CONFORMAL PREDICTION

IA Frameworks

Joseba Dalmau

Uncertainty Quantification In Deep Learning

- Can we trust the predictions of our models ?
- Under what circumstances ?
- To what extent ?

Classical UQ Techniques

- Bayesian methods
- Ensembling
- Dropout
- ...

Non Post-hoc

Without Guarantees

Conformal Prediction:

Objective and Guarantee

Given: a predictor $\hat{f} : \mathcal{X} \longrightarrow \mathcal{Y}$
and a nominal error rate α.

Build: $\hat{C}_\alpha : \mathcal{X} \longrightarrow \mathcal{P}(\mathcal{Y})$

With the following guarantee:

$$\mathbb{P}\left(Y_{test} \in \hat{C}_\alpha(X_{test})\right) \geq 1-\alpha$$

Outline

→ Conformal Regression

- Theorem
- Jackknife+ / CV+
- CQR

→ Conformal Classification

- LAC
- APS / RAPS

Conformal Regression

Given: $\hat{f} : \mathcal{X} \to \mathcal{Y}$ and α.

Build: $\hat{C}_\alpha : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$

With the following guarantee:

$$\mathbb{P}\left(Y_{test} \in \hat{C}_\alpha(X_{test})\right) \geq 1 - \alpha$$

Conformal Regression

Given: $\hat{f} : \mathcal{X} \to \mathbb{R}$ and α.

Build: $\hat{C}_\alpha : \mathcal{X} \to \mathcal{P}(\mathbb{R})$

usually an interval!

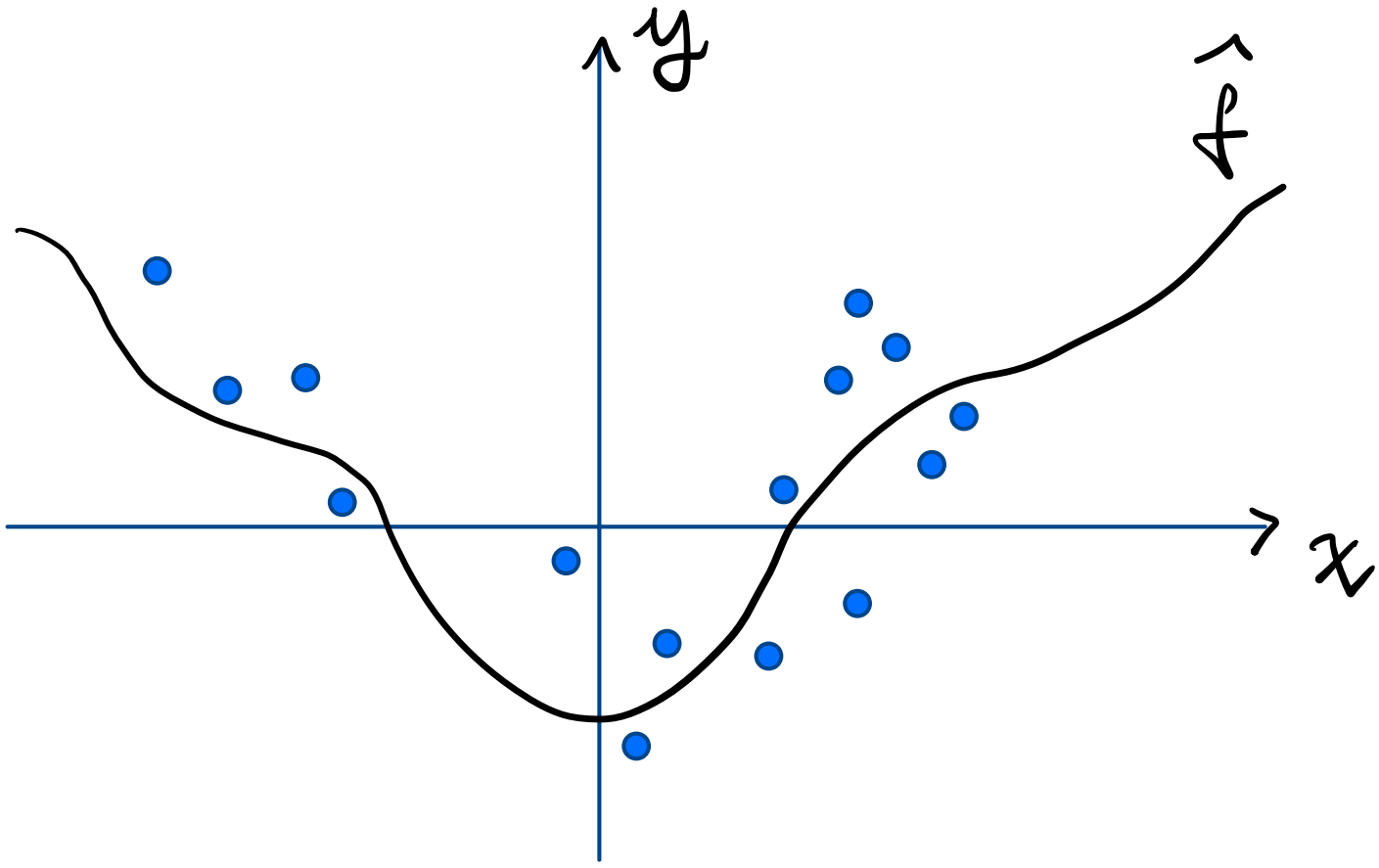
With the following guarantee:

$$\mathbb{P}\left(Y_{test} \in \hat{C}_\alpha(X_{test})\right) \geq 1 - \alpha$$

Calibration

We use a calibration set

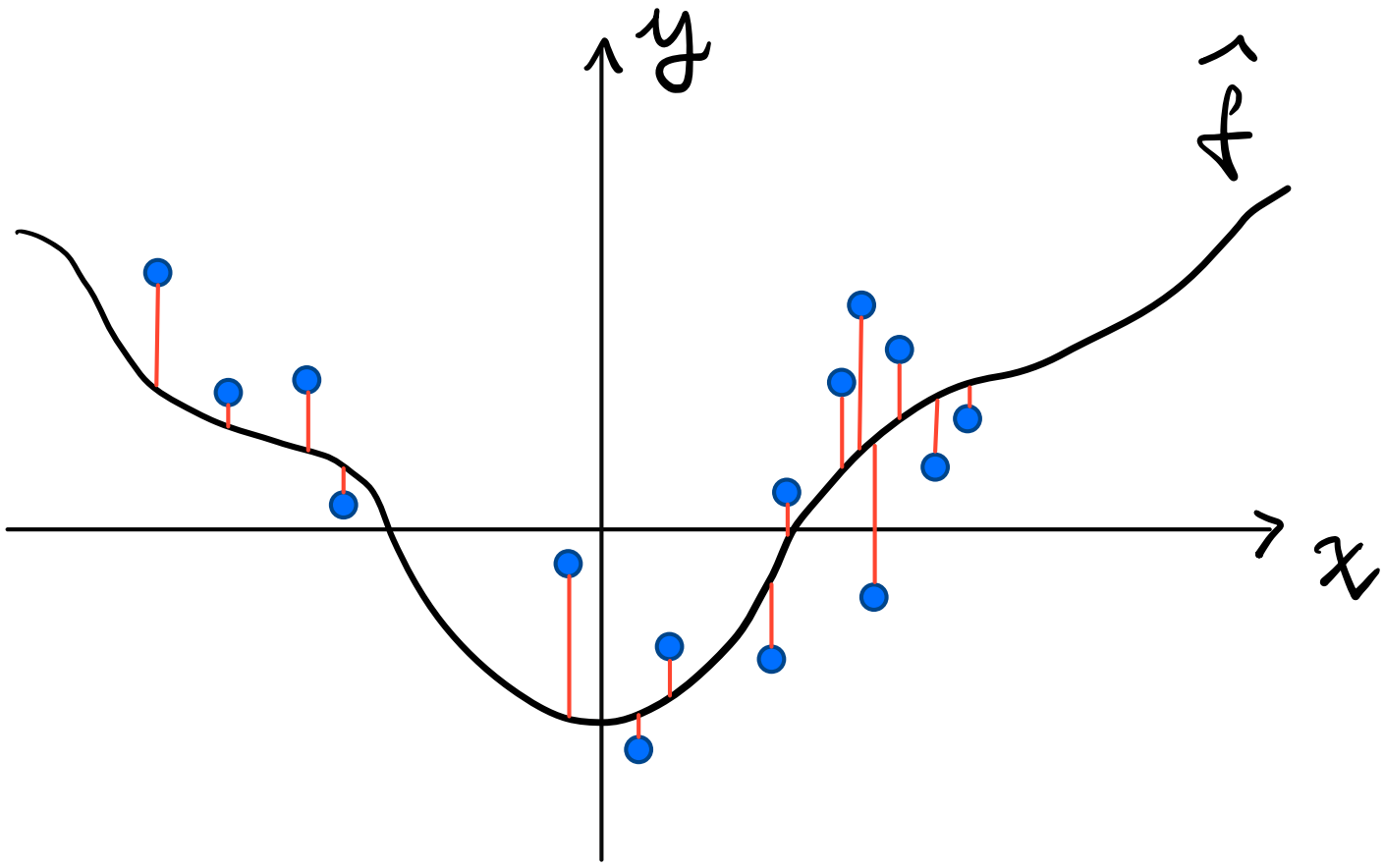
$$D_{calib} = \{(x_i, y_i)\}_{i=1}^{n}$$



CALIBRATION

We measure the scores (errors)

$$s_i = |y_i - \hat{f}(x_i)|$$



Calibration

We compute:

$\hat{s}_\alpha :=$ the $\frac{\lceil (n+1)(1-\alpha) \rceil}{n}$-th

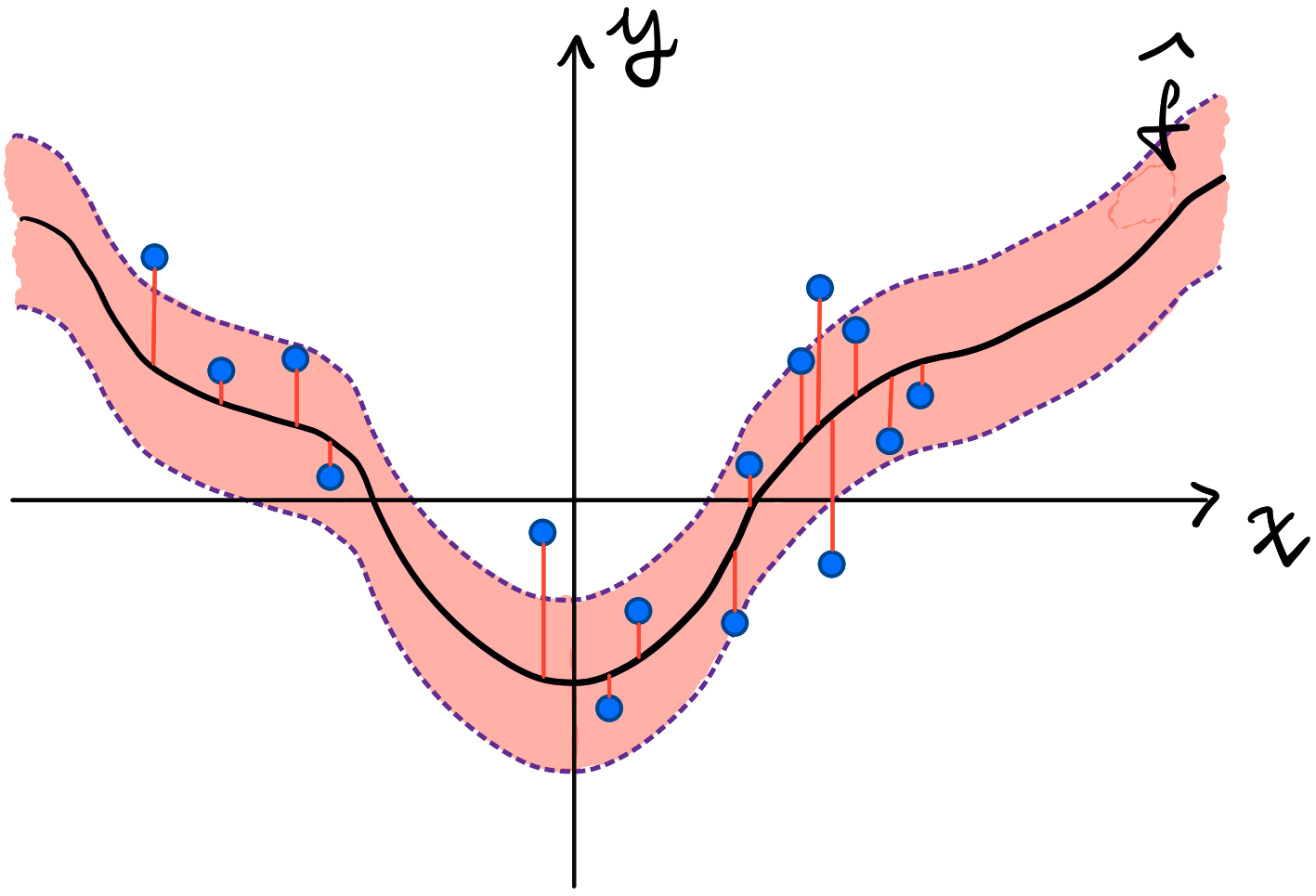
quantile of the scores $s_1, \ldots, s_n$

i.e. the $\lceil (n+1)(1-\alpha) \rceil$-th

smallest score.

CALIBRATION

We predict $\hat{C}_\alpha(x) = [\hat{f}(x) - \delta_\alpha, \hat{f}(x) + \delta_\alpha]$



Exercise

1. Write down the pseudocode of the Split Conformal Regression algorithm

2. Should we add any constraints on the nominal error rate α?

Hint: there may be a problem when choosing the "$\lceil (n+1)(1-\alpha) \rceil$-th smallest score"

GUARANTEE

Theorem.- Given a calibration set $\{(X_i, Y_i)\}_{i=1}^{n}$ and a test point (X_{n+1}, Y_{n+1}), if the scores $S_i = |Y_i - \hat{f}(X_i)|$ are exchangeable, then the interval

$$\hat{C}_\alpha(X_{n+1}) = \left[\hat{f}(X_{n+1}) - \delta_\alpha \,,\; \hat{f}(X_{n+1}) + \delta_\alpha\right]$$

satisfies:

$$\mathbb{P}\left(Y_{n+1} \in \hat{C}_\alpha(X_{n+1})\right) \geq 1-\alpha$$

EXCHANGEABILITY

The random variables $(X_1, X_2, \dots, X_n)$ are exchangeable if:

For any $\pi \in \{$ permutations of $\{1, \dots, n\}\}$

$$X_{\pi(1)}, \dots, X_{\pi(n)} \overset{\text{law}}{=} X_1, \dots, X_n$$

Exercise.- Give an example of an array of r.v.s that are exchangeable but not i.i.d.

EXCHANGEABILITY

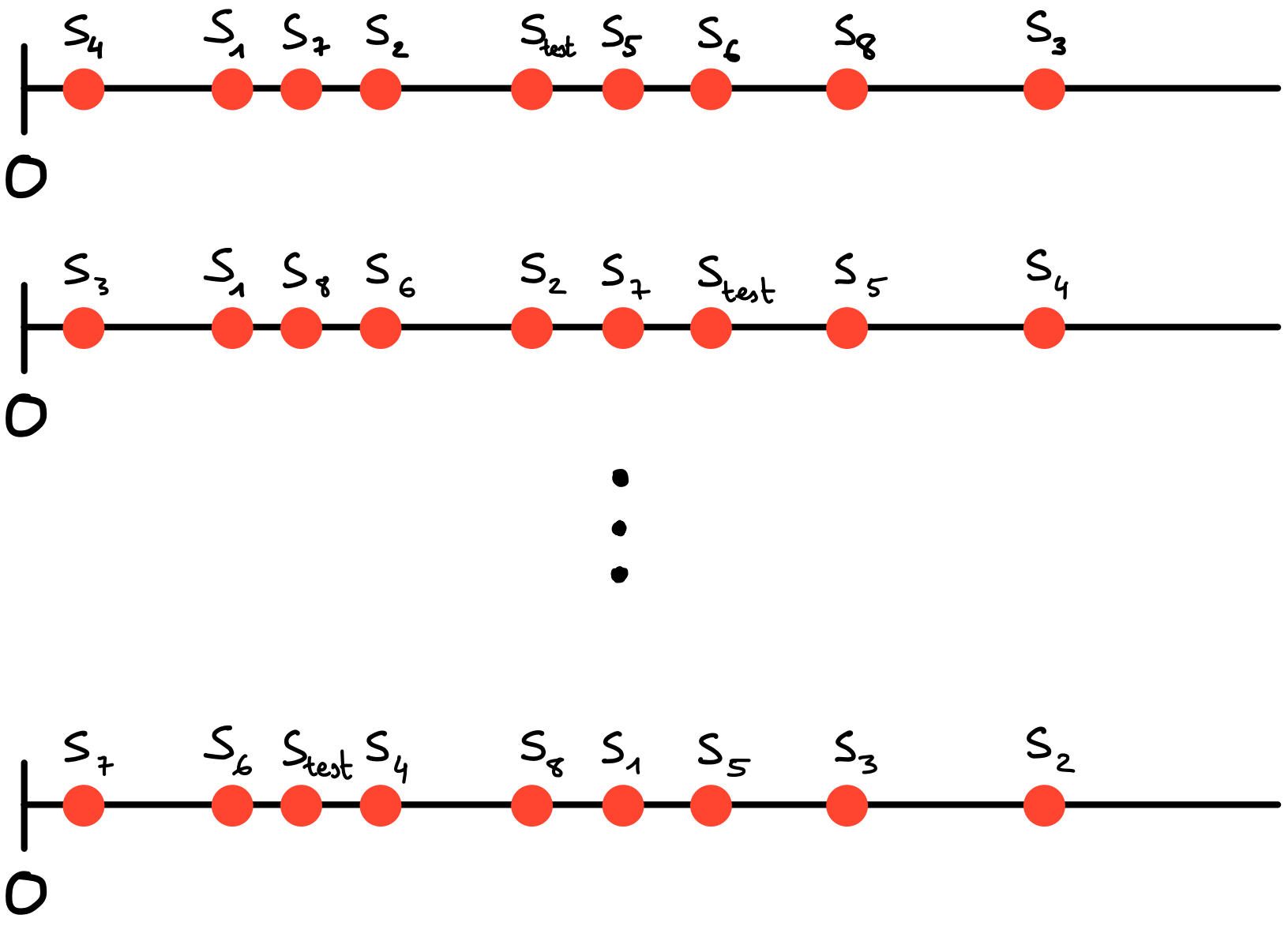
The random variables $(X_1, X_2, \dots, X_n)$ are exchangeable if:

For any $\pi \in \{$ permutations of $\{1, \dots, n\}\}$

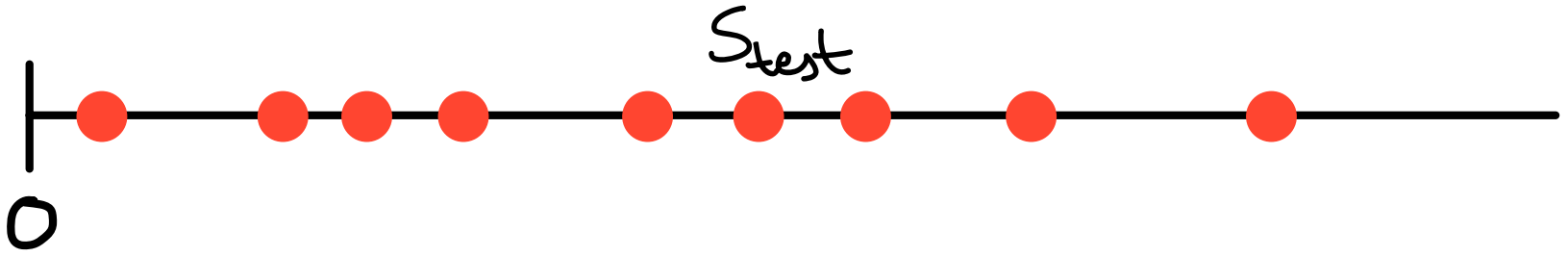
$$X_{\pi(1)}, \dots, X_{\pi(n)} \overset{\text{law}}{=} X_1, \dots, X_n$$

Example: $Z_1, \dots, Z_n$ i.i.d. $\Rightarrow X_i = \dfrac{Z_i}{\sum_{j=1}^{n} |Z_j|}$ exchangeable

Proof


s_4 s_1 s_7 s_2 s_{test} s_5 s_6 s_8 s_3
0
s_3 s_1 s_8 s_6 s_2 s_7 s_{test} s_5 s_4
0
s_7 s_6 s_{test} s_4 s_8 s_1 s_5 s_3 s_2
0

Proof



$$\mathbb{P}(\text{Rank of } S_{test} = k)$$

$$= \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

$$\Rightarrow \text{Rank of } S_{test} \sim \text{Unif}(\{1, \ldots, n+1\})$$

Proof

Rank of $S_{test} \sim \text{Unif}(\{1, \dots, n+1\})$

$$\Rightarrow \mathbb{P}\left(\begin{matrix}\text{Rank of} \\ S_{test}\end{matrix} \leq K\right) = \frac{K}{n+1}$$

We choose the smallest K s.t.

$$\frac{K}{n+1} \geq 1-\alpha \quad \text{i.e.} \quad K = \lceil (n+1)(1-\alpha) \rceil$$

Too Good To Be True?

$\hat{f}$ bad

$\Longrightarrow \hat{C}_\alpha$ very large

Advantages

- Post-hoc
- Distribution-free
- Minimal hypotheses
- Finite-sample guarantee

LIMITATIONS

Calibration-marginal guarantee:

$$\mathbb{P}\left(Y_{n+1} \in \underbrace{\hat{C}_\alpha(X_{n+1})}\right) \geq 1-\alpha$$

how is this set constructed?

LIMITATIONS

Calibration-marginal guarantee:

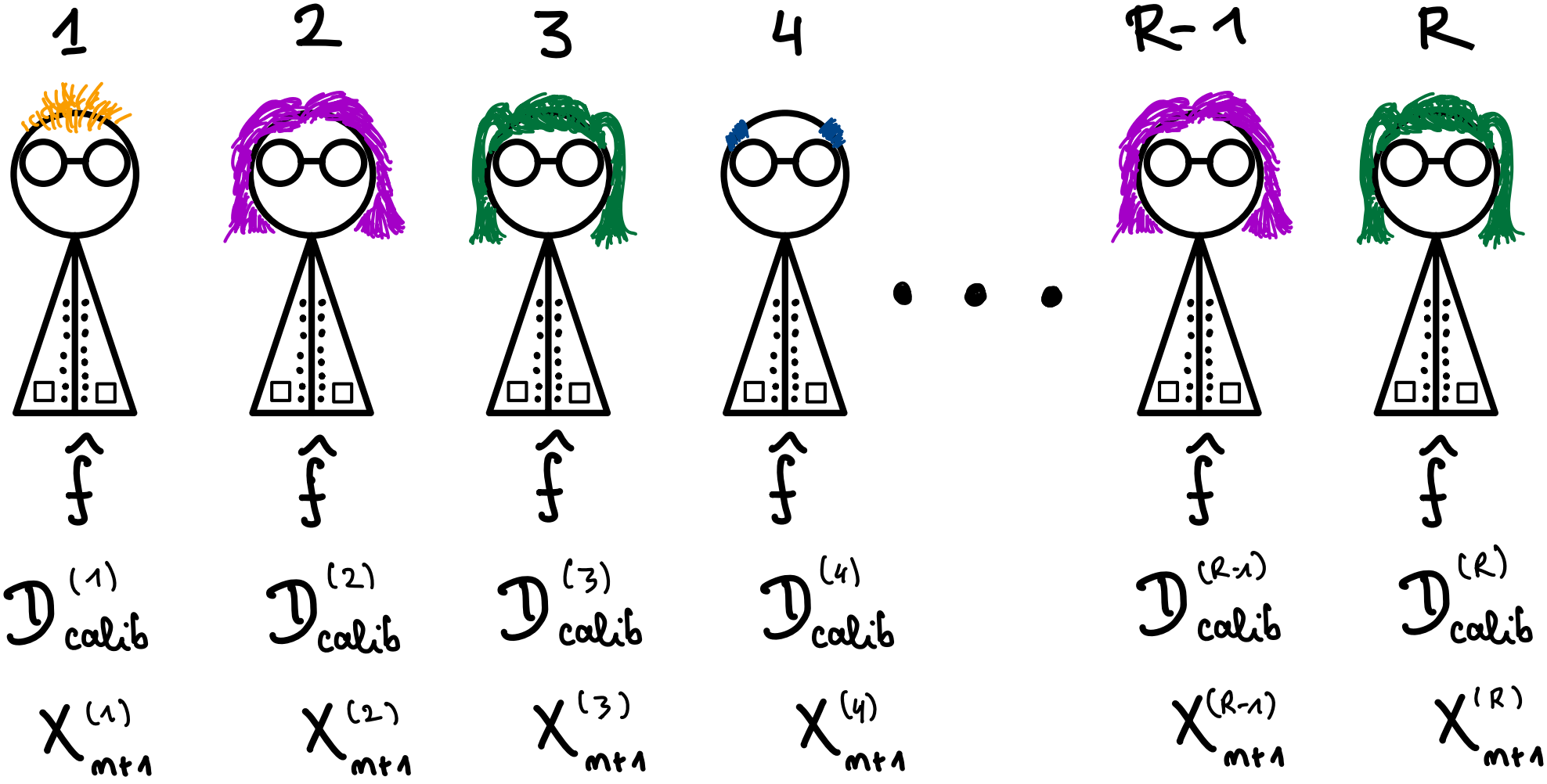
$$\mathbb{P}\left(Y_{n+1} \in \hat{C}_\alpha(X_{n+1})\right) \geq 1-\alpha$$

how is this set constructed?

$$\mathbb{P}\left(Y_{n+1} \in \hat{C}(X_{n+1}) \mid X_1, \dots, X_n\right) \geq 1-\alpha \; ?$$

LIMITATIONS

Calibration-marginal guarantee:

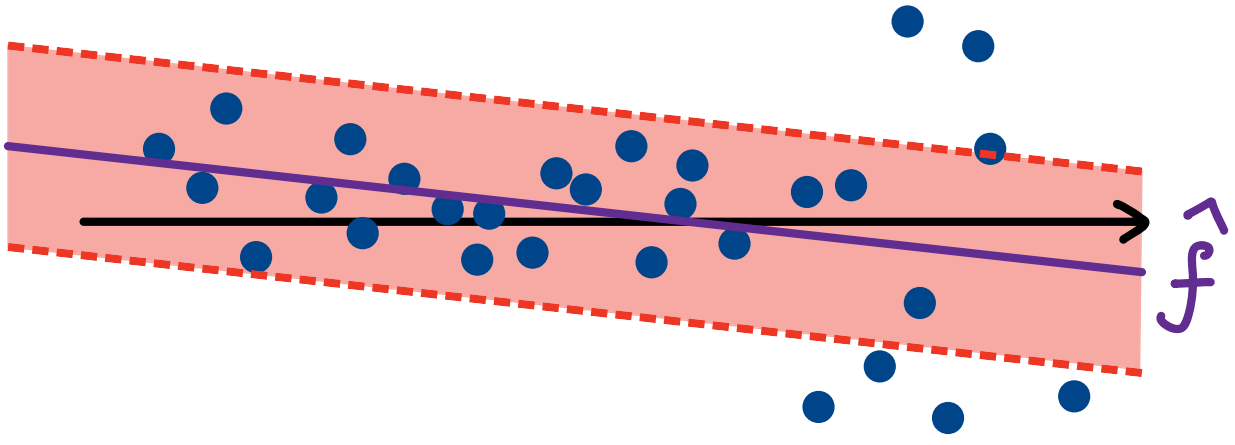


Limitations

Non-conditional guarantee

$$\mathbb{P}\left(Y_{n+1} \in \hat{C}_\alpha(X_{n+1})\right) \geq 1-\alpha$$ ✓

$$\mathbb{P}\left(Y_{n+1} \in \hat{C}_\alpha(X_{n+1}) \,\middle|\, X_{n+1} = x\right) \geq 1-\alpha$$ ✗



Split Conformal

Theorem.- Given a calibration set $\{(X_i, Y_i)\}_{i=1}^{n}$ and a test point (X_{n+1}, Y_{n+1}), if the scores $S_i = |Y_i - \hat{f}(X_i)|$ are exchangeable, then the interval

$$\hat{C}_\alpha(X_{n+1}) = \left[\hat{f}(X_{n+1}) - \delta_\alpha \,,\; \hat{f}(X_{n+1}) + \delta_\alpha\right]$$

satisfies:

$$\mathbb{P}\left(Y_{n+1} \in \hat{C}_\alpha(X_{n+1})\right) \geq 1-\alpha$$

Split Conformal

$\hat{f}$ model (obtained by optimizing on D^{train})

$D^{calib} = \{(X_i, Y_i)\}_{i=1}^{n}$ calibration data

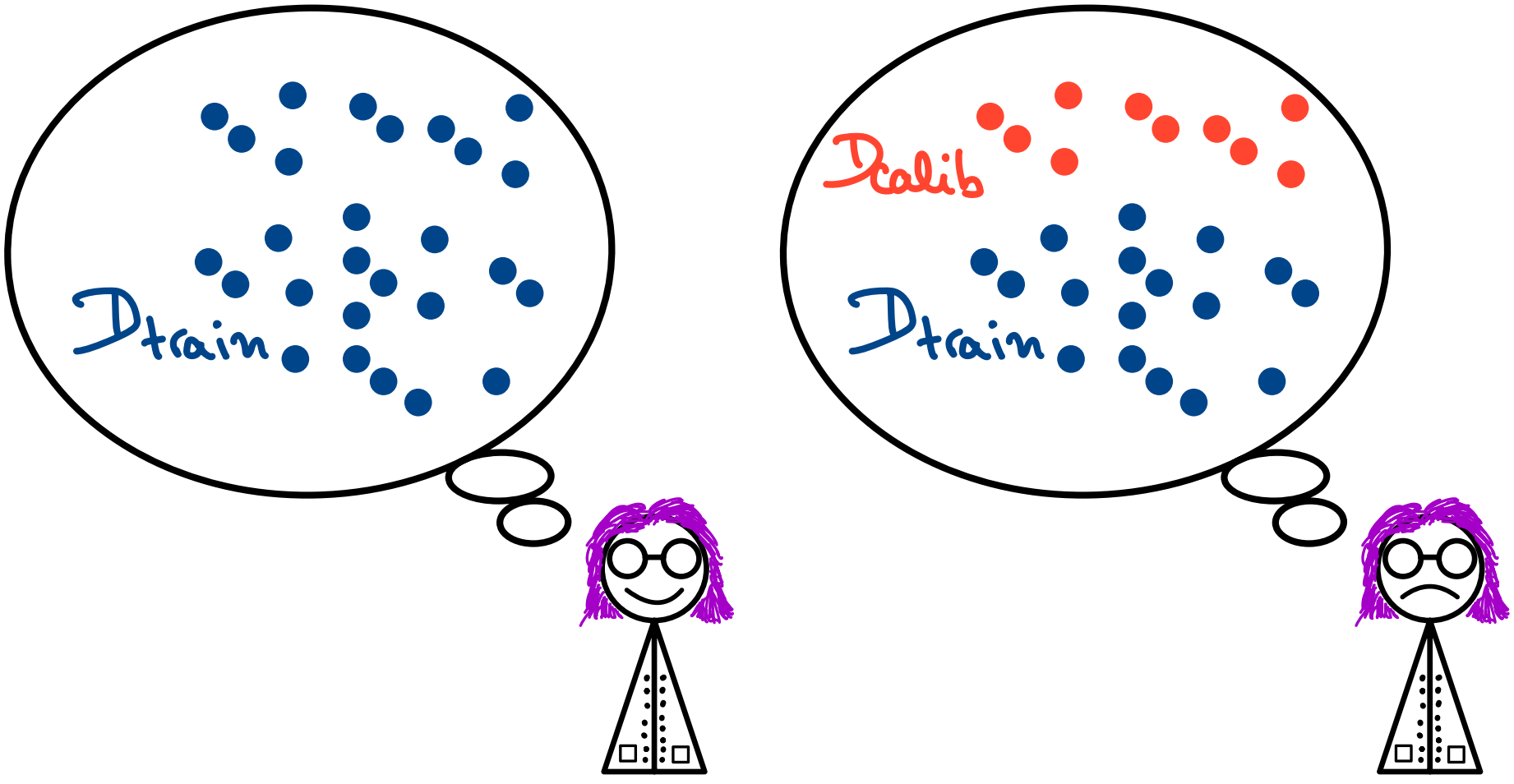
(X_{n+1}, Y_{n+1}) test point.

$S_i = |Y_i - \hat{f}(X_i)|, \quad i = 1, \ldots, n+1$

exchangeable

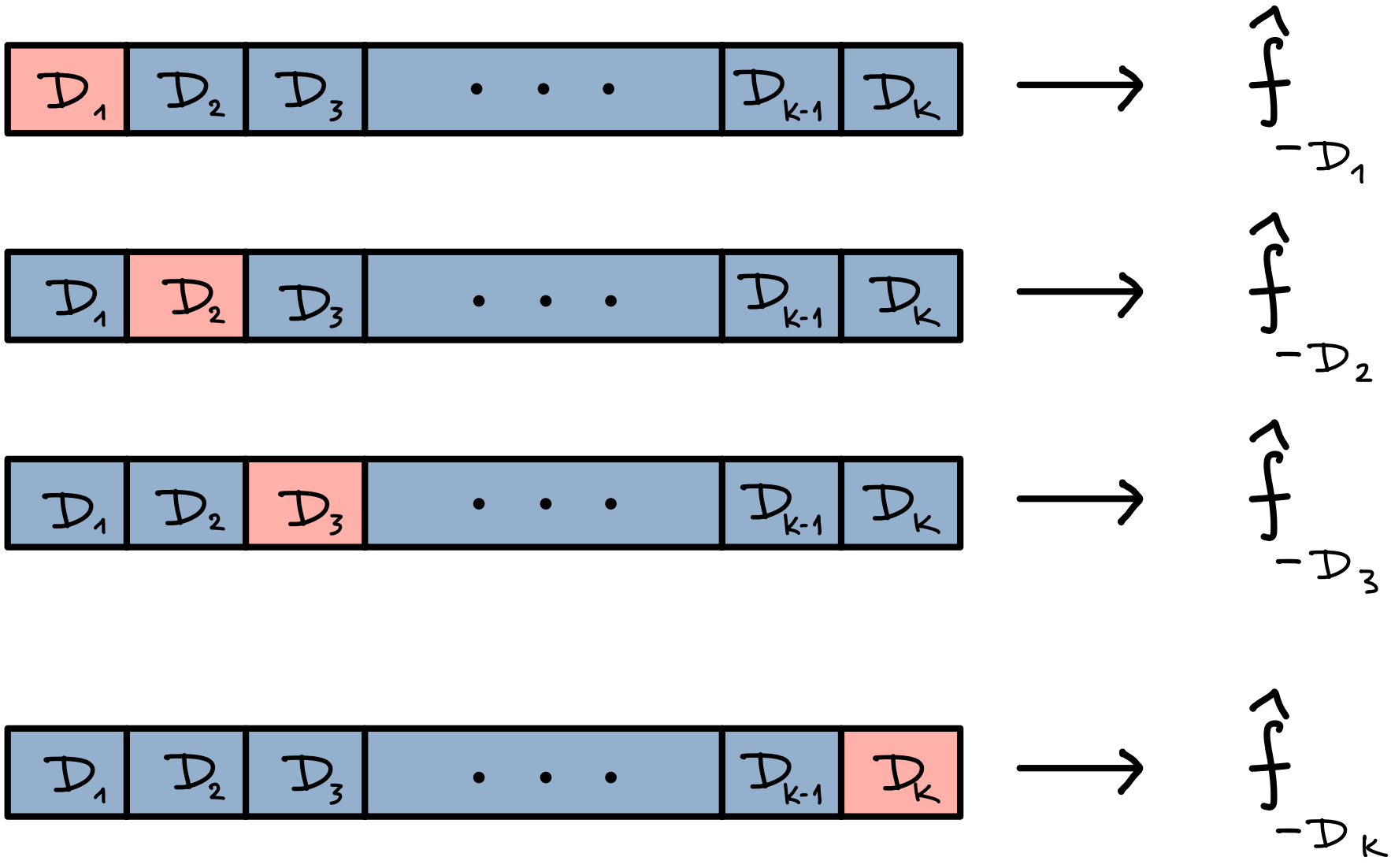
Limitations

Need for calibration data



JACKNIFE+, CROSS-VALIDATION+

Partition the data:



JACKNIFE+, CROSS-VALIDATION+

Calibration:

$$S_i^{CV} = \left| Y_i - \hat{f}_{-D_{ind(i)}}(X_i) \right|, \quad i = 1, \ldots, n$$

JACKNIFE+ , CROSS-VALIDATION+

Inference:

$\hat{\ell}_\alpha(x) := \lfloor \alpha(n+1) \rfloor$ - th smallest value of

$$\hat{f}_{-S_{ind(1)}}(x) - S^{cv}_1, \ldots, \hat{f}_{-S_{ind(n)}}(x) - S^{cv}_n$$

$\hat{u}_\alpha(x) := \lceil (1-\alpha)(n+1) \rceil$ - th smallest value of

$$\hat{f}_{-S_{ind(1)}}(x) + S^{cv}_1, \ldots, \hat{f}_{-S_{ind(n)}}(x) + S^{cv}_n$$

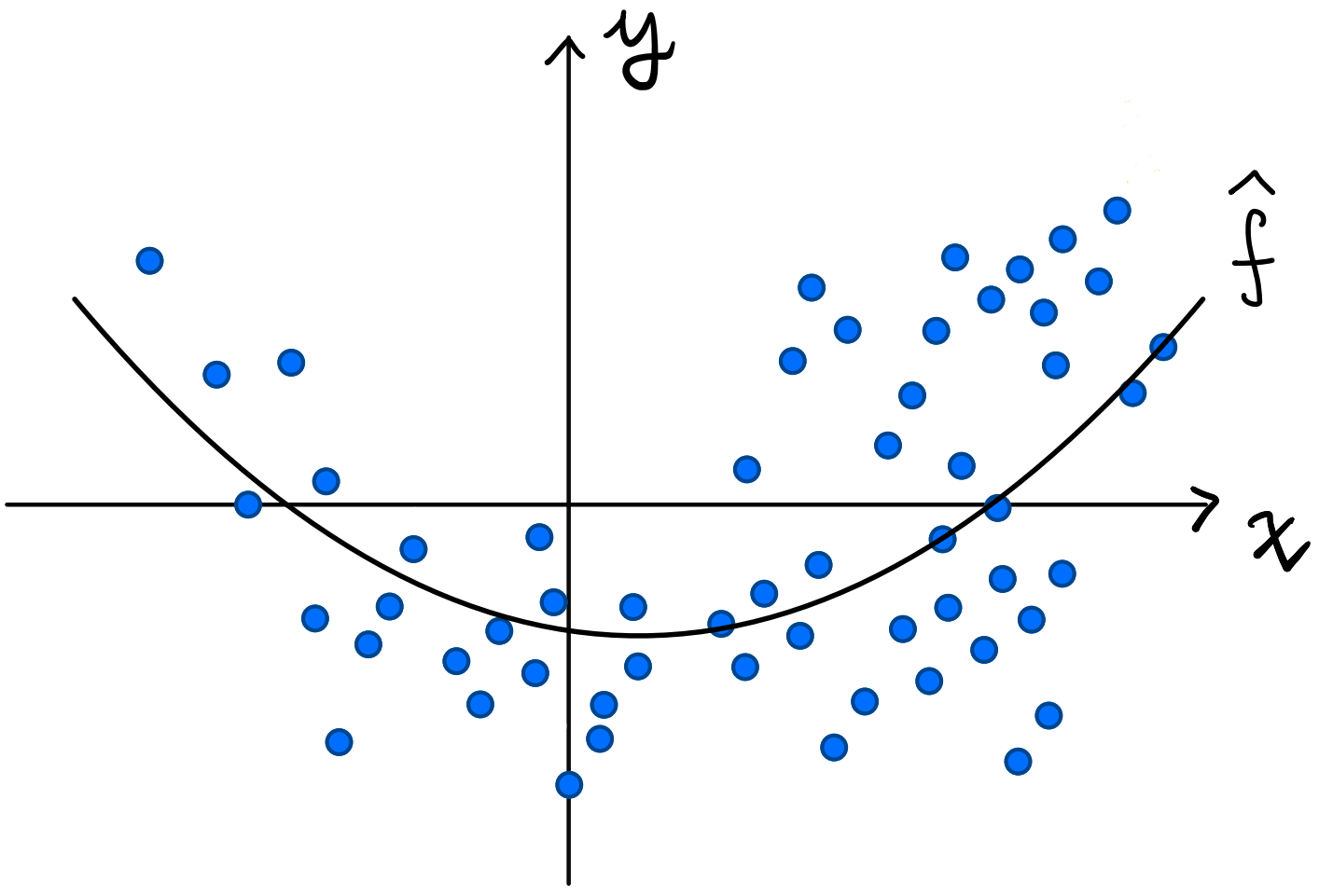
$$\hat{C}_\alpha(x) = \left[\hat{\ell}_\alpha(x), \hat{u}_\alpha(x) \right]$$

JACKNIFE+ , CROSS-VALIDATION+

Guarantee:

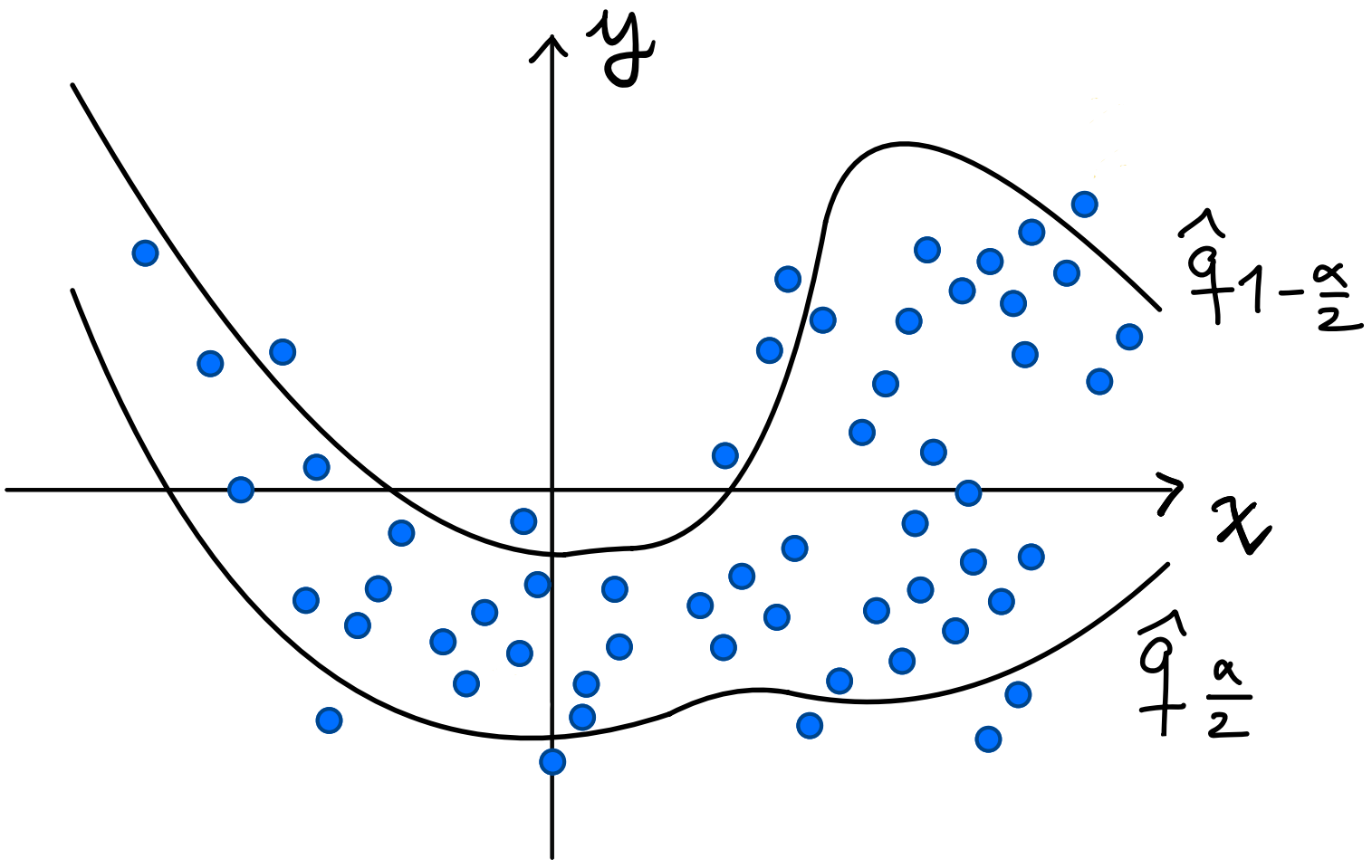
$$\mathbb{P}\left(Y_{n+1} \in \widehat{C}_\alpha(X_{n+1})\right) \geq 1-2\alpha$$

CQR: Conformal Quantile Regression



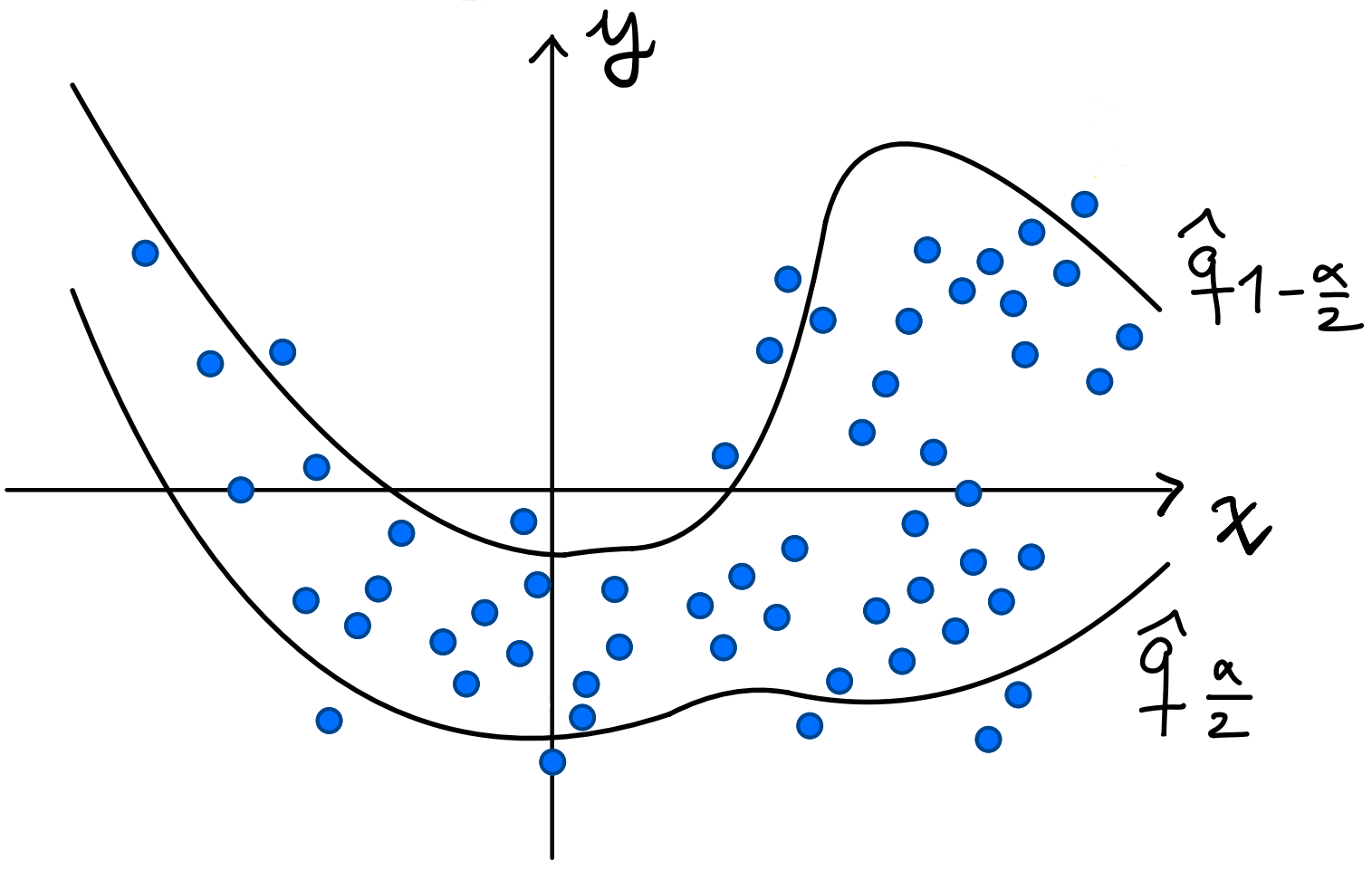
CQR

$$\hat{q}_t(x) \simeq \mathbb{P}(Y \leq t \mid X = x)$$



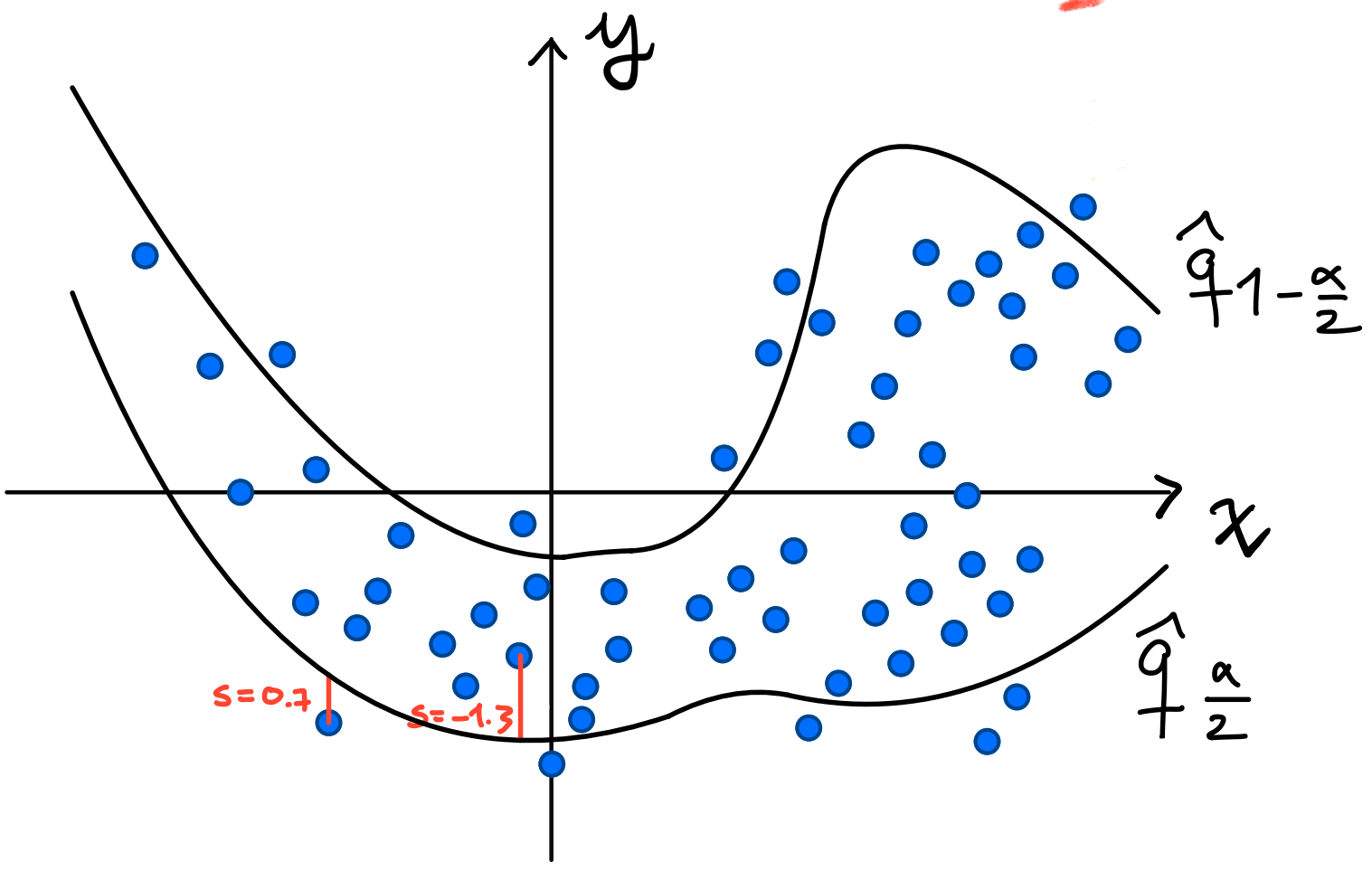
CQR

$$\mathbb{P}\left(Y \in \left[q_{\frac{\alpha}{2}}(x), q_{1-\frac{\alpha}{2}}(x)\right]\right) = ?$$



CQR: Calibration

$$s_i = \max\left\{ q_{\frac{\alpha}{2}}(x_i) - y_i,\ y_i + q_{1-\frac{\alpha}{2}}(x_i) \right\}$$



CQR: Calibration

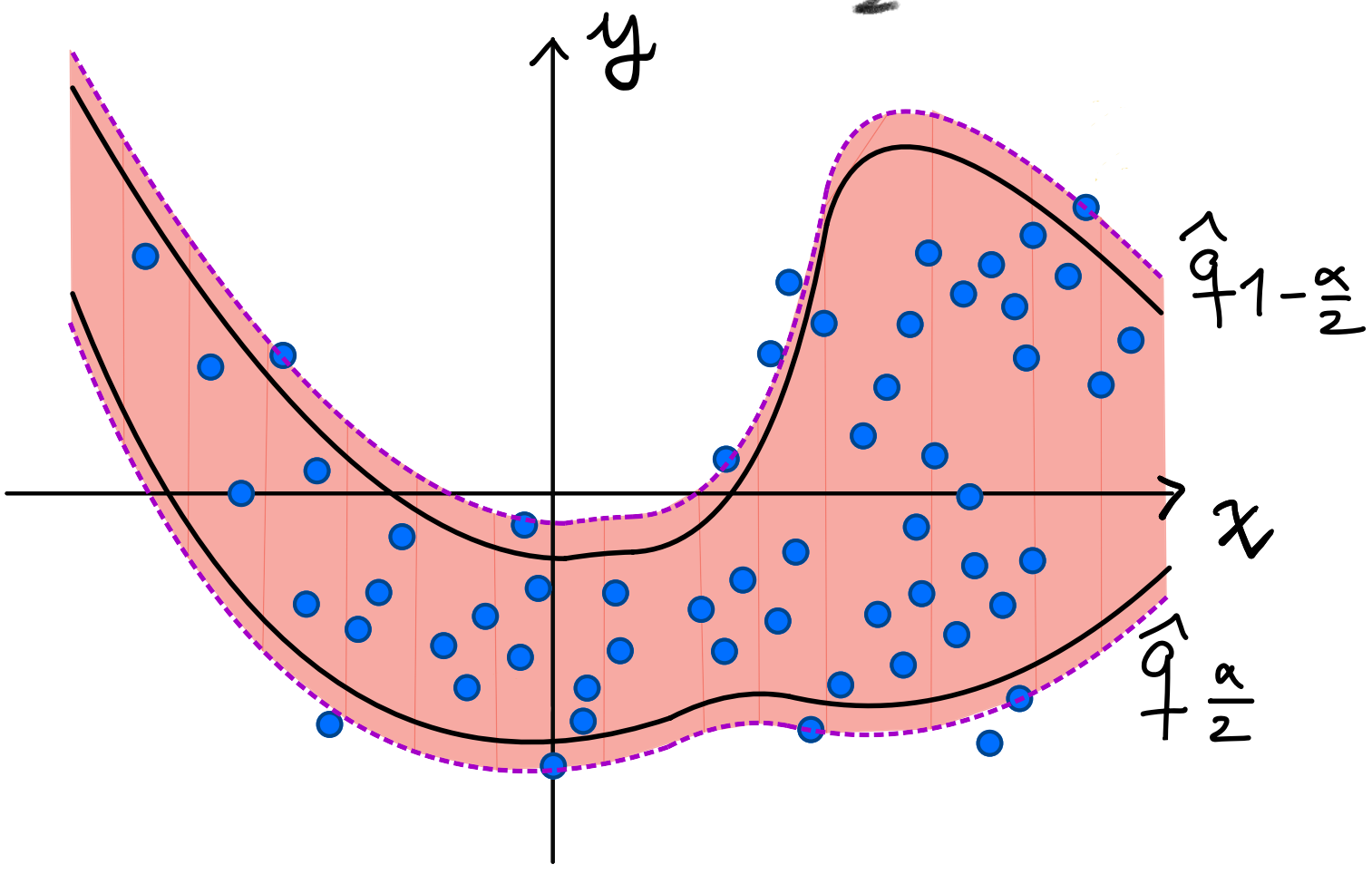
Calibration:

$$S_\alpha := \text{the } \left\lceil \frac{(n+1)(1-\alpha)}{n} \right\rceil \text{-th}$$

quantile of the scores $S_1, \ldots, S_n$

CQR : Calibration

$$C_\alpha(x) = \left[\hat{q}_{\frac{\alpha}{2}}(x) - \delta_\alpha,\ \hat{q}_{1-\frac{\alpha}{2}}(x) + \delta_\alpha\right]$$



General Formulation

Given : a predictor $\hat{f} : \mathcal{X} \longrightarrow \mathcal{Y}$

a nominal error rate α

a scoring function $s : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}^+$

e.g. in regression $s(y, y') = |y - y'|$

Split conformal : Calibration set $\{(X_i, Y_i)\}_{i=1}^n$

$\Rightarrow$ non-conformity scores $R_i = s(\hat{Y}_i, Y_i)$

$\Rightarrow$ $q_\alpha = \left\lceil \frac{(1-\alpha)(n+1)}{n} \right\rceil$-quantile of the $\{R_i\}_{i=1}^n$

$\Rightarrow$ $\mathcal{C}(x) := \{y : s(\hat{y}, y) \leq q_\alpha\}$

Conformal Classification

- Least Ambiguous Set-Valued Classifiers (LAC)
- Adaptive Prediction Sets (APS)
- Regularized Adaptive Prediction Sets (RAPS)

LAC

Given : $\hat{\pi}$ softmax classifier

α nominal error rate

Use the nonconformity scores

$$s(\hat{\pi}, y) = \underbrace{1 - \hat{\pi}_y}$$

i.e. 1 - the probability of the true label.

APS

Given : $\hat{\pi}$ softmax classifier

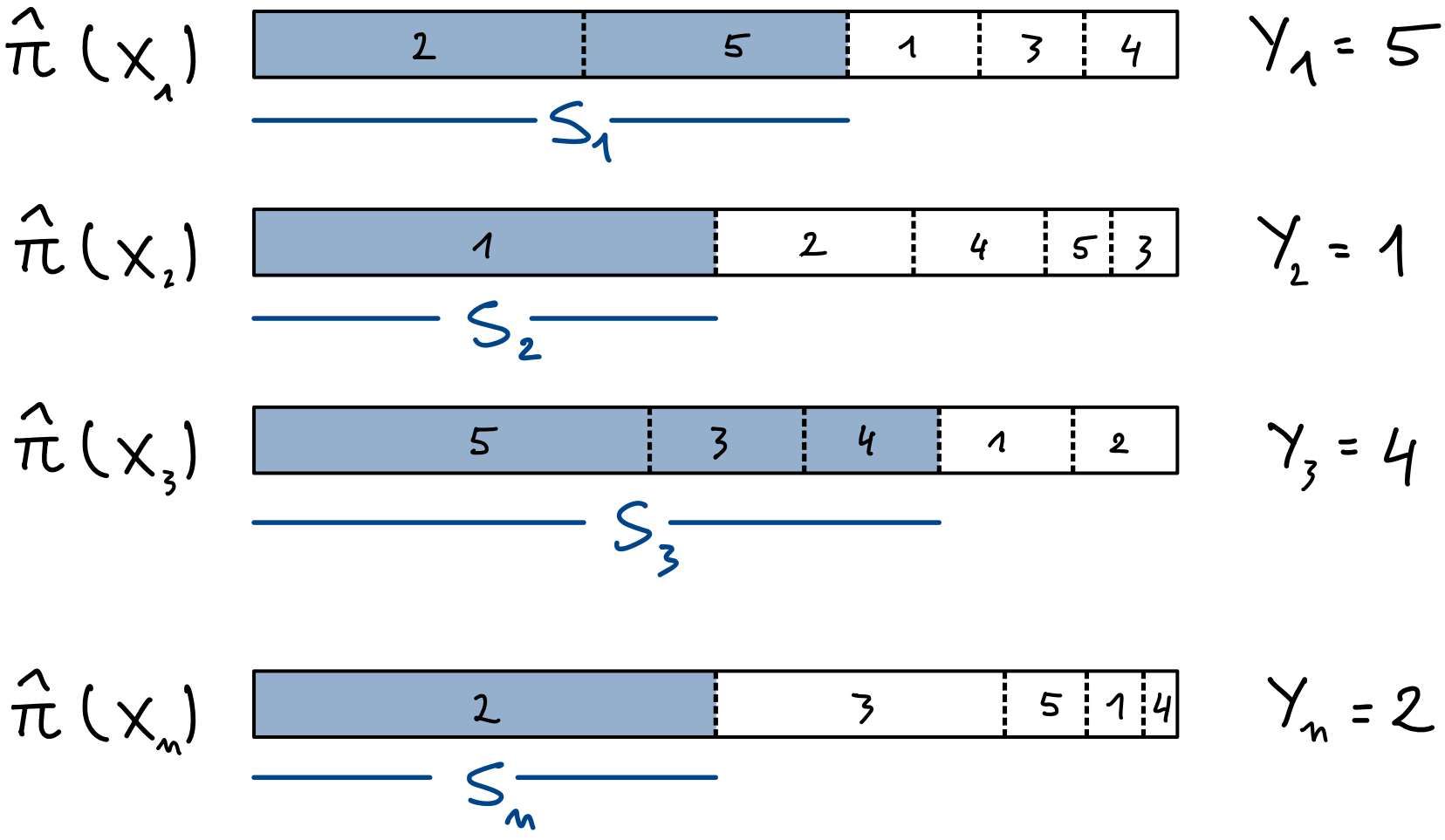
α nominal error rate

Rank: $\hat{\pi}_{(1)}(x) \geq \hat{\pi}_{(2)}(x) \geq \cdots \geq \hat{\pi}_{(K)}(x)$

$$L(x, \hat{\pi}, z) := \min_{c \in \{1, \ldots, K\}} \left\{ \hat{\pi}_{(1)}(x) + \cdots + \hat{\pi}_{c}(x) \geq z \right\}$$

APS

Calibration:



APS

Calibration:

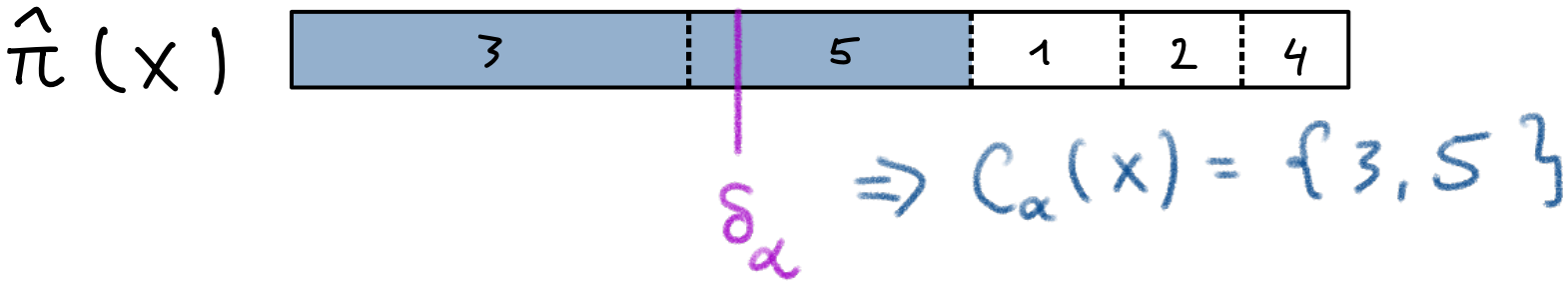
$$S_\alpha := \text{the } \left\lceil \frac{(n+1)(1-\alpha)}{n} \right\rceil \text{- th}$$

quantile of the scores $S_1, \ldots, S_n$

APS

Inference: $\hat{C}_\alpha(x) = \{(1), (2), \ldots, (k)\}$

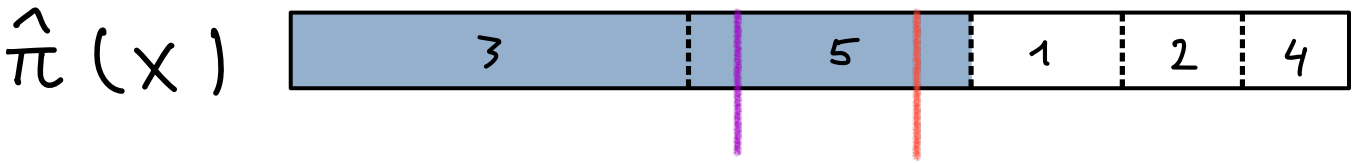
where $k = \min\{i : \hat{\pi}_{(1)} + \cdots + \hat{\pi}_{(i)} \geq \delta_\alpha\}$



Guarantee:

$$\mathbb{P}(Y_{n+1} \in \hat{C}_\alpha(X_{n+1})) \geq 1 - \alpha$$

APS WITH RANDOMIZATION



$$\Rightarrow \hat{\mathcal{C}}_\alpha(x) := \begin{cases} \{3\} & \text{if} \quad u \leq \delta_\alpha \\ \{3, 5\} & \text{if} \quad u > \delta_\alpha \end{cases}$$

Same guarantee, tighter prediction sets!

RAPS

Given : $\hat{\pi}$ softmax classifier

α nominal error rate

Rank: $\hat{\pi}_{(1)}(x) \geq \hat{\pi}_{(2)}(x) \geq \cdots \geq \hat{\pi}_{(\mathcal{Y})}(x)$

$$S_i := \hat{\pi}_{(1)} + \cdots + \hat{\pi}_{(k)} + \lambda(k - k_{reg} + 1)$$

k such that $(k) = Y_i$

rank of Y_i

hyper-parameters

RAPS

Prediction set:

$$\hat{\mathcal{C}}_\alpha(x) = \{(1), \dots, (k)\}$$

where

$$k = \max\{i : \pi_{(1)} + \dots + \pi_{(i)} + \lambda(i - k_{reg} + 1)^+ \leq \delta_\alpha\} + 1$$

More stable than APS if softmax is "noisy" i.e. many classes w. low proba.