

PROTEIN DESIGN

undoing some of the work already done. Checking a step in the sequence for feasibility is a difficult geometrical search problem closely related to robot navigation. Thus, the generation of legal actions is the expensive part of assembly sequencing. Any practical algorithm must avoid exploring all but a tiny fraction of the state space. Another important assembly problem is **protein design**, in which the goal is to find a sequence of amino acids that will fold into a three-dimensional protein with the right properties to cure some disease.

### 3.3 SEARCHING FOR SOLUTIONS

SEARCH TREE

NODE

EXPANDING

GENERATING

PARENT NODE

CHILD NODE

LEAF NODE

FRONTIER

OPEN LIST

SEARCH STRATEGY

REPEATED STATE

LOOPY PATH

Having formulated some problems, we now need to solve them. A solution is an action sequence, so search algorithms work by considering various possible action sequences. The possible action sequences starting at the initial state form a **search tree** with the initial state at the root; the branches are actions and the **nodes** correspond to states in the state space of the problem. Figure 3.6 shows the first few steps in growing the search tree for finding a route from Arad to Bucharest. The root node of the tree corresponds to the initial state, *In(Arad)*. The first step is to test whether this is a goal state. (Clearly it is not, but it is important to check so that we can solve trick problems like “starting in Arad, get to Arad.”) Then we need to consider taking various actions. We do this by **expanding** the current state; that is, applying each legal action to the current state, thereby **generating** a new set of states. In this case, we add three branches from the **parent node** *In(Arad)* leading to three new **child nodes**: *In(Sibiu)*, *In(Timisoara)*, and *In(Zerind)*. Now we must choose which of these three possibilities to consider further.

This is the essence of search—following up one option now and putting the others aside for later, in case the first choice does not lead to a solution. Suppose we choose Sibiu first. We check to see whether it is a goal state (it is not) and then expand it to get *In(Arad)*, *In(Fagaras)*, *In(Oradea)*, and *In(Rimnicu Vilcea)*. We can then choose any of these four or go back and choose Timisoara or Zerind. Each of these six nodes is a **leaf node**, that is, a node with no children in the tree. The set of all leaf nodes available for expansion at any given point is called the **frontier**. (Many authors call it the **open list**, which is both geographically less evocative and less accurate, because other data structures are better suited than a list.) In Figure 3.6, the frontier of each tree consists of those nodes with bold outlines.

The process of expanding nodes on the frontier continues until either a solution is found or there are no more states to expand. The general TREE-SEARCH algorithm is shown informally in Figure 3.7. Search algorithms all share this basic structure; they vary primarily according to how they choose which state to expand next—the so-called **search strategy**.

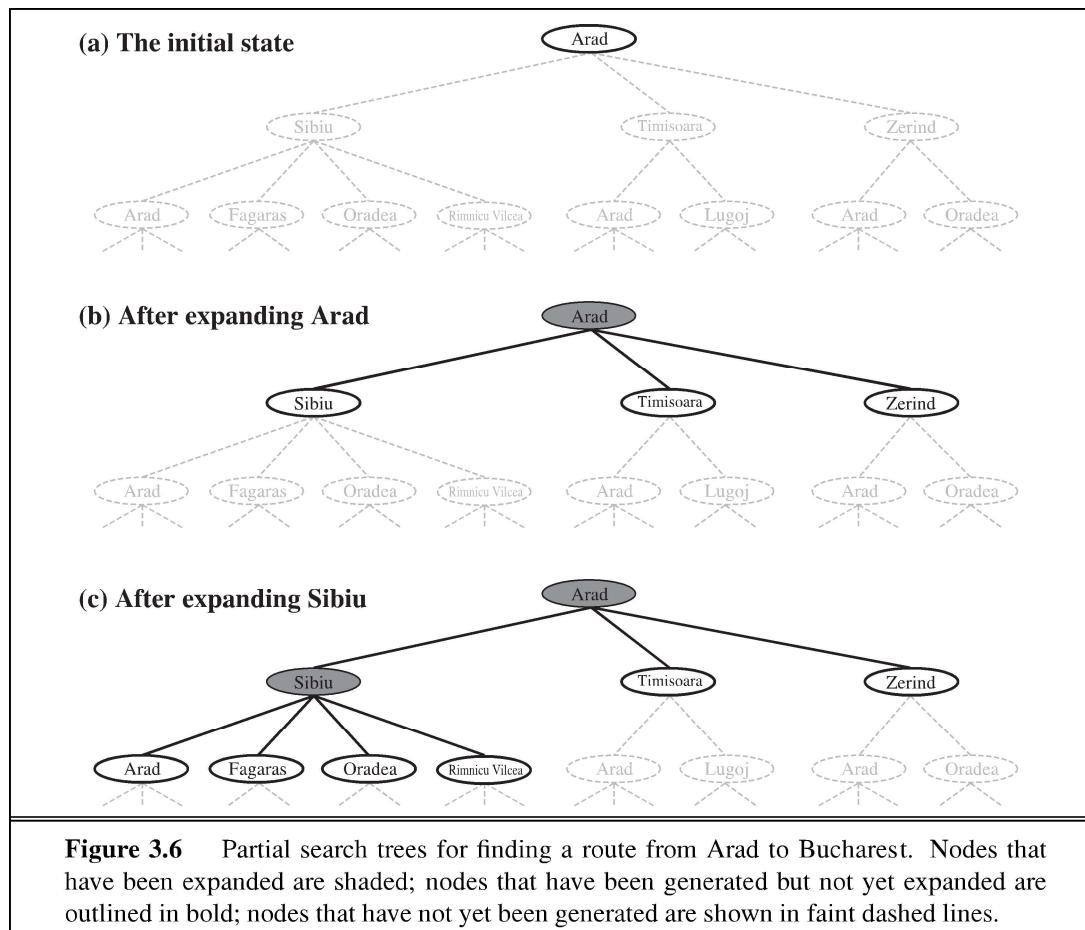
The eagle-eyed reader will notice one peculiar thing about the search tree shown in Figure 3.6: it includes the path from Arad to Sibiu and back to Arad again! We say that *In(Arad)* is a **repeated state** in the search tree, generated in this case by a **loopy path**. Considering such loopy paths means that the complete search tree for Romania is *infinite* because there is no limit to how often one can traverse a loop. On the other hand, the state space—the map shown in Figure 3.2—has only 20 states. As we discuss in Section 3.4, loops can cause

certain algorithms to fail, making otherwise solvable problems unsolvable. Fortunately, there is no need to consider loopy paths. We can rely on more than intuition for this: because path costs are additive and step costs are nonnegative, a loopy path to any given state is never better than the same path with the loop removed.

## REDUNDANT PATH

Loopy paths are a special case of the more general concept of **redundant paths**, which exist whenever there is more than one way to get from one state to another. Consider the paths Arad–Sibiu (140 km long) and Arad–Zerind–Oradea–Sibiu (297 km long). Obviously, the second path is redundant—it's just a worse way to get to the same state. If you are concerned about reaching the goal, there's never any reason to keep more than one path to any given state, because any goal state that is reachable by extending one path is also reachable by extending the other.

In some cases, it is possible to define the problem itself so as to eliminate redundant paths. For example, if we formulate the 8-queens problem (page 71) so that a queen can be placed in any column, then each state with  $n$  queens can be reached by  $n!$  different paths; but if we reformulate the problem so that each new queen is placed in the leftmost empty column, then each state can be reached only through one path.



```

function TREE-SEARCH(problem) returns a solution, or failure
  initialize the frontier using the initial state of problem
  loop do
    if the frontier is empty then return failure
    choose a leaf node and remove it from the frontier
    if the node contains a goal state then return the corresponding solution
    expand the chosen node, adding the resulting nodes to the frontier

function GRAPH-SEARCH(problem) returns a solution, or failure
  initialize the frontier using the initial state of problem
  initialize the explored set to be empty
  loop do
    if the frontier is empty then return failure
    choose a leaf node and remove it from the frontier
    if the node contains a goal state then return the corresponding solution
    add the node to the explored set
    expand the chosen node, adding the resulting nodes to the frontier
    only if not in the frontier or explored set

```

**Figure 3.7** An informal description of the general tree-search and graph-search algorithms. The parts of GRAPH-SEARCH marked in bold italic are the additions needed to handle repeated states.

In other cases, redundant paths are unavoidable. This includes all problems where the actions are reversible, such as route-finding problems and sliding-block puzzles. Route-finding on a **rectangular grid** (like the one used later for Figure 3.9) is a particularly important example in computer games. In such a grid, each state has four successors, so a search tree of depth  $d$  that includes repeated states has  $4^d$  leaves; but there are only about  $2d^2$  distinct states within  $d$  steps of any given state. For  $d = 20$ , this means about a trillion nodes but only about 800 distinct states. Thus, following redundant paths can cause a tractable problem to become intractable. This is true even for algorithms that know how to avoid infinite loops.

As the saying goes, *algorithms that forget their history are doomed to repeat it*. The way to avoid exploring redundant paths is to remember where one has been. To do this, we augment the TREE-SEARCH algorithm with a data structure called the **explored set** (also known as the **closed list**), which remembers every expanded node. Newly generated nodes that match previously generated nodes—ones in the explored set or the frontier—can be discarded instead of being added to the frontier. The new algorithm, called GRAPH-SEARCH, is shown informally in Figure 3.7. The specific algorithms in this chapter draw on this general design.

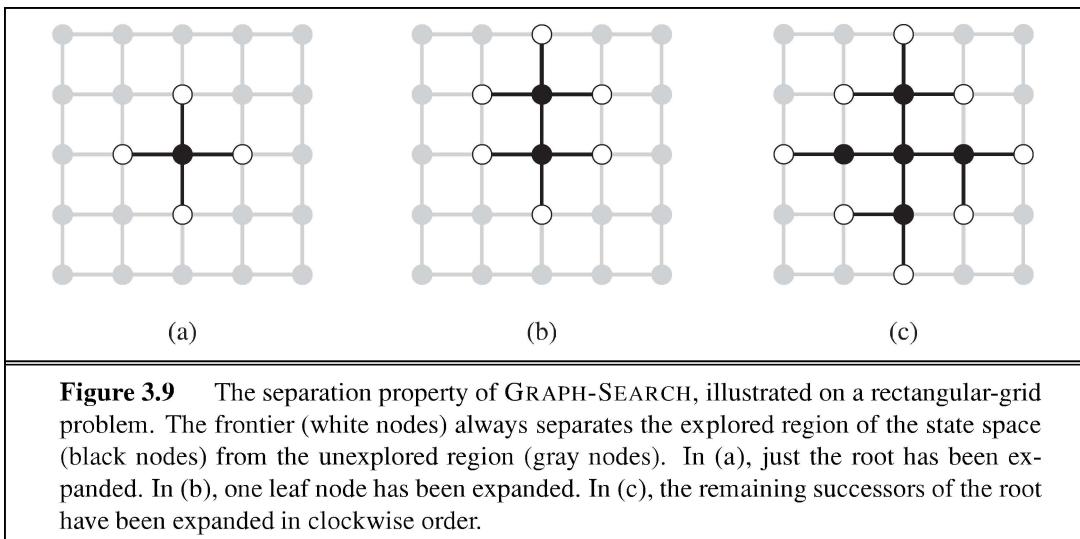
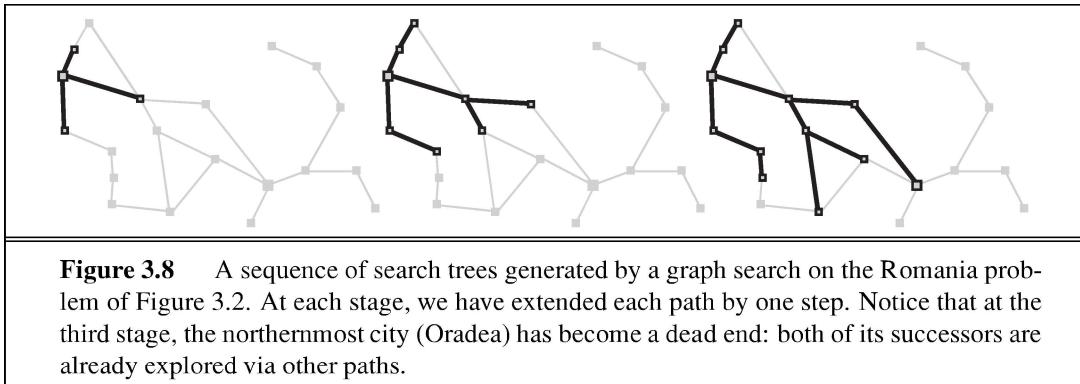
Clearly, the search tree constructed by the GRAPH-SEARCH algorithm contains at most one copy of each state, so we can think of it as growing a tree directly on the state-space graph, as shown in Figure 3.8. The algorithm has another nice property: the frontier **separates** the state-space graph into the explored region and the unexplored region, so that every path from

RECTANGULAR GRID



EXPLORED SET  
CLOSED LIST

SEPARATOR

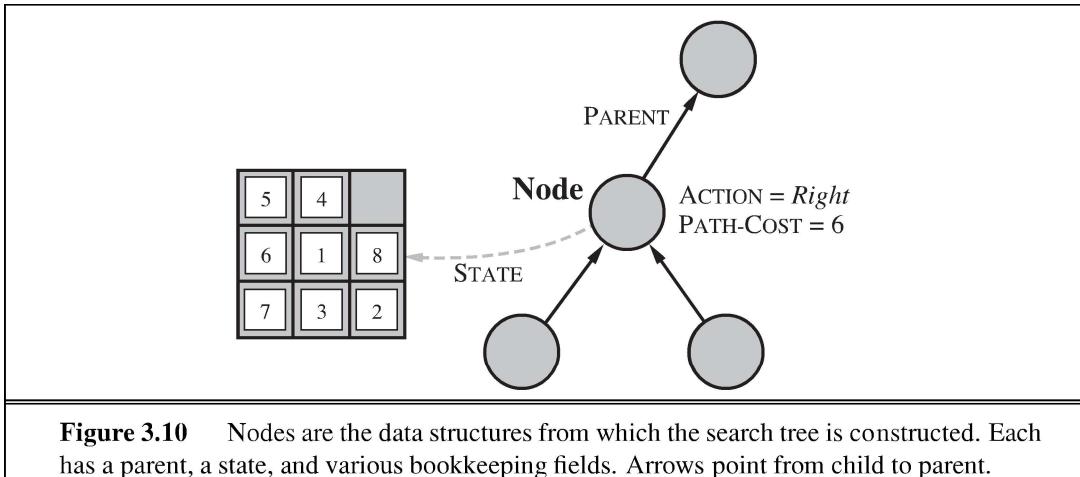


the initial state to an unexplored state has to pass through a state in the frontier. (If this seems completely obvious, try Exercise 3.13 now.) This property is illustrated in Figure 3.9. As every step moves a state from the frontier into the explored region while moving some states from the unexplored region into the frontier, we see that the algorithm is *systematically* examining the states in the state space, one by one, until it finds a solution.

### 3.3.1 Infrastructure for search algorithms

Search algorithms require a data structure to keep track of the search tree that is being constructed. For each node  $n$  of the tree, we have a structure that contains four components:

- $n.\text{STATE}$ : the state in the state space to which the node corresponds;
- $n.\text{PARENT}$ : the node in the search tree that generated this node;
- $n.\text{ACTION}$ : the action that was applied to the parent to generate the node;
- $n.\text{PATH-COST}$ : the cost, traditionally denoted by  $g(n)$ , of the path from the initial state to the node, as indicated by the parent pointers.



Given the components for a parent node, it is easy to see how to compute the necessary components for a child node. The function **CHILD-NODE** takes a parent node and an action and returns the resulting child node:

```

function CHILD-NODE(problem, parent, action) returns a node
  return a node with
    STATE = problem.RESULT(parent.STATE, action),
    PARENT = parent, ACTION = action,
    PATH-COST = parent.PATH-COST + problem.STEP-COST(parent.STATE, action)
  
```

The node data structure is depicted in Figure 3.10. Notice how the PARENT pointers string the nodes together into a tree structure. These pointers also allow the solution path to be extracted when a goal node is found; we use the **SOLUTION** function to return the sequence of actions obtained by following parent pointers back to the root.

Up to now, we have not been very careful to distinguish between nodes and states, but in writing detailed algorithms it's important to make that distinction. A node is a bookkeeping data structure used to represent the search tree. A state corresponds to a configuration of the world. Thus, nodes are on particular paths, as defined by PARENT pointers, whereas states are not. Furthermore, two different nodes can contain the same world state if that state is generated via two different search paths.

Now that we have nodes, we need somewhere to put them. The frontier needs to be stored in such a way that the search algorithm can easily choose the next node to expand according to its preferred strategy. The appropriate data structure for this is a **queue**. The operations on a queue are as follows:

- **EMPTY?(queue)** returns true only if there are no more elements in the queue.
- **POP(queue)** removes the first element of the queue and returns it.
- **INSERT(element, queue)** inserts an element and returns the resulting queue.

FIFO QUEUE  
LIFO QUEUE  
PRIORITY QUEUE

Queues are characterized by the *order* in which they store the inserted nodes. Three common variants are the first-in, first-out or **FIFO queue**, which pops the *oldest* element of the queue; the last-in, first-out or **LIFO queue** (also known as a **stack**), which pops the *newest* element of the queue; and the **priority queue**, which pops the element of the queue with the highest priority according to some ordering function.

CANONICAL FORM

The explored set can be implemented with a hash table to allow efficient checking for repeated states. With a good implementation, insertion and lookup can be done in roughly constant time no matter how many states are stored. One must take care to implement the hash table with the right notion of equality between states. For example, in the traveling salesperson problem (page 74), the hash table needs to know that the set of visited cities {Bucharest,Urziceni,Vaslui} is the same as {Urziceni,Vaslui,Bucharest}. Sometimes this can be achieved most easily by insisting that the data structures for states be in some **canonical form**; that is, logically equivalent states should map to the same data structure. In the case of states described by sets, for example, a bit-vector representation or a sorted list without repetition would be canonical, whereas an unsorted list would not.

COMPLETENESS  
OPTIMALITY  
TIME COMPLEXITY  
SPACE COMPLEXITY

### 3.3.2 Measuring problem-solving performance

Before we get into the design of specific search algorithms, we need to consider the criteria that might be used to choose among them. We can evaluate an algorithm's performance in four ways:

- **Completeness:** Is the algorithm guaranteed to find a solution when there is one?
- **Optimality:** Does the strategy find the optimal solution, as defined on page 68?
- **Time complexity:** How long does it take to find a solution?
- **Space complexity:** How much memory is needed to perform the search?

BRANCHING FACTOR  
DEPTH

Time and space complexity are always considered with respect to some measure of the problem difficulty. In theoretical computer science, the typical measure is the size of the state space graph,  $|V| + |E|$ , where  $V$  is the set of vertices (nodes) of the graph and  $E$  is the set of edges (links). This is appropriate when the graph is an explicit data structure that is input to the search program. (The map of Romania is an example of this.) In AI, the graph is often represented *implicitly* by the initial state, actions, and transition model and is frequently infinite. For these reasons, complexity is expressed in terms of three quantities:  $b$ , the **branching factor** or maximum number of successors of any node;  $d$ , the **depth** of the shallowest goal node (i.e., the number of steps along the path from the root); and  $m$ , the maximum length of any path in the state space. Time is often measured in terms of the number of nodes generated during the search, and space in terms of the maximum number of nodes stored in memory. For the most part, we describe time and space complexity for search on a tree; for a graph, the answer depends on how “redundant” the paths in the state space are.

SEARCH COST  
TOTAL COST

To assess the effectiveness of a search algorithm, we can consider just the **search cost**—which typically depends on the time complexity but can also include a term for memory usage—or we can use the **total cost**, which combines the search cost and the path cost of the solution found. For the problem of finding a route from Arad to Bucharest, the search cost is the amount of time taken by the search and the solution cost is the total length of the path

in kilometers. Thus, to compute the total cost, we have to add milliseconds and kilometers. There is no “official exchange rate” between the two, but it might be reasonable in this case to convert kilometers into milliseconds by using an estimate of the car’s average speed (because time is what the agent cares about). This enables the agent to find an optimal tradeoff point at which further computation to find a shorter path becomes counterproductive. The more general problem of tradeoffs between different goods is taken up in Chapter 16.

## 3.4 UNINFORMED SEARCH STRATEGIES

This section covers several search strategies that come under the heading of **uninformed search** (also called **blind search**). The term means that the strategies have no additional information about states beyond that provided in the problem definition. All they can do is generate successors and distinguish a goal state from a non-goal state. All search strategies are distinguished by the *order* in which nodes are expanded. Strategies that know whether one non-goal state is “more promising” than another are called **informed search** or **heuristic search** strategies; they are covered in Section 3.5.

### 3.4.1 Breadth-first search

**Breadth-first search** is a simple strategy in which the root node is expanded first, then all the successors of the root node are expanded next, then *their* successors, and so on. In general, all the nodes are expanded at a given depth in the search tree before any nodes at the next level are expanded.

Breadth-first search is an instance of the general graph-search algorithm (Figure 3.7) in which the *shallowest* unexpanded node is chosen for expansion. This is achieved very simply by using a FIFO queue for the frontier. Thus, new nodes (which are always deeper than their parents) go to the back of the queue, and old nodes, which are shallower than the new nodes, get expanded first. There is one slight tweak on the general graph-search algorithm, which is that the goal test is applied to each node when it is *generated* rather than when it is selected for expansion. This decision is explained below, where we discuss time complexity. Note also that the algorithm, following the general template for graph search, discards any new path to a state already in the frontier or explored set; it is easy to see that any such path must be at least as deep as the one already found. Thus, breadth-first search always has the shallowest path to every node on the frontier.

Pseudocode is given in Figure 3.11. Figure 3.12 shows the progress of the search on a simple binary tree.

How does breadth-first search rate according to the four criteria from the previous section? We can easily see that it is *complete*—if the shallowest goal node is at some finite depth  $d$ , breadth-first search will eventually find it after generating all shallower nodes (provided the branching factor  $b$  is finite). Note that as soon as a goal node is generated, we know it is the shallowest goal node because all shallower nodes must have been generated already and failed the goal test. Now, the *shallowest* goal node is not necessarily the *optimal* one;

```

function BREADTH-FIRST-SEARCH(problem) returns a solution, or failure
  node  $\leftarrow$  a node with STATE = problem.INITIAL-STATE, PATH-COST = 0
  if problem.GOAL-TEST(node.STATE) then return SOLUTION(node)
  frontier  $\leftarrow$  a FIFO queue with node as the only element
  explored  $\leftarrow$  an empty set
  loop do
    if EMPTY?(frontier) then return failure
    node  $\leftarrow$  POP(frontier) /* chooses the shallowest node in frontier */
    add node.STATE to explored
    for each action in problem.ACTIONS(node.STATE) do
      child  $\leftarrow$  CHILD-NODE(problem, node, action)
      if child.STATE is not in explored or frontier then
        if problem.GOAL-TEST(child.STATE) then return SOLUTION(child)
        frontier  $\leftarrow$  INSERT(child, frontier)
  
```

**Figure 3.11** Breadth-first search on a graph.

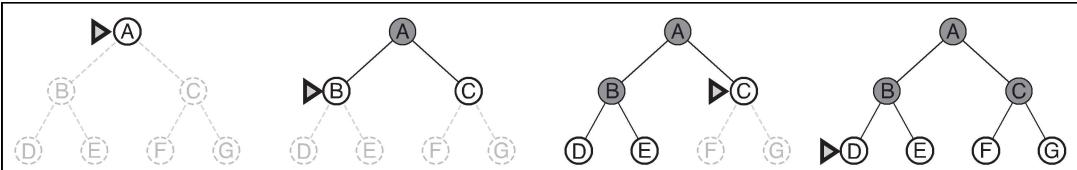
technically, breadth-first search is optimal if the path cost is a nondecreasing function of the depth of the node. The most common such scenario is that all actions have the same cost.

So far, the news about breadth-first search has been good. The news about time and space is not so good. Imagine searching a uniform tree where every state has  $b$  successors. The root of the search tree generates  $b$  nodes at the first level, each of which generates  $b$  more nodes, for a total of  $b^2$  at the second level. Each of *these* generates  $b$  more nodes, yielding  $b^3$  nodes at the third level, and so on. Now suppose that the solution is at depth  $d$ . In the worst case, it is the last node generated at that level. Then the total number of nodes generated is

$$b + b^2 + b^3 + \dots + b^d = O(b^d).$$

(If the algorithm were to apply the goal test to nodes when selected for expansion, rather than when generated, the whole layer of nodes at depth  $d$  would be expanded before the goal was detected and the time complexity would be  $O(b^{d+1})$ .)

As for space complexity: for any kind of graph search, which stores every expanded node in the *explored* set, the space complexity is always within a factor of  $b$  of the time complexity. For breadth-first graph search in particular, every node generated remains in memory. There will be  $O(b^{d-1})$  nodes in the *explored* set and  $O(b^d)$  nodes in the frontier,



**Figure 3.12** Breadth-first search on a simple binary tree. At each stage, the node to be expanded next is indicated by a marker.

so the space complexity is  $O(b^d)$ , i.e., it is dominated by the size of the frontier. Switching to a tree search would not save much space, and in a state space with many redundant paths, switching could cost a great deal of time.

An exponential complexity bound such as  $O(b^d)$  is scary. Figure 3.13 shows why. It lists, for various values of the solution depth  $d$ , the time and memory required for a breadth-first search with branching factor  $b = 10$ . The table assumes that 1 million nodes can be generated per second and that a node requires 1000 bytes of storage. Many search problems fit roughly within these assumptions (give or take a factor of 100) when run on a modern personal computer.

Depth	Nodes	Time	Memory
2	110	.11 milliseconds	107 kilobytes
4	11,110	11 milliseconds	10.6 megabytes
6	$10^6$	1.1 seconds	1 gigabyte
8	$10^8$	2 minutes	103 gigabytes
10	$10^{10}$	3 hours	10 terabytes
12	$10^{12}$	13 days	1 petabyte
14	$10^{14}$	3.5 years	99 petabytes
16	$10^{16}$	350 years	10 exabytes

**Figure 3.13** Time and memory requirements for breadth-first search. The numbers shown assume branching factor  $b = 10$ ; 1 million nodes/second; 1000 bytes/node.



Two lessons can be learned from Figure 3.13. First, *the memory requirements are a bigger problem for breadth-first search than is the execution time*. One might wait 13 days for the solution to an important problem with search depth 12, but no personal computer has the petabyte of memory it would take. Fortunately, other strategies require less memory.



The second lesson is that time is still a major factor. If your problem has a solution at depth 16, then (given our assumptions) it will take about 350 years for breadth-first search (or indeed any uninformed search) to find it. In general, *exponential-complexity search problems cannot be solved by uninformed methods for any but the smallest instances*.

### 3.4.2 Uniform-cost search

UNIFORM-COST  
SEARCH

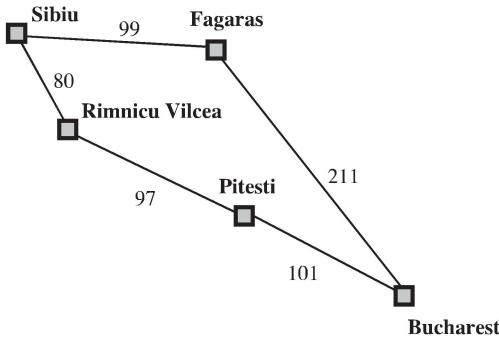
When all step costs are equal, breadth-first search is optimal because it always expands the *shallowest* unexpanded node. By a simple extension, we can find an algorithm that is optimal with any step-cost function. Instead of expanding the shallowest node, **uniform-cost search** expands the node  $n$  with the *lowest path cost*  $g(n)$ . This is done by storing the frontier as a priority queue ordered by  $g$ . The algorithm is shown in Figure 3.14.

In addition to the ordering of the queue by path cost, there are two other significant differences from breadth-first search. The first is that the goal test is applied to a node when it is *selected for expansion* (as in the generic graph-search algorithm shown in Figure 3.7) rather than when it is first generated. The reason is that the first goal node that is *generated*

```

function UNIFORM-COST-SEARCH(problem) returns a solution, or failure
  node  $\leftarrow$  a node with STATE = problem.INITIAL-STATE, PATH-COST = 0
  frontier  $\leftarrow$  a priority queue ordered by PATH-COST, with node as the only element
  explored  $\leftarrow$  an empty set
  loop do
    if EMPTY?(frontier) then return failure
    node  $\leftarrow$  POP(frontier) /* chooses the lowest-cost node in frontier */
    if problem.GOAL-TEST(node.STATE) then return SOLUTION(node)
    add node.STATE to explored
    for each action in problem.ACTIONS(node.STATE) do
      child  $\leftarrow$  CHILD-NODE(problem, node, action)
      if child.STATE is not in explored or frontier then
        frontier  $\leftarrow$  INSERT(child, frontier)
      else if child.STATE is in frontier with higher PATH-COST then
        replace that frontier node with child
  
```

**Figure 3.14** Uniform-cost search on a graph. The algorithm is identical to the general graph search algorithm in Figure 3.7, except for the use of a priority queue and the addition of an extra check in case a shorter path to a frontier state is discovered. The data structure for *frontier* needs to support efficient membership testing, so it should combine the capabilities of a priority queue and a hash table.



**Figure 3.15** Part of the Romania state space, selected to illustrate uniform-cost search.

may be on a suboptimal path. The second difference is that a test is added in case a better path is found to a node currently on the frontier.

Both of these modifications come into play in the example shown in Figure 3.15, where the problem is to get from Sibiu to Bucharest. The successors of Sibiu are Rimnicu Vilcea and Fagaras, with costs 80 and 99, respectively. The least-cost node, Rimnicu Vilcea, is expanded next, adding Pitesti with cost  $80 + 97 = 177$ . The least-cost node is now Fagaras, so it is expanded, adding Bucharest with cost  $99 + 211 = 310$ . Now a goal node has been generated, but uniform-cost search keeps going, choosing Pitesti for expansion and adding a second path

to Bucharest with cost  $80 + 97 + 101 = 278$ . Now the algorithm checks to see if this new path is better than the old one; it is, so the old one is discarded. Bucharest, now with  $g$ -cost 278, is selected for expansion and the solution is returned.

It is easy to see that uniform-cost search is optimal in general. First, we observe that whenever uniform-cost search selects a node  $n$  for expansion, the optimal path to that node has been found. (Were this not the case, there would have to be another frontier node  $n'$  on the optimal path from the start node to  $n$ , by the graph separation property of Figure 3.9; by definition,  $n'$  would have lower  $g$ -cost than  $n$  and would have been selected first.) Then, because step costs are nonnegative, paths never get shorter as nodes are added. These two facts together imply that *uniform-cost search expands nodes in order of their optimal path cost*. Hence, the first goal node selected for expansion must be the optimal solution.



Uniform-cost search does not care about the *number* of steps a path has, but only about their total cost. Therefore, it will get stuck in an infinite loop if there is a path with an infinite sequence of zero-cost actions—for example, a sequence of *NoOp* actions.<sup>6</sup> Completeness is guaranteed provided the cost of every step exceeds some small positive constant  $\epsilon$ .

Uniform-cost search is guided by path costs rather than depths, so its complexity is not easily characterized in terms of  $b$  and  $d$ . Instead, let  $C^*$  be the cost of the optimal solution,<sup>7</sup> and assume that every action costs at least  $\epsilon$ . Then the algorithm's worst-case time and space complexity is  $O(b^{1+\lfloor C^*/\epsilon \rfloor})$ , which can be much greater than  $b^d$ . This is because uniform-cost search can explore large trees of small steps before exploring paths involving large and perhaps useful steps. When all step costs are equal,  $b^{1+\lfloor C^*/\epsilon \rfloor}$  is just  $b^{d+1}$ . When all step costs are the same, uniform-cost search is similar to breadth-first search, except that the latter stops as soon as it generates a goal, whereas uniform-cost search examines all the nodes at the goal's depth to see if one has a lower cost; thus uniform-cost search does strictly more work by expanding nodes at depth  $d$  unnecessarily.

### 3.4.3 Depth-first search

DEPTH-FIRST  
SEARCH

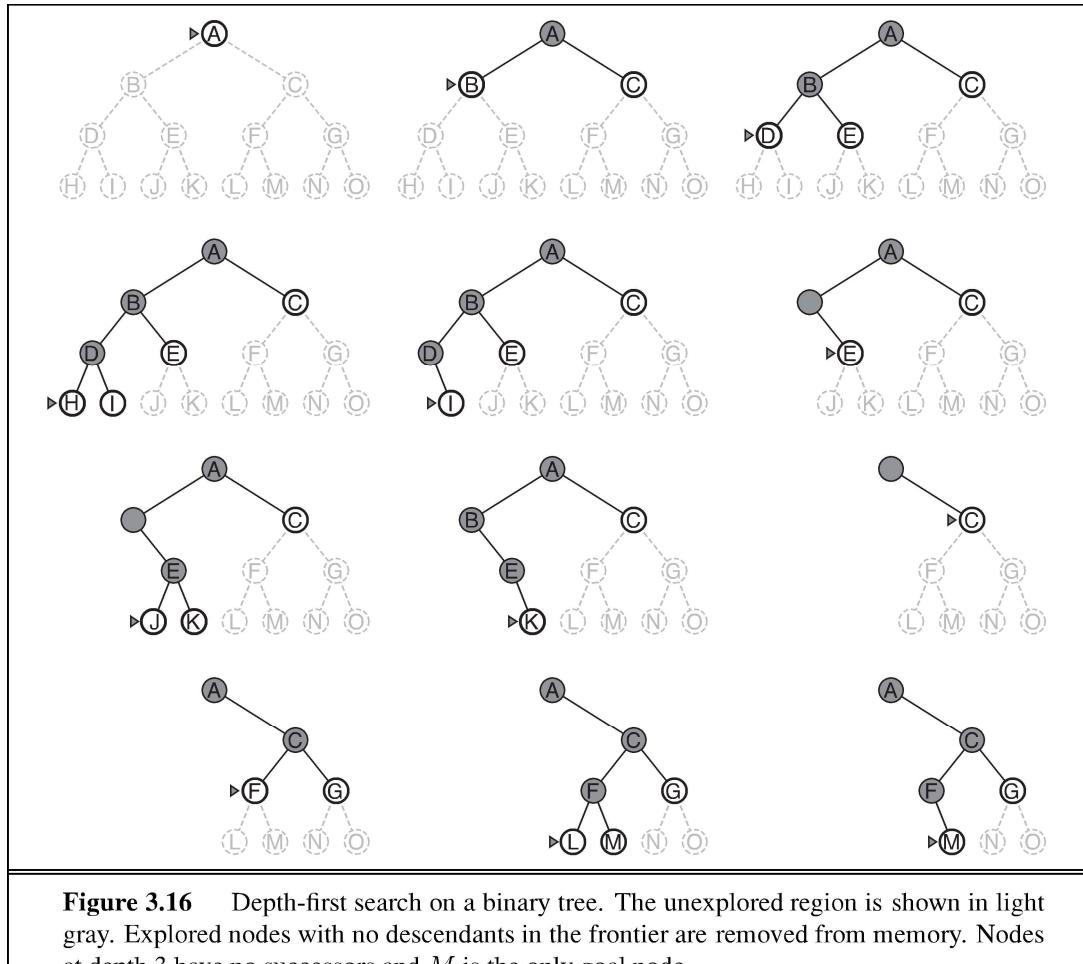
**Depth-first search** always expands the *deepest* node in the current frontier of the search tree. The progress of the search is illustrated in Figure 3.16. The search proceeds immediately to the deepest level of the search tree, where the nodes have no successors. As those nodes are expanded, they are dropped from the frontier, so then the search “backs up” to the next deepest node that still has unexplored successors.

The depth-first search algorithm is an instance of the graph-search algorithm in Figure 3.7; whereas breadth-first-search uses a FIFO queue, depth-first search uses a LIFO queue. A LIFO queue means that the most recently generated node is chosen for expansion. This must be the deepest unexpanded node because it is one deeper than its parent—which, in turn, was the deepest unexpanded node when it was selected.

As an alternative to the GRAPH-SEARCH-style implementation, it is common to implement depth-first search with a recursive function that calls itself on each of its children in turn. (A recursive depth-first algorithm incorporating a depth limit is shown in Figure 3.17.)

<sup>6</sup> *NoOp*, or “no operation,” is the name of an assembly language instruction that does nothing.

<sup>7</sup> Here, and throughout the book, the “star” in  $C^*$  means an optimal value for  $C$ .



The properties of depth-first search depend strongly on whether the graph-search or tree-search version is used. The graph-search version, which avoids repeated states and redundant paths, is complete in finite state spaces because it will eventually expand every node. The tree-search version, on the other hand, is *not* complete—for example, in Figure 3.6 the algorithm will follow the Arad–Sibiu–Arad–Sibiu loop forever. Depth-first tree search can be modified at no extra memory cost so that it checks new states against those on the path from the root to the current node; this avoids infinite loops in finite state spaces but does not avoid the proliferation of redundant paths. In infinite state spaces, both versions fail if an infinite non-goal path is encountered. For example, in Knuth’s 4 problem, depth-first search would keep applying the factorial operator forever.

For similar reasons, both versions are nonoptimal. For example, in Figure 3.16, depth-first search will explore the entire left subtree even if node  $C$  is a goal node. If node  $J$  were also a goal node, then depth-first search would return it as a solution instead of  $C$ , which would be a better solution; hence, depth-first search is not optimal.

The time complexity of depth-first graph search is bounded by the size of the state space (which may be infinite, of course). A depth-first tree search, on the other hand, may generate all of the  $O(b^m)$  nodes in the search tree, where  $m$  is the maximum depth of any node; this can be much greater than the size of the state space. Note that  $m$  itself can be much larger than  $d$  (the depth of the shallowest solution) and is infinite if the tree is unbounded.

So far, depth-first search seems to have no clear advantage over breadth-first search, so why do we include it? The reason is the space complexity. For a graph search, there is no advantage, but a depth-first tree search needs to store only a single path from the root to a leaf node, along with the remaining unexpanded sibling nodes for each node on the path. Once a node has been expanded, it can be removed from memory as soon as all its descendants have been fully explored. (See Figure 3.16.) For a state space with branching factor  $b$  and maximum depth  $m$ , depth-first search requires storage of only  $O(bm)$  nodes. Using the same assumptions as for Figure 3.13 and assuming that nodes at the same depth as the goal node have no successors, we find that depth-first search would require 156 kilobytes instead of 10 exabytes at depth  $d = 16$ , a factor of 7 trillion times less space. This has led to the adoption of depth-first tree search as the basic workhorse of many areas of AI, including constraint satisfaction (Chapter 6), propositional satisfiability (Chapter 7), and logic programming (Chapter 9). For the remainder of this section, we focus primarily on the tree-search version of depth-first search.

BACKTRACKING  
SEARCH

A variant of depth-first search called **backtracking search** uses still less memory. (See Chapter 6 for more details.) In backtracking, only one successor is generated at a time rather than all successors; each partially expanded node remembers which successor to generate next. In this way, only  $O(m)$  memory is needed rather than  $O(bm)$ . Backtracking search facilitates yet another memory-saving (and time-saving) trick: the idea of generating a successor by *modifying* the current state description directly rather than copying it first. This reduces the memory requirements to just one state description and  $O(m)$  actions. For this to work, we must be able to undo each modification when we go back to generate the next successor. For problems with large state descriptions, such as robotic assembly, these techniques are critical to success.

### 3.4.4 Depth-limited search

DEPTH-LIMITED  
SEARCH

The embarrassing failure of depth-first search in infinite state spaces can be alleviated by supplying depth-first search with a predetermined depth limit  $\ell$ . That is, nodes at depth  $\ell$  are treated as if they have no successors. This approach is called **depth-limited search**. The depth limit solves the infinite-path problem. Unfortunately, it also introduces an additional source of incompleteness if we choose  $\ell < d$ , that is, the shallowest goal is beyond the depth limit. (This is likely when  $d$  is unknown.) Depth-limited search will also be nonoptimal if we choose  $\ell > d$ . Its time complexity is  $O(b^\ell)$  and its space complexity is  $O(b\ell)$ . Depth-first search can be viewed as a special case of depth-limited search with  $\ell = \infty$ .

Sometimes, depth limits can be based on knowledge of the problem. For example, on the map of Romania there are 20 cities. Therefore, we know that if there is a solution, it must be of length 19 at the longest, so  $\ell = 19$  is a possible choice. But in fact if we studied the

```

function DEPTH-LIMITED-SEARCH(problem, limit) returns a solution, or failure/cutoff
  return RECURSIVE-DLS(MAKE-NODE(problem.INITIAL-STATE), problem, limit)

function RECURSIVE-DLS(node, problem, limit) returns a solution, or failure/cutoff
  if problem.GOAL-TEST(node.STATE) then return SOLUTION(node)
  else if limit = 0 then return cutoff
  else
    cutoff_occurred?  $\leftarrow$  false
    for each action in problem.ACTIONS(node.STATE) do
      child  $\leftarrow$  CHILD-NODE(problem, node, action)
      result  $\leftarrow$  RECURSIVE-DLS(child, problem, limit - 1)
      if result = cutoff then cutoff_occurred?  $\leftarrow$  true
      else if result  $\neq$  failure then return result
      if cutoff_occurred? then return cutoff else return failure

```

**Figure 3.17** A recursive implementation of depth-limited tree search.

DIAMETER

map carefully, we would discover that any city can be reached from any other city in at most 9 steps. This number, known as the **diameter** of the state space, gives us a better depth limit, which leads to a more efficient depth-limited search. For most problems, however, we will not know a good depth limit until we have solved the problem.

Depth-limited search can be implemented as a simple modification to the general tree- or graph-search algorithm. Alternatively, it can be implemented as a simple recursive algorithm as shown in Figure 3.17. Notice that depth-limited search can terminate with two kinds of failure: the standard *failure* value indicates no solution; the *cutoff* value indicates no solution within the depth limit.

### 3.4.5 Iterative deepening depth-first search

ITERATIVE DEEPENING SEARCH

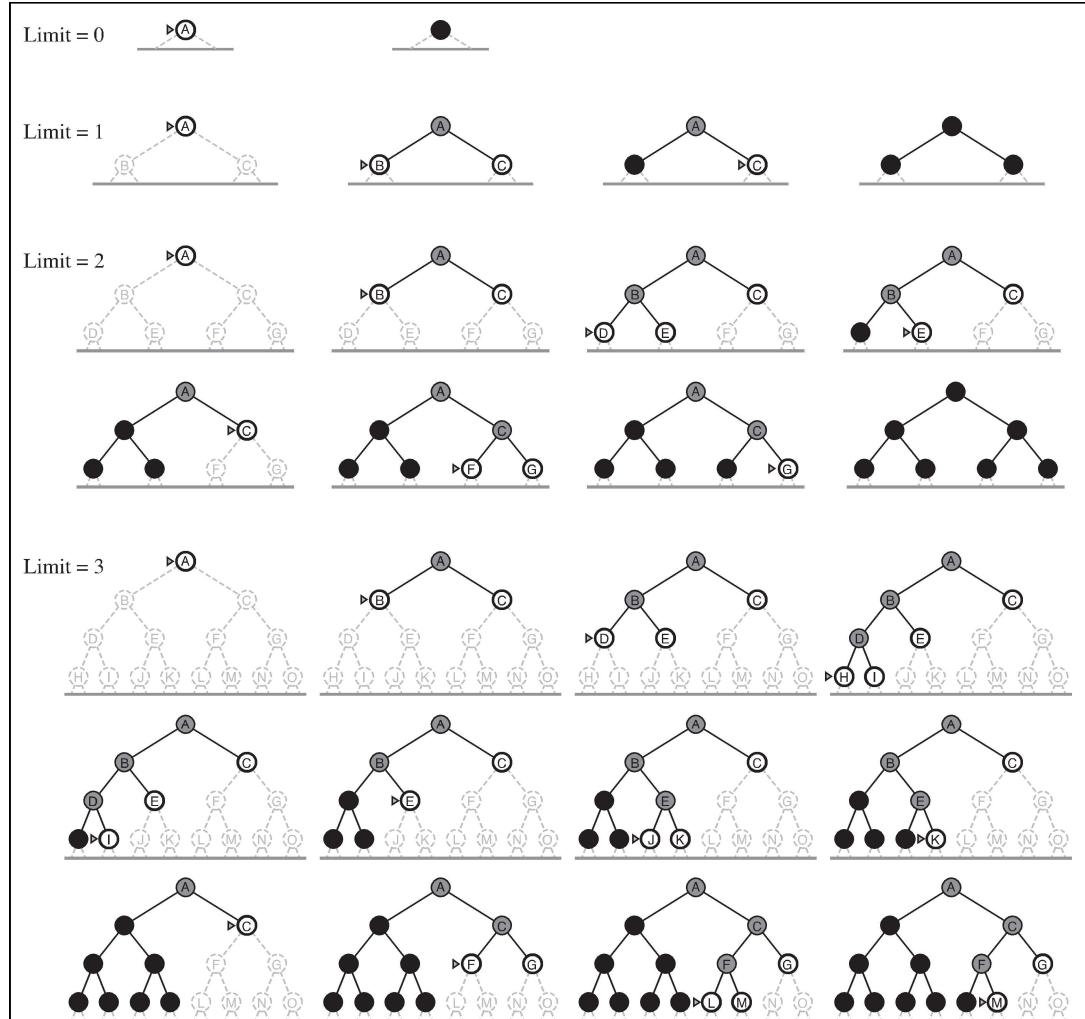
**Iterative deepening search** (or iterative deepening depth-first search) is a general strategy, often used in combination with depth-first tree search, that finds the best depth limit. It does this by gradually increasing the limit—first 0, then 1, then 2, and so on—until a goal is found. This will occur when the depth limit reaches  $d$ , the depth of the shallowest goal node. The algorithm is shown in Figure 3.18. Iterative deepening combines the benefits of depth-first and breadth-first search. Like depth-first search, its memory requirements are modest:  $O(bd)$  to be precise. Like breadth-first search, it is complete when the branching factor is finite and optimal when the path cost is a nondecreasing function of the depth of the node. Figure 3.19 shows four iterations of ITERATIVE-DEEPENING-SEARCH on a binary search tree, where the solution is found on the fourth iteration.

Iterative deepening search may seem wasteful because states are generated multiple times. It turns out this is not too costly. The reason is that in a search tree with the same (or nearly the same) branching factor at each level, most of the nodes are in the bottom level, so it does not matter much that the upper levels are generated multiple times. In an iterative deepening search, the nodes on the bottom level (depth  $d$ ) are generated once, those on the

```

function ITERATIVE-DEEPENING-SEARCH(problem) returns a solution, or failure
  for depth = 0 to  $\infty$  do
    result  $\leftarrow$  DEPTH-LIMITED-SEARCH(problem, depth)
    if result  $\neq$  cutoff then return result
  
```

**Figure 3.18** The iterative deepening search algorithm, which repeatedly applies depth-limited search with increasing limits. It terminates when a solution is found or if the depth-limited search returns *failure*, meaning that no solution exists.



**Figure 3.19** Four iterations of iterative deepening search on a binary tree.

next-to-bottom level are generated twice, and so on, up to the children of the root, which are generated  $d$  times. So the total number of nodes generated in the worst case is

$$N(\text{IDS}) = (d)b + (d-1)b^2 + \cdots + (1)b^d,$$

which gives a time complexity of  $O(b^d)$ —asymptotically the same as breadth-first search. There is some extra cost for generating the upper levels multiple times, but it is not large. For example, if  $b = 10$  and  $d = 5$ , the numbers are

$$N(\text{IDS}) = 50 + 400 + 3,000 + 20,000 + 100,000 = 123,450$$

$$N(\text{BFS}) = 10 + 100 + 1,000 + 10,000 + 100,000 = 111,110.$$

If you are really concerned about repeating the repetition, you can use a hybrid approach that runs breadth-first search until almost all the available memory is consumed, and then runs iterative deepening from all the nodes in the frontier. *In general, iterative deepening is the preferred uninformed search method when the search space is large and the depth of the solution is not known.*



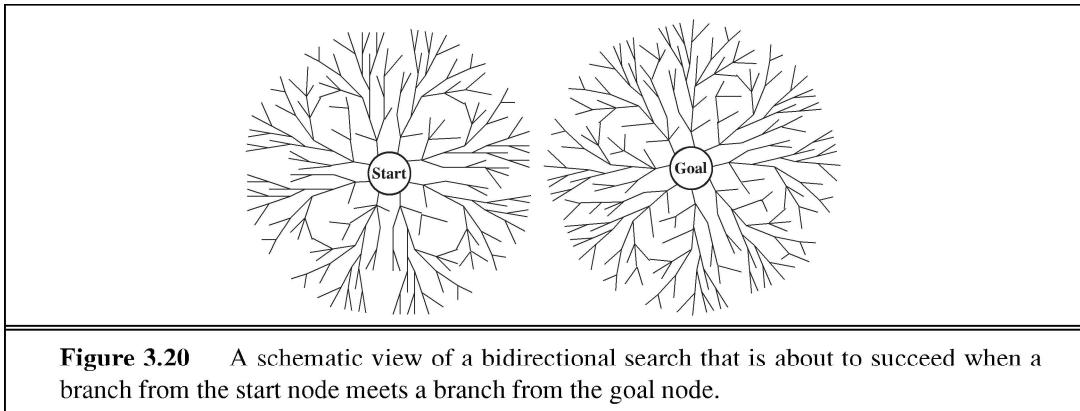
ITERATIVE LENGTHENING SEARCH

Iterative deepening search is analogous to breadth-first search in that it explores a complete layer of new nodes at each iteration before going on to the next layer. It would seem worthwhile to develop an iterative analog to uniform-cost search, inheriting the latter algorithm's optimality guarantees while avoiding its memory requirements. The idea is to use increasing path-cost limits instead of increasing depth limits. The resulting algorithm, called **iterative lengthening search**, is explored in Exercise 3.17. It turns out, unfortunately, that iterative lengthening incurs substantial overhead compared to uniform-cost search.

### 3.4.6 Bidirectional search

The idea behind bidirectional search is to run two simultaneous searches—one forward from the initial state and the other backward from the goal—hoping that the two searches meet in the middle (Figure 3.20). The motivation is that  $b^{d/2} + b^{d/2}$  is much less than  $b^d$ , or in the figure, the area of the two small circles is less than the area of one big circle centered on the start and reaching to the goal.

Bidirectional search is implemented by replacing the goal test with a check to see whether the frontiers of the two searches intersect; if they do, a solution has been found. (It is important to realize that the first such solution found may not be optimal, even if the two searches are both breadth-first; some additional search is required to make sure there isn't another short-cut across the gap.) The check can be done when each node is generated or selected for expansion and, with a hash table, will take constant time. For example, if a problem has solution depth  $d = 6$ , and each direction runs breadth-first search one node at a time, then in the worst case the two searches meet when they have generated all of the nodes at depth 3. For  $b = 10$ , this means a total of 2,220 node generations, compared with 1,111,110 for a standard breadth-first search. Thus, the time complexity of bidirectional search using breadth-first searches in both directions is  $O(b^{d/2})$ . The space complexity is also  $O(b^{d/2})$ . We can reduce this by roughly half if one of the two searches is done by iterative deepening, but at least one of the frontiers must be kept in memory so that the intersection check can be done. This space requirement is the most significant weakness of bidirectional search.



PREDECESSOR

The reduction in time complexity makes bidirectional search attractive, but how do we search backward? This is not as easy as it sounds. Let the **predecessors** of a state  $x$  be all those states that have  $x$  as a successor. Bidirectional search requires a method for computing predecessors. When all the actions in the state space are reversible, the predecessors of  $x$  are just its successors. Other cases may require substantial ingenuity.

Consider the question of what we mean by “the goal” in searching “backward from the goal.” For the 8-puzzle and for finding a route in Romania, there is just one goal state, so the backward search is very much like the forward search. If there are several *explicitly listed* goal states—for example, the two dirt-free goal states in Figure 3.3—then we can construct a new dummy goal state whose immediate predecessors are all the actual goal states. But if the goal is an abstract description, such as the goal that “no queen attacks another queen” in the  $n$ -queens problem, then bidirectional search is difficult to use.

### 3.4.7 Comparing uninformed search strategies

Figure 3.21 compares search strategies in terms of the four evaluation criteria set forth in Section 3.3.2. This comparison is for tree-search versions. For graph searches, the main differences are that depth-first search is complete for finite state spaces and that the space and time complexities are bounded by the size of the state space.

Criterion	Breadth-First	Uniform-Cost	Depth-First	Depth-Limited	Iterative Deepening	Bidirectional (if applicable)
Complete?	Yes <sup>a</sup>	Yes <sup>a,b</sup>	No	No	Yes <sup>a</sup>	Yes <sup>a,d</sup>
Time	$O(b^d)$	$O(b^{1+\lfloor C^*/\epsilon \rfloor})$	$O(b^m)$	$O(b^\ell)$	$O(b^d)$	$O(b^{d/2})$
Space	$O(b^d)$	$O(b^{1+\lfloor C^*/\epsilon \rfloor})$	$O(bm)$	$O(b\ell)$	$O(bd)$	$O(b^{d/2})$
Optimal?	Yes <sup>c</sup>	Yes	No	No	Yes <sup>c</sup>	Yes <sup>c,d</sup>

**Figure 3.21** Evaluation of tree-search strategies.  $b$  is the branching factor;  $d$  is the depth of the shallowest solution;  $m$  is the maximum depth of the search tree;  $\ell$  is the depth limit. Superscript caveats are as follows: <sup>a</sup> complete if  $b$  is finite; <sup>b</sup> complete if step costs  $\geq \epsilon$  for positive  $\epsilon$ ; <sup>c</sup> optimal if step costs are all identical; <sup>d</sup> if both directions use breadth-first search.