UNIT 3 INTEGRATION

Structure

3.0 INTRODUCTION

In Unit 1, we were primarily concerned with the problem of **finding the derivative of given function.** In this unit, we take up the inverse problem, that of finding the original function when we are given the derivative of a function. For instance, we are interested in finding the function F if we know that $F'(x) = 4x^3$. From our knowledge of derivative, we can say that

$$F(x) = x^4 \text{ because } \frac{d}{dx}[x^4] = 4x^3$$

We call the function F an antiderivative of F′ or F(x) is an antiderivative of f. Note that antiderivative of a function is not unique. For instance, x^4+1, x^4+23 are also antiderivatives of $4x^3$. In general, if f(x) is an antiderivative of f (x), then F(x) + c, where C is an arbitrary constant is also an antiderivative of *f*.

3.1 OBJECTIVES

After studying this Unit, you should able to:

- define antiderivative of a function;
- use table of integration to obtain antiderivative of some simple functions;
- use substitution to integrate a function; and
- use formula for integration by parts.

3.2 BASIC INTEGRATION RULES

If F(x) is an antiderivative of f(x) we write

$$\int \underbrace{f(x)}_{\text{Integrand}} dx = F(x) + C$$

Constant of Integration

Variable of Integration

We read $\int f(x)dx$ is the antiderivative of f with respect to x. The differential dx serves to identify x as the variable of integration. The term **indefinite integral** is a synonym for antiderivative.

Note that

$$\int F'(x)dx = F(x) + c \quad \text{and}$$

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

In this sense the integration is the inverse of the differentiation and differentiation is the inverse of integration.

We use the above observations to obtain the following basic rules of integration.

Basic Integration Rules

Table

Differentiation Formula	**Integration Formula**
1. $\frac{d}{dx} k = 0$	1. $\int 0 \; dx = k$
2. $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$	2. $\int x^n \; dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$
3. $\frac{d}{dx} In\,\lvert x \rvert = \frac{1}{x}$	3. $\int \frac{1}{x} \; dx = In\,\lvert x \rvert + c$
4. $\frac{d}{dx}\left[e^x\right] = e^x$	4. $\int e^x dx = e^x + c$
5. $\frac{d}{dx}\left[a^x\right] = a^x \ln a$	5. $\int a^x dx = \frac{a^x}{\ln a}, a > 0, a \neq 1$
6. $\frac{d}{dx} kf(x) = kf'(x)$	6. $\int kf(x) = k\int f(x) + c$
7. $\frac{d}{dx} f(x) \pm g(x)$ $= f'(x) \pm g(x)$	7. $\int f(x) \pm k\int f(x) + c =$ $\int f(x)dx \pm \int g(x)dx$

The general pattern of integration is as follows:

$$\boxed{\text{Given integral}} \downarrow \boxed{\text{Rewrite}} \downarrow \boxed{\text{Integrate}} \downarrow \boxed{\text{Simplify}}$$

Illustration

$$\int \left(\frac{3}{x^4} + \frac{2}{x^2} - \frac{4}{x}\right)\ dx$$

$$= 3\int x^{-4}dx + 2\int x^{-2}dx - 4\int \frac{1}{x}dx \qquad \text{[Rewrite]}$$

$$= \frac{3x^{-4+1}}{-4+1} + 2\frac{x^{-2+1}}{-2+1} - 4\ln|x| + \text{c} \qquad \text{[Integrate]}$$

$$= -\frac{1}{x^3} - \frac{2}{x} - 4\ln|x| + c \qquad \text{[Simplify]}$$

Solved Examples

Example 1: Evalutate

$$\int (2x^{1/2} + 3x^{1/3} - 4x^{1/4})\ dx$$

Solution :

$$\int (2x^{1/2} + 3x^{1/3} - 4x^{1/4})\ dx$$

$$= 2\int x^{\frac{1}{2}}dx + 3\int x^{\frac{1}{3}}dx - 4\int x^{\frac{1}{4}}dx \qquad \text{[Rewrite]}$$

$$= 2\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 3\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} - 4\frac{x^{\frac{1}{4}+1}}{\frac{1}{4}+1} + c \qquad \text{[Integrate]}$$

$$= \frac{4}{3}x^{\frac{3}{2}} + \frac{9}{4}x^{\frac{4}{3}} - \frac{16}{5}x^{\frac{5}{4}} + c \qquad \text{[Simplify]}$$

Example 2 : Evalutate

$$\int \frac{\left(\sqrt{x} + x^{1/3}\right)^2}{x} dx$$

Solution :

$$\int \frac{\left(\sqrt{x} + x^{1/3}\right)^2}{x} dx$$

$$= \int \frac{1}{x}\left[\left(\sqrt{x}\right)^2 + 2\left(\sqrt{x}\right)(x^{\frac{1}{3}}) + \left(x^{\frac{1}{3}}\right)^2\right] dx$$

$$= \int \frac{1}{x}\left[x + 2x^{\frac{1}{2}+\frac{1}{3}} + \ x^{\frac{2}{3}}\right] dx$$

$$= \int \left[1 + 2x^{\frac{5}{6}-1} + \ x^{\frac{2}{3}-1}\right] dx$$

$$= \int [1 + 2x^{-1/6} + \ x^{-1/3}] dx$$

$$= \ x + \frac{2x^{--\frac{1}{6}+1}}{(-\frac{1}{6}+1)} + \frac{x^{-\frac{1}{3}+1}}{(-\frac{1}{3}+1)} + \ c$$

$$= \ x + \frac{12}{5}x^{5/6} + \frac{3}{2}x^{2/3} + \ c$$

Example 3 : Evalaute

$$\int \frac{2^x + \ 3^x}{5^x} dx$$

Solution:

$$\int \frac{2^x + \ 3^x}{5^x} dx \ = \int \left(\frac{2^x}{5^x} + \frac{3^x}{5^x}\right) \ dx$$

$$= \ \int \left[\left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x \right] \ dx$$

$$= \frac{\left({}^2/_5\right)^x}{\ln\left({}^2/_5\right)} + \frac{\left({}^3/_5\right)^x}{\ln\left({}^3/_5\right)} + c$$

Example 4 : Evaluate

$$\int \frac{(a^x + b^x)^2}{a^x b^x}\, dx$$

Solution We have

$$\frac{(a^x + b^x)^2}{a^x b^x} = \frac{(a^x)^2 + (b^x)^2 + 2a^x b^x}{a^x b^x}$$

$$= \frac{(a^x)^2}{a^x b^x} + \frac{(b^x)^2}{a^x b^x} + \frac{2a^x b^x}{a^x b^x}$$

$$= \frac{a^x}{b^x} + \frac{b^x}{a^x} + 2$$

$$= \left(\frac{a}{b}\right)^x + \left(\frac{b}{a}\right)^x + 2$$

Thus,

$$\int \frac{(a^x + b^x)^2}{a^x b^x} dx = \int \left[\left(\frac{a}{b}\right)^x + \left(\frac{b}{a}\right)^x + 2\right] dx$$

$$= \frac{\left(\frac{a}{b}\right)^x}{\ln\left(\frac{a}{b}\right)} + \frac{\left(\frac{b}{a}\right)^x}{\ln\left(\frac{b}{a}\right)} + 2x + c$$

Example 5 : Evaluate

$$\int (e^{alnx} + e^{xlna})\, dx$$

Solution : We know that

$$e^{alnx} = x^a$$

and $e^{xlna} = x^a$

Thus, $\int (e^{alnx} + e^{xlna})\, dx$

$$= \int (x^a + a^x) dx$$

$$= \frac{x^{a+1}}{a+1} + \frac{a^x}{lna} + c$$

Check Your Progress – 1

Integrate the following functions.

1. $x^3 + 2^x$ 2. $x^e + e^x$

3. $(\sqrt{x} + x^2)/x^2$ 4. $(2^x + 3^x)^2/5^x$

5. $3^x + x^7 - 2/x^4$ 6. $(3^x + 5^x)/7^x$

Answers

1. $\frac{1}{4}x^4 + \frac{2^x}{ln2} + c$ 2. $\frac{x^{e+1}}{e+1} + e^x + c$

3. $\ln|x| + x + 4x^{1/2} + c$

4. $\left(\frac{4}{5}\right)^x \frac{1}{\ln(4/5)} + \left(\frac{9}{5}\right)^x \frac{1}{\ln(9/5)} + 2(6/5)^x \frac{1}{\ln(6/5)} + c$

5. $\frac{3^x}{\ln 3} + \frac{x^8}{8} + \frac{2}{3x^3} + c$ 6. $\left(\frac{3}{7}\right)^x \frac{1}{\ln(3/7)} + \left(\frac{5}{7}\right)^x \frac{1}{\ln(5/7)} + c$

3.3 INTEGRATION BY SUBSTITUTION

If the integrand is of the form $\int f(g(x))g'(x)dx$, we can integrate it by substituting g(x) = t. We illustrate the technique in the following illustration.

Illustration: Integrate $e^x(e^x + 2)^7$ To integrate this function, we put

$e^x + 2 = t \Rightarrow e^x dx = dt$

Thus,

$$\int e^x(e^x + 2)^7 dx = \int t^7 \, dt$$

$$= \frac{1}{8}t^8 + c$$

$$= \frac{1}{8}(e^x + 2)^8 + c$$

Solved Examples

Example 6 : Evaluate

$$\int \sqrt{7x - 2} \; dx$$

Solution : To evaluate this integral,

We put $7x - 2 = t^2$

$$\Rightarrow\ 7dx = 2tdt\ or\ dx\ = \frac{2}{7}\ t\ dt$$

$$\therefore \quad \int \sqrt{7x-2}\ dx \quad = \int \sqrt{t^2}\,\frac{2}{7} t\ dt = \frac{2}{7}\int t^2\ dt$$

$$= \frac{2}{7}\left(\frac{1}{3}\right)\ t^3 +\ c = \frac{2}{21}\ \ t^3 +\ c$$

$$= \frac{2}{21}(7x-2)^{\ 3/2} +\ c$$

Example 7 : Evaluate

$$\int x^2\sqrt{5x-3}\,dx$$

Solution : In this case, again, we put

5x – 3 = t^2 ⇒ 5 dx = 2tdt

$$\therefore dx\ = \frac{2}{5}\ \ t\ dt$$

Also, $x\ = \frac{1}{5}\ (t^2\ \ +3)$

Thus,

$$\int x^2\sqrt{5x-3}\,dx = \frac{1}{5}\int (t^2 +\ 3)\sqrt{t^2}\ \frac{2}{5}\ t\ dt$$

$$= \frac{2}{25}\int (t^2 +\ 3)t^2 dt$$

$$= \frac{2}{25}(t^4 +\ 3t^2)dt$$

$$= \frac{2}{25}\left(\frac{1}{5}t^5 + \frac{3t^3}{3}\right) +\ c$$

$$= \frac{2}{125}(t^5 +\ 5t^3) + c$$

$$= \frac{2}{125}\left[(5x-3)^{5/2} +\ 5(5x-3)^{3/2}\right] +\ c$$

Example 8 : Evaluate

$$\mathrm{I} = \int \frac{dx}{(3x-2)^2}$$

Solution : Put $3x - 2 = t \Rightarrow 3dx = dt$, so that

$$I = \frac{1}{3}\int \frac{dt}{t^2} = \frac{1}{3}\int t^{-2}\ dt$$

$$= \frac{1}{3}\int \frac{t^{-2+1}}{-2+1} + c = -\frac{1}{3t} + c$$

$$= -\frac{1}{3(3x-2)} + c$$

Example 9 : Evaluate

$$\int (x+1)\, e^x (xe^x + 3)^4 dx$$

Solution : Put $x\, e^x + 3 = t$

$\Rightarrow (x\, e^x + e^x)\ dx = dt$
or $(x + 1)\ e^x\ dx = dt$

Thus,

$$\int (x+1)e^x\ (xe^x + 3)^4 dx$$

$$= \int t^4\ \mathrm{dt} = \frac{1}{5}t^5 + \mathrm{c} = \frac{1}{5}(x\, e^x + 3)^5 + c$$

Example 10 : Evaluate the integral

$$\int \frac{2e^x + 3e^{-x}}{3\, e^x + 4\, e^{-x}}\ dx$$

Solution : Remark To evaluate an integral of

$$\int \frac{ae^x + be^{-x}}{c\, e^x + d\, e^{-x}}\ dx$$

We write

Numerator = α (Denominator) $+ \beta \frac{d}{dx}$ (Denominator)

and obtain values of α and β, by equating coefficients of e^x *and* e^{-x}

In the present case, we write

$2\, e^x + 3\, e^{-x} = \alpha\, (3e^x + 4\, e^{-x}) + \beta \frac{d}{dx}(3\, e^x - 4\, e^{-x})$

$\Rightarrow 2\, e^x + 3\, e^{-x} = \alpha\, (3e^x + 4\, e^{-x}) + \beta\, (3\, e^x - 4\, e^{-x})$

Equating coefficients of e^x and e^{-x}, we obtain

$$2 = 3\alpha + 3\beta$$

and $3 = 4\alpha - 4\beta$

$\Rightarrow$ $\alpha + \beta = 2/3$ and $\alpha - \beta = 3/4$

Adding, we obtain

$$2\alpha = \frac{2}{3} + \frac{3}{4} \quad or\ \alpha = \frac{17}{24}$$

$$\therefore \quad \beta = \frac{2}{3} - \alpha = \frac{2}{3} - \frac{17}{24} = -\frac{1}{24}$$

Thus,

$$\int \frac{3e^x - 4e^{-x}}{3e^x + 4e^{-x}}\,dx = \int \frac{\left(\frac{17}{24}\right)(3e^x + 4e^{-x}) + \left(-\frac{1}{24}\right)(3e^x - 4e^{-x})}{3e^x + 4e^{-x}}\,dx$$

$$= \left(\frac{17}{24}\right)\int dx - \left(\frac{1}{24}\right)\int \frac{3e^x - 4e^{-x}}{3e^x + 4e^{-x}}\,dx$$

$$= \frac{17}{24}x - \frac{1}{24}I_1$$

Where $I_1 = \int \frac{2e^x + 3e^{-x}}{3e^x + 4e^{-x}}\,dx$

$Put\ \ 3e^x + 4e^{-x} = t$

$\Rightarrow (3e^x + 4e^{-x})dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{t} = \text{In}\,|t|$

$= \ln(3e^x + 4e^{-x})$.

$$\text{Hence,} \int \frac{2e^x + 3e^{-x}}{3e^x + 4e^{-x}}\,dx = \frac{17}{24}x - \frac{1}{24}\ln(3e^x + 4e^{-x}) + c$$

Check Your Progress – 2

Evaluate the following integrals.

1. $\int \frac{x}{\sqrt{x+1}}\,dx$

2. $\int \frac{e^{3x}}{e^{3x}+4}\,dx$

3. $\int \frac{4x-7}{(2x^2-7x+8)^2}\,dx$

4. $\int x\sqrt{x+1}\,dx$

5. $\int \frac{dx}{\sqrt{x}+x}$

6. $\int 2^{4-5x}\,dx$

7. $\int \frac{e^x + 3e^{-x}}{2e^x + e^{-x}}\,dx$

8. $\int \frac{x^3}{\sqrt{x^2-1}}$

Answers

1. $\frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c$

2. $\frac{1}{3}\ln(e^{3x}+4) + c$

3. $\frac{-1}{(2x^2-7x+8)^2} + c$

4. $\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$

5 $2\ln(\sqrt{x} + 1) + c$

6. $-\frac{1}{5\ln 2}2^{4-5x} + c$

7. $\frac{5}{4}x + \frac{7}{4}\ln|2e^x - e^{-x}| + c$

8. $\frac{1}{3}(x^2-1)^{\frac{3}{2}} + \sqrt{x^2-1} + c$

3.4 INTEGRATION OF RATONAL FUNCTIONS

A function R$(x)$is said to be rational if R$(x)$is of the form $\frac{P(x)}{Q(x)}$ where P(x) and Q(x) are polynomials in x. For instance, $\frac{x-3}{x^2+1}$ and $\frac{2x+1}{x^2-3x+5}$ are rational functions.

A rational function R$(x) = \frac{P(x)}{Q(x)}$ is said to be **proper** if deg(p(x)) $< deg\ (Q(x))$ and is said to be **improper** if deg (P(x)) ≥ deg(Q(x)).

In case R$(x) = \frac{P(x)}{Q(x)}$ is **improper** rational function, we can write it as

$$R(x) = A(x) + \frac{B(x)}{Q(x)}$$

where A$(x)$is a polynomial and $\frac{B(x)}{Q(x)}$ is a proper rational functions

Partial Fractions

Recall when we add two rational functions, we get a rational function. For instance, when we add

$$\frac{2}{2x-3} \text{ and } \frac{1}{1-x}$$

we get $\frac{2}{2x-3}+\frac{1}{1-x} = \frac{2(1-x)+\ 2x-3}{(2x-3)(1-x)} = \frac{-1}{(2x-3)(1-x)}$

We call $\frac{2}{2x-3}$ and $\frac{1}{1-x}$

as partial fractions of $\frac{-1}{(2x-3)(1-x)}$

Methods of Splitting a Rational Function into Partial Fractions

Case 1 : When denominator consists of distinct Linear factors

We illustrate the method in the following illustration.

Illustraton: Resolve

$$\frac{x}{(2x-1)(x+1)(x-2)}$$

into partial fractions.

We write

$$\frac{x}{(2x-1)(x+1)(x-2)} = \frac{A}{2x-1}+\frac{B}{x+1}+\frac{C}{x-2}$$

where A, B and C are constants.

$\Rightarrow$ x = A(x + 1) (x−2) + B (2x−1) (x−2) + C(2x−1)(x + 1)

Put x = ½, –1 and 2 to obtain

$\frac{1}{2} = \mathrm{A}\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right) \Rightarrow \mathrm{A} = -\frac{2}{9};$

$-1 = \mathrm{B}(-3)(-3) \Rightarrow \mathrm{B} = -\frac{1}{9};$

$2 = \mathrm{C}(3)(3) \Rightarrow \mathrm{C} = \frac{2}{9}$

Thus

$$\frac{x}{(2x-1)(x+1)(x-2)} = -\frac{2}{9}\frac{1}{2x-1}-\frac{1}{9}\frac{1}{x+1}+\frac{2}{9}\frac{1}{x-2}$$

Case 2: When Denominator consists of repeated Linear Factors

Illustration: Resolve

$$\frac{x}{(2x-1)(x+1)^2}$$

into partial fractions.

Write

$$\frac{x}{(x-1)(x+1)^2} = \frac{A}{2x-1} + \underbrace{\frac{B}{x+1} + \frac{C}{(x+1)^2}}_{\text{Note carefully}}$$

where A, B and C are constants.

$\Rightarrow x$ = A$(x+1)^2$ + B($x-1$)(x+1) + C ($x-1$)

Put x =1 and -1, to obtain

1 = 4A $\Rightarrow$A = 1/4; and

$-1 = -2C$ $\Rightarrow$ C = 1/2.

Next, we compare coefficients of x^2 on both the sides to obtain

$$0 = A + B \Rightarrow B = -A = -\frac{1}{4}.$$

Case 3 : When the Denominator consists of irreducible Quadratic Factor.

Illustration : Resolve

$$\frac{x}{(x+1)(x^2+x+1)}$$

Into partial fractions.

Write

$$\frac{x}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$

where A, B and C are constants.

$\Rightarrow x$ = A(x^2+x+1) +(Bx + C) (x + 1)

Put x = -1 to obtain A = -1. Comparing coefficients, we obtain

O = A + B $\Rightarrow$ B = $-A$ = 1

Next, put x = 0 to obtain

0 = A + C $\Rightarrow$C = $-A$ = 1

Thus,

$$\frac{x}{(x+1)(x^2+x+1)} = \frac{-1}{x+1} + \frac{x+1}{x^2+x+1}$$

Solved Examples

Example 11 : Evaluate the integral

$$\int \frac{x}{(x+1)(2x-1)}\, dx$$

Solution : We first resolve the integrand into partial fractions. Write

$$\frac{x}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1}$$

$\Rightarrow x = \mathrm{A}(2x - 1) + \mathrm{B}(x + 1)$

Put $x = ½$ and -1 to obtain

$\frac{1}{2} = \mathrm{B}\left(\frac{1}{2} + 1\right) \Rightarrow \mathrm{B} = \frac{1}{3}$

$-1 = \mathrm{A}\,(-3) \Rightarrow \mathrm{A} = \frac{1}{3}$

Thus,

$$\int \frac{x}{(x+1)(2x-1)}\, dx = \frac{1}{3}\int \frac{dx}{(x+1)} + \frac{1}{3}\int \frac{dx}{(2x-1)}$$

$$= \frac{1}{3}\log|x+1| + \frac{1}{3}.\frac{1}{2}\log|2x-1| + \mathrm{c}$$

$$= \frac{1}{3}\log|x+1| + \frac{1}{6}\log|2x-1| + \mathrm{c}$$

Example 12: Integrate

(i) $\frac{1}{x^2 - a^2}$ (ii) $\frac{1}{a^2 - x^2}$

Solution: (i) We write $\frac{1}{x^2 - a^2}$ as $\frac{1}{(x-a)(x+a)}$ and split into partial fractions.

Write

$$\frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

$\Rightarrow 1 = \mathrm{A}(x + a) + \mathrm{B}\,(x - a)$

Put $x = a$ and $-a$ to obtain

$1 = \mathrm{A}(2a) \Rightarrow \mathrm{A} = \frac{1}{2a}$;

$1 = \mathrm{B}(-2a) \Rightarrow B = \frac{1}{2a}$;

Thus,

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \int \left| \frac{1}{x+a} - \frac{1}{x-a} \right| dx$$

$$= \frac{1}{2a} [\log|x+a| - \log|x-a|] + c$$

$$= \frac{1}{2a} log \left[\frac{x+a}{x-a} \right] + c$$

(ii) Note that

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} = \int \left[\frac{1}{a+x} + \frac{1}{a-x} \right] dx$$

$$= \frac{1}{2a} [\log|a+x| - \log |a-x|] + c$$

$$= \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

Two Important Formulae
1. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x+a}{x-a} \right| + \text{c}$
2. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + \text{c}$

Remark : Above two formulae may be used as standard formulae.

Example 13 : Evaluate the integral.

$$\int \frac{x\, dx}{(x-1)(x+5)(2x-1)}$$

Solution : We write

$$\frac{x}{(x-1)(x+5)(2x-1)} = \frac{A}{x-1} + \frac{B}{x+5} + \frac{C}{2x-1}$$

$\Rightarrow x = A(x+5)(2x-1) + B(x-1)(2x-1) + C(x-1)(x+5)$

Put $x = 1, -5$ and $\frac{1}{2}$ to obtain

$1 = 6A \Rightarrow A = 1/6$

$-5 = 66B \Rightarrow B = -5/66$

$$\frac{1}{2} = -\frac{11}{4}\,\mathrm{C} \Rightarrow \mathrm{C} = -2/11$$

Thus,

$$\int \frac{x}{(x-1)(x+5)(2x-1)} = \frac{1}{6}\int \frac{dx}{x-1} - \frac{5}{66}\int \frac{dx}{x+5} - \frac{2}{11}\int \frac{dx}{2x-1}$$

$$= \frac{1}{6}\log|x-1| - \frac{5}{66}\log|x+5| - \frac{1}{11}\log|2x-1| + c$$

Example 14 : Evaluate the integral

$$\mathrm{I} = \int \frac{dx}{1+3e^x+2e^{2x}}$$

Solution: Put $e^x = t$, so that $e^x dx = dt$, and

$$\mathrm{I} = \int \frac{dt}{t(1+3t+2t^2)}$$

$$= \int \frac{dt}{t(1+t)(1+2t)}$$

We now split

$$\frac{1}{t(1+t)(1+2t)}$$

into partial fractions, to obtain

$$\frac{1}{t(1+t)(1+2t)} = \frac{A}{t} + \frac{B}{1+t} + \frac{C}{1+2t}$$

$\Rightarrow$ 1= A(1 + t)(1 + 2t) + Bt(1 + $2t$) + Ct(1 + t)

Put t = 0, –1 and –1/2 to obtain

1 = A $\Rightarrow$ A = 1;

1 = B $\Rightarrow$ B = 1;

1 = −C/4 $\Rightarrow$C = −4

Thus,

$$\int \frac{dt}{t(1+t)(1+2t)} = \int \frac{dt}{t} + \int \frac{dt}{(1+t)} - 4\int \frac{dt}{(1+2t)}$$

$= \log|t| + \log|1+t| - 2\log|1+2t| + c$

$= \log(e^x) + \log(e^x+1) - 2\log(2e^x+1) + c$

$$= x + log\frac{e^x+1}{(2e^x+1)^2} + c$$

Example 15 : Evalaute the integral

$$\mathrm{I} = \int \frac{x^2}{(x+1)^3}\, dx$$

Solution : To evaluate an integral of the form

$$\int \frac{P(x)}{(a+bx)^r}\, dx, \text{we put } a + bx = t.$$

$$\text{So, we put } x + 1 = t \Rightarrow dx = dt$$

$$\text{and I} = \int \frac{(t+1)^2}{t^3}\, dt = \int \frac{t^2+2t+1}{t^3}\, dt$$

$$= \int \left(\frac{1}{t} - 2t^{-2} + t^{-3}\right) dt$$

$$= \log|\mathrm{t}| - \frac{2t^{-1}}{-1} + \frac{t^{-2}}{-2} + c$$

$$= \log|\mathrm{t}| + \frac{2}{t} - \frac{1}{2t^2} + c$$

$$= \log|x+1| - \frac{2}{x+1} + \frac{1}{2(x+1)^2} + c$$

Example 16 : Evaluate the integral

$$\mathrm{I} = \int \frac{(x+1)^2}{(x-1)^2}\, dx$$

Solution : Put $x - 1 = \mathrm{t}$, so that

$$\mathrm{I} = \int \frac{(t+1+1)^2}{t^2} dt = \int \frac{(t+2)^2}{t^2}\, dt$$

$$= \int \frac{(t+4t+4)^2}{t^2}\, dt$$

$$= \int \left[1 + \frac{4}{t} + 4t^{-2}\right] dt$$

$$= t + 4log|t| - \frac{4}{t} + c$$

$$= x - 1 + 4log|x-1| - \frac{4}{x-1} + c$$

$$= x + 4log|x-1| - \frac{4}{x-1} + c \quad [\text{absorb} - 1 \text{ in the constant of integration}]$$

Example 17 : Evaluate the integral

$$\mathrm{I} = \int \frac{3x-1}{(x+1)^2(2x-1)}\,dx$$

Solution :

We write

$$\frac{3x-1}{(x+1)^2(2x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{2x-1}$$

$\Rightarrow 3x-1 = \mathrm{A}(x+1)(2x-1) + \mathrm{B}(2x-1) + \mathrm{C}\,(x+1)^2$

Put $x = -1$ and ½ to obtain

$-4 = -3\mathrm{B} \Rightarrow \mathrm{B} = 4/3$

$$\frac{3}{2} - 1 == \mathrm{C}\,(-1/2+1)^2 \Rightarrow \frac{1}{2} = \frac{1}{4}C \Rightarrow \mathrm{C} = 2$$

Comparing coefficient of x^2, we get

$0 = 2\mathrm{A} + \mathrm{C} \Rightarrow 2\mathrm{A} = -\mathrm{C} = -2$

$\Rightarrow \mathrm{A} = -1$

Thus,

$$\int \frac{3x-1}{(x+1)^2(2x-1)}\,dx = -\int \frac{dx}{x+1} + \frac{4}{3}\int (x+1)^{-2}\,dx + 2\int \frac{dx}{2x-1}$$

$$= -\log|x+1| + \frac{4}{3}\,\frac{(x+1)^{-2+1}}{(-2+1)} + \frac{2log|2x-1|}{2} + c$$

$$= \log\left|\frac{2x-1}{x+1}\right| - \frac{4}{3}\,\frac{1}{x+1} + \mathrm{c}$$

Example 18 : Evaluate the integral

$$\mathrm{I} = \int \frac{dx}{(e^x-1)^2}$$

Solution :

Put $e^x - 1 = t$, so that $e^x dx = dt$, and

$$\mathrm{I} = \int \frac{dt}{t^2(t+1)}$$

We split $\frac{1}{t^2(t+1)}$ into partial fractions

$$\frac{1}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1}$$

$\Rightarrow 1 = \text{At}\,(t+1) + B(t+1) + Ct^2$

Put $t = 0$, $t = -1$ to obtain

$1 = \text{B} \Rightarrow \text{B} = 1$

$1 = \text{C} \Rightarrow \text{C} = 1$

Comapring coefficient at t^2, we obtain

$0 = \text{A} + \text{C} \Rightarrow \text{A} = -\text{C} = -1$

Thus,

$$\text{I} = \int \left[-\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t+1}\right] dt$$

$$= -\log|\text{t}| - \frac{1}{t} + \log|t+1| + c$$

$$= \log\left|\frac{t+1}{t}\right| - \frac{1}{t} + c$$

$$= log\left(\frac{e^x+1}{e^x}\right) - \frac{1}{e^x} + \text{c}$$

Check Your Progress 3

Integrate the following functions

1. $\frac{x^2+1}{(2x+1)(x-1)(x+1)}$
2. $\frac{x^2+1}{x(x^2-1)}$
3. $\frac{2x-3}{(x^2-1)(2x+3)}$
4. $\frac{x}{x(1+4x^3+3x^6)}$
5. $\frac{e^x}{e^x-3e^{-x}+2}$
6. $\frac{x^2}{(x+2)^3}$
7. $\frac{x^2}{(x-1)^3(x+1)}$
8. $\frac{e^x}{(e^x-1)^3}$

Answers

1. $-\frac{5}{6}\log|2x+1|+\frac{1}{3}\log|x-1|+\log|x+1|+c$

2. $\log\left|\frac{x^1-1}{x}\right|+c$

3. $\frac{5}{2}\log|x+1|+\frac{1}{10}\log|x-1|-\frac{12}{5}\log|2x+3|+c$

4. $\log|x|+\frac{1}{6}\log|1+x^3|-\frac{1}{2}\log|1+3x^3|+c$

5. $\frac{1}{4}\log\ \frac{|e^x-1|}{(e^x+1)^3}\ +\ \mathrm{c}$

6. $\log|x+2|+\frac{4}{x+2}-\frac{2}{(x+1)^2}+c$

7. $\frac{1}{8}\log\frac{|x-1|}{|x+1|}-\frac{3}{4}\frac{2}{x-1}-\frac{1}{4}\frac{1}{(x+1)^2}+c$

8. $-\frac{1}{2}\frac{1}{(e^x-1)^2}+c$

3.5 INTEGRATION BY PARTS

Recall the product rule for the derivative

$$\frac{d}{dx}[uv] = uv' + vu'$$

$$\Rightarrow uv = \int uv'+vu'\,dx$$

$$\Rightarrow= \int uv'dx = uv - \int vu'dx$$

We can write the above formula as

$$\int u(x)v(x)dx = u(x)\int v(x)dx - \int\left[\frac{du}{dx}\int v(x)dx\right]dx$$

In words, the above formula state

Integral of the product of two functions

= First function × integral of the second function – Integral of (the derivative of the first function × integral of the second function)

For instance, to evaluate $\int x.e^{x}$, we take e^{x} as second function and x as the first function.

By the above formula

$$\int xe^{x}\ dx\ = xe^{x} - \int \frac{d}{dx}[x]\ \mathrm{e}^{x}\mathrm{dx}$$

$$= xe^{x} - \int 1.e^{x} = xe^{x} - e^{x} + \mathrm{c}$$

Solved Examples

Example 18 Integrate $x \log x$

Soluton : We take x as the second function and log x as the first function.

$$\int x\ logx\ dx = \int (logx)x\ dx$$

$$= (\log x)\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}dx$$

$$= \frac{1}{2}x^2 \log x - \frac{1}{2}\int x dx$$

$$= \frac{1}{2}x^2 \log x - \frac{1}{4}x^2 + c$$

Example 19 Evaluate

$$\int \sqrt{x}\ logx\ dx$$

Solution : We take $\sqrt{x}$ as the second function and log x as the first function. We have

$$\int \sqrt{x}\log x dx = \int (\log x)x^{1/2}dx$$

$$= (\log x)\frac{x^{3/2}}{3/2} - \int \frac{1}{x}\frac{x^{3/2}}{3/2}dx$$

$$= \frac{2}{3}x^{3/2}\log x - \frac{2}{3}\int x^{1/2}dx$$

$$= \frac{2}{3}x^{3/2}\log x - \frac{2}{3}\int \frac{x^{3/2}}{3/2} + c$$

$$= \frac{2}{3}x^{3/2}\log x - \frac{4}{9}\int x^{3/2} + c$$

Example 20 : Evaluate

$$\int \frac{logx}{x^2}\,dx$$

Solution : We take x^{-2} as the second function and log x as the first function.

$$\mathrm{I} = \int x^{-2}\,logx\,dx$$

$$= \frac{x^{-2+1}}{-2+1}\log x - \int \frac{1}{x}\cdot\left(\frac{x^{-1}}{-1}\right)dx$$

$$= -\frac{1}{x}\log x + \int x^{-2}\,dx$$

$$= -\frac{1}{x}\log x + \frac{x^{-1}}{-1} + c$$

$$= -\frac{1}{x}\log x - \frac{1}{x} + c$$

Evaluate 21: Evaluate

$$\int xe^{-x}\,dx$$

Solution: We take e^{-x} as the second function and x as the first function. We have

$$\int xe^{-x} = x\left(\frac{e^{-x}}{-1}\right) - \int (1)\frac{e^{-x}}{-1}\,dx$$

$$= -xe^{-x} + \int e^{-x}\,dx$$

$$= -x\,e^{-x} - e^{-x} + c$$

$$= -(x+1)\,e^{-x} + c$$

Example 22 : Evaluate

$$\int \log(1+x)^{1+x}\,dx$$

Solution : We write $\log(1+x)^{1+x}$ = (1+ x) log (1+ x) and (1+ x) as the second function. We have

$$\mathrm{I} = \int (1+x)\log(1+x)\quad dx$$

$$= \frac{1}{2}(1+x)^2\log(1+x) - \int (1+x)^2\frac{1}{1+x}dx$$

$$= \frac{1}{2}(1+x)^2 \log(1+x) - \int (1+x)dx$$

$$= \frac{1}{2}(1+x)^2 \log(1+x) - \frac{1}{4}(1+x)^2 + c$$

Example 23 : Evaluate

$$\int \log x \, dx$$

Solution : We write $\log x = 1.\ \log x$ and take 1 as the 2nd function and log$x$ as the first function.

$$\int \log x dx = \int 1.\log x dx$$

$$= x \log x - \int (x)\frac{1}{x} dx$$

$$= x \log x - \int dx$$

$$= x \log x - x + c$$

$$= x(\log x - 1) + c$$

Example 24 ; Evaluate

$$\int x^3 (\log x)^2 \, dx$$

Solution : We take x^3 as the second function. We have

$$\int x^3 (\log x)^2 \, dx = \frac{1}{4} x^4(\log x)^2 - \frac{1}{4}\int x^4 2(\log x)\frac{1}{x} dx$$

$$= \frac{1}{4} x^4(\log x)^2 - \frac{1}{2}\int x^3(\log x)\, dx$$

$$= \frac{1}{4} x^4(\log x)^2 - \frac{1}{2}\left[\frac{1}{4}x^4\log x - \frac{1}{4}\int x^4\frac{1}{x}\, dx\right]$$

$$= \frac{1}{4} x^4(\log x)^2 - \frac{1}{2}\left[\frac{1}{4}x^4\log x - \frac{1}{4}\int x^3\, dx\right]$$

$$= \frac{1}{4} x^4(\log x)^2 - \frac{1}{2}\left[\frac{1}{4}x^4\log x - \frac{1}{16}x^4\right] + c$$

$$= \frac{1}{4} x^4(\log x)^2 - \frac{1}{8}\, x^4 \log x + \frac{1}{32}x^4 + c$$

$$= \frac{1}{32}\, x^4[8(\log x)^2 - 4\log x + 1] + c$$

Remark : If an integrand is of the form $e^x(f(x) + f'(x))$, we write it as $e^x f(x) + e^x f'(x)$, and just integrate the first function. We have

$$\text{I} = \int e^x f(x) + f'(x))dx$$

$$= \int e^x (f(x))dx + \int e^x f'(x)dx$$

$$= e^x f(x) - \int e^x f'(x) + \int e^x f'(x)dx$$

$$= e^x f(x) + c$$

Example 25 : Evaluate the integral

$$\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

Solution : We write

$$\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx = \int e^x \frac{1}{x}\, dx - \int e^x \frac{1}{x^2} dx$$

$$= e^x \frac{1}{x} - \int e^x \frac{(-1)}{x^2}\, dx - \int e^x \frac{1}{x^2}\, dx$$

$$= \frac{e^x}{x} + c$$

Check Your Progress 4

Integrate the followings:

1. $x^2 e^x$ 2. $x\log(1+x)\, dx$

3. $e^{\sqrt{x}}$ 4. $e^x\ (\log x + \frac{1}{x})$

5. $e^x \frac{x+1}{(x+2)^2}$ 6. $\log\sqrt{x}$

7. $\log(1 + x)$ 8. $(1 - x)^2 \log x$

Answers

1. $(x^2 - 2x + 2)\ e^x + c$

2. $\frac{1}{2}(x^2 - x)\log(1 + x) - \frac{1}{4}x^2 + \frac{1}{2}x + c$

3. $2(\sqrt{x} - 1)\ e^{\sqrt{x}} + c$

4. $e^x\ logx + c$

5. $\frac{e^x}{x + 2} + c$

6. $\frac{1}{2}\,(xlogx - x) + c$

7. $(x+1)\log(1+x) - x + c$

8. $\left(x - \frac{1}{3}x^3\right)\log x - x + \frac{1}{9}x^3 + c$

3.6 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

1. $\int (x^3 + 2^x)dx = \frac{1}{4}x^4 + \frac{1}{ln2}2^x + c$

2. $\int (x^e + e^x)\, dx = \frac{1}{e+1}e^x + e^x + c$

3. $\int \frac{(\sqrt{x} + x)^2}{x^2}\, dx = \int \frac{x + 2x^{1/2}x + x^2}{x^2}dx$

$$= \int (\frac{1}{x} + 2x^{-\frac{1}{2}} + 1)\, dx$$

$$= \ln|x| + 4x^{1/2} + x + c$$

4. $\int \frac{(2^x + 3^x)^2}{5^x}dx = \int \frac{(2^x)^2 + 2(2^x) + 3^x\,(3^x)^2}{5^x}\, dx$

$$= \int \left[\left(\frac{4}{5}\right)^x + 2\left(\frac{6}{5}\right)^x + \left(\frac{9}{5}\right)^x\right]dx$$

$$= \frac{(4/5)^x}{\ln(4/5)} + 2\,\frac{(6/5)^x}{\ln(6/5)} + \frac{(9/5)^x}{\ln(9/5)} + c$$

5. $\int (3^x + x^7 - 2\,x^{-4})\, dx = \frac{3^x}{\ln 3} + \frac{x^8}{8} - \frac{2x^{-3}}{(-3)} + c$

$$= \frac{3^x}{\ln 3} + \frac{1}{8}x^8 + \frac{2}{3x^3} + c$$

6. $\int \left(\frac{3^x + 5^x}{7^x}\right) dx = \int \left[\left(\frac{3}{7}\right)^x + \left(\frac{5}{7}\right)^x\right]dx$

$$= \frac{(3/7)^x}{\ln(3/7)} + \frac{(5/7)^x}{\ln(5/7)} + c$$

Check Your Progress 2

1. Put $x + 1 = t^2$ So that $x = t^2 - 1$ and $dx = 2t\,dt$

$$\therefore \int \frac{x}{\sqrt{x+1}} = \int \frac{t^2-1}{t} 2tdt = 2\left[\frac{t^3}{3} - t\right] + c$$

$$= \frac{2}{3}(x+1)^{\frac{3}{2}} + 2\sqrt{x+1} + c$$

2. Put $e^{3x} + 4 = t$, so that $3e^{3x}dx = dt$

$$\int \frac{e^{3x}}{e^{3x}+4} dx = \frac{1}{3} \int \frac{dt}{t} = \frac{1}{3}\ln|t| + c$$

$$= \frac{1}{3}\ln(e^{3x} + 4) + c$$

3. Put $2x^2 - 7x + 8 = t$, so that $(4x - 7)dx = dt$

$$\therefore \int \frac{4x-7}{(2x^2-7x+8)^2} dx = \int \frac{dt}{t^2} = \int t^{-2}\,dt$$

$$= -\frac{1}{t} + c = \frac{-1}{(2x^2-7x+8)} + c$$

4. Put $\sqrt{x+1} = t^2$ so that $x = t^2 - 1$, $dx = 2tdt$

$$\therefore \int x\sqrt{x+1}\,dx = \int (t^2-1)t.2t\,dt$$

$$= \frac{2}{5}t^5 - \frac{2}{3}t^3 + c$$

$$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + c$$

5. Put $\sqrt{x} = t$ or $x = t^2$, so that $dx = 2tdt$

$$\therefore \int \frac{dx}{\sqrt{x}+x} = \int \frac{2tdt}{t+t^2} = 2\int \frac{dt}{t+1}$$

$$= 2\ln(t+1) + c = 2\ln(\sqrt{x} + 1) + c$$

6. Put $4 - 5x = t$, so that $-5dx = dt$

Thus, $\int 2^{4-5x}\,dx = -\frac{1}{5}\int 2^t dt = -\frac{1}{5}\frac{2^t}{ln2} + c$

$$= -\frac{1}{5}\frac{2^{4-5x}}{ln2} + c$$

7. Write

$e^x + 3e^{-x} = \alpha\,(2\,e^x - e^{-x}) + \beta\,\frac{d}{dx}(2\,e^x - e^{-x})$

$\Rightarrow e^x + 3e^{-x} = \alpha\,(2e^x - e^{-x}) + \beta\,(2\,e^x + e^{-x})$

Equating coefficients of $e^x\ and\ e^{-x}$, we obtain

$1 = 2\alpha + 2\beta$ and $3 = -\alpha + \beta$

$\Rightarrow \alpha + \beta = \frac{1}{2} and\ -\alpha + \beta = 3$

Solving, we obtain $\alpha = -\frac{5}{4},\ \beta = \frac{7}{4}$

Thus,

$$\int \frac{e^x + 3e^{-x}}{2e^x + e^{-x}}\,dx = \int \frac{\left(-\frac{5}{4}\right)(2e^x - e^{-x}) + \left(\frac{7}{4}\right)(2e^x + e^{-x})}{(2e^x - e^{-x})}\,dx$$

$$= \left(-\frac{5}{4}\right)\int dx + \frac{7}{4}\int \frac{2e^x + e^{-x}}{2e^x - e^{-x}}\,dx$$

$$= -\frac{5}{4}x + \frac{7}{4}\,I_1$$

where $I_1 = \int \frac{2e^x + e^{-x}}{2e^x - e^{-x}}\,dx$

Put $2e^x + e^{-x} = t,\ so\ that$

$(2e^x + e^{-x})dx = dt$

$\therefore I_1 = \int \frac{dt}{t} = \ln|t|$

$= \ln|2e^x - e^{-x}|$

Thus,

$$\int \frac{e^x + 3e^{-x}}{2e^x - e^{-x}}\,dx$$

$$= -\frac{5}{4}\,x + \frac{7}{4}\ln|2e^x - e^{-x}| + c$$

8. Put $x^2 - 1 = t^2$, so that $2x dx = 2t dt$.

Now,

$$\int \frac{x^3}{\sqrt{x^2-1}}\,dx = \int \frac{x^2 x\,dx}{\sqrt{x^2-1}} = \int \frac{(t^2+1)t\,dt}{t}$$

$$= \int (t^2+1)dt = \frac{1}{3}t^3 + t + c$$

$$= \frac{1}{3}(x^2+1)^{3/2} + (x^2+1)^{1/2} + c$$

Check Your Progress 3

1. We split $\frac{x^2+1}{(2x+1)(x-1)(x+1)}$ into partial fractions.

We write

$$\frac{x^2+1}{(2x+1)(x-1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{x+1}$$

$\Rightarrow x^2 + 1 = A(x-1)(x+1) + B(2x+1)(x+1) + C(2x+1)(x-1)$

Put $x = -1/2, 1, -1$ to obtain

$$\frac{1}{4} + 1 = A\left(\frac{-3}{2}\right)\left(\frac{1}{2}\right) \Rightarrow A = \frac{-5}{3};$$

$2 = B(3)(2) \Rightarrow B = 1/3$; and

$2 = C(-1)(-2) \Rightarrow C = 1$

Thus,

$$\int \frac{x^2+1}{(2x+1)(x-1)(x+1)}\,dx = -\frac{5}{3}\int \frac{dx}{2x+1} + \frac{1}{3}\int \frac{dx}{x-1} + \int \frac{dx}{x+1}$$

$$= -\frac{5}{3} \times \frac{1}{2}\log|2x+1| + \frac{1}{3}\log|x-1| + \log|x+1| + c$$

$$= -\frac{5}{6}\log|2x+1| + \frac{1}{3}\log|x-1| + \log|x+1| + c$$

2. Write

$$\frac{x^2+1}{x(x^2-1)} = \frac{x^2+1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$\Rightarrow x^2+1 = A(x^2-1) + Bx\,(x+1) + Cx(x-1)$

Put x = 0, 1, –1 to obtain

1= –A $\Rightarrow$ A = –1;

2 = 2B $\Rightarrow$B = 1; and

2 = 2C $\Rightarrow$C = 1

$$\int \frac{x^2+1}{x(x^2-1)}\,dx = -\int \frac{dx}{x} + \int \frac{dx}{x-1} + \int \frac{dx}{x+1}$$

= log|x| + log |x – 1| + | log |x + 1| + c

$$= log\left|\frac{x^2-1}{x}\right| + c$$

3. Write

$$\frac{2x-3}{(x^2-1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(2x+3)}$$

$\Rightarrow$2x – 3 = A(x + 1)(2 x+ 3) + B(x – 1)(2x + 3) + C (x^2-1)

Put x = 1, −1 and −3/2 to obtain

−1 = A(2)(5) $\Rightarrow$ A = −1/10
−5 = −2B $\Rightarrow$ B = 5/2
−6 = 5C/4 $\Rightarrow$ C= −24/5

Thus,

$$\int \frac{2x-3}{(x^2-1)(2x+3)}\,dx = -\frac{1}{10}\int \frac{dx}{x-1} + \frac{5}{2}\int \frac{dx}{x+1} - \frac{24}{5}\int \frac{dx}{2x+3}dx$$

$$= -\frac{1}{10}\log|x-1| + \frac{5}{2}\log|x+1| - \frac{12}{5}\log|2x+3| + c$$

4. Multiply the numerator and denominator by x^2 to obtain

$$\mathrm{I} = \int \frac{x^2}{x(1+4x^3+3x^6)}dx = \int \frac{x^2}{x^3(1+4x^3+3x^6)}dx$$

Put $x^3 = t,$ so that

$$\mathrm{I} = \frac{1}{3}\int \frac{dt}{t(1+4t+3t^2)} = \frac{1}{3}\int \frac{dt}{t(1+t)(1+3t)}$$

Now, write

$$\frac{1}{t(1+t)(1+3t)} = \frac{A}{t} + \frac{B}{1+t} + \frac{C}{1+3t}$$

$\Rightarrow$1 = A(1+ t) (1+ 3t) + Bt(1 + 3t) + Ct(1 + t)

Put t = 0, −1 and −1/3 to obtain

A = 1, B=1/2, C= −9/2

Thus,

$$\frac{1}{3}\int \frac{dt}{t(1+t)(1+3t)} = \frac{1}{3}\int \left[\frac{1}{t} + \frac{1}{2(1+t)} - \frac{9}{2(1+3t)}\right] dt$$

$$= \frac{1}{3}\left[\log|t| + \frac{1}{2}\log|1+t| - \frac{9}{2} \times \frac{1}{3}\log\ |1+3t\ |\right] + c$$

$$= \left[\log|x| + \tfrac{1}{6}\log|1+x^3| - \tfrac{1}{2}\log\ |1+3x^3\ |\right] + \mathrm{c}$$

5. Write

$$\frac{e^x}{e^x - 3e^{-x} + 2} = \frac{e^x}{e^x - 3/e^x + 2} = \frac{e^{2x}}{e^{2x} + 2e^x - 3}$$

$$\text{Let I } = \int \frac{e^x e^x}{e^{2x} + 2e^x - 3} dx$$

Put $e^x = t$, so that $e^x dx$ and

$$\text{I} = \int \frac{t}{t^2 + 2t - 3}\ dt = \int \frac{t}{(t-1)(t+3)} dt$$

Split $\frac{t}{(t-1)(t+3)}$ into portial fractions, to obtain

$$\frac{t}{(t-1)(t+3)} = \frac{1}{4}\frac{1}{t-1} - \frac{3}{4}\frac{1}{t+3}$$

$$\Rightarrow \int \frac{t}{(t-1)(t+3)} dt = \frac{1}{4}\log|t-1| - \frac{3}{4}|t+3| + c$$

$$= \frac{1}{4} log \left|\frac{t-1}{(t+3)^3}\right| + c$$

$$\text{Thus, } \ \text{I} = \frac{1}{4}\log\left|\frac{e^x - 1}{(e^x+3)^3}\right| + c$$

6. Put $x + 2 = t$, so that

$$I = \int \frac{x^2}{(x+2)^2}dx = \frac{(t-2)^2}{t^3}\,dt$$

$$= \int \frac{t^2-4t+4}{t^3}dt$$

$$= \int \left[\frac{1}{t} - \frac{4}{t^2} + \frac{4}{t^3}\right]dt$$

$$= \log|t| + \frac{4}{t} - \frac{2}{t^2} + c$$

$$= \log|x+2| + \frac{4}{x+2} - \frac{2}{(x+2)^2} + c$$

7. Write

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1}$$

$$\Rightarrow x^2 = A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3$$

Put x =1 and –1 to obtain

1 = 2 C $\Rightarrow$C=1/2 and 1 = – 8D $\Rightarrow$D = –1/8

Comparing coefficient of x^3, we obtain

0 = A + D $\Rightarrow$A = – D = 1/8

Next, put $x= 0$ to obtain

0 = A – B+C – D $\Rightarrow$ B = A + C – D = ¾

Thus

$$I = \int \frac{x^2\,dx}{(x-3)^3(x+1)} = \frac{1}{8}\log\left|\frac{x-1}{x+1}\right| - \frac{3}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{(x-1)^2} + c$$

8. Put $e^x - 1 = t,$ so that

$$I = \int \frac{dt}{t^3} = \int t^{-3}dt = \frac{t^{-3+1}}{-3+1} + C$$

$$= -\frac{1}{2t^2} + c = -\frac{1}{2}\frac{1}{(e^{x-1})^2} + c$$

Check Your Progress 4

1. $\int x^2\,e^x dx \;=\; x^2 e^x - \int 2xe^x$

$$= x^2e^x - 2[xe^x - \int (1)\, e^x dx]$$

$$= x^2e^x - 2[xe^x - e^x] + c$$

$$= (x^2 - 2x + 2)e^x + c$$

2. $\displaystyle\int x \log(1+x)dx = \frac{1}{2}x^2\log(1+x) - \frac{1}{2}\int \frac{x^2}{1+x}\, dx$

$$\text{Let } I_1 = \int \frac{x^2}{1+x}\, dx = \int \frac{x^2-1+1}{1+x}\, dx$$

$$= \int \left[x - 1 + \frac{1}{1+x}\right] dx$$

$$= \frac{1}{2}x^2 - x + \log(1+x)$$

Thus,

$$\int x \log(1+x)dx = \frac{1}{2}x^2\log(1+x) - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{2}x\log(1+x) + c$$

3. Put $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2t\, dt$

Thus,

$$I = \int e^{\sqrt{x}} dx = 2\int t\, e^t dt$$

$$= 2[te^t - \int (1)\, e^t dt]$$

$$= 2[te^t - e^t] + c$$

$$= 2(\sqrt{x} - 1)\, e^{\sqrt{x}} + c$$

4. $\displaystyle I = \int e^x \log x + \int e^x \frac{1}{x}\, dx$

$$= e^x\log x - \int e^x \frac{1}{x}\, dx + \int e^x \frac{1}{x} dx$$

$$= e^x\log x + c$$

5. We write $\displaystyle\frac{x+1}{(x+2)^2}\quad \frac{x+2-1}{(x+2)^2} = \frac{1}{x+2} - \frac{1}{(x+2)^2}$

We have

$$= \int e^x \frac{x+1}{(x+1)^2} dx \quad \int e^x \left[\frac{1}{x+2} - \frac{1}{(x+1)^2}\right] dx$$

$$= \int e^x (x+2)^{-1}\ dx - \int e^x \frac{1}{(x+2)^2}\ dx$$

$$= e^x (x+2)^{-1}\, dx - \int e^x\, (-1)(x+2)^{-2}\ dx - \int e^x \frac{1}{(x+2)^2}\ dx$$

$$= \frac{e^x}{x+2} + \int \frac{e^x}{(x+2)^2}\ dx - \int e^x \frac{1}{(x+2)^2}\ dx$$

$$= \frac{e^x}{x+2} + c$$

6. $\int \log\sqrt{x}\ dx = \frac{1}{2}\int \log x\, dx = \frac{1}{2}\int (1)\log x\, dx$

$$= \frac{1}{2}[x\log x - \int (x)\frac{1}{x} dx]$$

$$= \frac{1}{2}\ [x\log x - x] + c$$

7. $\int \log(1+x)\, dx = \int (1)\log(1+x)\, dx$

$$= x\log(1+x)\int x\ \frac{1}{1+x} dx$$

$$= x\log(1+\ x) - \int \frac{x+1-1}{x+1}\ dx$$

$$= x\log(1+\ x) - \int \left[1 - \frac{1}{1+x}\right] dx$$

$= x\log(1+ x) – [x – \log(1+x)] + c$

$= (x + 1)\log(1+ x) – x + c$

8. $\int (1-x^2)\ \log x\, dx$

$$= \left(x - \frac{x^3}{3}\right)\log x - \int \left(x - \frac{x^3}{3}\right)\frac{1}{x} dx$$

$$= \left(x - \frac{1}{3}x^3\right)\log x - \int \left(1 - \frac{x^2}{3}\right)\ dx$$

$$= \left(x - \frac{1}{3}x^3\right)\log x - \left(x - \frac{x^3}{9}\right) + c$$

$$= \left(x - \frac{1}{3}x^3\right)\log x - \ x + \frac{x^3}{9}\ + c$$

3.7 SUMMARY

The unit discusses integration of a function as inverse of the derivative of the function. In **section 3.2**, basic integration rules are derived using corresponding differentiation rules. A number of examples are included to explain application of the rules. In **section 3.3**, for finding integral of complex functions in terms of simpler functions, the method of substitution is discussed through suitable examples. In **section 3.4**, methods for integration of rational functions, are introduced and explained. In **section 3.5**, method of integration by parts for finding integral of product of two functions in terms of the integrals of the functions is discussed.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 3.6**.