UNIT 3 THREE-DIMENSIONAL GEOMETRY – 1

Structure

3.0 INTRODUCTION

Let us recall the concepts of two dimensional geometry. A point in plane is represented by an order pair of real numbers by a two-dimensional Cartesian coordinate system. In this unit, we shall see that a point in space is uniquely determined by an order triple of real numbers by a three dimensional Cartesian coordinate system. Once we have established a one to one correspondence between points in space and ordered triple of real numbers, we can extend the concept of distance between two points in space. We shall study distance formula for finding distance between two given points in space.

Also recall that we introduced the concepts of direction cosines and direction ratios of a vector. Infact, the same concept are valid for a directed line also. Infact, the concept of slope of a line in two dimensional plane is extended by direction cosines and direction ratios of a line in three dimensional plane.

We shall also study various forms of equation of a straight line in space. A straight line in space is uniquely determined if we know a point on the line and direction of the line or if we know two points on the line. Thus, we shall obtain equation of a straight line with a given point and parallel to a given direction, and equation of a straight line passing, through two given points. These equations are obtained both in vector and Cartesian forms.

Also recall that two lines in a plane either intersect or are parallel (or are coincident). But in space, we may have two non-intersecting and non-parallel lines. Such lines are called skew lines. In the last section, we shall introduce the concept of distance between two lines and find formula for calculating the shortest distance between two skew lines.

3.1 OBJECTIVES

After completing this unit, will be able to :

- find the distance between two points in space;
- find the direction cosines and direction ratios of a line passing through two points;
- find the equation of a straight line passing through a given point and parallel to a given vector in vector form and in Cartesian form;
- find the vector and Cartesian equations of a line passing through two given points;
- define the terms skew lines and coplanar lines; and
- find the distance between two skew lines.

3.2 THREE DIMENSIONAL SPACE

Let us recall that a point in plane is uniquely determined by an ordered pair of real numbers through a two dimensional Cartesian coordinate system. Similarly, there is one to one correspondence between points in space and ordered triplets of real numbers through a three dimensional Cartesian coordinate system, by fixing a point 0 as origin 0 and three mutually perpendicular lines through O as x-axis, y-axis and z-axis. The three axes are taken in such a way that they form a right-handed system. With any point P, we associate a triple of real number (x,y,z) in the following manner :

We drop a perpendicular from P to the xy–plane meeting it at M and take PM = z. From M drop perpendiculars on x-axis and y-axis meeting them at A and B respectively. Take MB = x and MA = y

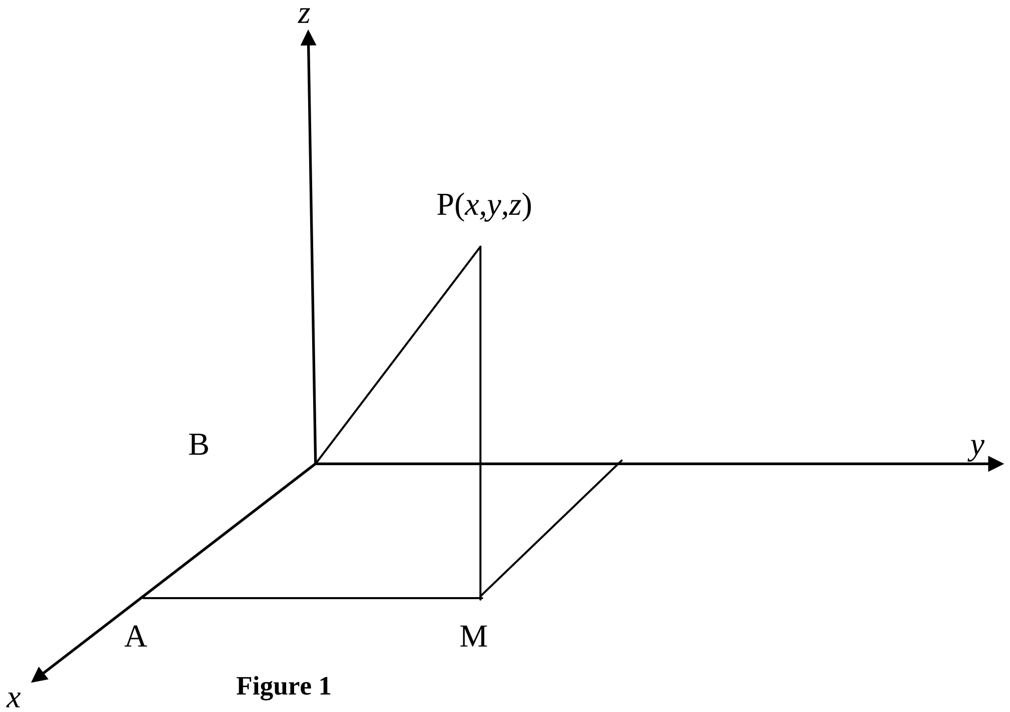

Figure 1

Also, given a triple (x,y,z) we can locate a point P in space uniquely whose Cartesian coordinates are (x,y,z)

Distance between two Points

Recall from the Unit 'Vectors – I' that if P is any point with coordinate (x,y,z) and position vector $\vec{r}$ then

$$\overrightarrow{OP} = \vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$$

The distance OP of any point P(x,y,z) from the origin is given by

OP = $|\overrightarrow{OP}| = |\vec{r}\,| = \sqrt{x^2 + y^2 + z^2}$

Similarly, if P $(x_1,\ y_1, z_1)$ *and* $Q(x_2,\ y_2, z_2)$ are two points with postion $\vec{r}_1$ vectors $\vec{r}_2$

Then

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r}_2 - \vec{r}_1$

$= (x_2\,\hat{\imath} + y_2\,\hat{\jmath} + z_2\,\hat{k}) - (x_1\,\hat{\imath} + y_1\,\hat{\jmath} + z_1\,\hat{k})$

$= (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\,\hat{\jmath} + (z_2 - z_1)\,\hat{k}$

and PQ = $|\overrightarrow{PQ}| = |\vec{r}_2 - \vec{r}_1|$

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

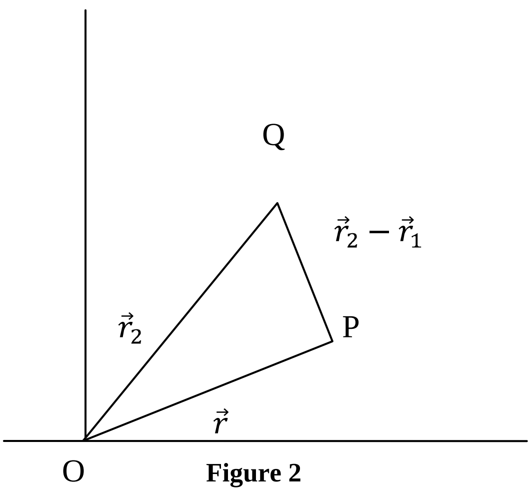

Figure 2

Thus, the distance between two points P $(x_1,\ y_1, z_1)$ *and* $Q(x_2,\ y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

We have studies section formula (in Unit – 1 on Vectors) to find position vector of a point R which divides the line segment joining the points P and Q with position vectors $\vec{r}_1$and $\vec{r}_2$ respectively. If $\vec{r}_1$ is the position vector of R, then

$$\vec{r}_1 = \frac{m\,\vec{r}_2 + n\,\vec{r}_1}{m+n}$$

If P has coordinates P $(x_{1,}\ y_{1,}z_{1,})$and Q has coordiantes $Q(x_{2,}\ y_{2,}z_{2,})$and R has coordinates (x,y,z), then

$$x = \frac{m\,x_2 + n\,x_1}{m+n}$$

$$y = \frac{m\,y_2 + n\,y_1}{m+n}$$

$$z = \frac{m\,z_2 + n\,z_1}{m+n}$$

If we put $m = n = 1$, we get the coordinate of mid-point of PQ given as

$$x = \frac{x_2 + x_1}{2}$$

$$y = \frac{y_2 + y_1}{2}$$

$$z = \frac{z_2 + z_1}{2}$$

Example 1 : Find the distance between the points $P(1, –1, 0)$ and Q (2, 3, –1)

Solution : $PQ = \sqrt{(2-1)^2 + (3+1)^2 + (-1-0)^2}$

$= \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$

Direction Cosines and Direction Ratios

If α, β *and* γ are the angles which a non-zero vector $\vec{r}$ makes with positive x–axis, y–axis and z– axis respectively, then cos α, cos β and cos γ are called the direction cosines of $\vec{r}_1$ we write

$l = \cos\alpha$, $m = \cos\beta$ *and* $n = \cos\gamma$

Also, $l^2 + m^2 + n^2 = 1$

If a, b, c are numbers proportional to l, m and n respectively, then a, b, c are called the direction ratios of $\vec{r}_1$

$$= \frac{a}{l} = \frac{b}{m} = \frac{c}{n}$$

with $l^2 + m^2 + n^2 = 1$.

Which gives

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

If $\vec{V} = a\,\hat{\imath} + b\,\hat{\jmath} + c\,\hat{k}$ is any vector, its direction ratios are *a*, *b*, *c* and its direction cosines are

$$l = \frac{a}{|\vec{V}|}$$

$$m = \frac{b}{|\vec{V}|}$$

$$n = \frac{c}{|\vec{V}|}$$

Consider the line joining the points P $(x_{1,}\ y_{1,} z_{1,}) and\ Q(x_{2,}\ y_{2,} z_{2,})$ whose position vectors

are $\vec{r}_1$ and $\vec{r}_2$

Now $\overrightarrow{AB} = \vec{r}_1 - \vec{r}_1$

$= (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\,\hat{\jmath} + (z_2 - z_1)\hat{k})$

Therefore, direction ratios of AB are $x_1 - x_2, y_2 - y_1\ and\ z_2 - z_{1.}$ And the direction cosines of *AB* are

$$l = \frac{x_2 - x_1}{|\vec{r}\ \ |}$$

$$m = \frac{y_2 - y_1}{|\vec{r}\ \ |}$$

$$n = \frac{z_2 - z_1}{|\vec{r}\ \ |}$$

Where $\vec{r} = \vec{r}_1 - \vec{r}_1$

Thus, the direction cosines of a line joining points A$(x_{1,}\ y_{1,} z_{1,}) and\ B(x_{2,}\ y_{2,} z_{2,})$ are given by

$\frac{x_2 - x_1}{AB}$

$\frac{y_2 - y_1}{AB}$

$\frac{z_2 - z_1}{AB}$

Where *AB* is the distance between *A* and *B*

Example 2 : Find the direction cosines of a line which makes equal angles with the axes.

Solution : Let *l*, *m*, *n* be the direction cosines of the given line. Then

$$l^2 + m^2 + n^2 = 1$$

Since the given line makes equal angles with the axes, therefore, we have
$l = m = n$

So, $l^2 + l^2 + l^2 = 1$

$or \quad 1 = \pm \frac{1}{\sqrt{3}}$

Thus $l = m = n \pm \frac{1}{\sqrt{3}}$

Example 3: Find the direction cosines of the line passing through the two points (1,2,3) and (–1,1,0)

Solution : We know that the direction cosines of the line passing through two points $A(x_{1,}\ y_{1,} z_{1,}) and\ B(x_{2,}\ y_{2,} z_{2,})$ are given by

$$l = \frac{x_2 - x_1}{AB}$$

$$m = \frac{y_2 - y_1}{AB}$$

$$n = \frac{z_2 - z_1}{AB}$$

$$AB = \sqrt{(x_2 - x_2)^2 + (y_{2,} - y_2)^2 + (z_2 - z_1)^2}$$

Here A is (1,2,3) and B is (–1,1,0)

So, *AB*,

where $AB = \sqrt{(-1 - 1)^2 + (1 - 2)^2 + (0 - 3)^2}$

$= \sqrt{4 + 1 + 9} = \sqrt{14}$

Thus, the direction cosines of line joining A and B are

$$l = \frac{-2}{\sqrt{14}}$$

$$m = \frac{-1}{\sqrt{14}}$$

$$n = \frac{-3}{\sqrt{14}}$$

Check Your Progress 1

1. Prove by finding distances that the three points (–2,3,5), (1,2,3) and (7,0,−1) are collinear.
2. Show that the points (0,7,10) (–1,6,6) and (–4,9,6) from an isosceles right angled triangle.
3. The direction ratios of a line are 1, –2, –2. What are its direction cosines ?
4. A line makes angles of 45° and 60° with the positive axes of x and y respectively. What angle does it make with the positive axis of x ?
5. Find the direction cosines of the line passing through the two points (–2, 4, 5) and (1,2,3).

3.3 EQUATION OF A STRAIGHT LINE IN SPACE

A line in space is completely determined once we know one of its points and its direction. We shall use vectors to measure direction and find equations of straight line in space.

(a) **Equation of a straight line passing through a fixed point A and parallel to the vector $\vec{b}$.**

Let $\vec{a}$ be the position vector of the given point A and let $\vec{r}$ be the position vector of any point P on the given line.

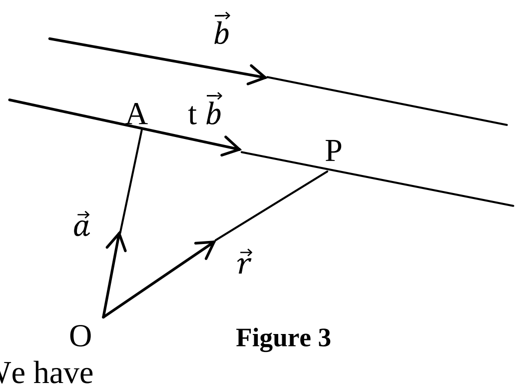

Figure 3

We have

$$\vec{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \vec{a} + \overrightarrow{AP}$$

Since $\overrightarrow{AP}$ is parallel to $\vec{b}$, we must have $\overrightarrow{AP}$ = t $\vec{b}$ for some scalar *t*.

$\therefore \quad \vec{r} = \vec{a} + t\vec{b}$

Thus, each point *P* on the line has position vector $\vec{a} + t\vec{b}$ for some scalar *t*. Conversely, for each value of the scalar *t*, $\vec{a} + t\vec{b}$ is the position vector of a point of the line.

Hence, the vector equation of the line is

$$\vec{r} = \vec{a} + t\vec{b} \qquad (1)$$

Where t is a parameter.

Let us now write (1) in Cartesian form.

Let the coordinates of point A be (x_1, y_1, z_1)

So that $\vec{a} = x_1\,\hat{\imath} + y_1\hat{\jmath} + z_1\,\hat{k}$

Suppose the line has direction ratios a, b and c, then $\vec{b} = a\,\hat{\imath} + b\hat{\jmath} + c\hat{k}$

Let $\vec{r} = x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k}$

Substituting in (1), we get

$$x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k} = x_1\,\hat{\imath} + y_1\hat{\jmath} + z_1\,\hat{k} + t\,(a\,\hat{\imath} + b\hat{\jmath} + c\,\hat{k})$$

Equating the coefficients of $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ we get

$$x = x_1 + ta,\ \ y = y_1 + tb \ \ and\ z = z_1 + tc \qquad (2)$$

These are the parametric equations of the line.

Eliminating the parameter *t*, from (2), we get

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad (3)$$

This the Cartesian equation of a straight line.

Cartesian Equation of a straight line passing through the point (x_1, y_1, z_1) and having direction cosines *l, m, n* is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad (4)$$

Equations (1), (2), (3) and (4) are different forms of equations of a straight line passing through a given point and parallel to a given direction.

(b) Equation of a straight line passing through two given points

Let us now find equation of a straight line passing through two distinct points A $(x_{1,}\ y_{1,}z_{1,})$ B $(x_{2,}\ y_{2,}z_{2,})$. Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A and B respectively.

Since the points A and B lie on the line therefore $\overrightarrow{AB}$ is parallel to the line.

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \vec{b} - \vec{a}$

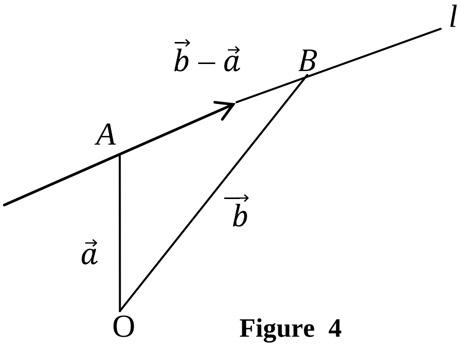

Figure 4

Thus, the vector equation of line passing through A and parallel to $\overrightarrow{AB}$ is

$\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$ (5)

Equation (5) is the vector equation of a line passing through two points with position vectors $\vec{a}$ and $\vec{b}$.

Equation (5) can also be written as

$\vec{r} = (1 - t)\,\vec{a} + t\,\vec{b},\ \ t \in$ R (6)

Let us now derive the Cartesian form the vector equation (5)

We have

$\vec{a} = x_1\,\hat{\imath} + y_{1,}\hat{\jmath} + z_1\,\hat{k}$

$\vec{b} = x_2\,\hat{\imath} + y_{2,}\hat{\jmath} + z_2\,\hat{k}$

Let $\vec{r} = x_1\,\hat{\imath} + y_{1,}\hat{\jmath} + z_1\,\hat{k}$

Substituting these values in (5), we get

$x\,\hat{\imath} + y\hat{\jmath} + z\,\hat{k} = x_1\,\hat{\imath} + y_{1,}\hat{\jmath} + z_1\,\hat{k} + t\,[(x_2 - x_1)\,\hat{\imath} + (y_2 - y_{1,})\hat{\jmath} + (z_2 - z_1)\hat{k}$

Equating the coefficients of $\hat{\imath}, \hat{\jmath}, \hat{k}$ we get

$x = x_1 + t\,(x_2 - x_1)$
$y = y_1 + t\,(y_1 - y_1)$
$z = z_1 + t\,(z_2 - z_1)$

Eliminating t, we obtain

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \qquad (7)$$

Equation (7) is the Cartesian form of equation of a straight line passing through two points $(x_{1,}\ y_{1,}z_{1,})$ and $(x_{2,}\ y_{2,}z_{2,})$.

Example 5 : Find the equations of the line (both Vector and Cartesian) passing through the point (1, –1,–2) and parallel to the vector $3\,\hat{\imath} - 2\hat{\jmath} + 5\,\hat{k}$.

Solution : We have

$\vec{a} = \hat{\imath} - \hat{\jmath} - 2\hat{k}$ and

$\vec{b} = 3\,\hat{\imath} - 2\hat{\jmath} + 5\,\hat{k}$

Therefore, the vector equation of the line is

$\vec{r} = (\hat{\imath} - \hat{\jmath} - 2\hat{k}) + t\,(3\,\hat{\imath} - 2\hat{\jmath} + 5\,\hat{k})$

Now $\vec{r}$ is the position vector of any point P (x,y,z) on the line.
So, $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$(\hat{\imath} - \hat{\jmath} - 2\hat{k}) + t\,(3\,\hat{\imath} - 2\hat{\jmath} + 5\,\hat{k})$

$= (1 + 3t)\,\hat{\imath} - (1 + 2t)\,\hat{\jmath} + (-2 + 5t)\,\hat{k}$

Thus $x = 1 + 3t$, $y = -1 - 2t$, $z = -2 + 5t$

Eliminating t, we get

$$\frac{x-1}{3} = \frac{y+1}{-2} = \frac{z+2}{5}$$

Example 6 : Find the Vector and Cartesian equation of the line passing through the points (–2, 0, 3) and (3,5, –2)

Solution : We have

$\vec{a} = -2\hat{\imath} + 3\hat{k}$ and

$\vec{b} = 3\hat{\imath} + 5\,\hat{\jmath} - 2\hat{k}$

So, $\vec{b} - \vec{a} = 5\hat{\imath} + 5\,\hat{\jmath} - 5\hat{k}$

The vector equation of a line passing through two points with position vectors is given by

$\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

$\therefore\ \vec{r} = (-2\hat{\imath} + 3\hat{k})$ and $t\,(5\hat{\imath} + 5\,\hat{\jmath} - 5\hat{k})$

If $\vec{r}$ is the position vector of the point P (*x*, *y*, *z*) then we have

$x\hat{\imath} + y\,\hat{\jmath} + z\hat{k} = (-2\hat{\imath} + 3\hat{k}) + t\,(5\hat{\imath} + 5\,\hat{\jmath} - 5\hat{k}\,)$

$= (-2 + 5t) + 5\,t\,\hat{\jmath} + (3 - 5t)\hat{k}$

So, $x = -2 + 5t$, $y = 5t$, $z = 3 - 5t$

Eliminating *t*, we get

$$\frac{x+2}{5} = \frac{y}{5} = \frac{z-3}{-5}$$

Example 7 : The Cartesian equation of a line is

$$\frac{x+3}{4} = \frac{y-2}{5} = \frac{z+5}{1}$$

Find the Vector equation of the line.

Solution : Comparing the given equation with the standard form

$$\frac{x-x_{1,}}{a} = \frac{y-y_{1,}}{b} = \frac{z-z_{1,}}{c}$$

We get

$x_{1,} = -3$, $y_{1,} = 2$ $z_{1,} = -5$ $a = 4$, $b = 5$, $c = 1$

Thus, the required line passes through the point (–3, 2, –5) and is parallel to the vector $4\hat{\imath} + 5y\,\hat{\jmath} + \hat{k}$.

Thus, the equation of line in vector form is

$\vec{r} = (\,-3\hat{\imath} + 5\,\hat{\jmath} - 6\hat{k}\,) + t\,(\,4\hat{\imath} + 5\,\hat{\jmath} + \hat{k}\,)$

Angle between two lines

Consider the two lines with vector equations

$\vec{r} = \vec{a} + t\,\vec{b}$ (8)

$\vec{r} = \overrightarrow{a'} + t\,\vec{b}'$ (9)

The angle θ between these lines is defined as the angle between the directions of $\vec{a}$ and $\vec{b}$ Also, we know that the angle between the vectors $\vec{b}$ and $\vec{b}'$ is given by

$$cos\theta = \left|\frac{\vec{b}\vec{b}'}{|\vec{b}||\vec{b}'|}\right|. \qquad (10)$$

Thus, (10) gives the angle between the lines (8) and (9). If the equation of two lines are given in Cartesian form

$$\frac{x - x_{1,}}{a_{1,}} = \frac{y - y_{1,}}{b_{1,}} = \frac{z - z_{1,}}{c_{1,}} \text{ and}$$

$$\frac{x - x_{2,}}{a_{2,}} = \frac{y - y_{2,}}{b_{2,}} = \frac{z - z_{2,}}{c_{2,}}$$

then the angle θ between them is given by

$$\text{Cos } \theta = \pm \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1{}^2 + b_1{}^2 + c_1{}^2}\sqrt{a_2{}^2 + a_2{}^2 + a_2{}^2}} \qquad (11)$$

as $\sin^2 \theta = 1 - \cos^2 \theta$, therefore, we also have

$$\sin \theta = \pm \frac{\sqrt{(a_1 b_2 - a_2 b_1)^2 + (b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2}}{\sqrt{a_1{}^2 + b_1{}^2 + c_1{}^2}\sqrt{a_2{}^2 + a_2{}^2 + a_2{}^2}} \qquad (12)$$

Thus, the angle θ between two lines whose direction ratios are a_1, b_1, c_1 and a_2, b_2, c_2 is given by (11) or (12).

If instead of direction ratios, we take direction cosines *l*, *m*, *n* and l_2, m_2, n_2 of the two lines, then (10) and (11) can be written as

cos θ= and $l_1 l_2 +, m_1 m_2 + n_1 n_2$

and sin θ = $\sqrt{(l_1 m_2 - l_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2}$

It is clear from these relations that two lines are perpendicular to each other if and only if

$l_1 l_2 +, m_1 m_2 + n_1 n_2 = 0$

and parallel to each other if and only if

$l_1 l_2 +, m_1 m_2 + n_1 n_2 = 0$

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$$

Similarly, two lines with direction ratios $a_1, b_{1,} c_1$ and $a_2, b_{2,} c_2$ are

(i) perpendicular iff $a_1, a_{2+}, b_{2,} b_{1+} c_2 c_2$ and

(ii) parallel iff

$$\frac{a_1}{a_1} = \frac{b_1}{b_1} = \frac{c_1}{c_1}$$

Example 8: Find the angle between the lines

$$\vec{r} = 2\hat{\imath} + 3\ \hat{\jmath} - 4\hat{k} + t\ (\ \hat{\imath} - 2\ \hat{\jmath} + 2\hat{k}\ )$$
$$\vec{r} = 3\hat{\imath} - 5\ \hat{k}\ + s\ (\ 3\hat{\imath} - 2\ \hat{\jmath} + 6\hat{k}\ )$$

Solution : Here $\vec{b} = \hat{\imath} - 2\ \hat{\jmath} + 2\hat{k}$ and $\vec{b} = 3\hat{\imath} - 2\ \hat{\jmath} + 6\hat{k})$

If θ is the angle between the two lines, then

$$cos\ \theta = \left|\frac{\vec{b}\vec{b}'}{|\vec{b}||\vec{b}'|}\right|$$

$$\left|\frac{(\hat{\imath} - 2\ \hat{\jmath} + 2\hat{k}\ ).(3\hat{\imath} - 2\ \hat{\jmath} + 6\hat{k}\ )}{\sqrt{1+4+4}\ \sqrt{9+4+36}}\right|$$

$$= \left|\frac{3+4+12}{3 \times 7}\right| = \frac{19}{21}$$

Hence, $\theta = Cos^{-1}\left(\frac{19}{21}\right)$

Example 9 : Find the angle between the pair of lines

$$\frac{x-5}{2} = \frac{y-3}{1} = \frac{z-1}{-3} \text{ and}$$

$$\frac{x}{3} = \frac{y-1}{2} = \frac{z+5}{-1}$$

Solution : The direction ratios of the line

$$\frac{x-5}{2} = \frac{y-3}{1} = \frac{z-1}{-3}$$

are 2, 1, and –3. similarly, the direction ratios of the line

$$\frac{x}{3} = \frac{y-1}{2} = \frac{z+5}{-1} \text{ are } 3, 2, \text{and} - 1$$

Therefore, the angle θ between them is given by

$$\cos\theta = \left|\frac{2.3+1.2+(-3)(-1)}{\sqrt{2^2+1^2+(-3)^2}\sqrt{3^2+2^2+(-1)^2}}\right|$$

$$= \frac{11}{\sqrt{14}\sqrt{14}} = \frac{11}{14}$$

Hence, $\theta = cos^{-1}\left(\frac{11}{14}\right)$

Example 10: Find the equation of the line passing through the point (–1,3,–2) and perpendicular to the two lines

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \text{ and } \frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}.$$

Solution : Let the equation of the given line be

$$\frac{x+2}{l} = \frac{y-3}{m} = \frac{z+2}{n}$$

Since (i) is perpendicular to both the lines therefore,

$l + 2m + 3n = 0$ ……………………(ii)

$-3l + 2m + 5n = 0$ …………………(iii)

Solving (ii) and (iii) we get

$$\frac{l}{10-6} = \frac{m}{-9-5} = \frac{n}{2+6}$$

Or

$$\frac{l}{4} = \frac{m}{-14} = \frac{n}{8}$$

$$\frac{l}{4} = \frac{m}{-7} = \frac{n}{8}$$

∴ The required equation is

$$\frac{x+2}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$$

Check Your Progress 2

1 Find the Vector equation of the line passing through the point (3,1,4) and parallel to the vector $-\hat{\imath} + \hat{\jmath} - 2\hat{k}.$ Also find the Cartesian equation of the line.

2 Find the vector and Cartesian equation of the line passing through (1,0,–4) and is parallel to the line

$$\frac{x+1}{3} = \frac{z+2}{4} = \frac{z-2}{2}$$

3. Find the vector equation for the line through the points (3,4,–7) and (1, –1,6) Also find the Cartesian equation.

4. Find the angle between the following pairs of lines.

(i) $\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$ *and* $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$

(ii) $\vec{r} = 3\hat{\imath} + 2\hat{\jmath} - 4\hat{k} + t(\hat{\imath} - 2\hat{\jmath} + 2\hat{k})$

$\vec{r} = 5\hat{\jmath} - 2\hat{k} + S(\widehat{4\imath} - \hat{\jmath} + 8\hat{k})$

5. Find k so that the lines

$$\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}\ and\ \frac{x-1}{3k}=\frac{y-5}{1}=\frac{z-6}{-5}$$

are at right angles.

3.4 SHORTEST DISTANCE BETWEEN TWO LINES

Two lines in space may either lie in the same plane or in different planes. In the former case, such lines are called coplanar lines. Clearly line which do not lie in the same planes are called non coplanar lines. We know that two lines in plane are either intersecting or parallel. But in space, two non coplanar lines may neither intersect, nor be parallel to each other . Such lines are called skew lines.

Definition : Two non coplanar liens are called skew lines if they are neither parallel nor intersecting.

The shortest distance between two lines is the join of a point in one line with one point on the other so that the length of the segment so obtained is the smallest.

Clearly, the shortest distance between two intersecting lines is zero and shortest distance between two parallel lines is the distance by which the two lines are separated. For skew lines, the direction of shortest distance is perpendicular to both the lines.

Let us now find an expression for the shortest distance between two skew lines.

Consider two skew liens l_1 and l_2 (Fig. 5) with vector equation

$$\vec{r}=\vec{a}+t\vec{b}$$
$$\text{and } \vec{r}=\vec{c}+p\,\vec{d}$$

Let A and C be points on $\vec{r}$ and $\vec{r}$ *respectively*.

Let A and C be points on l_1 and l_2 with position vectors $\vec{a}$ *and* $\vec{c}$ respectively.

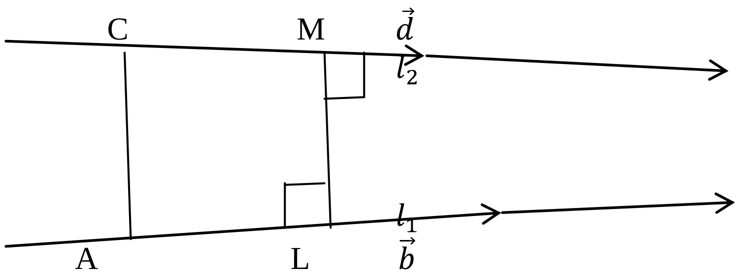

Figure 5

The line LM of shortest distance is perpendicular to both l_1 and l_2 therefore is parallel to $\vec{b}\times\vec{d}$.

The length of LM is the project of $\overrightarrow{AC}$ on $\overrightarrow{LM}$

So, LM = AC |Cos θ |, when θ is the angle between $\overrightarrow{AC}$ on $\overrightarrow{LM}$

Now a unit vector $\hat{n}$ along by $\overrightarrow{LM}$ *is given by*

$$\hat{n} = \frac{\vec{b} \times \vec{d}}{|\vec{b} \times \vec{d}\,|}$$

$$\text{So}, \overrightarrow{LM} = \text{LM}\,\hat{n} = \text{LM}\left(\frac{\vec{b} \times \vec{d}}{|\vec{b} \times \vec{d}\,|}\right)$$

$$\text{Also } Cos\,\theta \quad \frac{\overrightarrow{AC}\,.\,\overrightarrow{LM}}{(AC)(LM)} = \frac{(\text{LM})[\,\overrightarrow{AC}.(\,\vec{b} \times \vec{d}\,)}{(AC)(LM)\left|\vec{b} \times \vec{d}\,\right|}$$

$$= \frac{\overrightarrow{AC}\,.\,(\,\vec{b} \times \vec{d}\,)}{(AC)|\vec{b} \times \vec{d}\,|}$$

$$= \frac{\overrightarrow{(c -}\,\vec{a}\,)\,(\,\vec{b} \times \vec{d}\,)}{(AC)|\vec{b} \times \vec{d}\,|}$$

Hence, LM = AC |Cos θ|

$$= \frac{\overrightarrow{(c -}\,\vec{a}\,)\,.(\,\vec{b} \times \vec{d}\,)}{|\vec{b} \times \vec{d}\,|} \qquad \text{... (i)}$$

$$= \frac{\overrightarrow{(c}\,\vec{b}\,\vec{d}\,)\,(\vec{a}\,\vec{b}\,\,\vec{d}\,)}{|\vec{b} \times \vec{d}\,|}$$

Remarks :

1. Two lines will intersect if and only if the shortest distance between them is zero i.e. LM = 0

   or $(\vec{c} – \overrightarrow{a})\,(\vec{b} – \overrightarrow{d}) = 0$

2. If two lines are parallel and are given by $\overrightarrow{r} = \overrightarrow{a} + t\,\overrightarrow{b}$ and $\overrightarrow{r} = \overrightarrow{c} + p\,\overrightarrow{b}$, then they are coplanar. Then the distance between these two lines is given by

$$d = \left|\frac{\overrightarrow{b} \times (\overrightarrow{c} - \overrightarrow{a})}{\overrightarrow{|\,b|}}\right|$$

   This is because d = LM = AC |cos $\frac{\pi}{2} - \theta$ | (see. Fig. 6) so that

$$\text{d} = \text{AC}\,|\sin\theta\,|\,\vec{c} - \vec{a}|\left|\frac{\overrightarrow{b} \times (\overrightarrow{c} - \overrightarrow{a})}{\overrightarrow{|\,b\,|} \times |\overrightarrow{c} - \overrightarrow{a}|}\right|$$

$$= \left| \frac{\vec{b} \times (\vec{c} - \vec{a})}{|\vec{b}|} \right|$$

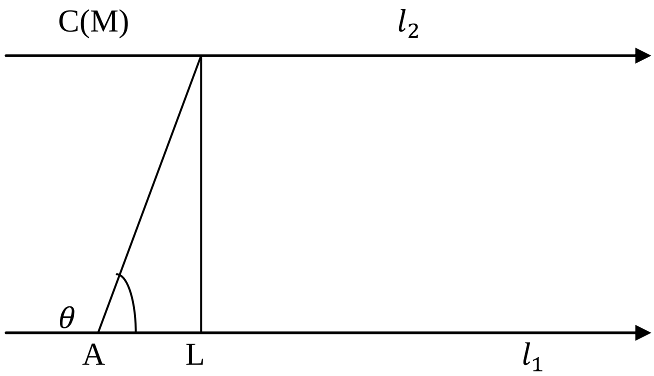

Figure 6

Cartesian form the distance between skew lines

Let the equation of two skew lines l_1 and l_2

$$l_1: \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

$$\text{and } l_2: \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

The corresponding vector from equation of l_1 and l_2 respectively are

$\vec{r} = (x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}) + t(a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k})$

and $\vec{r} = (x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}) + t(a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k})$.
Comparing these equations with $\vec{r} = \vec{a} + t\,\vec{b}$ and $\vec{r} = \vec{c} + t\,\vec{d}$ and substituting the values in (i), we see that the distance between l_1 and l_2 is

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

Example 11 : Find the shortest distance between the lines

$$\vec{r} = (3\hat{\imath} + 4\hat{\jmath} - 2\hat{k}) + t(-\hat{\imath} + 2\hat{\jmath} + \hat{k})$$

$$\text{and } \vec{r} = (\hat{\imath} - 7\hat{\jmath} + 2\hat{k}) + (\hat{\imath} + 3\hat{\jmath} - 2\hat{k})$$

Solution : Comparing with equations $\vec{r} = \vec{a} + t\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$, we have

$$\vec{a} = 3\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$$

$$\vec{b} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}$$

$\vec{c} = \hat{\imath} - 7\ \hat{\jmath} + 2\hat{k}$

$\vec{d} = \hat{\imath} + 3\ \hat{\jmath} - 2\hat{k}$

$\therefore\ \vec{a} - \vec{c}\ = 2\ \hat{\imath} - 11\ \hat{\jmath}$

and $\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix}$

$= \hat{\imath} + 3\ \hat{\jmath} - 5\hat{k}$

So $|\vec{b} \times \vec{d}|\ = \sqrt{1 + 9 + 25} = \sqrt{35}$

Thus, Shortest distance

$$= \left|\frac{(\vec{c} - \vec{a}).(\vec{b}\ \times\ \vec{d})}{|\vec{b}\ \times\ \vec{d}|}\right|$$

$$= \left|\frac{(-2\hat{\imath} - 11\ \hat{\jmath}).(\hat{\imath} + 3\ \hat{\jmath} - 5\hat{k})}{\sqrt{35}}\right|$$

$$= \frac{35}{\sqrt{35}} = \sqrt{35}$$

Example 12 : Find the shortest distance between the lines whose vector equations are :

$$\vec{r} = (1 + \lambda)\hat{\imath} + (2 - \lambda)\ \hat{\jmath} + (1 + \lambda)\hat{k}$$

and $\vec{r} = 2\left(1 +\ \mu\right)\hat{\imath} + \left(1 -\ \mu\right)\hat{\jmath} + (-1 + 2\,\mu)\hat{k}$

Solution : The two equations are:

$$\vec{r} = \left(\hat{\imath} + 2\,\hat{\jmath} + \hat{k}\right) +\ \lambda\left(\hat{\imath} -\ \hat{\jmath} + \hat{k}\right)$$

and $\vec{r} = \left(2\hat{\imath} - \hat{\jmath} - \hat{k}\right) +\ \mu\left(2\hat{\imath} +\ \hat{\jmath} + 2\hat{k}\right)$

Here, we have

$\vec{a}\ = \hat{\imath} + 2\ \hat{\jmath} + \hat{k}$

$\vec{b}\ = \hat{\imath} -\ \hat{\jmath} + \hat{k}$

$\vec{c}\ = 2\ \hat{\imath} -\ \hat{\jmath} - \hat{k}$

$\vec{d}\ = 2\hat{\imath} +\ \hat{\jmath} + 2\hat{k}$

So, $\vec{c}\ - \vec{a} =\ \hat{\imath} - 3\,\hat{\jmath} - 2\hat{k}$

$$\text{and} \quad \vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$= -3\hat{\imath} + 3\,\hat{k}$$

$$\text{So} \quad |\vec{b} \times \vec{d}| = \sqrt{9+9} = 3\sqrt{2}$$

Hence, the shortest distance between the two lines =

$$= \left| \frac{(\vec{b} \times \vec{d}).(\vec{c} - \vec{a})}{|\vec{b} \times \vec{d}|} \right|$$

$$= \left| \frac{(-3\hat{\imath} - 3\,\hat{k}).(\hat{\imath} - 3\,\hat{\jmath} - 2\hat{k})}{\sqrt[3]{2}} \right|$$

$$= \left| \frac{-3-6}{\sqrt[3]{2}} \right| = \frac{3}{\sqrt{2}} = \frac{\sqrt[3]{2}}{2}$$

Example 13 : Show that the lines

$$\frac{x-5}{4} = \frac{y-7}{4} = \frac{z-3}{-5}$$

$$\text{and } \frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$$

intersect.

Solution : Given lines are :

$$\frac{x-5}{4} = \frac{y-7}{4} = \frac{z-3}{-5}$$

$$\text{and } \frac{x-5}{4} = \frac{y-7}{4} = \frac{z-5}{4}$$

converting these equations into vectors, we get

$$\vec{r} = (5\hat{\imath} + 7\,\hat{\jmath} - 3\hat{k}) + t\,(4\hat{\imath} + 4\,\hat{\jmath} - 5\hat{k})$$

$$\text{and } \vec{r} = (8\,\hat{\imath} + 4\,\hat{\jmath} + 5\hat{k}) + s\,(7\,\hat{\imath} + \hat{\jmath} + 3\hat{k})$$

Comparing with equations $\vec{r} = \vec{a} + t\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ respectively, we have

$\vec{a} = 5\hat{\imath} + 7\,\hat{\jmath} - 3\hat{k}$

$\vec{b} = 4\hat{\imath} + 4\,\hat{\jmath} - 5\hat{k}$

$\vec{c} = 8\hat{\imath} + 4\,\hat{\jmath} + 5\hat{k}$

$\vec{d} = 7\,\hat{\imath} + \hat{\jmath} + 3\hat{k}$

$\therefore \quad \vec{c} - \vec{a} = 3\,\hat{\imath} - 3\,\hat{\jmath} + 8\hat{k}$

and $\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$

$= 17\hat{\imath} + 47\ \hat{\jmath} - 24\hat{k}.$

Now $(\vec{c} - \vec{a}\ ).(|\vec{b} \times \vec{d}) = \left(3\hat{\imath} - 3\ \hat{\jmath} + 8\hat{k}\right).\left(17\hat{\imath} - 47\ \hat{\jmath} - 24\hat{k}\right) = 51 + 141 - 192 = 0.$

Thus, the shortest distance between the two lines =

$$\left|\frac{(\vec{c} - \vec{a}).(\vec{b}\ \times\ \vec{d})}{|\vec{b}\ \times\ \vec{d}|}\right| = 0$$

Hence, the two lines intersect.

Check Your Progress – 3

1. Find the shortest distance between the two lines whose vector equations are
   $\vec{r} = \hat{\imath} - \ \hat{\jmath}\ + t\ (\ 2\hat{\imath} + \hat{k}\ )$
   and $\vec{r} = (2\hat{\imath} - \ \hat{\jmath}\ ) + s\ (\ \hat{\imath} + \hat{\jmath} - \hat{k}\ )$

2. Find the shortest distance between the two lines
   $\vec{r} = (1 - t)\hat{\imath} + (t - 2)\ \hat{\jmath} + (3 - 2t)\hat{k}$
   and $\vec{r} = (s + 1)\hat{\imath} + (2s - 1)\ \hat{\jmath} + (2s + 1)\hat{k}$

3. Find the distance between the lines
   $\dfrac{x - 1}{2} = \dfrac{y + 1}{2} = z$ and

   $\dfrac{x + 1}{5} = \dfrac{y - 2}{2}; z = 2$

4. Determine whether the following pair of lines interest
   $\vec{r} = (2\lambda + 1)\ \hat{\imath} - (\lambda + 1)\ \hat{\jmath} + (\lambda + 1)\ \hat{k}$

   $\vec{r} = (3\ \mu\ + 2)\ \hat{\imath} - (5\ \mu\ + 5)\ \hat{\jmath} + (2\ \mu\ - 1)\ \hat{k}$

3.5 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. Let A, B, C denote the points (–2,3,5), (1,2,3) and (7,0,–1) respectively

   Then AB = $\sqrt{(1 + 2)^2 + \ (2 - 3)^2 + (3 - 5)^2}$

   $= \sqrt{9 + 1 + 4} = \sqrt{14}$

   BC = $\sqrt{(7 - 1)^2 + \ (0 - 2)^2 + (-1 - 3)^2}$

AB = $\sqrt{36+4+16}$ = $\sqrt{56}$ = $3\sqrt{14}$

AC = $\sqrt{(7+2)^2+\ (0-3)^2+(-1-5)^2}$

$\sqrt{81+9+36}$ = $\sqrt{126}$ $3\sqrt{14}$

Therefore, AB + BC = AC

Hence, the three points are Colliner.

2. Let A (0,7,10), B (–1,6,6) and C(–4,9,6) denote the given points. Then

AB = $\sqrt{(-1+0)^2+\ (6-7)^2+(6-10)^2}$

AB = $\sqrt{1+1+16}$ = $3\sqrt{2}$

BC= $\sqrt{(-4+0)^2+\ (9-6)^2+(6-6)^2}$

= $\sqrt{9+9}$ = $3\sqrt{2}$

AC = $\sqrt{(-4-0)^2+\ (9-7)^2+(6-10)^2}$

AC = $\sqrt{16+4+16}$ = 6

Since AB = BC, therefore, the triangle $\Delta\, ABC\ is\ isosceles.$

Further, AB^2 + BC^2 = $(3\sqrt{2})^2$ +$(3\sqrt{2})^2$ $= 18+18 = 36$

Also $BC^2\ = 36$

$\therefore\ AB^2+\ BC^2 =\ AC^2$

Hence, $\Delta\, ABC$ is a right angled triangle.

3. Let *a* = 1, *b* = –2, *c* = –2

The direction cosines are

$$l = \frac{a}{\sqrt{a^2+\ b^2+c^2}} = \frac{1}{\sqrt{1+4+4}} = \frac{1}{9}$$

$$l = \frac{a}{\sqrt{a^2+\ b^2+c^2}} = \frac{1}{\sqrt{1+4+4}} = \frac{1}{9}$$

$$m = \frac{b}{\sqrt{a^2+\ b^2+c^2}} = \frac{-2}{\sqrt{1+4+4}} = \frac{-2}{9}$$

$$n = \frac{c}{\sqrt{a^2+\ b^2+c^2}} = \frac{1}{\sqrt{1+4+4}} = \frac{-2}{9}$$

4. Let α, β, γ be the angles which the line makes with the x-axis, y-axis, y-axis and z-axis respectively.

and cos β = cos 60° $=\frac{1}{2}$

Now, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

$$\frac{1}{2} + \frac{1}{4} + cos^2 r = 1$$

$$\frac{3}{4} + cos^2 r = 1$$

$$cos^2 r = \frac{1}{4}$$

$$cos\ r = \pm \frac{1}{2}$$

Or $\cos^2 \gamma = \frac{1}{4}$ i.e., $\cos^2 \gamma = \pm \frac{1}{2}$

$\Rightarrow$ $\gamma = 60^\circ$ or 120°

5. Let A(–2, 4, –5 and B(1, 2, 3) denote the given points. Then

AB = $\sqrt{(1+2)^2 + (2-4)^2 + (3+5)^2} = \sqrt{9+4+64} = \sqrt{77}$

The direction cosines of the line joining the points A and B are given by

$$l = \frac{x_2 - x_1}{AB}, m = \frac{y_2 - y_1}{AB}, and\ n = \frac{z_2 - z_1}{AB}$$

Hence, the direction cosines are

$$l = \frac{3}{\sqrt{77}}, m = \frac{-2}{\sqrt{77}}, n = \frac{8}{\sqrt{77}}$$

Check Your Progress – 2

1. We have

$\vec{a} = 3\hat{\imath} + \hat{\jmath} + 4\hat{k}$ and

$\vec{b} = -\hat{\imath} + \hat{\jmath} - 2\hat{k}$

Therefore, the vector equation of the line is

$\vec{r} = (3\hat{\imath} + \hat{\jmath} + 4\hat{k}) + t\,(-\hat{\imath} + \hat{\jmath} - 2\hat{k})$

If $\vec{r}$ is the position vector of point (x, y, z), then

$x\hat{\imath} + y\hat{\jmath} + z\hat{k} = (3\hat{\imath} + \hat{\jmath} + 4\hat{k}) + t(-\hat{\imath} + \hat{\jmath} - 2\hat{k})$

$= (3–t)\,\hat{\imath} + (1–t)\,\hat{\jmath} + (4–2t)\,\hat{k}$

$\therefore\ x = 3 – t, y = 1 + t, z = 4–2t$

Eliminating t, we get

$$\frac{x-3}{-1} = \frac{y-1}{1} = \frac{z-4}{-2}$$

which is the equation of line in Cartesian form.

2. A vector parallel to the line

$$\frac{x+2}{3} = \frac{y-4}{1} = \frac{z-2}{2}$$

is $3\hat{\imath} + \ \hat{\jmath} + 2\hat{k}$

Thus we have to find equation of a line passing through (1, 0, 4) & parallel to the vector $3\hat{\imath} + \ \hat{\jmath} + 4\hat{k}$

So, $\vec{a} = \hat{\imath} - 4\hat{k}$ and $\vec{b} = 3\hat{\imath} + \ \hat{\jmath} + 2\hat{k}$

$\therefore$ the vector eqatuon of required line is

$\vec{r} = (\hat{\imath} - 4\hat{k}) + t\,(3\hat{\imath} + \ \hat{\jmath} + 2\hat{k})$

Also, the Cartesian equation of the line is

$$\frac{x-1}{3} = \frac{y-0}{1} = \frac{z+4}{2}$$

3. Let A (3, 4,–7) and B(1, –1, 6) denote the given points. The direction ratios of AB are 1–3, –1–4, 6+7 or –2, –5, 13

$\therefore \quad \vec{b} = -2\hat{\imath} - 5\ \hat{\jmath} + 13\hat{k}$

Also as A (3, 4, –7) lies on the line,

$\therefore \quad \vec{a} = \overrightarrow{OA} = 3\hat{\imath} + 4\ \hat{\jmath} - 7\hat{k}$

Hence, the vector equation is

$\vec{r} = \vec{a} + t\,\vec{b}$

i.e., $\vec{r} = (3\hat{\imath} + 4\ \hat{\jmath} - 7\hat{k}) + t\ (-2\hat{\imath} - 5\ \hat{\jmath} + 13\hat{k})$

Also, Cartesian equation of a line passing through the points $(x_1\ y_1\ , z_1)\ \ (\ x_2\ , x_2, x_2)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$$

So, the required Cartesian equation is

$$\frac{x-3}{-2} = \frac{y-4}{-5} = \frac{z-13}{-7}$$

4(i) The given liens are

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{-1} \qquad .....(1)$$

$$\text{and } \frac{x}{1} = \frac{y}{0} = \frac{z}{-1} \qquad .....(2)$$

Therefore, the direction ratios of line (1) are 1, 0, –1 and the direction ratios of line (2) are 3,4,5

If θ is the angle between the two lines, then

Cos θ =

$$\text{Cos}\theta = \left|\frac{1.3 + 0.4 + (-1)5}{\sqrt{1^2 + 0^2 + 1^2}\,\sqrt{3^2 + 4^2 + 5^2}}\right|$$

$$= \left|\frac{3-5}{\sqrt{2}\sqrt{50}}\right| = \left|\frac{-2}{\sqrt{100}}\right| = \left|\frac{-2}{10}\right| = \left|\frac{-1}{5}\right| = \frac{1}{5}$$

Hence $\theta = \cos^{-1}\left(\frac{1}{5}\right)$

(i) Here, $\vec{b} = \hat{\imath} - 2\,\hat{\jmath} + 2\hat{k}$ and $\overrightarrow{b'} = 4\hat{\imath} + \hat{\jmath} + 8\hat{k}$

If θ is the angle between the two lines,

$$\text{Then Cos}\theta = \left|\frac{\overrightarrow{b}.\overrightarrow{b'}}{|\vec{b}||\vec{b'}|}\right|$$

$$= \left|\frac{(\hat{\imath} - 2\,\hat{\jmath} + 2\hat{k})(4\hat{\imath} + \hat{\jmath} + 8\hat{k})}{\sqrt{1^2 + (-2)^2 + 2^2}\sqrt{4^2 + 1^2 + 8^2}}\right|$$

$$= \left|\frac{4 - 2 + 16}{\sqrt{9}\sqrt{81}}\right| = \frac{18}{27} = \frac{2}{3}$$

Hence $\theta = \cos^{-1}\left(\frac{2}{3}\right)$

5. The given lines are

$$= \frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2} \qquad ...(1)$$

$$\text{and } \frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5} \qquad ...(2)$$

The direction ratios of line (1) are –3, 2*k*, 2 and direction ratios of line (2) are 3*k*, 1, –5. Since the lines (1) and (2) are at right angles, therefore

(–3) (3*k*) + (2*k*)(1) + 2(–5) = 0

or –9*k* + 2*k* –10 = 0

or –7*k* = 10

$$or \quad k = \frac{10}{7}$$

Check Your Progress – 3

1. Here

$\vec{a} = \hat{\imath} - \hat{\jmath}$

$\vec{b} = 2\hat{\imath} + \hat{k}$

$\vec{c} = 2\hat{\imath} - \hat{\jmath}$ and

$\vec{d} = \hat{\imath} + \hat{\jmath} - \hat{k}$

$\therefore\ \vec{c} - \vec{a} = (2\,\hat{\imath} - \hat{\jmath}) + (\hat{\imath} - \hat{\jmath}) = \hat{\imath}$

and $\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

$= -\hat{\imath} + 3\,\hat{\jmath} + 2\hat{k} =$

$\therefore |\vec{b} \times \vec{d}\,| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{14}$

The shortest distance between the two lines is

$$= \left|\frac{(\vec{c} - \vec{a}).(\vec{b} \times \vec{d})}{|\vec{b} \times \vec{d}|}\right|$$

$$\left|\frac{\hat{\imath}\left(-\hat{\imath} + 3\,\hat{\jmath} + 2\hat{k}\right)}{\sqrt{14}}\right| = \left|\frac{-1}{\sqrt{14}}\right| = \frac{-1}{\sqrt{14}}$$

2. The given equation can be written as

$\vec{r} = (\hat{\imath} - 2\,\hat{\jmath} + 3\hat{k}) + t\,(-\hat{\imath} + \hat{\jmath} - 2\hat{k})$ and

$\vec{r} = (\hat{\imath} - \hat{\jmath} - \hat{k}) + s\,(\hat{\imath} + 2\,\hat{\jmath} - 2\hat{k})$

Here, $\vec{a} = \hat{\imath} - 2\,\hat{\jmath} + 3\hat{k}$

$\vec{b} = \hat{\imath} + \hat{\jmath} - 2\hat{k}$

$\vec{c} = \hat{\imath} - \hat{\jmath} - \hat{k}$

$\vec{d} = \hat{\imath} + 2\,\hat{\jmath} - 2\hat{k}$

$\vec{c} - \vec{a} = (\hat{\imath} - \hat{\jmath} - \hat{k}) - (\hat{\imath} - 2\,\hat{\jmath} + 3\hat{k})$

$= \hat{\jmath} - 4\hat{k}$

Also, $\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$

$= 2\hat{\imath} - 4\,\hat{\jmath} - 3\hat{k}$

$\therefore \ |\vec{b} \times \vec{d}\,| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{29}$

The shortest distance between the two lines is

$$= \left|\frac{(\vec{b} \times \vec{d}).(\vec{c} - \vec{a})}{|\vec{b} \times \vec{d}|}\right| = \left|\frac{(2\hat{\imath} - 4\hat{\jmath} - 3\hat{k}).(\hat{\jmath} - 4\hat{k})}{\sqrt{29}}\right| = \left|\frac{-4 + 12}{\sqrt{29}}\right| = \frac{8}{\sqrt{29}}$$

3. The given lines are

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{1} \qquad \dots(1)$$

and $\dfrac{x+1}{5} = \dfrac{y-2}{1} = \dfrac{z-2}{0}$

Converting these equations into vector form, we have

$\vec{r} = (\hat{\imath} - \hat{\jmath}) + t\,(2\hat{\imath} + 3\hat{\jmath} + \hat{k})$

and $\vec{r} = (-\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) + s\,(5\hat{\imath} + \hat{\jmath})$

Comparing these equation with $\vec{r} = \vec{a} + t\vec{b}$

and $\vec{r} = \vec{c} + s\,\vec{d}$, we have

$\vec{a} = (\hat{\imath} - \hat{\jmath})$

$\vec{b} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$

$\vec{c} = -\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

$\vec{b} = 5\hat{\imath} + \hat{\jmath}$

Now, $\vec{c} - \vec{a} = (-\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) - (\hat{\imath} - \hat{\jmath}) = 2\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$

and

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 1 \\ 5 & 1 & 0 \end{vmatrix}$$

$= -\hat{\imath} + 5\hat{\jmath} - 13\hat{k}$

$\therefore \ |\vec{b} \times \vec{d}\,| = \sqrt{(-1)^2 + (5)^2 + (-13)^2} = \sqrt{1 + 25 + 169} = \sqrt{195}$

$$\text{Shortest distance} = \left|\frac{(\vec{c} - \vec{a}).(\vec{b} \times \vec{d})}{|\vec{b} \times \vec{d}|}\right|$$

$$= \left|\frac{(-\widehat{2\imath} + 3\hat{\jmath} + 2\hat{k}).(-\hat{\imath} + 5\hat{\jmath} - 13\hat{k})}{\sqrt{195}}\right| = \left|\frac{2 + 15 - 26}{\sqrt{14}}\right| = \frac{9}{\sqrt{195}}$$

4. The given lines are

$\vec{r} = \hat{\imath} - \hat{\jmath} + \hat{k}$ + ($2\hat{\imath} - \hat{\jmath} + \hat{k}$) and

$\vec{r} = 2\hat{\imath} - 5\hat{\jmath} - \hat{k}$ +μ ($3\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$)

Here, $\vec{a} = \hat{\imath} - \hat{\jmath} + \hat{k}$

$\vec{b} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$

$\vec{c} = 2\hat{\imath} - 5\hat{\jmath} + \hat{k}$

$\vec{d} = 3\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$

Now, $\vec{c} - \vec{a} = -\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$

and $\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$

$= 3\hat{\imath} - \hat{\jmath} - 7\hat{k}$

$\therefore\ |\vec{b} \times \vec{d}| = \sqrt{3^2 + (-1)^2 + (-7)^2} = \sqrt{59}$

$$\text{Shortest Distance} = \left|\frac{(\vec{c} - \vec{a}).(\vec{b} \times \vec{d})}{|\vec{b} \times \vec{d}|}\right|$$

$$= \left|\frac{(\hat{\imath} - 4\hat{\jmath} - 2\hat{k}).(3\hat{\imath} - \hat{\jmath} - 7\hat{k})}{\sqrt{59}}\right| = \left|\frac{3 + 4 + 14}{\sqrt{59}}\right| = \frac{21}{\sqrt{59}}$$

3.6 SUMMARY

The unit, as the title suggests, is about three-dimensional geometry. In **section 3.2**, first, the concept of three dimensional space is illustrated. Then formula for distance between two points in three-dimensional space, is derived. Then, the concepts of direction cosines and direction ratios are explained. In **section 3.3**, formulae for finding equations of a straight line are derived when (i) a point of the line and a vector parallel to the required line are given and when (ii) a pair of points is given. In **section 3.4**, formula for finding shortest distance between pair of straight lines in three-dimensional space, is first derived and then is used in solving problems.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 3.5**.