]=-2y[3]-2\delta_d[3]-\delta_d[2]+6\delta_d[1]=0$

$n\geq 5$: $y[n]=0$

Thus the impulse response of the system is $h[0]=0$, $h[1]=-2$, $h[2]=3$, $h[n]=0$ for $n\geq 3$ and $n<0$. It is FIR.

7.9 Consider

$$y[n]=\sum_{k=0}^{\infty}h[n-k]u[k]$$

with $h[0]=3$, $h[1]=-2$, $h[2]=0$, $h[3]=5$, and $h[n]=0$ for all $n\geq 4$ and all $n<0$. Note that if a system is causal, then $h[n]=0$ for $n<0$ which implies $h[n-k]=0$ for all $k>n$. Thus the infinite summation can be reduced as

$$y[n]=\sum_{k=0}^{n}h[n-k]u[k]$$

for all causal systems. We write it expressively, starting from $k=n, n-1, \cdots, 1, 0$, as

$y[n]=h[0]u[n]+h[1]u[n-1]+h[2]u[n-2]+h[3]u[n-3]+h[4]u[n-4]+\cdots+h[n-1]u[1]+h[n-0]u[0]$

which yields, by direct substitution,

$y[n]=3u[n]-2u[n-1]+0\cdot u[n-2]+5u[n-3]+0\cdot u[n-4]+\cdots+0\cdot u[0]$

$=3u[n]-2u[n-1]+5u[n-3]$

This is Eq. (7.5).

7.10 Let $y_i[n]=\sum_{k=0}^{\infty}h[n-k]u_i[k]$, $i=1,2$.

If $u_3[n]=u_1[n]+u_2[n]$, for all $n\geq 0$, then

$$y_3[n]=\sum_{k=0}^{\infty}h[n-k]u_3[k]=\sum_{k=0}^{\infty}h[n-k](u_1[k]+u_2[k])$$

$$=\sum_{k=0}^{\infty}h[n-k]u_1[k]+\sum_{k=0}^{\infty}h[n-k]u_2[k]$$

$=y_1[n]+y_2[n]$ (Additivity)

If $u_4[n]=\beta u_1[n]$, then

$$y_4[n]=\sum_{k=0}^{\infty}h[n-k]u_4[k]=\sum_{k=0}^{\infty}h[n-k]\beta u_1[n]$$

$$=\beta\sum_{k=0}^{\infty}h[n-k]u_1[k]=\beta y_1[n]$$ (homogeneity)

7.11 $$y[n]=\sum_{k=0}^{\infty}h[n-k]u[k]$$

Let $u_1[n]=u[n-n_1]$. Here we require $u_1[n]=0$ for $n<n_1$. Thus we have

$$y_1[n] = \sum_{k=0}^{\infty} h[n-k]u_1[k]$$

$$= \sum_{k=n_1}^{\infty} h[n-k]u[k-n_1]$$

Define $\bar{k} = k - n_1$. Then $k = n_1 + \bar{k}$, and

$$y_1[n] = \sum_{\bar{k}=0}^{\infty} h[n-n_1-\bar{k}]u[\bar{k}] = y[n-n_1]$$

This shows the shifting property.

7.12 If $h[n] = 1.00015^n$, for $n \geq 0$, and $h[n]=0$, for $n<0$, then $h[n-k]=0$, for all $k>n$. Thus we have

$$y[n] = \sum_{k=0}^{\infty} h[n-k]u[k] = \sum_{k=0}^{n} (1.00015)^{n-k} u[k]$$

The equation holds for all n and we can write

$$y[n-1] = \sum_{k=0}^{n-1} (1.00015)^{n-1-k} u[k]$$

$$= (1.00015)^{-1} \sum_{k=0}^{n-1} (1.00015)^{n-k} u[k]$$

Substituting this equation into

$$y[n] = \sum_{k=0}^{n} (1.00015)^{n-k} u[k]$$

$$= \sum_{k=0}^{n-1} (1.00015)^{n-k} u[k] + (1.00015)^{n-n} u[n]$$

$$= 1.00015\, y[n-1] + u[n]$$

yields

$$y[n] - 1.00015\, y[n-1] = u[n]$$

7.13 The 90-day moving average is defined by

$$y[n] = \frac{1}{90}\left(u[n] + u[n-1] + \cdots + u[n-89]\right)$$

It is a nonrecursive difference equation of order 89. Its impulse response is

$$h[n] = \begin{cases} 1/90 & \text{for } n = 0:89 \\ 0 & \text{for } n<0 \text{ and } n \geq 90 \end{cases}$$

Its convolution description is

$$y[n] = \sum_{k=0}^{\infty} h[n-k]u[k] = \sum_{k=n-89}^{n} \frac{1}{90} u[k]$$

$$= \frac{1}{90} \sum_{k=n-89}^{n} u[k]$$

which is the same as the preceding equation. The equation holds for all n and we can write

$$y[n-1] = \frac{1}{90}\left(u[n-1] + u[n-2] + \cdots + u[n-89] + u[n-90]\right)$$

Thus we have

$$y[n] - y[n-1] = \frac{1}{90}\left(u[n] - u[n-90]\right)$$

This is a recursive difference equation of order 90. It requires less computation.

7.14 $\mathbf{c} = [1\ 1\ 1]$, a 1×3 row vector.

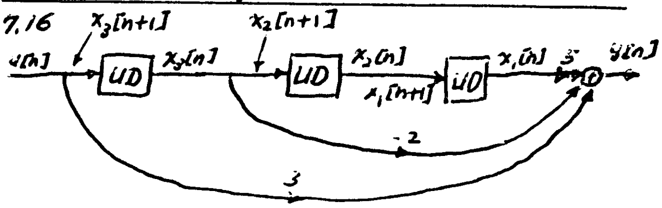
$$y[n] = \mathbf{c}\mathbf{u}[n] = [1\ \ 1\ \ 1]\begin{bmatrix} u_1[n] \\ u_2[n] \\ u_3[n] \end{bmatrix}$$

$$= u_1[n] + u_2[n] + u_3[n]$$

It is linear and time invariant.

7.15 Direct verification

7.16



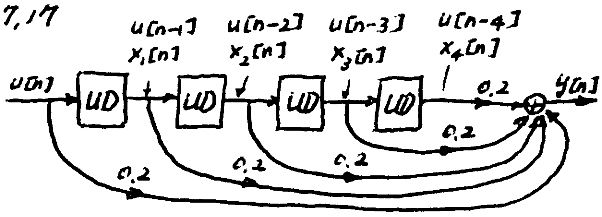
$$x_1[n+1] = x_2[n]$$
$$x_2[n+1] = x_3[n]$$
$$x_3[n+1] = u[n]$$
$$y[n] = 5x_1[n] - 2x_3[n] + 3u[n]$$

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[n]$$

$$y[n] = [5\ \ 0\ \ -2]\begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix} + 3u[n]$$

They differ from (7.18) and (7.19). But the two sets of equations are equivalent.

7.17



This is the basic block diagram. Now if we assign the output of each unit-delay element as a state variable as shown, then we have

$$x_1[n+1] = u[n]$$
$$x_2[n+1] = x_1[n]$$
$$x_3[n+1] = x_2[n]$$
$$x_4[n+1] = x_3[n]$$
$$y[n] = 0.2x_1[n] + 0.2x_2[n] + 0.2x_3[n] + 0.2x_4[n] + 0.2u[n]$$

Thus its ss equation is

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \\ x_4[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \\ x_4[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u[n]$$

$$y[n] = [0.2 \;\; 0.2 \;\; 0.2 \;\; 0.2]\,\mathbf{x}[n] + 0.2u[n]$$

7.18

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \end{bmatrix} = \begin{bmatrix} 2 & -2.3 \\ -1.3 & 1.6 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \end{bmatrix} u[n]$$

$$y[n] = [-1.5 \;\; -3.1] \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + 4u[n]$$

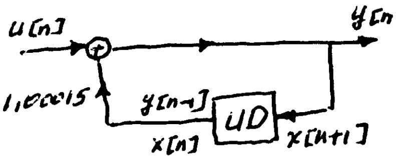
7.19 $y[n] - 1.00015\,y[n-1] = u[n]$

Define $x[n] = y[n-1]$. Then $y[n] = x[n+1]$. Thus we have

$$x[n+1] = 1.00015\,x[n] + u[n]$$
$$y[n] = 1.00015\,x[n] + u[n]$$

This is a ss equation of dimension 1.

7.20



Its ss equation is the same as the one in Problem 7.19.

7.21 $\mathcal{Z}[\delta_d[n-1]] = z^{-1}$

$$\mathcal{Z}[2\delta_d[n] - 3\delta_d[n-4]] = 2 - 3z^{-4}$$

7.22 If the savings account is initially relaxed at $n_0 = 0$, we may assume $y[n] = 0$ and $u[n] = 0$ for all $n < 0$. Thus $y[n]$ and $u[n]$ are positive-time sequences. Using the linearity property of the z-transform, we have

$$Y(z) - 1.00015z^{-1}Y(z) = U(z)$$

or $(1 - 1.00015z^{-1})Y(z) = U(z)$

which implies

$$\frac{Y(z)}{U(z)} = \frac{1}{1 - 1.00015z^{-1}} = \frac{z}{z - 1.00015}$$

7.23 $y[n] = 2u[n-1] - 4u[n-3]$.

Its impulse response was computed in Problem 7.5 as $h[1] = 2$, $h[3] = -4$ and $h[n] = 0$ for all n other than 1 and 3. Thus we have

$$h[n] = 2\delta_d[n-1] - 4\delta_d[n-3]$$
$$H(z) = \mathcal{Z}[h[n]] = 2z^{-1} - 4z^{-3} = \frac{2z^2 - 4}{z^3}$$

Applying the z-transform and assuming initial relaxedness ($u[n] = 0$, $y[n] = 0$, for all $n < 0$), we have

$$Y(z) = 2z^{-1}U(z) - 4z^{-3}U(z)$$
$$= (2z^{-1} - 4z^{-3})\,U(z)$$

or $$\frac{Y(z)}{U(z)} = 2z^{-1} - 4z^{-3} = \frac{2z^2 - 4}{z^3}$$

7.24 $y[n] = \frac{1}{5}(u[n]+u[n-1]+u[n-2]+u[n-3]+u[n-4])$

Applying the z-transform and assuming initial relaxedness, we have

$$Y(z) = \frac{1}{5}\left(U(z)+z^{-1}U(z)+z^{-2}U(z)+z^{-3}U(z)+z^{-4}U(z)\right)$$

$$= \frac{1}{5}(1+z^{-1}+z^{-2}+z^{-3}+z^{-4})U(z)$$

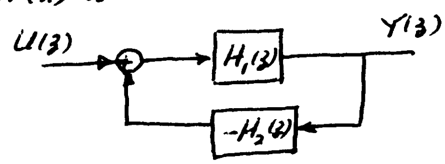
Thus we have

$$\frac{Y(z)}{U(z)} = \frac{1+z^{-1}+z^{-2}+z^{-3}+z^{-4}}{5}$$

$$= \frac{z^4+z^3+z^2+z+1}{5z^4}$$

7.25 Consider two systems with transfer functions $H_1(z)$ and $H_2(z)$. The transfer function of the tandem connection of $H_1(z)$ followed by $H_2(z)$ is $H_2(z)H_1(z)$. The transfer function of the tandem connection of $H_2(z)$ followed by $H_1(z)$ is $H_1(z)H_2(z)$. If both systems are single input and single out, then both $H_1(z)$ and $H_2(z)$ are 1×1 and we have $H_1(z)H_2(z) = H_2(z)H_1(z)$. Thus for SISO systems, we can interchange the order of tandem connection. For MIMO systems, generally we cannot interchange their order of connection. For example, if $H_1(z)$ is 2×2 and if $H_2(z)$ is 2×1, then $H_1(z)H_2(z)$ is defined but $H_2(z)H_1(z)$ is not defined. Even if both are 2×2 and both $H_1(z)H_2(z)$ and $H_2(z)H_1(z)$ are defined, generally we have $H_1(z)H_2(z) \neq H_2(z)H_1(z)$.

7.26 Using Fig. 7.10, we can draw Fig. 7.11(a) as



Using Fig. 7.8(c), we have

$$\frac{Y(z)}{U(z)} = \frac{H_1(z)}{1-H_1(z)(-H_2(z))} = \frac{1}{1+H_1(z)H_2(z)}$$

Chapter 8

8.1 Equations (8.10), (8.11), and (8.12) are directly applicable. The voltage across the 5H inductor is, from Fig. 5.1(a),

$$y(t) = 5\dot{x}_1(t) = -x_3(t) + u(t)$$

where we have used (8.10). Thus its ss equation is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 & -0.2 \\ 0 & -0.15 & 0.25 \\ 0.5 & -0.5 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0.2 \\ 0 \\ 0 \end{bmatrix}u(t)$$

$$y(t) = [0 \quad 0 \quad -1]\,\mathbf{x}(t) + u(t)$$

Note that the state equation is the same as (8.14). The output equation however is different from (8.15)

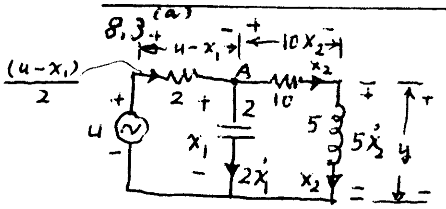
8.2 From Fig. 5.1(a), we see that the current passing through the 2-F capacitor is $2\dot{x}_3(t)$. Thus we have, using (8.12),

$$y = 2\dot{x}_3(t) = x_1(t) - x_2(t)$$

$$= [1 \quad -1 \quad 0]\,\mathbf{x}(t) + 0\times u(t)$$

This is the output equation. Its state equation is the same as (8.14) or the one in Problem 8.1.

8.3 (a)



If we select the capacitor voltage as x_1, then its current is $2\dot{x}_1$. If the inductor current is assigned as x_2, then its voltage is $5\dot{x}_2$, with polarity shown. The voltage across the 10 Ω resistor is $x_1 - 5\dot{x}_2$. But its current is simply x_2. If we select its current x_2, then the voltage across the 10-Ω resistor is $10x_2$ as shown. The 2-Ω resistor has current $2\dot{x}_1 + x_2$ and voltage $u - x_1$. We select the voltage because it contains no derivative. Then the current passing through the 2-Ω resistor is $(u-x_1)/2$. Applying the KCL at node A yields

$$2\dot{x}_1 = \frac{u - x_1}{2} - x_2$$

which implies

$$\dot{x}_1 = -\frac{1}{4}x_1 - \frac{1}{2}x_2 + \frac{1}{4}u$$

Applying the KVL along the right-hand-side loop yields

$$5\dot{x}_2 = x_1 - 10x_2$$

which implies

$$\dot{x}_2 = \frac{1}{5}x_1 - 2x_2$$

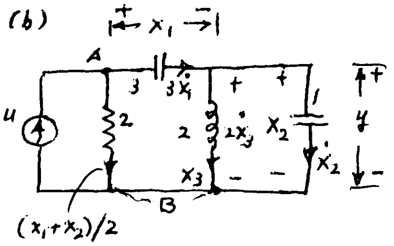
$$y = 5\dot{x}_2 = x_1 - 10x_2$$

Thus the circuit can be described by

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}-0.25 & -0.5\\ 0.2 & -2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0.25\\ 0\end{bmatrix}u(t)$$

$$y(t) = [1 \quad -10]\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + 0 \times u(t)$$

(b)



The voltage across the 2-Ω resistor is $x_1 + x_2$. Thus its current is $(x_1+x_2)/2$.

At node A: $3\dot{x}_1 = u - (x_1 + x_2)/2$

$$\therefore \dot{x}_1 = -\frac{1}{6}x_1 - \frac{1}{6}x_2 + \frac{1}{3}u$$

At node B: $u = \frac{x_1+x_2}{2} + x_3 + \dot{x}_2$

$$\therefore \dot{x}_2 = \frac{-1}{2}x_1 - \frac{1}{2}x_2 - x_3 + u$$

Right-hand-side loop: $2\dot{x}_3 = x_2$

$$\therefore \dot{x}_3 = \frac{1}{2}x_2$$

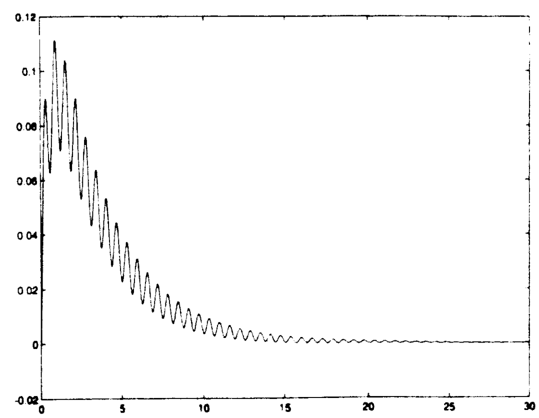
We also have $y = x_2$. Thus the circuit is described by

$$\dot{x}(t) = \begin{bmatrix}-1/6 & -1/6 & 0\\ -1/2 & -1/2 & -1\\ 0 & 1/2 & 0\end{bmatrix}x(t) + \begin{bmatrix}1/3\\ 1\\ 0\end{bmatrix}u(t)$$

$$y(t) = [0 \quad 1 \quad 0]\,x(t) + 0 \cdot u(t)$$

8.4

```
a=[-0.25 -0.5;0.2 -2];b=[0.25;0];
c=[1 -10];d=0;dog=ss(a,b,c,d);
t1=0:0.01:30;
u1=1+exp(-0.2*t1).*sin(10*t1);
y1=lsim(dog,u1,t1);
t2=0:0.001:30;
u2=1+exp(-0.2*t2).*sin(10*t2);
y2=lsim(dog,u2,t2);
plot(t1,y1,t2,y2,':')
```

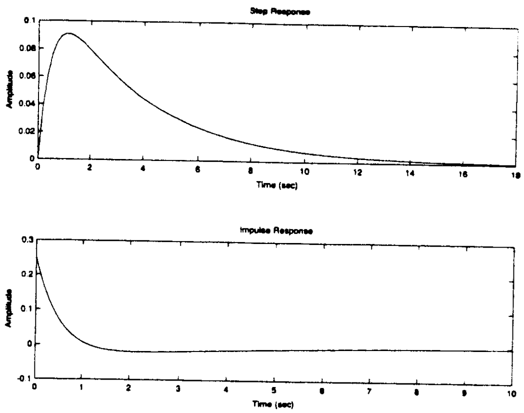


The program generates two plots, one with step size 0.01 (solid line) and the other with 0.001 (dotted line). The two plots overlap. Thus the selection of 0.01 is small enough and the generated response is close to the exact one.

8.5

```
a=[-0.25 -0.5;0.2 -2];b=[0.25;0];
c=[1 -10];d=0;dog=ss(a,b,c,d);
subplot(2,1,1)
step(dog)
subplot(2,1,2)
impulse(dog)
```

The program generates



8.6 All we need to show is that the input $u(t) = \delta(t)$ will move the initial state from $x(0) = 0$ to $x(0) = b$. Indeed, the integration of (8.16) yields, for $\varepsilon \to 0$,

$$\int_0^{\varepsilon} \dot x(t)\,dt = x(\varepsilon) - x(0)$$

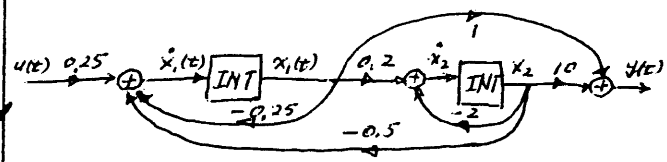
$$= \int_0^{\varepsilon} A x(\tau)\,d\tau + \int_0^{\varepsilon} b\,u(\tau)\,d\tau = 0 + b\int_0^{\varepsilon}\delta(\tau)\,d\tau$$

$$= b$$

Thus the impulse response of (8.16) and (8.17) with $d = 0$ (output excited by $u(t) = \delta(t)$ and $x(0) = 0$) equals the zero-input response (output excited by $u(t) = 0$ for all $t \geq 0$ and $x(0) = b$.)

8.7 (a) $\dot x_1 = -0.25x_1 - 0.5x_2 + 0.25u$

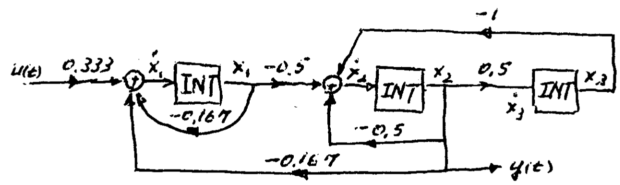
$\dot x_2 = 0.2x_1 - 2x_2$, $y = x_1 + 10x_2$



(b) $\dot x_1 = -0.167x_1 - 0.167x_2 + 0.333u$

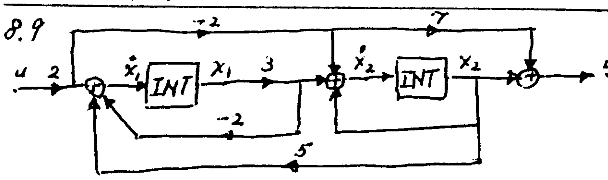
$\dot x_2 = -0.5x_1 - 0.5x_2 - x_3$

$\dot x_3 = 0.5x_2$, $y = x_2$



8.8 The basic block diagram in Prob 8.7(a) has 3 adders (each needs 2 op amps), 2 integrators (each needs 2 op amps), 3 positive gains (each needs 2 op amps), and 3 negative gains (each needs 1 op amp). Thus the diagram requires a total of 19 op amps.

8.9



$\dot x_1 = -6x_1 + 5x_2 + 2u$

$\dot x_2 = 3x_1 + x_2 - 4u$

$y = x_2 - 28u$

$$\dot x = \begin{bmatrix} -6 & 5 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ -4 \end{bmatrix} u$$

$$y = [0 \quad 1]x - 28u$$

8.10 $e_- = e_+ = 0$, $i_1 = \frac{v_1 - e_-}{R/a} = \frac{av_1}{R}$, $i_2 = \frac{bv_2}{R}$, $i_3 = \frac{cv_3}{R}$. The current flowing from the output into the inverting terminal is $C\dot x$. Thus we have

$$\frac{av_1}{R} + \frac{bv_2}{R} + \frac{cv_3}{R} + C\dot x = 0$$

or

$$\dot x = -\frac{1}{RC}(av_1 + bv_2 + cv_3)$$

If $RC=1$, then

$$\dot{x} = -(av_1 + bv_2 + cv_3)$$

8.11 If we assign the output as $-x(t)$, then the current flowing from the output into the inverting terminal is $-C\dot{x}(t)$. Thus we have

$$\frac{av_1}{R} + \frac{bv_2}{R} + \frac{cv_3}{R} - C\dot{x} = 0$$

or, using $RC=1$,

$$\dot{x} = av_1 + bv_2 + cv_3$$

8.12 From Fig. 8.17 and using the results in Problem 8.10 and 8.11, we have

$$\dot{x}_1 = 0.25u + 0.5(-x_2) + 0.25(-x_1)$$

$$\dot{x}_2 = -(0.2(-x_1) + 2x_2)$$

$$y = -(-x_1 + 10x_2)$$

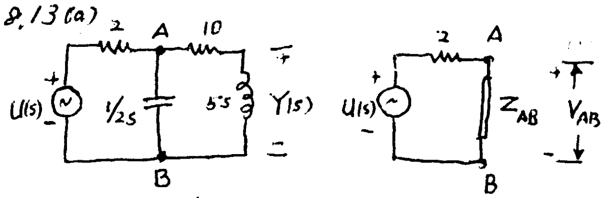
or
$$\dot{x}_1 = -0.25x_1 - 0.5x_2 + 0.25u$$

$$\dot{x}_2 = 0.2x_1 - 2x_2$$

$$y = x_1 - 10x_2$$

This set of equations is identical to the one in Problem 8.3(a). Thus the op-amp circuit in Fig. 8.17 implements the ss equation in Problem 8.3(a) and uses 4 op amps. If the ss equation is implemented as a basic block diagram and then replace every basic element by its op-amp circuit implementation, then it requires as discussed in Prob. 8.8 19 op amps. Thus there are many ways of implementing an ss equation using op-amp circuits. In practice, we should search the one which uses the smallest number of components.

8.13 (a)



$$Z_{AB}(s) = \frac{\frac{1}{2s}\times(10+5s)}{\frac{1}{2s} + 10s + 5s} = \frac{5s+10}{10s^2+20s+1}$$

$$V_{AB}(s) = \frac{Z_{AB}}{Z_{AB}+2}U(s) = \frac{\frac{5s+10}{10s^2+20s+1}}{\frac{5s+10}{10s^2+20s+1}+2}U(s)$$

$$= \frac{5s+10}{20s^2+45s+12}U(s)$$

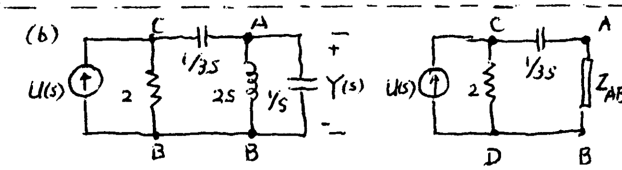
$$Y(s) = \frac{5s}{5s+10}V_{AB}(s) = \frac{5s}{5s+10}\times\frac{5s+10}{20s^2+45s+12}U(s)$$

$$= \frac{5s}{20s^2+45s+12}U(s)$$

Thus the transfer function from u to y is

$$H(s) = \frac{Y(s)}{U(s)} = \frac{5s}{20s^2+45s+12}$$

(b)



$$Z_{AB} = \frac{2s\times\frac{1}{s}}{2s+\frac{1}{s}} = \frac{2s}{2s^2+1}$$

$$Z_{CD} = \frac{2\times(\frac{1}{3s}+Z_{AB})}{2+\frac{1}{3s}+Z_{AB}} = \frac{2\left(\frac{2s^2+1+6s^2}{3s(2s^2+1)}\right)}{\frac{6s(2s^2+1)+2s^2+1+6s^2}{3s(2s^2+1)}}$$

$$= \frac{2(8s^2+1)}{12s^3+8s^2+6s+1}$$

$$V_{CD} = Z_{CD}\,U(s)$$

$$Y(s) = V_{AB}(s) = \frac{Z_{AB}}{\frac{1}{3s}+Z_{AB}}V_{CD} = \frac{\frac{2s}{2s^2+1}}{\frac{1}{3s}+\frac{2s}{2s^2+1}}V_{CD}$$

$$= \frac{6s^2}{8s^2+1}\cdot\frac{2(8s^2+1)}{12s^3+8s^2+6s+1}\cdot U(s)$$

$$= \frac{12s^2}{12s^3+8s^2+6s+1}U(s)$$

Thus the transfer function from u to y is

$$H(s) = \frac{Y(s)}{U(s)} = \frac{12s^2}{12s^3+8s^2+6s+1}$$

8.14 -10 : proper, biproper

$\frac{2s^2+1}{3s-1}$ improper

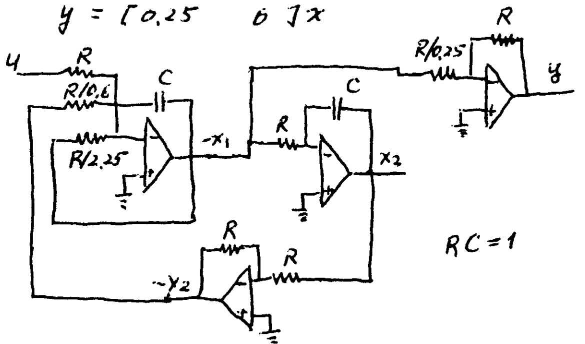
$\frac{2s^2+1}{3s^2-1}$ proper, biproper

$\frac{2s^2+1}{s^{10}}$ proper, strictly proper

8.15 $H(s) = \frac{5s}{20s^2+45s+12} = \frac{0.25s}{s^2+2.25s+0.6}$

$$\dot{x} = \begin{bmatrix} -2.25 & -0.6 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [0.25 \quad 0]x$$



The ss equation looks different from the one in Prob. 8.3(a). They are however equivalent and describe the same RLC circuit. The numbers of capacitors and op amps used in this implementation and Fig 8.17 are the same. However the former uses 8 resistors, the latter uses 10 resistors. The state variables in Prob. 8.3(a) are associated with the capacitor voltage and the inductor current. The meaning of the state variables in this implementation however are not transparent.

8.16 $\frac{Y(s)}{U(s)} = \frac{12s^2}{12s^3+8s^2+6s+1}$

$$(12s^3+8s^2+6s+1)Y(s) = 12s^2 U(s)$$

$$12s^3 Y(s) + 8s^2 Y(s) + 6s Y(s) + Y(s) = 12s^2 U(s)$$

$$12y^{(3)}(t) + 8\ddot{y}(t) + 6\dot{y}(t) + y(t) = 12\ddot{u}(t)$$

This third-order differential equation describes the circuit in Fig 8.14(b).

8.17 $H_1(s) = \frac{3s^2+1}{2s^2+4s+5} = \frac{1.5s^2+0.5}{s^2+2s+2.5}$

$$= 1.5 + \frac{-3s-3.25}{s^2+2s+2.5}$$

$$\dot{x} = \begin{bmatrix} -2 & -2.5 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [-3 \quad -3.25]x + 1.5u$$

$H_2(s) = \frac{1}{2s^3} = \frac{0.5}{s^3} = \frac{0 \cdot s^2 + 0 \cdot s + 0.5}{s^3+0 \cdot s^2+0 \cdot s+0}$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 0 \quad 0.5]x + 0 \cdot u$$

8.18 Taking the Laplace transform and assuming zero initial condition or $x(0)=0$, we have

$$sX(s) = -aX(s) + U(s)$$
$$Y(s) = kX(s) + dU(s)$$

From the first equation, we have

$$(s+a)X(s) = U(s) \quad \text{or} \quad X(s) = \frac{1}{s+a}U(s)$$

Thus we have

$$Y(s) = \frac{k}{s+a}U(s) + dU(s) = \left(\frac{k}{s+a} + d\right)U(s)$$

and the transfer function from u to y is

$$H(s) = \frac{Y(s)}{U(s)} = \frac{k}{s+a} + d$$

8.19 $y^{(3)}(t) + a_2\ddot{y}(t) + a_3\dot{y}(t) + a_4 y(t) = bu(t)$.

Let us define $x_1(t) = \ddot{y}(t)$, $x_2(t) = \dot{y}(t)$, $x_3(t) = y(t)$

Then we have

$$\dot{x}_3(t) = \dot{y}(t) = x_2(t)$$
$$\dot{x}_2(t) = \ddot{y}(t) = x_1(t)$$
$$\dot{x}_1(t) = \dddot{y}(t) = -a_2\ddot{y} - a_3\dot{y} - a_4 y + bu$$
$$= -a_2 x_1 - a_3 x_2 - a_4 x_3 + bu$$

or

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 0 \quad 1]\underline{x}(t) + 0 \cdot u(t)$$

Let us define $x_1(t)=y(t)$, $x_2(t)=\dot{y}(t)$, $x_3(t)=\ddot{y}(t)$

The we have

$$\dot{x}_1(t)=\dot{y}(t)=x_2(t)$$
$$\dot{x}_2(t)=\ddot{y}(t)=x_3(t)$$
$$\dot{x}_3(t)=\dddot{y}(t)=-a_2x_3(t)-a_3x_2(t)-a_4x_1(t)+bu(t)$$

Thus we have

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-a_4&-a_3&-a_2\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}+\begin{bmatrix}0\\0\\b\end{bmatrix}u(t)$$

$$y(t)=[1\quad 0\quad 0]\,\underset{\sim}{x}(t)+0\cdot u(t)$$

8.20 $$\frac{Y(s)}{U(s)}=\frac{b_1s^2+b_2s+b_3}{s^3+a_2s^2+a_3s+a_4}$$

Define $$\frac{U(s)}{V(s)}:=s^3+a_2s^2+a_3s+a_4 \qquad (1)$$

Then we have

$$\frac{Y(s)}{V(s)}=\frac{Y(s)}{U(s)}\cdot\frac{U(s)}{V(s)}=b_1s^2+b_2s+b_3 \quad (2)$$

In the time domain, (1) and (2) becomes

$$v^{(3)}(t)+a_2\ddot{v}(t)+a_3\dot{v}(t)+a_4v(t)=u(t)$$

$$y(t)=b_1\ddot{v}(t)+b_2\dot{v}(t)+b_3v(t)$$

Define $x_1=\ddot{v}$, $x_2=\dot{v}$, $x_3=v$. Then we have

$$\dot{x}_1=\dddot{v}=-a_2x_1-a_3x_2-a_4x_3+u$$
$$\dot{x}_2=\ddot{v}=x_1$$
$$\dot{x}_3=\dot{v}=x_2$$
$$y=b_1x_1+b_2x_2+b_3x_3$$

or

$$\dot{\underset{\sim}{x}}=\begin{bmatrix}-a_2&-a_3&-a_4\\1&0&0\\0&1&0\end{bmatrix}\underset{\sim}{x}+\begin{bmatrix}1\\0\\0\end{bmatrix}u$$

$$y=[b_1\quad b_2\quad b_3]\,\underset{\sim}{x}$$

where $\underset{\sim}{x}=[x_1\ x_2\ x_3]'$. Note that it is difficult to develop directly from (8.83), without defining (1), the ss equation, because the relationships between x_i and $\{u, y\}$ are not transparent. Recall from Prob. 8.19, if $b_1=0$ and $b_2=0$, then we have $x_1=\ddot{y}$, $x_2=\dot{y}$, and $x_3=y$. If $b_1\neq 0$ and/or $b_2\neq 0$, then we can no longer use $x_1=\ddot{y}$, $x_2=\dot{y}$, and $x_3=y$.

8.21 $$H_1=\frac{2(s-1)(s+3)}{s^3+5s^2+8s+6}=\frac{2s^2+4s-6}{s^3+5s^2+8s+6}=\frac{N_1(s)}{D_1(s)}$$

Its realization is

$$\dot{x}=\begin{bmatrix}-5&-8&-6\\1&0&0\\0&1&0\end{bmatrix}x+\begin{bmatrix}1\\0\\0\end{bmatrix}u$$

$$y=[2\quad 4\quad -6]\,x$$

It has dimension 3. $N_1(s)$ has roots 1 and -3. We compute

$$D_1(1)=1+5+8+6\neq 0$$
$$D_1(-3)=(-3)^3+5(-3)^2+8(-3)+6$$
$$=-27+45-24+6=0$$

Thus $D_1(s)$ has the root -3 and $N_1(s)$ and $D_1(s)$ are not coprime. We compute

$$\begin{array}{r|l} & s^2+2s+2\\ \hline s+3 & s^3+5s^2+8s+6\\ & s^3+3s^2\\ \hline & \quad 2s^2+8s+6\\ & \quad 2s^2+6s\\ \hline & \qquad 2s+6\\ & \qquad 2s+6\\ \hline & \qquad\quad 0\end{array}$$

Thus we have

$$H_1(s)=\frac{2(s-1)(s+3)}{(s+3)(s^2+2s+2)}=\frac{2(s-1)}{s^2+2s+2}$$

$H_1(s)$ has degree 2, Thus the preceding realization with dimension 3 is not a minimal realization. The following

$$\dot{x}=\begin{bmatrix}-2&-2\\1&0\end{bmatrix}x+\begin{bmatrix}1\\0\end{bmatrix}u$$

$$y=[2\quad -2]\,x$$

of dimension 2 is a minimal realization.

$$H_2(s)=\frac{s^3}{s^3+2s-1}=\frac{N_2(s)}{D_2(s)}$$

$N_2(s)$ has roots 0, 0, 0. $D_2(s)$ has no root at 0 because $D_2(0)=-1\neq 0$. Thus $N_2(s)$ and $D_2(s)$ are coprime and $H_2(s)$ has degree 3.

$$H_2(s)=\frac{s^3}{s^3+2s-1}=1+\frac{-2s+1}{s^3+2s-1}$$
$$=\frac{0\cdot s^2-2s+1}{s^3+0\cdot s^2+2s-1}+1$$

Its minimal realization is

$$\dot{x}=\begin{bmatrix}0&-2&1\\1&0&0\\0&1&0\end{bmatrix}x+\begin{bmatrix}1\\0\\0\end{bmatrix}u$$

$$y=[0\ \ -2\ \ 1]x+1\times u$$

8.22 $$\dot{x}=\begin{bmatrix}-2&-10\\1&0\end{bmatrix}x+\begin{bmatrix}1\\0\end{bmatrix}u$$

$$y=[3\ \ 4]x-2u(t)$$

$$(sI-A)^{-1}=\left(\begin{bmatrix}s&0\\0&s\end{bmatrix}-\begin{bmatrix}-2&-10\\1&0\end{bmatrix}\right)^{-1}$$

$$=\begin{bmatrix}s+2&10\\-1&s\end{bmatrix}^{-1}=\frac{1}{s(s+2)+10}\begin{bmatrix}s&-10\\1&s+2\end{bmatrix}$$

$$H(s)=c(sI-A)^{-1}b+d$$

$$=[3\ \ 4]\begin{bmatrix}s&-10\\1&s+2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\cdot\frac{1}{s^2+2s+10}+(-2)$$

$$=[3\ \ 4]\begin{bmatrix}s\\1\end{bmatrix}\cdot\frac{1}{s^2+2s+10}-2$$

$$=\frac{3s+4}{s^2+2s+10}-2=\frac{-2s^2-s-16}{s^2+2s+10}$$

8.23 (a) $Y(s)=\frac{R}{R+R}U(s)=0.5U(s)$

Its transfer $H_a(s)=0.5$ has degree 0 and the resistive voltage divider has no energy storage element. Thus the divider is completely characterized by its transfer function.

(b) $Y(s)=\frac{Ls}{Ls+Ls}U(s)=0.5U(s)$

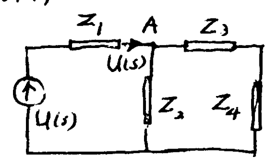
$H_b(s)=0.5$ does not characterize the circuit.

(c) $Y(s)=\frac{1/cs}{1/cs+1/cs}=0.5U(s)$

$H_c(s)=0.5$ does not characterize completely the circuit.

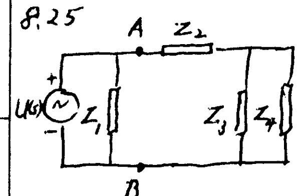
Which voltage divider to use depends on the type of signals to be processed. It involves the issues of efficiency, cost, and availability. Thus the answer to the question is not a simple one and it requires a great deal of investigation in practice.

8.24



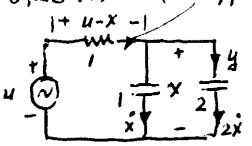
Because the current entering node A is always $U(s)$ no matter what $Z_1(s)$ is. Therefore the transfer function from u to y_1 or y_2 will not involve Z_1 and will not describe the response involving Z_1. If Z_1 contains L and/or C then the transfer function will not characterize completely the circuit. In practice we should set $Z_1(s)=0$ (short circuit).

8.25



The voltage across notes A and B is always $U(s)$. Thus Z_1 will not appear in the transfer function from u to y_1 or y_2. Thus if Z_1 contains L and/or C, then the transfer function will not characterize completely the circuit. In practice, we should set $Z_1=\infty$ (open circuit).

8.26 (a)

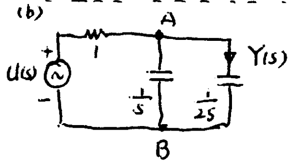


$$\frac{u-x}{1} = \dot x + 2\dot x = 3\dot x$$

$$\dot x = \frac{-1}{3}x + \frac{1}{3}u$$

$$y = 2\dot x = \frac{-2}{3}x + \frac{2}{3}u$$

(b)



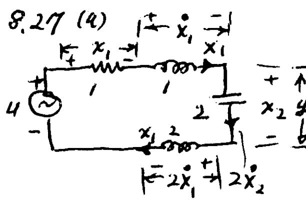
$$Z_{AB} = \frac{\frac{1}{s}\cdot\frac{1}{2s}}{\frac{1}{s}+\frac{1}{2s}} = \frac{\frac{1}{2s^2}}{\frac{2+1}{2s}} = \frac{1}{3s}$$

$$V_{AB} = \frac{Z_{AB}}{1+Z_{AB}}U(s) = \frac{\frac{1}{3s}U(s)}{1+\frac{1}{3s}} = \frac{1}{3s+1}U(s)$$

$$Y(s) = \frac{V_{AB}}{1/2s} = \frac{2s}{3s+1}U(s)$$

The transfer function is $2s/(3s+1)$. It does not characterize completely the circuit because it has degree 1, whereas the circuit has two energy storage elements.

8.27 (a)



$$u = x_1 + \dot x_1 + x_2 + 2\dot x_1$$

$$\dot x_1 = \frac{-1}{3}x_1 - \frac{1}{3}x_2 + \frac{1}{3}u$$

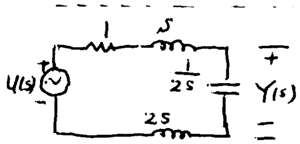
$$2\dot x_2 = x_1$$

$$\dot x_2 = 0.5x_1$$

$$y = x_2$$

$$\dot x = \begin{bmatrix} \frac{-1}{3} & \frac{-1}{3} \\ \frac{1}{2} & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 1]x$$



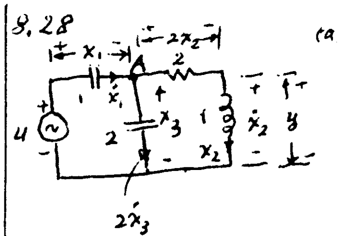
$$Y(s) = \frac{1/2s}{1+s+\frac{1}{2s}+2s}U(s) = \frac{1/2s}{\frac{6s^2+2s+1}{2s}}U(s) = \frac{1}{6s^2+2s+1}U(s)$$

Its transfer function

$$H(s) = \frac{1}{6s^2+2s+1}$$

has degree 2 and does not characterized completely the circuit because the circuit has 3 energy storage elements.

8.28



(a) The right-hand-side loop implies

$$\dot x_2 = x_3 - 2x_2$$

The left-hand-side loop implies

$$u = x_1 + x_3$$

From node A, we have $\dot x_1 = 2\dot x_3 + x_2$ (1)

$\dot u = \dot x_1 + \dot x_3$ (2)

(1) and (2) imply $\dot x_1 = \frac{1}{3}x_2 + \frac{2}{3}\dot u$

$$\dot x_3 = \dot u - \dot x_1 = \frac{-1}{3}x_2 + (\dot u - \frac{2}{3}\dot u) = \frac{-1}{3}x_2 + \frac{1}{3}\dot u$$

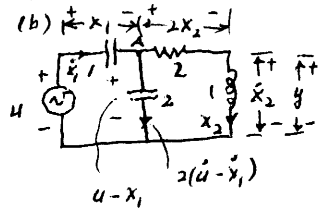
Thus we have

$$\dot x = \begin{bmatrix} 0 & 1/3 & 0 \\ 0 & -2 & 1 \\ 0 & -1/3 & 0 \end{bmatrix} x + \begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix} \dot u$$

$$y = \dot x_2 = [0 \quad -2 \quad 1]x$$

It is not in the standard form because of $\dot u(t) = du(t)/dt$

(b)



Node A: $\dot x_1 = 2(\dot u - \dot x_1) + x_2$

$$\dot x_1 = \frac{1}{3}x_2 + \frac{2}{3}\dot u$$

Right-hand-side loop

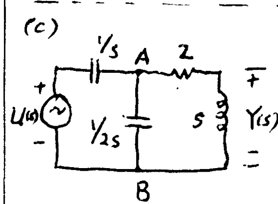
$$\dot x_2 = u - x_1 - 2x_2$$

$$\begin{bmatrix} \dot x_1 \\ \dot x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 2/3 \\ 0 \end{bmatrix} \dot u$$

$$y = [-1 \quad -2]x + u$$

It is not in the standard form.

(c)



$$Z_{AB} = \frac{(s+2)\times\frac{1}{2s}}{s+2+\frac{1}{2s}} = \frac{s+2}{2s^2+4s+1}$$

$$V_{AB} = \frac{Z_{AB}}{\frac{1}{s}+Z_{AB}}U(s) = \frac{\frac{s+2}{2s^2+4s+1}}{\frac{1}{s}+\frac{s+2}{2s^2+4s+1}}U(s) = \frac{s(s+2)}{3s^2+6s+1}U(s)$$

$$Y(s) = \frac{s}{s+2}V_{AB} = \frac{s^2}{3s^2+6s+1}U(s)$$

Thus the transfer function of the circuit is

$$H(s) = \frac{s^2}{3s^2+6s+1}$$

It has degree 2 and does not characterize completely the circuit with 3 energy storage elements.

8.29

$$(D_2s^2+D_1s+D_0)(B_1s+B_0) = D_2B_1s^3+(D_2B_0+D_1B_1)s^2 + (D_1B_0+D_0B_1)s + D_0B_0$$

$$(N_2s^2+N_1s+N_0)(A_1s+A_0) = N_2A_1s^3+(N_2A_0+N_1A_1)s^2 + (N_1A_0+N_0A_1)s + N_0A_0$$

Equating the coefficients of s^0, s^1, s^2 and s^3 yield

$D_0B_0 = N_0A_0$ or $D_0B_0 - N_0A_0 = 0$

$D_1B_0 + D_0B_1 = N_1A_0 + N_0A_1$

or $D_1B_0 - N_1A_0 + D_0B_1 - N_0A_1 = 0$

$D_2B_0 + D_1B_1 = N_2A_0 + N_1A_1$

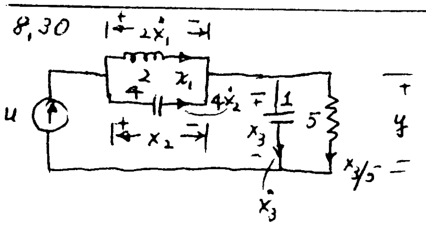
or $D_2B_0 - N_2A_0 + D_1B_1 - N_1A_1 = 0$

$D_2B_1 = N_2A_1$ or $D_2B_1 - N_2A_1 = 0$

These four equations can be expressed in matrix form as

$$\left[\begin{array}{cc:cc} D_0 & N_0 & 0 & 0 \\ D_1 & N_1 & D_0 & N_0 \\ D_2 & N_2 & D_1 & N_1 \\ 0 & 0 & D_2 & N_2 \end{array}\right]\left[\begin{array}{c} B_0 \\ -A_0 \\ \hdashline B_1 \\ -A_1 \end{array}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

8.30



$2\dot{x}_1 = x_2$

$\dot{x}_1 = 0.5x_2$

$4\dot{x}_2 = u - x_1$

$\dot{x}_3 + \frac{x_3}{5} = u$

$\dot{x}_3 = -0.2x_3 + u$

$$\dot{x} = \begin{bmatrix} 0 & 0.5 & 0 \\ -0.25 & 0 & 0 \\ 0 & 0 & -0.2 \end{bmatrix}x + \begin{bmatrix} 0 \\ 0.25 \\ 1 \end{bmatrix}u$$

$$y = x_3 = [0 \ \ 0 \ \ 1]x$$

8.31 $n = [0 \ 1 \ 0 \ 0.125]$

$d = [1 \ 0.2 \ 0.125 \ 0.025]$

Thus the transfer function is

$$H(s) = \frac{0\cdot s^3 + 1\cdot s^2 + 0\cdot s + 0.125}{1\cdot s^3 + 0.2\cdot s^2 + 0.125\cdot s + 0.025}$$

$$= \frac{s^2+0.125}{s^3+0.2s^2+0.125s+0.025}$$

$$= \frac{s^2+0.125}{(s^2+0.125)(s+0.2)} = \frac{1}{s+0.2}$$

Its degree is 1. The dimension of the ss equation is 3; its degree however is 1. Thus the ss equation has some deficiency (cannot be both controllable and observable) and is not used in design.

8.32 $$y(t) = \int_{\tau=0}^{t} h(t-\tau)u(\tau)\,d\tau$$

Define $\bar{\tau} = t-\tau$. Note that t is fixed and τ is the variable. Then $\tau = t-\bar{\tau}$ and $d\bar{\tau} = -d\tau$. Thus we have

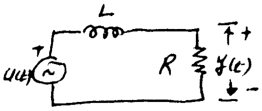
$$y(t) = \int_{\bar{\tau}=t}^{0} h(\bar{\tau})u(t-\bar{\tau})(-d\bar{\tau})$$

$$= \int_{\bar{\tau}=0}^{t} h(\bar{\tau})u(t-\bar{\tau})\,d\bar{\tau} = \int_{\tau=0}^{t} h(\tau)u(t-\tau)\,d\tau$$

8.33 Let $y_q(t)$ be the output excited by $u(t) = q(t) = 1$ for $t \geq 0$. Then we have

$$y_q(t) = \int_{\tau=0}^{t} h(\tau)\cdot 1 \cdot d\tau \Rightarrow h(t) = \frac{dy_q(t)}{dt}$$

8.34 A diode is a nonlinear memoryless element. Because the diodes are ideal, the circuit is equivalent to



It is a LTI system. Its transfer function is

$$H(s) = \frac{Y(s)}{U(s)} = \frac{R}{Ls+R}.$$

Chapter 9

9.1

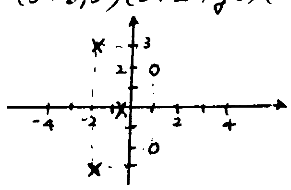
$$H_1(s) = \frac{s^2-1}{3(s^2+s-2)} = \frac{(s+1)(s-1)}{3(s+2)(s-1)} = \frac{s+1}{3(s+2)}$$

$H_1(s)$ has a zero at -1 and a pole at -2.

$$H_2(s) = \frac{2s+5}{3(s^2+3s+2)} = \frac{2(s+2,5)}{3(s+1)(s+2)}$$

$H_2(s)$ has a zero at $-2,5$ and poles at -1 and -2.

$$H_3(s) = \frac{s^2-2s+5}{(s+0,5)(s^2+4s+13)} = \frac{(s-1)^2+4}{(s+0,5)[(s+2)^2+9]} = \frac{(s-1+j2)(s-1-j2)}{(s+0,5)(s+2+j3)(s+2-j3)}$$



It has three poles at $-0,5$ and $-2\pm j3$ and two zeros at $1\pm j2$

9.2 $H(s) = \dfrac{2s^2-10s+1}{(s+1)(s+2)}$

Impulse response: $u(t)=\delta(t)$, $U(s)=1$

$$Y(s) = H(s)\cdot 1 = \frac{2s^2-10s+1}{(s+1)(s+2)} = k_0 + \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

with $k_0 = Y(\infty) = 2$

$$k_1 = \frac{2s^2-10s+1}{s+2}\Big|_{s=-1} = \frac{2+10+1}{-1+2} = 13$$

$$k_2 = \frac{2s^2-10s+1}{s+1}\Big|_{s=-2} = \frac{8+20+1}{-2+1} = -29$$

Impulse response $= 2\delta(t) + 13e^{-t} - 29e^{-2t}$, for $t \geq 0$.

Step response: $u(t)=1$, $t\geq 0$, $U(s)=\frac{1}{s}$

$$Y(s) = H(s)\cdot\frac{1}{s} = \frac{2s^2-10s+1}{s(s+1)(s+2)} = k_0 + \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

with $k_0 = Y(\infty) = 0$

$$k_1 = \frac{2s^2-10s+1}{(s+1)(s+2)}\Big|_{s=0} = \frac{1}{2} = 0,5$$

$$k_2 = \frac{2s^2-10s+1}{s(s+2)}\Big|_{s=-1} = \frac{2+10+1}{(-1)(-1+2)} = -13$$

$$k_3 = \frac{2s^2-10s+1}{s(s+1)}\Big|_{s=-2} = \frac{8+20+1}{(-2)(-1)} = \frac{29}{2} = 14,5$$

$\therefore$ Step response $= 0,5 - 13e^{-t} + 14,5e^{-2t}$, $t \geq 0$

9.3 The impulse response of $H(s)$ is $\mathcal{L}^{-1}[H(s)]$

The step response of $sH(s)$ is $\mathcal{L}^{-1}[sH(s)\cdot\frac{1}{s}]$
$= \mathcal{L}^{-1}[H(s)]$.

This shows the assertion.

9.4 Applying the Laplace transform yields, using (8.43),

$$sY(s) - y(0) + 0,0001Y(s) = U(s)$$

If $u(t)=0$, $y(0)=80$, then we have

$$(s+0,0001)Y(s) = y(0) = 80$$

or $$Y(s) = \frac{80}{s+0,0001}$$

Its inverse Laplace Transform is

$$y(t) = 80e^{-0,0001t}$$

Let t_1 be the time for $y(t)$ to drop to 70, that is,

$$70 = 80e^{-0,0001t_1}$$

Then we have

$$-0,0001t_1 = \ln(70/80) = -0,134$$

or $t_1 = 1340\,s$ or 22 min 20 sec

9.5 Step response

$$Y(s) = H(s)\cdot\frac{1}{s} = \frac{10(s-1)}{(s+1)^3(s+0,1)s} = \frac{k_1}{s+1} + \frac{k_2}{(s+1)^2} + \frac{k_3}{(s+1)^3} + \frac{k_4}{s+0,1} + \frac{k_5}{s}$$

$$y(t) = k_1e^{-t} + k_2te^{-t} + k_3t^2e^{-t} + k_4e^{-0,1t} + k_5$$

for $t \geq 0$, with $k_5 = H(0) = \frac{-10}{1\times 0,1} = -100$.

9.6 $Y(s) = \dfrac{N(s)}{(s+2)^4(s+0.2)(s+1+j3)(s+1-j3)\cdot s}$

$$y(t) = k_1 e^{-2t} + k_2 t e^{-2t} + k_3 t^2 e^{-2t} + k_4 t^3 e^{-2t} + k_5 e^{-0.2t} + k_6 e^{-t}\sin(3t+k_7) + k_8,$$

for $t \geq 0$, with

$$k_8 = H(0) = \frac{N(0)}{D(0)} = \frac{320}{2^4 \times 0.2 \times 10} = \frac{320}{16 \times 2} = 10$$

9.7 $Y(s) = \dfrac{s+3}{(s+1)(s-1)} \cdot \dfrac{s-1}{s(s+3)} = \dfrac{1}{s(s+1)}$

$$= \frac{1}{s} + \frac{-1}{s+1}$$

$$y(t) = 1 - e^{-t}, \quad t \geq 0$$

The pole at $+1$ of $H(s)$ is not excited by the input. In practice, this type of cancellation rarely occurs and all poles of $H(s)$ will be excited.

9.8 The definition is not acceptable because for any stable or unstable system, we can always find a bounded input which excites a bounded output as shown in Prob. 9.7. Thus we require every bounded input not just a bounded input.

9.9 Not true. Even if all poles except one lie inside the left-half s-plane, the system is not stable.

9.10 (1) $s^5 + 3s^3 + 2s^2 + s + 1$ is not CT stable because the term s^4 is missing

(2) Not CT stable because it has a negative coefficient

(3) $s^5 + 4s^4 + 3s^3 + 2s^2 + s + 1$

	s^5	1	3	1	
1/4	s^4	4	2	1	[0 2.5 0.75]
4/2.5	s^3	2.5	0.75		[0 0.8 1]
2.5/0.8	s^2	0.8	1		[0 −2.375]
	s	−2.375			
	1	1			

Not CT stable because of the negative number.

(4) $s^5 + 6s^4 + 23s^3 + 52s^2 + 54s + 20$

	s^5	1	23	54	
1/6	s^4	6	52	20	[0 14.3 50.7]
6/14.3	s^3	14.3	50.7		[0 30.7 20]
14.3/30.7	s^2	30.7	20		[0 41.38]
	s	41.38			
	1	20			

The polynomial is CT stable.

9.11 The root of $a_0 s + a_1$ is $-a_1/a_0$. It is negative if and only if both a_0 and a_1 are positive or negative. Thus if a_0 and a_1 are of the same sign, then $a_0 s + a_1$ is CT stable.

Consider $D(s) = a_0 s^2 + a_1 s + a_2$ with $a_0 > 0$

	s^2	a_0	a_2	
a_0/a_1	s	a_1		$[0 \;\; a_2]$
	1	a_2		

If $a_0 > 0$, $D_0(s)$ is CT stable if and only if $a_1 > 0$ and $a_2 > 0$. Because $-D_0(s)$ is CT stable if and only if $D_0(s)$ is CT stable. Thus we conclude $D_0(s)$ is CT stable if and only if a_i, $i = 0, 2$, are of the same sign.

9.12 $s^3 + a_1 s^2 + a_2 s + a_3$

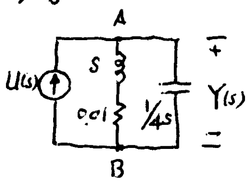
	s^3	1	a_2	
$1/a_1$	s^2	a_1	a_3	$\left[0 \;\; a_2 - \frac{a_3}{a_1}\right] = \left[0 \;\; \frac{a_1 a_2 - a_3}{a_1}\right]$
	s	$\frac{a_1 a_2 - a_3}{a_1}$		
	1	a_3		

CT stable $\Leftrightarrow$ $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and

$$a_1 a_2 - a_3 > 0$$

or $a_1 > 0$, $a_2 > 0$, $a_1 a_2 > a_3 > 0$

9.13



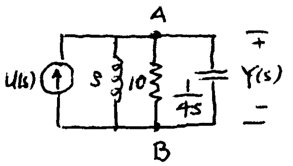
$$Z_{AB} = \frac{(s+0.01)\cdot\frac{1}{4s}}{s+0.01+\frac{1}{4s}} = \frac{s+0.01}{4s^2+0.04s+1}$$

$$Y(s) = Z_{AB}\cdot U(s)$$

Thus the transfer function from u to y is

$$H_1(s) = \frac{s+0.01}{4s^2+0.04s+1}$$

Its denominator is CT stable because its three coefficients are positive (Prob. 9.11) Thus $H_1(s)$ or the circuit is stable.



The impedance of the parallel connection of s and 10 is $\frac{10s}{s+10}$.

Thus we have

$$Z_{AB} = \frac{\frac{10s}{s+10}\cdot\frac{1}{4s}}{\frac{10s}{s+10}+\frac{1}{4s}} = \frac{10s}{40s^2+s+10}$$

It is stable.

9.14 $H(s) = \dfrac{10(s-1)}{(s+1)^3(s+0.1)}$

As shown in Prob. 9.5 its step response is

$$y(t) = k_1e^{-t}+k_2te^{-t}+k_3t^2e^{-t}+k_4e^{-0.1t} - 100, \quad t \geq 0$$

$$y_{ss}(t) = \lim_{t\to\infty} y(t) = -100$$

$$y_{tr}(t) = k_1e^{-t}+k_2te^{-t}+k_3t^2e^{-t}+k_4e^{-0.1t}$$

$H(s)$ is stable and its time constant is $1/0.1 = 10$. Thus it takes 50 seconds for the transient response to die out and for the response to reach steady state.

9.15 $H(s) = \dfrac{N(s)}{(s+2)^4(s+0.2)(s+1+j3)(s+1-j3)}$

As shown in Prob. 9.6, we have

$$y_{ss}(t) = \lim_{t\to\infty} y(t) = 10$$

and

$$y_{tr}(t) = k_1e^{-2t}+k_2te^{-2t}+k_3t^2e^{-2t}+k_4t^3e^{-2t} + k_5e^{-0.2t}+k_6e^{-t}\sin(3t+k_7)$$

$H(s)$ is stable and its time constant is $1/0.2 = 5s$. Thus it takes 25 seconds for the response to reach steady state.

9.16 $H(s) = \dfrac{1}{(s+1)^3}$ Its time constant is $1s$.

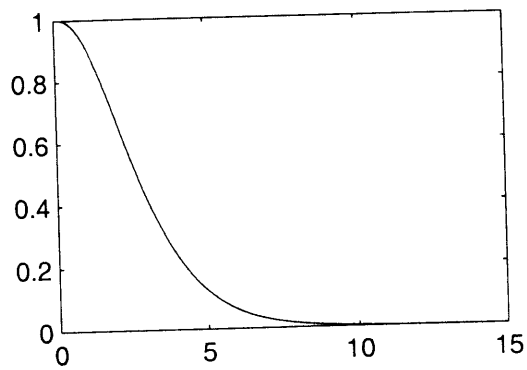
$$Y(s) = H(s)\cdot\frac{1}{s} = \frac{1}{(s+1)^3s} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} - \frac{1}{(s+1)^3}$$

$$\therefore\ y(t) = 1 - e^{-t} - te^{-t} - 0.5t^2e^{-t}$$

$$y_{ss}(t) = 1$$

$$y_{tr}(t) = -(1+t+0.5t^2)e^{-t}$$

We plot $|y_{tr}(t)|$ in the following



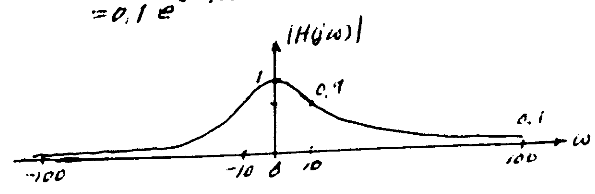
where $|y_{tr}(5)| = 0.125$ and $|y_{tr}(9)| = 0.006$. Thus the transient response reaches zero or stays within 1% of its peak value in nine time constants.

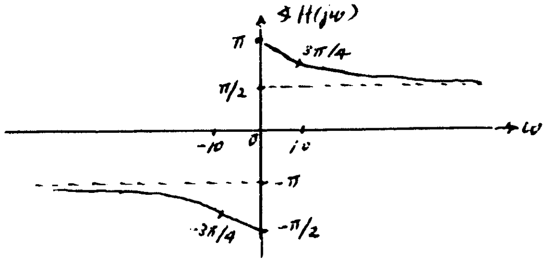
9.17 Consider $H(s) = \dfrac{-10}{s+10}$ We compute

$$H(j0) = \frac{-10}{0+10} = -1 = 1e^{j\pi}$$

$$H(j10) = \frac{-10}{j10+10} = \frac{-1}{1+j} = \frac{1e^{j\pi}}{1.414e^{j\pi/4}} = 0.7e^{j3\pi/4}$$

$$H(j100) = \frac{-10}{j100+10} = \frac{-1}{1+j10} \approx \frac{-1}{j10} = \frac{e^{j\pi}}{10e^{j\pi/2}} = 0.1e^{j\pi/2}$$





Note that the magnitude response is even and the phase response is odd. The phase approaches $\pi/2$ rad. as $\omega \to \infty$.

9.18 $H_1(s) = \dfrac{2}{s+2}$, $H_2(s) = \dfrac{2}{s-2}$

$$|H_1(j\omega)| = \left|\frac{2}{j\omega+2}\right| = \frac{2}{\sqrt{\omega^2+4}}$$

$$|H_2(j\omega)| = \left|\frac{2}{j\omega-2}\right| = \frac{2}{\sqrt{\omega^2+(-2)^2}} = \frac{2}{\sqrt{\omega^2+4}}$$

Thus we have $|H_1(j\omega)| = |H_2(j\omega)|$ for all ω.

9.19 $H(s) = \dfrac{s-0.2}{s^2+s+100}$

$H(s)$ is stable. In order to apply Theorem 9.5, we compute

$$H(0) = \frac{-0.2}{100} = -0.002 = 0.002\,e^{j\pi}$$

$$H(j10) = \frac{j10-0.2}{(j10)^2+j10+100} \approx \frac{j10-0.2}{-100+j10+100} \approx \frac{j10}{j10} = 1 = 1e^{j0}$$

$$H(j100) = \frac{j100-0.2}{-10000+j100+100} \approx \frac{j100}{-10000} = \frac{100e^{j\pi/2}}{10000e^{j\pi}} = 0.01e^{-j\pi/2}$$

Thus the steady-state response of $H(s)$ excited by $u(t) = 2 + \cos 10t - \sin 100t$ is

$$y_{ss}(t) = 2\times(-0.0002) + 1\times\cos(10t+0) - 0.01\sin(100t - \pi/2) \approx \cos 10t.$$

$H(s)$ attenuates greatly dc and high-frequency sinusoids, thus it is bandpass filter, we write

$$s^2+s+100 = (s+0.5)^2+100-0.25$$

Thus $H(s)$ has poles at $-0.5 \pm j\sqrt{99.75}$ and its time constant is $1/0.5 = 2$ s. Thus it takes roughly 10 seconds for the response to reach steady state.

9.20 $H(\omega) = \begin{cases} e^{-j\omega t_0} & |\omega| \le \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$

$$h(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} H(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{\omega=-\omega_c}^{\omega_c} e^{-j\omega t_0}e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j(t-t_0)\omega}\,d\omega = \frac{1}{2\pi j(t-t_0)}e^{j(t-t_0)\omega}\Big|_{\omega=-\omega_c}^{\omega_c}$$

$$= \frac{e^{j(t-t_0)\omega_c} - e^{-j(t-t_0)\omega_c}}{2\pi j(t-t_0)} = \frac{2j\sin[(t-t_0)\omega_c]}{2\pi j(t-t_0)}$$

$$= \frac{\sin[(t-t_0)\omega_c]}{\pi(t-t_0)}$$

for all t in $(-\infty,\infty)$. Because $h(t)$ is not identically zero for all $t<0$, the ideal lowpass filter is not causal and cannot be built in the real world.

9.21 $H(s) = \dfrac{k}{s+a}$

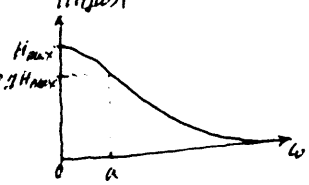
$$H(0) = \frac{k}{a}, \quad |H(0)| = |k|/a =: H_{max}$$

$$H(ja) = \frac{k}{ja+a} = \frac{k}{a}\cdot\frac{1}{1+j1} = \frac{k}{a}\cdot\frac{1}{1.414e^{j\pi/4}} = 0.707(k/a)e^{-j\pi/4}$$

$$|H(ja)| = 0.707|k|/a = 0.707H_{max}$$

$$|H(j\omega)| \to 0 \text{ as } \omega \to \infty$$

Thus the magnitude response of $H(s)$ is as shown



It is a lowpass filter with 3dB bandwidth a. If $a<0$, $H(s)$ is not stable and cannot be used as a filter.

k can be negative, but it will introduce phase π and, possibly, more time delay.

9.22 The transfer function of Fig. 9.21(a) is

$$H_1(s) = \frac{1/0.1s}{1+1/0.1s} = \frac{1}{0.1s+1} = \frac{10}{s+10}$$

It is of the form in Prob. 9.21. Thus it is lowpass with 3-dB bandwidth 10. We compute

$$H_1(j0.1) = \frac{10}{j0.1+10} \approx \frac{10}{10} = 1$$

$$H_1(j100) = \frac{10}{j100+10} \approx \frac{10}{j100} = 0.1e^{-j\pi/2}$$

Thus the steady-state response is

$$y_{ss}(t) = \sin 0.1t + 0.1\sin(100t - \pi/2)$$

The high-frequency sinusoid is attenuated.

9.23 The transfer function of Fig. 9.21(b) is

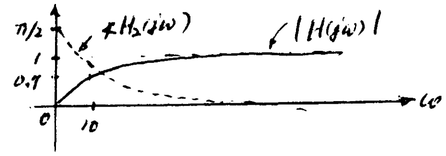
$$H_2(s) = \frac{1}{1+1/0.1s} = \frac{0.1s}{0.1s+1} = \frac{s}{s+10}$$

$$H_2(j\omega) = \frac{j\omega}{j\omega+10}$$

$$H_2(0) = 0, \quad H(j0.1) = \frac{j0.1}{j0.1+1} \approx \frac{j0.1}{1} = 0.1e^{j\pi/2}$$

$$H_2(j10) = \frac{j10}{j10+10} = \frac{j}{1+j} = \frac{e^{j\pi/2}}{1.4e^{j\pi/4}} = 0.7e^{j\pi/4}$$

$$H_2(j100) = \frac{j100}{j100+10} \approx \frac{j100}{j100} = 1$$



It is highpass with 3-dB passband edge frequency 10 rad/s. Its passband is $[10, \infty)$. Thus its bandwidth is ∞. The steady-state response is

$$y_{ss}(t) = 0.1\sin(0.1t + \pi/2) + \sin(100t)$$

9.24 Because

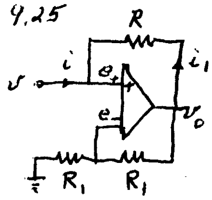
$$|H_1(j\omega)| \to \begin{cases} |b|/a_3 & \text{for } \omega = 0 \\ 0 & \text{for } \omega \to \infty \end{cases}$$

$$|H_3(j\omega)| \to \begin{cases} |d|/a_3 \approx 0 & \text{for } \omega = 0 \\ |b|/a_2 & \text{for } \omega \approx \sqrt{a_3} \\ 0 & \text{for } \omega \to \infty \end{cases}$$

$$|H_4(j\omega)| \to \begin{cases} |d|/a_3 \approx 0 & \text{for } \omega = 0 \\ |b| & \text{for } \omega \to \infty \end{cases}$$

the assertions follow. a_2 and a_3 must be positive in order for the filter to be stable.

9.25



$$e_- = \frac{v_o}{R_1+R_1}\cdot R_1 = \frac{v_o}{2}$$

$$e_+ = e_- = 0.5v_o = v$$

$$i_1 = \frac{v_o - e_+}{R} = \frac{v_o - 0.5v_o}{R} = \frac{0.5v_o}{R} = \frac{v}{R}$$

$$i = -i_1 = -\frac{v}{R} \qquad \therefore\ v = -Ri$$

Note that we have used $i_- = 0$ and $i_+ = 0$.

9.26

$$\mathcal{F}[x(t)] = \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{t=-\infty}^{0} x_-(t)e^{-j\omega t}dt + \int_{0}^{\infty} x_+(t)e^{-j\omega t}dt$$

$\leftarrow$ this term equals $\mathcal{L}[x_+(t)]\big|_{s=j\omega}$

Define $\bar{t} = -t$. Then we have

$$\int_{t=-\infty}^{0} x_-(t)e^{-j\omega t}dt = \int_{\bar{t}=\infty}^{0} x_-(-\bar{t})e^{j\omega\bar{t}}(-d\bar{t})$$

$$= \int_{\bar{t}=0}^{\infty} x_-(-\bar{t})e^{j\omega\bar{t}}d\bar{t} = \int_{t=0}^{\infty} x_-(-t)e^{j\omega t}dt$$

$$= \mathcal{L}[x_-(-t)]\big|_{s=-j\omega}$$

Thus we have

$$\mathcal{F}[x(t)] = \mathcal{L}[x_+(t)]_{s=j\omega} + \mathcal{L}[x_-(-t)]\big|_{s=-j\omega}$$

This holds only if $x(t)$ is absolutely integrable in $(-\infty, \infty)$.

9.27 If $X(s)$ has a simple pole at $s=0$, it can be expanded as

$$X(s) = \frac{k}{s} + \text{terms due to other poles}$$

with $k = X(s)\cdot s\big|_{s=0}$

If all other poles lie inside the left half s-plane or have negative real parts, then their time responses all approach 0 as

$t \to \infty$. Thus we have

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s), \text{ a nonzero constant.}$$

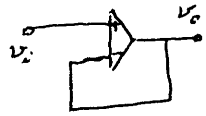
If $X(s)$ has no pole at $s=0$ and all other poles have negative real parts, then $x(t) \to 0$ as $t\to\infty$. In this case the equation still holds because there is no pole of $X(s)$ to cancel s and $\lim_{s\to 0} sX(s) = 0$.

If $X(s)$ has one or more poles with positive real parts, or repeated pole at $s=0$, then $|x(t)| \to \infty$ as $t\to\infty$.

If $X(s)$ has simple poles on the $j\omega$-axis, then $x(t)$ contains sustained oscillation. For example, if $X(s) = \frac{s}{s^2+100}$, then $x(t) = \cos 10t$. Although $\lim_{s\to 0} sX(s) = 0$, we do not have $\lim_{t\to\infty} x(t) = 0$. Thus in using the equation, we must first check the applicability conditions.

Chapter 10

10.1



$$V_o(s) = A(s)[E_+(s) - E_-(s)]$$

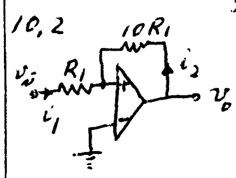
$$= \frac{10^5}{s+100}[V_i(s) - V_o(s)]$$

$$\left(1 + \frac{10^5}{s+100}\right) V_o(s) = \frac{10^5}{s+100} V_i(s)$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{10^5}{s+100+10^5} \approx \frac{10^5}{s+10^5}$$

It is lowpass with passband $[0, 10^5]$. See Prob. 9.21. For signals whose spectra lying inside $[0, 10^5]$, $H(s)$ can be reduced as 1. Moreover, because of the time constant 10^{-5}, $H(s)$ can respond almost instantaneously.

10.2



Ideal model: $e_+ = e_- = 0$, $i_+ = 0$

$$i_1 = \frac{v_i - e_+}{R_1} = \frac{v_i}{R_1}$$

$$i_2 = \frac{v_o - e_+}{10R_1} = \frac{v_o}{10R_1}$$

$$i_1 = -i_2 \quad \text{or} \quad \frac{v_i}{R_1} = \frac{-v_o}{10R_1}$$

Thus we have $v_o(t) = -10 v_i(t)$ and $V_o(s) = -10 V_i(s)$. Thus the transfer function is $H_s(s) = -10$.

$$V_o(s) = A(s)[E_+(s) - E_-(s)] = \frac{10^7}{s+50.3} E_+(s)$$

or $E_+(s) = \frac{s+50.3}{10^7} V_o(s)$

$$I_1(s) = \frac{V_i(s) - E_+(s)}{R_1}, \quad I_2(s) = \frac{V_o(s) - E_+(s)}{10R_1}, \quad I_+(s) = 0$$

$$I_1(s) + I_2(s) - I_+(s) = 0 \quad \therefore I_1(s) = -I_2(s)$$

$$\frac{V_i(s) - E_+(s)}{R_1} = -\frac{V_o(s) - E_+(s)}{10R_1}$$

or $10V_i(s) - 10E_+(s) = -V_o(s) + E_+(s)$

$$10V_i(s) = -V_o(s) + 11E_+(s) = -V_o(s) + \frac{11(s+50.3)}{10^7}V_o(s)$$

$$= V_o(s)\left[\frac{11s+553.3}{10^7} - 1\right] = V_o(s)\cdot\frac{11s+553.3-10^7}{10^7}$$

Thus we have

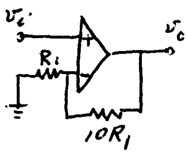
$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{10^8}{11s+553.3-10^7} \approx \frac{10^8}{11s-10^7}$$

For ω in $[0, 1000]$, $H(s)$ and $H_s(s)$ have almost identical frequency response for

$$H(j\omega)=\frac{10^8}{11\omega-10^7}\approx\frac{10^8}{-10^7}=-10=H_s(j\omega)$$

Even so, $H(s)$ cannot be reduced to -10 for any signal because $H(s)$ is not stable. For example, the step response of $H(s)$ grows unbounded; whereas, the step response of $H_s(s)$ is -10 for all $t\geq 0$. Thus the circuit cannot be used as an inverting amplifier.

10.3



Ideal model $e_+=e_-$, $i_-=0$

$$e_-=\frac{v_o}{R_1+10R_1}\cdot R_1=\frac{1}{11}v_o$$

$$v_i=e_+=e_-=\frac{1}{11}v_o$$

$\therefore\ v_o(t)=11\,v_i(t)$, $V_o(s)=11V_i(s)$

$$H_s(s)=\frac{V_o(s)}{V_i(s)}=11$$

- - - - - - - - - - - - - - - - - - - -

$V_o(s)=A(s)[E_+(s)-E_-(s)]$ with

$$A(s)=\frac{10^7}{s+50.3},\quad E_+(s)=V_i(s),\quad E_-(s)=\frac{1}{11}V_o(s)$$

$$(s+50.3)V_o(s)=10^7V_i(s)-\frac{10^7}{11}V_o(s)$$

$$(s+50.3+10^7/11)V_o(s)=10^7V_i(s).$$

Thus the transfer function is

$$H(s)=\frac{V_o(s)}{V_i(s)}=\frac{10^7}{s+50.3+10^7/11}\approx\frac{10^7}{s+10^7/11}$$

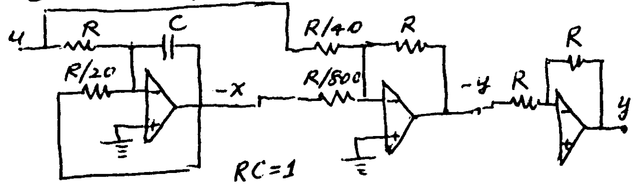
It is stable with time constant $11/10^7$ $=11\times10^{-7}=1.1\times10^{-6}$. It is lowpass with 3-dB passband $[0,\ 10^7/11]$ which can be considered as its operational frequency range. For low-frequency signals or, to be more precise, for signals whose spectra lying inside $[0,\ 10^7/11]$, $H(s)$ can be reduced as 11.

10.4 $$H(s)=\frac{2s}{1+s/20}=\frac{40s}{s+20}=40+\frac{-800}{s+20}$$

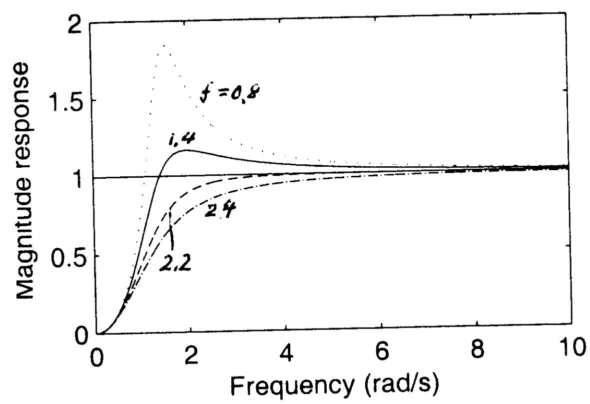
Its ss equation realization is

$$\dot{x}(t)=-20x(t)+u(t)$$
$$y(t)=-800x(t)+40u(t)$$

It can be implemented as

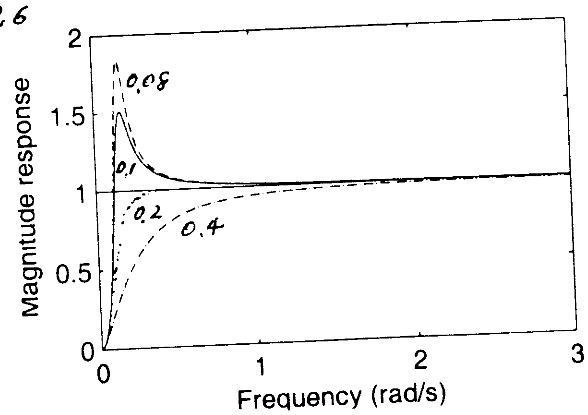


10.5 Using the MATLAB program following (10.13) with $d=[1\ f\ 2]$ for $f=0.8, 1.4, 2.2$ and 2.4, we obtain the magnitude responses as shown



For $m=1$ and $k=2$, if $f=2.2$, then the operational frequency range of the seismometer is the largest. The range is $[3.8,\infty)$. It is smaller than $[1.25,\infty)$ obtained for $m=1$, $k=0.2$ and $f=0.63$

10.6



The preceding plot is obtained for $m=1$, $b=0.02$ and various f. If $f=0.2$, the operational frequency range is the largest. It is roughly $[0.4, \infty)$ which is larger than $[1.25, \infty)$ obtained for $m=1$ and $b=0.2$.

10.7 $H_1 = \frac{10}{s+1}$ and $H_2 = -2$ are stable.

$$H = \frac{H_1}{1+H_1H_2} = \frac{\frac{10}{s+1}}{1+\frac{10}{s+1}\times(-2)} = \frac{\frac{10}{s+1}}{\frac{s+1-20}{s+1}}$$

$$= \frac{10}{s-19} \quad \text{not stable.}$$

Thus the negative feedback of two stable systems can be unstable. Thus it is not necessarily true that negative feedback can stabilize a system.

10.8 $H_1 = \frac{-2}{s-1}$ and $H_2 = \frac{3s+4}{s-2}$ are unstable.

$$H = \frac{H_1}{1-H_1H_2} = \frac{\frac{-2}{s-1}}{1-\frac{-2}{s-1}\cdot\frac{3s+4}{s-2}} = \frac{\frac{-2}{s-1}}{\frac{s^2-3s+2+6s+8}{(s-1)(s-2)}}$$

$$= \frac{-2(s-2)}{s^2+3s+10} \quad \text{stable.}$$

Thus the positive feedback of two unstable systems can be stable. Thus it is not necessarily true that positive feedback can destabilize a system.

10.9 The output e of the left-most op amp in Fig. 10.13 (c) is, if $R_f = R/\beta$,

$$e = -\frac{10R}{R}u - \frac{10R}{R_f}y = -10u - 10\beta y$$

$$= -10(u+\beta y)$$

From Fig. 10.13 (b), we have, if $-A = -10$,

$$e = -A(u+\beta y) = -10(u+\beta y).$$

This verifies the assertion.

10.10 The transfer function of Fig. 10.14 is

$$H_f(s) = A \cdot \frac{C(s)P(s)}{1+C(s)P(s)}$$

If $C(s) = k$ and $P(s) = 2/s(s+1)$, then

$$H_f(s) = A \cdot \frac{k\frac{2}{s(s+1)}}{1+k\frac{2}{s(s+1)}} = A \cdot \frac{2k}{s^2+s+2k}$$

If the poles of $H_f(s)$ are located at $-0.5 \pm j2$, then its denominator should equal

$$(s+0.5-j2)(s+0.5+j2) = s^2+s+4.25$$

Thus if $2k = 4.25$ or $k = 2.125$, then the unity feedback system has poles at $-0.5 \pm j2$.

If the poles of $H_f(s)$ are located at $-1 \pm j2$, then its denominator should equal

$$(s+1-j2)(s+1+j2) = s^2+2s+5$$

No k exists to make $s^2+s+2k = s^2+2s+5$. Thus if $C(s) = k$, a compensator of degree 0, we cannot place the poles of $H_f(s)$ at $-1 \pm j2$.

10.11 If $C(s) = \frac{N_1 s + N_0}{D_1 s + D_0}$, a proper compensator of degree 1, then

$$H_f(s) = A \cdot \frac{\frac{N_1 s+N_0}{D_1 s+D_0}\cdot\frac{2}{s(s+1)}}{1+\frac{N_1 s+N_0}{D_1 s+D_0}\cdot\frac{2}{s(s+1)}}$$

$$= A \cdot \frac{2(N_1 s+N_0)}{(D_1 s+D_0)s(s+1)+2(N_1 s+N_0)}$$

$$= A \cdot \frac{2(N_1 s+N_0)}{D_1 s^3+(D_0+D_1)s^2+(D_0+2N_1)s+2N_0}$$

If the poles of $H_f(s)$ are located at -2 and $-1 \pm j2$, then its denominator should equal

$$(s+2)(s+1-j2)(s+1+j2) = (s+2)(s^2+2s+5)$$

$$= s^3+4s^2+9s+10$$

Equating the coefficients yields

$D_1 = 1$, $D_0 + D_1 = 4 \Rightarrow D_0 = 3$

$D_0 + 2N_1 = 9 \Rightarrow N_1 = 3$,

$2N_0 = 10 \Rightarrow N_0 = 5$

Thus if $C(s) = \frac{3s+5}{s+3}$, then

$$H_f(s) = A \cdot \frac{2(3s+5)}{s^3+4s^2+9s+10}$$

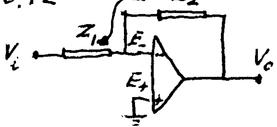
It has poles at -2 and $-1 \pm j2$.

Because $H_f(s)$ is stable, if we apply $u(t) = a$, for $t \geq 0$, then the output approaches $H_f(0) \cdot a$ (Theorem 4.5). Thus we require

$$H_f(0) = A \cdot \frac{10}{10} = A = 1$$

In other words, no gain is needed. Direct connection will achieve asymptotic tracking of any step reference input.

10.12



Because $I_- = 0$, the same current I passing through Z_2 and Z_1 and

$$I = \frac{V_o - V_i}{Z_1 + Z_2}$$

Thus $E_- = IZ_1 + V_i$

$$= \frac{V_o - V_i}{Z_1 + Z_2} \cdot Z_1 + V_i = \frac{V_o Z_1 + V_i Z_2}{Z_1 + Z_2}$$

$$V_o = A(E_+ - E_-) = -AE_- = -A \cdot \frac{V_o Z_1 + V_i Z_2}{Z_1 + Z_2}$$

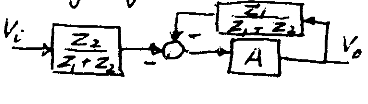
$$\therefore (Z_1 + Z_2)V_o = -AZ_1 V_o - AZ_2 V_i$$

$$(Z_1 + Z_2 + AZ_1)V_o = -AZ_2 V_i$$

Thus we have

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{-AZ_2}{Z_1 + Z_2 + AZ_1}$$

Using Fig. 10.20(b) and Fig. 10.10(d)



$$\frac{V_o(s)}{V_i(s)} = \frac{-Z_2}{Z_1+Z_2} \cdot \frac{A}{1 + \frac{Z_1}{Z_1+Z_2}A}$$

$$= \frac{-Z_2}{Z_1+Z_2} \cdot \frac{A(Z_1+Z_2)}{Z_1+Z_2+Z_1 A} = \frac{-AZ_2}{Z_1+Z_2+AZ_1}$$

10.13 $Z_1 = R$, $Z_2 = 10R$, $A_1 = 10^5$

$$H_1(s) = \frac{-10^5 \cdot 10R}{R + 10R + 10^5 R} = \frac{-10^6}{1+10+10^5}$$

$$= \frac{-1000000}{100011} = -9.9989$$

If $A_2 = 2 \times 10^5$, then

$$H_2(s) = \frac{-2 \times 10^5 \times 10R}{R + 10R + 2\times 10^5 R} = \frac{-2\times 10^6}{1+10+2\times 10^5}$$

$$= \frac{-2000000}{200011} = -9.9995$$

The open-loop gains differ by

$$\left|\frac{A_2 - A_1}{A_1}\right| = \frac{10^5}{10^5} = 1 \text{ or } 100\%$$

The transfer functions, which happen to be gains, differ by

$$\left|\frac{H_2 - H_1}{H_1}\right| = \frac{0.0006}{9.9989} = 0.00006 \text{ or } 0.006\%$$

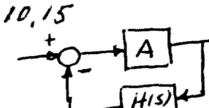
Thus the transfer function is insensitive to the variation of A.

10.14 $s^3 + 2s^2 + s + 1 + A$

	s^3	1	1	
$1/2$	s^2	2	$1+A$	$[0 \quad 1-(1+A)/2]$
	s	$\frac{1-A}{2}$		
	1	$1+A$		

The polynomial is CT stable $\Leftrightarrow$ $1+A > 0$ and $1-A > 0$ $\Leftrightarrow$ $-1 < A < 1$

10.15



If $H(s) = \frac{s+1}{s^2+2s+5}$, then

$$H_o(s) = \frac{A}{1 + \frac{A(s+1)}{s^2+2s+5}}$$

or $$H_o(s) = \frac{A(s^2+2s+5)}{s^2+2s+5+A(s+1)}$$

Its denominator has degree 2 and is CT stable for all $A > 0$. As A becomes very large and if $|(j\omega)^2 + 2j\omega + 5| \ll A|j\omega + 1|$

then $$H_o(s) \approx \frac{A(s^2+2s+5)}{A(s+1)} = \frac{s^2+2s+5}{s+1}$$

Thus the feedback system functions as the inverse of $H(s)$. But it holds only for low-frequency signals.

If $H(s) = \frac{s-1}{s^2+2s+5}$, then

$$H_o(s) = \frac{A}{1+AH(s)} = \frac{A(s^2+2s+5)}{s^2+2s+5+A(s-1)}$$

or $H_o(s) = \dfrac{A(s^2+2s+5)}{s^2+(2+A)s+(5-A)}$

The system is not stable if $A>5$. Thus the inverse of $H(s)$ cannot be implemented as the feedback system with A very large.

10.16 $H(s) = \dfrac{s-2}{s+1}$

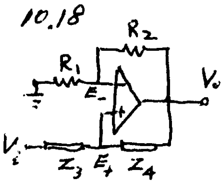
Its inverse $H^{-1}(s) = \dfrac{s+1}{s-2}$ is not stable. If it is implemented as in Fig. 10.16, then

$$H_o(s) = \frac{A}{1+A\cdot\frac{s-2}{s+1}} = \frac{A(s+1)}{s+1+A(s-2)} = \frac{A(s+1)}{(A+1)s+(1-2A)}$$

$H_o(s)$ is not stable for A large. Thus the inverse of $H(s)$ cannot be implemented as in Fig. 10.16.

10.17 Direct substitution

10.18



$$Z_3 = \frac{R\cdot\frac{1}{Cs}}{R+\frac{1}{Cs}} = \frac{R}{RCs+1}$$

$$Z_4 = R+\frac{1}{Cs} = \frac{RCs+1}{Cs}$$

Use ideal model $E_- = E_+$, $I_-=0$, $I_+=0$

$$E_- = \frac{R_1}{R_1+R_2}V_o$$

$$E_+ = \frac{V_o-V_i}{Z_3+Z_4}\cdot Z_3 + V_i = \frac{Z_3}{Z_3+Z_4}V_o + \frac{Z_4}{Z_3+Z_4}V_i$$

$$\frac{R_1}{R_1+R_2}V_o = \frac{Z_3}{Z_3+Z_4}V_o + \frac{Z_4}{Z_3+Z_4}V_i$$

$$\left(\frac{R_1}{R_1+R_2} - \frac{Z_3}{Z_3+Z_4}\right)V_o = \frac{Z_4}{Z_3+Z_4}V_i$$

$$H(s) = \frac{V_o}{V_i} = \frac{\frac{Z_4}{Z_3+Z_4}}{\frac{R_1}{R_1+R_2} - \frac{Z_3}{Z_3+Z_4}}$$

$$= \frac{(R_1+R_2)Z_4}{R_1Z_4 - R_2Z_3}$$

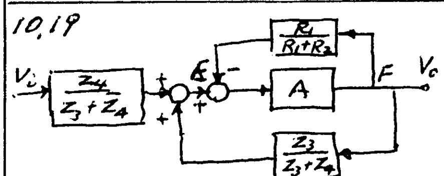
$$= \frac{(R_1+R_2)\cdot\frac{RCs+1}{Cs}}{R_1\cdot\frac{RCs+1}{Cs} - R_2\cdot\frac{R}{RCs+1}}$$

$$= \frac{(R_1+R_2)(RCs+1)^2}{R_1(RCs+1)^2 - R_2RCs}$$

$$= \frac{(R_1+R_2)(RCs+1)^2}{R_1(RCs)^2+(2R_1-R_2)RCs+R_1}$$

Its denominator is the same as the one in (10.45). The condition for $H(s)$ to have poles at $\pm j\omega_0 = \pm j\left(\frac{1}{RC}\right)$ is $2R_1 = R_2$. This result is the same as the one obtained in Section 10.7

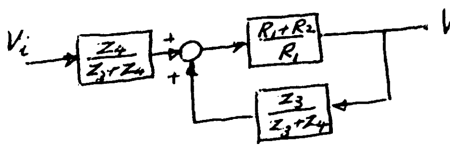
10.19



For Fig. 10.23, Fig. 10.20(b) becomes as shown. The transfer function from E to F is

$$\frac{A}{1+A\cdot\frac{R_1}{R_1+R_2}}$$

which becomes $(R_1+R_2)/R_1$ as $A\to\infty$. Note that it is memoryless and is always stable. It is actually the implementation of inverse systems discussed in Fig 10.16. Thus the block diagram becomes, for ideal op amp,



Its transfer function is

$$\frac{V_o}{V_i} = \frac{Z_4}{Z_3+Z_4}\cdot\frac{\frac{R_1+R_2}{R_1}}{1-\frac{R_1+R_2}{R_1}\cdot\frac{Z_3}{Z_3+Z_4}}$$

$$= \frac{(R_1+R_2)Z_4}{R_1(Z_3+Z_4)-R_1Z_3-R_2Z_3}$$

$$= \frac{(R_1+R_2)Z_4}{R_1Z_4-R_2Z_3}$$

This is the same as the one computed in Prob. 10.18.

Chapter 11

11.1 $H(z) = \mathcal{Z}[h[n]] = \sum_{n=0}^{\infty} (1.00015)^n z^{-n}$

$= \sum_{n=0}^{\infty} (1.00015 z^{-1})^n = \frac{1}{1-1.00015 z^{-1}}$

$= \frac{z}{z-1.00015}.$

11.2 (1) Applying the z-transform to

$2y[n] - 3y[n-1] = 4u[n]$

yields $2Y(z) - 3z^{-1}Y(z) = 4U(z)$

Thus the transfer function is

$H(z) = \frac{Y(z)}{U(z)} = \frac{4}{2-3z^{-1}} = \frac{4z}{2z-3} = \frac{2z}{z-1.5}$

Its inverse z-transform or the impulse response is

$h[n] = 2 \times (1.5)^n, \quad n \geq 0$

IIR. Same as Prob. 7.8(1). Prob. 11.2(2) in Page 45

11.3 Let $y[n]$ and $u[n]$ be positive time. Applying the z-transform to (7.12) yields

$Y(z) = 0.05[U(z) + z^{-1}U(z) + z^{-2}U(z) + \cdots + z^{-19}U(z)]$

$= 0.05(1 + z^{-1} + z^{-2} + \cdots + z^{-19})U(z)$

Thus we have

$H_1(z) = \frac{Y(z)}{U(z)} = 0.05(1 + z^{-1} + z^{-2} + \cdots + z^{-19})$

$= \frac{0.05(z^{19} + z^{18} + \cdots + z + 1)}{z^{19}}$

Applying the z-transform to (7.13) yields

$Y(z) - z^{-1}Y(z) = 0.05(U(z) - z^{-20}U(z))$

$(1 - z^{-1})Y(z) = 0.05(1 - z^{-20})U(z)$

Thus we have

$H_2(z) = \frac{Y(z)}{U(z)} = \frac{0.05(1-z^{-20})}{1-z^{-1}}$

$= \frac{0.05(z^{20}-1)}{z^{19}(z-1)}$

Note that $H_2(z) = H_1(z) \cdot \frac{z-1}{z-1}$. They have the same sets of poles and zeros and are basically the same. See Prob. 11.6.

11.4 $\frac{Y(z)}{U(z)} = \frac{2z^2+5z+3}{4z^2+3z+1} = \frac{2+5z^{-1}+3z^{-2}}{4+3z^{-1}+z^{-2}}$

$(4 + 3z^{-1} + z^{-2})Y(z) = (2 + 5z^{-1} + 3z^{-2})U(z)$

Its difference equation is

$4y[n] + 3y[n-1] + y[n-2] = 2u[n] + 5u[n-1] + 3u[n-2].$

11.5 (1) $H_1(z) = \frac{3z+6}{2z^2+2z+1} = \frac{3(z+2)}{2(z^2+z+0.5)}$

$= \frac{1.5(z+2)}{(z+0.5)^2+0.25}$

$= \frac{1.5(z+2)}{(z+0.5+j0.5)(z+0.5-j0.5)}$

$H_1(z)$ has one zero at -2 and two poles at $-0.5 \pm j0.5$

(2) $H_2(z) = \frac{z^{-1}-z^{-2}-6z^{-3}}{1+2z^{-1}+z^{-2}} = \frac{z^2-z-6}{z^3+2z^2+z}$

$= \frac{(z-3)(z+2)}{z(z+1)^2}$

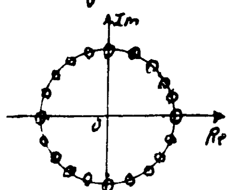
$H_2(z)$ has two zeros at 3 and -2 and three poles at 0, -1, and -1.

11.6 $H_1(z)$ has all poles at $z=0$, a repeated poles with multiplicity 19. Instead of computing the zeros of $H_1(z)$ directly, we consider

$H_2(z) = \frac{0.05(z^{20}-1)}{z^{19}(z-1)}$

The polynomial $z^{20}-1$ has 20 roots. They are the solutions of $z^{20}-1=0$ or

$z^{20} = 1 = e^{jk2\pi}$ or $z = e^{jk2\pi/20}$, It



yields 20 distinct roots for $k = -9:10$; they are equally spaced 20 points on the unit circle as shown with spacing $\pi/10$ rad or $18°$. Because the root at $z=1$ or $(z-1)$ is canceled by the factor $(z-1)$ in the denominator of $H_2(z)$, $z=1$ is not a zero nor a pole of $H_2(z)$. All other 19 roots of $z^{20}-1$ are the zeros of $H_2(z)$

and $H_1(z)$. All 19 poles of $H_2(z)$ and $H_1(z)$ are located at $z=0$.

11.7 $H(z) = \frac{0.9z}{(z+1)(z-0.8)}$

Impulse response: $u[n] = \delta_d[n]$, $U(z) = 1$.

$$Y(z) = H(z)U(z) = \frac{0.9z}{(z+1)(z-0.8)}$$

To find its time sequence or, equivalently, its inverse z-transform, we expand

$$\frac{Y(z)}{z} =: \bar{Y}(z) = \frac{0.9}{(z+1)(z-0.8)}$$

$$= k_0 + k_1 \frac{1}{z+1} + k_2 \cdot \frac{1}{z-0.8}$$

with $k_0 = \bar{Y}(\infty) = 0$

$$k_1 = \frac{0.9}{(z-0.8)}\Big|_{z+1=0} = \frac{0.9}{-1.8} = -0.5$$

$$k_2 = \frac{0.9}{z+1}\Big|_{z-0.8=0} = \frac{0.9}{1.8} = 0.5$$

Thus we have

$$\frac{Y(z)}{z} = \frac{-0.5}{z+1} + \frac{0.5}{z-0.8}$$

which implies

$$Y(z) = -0.5 \cdot \frac{z}{z+1} + 0.5 \frac{z}{z-0.8}$$

Thus the impulse response is

$$h[n] = -0.5(-1)^n + 0.5(0.8)^n, \quad n \geq 0$$

Step response: $u[n] = 1$, $n \geq 0$, $U(z) = \frac{z}{z-1}$

$$Y(z) = H(z)U(z) = \frac{0.9z}{(z+1)(z-0.8)} \cdot \frac{z}{z-1}$$

We expand $Y(z)/z$ as

$$\frac{Y(z)}{z} = \bar{Y}(z) = \frac{0.9z}{(z+1)(z-0.8)(z-1)}$$

$$= k_0 + k_1 \frac{1}{z+1} + k_2 \frac{1}{z-0.8} + k_3 \frac{1}{z-1}$$

with $k_0 = \bar{Y}(\infty) = 0$

$$k_1 = \frac{0.9z}{(z-0.8)(z-1)}\Big|_{z+1=0} = \frac{-0.9}{(-1.8)(-2)} = -0.25$$

$$k_2 = \frac{0.9z}{(z+1)(z-1)}\Big|_{z-0.8=0} = \frac{0.9 \times 0.8}{1.8 \times (-0.2)} = -2$$

$$k_3 = \frac{0.9z}{(z+1)(z-0.8)}\Big|_{z-1=0} = \frac{0.9}{2 \times 1.8} = 0.25$$

Thus we have

$$\frac{Y(z)}{z} = -0.25 \frac{1}{z+1} - 2\frac{1}{z-0.8} + 0.25 \frac{1}{z-1}$$

or $Y(z) = -0.25 \frac{z}{z+1} - 2 \frac{z}{z-0.8} + 0.25 \frac{z}{z-1}$

Thus the step response is

$$y[n] = -0.25(-1)^n - 2(0.8)^n + 0.25, \quad n \geq 0$$

11.8 $H(z) = \frac{z+1}{-8z+10} = \frac{z+1}{-8(z-1.25)}$ $\quad u[n] = \sin 0.1n$

$$U(z) = \frac{z \cdot \sin 0.1}{z^2 - 2(\cos 0.1)z + 1} = \frac{0.0998z}{z^2 - 1.99z + 1}$$

$$= \frac{0.0998z}{(z - e^{j0.1})(z - e^{-j0.1})}$$

$$Y(z) = H(z)U(z) = \frac{(z+1) \times 0.0998z}{-8(z-1.25)(z-e^{j0.1})(z-e^{-j0.1})}$$

$$\frac{Y(z)}{z} = \frac{-0.3742}{z-1.25} + \frac{0.4555e^{-j1.1495}}{z-e^{j0.1}} + \frac{0.4555e^{j1.1495}}{z-e^{-j0.1}}$$

$$\therefore y[n] = -0.374(1.25)^n + 0.455e^{-j1.1495}(e^{j0.1})^n + 0.455e^{j1.1495}(e^{-j0.1})^n$$

$$= -0.374(1.25)^n + 0.455\left[e^{j(0.1n-1.15)} + e^{-j(0.1n-1.15)}\right]$$

$$= -0.374(1.25)^n + 0.91 \cos(0.1n - 1.15)$$

$$= -0.374(1.25)^n + 0.91 \sin(0.1n - 1.15 + \pi/2)$$

$$= -0.374(1.25)^n + 0.91 \sin(0.1n + 0.42), \quad n \geq 0$$

11.9 $H(z) = \frac{b_1 z + b_0}{z - p}$

We expand $H(z)/z$ as

$$\frac{H(z)}{z} = \frac{b_1 z + b_0}{z(z-p)} = \bar{k}_0 + \bar{k}_1 \frac{1}{z} + \bar{k}_2 \frac{1}{z-p}$$

with $\bar{k}_0 = 0$, $\bar{k}_1 = \frac{b_1 z + b_0}{z-p}\Big|_{z=0} = \frac{b_0}{-p} = \frac{-b_0}{p}$,

$$\bar{k}_2 = \frac{b_1 z + b_0}{z}\Big|_{z-p=0} = \frac{b_1 p + b_0}{p}$$

Thus we have

$$H(z) = \frac{-b_0}{p} \cdot \frac{z}{z} + \frac{b_1 p + b_0}{p} \cdot \frac{z}{z-p}$$

which implies

$$\mathcal{Z}^{-1}[H(z)] = \frac{-b_0}{p}\delta_d[n] + \frac{b_1 p + b_0}{p}\cdot p^n, \quad n \geq 0$$

11.10 $u[n] = 1$, $n \geq 0$. $U(z) = \frac{z}{z-1}$

$$Y(z) = H(z)U(z) = \frac{z(z^2+2z+1)}{(z-1)^2(z-0.5+j0.6)(z-0.5-j0.6)}$$

For DT systems, if complex conjugate poles are expressed in polar form, then its magnitude dictates the envelope and its phase dictates the frequency. Recall that we have assumed $T=1$ and its frequency range is $(-\pi, \pi]$. We have $0.5 \pm j0.6 = 0.78e^{\pm j0.88}$. Thus the general form of the step response is

$$y[n] = k_0\delta_d[n] + k_1 + k_2 n + k_3(0.78)^n \sin(0.88n + k_4)$$

for $n \geq 0$. Note that the first three terms are due to the repeated pole at $z=1$ or due to

$$\mathcal{Z}^{-1}\left[\frac{b_2 z^2 + b_1 z + b_0}{(z-1)^2}\right] = k_0\delta_d[n] + k_1(1)^n + k_2 n(1)^n$$

The general form is mainly used to study the response as $n \to \infty$. In this case, the term $k_0\delta_d[n]$ can be ignored because $\delta_d[n] = 0$ for all $n > 0$.

11.11 The step response is

$$Y(z) = \frac{N(z)}{(z+0.6)^3(z-0.5)(z^2+z+0.61)}\cdot\frac{z}{z-1}$$

$$z^2+z+0.61 = (z+0.5+j0.6)(z+0.5-j0.6)$$
$$= (z-0.78e^{j2.27})(z-0.78e^{-j2.27})$$

Thus we have

$$y[n] = k_0\delta_d[n] + k_1(-0.6)^n + k_2 n(-0.6)^n + k_3 n^2(-0.6)^n$$
$$+ k_4(0.5)^n + k_5(0.78)^n \sin(2.27n + k_6) + k_u(1)^n$$

for $n \geq 0$ with $k_u = H(1) = 10$.

$y[n] \to 10$ as $n \to \infty$

The degree difference of $Y(z)$ is $M = 2$. Thus $y[0] = 0$, $y[1] = 0$, $y[2] = 2/1 = 2$.

11.12 A polynomial is CT stable if all its roots have negative real parts. It is DT stable if all its roots have magnitudes less than 1. A polynomial can be defined for variable s, z, x, or others.

$x + 0.5$ is CT and DT stable.

$x + 2$ is CT stable but not DT stable.

$x - 0.5$ is DT stable but not CT stable.

$x - 2$ is not CT stable nor DT stable.

11.13

(1) $$H_1(z) = \frac{z+1}{(z-0.6)^2(z+0.8+j0.6)(z+0.8-j0.6)}$$

The magnitude of the pole at $-0.8-j0.6$ is $\sqrt{(-0.8)^2+(0.6)^2} = \sqrt{0.64+0.36} = \sqrt{1} = 1$.

It is not less than 1. Thus $H_1(z)$ is not stable.

(2) $$H_2(z) = \frac{3z-6}{(z-2)(z+0.2)(z-0.6+j0.7)(z-0.6-j0.7)}$$

Because the numerator $3(z-2)$ has the same factor as the denominator, the poles of $H_2(z)$ are -0.2, $0.6 \pm j0.7$ (2 is not a pole). The magnitude of $0.6 \pm j0.7$ is $|\sqrt{0.36+0.49}| = |\sqrt{0.85}| < 1$. Thus H_2 is stable.

(3) $$H_3(z) = \frac{z-10}{z^2(z+0.95)}$$

It has poles at 0, 0, and -0.95. Their magnitudes are less than 1. Thus H_3 is stable.

11.14 (1) $z^3 + 4z^2 + 0\cdot z + 2$

$$\begin{array}{cccc} 1 & 4 & 0 & 2 \\ 2 & 0 & 4 & 1 \\ \hline -3 & & & \end{array} \quad k_1 = 2/1 = 2$$

A negative leading coefficient appears, thus the polynomial is not DT stable.

(2) $D_2(z) = z^3 - z^2 + 2z - 0.7$

1	−1	2	−0.7	
−0.7	2	−1	1	$k_1 = -0.7$
0.51	0.4	1.3	0	
1.3	0.4	0.51		$k_2 = 1.3/0.51 = 2.25$
−2.81				

$D_2(z)$ is not DT stable.

(3) $D_3(z) = 2z^4 + 1.6z^3 + 1.42z^2 + 0.64z + 0.32$

2	1.6	1.42	0.64	0.32	
0.32	0.64	1.42	1.6	2	$k_1 = 0.16$
1.95	1.5	1.62	0.38	0	
0.38	1.62	1.5	1.95		$k_2 = 0.195$
1.9	1.2	1.33	0		
1.33	1.2	1.9			$k_3 = 0.68$
0.996	0.38	0			
0.38	0.996				$k_4 = 0.386$
0.85					

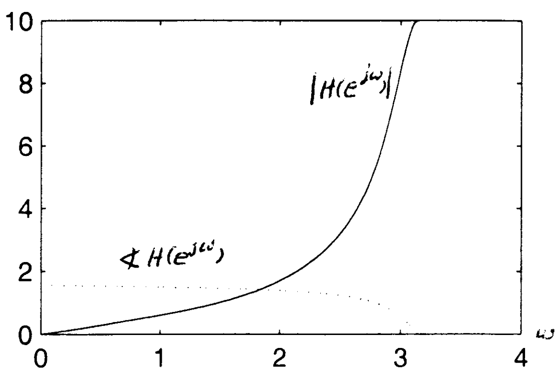
The four subsequent leading coefficients are all positive. Thus $D_3(z)$ is DT stable.

11.15 The polynomial $z^2 - 0.64$ has a missing term and a negative coefficient, it is however DT stable because its roots are ± 0.8 with magnitudes less than 1.

11.16 $H(z) = \frac{z-1}{z+0.8}$. The MATLAB program

```
n=[1 -1];d=[1 0.8];
w=0:0.001:pi;
H=freqz(n,d,w);
plot(w,abs(H),w,angle(H),':')
```

generates



We compute $H(e^{j\omega}) = \frac{e^{j\omega}-1}{e^{j\omega}+0.8}$

$\omega = 0$: $e^{j\omega} = 1 \Rightarrow H(1) = \frac{1-1}{1+0.8} = 0$

$\omega = \pi/4$: $e^{j\omega} = 0.7 + j0.7 \Rightarrow H(e^{j\pi/4}) = \frac{0.7+j0.7-1}{0.7+j0.7+0.8}$

$= \frac{-0.3+j0.7}{1.5+j0.7} = \frac{0.76e^{j1.98}}{1.66e^{j0.44}} = 0.46e^{j1.54}$

$\omega = \pi/2$: $e^{j\omega} = j \Rightarrow H(j) = \frac{j-1}{j+0.8} = \frac{1.4e^{j2.36}}{1.3e^{j0.9}}$

$= 1.08e^{j1.46}$

We see that hand computation is very complex. The computed values at $\omega = \pi/4 = 0.78$ and $\omega = \pi/2 = 1.57$ are consistent with the plot. However the phase at $\omega = 0$ is $\pi/2$ in the plot. This can be explained as follows. For ω very small, we have $e^{j\omega} = 1 + j\omega$ and $H(e^{j\omega}) = \frac{1+j\omega-1}{1+j\omega+0.8}$

$= \frac{j\omega}{1.8+j\omega} \approx \frac{j\omega}{1.8}$. Thus its phase is $90°$ or $\pi/2$ (rad)

Recall that we have assumed $T=1$ and the Nyquist frequency rang is $(-\pi/T, \pi/T] = (-\pi, \pi]$. Thus the highest frequency is π. From the magnitude response shown, we see that the system is a highpass filter.

11.17 From the plot in Prob. 11.16, we can read roughly $H(e^{j\cdot 0}) = H(1) = 0\cdot e^{j\pi/2}$, $H(e^{j0.1}) = 0.06e^{j1.6}$ and $H(e^{j3}) = 8.4e^{j0.6}$. Thus the steady-state response is, using Theorem 11.4,

$$y_{ss}[n] = 0 \times 2 + 0.06 \sin(0.1n + 1.6) + 8.4 \cos(3n + 0.6) \approx 8.4 \cos(3n + 0.6)$$

It passes the high-frequency sinusoid and attenuates greatly low frequency sinusoids.

$H(s) = (z-1)/(z+0.8)$ has only one pole at -0.8. Its time constant is

$$n_c = \frac{1}{\ln(0.8)} = 4.48$$

Thus it takes roughly $5n_c = 22.4$ samples to reach steady-state. Recall that we disregard the fact that the number of samples must be an integer, because we are interested in only an estimate.

11.18 $H_1(z) = \frac{z+1}{10z-8}$, $H_2(z) = \frac{z+1}{8z-10}$

To show $|H_1(e^{j\omega})| = |H_2(e^{j\omega})|$, we show

$$|10e^{j\omega} - 8| = |8e^{j\omega} - 10|$$

We have

$$|10e^{j\omega} - 8| = |10\cos\omega + j10\sin\omega - 8|$$
$$= \sqrt{(10\cos\omega - 8)^2 + (10\sin\omega)^2} \quad (1)$$

We compute

$$|8e^{j\omega} - 10| = |e^{j\omega}(8 - 10e^{-j\omega})|$$
$$= |-e^{j\omega}|\,|10e^{-j\omega} - 8| = |10e^{-j\omega} - 8|$$
$$= |10\cos\omega - j10\sin\omega - 8|$$
$$= \sqrt{(10\cos\omega - 8)^2 + (-10\sin\omega)^2}$$

which equals (1). Thus $H_1(z)$ and $H_2(z)$ have the same magnitude response. Their phase responses however are different.

11.19 $H(z) = \frac{1}{z+1}$

has pole at $z=-1$ and is not stable. Its steady state response excited by $\sin 0.6n$ is of the form

$$y_{ss}[n] = |H(e^{j0.6})| \sin(0.6n + \measuredangle H(e^{j0.6})) + k_1(-1)^n$$

The term $k_1(-1)^n$ is due to the pole of $H(z)$, and can be expressed as $k_1 \cos n\pi$. Thus it is a sinusoid with the highest possible frequency π. It will not approach zero and Theorem 11.4 does not hold.

11.20 $H(e^{j\omega}) = \begin{cases} 1 \times e^{-j\omega n_0} & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$
$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega n_0} e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-n_0)} d\omega$$
$$= \frac{1}{2\pi j(n-n_0)} e^{j\omega(n-n_0)}\Big|_{\omega=-\omega_c}^{\omega_c}$$
$$= \frac{e^{j(n-n_0)\omega_c} - e^{-j(n-n_0)\omega_c}}{2\pi j(n-n_0)}$$
$$= \frac{\sin[(n-n_0)\omega_c]}{\pi(n-n_0)}$$

for all n.

11.21 $H(z) = \frac{2z^2+5z+3}{4z^2+3z+1} = \frac{0.5z^2+1.25z+0.75}{z^2+0.75z+0.25}$

$$= 0.5 + \frac{0.875z + 0.625}{z^2+0.75z+0.25}$$

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \end{bmatrix} = \begin{bmatrix} -0.75 & -0.25 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[n]$$

$$y[n] = [0.875 \quad 0.625] \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + 0.5u[n]$$

11.22 $H(z) = \frac{z}{z-1.00015} = 1 + \frac{1.00015}{z-1.00015}$

$$x[n+1] = 1.00015\,x[n] + u[n]$$
$$y[n] = 1.00015\,x[n] + u[n]$$

The result is the same.

11.23 $H(z) = \frac{z}{z-0.8} = \frac{0.8}{z-0.8} + 1$

Its one-dimensional or minimal realization is

$$x[n+1] = 0.8\,x[n] + u[n]$$
$$y[n] = 0.8\,x[n] + u[n]$$

To find a 2-dimensional realization, we consider

$$H(z) = \frac{0.8(z-0.5)}{(z-0.8)(z-0.5)} + 1$$
$$= \frac{0.8z - 0.4}{z^2 - 1.3z + 0.4} + 1$$

Its realization is

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \end{bmatrix} = \begin{bmatrix} 1.3 & -0.4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[n]$$

$$y[n] = [0.8 \quad -0.4] \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + u[n]$$

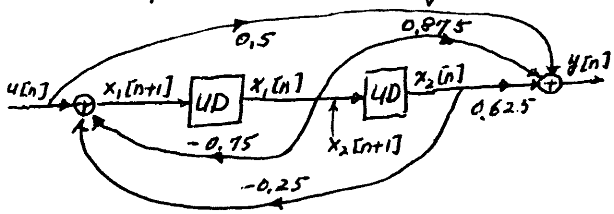
11.24 We use the realization in Prob. 11.21

or $x_1[n+1] = -0.75x_1[n] - 0.25x_2[n] + u[n]$

$x_2[n+1] = x_1[n]$

$y[n] = 0.875x_1[n] + 0.625x_2[n] + 0.5u[n]$

to develop a basic block diagram



11.25 If $X(z)$ has a simple pole at $z=1$, it can be expanded as

$$X(z) = k \cdot \frac{z}{z-1} + \text{Terms due to other poles}$$

with

$$k = \lim_{z \to 1} X(z) \cdot \frac{z-1}{z} = \lim_{z \to 1} (z-1)X(z)$$

If all other poles have magnitudes less than 1, then their time responses all approach zero as $n \to \infty$. Thus we have

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1)X(z) = k \text{ (a nonzero constant)}$$

If $X(z)$ has no pole at $z=1$ and all poles have magnitudes less than 1, then its time sequence $x[n] \to 0$ as $n \to \infty$. In this case the equation still holds because $X(z)$ has no pole to cancel $(z-1)$ and we have

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1)X(z) = 0$$

If $X(z)$ has one or more poles with magnitudes larger than 1 or repeated pole at $z=1$, then $x[n] \to \infty$ as $n \to \infty$. If $X(z)$ has simple complex-conjugate poles on the unit circle, then $x[n]$ contains sustained oscillation, and the equation is not applicable.

11.2 (2) Applying the z-transform to

$$y[n] + 2y[n-1] = -2u[n-1] - u[n-2] + 6u[n-3]$$
$$Y(z) + 2z^{-1}Y(z) = -2z^{-1}U(z) - z^{-2}U(z) + 6z^{-3}U(z)$$

Thus its transfer function is

$$H(z) = \frac{Y(z)}{U(z)} = \frac{-2z^{-1} - z^{-2} + 6z^{-3}}{1 + 2z^{-1}} = \frac{-2z^2 - z + 6}{z^3 + 2z^2}$$
$$= \frac{(-2z+3)(z+2)}{z^2(z+2)} = \frac{-2z+3}{z^2} = -2z^{-1} + 3z^{-2}$$

Its inverse z-transform or the impulse response is

$h[0]=0$, $h[1]=-2$, $h[2]=3$ and $h[n]=0$

for all n other than 0, 1 and 2. It is FIR. The result is the same as Prob. 7.8(2).