The Riemann zeta function and the prime number theorem

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For complex numbers s with $Re(s) > 1$ the **Riemann zeta function** is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

We can also define the Riemann zeta function as

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

This is called the **Euler product expansion** of the Riemann zeta function.

Proof: Recall that the geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, for $r \in \mathbb{C}$ such that $|r| < 1$. For any prime p, $|p^{-s}| < 1$ whenever $|s| > 1$. This condition is satisfied when $Re(s) > 1$, so

$$\begin{aligned}
\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} &= \prod_{p \text{ prime}} \sum_{n=0}^{\infty} \left(p^{-s}\right)^n \\
&= \prod_{p \text{ prime}} \sum_{n=0}^{\infty} \left(p^{n}\right)^{-s} \\
&= (1 + 2^{-s} + 4^{-s} + \cdots)(1 + 3^{-s} + 9^{-s} + \cdots)\cdots \\
&= 1 + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + 7^{-s} + 8^{-s} + 9^{-s} + \cdots .
\end{aligned}$$

The bases of the terms in this sum are the products of primes, with any multiplicity, each to the power of $-s$. The fundamental theorem of arithmetic gives that every positive integer has a unique prime factorization, so the bases of the terms of this sum are precisely the positive integers. We conclude that

$$\prod_{p \text{ prime}} \sum_{n=0}^{\infty} \left(p^{n}\right)^{-s} = \sum_{n=1}^{\infty} n^{-s} = \zeta(s).$$

$\square$

The reciprocal of the **Riemann zeta function** can be given as

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Proof: The Euler product expansion of the Riemann zeta function gives that

$$\begin{aligned}
\frac{1}{\zeta(s)} &= \prod_{p \text{ prime}} (1 - p^{-s}) \\
&= \left(1 - 2^{-s}\right)\left(1 - 3^{-s}\right)\left(1 - 5^{-s}\right)\cdots \\
&= 1 - 2^{-s} - 3^{-s} - 5^{-s} + 6^{-s} - 7^{-s} + 10^{-s} - 11^{-s} - 13^{-s} + 14^{-s} + 15^{-s} + \cdots .
\end{aligned}$$

The bases of the terms in this sum are the products of primes, with multiplicity one, each to the power of $-s$, with coefficients determined by the parity of the number of prime factors such that the coefficient of the term with base n is equals to $(-1)^{\omega(n)}$. More precisely, the bases of the terms of this sum are precisely the integers in which each prime factor appears at most once, i.e. the square-free integers. The Möbius function, $\mu(n)$, equals $(-1)^{\omega(n)}$ for square-free n and equals 0 for n divisible by a perfect square, so we conclude that

$$\frac{1}{\zeta(s)} = \sum_{1 \leq n \text{ square-free}} (-1)^{\omega(n)} n^{-s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

□

We can now show that
$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{1 \leq n \text{ square-free}} \frac{1}{n^s} = \sum_{n \geq 1} \frac{|\mu(n)|}{n^s}.$$

Proof:

$$\begin{aligned}
\frac{\zeta(s)}{\zeta(2s)} &= \left(\prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \right) \Bigg/ \left(\prod_{p \text{ prime}} \frac{1}{1-p^{-2s}} \right) \\
&= \prod_{p \text{ prime}} \frac{1-p^{-2s}}{1-p^{-s}} \\
&= \prod_{p \text{ prime}} \frac{(1+p^{-s})(1-p^{-s})}{1-p^{-s}} \\
&= \prod_{p \text{ prime}} \left(1+p^{-s}\right) \\
&= 1 + 2^{-s} + 3^{-s} + 5^{-s} + 6^{-s} + 7^{-s} + 10^{-s} + 11^{-s} + 13^{-s} + 14^{-s} + 15^{-s} + \cdots .
\end{aligned}$$

As in the previous proof, the bases of the terms in this sum are the square-free integers. The coefficient of each term in this sum is simply 1. The absolute value of the Möbius function, $|\mu(n)|$, equals 1 for square-free n and equals 0 for n divisible by a perfect square, so we conclude that

$$\frac{\zeta(s)}{\zeta(2s)} = \prod_{p \text{ prime}} \left(1+p^{-s}\right) = \sum_{1 \leq n \text{ square-free}} n^{-s} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}.$$

□

Recall that the proportion of positive integers that are 2^k-free is $(1-2^{-k})$. Moreover, the proportion of positive integers that are the free of the kth powers of relatively prime positive integers $n_1, \ldots, n_m$ is

$$\prod_{j=1}^{m} \left(1 - (n_m)^{-k}\right).$$

It follows that the proportion of positive integers that are kth power-free, i.e. are not divisible by any primes to the power of k, is

$$\prod_{p \text{ prime}} \left(1 - p^{-k}\right) = \frac{1}{\zeta(k)}.$$

For example, the proportion of square-free integers is

$$\frac{1}{\zeta(2)} = \frac{1}{\pi^2/6} = \frac{6}{\pi^2}.$$

We can use a truncated version of the Euler product expansion of the Riemann zeta function to compute the proportions of square-free integers up to certain integers.

Let $F(X)$ equal the Euler product expansion of the Riemann zeta function truncated to the Xth term, i.e.

$$F(X) := \prod_{p \text{ prime}}^{X} \left(1 - p^{-2}\right),$$

so $F(X)$ gives the proportion of square-free integers up to X. In the following table, we calculate various values of F and their percent errors relative to the proportion of all positive integers that are square-free, $\frac{6}{\pi^2} \approx 0.607927102$.

X	$F(X)$	Percent error
100	0.60903373	0.18%
200	0.60838189	0.075%
300	0.608225602	0.049%
400	0.608144905	0.036%
500	0.608093228	0.027%
600	0.608065267	0.023%
700	0.608041737	0.019%
800	0.608026443	0.016%
900	0.608013722	0.014%
1000	0.608004307	0.013%

We see that, as X increases, $F(X)$ quickly becomes an excellent approximation of the proportion of square-free integers. This finding demonstrates two things. First, that the density of square-free integers from 1 to 1,000 is very similar to the density of square-free integers over all positive integers. Second, that we can approximate $\frac{1}{\zeta(s)}$ fairly well by using the truncated version of the reciprocal of the Euler identity: $F(X)$ with a relatively small X.

We define the **von Mangoldt function** $\Lambda : \mathbb{Z}^+ \to \mathbb{R}$ as

$$\Lambda(n) = \begin{cases} 0 & \text{if } n = 1, \\ 0 & \text{if } n > 1 \text{ is not a prime power,} \\ \log p & \text{if } n = p^a. \end{cases}$$

Its first twelve values are
$\Lambda(1) = 0$,
$\Lambda(2) = \log 2$,
$\Lambda(3) = \log 3$,
$\Lambda(4) = \log 2$,
$\Lambda(5) = \log 5$,
$\Lambda(6) = 0$,
$\Lambda(7) = \log 7$,
$\Lambda(8) = \log 2$,
$\Lambda(9) = \log 3$,
$\Lambda(10) = 0$,
$\Lambda(11) = \log 11$,
$\Lambda(12) = 0$.

We can show that

$$\log n = \sum_{1 \le d | n} \Lambda(d).$$

Proof: Choose $n \in \mathbb{Z}^+$. In the case that $n = 1$, $\log n = 0 = \Lambda(1) = \sum_{1 \le d|1} \Lambda(d)$. Hence, we will suppose that $n > 1$. The fundamental theorem of arithmetic gives that n has a unique prime factorization of the form $n = \Pi_{i=1}^m p_i^{a_i}$, for $m \ge 1$, $p_1, \ldots, p_m$ distinct primes, and $a_1, \ldots, a_m \in \mathbb{Z}^+$. Let X be the set of prime powers greater than 1 that divide n, i.e. $X = \{p_i^a : 1 \le i \le m, 1 \le a \le a_i\}$. Λ evaluates to 0 on divisors of n not in X, so

$$\begin{aligned}
\sum_{1 \le d | n} \Lambda(d) &= \sum_{d \in X} \Lambda(d) \\
&= \sum_{i=1}^m \sum_{j=1}^{a_i} \Lambda(p_i^j) \\
&= \sum_{i=1}^m a_i \log p_i \\
&= \sum_{i=1}^m \log p_i^{a_i} \\
&= \log \Pi_{i=1}^m p_i^{a_i} \\
&= \log n.
\end{aligned}$$

n was chosen arbitrarily, so we are done. $\square$

We can apply the Möbius inversion formula to this result to show that, for $n \in \mathbb{Z}^+$,

$$\Lambda(n) = -\sum_{1 \le d | n} \mu(d) \log d.$$

Proof: Recall that the Möbius inversion formula gives that, for arithmetic functions f and g such that, $\forall n \in \mathbb{Z}^+$

$$g(n) = \sum_{1 \le d | n} f(d),$$

it holds that

$$f(n) = \sum_{1 \le d | n} \mu(d) g(n/d).$$

Thus,

$$\begin{aligned}
\Lambda(n) &= \sum_{1 \le d | n} \mu(d) \log(\frac{n}{d}) \\
&= \sum_{1 \le d | n} \mu(d)(\log n - \log d) \\
&= \sum_{1 \le d | n} \mu(d) \log n - \sum_{1 \le d | n} \mu(d) \log d \\
&= \log n \sum_{1 \le d | n} \mu(d) - \sum_{1 \le d | n} \mu(d) \log d \\
&= -\sum_{1 \le d | n} \mu(d) \log d.
\end{aligned}$$

Note that first term in the penultimate line vanishes because, in the case that $n = 1$, $\log n = 0$, and in the case that $n > 1$, $\sum_{1 \le d|n} \mu(d) = 0$. $\square$

The proof of the prime number theorem relies on the identity

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Proof: Recall the Euler product expansion:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} = \prod_{p \text{ prime}} \left(1-p^{-s}\right)^{-1}.$$

Applying the Mercator series

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k,$$

we get that

$$\begin{aligned}
\log \zeta(s) &= \log\left(\prod_{p \text{ prime}} (1-p^{-s})^{-1}\right) \\
&= \sum_{p \text{ prime}} \log\left((1-p^{-s})^{-1}\right) \\
&= -\sum_{p \text{ prime}} \log(1-p^{-s}) \\
&= -\sum_{p \text{ prime}} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (-p^{-s})^k\right) \\
&= -\sum_{p \text{ prime}} \left(-\sum_{k=1}^{\infty} \frac{(p^{-s})^k}{k}\right) \\
&= \sum_{p \text{ prime}} \sum_{k=1}^{\infty} \frac{p^{-ks}}{k} \\
&= \sum_{p \text{ prime}} \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{1}{p^{ks}} \\
&= \sum_{p^k \text{ prime power}} \frac{1}{k} \cdot \frac{1}{(p^k)^s} \\
&= \sum_{p^k \text{ prime power}} \frac{\log p}{k \log p} \cdot \frac{1}{(p^k)^s} \\
&= \sum_{p^k \text{ prime power}} \frac{\log p}{\log p^k} \cdot \frac{1}{(p^k)^s} \\
&= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n} \cdot \frac{1}{n^s}.
\end{aligned}$$

The last equality holds because $\Lambda(n)$ equals $\log p$ if n is an integer power of prime p and equals 0 if n is not a prime power. We can now take the logarithmic derivative of $\zeta(s)$ to get

$$\begin{aligned}
\frac{\zeta'}{\zeta}(s) &= \frac{d}{ds}\log(\zeta(s)) \\
&= \frac{d}{ds}\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\log n}\cdot\frac{1}{n^s}\right) \\
&= \sum_{n=1}^{\infty}\frac{d}{ds}\left(\frac{\Lambda(n)}{\log n}\cdot\frac{1}{n^s}\right) \\
&= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{\log n}\cdot\frac{d}{ds}\left(\frac{1}{n^s}\right) \\
&= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{\log n}\cdot\left(\frac{\log n}{-n^s}\right) \\
&= -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}.
\end{aligned}$$

Multiplying both sides of the equation by -1, we are done. $\square$

For $X \in \mathbb{R}$, the **Chebyshev function** is defined by

$$\Psi(x) := \sum_{p^k \leq x} \log p.$$

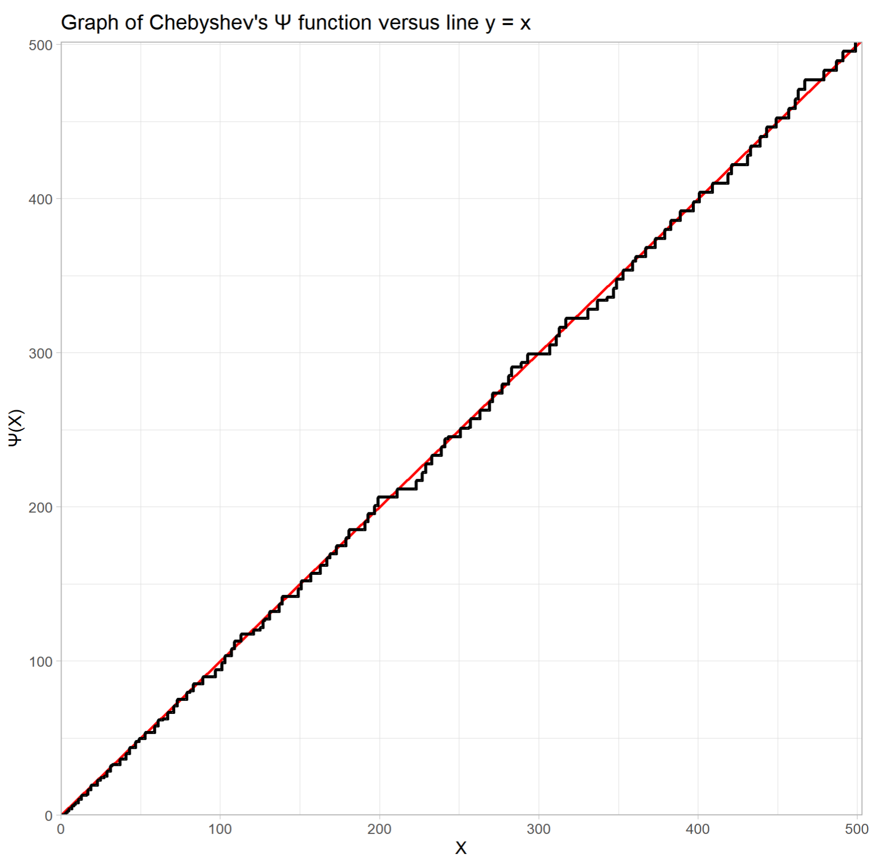

Figure 1: Graphs of $y = \Psi(X)$ (black) and $y = x$ (red), for $X \in [0, 500]$. Note the extreme similarity of the two graphs. $\Psi(x)$ is a step function and prime powers grow increasingly sparse in the natural numbers, so

the deviations between the two graphs increase as x increases.

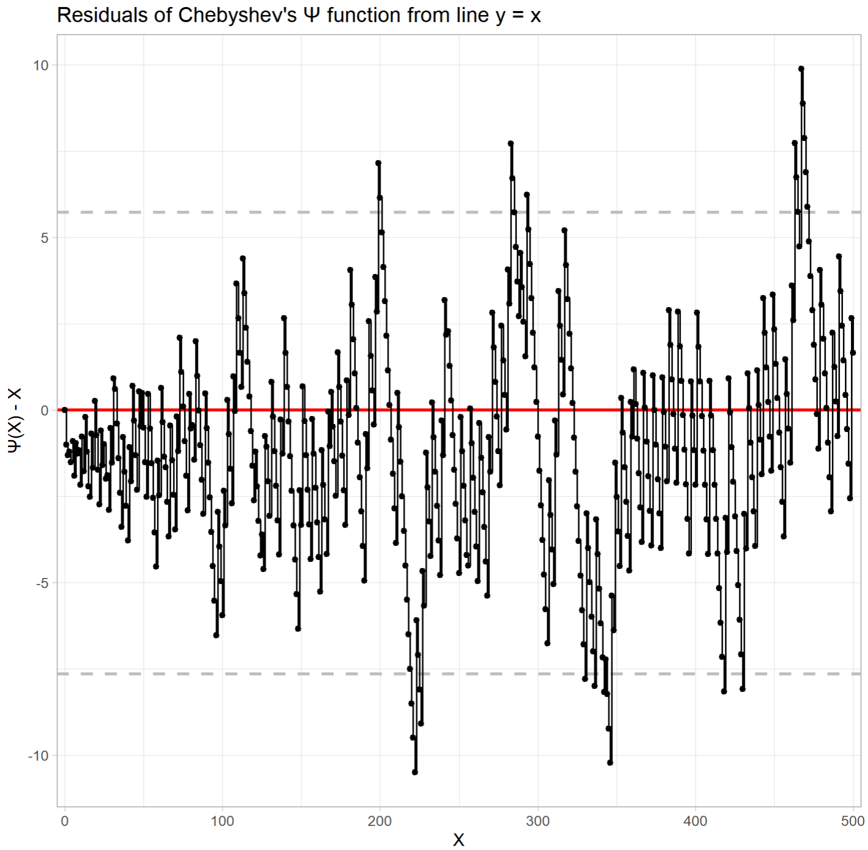

Figure 2: Graphs of $y = \Psi(X)$ (black) and $y = 0$ (red), for $X \in [0, 500]$, showing the deviation of $X \mapsto Psi(X)$ from the identity function. Points are placed at integer values of X to emphasize patterns in the residuals of $\Psi(X)$ from X. Dashed grey lines represent interval containing 95% of residuals.

We can use the Chebyshev function in combination with the Riemann hypothesis to prove the prime number theorem. We begin by providing some background about the analytic continuation of the Riemann zeta function over $\mathbb{C} \setminus \{1\}$:

1. $\zeta(s) \neq 0$ if $Re(s) > 1$.
2. $(s-1)\zeta(s)$ has an analytic continuation to $\{s \in \mathbb{C} : Re(s) > 0\}$ given by

$$(s-1)\zeta(s) = s - s(s-1)\int_1^\infty \frac{\{x\}}{x^{s+1}}dx,$$

   where $\{x\}$ denotes the fractional part of x.
3. $-\frac{\zeta'}{\zeta}(s)$ has an analytic continuation to $\{s \in \mathbb{C} : Re(s) \geq 1\}$, with only a simple pole at $s = 1$ with residue 1, where $-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^\infty \frac{\Lambda(n)}{n^s}$ by our identity shown earlier.

Proof:

1. This follows from the absolute convergence of the Euler product expansion.
2. Assuming the formula

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx,$$

   we see that the integral converges whenever $Re(s) > 0$. Multiplying both sides of the equation by $(s-1)$, we get the desired result.

3. When $Re(s) > 1$, $-\frac{\zeta'}{\zeta}(s)$ is analytic, since $\zeta(s)$ is non-vanishing and analytic whenever $Re(s) > 1$. By **(2)**, we have that $(s-1)\zeta(s) = sF(s)$, where $F(s)$ is analytic when $Re(s) > 0$. Differentiating both sides of the equation, we get that $\zeta(s) + (s-1)\zeta'(s) = F(s) + sF'(s)$. Dividing by $\zeta(s)$, we get

$$1 + (s-1)\frac{\zeta'(s)}{\zeta(s)} = \frac{F(s)}{\zeta(s)} + \frac{sF'(s)}{\zeta(s)}.$$

$\lim_{s\to 1^+} \zeta(s) = +\infty$ and $F(s)$ is analytic near $s = 1$, so we conclude that

$$\lim_{s\to 1^+} (s-1)\frac{\zeta'(s)}{\zeta(s)} = -1.$$

□

With these facts about the analytic continuation of the Riemann zeta function in mind, we assume an exact formula for $\Psi(x)$ in terms of the zeros of the Riemann zeta function:

$$\Psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2}),$$

where ρ iterates over the non-trivial zeros of the Riemann zeta function.
We can show that the Riemann hypothesis applied to this exact formula implies the prime number theorem. Recall that the prime number theorem gives:

$$\text{As } x \to \infty, \text{ we have that } \pi(x) \sim \frac{x}{\log x},$$

where $\pi(x) := |\{p \leq x : p \text{ is prime}\}|$ is the **prime-counting function**.
The first step in showing this result is to show that $\Psi(x) \sim x$.
As $x \to \infty$, we see that the last two terms of the exact formula for $\Psi(x)$ are negligible:

$$\lim_{x\to\infty} \frac{1}{2}\log(1 - x^{-2}) = \frac{1}{2}\log 1 = 0,$$

and $-\frac{\zeta'}{\zeta}(0)$ is a constant, while the remaining two terms grow indefinitely in magnitude, i.e. $x \to \infty$ and $-\Sigma_{\rho}\frac{x^{\rho}}{\rho} \to -\infty$.
Thus, as $x \to \infty$,

$$\Psi(x) \sim x - \sum_{\rho} \frac{x^{\rho}}{\rho}.$$

It remains to show that as $x \to \infty$, $\sum_{\rho} \frac{x^{\rho}}{\rho} \to 0$. Assuming the Riemann hypothesis, we have that, for any non-trivial zeros of the Riemann zeta function, ρ, $Re(\rho) = \frac{1}{2}$. Let ρ be a zero of the Riemann zeta function and let $\gamma_{\rho} := Im(\rho)$. Recall that the trivial zeros of the Riemann zeta function are precisely the negative even integers. Then

$$\begin{aligned}\left|\sum_{\rho} \frac{x^{\rho}}{\rho}\right| &= \left|\sum_{\rho} \frac{x^{\frac{1}{2}+i\gamma_{\rho}}}{\rho}\right| \\ &= x^{\frac{1}{2}} \left|\sum_{\rho} \frac{x^{i\gamma_{\rho}}}{\rho}\right| \\ &\leq \sqrt{x}\sum_{\rho} \frac{1}{|\rho|},\end{aligned}$$

where the inequality holds via the triangle inequality because $|x^{i\gamma_{\rho}}| = 1$, as any non-negative real number raised to an imaginary power is on the unit circle.

Define $N(T) := |\{\rho : \zeta(\rho) = 0,\, 0 < \gamma_\rho < T, \text{ and } T \in \mathbb{R}\}|$, i.e. $N(T)$ denotes the number of non-trivial zeros of the Riemann zeta function whose imaginary part is positive and less than some real number T. We state without proof that $N(T) \sim \left(\frac{T}{2\pi}\right) \log\left(\frac{T}{2\pi}\right)$.

Let (ρ_n) be the sequence of the nontrivial zero of the Riemann zeta function with positive imaginary part in order of increasing imaginary part. Let (b_n) be the sequence defined by $b_n = |\rho_n|$, $\forall n \in \mathbb{Z}^+$. Since the real parts of all elements of (ρ_n) are equal and the imaginary part of each element of (ρ_n) is greater than that of its predecessor, (b_n) is an increasing sequence. Recall that the Tth harmonic number is less than or equal to $\log T$, i.e.

$$\sum_{n=1}^{T} \frac{1}{n} \leq \log T.$$

We will state without proof that the partial sums of the sequence of the reciprocals of elements of (b_n) are asymptotically bounded above by the harmonic numbers, so, for sufficiently large T,

$$\begin{aligned}
\sum_{n=1}^{N(T)} \frac{1}{|\rho_n|} &= \sum_{n=1}^{N(T)} \frac{1}{b_n} \\
&\leq \sum_{n=1}^{N(T)} \frac{1}{n} \\
&\leq \log\left(N(T)\right) \\
&\sim \log\left(\left(\frac{T}{2\pi}\right) \log\left(\frac{T}{2\pi}\right)\right) \\
&= \log\left(\frac{T}{2\pi}\right) + \log^2\left(\frac{T}{2\pi}\right) \\
&= \log^2 T + \log T - \log^2(2\pi) - \log(2\pi) \\
&\sim O\left(\log^2(T)\right).
\end{aligned}$$

It follows that, for sufficiently large T,

$$\begin{aligned}
\left|\sum_{\rho} \frac{x^\rho}{\rho}\right| &\leq \sqrt{x} \sum_{\rho} \frac{1}{|\rho|} \\
&\leq \sqrt{x} \cdot O(\log^2 T) \\
&\leq O(\sqrt{x} \log^2 T),
\end{aligned}$$

which is what we wanted to show. We now have:

$$\text{As } x \to \infty,\ \Psi(x) = x - O(\sqrt{x} \log^2 T) \sim x.$$

Next, we consider

$$\frac{\Lambda(n)}{\log n} = \begin{cases} 0 & \text{if } n > 1 \text{ is not a prime power,} \\ 1 & \text{if } n \text{ is prime,} \\ \frac{\log p}{\log n} & \text{if } n = p^a, \end{cases}$$

where $\frac{\Lambda(n)}{\log n}$ is defined for $n > 1$. As $n \to \infty$, the $\frac{\log p}{\log n} \to 0$ because, on average, $n \gg p$, so

$$\pi(x) \sim \sum_{n=2}^{x} \frac{\Lambda(n)}{\log n}$$

because the summation adds 0 if n is not a prime power, adds $\frac{\log n}{\log n} = 1$ if n is prime, and adds $\frac{\log p}{\log n}$ if n is an integer power of prime p.

We will now introduce **Abel's summation formula**, or **partial summation**, in order to decompose this sum. Abel's summation formula gives that, for a (a_n) of complex numbers, $A(t) := \sum_{n=0}^{t} a_n$, $x, y \in \mathbb{R}$ such that $x < y$, and continuously differentiable function $\phi \in C^1([x, y])$ we have the following:

$$\sum_{n>x}^{y} a_n \phi(n) = A(y)\phi(y) - A(x)\phi(x) - \int_x^y A(u)\phi'(u)du.$$

The von Mangoldt function, $\Lambda(n)$, equals $\log p$ if n is an integer power of prime p and equals 0 otherwise, so

$$\Psi(x) = \sum_{n=1}^{x} \Lambda(n).$$

Then, applying Abel's summation formula with $(\Lambda(n))$ as the sequence of complex numbers and $\phi(n) = \frac{1}{\log n}$ as the function in $C^1([2, x])$, we get:

$$\begin{aligned}
\sum_{n=2}^{x} \frac{\Lambda(n)}{\log n} &= \frac{\Psi(x)}{\log x} - \frac{\Psi(2)}{\log 2} + \int_2^x \frac{\Psi(u)}{u \log^2 u} du \\
&= \frac{\Psi(x)}{\log x} - \frac{\log 2}{\log 2} + \int_2^x \frac{\Psi(u)}{u \log^2(u)} du \\
&\sim \frac{\Psi(x)}{\log x} + \int_2^x \frac{\Psi(u)}{u \log^2 u} du. \\
&\sim \frac{x + O(\sqrt{x}\log^2 x)}{\log x} + \int_2^x \frac{u + O(\sqrt{u}\log^2 u)}{u \log^2 u} du \\
&= \frac{x}{\log x} + \frac{O(\sqrt{x}\log^2 x)}{\log x} + \int_2^x \left(\frac{u + O(\sqrt{u}\log^2 u)}{u} \cdot \frac{1}{\log^2 u} \right) du \\
&= \frac{x}{\log x} + O\left(\frac{\sqrt{x}\log^2 x}{\log x} \right) + O\left(\int_2^x \frac{1}{\log^2 u} du \right) \\
&= \frac{x}{\log x} + O\left(\sqrt{x}\log x \right) + O\left(\int_2^x \frac{1}{\log^2 u} du \right) \\
&\sim \frac{x}{\log x}
\end{aligned}$$

We conclude that

$$\pi(x) \sim \sum_{n=2}^{x} \frac{\Lambda(n)}{\log n} \sim \frac{x}{\log x},$$

which is precisely what we set out to prove. □