UNIT 2 VECTORS – 2

Structure

2.0 INTRODUCTION

In the previous unit, we discussed vectors and scalars. We learnt how to add and subtract two vectors, and how to multiply a vector by a scalar. In this unit, we shall discuss multiplication of vectors. There are two ways of defining product of vectors. We can multiply two vectors to get a scalar or a vector. The former is called scalar product or dot product of vectors and the latter is called vector product or cross product of vectors. We shall learn many applications of dot product and cross product of vectors. We shall use dot product to find angle between two vectors. Two vectors are perpendicular if their dot product is zero. Dot product helps in finding projection of a vector onto another vector. The cross product of two vectors is a vector perpendicular to both the vectors.

If cross product of two vectors is zero then the two vectors are parallel (or collinear). Cross product of vectors is also used in finding area of a triangle or a parallelogram. Using the two kinds of products, we can also find product of three vectors. Many of these products will not be defined. In this unit, we shall discuss the two valid triple products, namely, the scalar triple product and the vector triple product.

2.1 OBJECTIVES

After studying this unit, you should be able to:

- define scalar product or dot product of vectors;
- find angle between two vectors;
- find projection of a vector on another vector;

- define cross product or vector product of vectors;
- use cross product to find area of a parallelogram vector product of vectors;
- define scalar triple product and vector triple product of vectors.

2.2 SCALAR PRODUCT OF TWO VECTORS

Definition : The scalar product or the dot product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}.\vec{b}$ is defined by

$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\,cos\,\theta$$

where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$.

Also the scalar product of any vector with the zero vector is, by definition, the scalar zero. It is clear from the definition that the dot product $\vec{a}.\vec{b}$ is a scalar quantity.

Sign of the Scalar Product

If $\vec{a}$ and $\vec{b}$ are two non zero vectors, then the scalar product

$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

is positive, negative or zero, according as the angle θ, between the vectors is acute, obtuse or right. In fact,

θ is acute $\Rightarrow$ $\cos\theta > 0$ $\Rightarrow$ $\vec{a}.\vec{b} > 0$

θ is right $\Rightarrow$ $\cos\theta = 0$ $\Rightarrow$ $\vec{a}.\vec{b} = 0$

θ is obtuse $\Rightarrow$ $\cos\theta < 0$ $\Rightarrow$ $\vec{a}.\vec{b} < 0$

Also, note that if and $\vec{a}$ and $\vec{b}$ are non zero vectors then $\vec{a}.\vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular (or orthogonal) to each other.

If $\vec{a}$. is any vector, then the dot product $\vec{a}.\vec{a}$., of $\vec{a}$ with itself, is given by
$\vec{a}.\vec{a} = |\vec{a}||\vec{a}| \cos 0 = |\vec{a}|^2$

Thus, the length $|\vec{a}|$ of any vector $\vec{a}$ is the non negative square root $\sqrt{\vec{a}.\vec{a}}$, i.e., of the scalar product $\vec{a}\,\vec{a}$, i.e.,

$|\vec{a}| = \sqrt{\vec{a}.\vec{a}}$,

Angle between two Vectors

If θ is an angle between two non zero vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$\Rightarrow \qquad \cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}$

or $\qquad \cos\theta = \dfrac{\vec{a}.\vec{b}}{\sqrt{\vec{b}.\vec{b}}\sqrt{\vec{a}\vec{a}}}$

So, the angle θ between two vectors is given by

$$\theta = \cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

Properties of Scalar Product

1. Scalar product is cumulative, i.e., $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ for every pair of vectors $\vec{a}$ and $\vec{b}$.
2. $\vec{a}.(-\vec{b}) = -(\vec{a}.\vec{b})$ and $(-\vec{a}).(-\vec{b}) = \vec{a}.\vec{b}$ for every pair of vectors $\vec{a}$ and $\vec{b}$.
3. $(\lambda_1 \vec{a}).(\lambda_2\vec{b}) = (\lambda_1 \lambda_2)(\vec{a}.\vec{b})$ where $\vec{a}$ and $\vec{b}$ are vectors and λ_1, λ_2 are scalars
4. (Distributivity) $\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$

The following identities can be easily proved using above properties.

(i) $\quad (\vec{a} + \vec{b}).(\vec{a} - \vec{b}) \quad = a^2 - b^2$ (here, $a^2 = \vec{a}.\vec{a} = |\vec{a}|^2$)

(ii) $\quad (\vec{a} + \vec{b})^2 \quad = a^2 + 2\vec{a}.\vec{b} + b^2$

(iii) $\quad (\vec{a} - \vec{b})^2 \quad = a^2 - 2\vec{a}.\vec{b} + b^2$

If $\hat{\imath}$, $\hat{\imath}$ and $\hat{k}$ are mutually perpendicular unit vectors, then

$\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$ and

$\hat{\imath}.\hat{\jmath}. = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0 \qquad (1)$

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ be two vectors in component form, then their scalar product is given by

$= a_1\hat{i}\ .(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_2\hat{i}(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_3\hat{k}(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$ (using distributivity)

$= a_2b_3(\hat{j}.\hat{k}) + a_3b_1(\hat{k}.\hat{i}) + a_3b_2(\hat{k}.\hat{j}) + a_3b_3(\hat{k}.\hat{k})$ (using properties)

$= a_1 b_1 + a_2 b_2 + a_3 b_3$ (using (1))

Thus $\vec{a}.\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Projection of a Vectors on another Vector

Let $\vec{a} = \overrightarrow{OQ}$ and $\vec{b} = \overrightarrow{OB}$ be two vectors

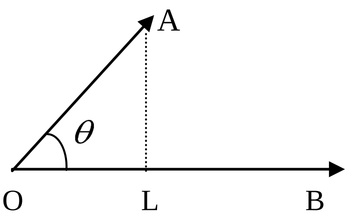


Figure 1

Drop a perpendicular form A on OB as shown in Figure 1. The projection of $\vec{a}$ on $\vec{b}$ is the vector $\overrightarrow{OL}$. If θ is the angle between $\vec{a}$ and $\vec{b}$, then projection of $\vec{a}$ and $\vec{b}$ has length $|\overrightarrow{OA}| \cos\theta$ and direction along unit vector $\vec{b}$. Thus, projection of $\vec{a}$ on $\vec{b}$ = ($|\vec{a}| \cos\theta$) $\vec{b}$

$$= \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|}\right)\hat{b} = \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

The scalar component of project of $\vec{a}$ on $\vec{b}$ is $\frac{\vec{a}\,.\vec{b}}{|\vec{b}|}$. Similarly, the scalar component of projection of $\vec{b}$ on $\vec{a}$ $\frac{\vec{a}\,.\vec{b}}{|\vec{a}|}$

Example 1: Find the angle θ between the following pair of vectors.

(a) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(b) $\vec{a} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$ $\qquad$ $\vec{b} = 2\hat{\imath} + 2\hat{\jmath} + \hat{k}$

Solution: (a) $\vec{a}\,.\,\vec{b}$ $= (2\hat{\imath} + 3\hat{k})\,.\,(\hat{\imath} + 4\hat{\jmath} - \hat{k})$

$= 2.1 + 0.4 + 3\,(-1)$

$= 2 - 3 = -1$

$|\vec{a}|^2$ $= \vec{a}\,.\,\vec{a} = (2\hat{\imath} + 3\hat{k})\,.\,(2\hat{\imath} + 3\hat{k})$

$= 2.2 + 0.0 + 3.3$

$= 13$

$\therefore$ $|a|$ $= \sqrt{13}$

and $|\vec{b}|^2$ $= \vec{b}.\vec{b}$ $=$ $(\hat{\imath} + 4\hat{\jmath} - 3\hat{k})\,.\,(\hat{\imath} + 4\hat{\jmath} - 3\hat{k})$

$= 1.1 + 4.4 + (-3)\,.\,(-3)$

$= 26$

$\therefore$ $|a| = \sqrt{26}$

$\therefore$ $\cos\theta$ $= \dfrac{\vec{a}\,.\vec{b}}{|\vec{a}|\,|\vec{b}|}$

$= \dfrac{-1}{\sqrt{13}\sqrt{26}} = \dfrac{-1}{13\sqrt{2}}$

$\therefore$ θ $= cos^{-1}\left(\dfrac{-1}{13\sqrt{2}}\right)$

(b) Here, $\vec{a}\,.\,\vec{b} = (\hat{\imath} + 2\hat{\jmath} + 2\hat{k}).\,(2\hat{\imath} + 2\hat{\jmath} + \hat{k})$

$= 1.2 + 2.(-2) + 2.1$

$= 0$

$|\vec{a}| = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$

$$\left|\vec{b}\right| = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\therefore \quad \cos\theta = \frac{\vec{a}\,.\,\vec{b}}{|\vec{a}\,||\vec{b}\,|} = \frac{0}{3.3} = 0$$

$$\therefore \quad \theta = \frac{\pi}{2}$$

.

Example 2 : Show that $|\vec{a}\,|\vec{b} + \left|\vec{b}\right|\vec{a}$ is perpendicular to $|\vec{a}\,|\vec{b} - \left|\vec{b}\right|\vec{a}$, for any two non zero vectors $\vec{a}$ and $\vec{b}$.

Solution : We know that two vectors are perpendicular if their scalar product is zero.

$$(|\vec{a}\,|\vec{b} + \left|\vec{b}\right|\vec{a}\,)\,.\,(|\vec{a}\,|\vec{b} - \left|\vec{b}\right|\vec{a}\,)$$

$$= |\vec{a}\,|\vec{b}.\,(|\vec{a}\,|\vec{b} - \left|\vec{b}\right|\vec{a}\,) + \left|\overline{A\vec{B}}\right|\vec{a}\,.\,(|\vec{a}\,|\vec{b} - \left|\vec{b}\right|\vec{a}\,)$$

(using distributivity)

$$= |\vec{a}\,|^2 \quad (\vec{b}.\vec{b}\,) - |\vec{a}\,|\,\left|\vec{b}\right|(\vec{b}.\vec{a}\,) + \left|\vec{b}\right|\,|\vec{a}\,|\,(\vec{a}.\vec{b}\,) - \left|\vec{b}\right|^2(\vec{a}.\vec{a})$$

$$= |\vec{a}\,|^2\left|\vec{a}\times\vec{b} = 0\right|^2 - |\vec{a}\,|\,\left|\vec{b}\right|(\vec{a}.\vec{b}\,) + |\vec{a}\,|\,\left|\vec{b}\right|(\vec{a}.\vec{b}\,) - \vec{b}\ \ |\vec{a}\,|^2 = 0$$

$$\left(\because \vec{a}.\vec{b} = \vec{b}.\vec{a},\ \vec{a}.\vec{a} = |\vec{a}\,|^2 \text{ and } \vec{b}.\vec{b} = \overrightarrow{|b|}^2\right)$$

so, the given vectors are perpendicular.

Example 3 : Find the scalar component of projection of the vector

$$\vec{a} = 2\hat{i} + 3\hat{j} + 5\hat{k} \text{ on the vector } \vec{b} = 2\hat{i} - 2\hat{j} - \hat{k}.$$

Solution : Scalar projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a}.\vec{b}}{\left|\vec{b}\right|}$

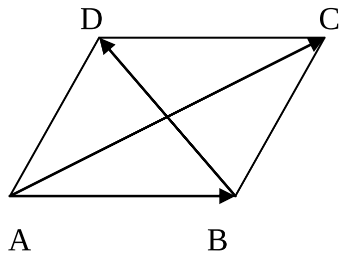
Here, $\vec{a}\,.\,\vec{b} = 2.2 + 3\,(-2) + 5(-1) = -7$

and $|\vec{b}| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$

$\therefore$ Scalar projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{-7}{3}$

Example 4 : Show that the diagonals of a rhombus are at right angles.

Solution : Let A B C D be a rhombus (Figure 2)



(Figure 2)

ABCD being a rhombus, we have

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and

$\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$

$$\therefore\ \overrightarrow{AC}.\overrightarrow{BD} = \left(\overrightarrow{AB} + \overrightarrow{BC}\right).\left(\overrightarrow{AD} - \overrightarrow{AB}\right)$$

$$= \left(\overrightarrow{AB} + \overrightarrow{AD}\right).\left(\overrightarrow{AD} - \overrightarrow{AB}\right) \qquad \left(\because \overrightarrow{AC} = \overrightarrow{AD}\right)$$

$$= \overrightarrow{AD}^2 - \overrightarrow{AB}^2 \qquad \left(\because \left(\vec{a} + \vec{b}\right).\left(\vec{a} - \vec{b}\right) = \vec{a}^2 - \vec{b}^2\right)$$

$$= \overrightarrow{AD}^2 - \overrightarrow{AD}^2 = 0$$

$$\Rightarrow \overrightarrow{AC} \perp \overrightarrow{BD} \qquad (\because \overrightarrow{AB}^2 = \overrightarrow{AD}^2)$$

Example 5 : For any vectors $\vec{a}$ and $\vec{b}$, prove the triangle inequality

$$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$$

Solution : If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then the inequality holds trivially.
So let $|\vec{a}| \neq 0 \neq |\vec{b}|$. Then,

$$|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})^2 = (\vec{a} + \vec{b}).(\vec{a} + \vec{b})$$

$$= \vec{a}\vec{a} + \vec{a}\vec{b} + \vec{b}\vec{a} + \vec{b}\vec{b}$$

$$= |\vec{a}|^2 + 2\vec{a}\vec{b} + |\vec{b}|^2 \qquad (\because \vec{a}\vec{b} = \vec{a}\vec{b})$$

$$= |\vec{a}|^2 + 2|\vec{a}||\vec{b}|\cos\theta + |\vec{b}|^2 \text{ where } \theta \text{ is the angle between } \vec{a} \text{ and } \vec{b}$$

$$\leq |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2 \qquad (\because \cos\theta \leq 1\ \forall\ \theta)$$

$$= (|\vec{a}| + |\vec{b}|)^2$$

Hence $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

Remark : Let $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{BC}$, then $\vec{a} + \vec{b} = \overrightarrow{AC}$

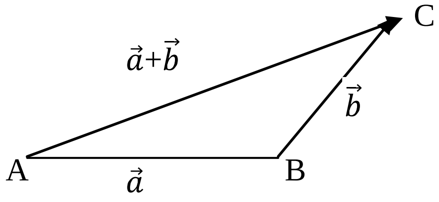


Figure 3

As shown in figure 3 inequality says that the sum of two sides of triangle is greater than the third side. If the equality holds in triangle inequality, i.e.,
$|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$

Then $|\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

showing that the points A, B and C are collinear.

Check Your Progress – 1

1. If $\vec{a} = 5\hat{\imath} - 3\hat{\jmath} - 3\hat{k}$ and $\vec{b} = \hat{\imath} + \hat{\jmath} - 5\hat{k}$ then show that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular.

2. Find the angle between the vectors

(a) $\vec{a} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$ $\quad \vec{b} = -4\hat{\imath} + 2\hat{k}$

(b) $\vec{a} = \hat{\imath} - \hat{\jmath} - 2\hat{k}$ $\quad \vec{b} = -2\hat{\imath} + 2\hat{\jmath} + 4\hat{k}$

3. Find the vector projection of $\vec{a}$ on $\vec{b}$ where $\vec{a} = 3\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$ and $\vec{b} = 7\hat{\imath} + \hat{\jmath} - 2\hat{k}$. Also find the scalar component of projection of vector $\vec{b}$ on $\vec{a}$.

4. Prove the Cauchy – Schwarz Inequality

$|\vec{a}.\ \vec{b}| \leq |\vec{a}|\,|\vec{b}|$

5. Prove that $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b}$

2.3 VECTOR PRODUCT (OR CROSS PRODUCT) OF TWO VECTORS

Definition : If $\vec{a}$ and $\vec{b}$ are two non zero and non parallel (or equivalently non collinear) vectors, then their vector product $\vec{a} \times \vec{b}$ is defined as

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\ \hat{n}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 < \theta < \pi)$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

If $\vec{a}$ and $\vec{b}$ are parallel (or collinear) i.e., when $\theta = 0$ or π, then we define the vector product of $\vec{a}$ and $\vec{b}$ to be the zero vector i.e., $\vec{a} \times \vec{b} = \vec{0}$. Also note that if either $\vec{a} = 0$ or $\vec{b} = 0$,then θ is not defined and we define $\vec{a} \times \vec{b} = 0$.

Properties of the Vector Product

1. $\vec{a} \times \vec{a} = \vec{0}$ since $\theta = 0$
2. Vector product is not commutative i.e., $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$. $\vec{a}$,
   However, $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.
   We have $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\ \hat{n}$ where $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system and $\vec{b} \times \vec{a} = |\vec{b}||\vec{a}|\,|\sin\theta\ \hat{n}|$ where $\vec{b}$, $\vec{a}$ and $\hat{n}\ a$ right handed system. So the direction of $\hat{n}|$ is opposite to that of $\hat{n}$.

   Hence, $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\ \hat{n}$
   $= -|\vec{a}||\vec{b}| \sin\theta\ \hat{n}$,
   $= -\vec{b} \times \vec{a}$.
3. Let $\hat{\imath}, \hat{\jmath}, \hat{k}$ from a right–handed system of mutually perpendicular unit vectors in three dimensional space. Then

   $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$ and
   $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\quad \hat{\jmath} \times \hat{k} = \hat{\imath}$, $\quad \hat{k} \times \hat{\imath} = \hat{\jmath}$
   Also, $\hat{\jmath} \times \hat{\imath} = -\hat{k}$, $\ \hat{k} \times \hat{\jmath} = -\hat{\imath}$, $\quad \hat{\imath} \times \hat{k} = -\hat{\jmath}$

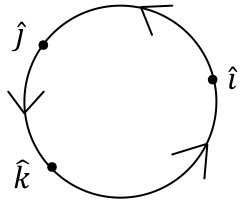


Figure 4

4. Two non zero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if

$\vec{a} \times \vec{b} = \vec{0}$

5. Vector product is distributors over addition i.e., if $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors, then

(i) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

(ii) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$.

6. If λ is a scalar and $\vec{a}$ and $\vec{b}$ are vectors, then

$(\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b}) = \lambda(\vec{a} \times \vec{b})$

Vector Product in the component form

Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and

$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$.

Then

$$\vec{a} \times \vec{b} = (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$$

$$= a_1 b_1(\hat{\imath} \times \hat{\imath}) + a_1 b_2(\hat{\imath} \times \hat{\jmath}) + a_1 b_3(\hat{\imath} \times \hat{k}) + a_2 b_1(\hat{\jmath} \times \hat{\imath}) + a_2 b_3 (\hat{\jmath} \times \hat{k}) + a_2 b_3(\hat{\jmath} \times \hat{k}) + a_3 b_1(\hat{k} \times \hat{\imath}) + a_3 b_2(\hat{k} \times \hat{\jmath}) + a_3 b_3 (\hat{k} \times \hat{k})$$

$$= a_1 b_2\hat{k} + a_1 b_3(-\hat{\jmath}) + a_2 b_1(-\hat{k}) + a_2 b_3\hat{\imath} + a_3 b_1\hat{\jmath} + a_3 b_2(-\hat{\imath}) \quad (using\ property\ 3)$$

$$= (a_2 b_3 - a_3 b_2)\hat{\imath} + (a_1 b_3 - a_3 b_1)\hat{\jmath} + (a_1 b_2 - a_2 b_1)\hat{k}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 6: Find $\vec{a} \times \vec{b}$ if $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

Solution : We have

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= (3-2)\hat{\imath} - (3-1)\hat{\jmath} + (2-1)\hat{k}$$

$$= \hat{\imath} - 2\hat{\jmath} + \hat{k}$$

Example 7: Find a unit vector perpendicular to both the vectors

$$\vec{a} = 4\hat{i} + \hat{j} + 3\hat{k} \text{ and } \vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$$

Solution : A vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix}$$

$$= (2-3)\hat{\imath} - (-8+6)\hat{\jmath} + (4-2)\hat{k}$$

$$= -\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{9} = 3.$$

So the desired unit vector is

$$\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{a}|} = \frac{1}{3}(-\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$$

Example 8 : Prove the distributive law

$$\vec{a} \times (\vec{b}+\vec{c}) = \vec{a} \times \vec{b}+\vec{a} \times \vec{c}$$

using component form of vectors.

Solution : Let $\vec{a} = a_1\,\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\vec{b} = b_1\,\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ and

$\vec{c} = c_1\,\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$.

So, $\vec{b} + \vec{c} = (b_1 + c_1)\,\hat{\imath} + (b_2 + c_2)\,\hat{\jmath} + (b_3 + c_3)\,\hat{k}$

Now,

$$\vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c}.$$

Example 9: Show that

$$|\vec{a} \times \vec{b}|^2 = (\vec{a}.\vec{a})(\vec{b}.\vec{b}) - (\vec{a}.\vec{b})^2$$

Solution : $|\vec{a} \times \vec{b}|^2 = |\vec{a}||\vec{b}| \sin\theta$

$\therefore\ |\vec{a} \times \vec{b}|^2 = (|\vec{a}||\vec{b}| \sin\theta)^2$

$= |\vec{a}|^2|\vec{b}|^2 (1 - \cos^2\theta)$

$= |\vec{a}|^2|\vec{b}|^2 - (|\vec{a}|^2|\vec{b}|^2 \cos^2\theta)$

$= (\vec{a}\vec{a})^2(\vec{b}\vec{b}) - (\vec{a}.\vec{b})^2$

Example 10 : Find $|\vec{a} \times \vec{b}|$ if

$|\vec{a}| = 10$, $|\vec{b}| = 2$, $\vec{a}.\vec{b} = 12$.

Solution : Here $\vec{a}.\vec{b} = 12$

$$\therefore \text{Cos}\,\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{a}\vec{b}|} = \frac{12}{(10)(2)} = \frac{3}{5}$$

$$\therefore \text{Sin}\,\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}|.|\vec{b}| sin\theta$$

$$= 10 \times 2 \times \frac{4}{5} = 16$$

Area of Triangle

Let $\vec{a}$, $\vec{b}$ be two vectors and let θ be the angle between them (O < θ < π). Let O be the origin for the vectors as shown in figures (Fig. 5) below and let
$\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$

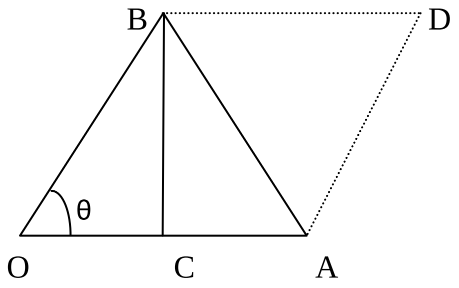


Figure 5

Draw BC⊥ OA.

Then BC = OB Sin θ = |b| sin θ

$$\therefore \text{ Area of } \Delta \text{ OAB } = \frac{1}{2} (\text{OA})(\text{BC})$$

$$= \frac{1}{2}|\vec{a}||\vec{b}| \sin\theta$$

$$= \frac{1}{2}|\vec{a} \times \vec{b}|$$

Thus, if $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle, then its area is given as $\frac{1}{2}|\vec{a} \times \vec{b}|$.

Area of a Parallelogram

In above figure, if D is the fourth vertex of the parallelogram formed by O, B, A then its area is twice the area of the triangle OBA.

Hence, area of a parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent sides = $|\vec{a} \times \vec{b}|$.

Example 11: Find the area of Δ ABC with vertices A (1,3,2), B (2, –1,1) and C(–1, 2, 3).

Solution : We have

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (2-1)\,\hat{\imath} + (-1-3)\,\hat{\jmath} + (1-2)\,\hat{k}$$
$$= \hat{\imath} - 4\hat{\jmath} - \hat{k}$$

$$\text{and } \overrightarrow{BC} = (-1-1)\,\hat{\imath} + (2-3)\,\hat{\jmath} + (3-2)\,\hat{k}$$
$$= -2\,\hat{\imath} - \hat{\jmath} + \hat{k}$$

$$\text{Vector Area of } \Delta \text{ ABC } = \frac{1}{2}(\overrightarrow{AB} \times \overrightarrow{BC})$$

$$= \frac{1}{2}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -4 & -1 \\ -2 & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\,[-5\,\hat{\imath} + \hat{\jmath} - 9\hat{k}\,]$$

$$\therefore \text{ Area of } \Delta\,\text{ABC} = \frac{1}{2}\sqrt{5^2 + 1^2 + 9^2} = \frac{1}{2}\sqrt{107}$$

Example 12 : Show that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$. Interpret the result geometrically.

Solution : $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$

$= (\vec{a} - \vec{b}) \times \vec{a} + (\vec{a} - \vec{b}) \times \vec{b}$

$= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b}$

$= 0 + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} + 0$

$= 2\,(\vec{a} \times \vec{b})$

Let ABCD be a parallelogram with $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = -\vec{b}$.

Then area of parallelogram = $\overrightarrow{AB} \times \overrightarrow{AD} = \vec{a} \times \vec{b}$

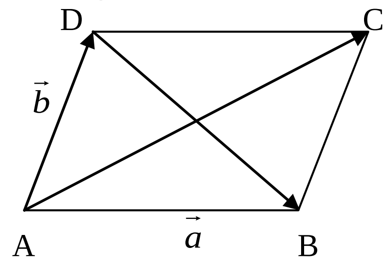


Figure 6

Also, diagonal $\overrightarrow{AC} = \vec{a} + \vec{b}$

and diagonal $\overrightarrow{DB} = \vec{a} - \vec{b}$

$\therefore\ (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \overrightarrow{AC} \times \overrightarrow{DB}$

= area of parallelogram formed by $\overrightarrow{AC}$ and $\overrightarrow{DB}$.

Thus, the above result shows that the area of a parallelogram formed by diagonals of a parallelogram is twice the area of the parallelogram.

Check Your Progress – 2

1. Find a unit vector perpendicular to each of the vector $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$, where $\vec{a} = \hat{\imath} + 2\hat{\jmath} - 4\,\hat{k}$ and $\vec{b} = \hat{\imath} - \hat{\jmath} + 2\,\hat{k}$

2. Show that
$$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$$

3. For Vectors

$\vec{a} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$, $\vec{b} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{c} = \hat{\imath} + \hat{\jmath} - 2\hat{k}$

Compute $(\vec{a} \times \vec{b}) \times \vec{c}$ and $\vec{a} \times (\vec{b} \times \vec{c})$.

is $(\vec{a}\times\vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$.

4. If the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy $\vec{a} + \vec{b} + \vec{c} = 0$, then prove that

$\vec{b} \times \vec{c} = \vec{c} \times \vec{a} = \vec{a} \times \vec{b}$.

5. Find the area of a parallelogram whose diagonals are

$\vec{a} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{b} = \hat{\imath} - 3\hat{\jmath} + 4\hat{k}$.

2.4 TRIPLE PRODUCT OF VECTORS

Product of three vectors may or may not have a meaning. For example, $(\vec{a}.\vec{b}).\vec{c}$ has no meaning as $\vec{a}.\vec{b}$ is a scalar and dot product is defined only for vectors. Similarly, $(\vec{a}.\vec{b}) \times \vec{c}$ has no meaning. The products of the type $(\vec{a} \times \vec{b}) \times \vec{c}$ and $\vec{a}.(\vec{b} \times \vec{c})$. are meaningful and called triple products. The former is a vector while the latter is a scalar.

Scalar Triple Product

Definition : Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors. The scalar product of $\vec{a}.(\vec{b} \times \vec{c})$ is called scalar triple product of $\vec{a}$, $\vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a}, \vec{b}, \vec{c}]$. Thus $[\vec{a}, \vec{b}, \vec{c}] = \vec{a}.(\vec{b} \times \vec{c})$.

It is clear from the definition that $[\vec{a}, \vec{b}\ \vec{c}]$. Is a scalar quantity.

Geometrically, the scalar triple product gives the volume of a parallelepiped formed by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ as adjacent sides.

Scalar Triple Product as a determinant

Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

and $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$

Then

$$\vec{a}.(\vec{b} \times \vec{c}) = \vec{a}.\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}).\left\{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}\hat{\imath} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\hat{k}\right\}$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Thus, $\vec{a}.(\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Note : We can omit the brackets in $\vec{a}.(\vec{b} \times \vec{c})$ and just write $\vec{a}.\vec{b} \times \vec{c}$ because $(\vec{a}.\vec{b}) \times \vec{c}$ is meaningless.

Properties of Scalar Triple Product

1. $[\vec{a},\vec{b},\vec{c}] = [\vec{b},\vec{c},\vec{a}] = [\vec{c},\vec{a},\vec{b}] - [\vec{b},\vec{a},\vec{c}]$
$= -[\vec{c},\vec{b},\vec{a}] = -[\vec{a},\vec{c},\vec{b}]$
This is clear if we note the properties of a determinant as $[\vec{a},\vec{b},\vec{c}]$ can be expressed as a determinant.
2. In scalar triple product $\vec{a}.(\vec{b} \times \vec{c})$, the dot and cross can be interchanged.
Indeed, $\vec{a}.(\vec{b} \times \vec{c}) = [\vec{a},\vec{b},\vec{c}] = [\vec{b},\vec{c},\vec{a}] = [\vec{c},\vec{a},\vec{b},]$
$= \vec{c}.(\vec{a} \times \vec{b}) = (\vec{a} \times \vec{b}).\vec{c}.$
3. $= [p\vec{a}, q\vec{b}, r\vec{c}] = pqr[\vec{a},\vec{b},\vec{c}]$ where p, q and r are scalars. Again it is clear using properties of determinants.
4. If any two of $\vec{a}.\vec{b}$ and $\vec{c}$ are the same then $[\vec{a}\vec{b}\vec{c}] = 0$ For example, $[\vec{a},\vec{b},\vec{c}] = 0$

Coplanarity of three vectors

Theorem : Three vectors $\vec{a}.\vec{b}$ and $\vec{c}$ are coplanar if and only if $[\vec{a},\vec{b},\vec{c} = 0]$

Proof : First suppose that $\vec{a}.\vec{b}$ and $\vec{c}$ are coplanar . If $\vec{b}$ and $\vec{c}$ are parallel vectors, then $\vec{b} \times \vec{c} = 0$ and so $\vec{a}.(\vec{b} \times \vec{c}) = 0$. If $\vec{b}$ and $\vec{c}$ are not parallel, then $\vec{b} \times \vec{c}$ being perpendicular to plane containing $\vec{b}$ and $\vec{a}$, is also perpendicular to $\vec{a}$ because $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar.

$\therefore \quad \vec{a}(\vec{b} \times \vec{c}) = 0.$

Conversely, suppose that $[\vec{a},\vec{b},\vec{c}\,] = 0$. If $\vec{a}$ and $\vec{b} \times \vec{c}$ are both non zero, then $\vec{a}$ and $\vec{b} \times \vec{c}$ are perpendicular as their dot product is zero. But $\vec{b} \times \vec{c}$ is perpendicular to both $\vec{b}$ and $\vec{c}$ and hence $\vec{a},\vec{b}$ and $\vec{c}$ must lie in a plane, i.e., they are coplanar. If $\vec{a} = \vec{0}$ then $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar as zero vector is coplanar with any two vectors. If $\vec{b} \times \vec{c} = \vec{0}$, then $\vec{b} \times \vec{c}$ are parallel vectors and hence $\vec{a},\vec{b}$and $\vec{c}$ must be coplanar.

Note : Four points A, B , C and D are coplanar if the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar.

Example 13 : If $\vec{a} = 7\hat{\imath} + a_2 - 2\hat{\jmath} + 3\,\hat{k}$, $\vec{b} = \hat{\imath} - 2\hat{\jmath} + 2\,\hat{k}$, and $\vec{c} = 2\hat{\imath} + 8\hat{\jmath}$ find $[\vec{a}, \vec{b}, \vec{c}]$

Solution : $[\vec{a}, \vec{b}, \vec{c}] = \vec{a}.\,(\vec{b} \times \vec{c})$

$$= \begin{vmatrix} 7 & -2 & 3 \\ 1 & -2 & 2 \\ 2 & 8 & 0 \end{vmatrix}$$

$= 7(0 - 16) + 2(0 - 4) + 3(8 + 4)$

$= -112 - 8 + 36$

$= -84$

Example 14 : Find the value of λ for which the vectors $\vec{a} = \hat{\imath} - 4\hat{\jmath} + \hat{k}$, $\vec{b} = \lambda\,\hat{\imath} - 2\hat{\jmath} + \hat{k}$, and $\vec{c} = 2\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$ are coplanar,

Solution : If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, we have

$= \vec{a}.\,(\vec{b} \times \vec{c}) = 0$ i.e.,

$$\begin{vmatrix} 1 & -4 & 1 \\ \lambda & -2 & 1 \\ 2 & 3 & 3 \end{vmatrix} = 0$$

$\Rightarrow\ 1(-6 - 3) + 4(3\lambda - 2) + (3\lambda + 4) = 0$

$$\Rightarrow\ \lambda = \frac{13}{15}$$

Example 15 : If $\vec{a}, \vec{b}, \vec{c}$ are coplanar then prove that $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are also coplanar.

Solution : Since $\vec{a}, \vec{b}, \vec{c}$ are coplanar,

$\therefore\ [\vec{a}, \vec{b}, \vec{c}] = 0$

Now $[\vec{a} + \vec{b}, \vec{b} + \vec{c},\ \vec{c} + \vec{a}] = (\vec{a} + \vec{b})\,.(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})$

$= (\vec{a} + \vec{b})\,.[(\vec{b} + \vec{c}) \times \vec{c} + (\vec{b} + \vec{c}) \times \vec{a}]$

$= (\vec{a} + \vec{b})\,.[\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}]$ (as $\vec{c} \times \vec{c} = \vec{0}$)

$= \vec{a}.(\vec{b} \times \vec{c}) + \vec{a}\,.(\vec{b} \times \vec{a}) + \vec{a}.(\vec{c} \times \vec{a}) + \vec{b}.(\vec{b} \times \vec{c}) + \vec{b}.\,(\vec{b} \times \vec{a}) + \vec{b}.\,(\vec{c} \times \vec{a})$

$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{a}] + [\vec{a}, \vec{c}, \vec{a}] + [\vec{b}, \vec{b}, \vec{c}] + [\vec{b}, \vec{b}, \vec{a}] + [\vec{b}, \vec{c}, \vec{a}]$

$= 2\,[\vec{a}, \vec{b}, \vec{c}]$ (using property (4))

$= 0$

$\therefore\ \vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar

Vector Triple Product

Definition : Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ any three vectors. Then, the vectors $\vec{a} \times (\vec{b} \times \vec{c})$ and $(\vec{a} \times \vec{b}) \times \vec{c}$ are called vector triple products.

It is clear that, in general
$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$.

Note that $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to both $\vec{a}$ and $\vec{b} \times \vec{c}$. And also $\vec{b} \times \vec{c}$ is perpendicular to both $\vec{b}$ and $\vec{c}$. Thus $\vec{a} \times (\vec{b} \times \vec{c})$ lies in a plane containing the vectors $\vec{b}$ and $\vec{c}$, i.e., $\vec{a} \times (\vec{b} \times \vec{c})$, $\vec{b}$ and $\vec{c}$ are coplanar vectors.

We now show that for any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, we have
$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Let $\vec{a} = a_1\hat{\imath} + a_2\,\hat{\jmath} + a_3\hat{k}$
$\vec{b} = b_1\,\hat{\imath} + b_2\,\hat{\jmath} + b_3\hat{k}$
and $\vec{c} = c_1\hat{\imath} + c_2\,\hat{\jmath} + c_3\hat{k}$. Then

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (b_2c_3 - b_3c_2)\,\hat{\imath} + (b_3c_1 - b_1c_3)\hat{\jmath} + (b_1c_2 - b_2c_1)\,\hat{k}$$

$$\therefore\ \vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

$= [a_2\,(b_1c_2 - b_2c_1) - a_3\,(b_3c_1 - b_1c_3)\,]\hat{\imath} + [a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1)]\,\hat{\jmath}$
$+ [a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)]\,\hat{k}$

Also, $(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\,\vec{c}$ (1)

$= (a_1c_1 + a_2c_2 + a_3c_3)\,(b_1\,\hat{\imath} + b_2\,\hat{\jmath} + b_3\hat{k}\,) - (a_1b_1 + a_2b_2 + a_3b_3)$
$(c_1\hat{\imath} + c_2\,\hat{\jmath} + c_3\hat{k})$

$= [b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_2(a_1b_1 + a_2b_2 + a_3b_3)\,]\,\hat{\imath}$
$+ [b_2(a_1c_1 + a_2c_2 + a_3c_3) - c_2\,(a_1b_1 + a_2b_2 + a_3b_3)\,]\,\hat{\jmath}$
$+ [b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3\,(a_1b_1 + a_2b_2 + a_3b_3)\,]\,\hat{k}$

$= [a_2(b_1c_2 - b_2c_1) - a_3(b_3\,c_1 - b_1c_3)]\,\hat{\imath}$
$+ [a_3(b_2c_3 - b_3c_2) - a_1(b_1\,c_2 - b_2c_1)]\,j$
$+ [a_1(b_3c_1 + b_1c_3) - a_2(b_2\,c_3 - b_3c_2)]\,\hat{k}$ (2)

From (1) and (2), we have

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c}) - (\vec{a}.\vec{b})\vec{c}$

It is also clear from this expression that $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector in the plane of $\vec{b}$ and $\vec{c}$.

Now $\vec{a} \times \vec{b} \times \vec{c} = -\vec{c} \times (\vec{a} \times \vec{b})$

$$= -[(\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}]$$

$$= (\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\,\vec{a}$$

$$= (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\,\vec{a}$$

So, $(\vec{a} \times \vec{b}) \times \vec{c}$ is a vector in the plane of $\vec{b}$ and $\vec{a}$.

Theorem : Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors. Then $\vec{a} \times (\vec{b} \times \vec{c}) = \left(\vec{a} \times \vec{b}\right) \times \vec{c}$ if and only if $\vec{a}$ and $\vec{c}$ are collinear.

Proof : First suppose $\vec{a} \times (\vec{b} \times \vec{c}) = \left(\vec{a} \times \vec{b}\right) \times \vec{c}$

Now, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - \left(\vec{a}.\vec{b}\right)\vec{c}$

and $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{c})\vec{a}$

$So,\ -(\vec{a}.\vec{b})\vec{c} = -(\vec{b}.\vec{c})\vec{a}$

$\Rightarrow \vec{c}$ and $\vec{a}$ are collinear vectors.

Conversely, suppose that $\vec{a}$ and $\vec{c}$ are collinear vectors.
Then there exist a scalar λ such that

$\vec{c} = \lambda\,\vec{a}$. Then

$-(\vec{a}.\vec{b})\vec{c} = -\left(\vec{a}.\vec{b}\right)(\lambda\,\vec{a}) = -\lambda\left(\vec{a}.\vec{b}\right)\vec{a}$

and $-(\vec{b}.\vec{c})\vec{a} = -\left(\vec{b}.\lambda\,\vec{a}\right)\vec{a} = -\lambda\left(\vec{b}.\vec{a}\right)\vec{a}$

$= -\lambda\left(\vec{a}.\vec{b}\right)\vec{a}$

So, $-(\vec{a}.\vec{b})\vec{c} = -\left(\vec{b}.\vec{c}\right)\vec{a}$

$\Rightarrow (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = (\vec{a}.\vec{c})\vec{b} - \left(\vec{b}.\vec{c}\right)\vec{a}$

i.e., $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$

Example 16 : Show that

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times \left(\vec{a} \times \vec{b}\right) = \vec{0}$$

Solution : $\vec{a} \times \left(\vec{b} \times \vec{c}\right) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times \left(\vec{a} \times \vec{b}\right)$

$= (\vec{a}.\vec{c})\overrightarrow{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a} + (\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}$

$= \vec{0}$ since dot product is commutative.

Example 17 : Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any four vectors. Then prove that

(i) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$

(ii) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$

Solution : (i) Let $\vec{a} \times \vec{b} = \vec{r}$ Then

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{r} \times (\vec{c} \times \vec{d})$$

$$= (\vec{r} \cdot \vec{d})\vec{c} - (\vec{r} \cdot \vec{c})\vec{d}$$

$$= (\vec{d} \cdot \vec{r})\vec{c} - (\vec{c} \cdot \vec{r})\vec{d}$$

$$= [\vec{d} \cdot \vec{a} \times \vec{b}]\vec{c} - [\vec{c} \cdot \vec{a} \times \vec{b}]\vec{d}$$

$$= [\vec{d}, \vec{a}, \vec{b}]\vec{c} - [\vec{c}, \vec{a}, \vec{b}]\vec{d}$$

$$= [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$$

(ii) Let $\vec{c} \times \vec{d} = \vec{r}$. Then

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \times \vec{r}$$

$$= -\vec{r} \times (\vec{a}, \times \vec{b})$$

$$= -[(\vec{r} \cdot \vec{b})\vec{a} - (\vec{r} \cdot \vec{a})\vec{b}]$$

$$= (\vec{r} \cdot \vec{a})\vec{b} - (\vec{r} \cdot \vec{b})\vec{a}$$

$$= (\vec{a} \cdot \vec{r})\vec{b} - (\vec{b} \cdot \vec{r})\vec{a}$$

$$= (\vec{a} \cdot (\vec{c} \times \vec{d})\,\vec{b} - (\vec{b} \cdot (\vec{c} \times \vec{d})\vec{a}$$

$$= [\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$$

Example 18 : Prove that

$$[\vec{a} \times \vec{b},\ \vec{b} \times \vec{c},\ \vec{c} \times \vec{a}] = [\vec{a}, \vec{b}, \vec{c}]^2$$

Solution : $[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}]$

$$= (\vec{a} \times \vec{b}) \cdot [(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})]$$

$$= (\vec{a} \times \vec{b})[(\vec{b} \times \vec{c}) \cdot \vec{a})\vec{c} - (\vec{b} \times \vec{c}) \cdot \vec{c})\vec{a}]$$

$$= (\vec{a} \times \vec{b}) \cdot [(\vec{b} \times \vec{c}) \cdot \vec{a})\vec{c})] \quad (\because (\vec{b} \times \vec{c}) \cdot \vec{c} = 0)$$

$$= [(\vec{b} \times \vec{c}) \cdot \vec{a})][(\vec{a} \times \vec{b}) \cdot \vec{c}] = [\vec{a} \cdot (\vec{b} \times \vec{c})][\vec{a} \cdot (\vec{b} \times \vec{c})]$$

$$= [\vec{a}, \vec{b}, \vec{c}]^2.$$

Example 19 : For vectors $\vec{a} = \hat{\imath} - 2\,\hat{\jmath} + \hat{k}$,

$\vec{b} = 2\,\hat{\imath} + \hat{\jmath} + \hat{k}$

and $\vec{c} = \hat{\imath} + 2\,\hat{\jmath} - \hat{k}$ verify that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Solution : We have

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$$

So, $\vec{a} \times (\vec{b} \times \vec{c})$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 1 \\ 3 & 3 & 3 \end{vmatrix}$$

$$= -9\hat{\imath} - 6\hat{\jmath} - 3\hat{k} \qquad (1)$$

Also,

$\vec{a}.\vec{c} \;= (\hat{\imath} - 2\hat{\jmath} + \hat{k}).(\hat{\imath} + 2\hat{\jmath} + \hat{k})$

$= 1 - 4 - 1 = -4$

$\therefore\ (\vec{a}.\vec{c}).\vec{b} = -4\ (2\hat{\imath} + \hat{\jmath} + \hat{k}) = -8\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$

and $\vec{a}.\vec{b} = (\hat{\imath} + 2\hat{\jmath} + \hat{k}).(2\hat{\imath} + \hat{\jmath} + \hat{k}) = 1$

$\therefore\ (\vec{a}.\vec{b})\vec{c} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$

Thus, $(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = -9\hat{\imath} - 6\hat{\jmath} - 3\hat{k}$ (2)

Hence, from (1) and (2)

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Check Your Progress – 3

1. For vectors $\vec{a} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$, $\vec{b} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}, and\ \vec{c} = 3\hat{\imath} + \hat{\jmath} + 2\hat{k}$ find $[\vec{a}, \vec{b}, \vec{c}]$.
2. Find the volume of the parallelepiped whose edges are represented by

   $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} - \hat{k}, \vec{c} = 3\hat{\imath} - \hat{\imath} + 2\hat{k}$
3. Show that the four points having position vectors

   $\hat{\imath} + \hat{\jmath} + 2\hat{k}$, $6\hat{\imath} + 11\hat{\jmath} + 2\hat{k}$, $\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$, $\hat{i} + \frac{1}{2}\hat{\jmath} + 4\hat{k}$, are coplanar.
4. Prove that

   $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c})$
5. For any vector $\vec{a}$ prove that

   $\hat{\imath} \times (\vec{a} \times \hat{\imath}) + \hat{\jmath} \times (\vec{a} \times \hat{\jmath}) + \hat{k} \times (\vec{a} \times \hat{k}) = 2\vec{a}$
6. Prove that

   $\vec{a} \times [\vec{a} \times (\vec{a} \times \vec{b})] = (\vec{a}.\vec{a})\ (\vec{b} \times \vec{a})$

2.5 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. Here, $\vec{a} + \vec{b} = 6\hat{\imath} + 2\hat{\jmath} - 8\hat{k}$ and

   $\vec{a} - \vec{b} = 4\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$

   So, $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = (6\hat{\imath} + 2\hat{\jmath} - 8\hat{k}).(4\hat{\imath} - 4\hat{\jmath} + 2\hat{k})$

   $= 24 - 8 - 16 = 0$

   Hence $\vec{a} + \vec{b}$ and $\vec{a} + \vec{b}$ are perpendicular vectors.

2. (a) Here, $\vec{a}\,.\,\vec{b} = 3\,(-4) + (-1).\,0 + 2.2$

$= -12 + 4 = -8$

$|\vec{a}| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}$

$|\vec{b}| = \sqrt{(-4)^2 + 2^2} = \sqrt{20}$

$$\therefore \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-8}{\sqrt{14}\sqrt{20}} = \frac{-4}{\sqrt{70}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-4}{\sqrt{70}}\right)$$

(b) Here, $\vec{a}.\vec{b} = 1(-2) + (-1)\,2 + (-2)\,4$

$= -12$

$|\vec{a}| = \sqrt{1^2 + (-1)^2 + (-2)^2} = \sqrt{6}$

$|\vec{b}| = \sqrt{(-2)^2 + 2^2 + 4^2} = \sqrt{24}$

$$\therefore \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-12}{\sqrt{6}\sqrt{24}} = -1$$

$\Rightarrow \theta = \pi$

3. Vector Projection of $\vec{a}$ on $\vec{b}$

$$= \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

$$= \left(\frac{12}{54}\right)\ (7\hat{\imath} + \hat{\jmath} - 2\hat{k})$$

$$= \frac{14}{9}\hat{\imath} + \frac{2}{9}\hat{\jmath} - \frac{4}{9}\hat{k}$$

Scalar project of $\vec{b}$ on $\vec{a}$ $= \dfrac{\vec{a}.\vec{b}}{|\vec{a}|} = \dfrac{12}{\sqrt{38}}$

4. The inequality holds trivially if $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c}) = \vec{0},\ \vec{b} = 0$

So, let $|\vec{a}| \neq 0$ or $|\vec{b}| \neq 0$

Now, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\therefore = \frac{|\vec{a}.\vec{b}|}{|\vec{a}||\vec{b}|} = |\cos\theta| \leq 1$$

Hence $|\vec{a}.\vec{b}| \leq |\vec{a}||\vec{b}|$

5. $|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}).(\vec{a} - \vec{b})$

$= \vec{a}.\vec{a} - \vec{a}.\vec{b} - \vec{b}.\vec{a} - \vec{b}.\vec{b}$

$= |\vec{a}|^2 - 2\vec{a}.\vec{b} + |\vec{b}|^2$

($\because$ dot product is commulative)

Check Your Progress – 2

1. Here $\vec{a}+\vec{b} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$

$\vec{a}-\vec{b} = 3\hat{\jmath} - 6\hat{k}$

Let $\vec{c} = (\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})$. Then $\vec{c}$ is vector perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

Now $\vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -2 \\ 0 & 3 & -6 \end{vmatrix}$

$= (-6+6)\,\hat{\imath} + (-12+0)\hat{\jmath} + (6+0)\,\hat{k}$

$= 12\hat{\jmath} + 6\hat{k}$

A unit vector in the direction of $\vec{c}$ is

$$\hat{c} = \frac{1}{|\vec{c}|}\,\vec{c} = \frac{1}{\sqrt{12^2+6^2}}(12\hat{\jmath} + 6\,\hat{k})$$

$$= \frac{1}{6\sqrt{5}}\,(12\hat{\jmath} + 6\,\hat{k})$$

$$= \frac{2}{\sqrt{5}}\hat{\jmath} + \frac{1}{\sqrt{5}}\hat{k}$$

So, $\vec{c}$ is a unit vector perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

2. $\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b})$

$= \vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a} + \vec{c}\times\vec{b}$ (using distributivity)

$= \vec{a}\times\vec{b} - \vec{c}\times\vec{a} + \vec{b}\times\vec{c} - \vec{a}\times\vec{b} + \vec{c}\times\vec{a} - \vec{b}\times\vec{c}$ $(\because \vec{b}\times\vec{a} = -\vec{a}\times\vec{b})$

$\vec{a}\times\vec{c} = -\vec{c}\times\vec{a}$ and $\vec{c}\times\vec{b} = -\vec{b}\times\vec{c} = 0$

3. $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 1 \\ 2 & -1 & 1 \end{vmatrix} = -\hat{\imath} + \hat{\jmath} + 3\hat{k}$

$\therefore (\vec{a}\times\vec{b})\times\vec{c} = \begin{vmatrix} i & \hat{\jmath} & \hat{k} \\ -1 & 1 & 3 \\ 1 & 1 & -2 \end{vmatrix} = -5\,\hat{\imath} + \hat{\jmath} - 2\hat{k}$

Also,

$(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix} = \hat{\imath} + 5\hat{\jmath} + 3\hat{k}$

So, $\vec{a}\times(\vec{b}\times\vec{c})$ $\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 1 \\ 1 & 5 & 3 \end{vmatrix} = -11\hat{\imath} - 2\,\hat{\jmath} + 7\hat{k}$

Clearly,

$(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\,(\vec{b}\times\vec{c})$

4. $(\vec{a} + \vec{b} + \vec{c}) = \vec{0}$

$\Rightarrow \vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0} = \vec{0}$

i.e., $\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$

i.e., $\vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$ $\quad (\because \vec{a} \times \vec{a} = \vec{0})$

i.e., $\vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$

i.e., $\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ ……………..(1)

Similarly,

$\vec{b} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{b} \times \vec{0} = \vec{0}$

i.e., $\vec{b} \times \vec{a} + \vec{b} \times \vec{c} = \vec{0}$

i.e., $\vec{b} \times \vec{c} = -\vec{b} \times \vec{a}$

i.e., $\vec{b} \times \vec{c} = \vec{a} \times \vec{b}$ ……………. (2)

From (1) and (2), we have

$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

5. Let ABCD be the parallelogram

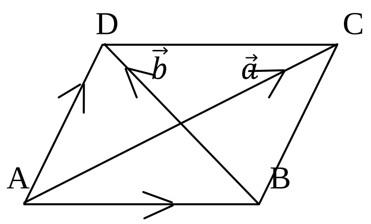


Figure 7

Now, $\overrightarrow{AB} = \frac{1}{2}\vec{a} - \frac{1}{2}\vec{b}$

$= \frac{1}{2}(3\hat{\imath} + \hat{\jmath} - 2\hat{k}) - \frac{1}{2}(\hat{\imath} - 3\hat{\jmath} + 4\hat{k})$

$= \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$

and, $\overrightarrow{AD} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$

$= \frac{1}{2}(3\hat{\imath} + \hat{\jmath} - 2\hat{k}) + \frac{1}{2}(\hat{\imath} - 3\hat{\jmath} + 4\hat{k})$

$= 2\hat{\imath} - \hat{\jmath} + \hat{k}$

So area of parallelogram ABCD

$= |\overrightarrow{AB} \times \overrightarrow{AD}|$

Now, $|\overrightarrow{AB} \times \overrightarrow{AD}| = \begin{vmatrix} \hat{\imath} & \hat{\imath} & \hat{k} \\ 1 & 2 & -3 \\ 2 & -1 & 1 \end{vmatrix}$

$= (2 - 3)\,\hat{\imath} - (1+ 6)\,\hat{\jmath} \;\; + (-1 - 4)\,\hat{k}$

$= -1\hat{\imath} - 7\,\hat{\jmath} \; - 5\,\hat{k}$

$\therefore |\;\overrightarrow{AB} \times \overrightarrow{\mathrm{AD}}\;| \;=\; \sqrt{(-1)^2 + (-7)^2 + (-5)^2}$

$= 5\sqrt{3}$

Hence, the area of parallelogram ABCD = $5\sqrt{3}$

Check Your Progress – 3

1. $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 3\,\hat{\imath} + 5\hat{\jmath} \;\; - 7\,\hat{k}$

$\therefore \quad \vec{a}.(\vec{b} \times \vec{c}) \quad = (2\hat{\imath} + \hat{\jmath} + 3\hat{k}).\,(3\hat{\imath} + 5\hat{\jmath}\; 3 - 7\hat{k})$

$= 6 + 5 - 21$

$= -10$

2. Volume of Parallelopiped = $|\vec{a}.(\vec{b} \times \vec{c})|$

Now, $\vec{a}.(\vec{b} \times \vec{c}) = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$

$= -7$

$\therefore \quad$ Volume $= |-7| = 7$

3. Let $\overrightarrow{OA} \;\; = \hat{\imath} - \hat{\jmath} + 2\,\hat{k}$

$\overrightarrow{OB} \;\; = \;\; 6\hat{\imath} + 11\,\hat{\jmath} + 2\,\hat{k}$

$\overrightarrow{OC} \;\; = \;\hat{\imath} + 2\hat{\jmath} + \; 6\,\hat{k}$

$\overrightarrow{OD} \;=\; \hat{\imath} + \dfrac{1}{2}\,\hat{\jmath} \;+\; 4\,\hat{k}$

So, $\overrightarrow{AB} \;= \overrightarrow{OB} \;- \overrightarrow{OA} \;\; = 5\hat{\imath} + 12\hat{\jmath}$

$\overrightarrow{BC} \;=\; \overrightarrow{OC} \;\; - \overrightarrow{OB} \;\; = -5\hat{\imath} - 9\hat{\jmath} \;\; + 4\,\hat{k}$

$\overrightarrow{OA} \;=\; \overrightarrow{OD} - \overrightarrow{OC} \qquad = -\dfrac{3}{2}\hat{\jmath} - 2\,\hat{k}$

Now, $\overrightarrow{AB} \cdot (\overrightarrow{BC} \times \overrightarrow{CD}) = \begin{vmatrix} 5 & 12 & 0 \\ -5 & -9 & 4 \\ 0 & -\frac{3}{2} & -2 \end{vmatrix} = 0$

$\therefore \overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{CD}$ are coplanar.

Hence A, B, C and D are coplanar.

4. Let $\vec{a} \times \vec{r} = \vec{r}$ Then

$(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = \vec{r}.(\vec{c} \times \vec{d})$

$= (\vec{r} \times \vec{c}).\vec{d}$

$= ((\vec{a} \times \vec{b}) \times \vec{c}).\vec{d}$

$= [(\vec{c}.\vec{a})\,\vec{b} - (\vec{c}.\vec{a})\vec{a}].\vec{d}$

$= (\vec{c}.\vec{a})\,(\vec{b}.\vec{d}) - (\vec{c}.\vec{b})(\vec{a}.\vec{d})$

$= (\vec{a}.\vec{c})\,(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d})$

5. L.H.S. $= \hat{\imath} \times (\vec{a} \times \hat{\imath}) + \hat{\jmath} \times (\vec{a} \times \hat{\jmath}) + \hat{k} \times (\vec{a} \times \hat{k})$

$= [(\hat{\imath}.\hat{\imath})\vec{a} - (\hat{\imath}.\vec{a})\hat{i}] + [(\hat{\jmath}.\hat{\jmath})\vec{a} - (\hat{\jmath}.\vec{a})\hat{\jmath}] + [(\hat{k}.\hat{k})\vec{a} - (\hat{k}.\vec{a})\hat{k}]$

$= \vec{a} - (\hat{\imath}.\vec{a})\hat{\imath} + \vec{a} - (\hat{\jmath}.\vec{a})\hat{\jmath} + \vec{a} - (\hat{k}.\vec{a})\hat{k}$

$= 3\vec{a} - [(\hat{\imath}.\vec{a})\hat{i} + (\hat{\jmath}.\vec{a})\hat{\jmath} + (\hat{k}.\vec{a})\hat{k}]$

Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

So, $\hat{\imath}.\vec{a} = \hat{\imath}.(a_1\hat{\imath} - a_2\hat{\jmath} + a_3\hat{k}) = a_1$

Similarly, $\hat{\jmath}.\vec{a} = a_2$, $\hat{k}.\vec{a} = a_3$

$\therefore$ L.H.S. $= 3\vec{a} - (a_1\hat{\imath} - a_2\hat{\jmath} + a_3\hat{k})$

$= 3\vec{a} - \vec{a} = 2\vec{a}$

6. $\vec{a} \times (\vec{a} \times (\vec{a} \times \vec{b}))$

$= \vec{a} \times [(\vec{a}.\vec{b})\vec{a} - (\vec{a}.\vec{a})\vec{b}]$

$= (\vec{a}.\vec{b})\,(\vec{a} \times \vec{a}) - (\vec{a}.\vec{a})\,(\vec{a} \times \vec{b})$

$= -(\vec{a}.\vec{a})\,(\vec{a}.\vec{b})$ $\quad (\because \vec{a} \times \vec{a} = \vec{0})$

$= (\vec{a}.\vec{a})\,(\vec{b} \times \vec{a})$ $\qquad (\because -\vec{a} \times \vec{b} = \vec{b} \times \vec{a})$

2.6 SUMMARY

This unit discusses various operations on vectors. The binary operation of scalar product is discussed in **section 2.2**. In the next section, the binary operation of vector product (also, called cross product) is illustrated. Finally in **section 2.4**, the ternary operation of tripe product of vectors is explained.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 2.5**.